PHL245 Notes

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Contents

1 Arguments

> Definition 1 **statement** : A *statement* is a truth-apt declarative sentence.

A declarative sentence is a one that states a fact, opinion, observation, or explanation. Sentences that are *not* declarative include interrogative sentences (which ask questions), imperative sentences (which give commands), and exclamatory sentences (which express strong emotions). Non-statements are useful in rhetorical persuasion, however serve no purpose in technical logic.

> Definition 2 **argument** : An *argument* is a collection of statements, partitioned into premises and a conclusion, such that the conclusion follows from the premises.

Deductive arguments are certain, meaning that whenever the premises hold, we know absolutely that the conclusion holds. Inductive and abductive arguments, however, are fallible.

> Definition 3 **validity & soundness** : An argument is *valid* if, whenever the premises are true, the conclusion is also true. An argument is *sound* when it is valid and its premises are true.

> Definition 4 **tautology** : A statement is a *tautology* when it is always true.

> Definition 5 **contradiction** : A statement is a *contradiction* when it is always false.

> Definition 6 **contingency** : A statement is *contingent* when there exist cases where it is true, and cases where it is false.

2 Semantics

2.1 Syntax

Definition 7 **logical connective** : We define a symbol to be a *logical connective* when it is an element of the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$. We furthermore call $\neg$ the *unary* connective, and the elements in $\mathcal{B} = \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ *binary* connectives.

A statement that has no logical connectives, and can therefore not be simplified any further, is called *atomic*. A statement that has logical connectives is called *molecular*.

Definition 8 **well formed formula** : A *propositional variable* is a symbol belonging to the set $\mathcal{P} = \{P, Q, R, ..., Z\}$ representing an atomic statement. With this definition, we may define the set $\mathcal{W}$ of *well-formed formulae* (WFF) inductively.

If $\mathcal{P}$ and $\mathcal{B}$ are the sets of propositional variables and binary connectives as defined above,
- (base) $\forall P \in \mathcal{P}$, P is a formula
- (unary) If φ is a formula, then $\neg\varphi$ is a formula
- (binary) If φ and ψ are formulae, then $(\varphi \cdot \psi)$ is a formula, where $\cdot \in \{\wedge, \vee \rightarrow, \leftrightarrow\}$

We may think of WFF as arrangements of our symbols in ways that make sense. Analogously, if we have a dictionary $\{\text{a}, \text{an}, \text{animal}, ...\}$ and a collection of punctuation $\{\ , ., !, ?, ...\}$, not all combinations of these elements will produce valid sentences.

Definition 9 **main connective** : Every molecular well-formed formula is by definition of form
- $\cdot\, \varphi$ for $\cdot \in \{\neg\}$; or
- $(\varphi \cdot \psi)$ for $\cdot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

We call $\cdot$ the *main connective* in either of these cases.

We may think of the well-formed formula as a parse tree by considering its structural definition. In this form, the main connective is the root node. In general, we may define sub-main connectives, sub-sub-main connectives, etc as internal nodes of level 2, level 3, etc. This way, each n-sub-main connective is in fact the main connective of an n-subtree.[1]

A formula is then parseable if and only if, for any subformula, there exists a clear main connective. But as parseability is how we evaluate whether a formula *makes sense*, the main connective is central to the understanding of formulae.

2.2 Semantics

We might notice that there is an excess use of parentheses within a well-formed formula. However, removal of these symbols runs the risk of ambiguity, as we can no longer identify the main connective. For instance, it is not immediately clear what the main connective of $P \wedge Q \rightarrow R$ is.

We may reduce the redundancy of parentheses by introducing *semantics* to formulae. We assign meaning to each symbol in order to provide a sensical heirarchy of connectives.

[1]Furthermore, all leaves are propositional variables.

Operator	Semantic Meaning	Order
$\leftrightarrow$	biconditional	3
$\rightarrow$	conditional[2]	3
$\wedge$	conjunction/and	2
$\vee$	disjunction/or	2
$\neg$	negation/not	1
$()$		0

Here, the highest order connective will implicitly become the main connective. So, in a formula such as $P \wedge \neg Q \rightarrow R$, we may read off $\rightarrow$ as the main connective.

We may further simplify formulae by asserting that $\wedge$ and $\vee$ are associative (indeed, in common language they are). Thus, there is no need for parentheses in chains of $P \wedge Q \wedge \ldots$ and $P \vee Q \vee \ldots$ *The main connective will be interpreted as the right-most operand in these chains.*

Using these rules, we may take some otherwise ambiguous formulae and parse them as we would well-formed formulae. It is worth noting that this does not cause *all* conceivable formulae to be parseable. For instance, $P \wedge Q \vee R$ still does not make sense. Indeed, any formula containing a level with two (non-associative) connectives of the same order cannot be parsed into a well-formed formula.

Definition 10 **official notation** : A formula φ is in *official notation* when $\varphi \in \mathcal{W}$. We say φ is instead in *informal* notation when $\varphi \notin \mathcal{W}$ but

1. φ contains no parentheses around unary connectives or propositional variables; and
2. φ can be parsed into a WFF by the heirarchy of connectives and associativity of $\wedge$ and $\vee$.

In this case, we also say φ is a *well-formed symbolic sentence.*

Example. We may simplify the formula $(((P \wedge Q) \wedge R) \wedge S)$ to $P \wedge Q \wedge R \wedge S$ by associativity.

Example. The formula $Q \rightarrow R \rightarrow S$ is not well formed.

Note 11 : From this point forward, all references to any formula φ will assume that φ is parseable, so that it is by extension either in official notation or in informal notation.

2.3 Truth Tables

We consider a formula's semantics to be defined when the formula has a truth value associated to each combination of truth values assigned to its propositional variables.

Definition 12 **truth value assignment (TVA)** : A *truth value assignment* to a formula φ is an assignment of truth values to the atomic statements, or propositional variables, of φ. It corresponds to a single row in the truth table of φ.

Definition 13 **truth table** : The *truth table* for a formula φ is a table listing all TVAs of φ, along with the truth value of φ for each TVA.

[2] Here, we call the left operand the *antecedent*, and the right operand the *consequent*.

Theorem 14 : There exist 2^n possible TVAs for a formula composed of n propositional variables.

In discussing truth tables, we now see that formulae are truth-apt and therefore sentences. We may therefore translate our prior definitions for statements. If φ is a formula, then

- φ is a *tautology* if φ is true on every TVA
- φ is a *contradiction* if φ is false on every TVA
- φ is *contingent* if there exists a TVA for which φ is true, and a TVA for which φ is false.

In fact, because statements are by definition truth-apt, they have corresponding truth tables in some collection of propositional variables. It can be shown (via CNF/DNF, for instance) that for every truth table T there exists a formula satisfying T. From this it is clear that all statements are also formulae. *With this we have established an equivalence between statements and formulae.*

Definition 15 **consistency** : A set of statements $\{\varphi, \psi, \ldots\}$ is *consistent* when there is at least one TVA for which all statements in the set are true. The set is *inconsistent* otherwise.

Definition 16 **logical equivalence** : The statements in a set $\{\varphi, \psi, \ldots\}$ of statements are *logically equivalent* when every statement has the same truth value on each TVA. That is, when the truth tables for all statements are equal.

We may thus translate our notions of *validity* also. Recall that an argument A is a collection of statements. We say A is valid when, for each TVA where all premises are true, the conclusion is true.

Note 17 **notation** : When a valid argument is composed of premise statements $\varphi, \psi, \ldots, \zeta$ and a conclusion $\mathcal{C}$, we denote the argument by $\varphi.\psi.\ldots.\zeta. \therefore \mathcal{C}$. The $\therefore$ symbol is read as "therefore." This is equivalent to writing $\varphi \land \psi \land \ldots \land \zeta \Rightarrow \mathcal{C}$.[3]

[3]Note that this is *different* from writing $\varphi \land \psi \land \ldots \land \zeta \to \mathcal{C}$. This latter expression is a formula and has its own truth table. It may or may not be the case that the formula evaluates to true for every TVA. However, when writing $\varphi \land \psi \land \ldots \land \zeta \Rightarrow \mathcal{C}$ or $\varphi.\psi.\ldots.\zeta. \therefore \mathcal{C}$, we are asserting that this argument is valid with certainty, and so the above formula *does* evaluate to true for every TVA.

3 Symbolization

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7 Multi-Place Symbolization in Predicate Logic

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9 Semantics in Predicate Logic