Welcome to
instats

The Session Will Begin Shortly
(At the top of the hour, Eastern USA time)

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Nonlinear Time Series Analysis, Part II: Modeling and Phenomenology

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Seminar Overview

- Day 1
  - Session 1 – Overview of Phenomenology
  - Session 2 – Dynamical Systems Analysis
- Day 2
  - **Session 3 – Sparse Identification of Nonlinear Dynamics**
  - Session 4 – Dynamic Mode Decomposition
- Day 3
  - Session 5 – Hidden Markov Models
  - Session 6 – Machine Learning Approaches
- Day 4
  - Session 7 – Putting it All Together: Lorenz
  - Session 8 – Putting it All Together: Infectious Diseases

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Sparse Identification of Nonlinear Dynamics

- Sparse Identification of Nonlinear Dynamics (SINDy)
  - Estimate derivatives
  - Construct a (large) library of possible functions
  - Regularized linear regression to choose functions from the library
- No theory needed, but in the end, there is an equation
  - Phenomenology
  - Fixed points

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SINDy

- All differential equations can be written like this:

$$\dot{x} = f(x)$$

  - *x* is a vector of *N* measurements
  - The trick: Let $\dot{x} = y$. Then $\ddot{x} = \dot{y}$ and we have a vector of *2N* measurements (*N* @ *x*, and *N* @ *y*)
- What is *f(x)*?
  - We don't know!
  - Create a library and let SINDy figure it out

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Derivative Estimation

- Lots of choices
- Important to stay as close to the original data as possible
  - Very important
- Computer science types like fourth-order centered finite differences
- GLLA, GOLD, FDA, Empirical Bayes change the 0^{th} order data.
- Regularized derivatives are great, provided…long and not too noisy

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Library of Functions

- What goes into the library?
  - We don't know!
- Choose all sorts of functions: polynomials (e.g., linear, quadratic, etc.), trigonometric, rational functions (e.g., (x+3)/x), etc., for a library
  - Hint: Much of the time, polynomials are sufficient to describe behaviors
- ~~Don't~~ Cross the streams
  - $\dot{x} = xy^2$
  - Multivariate is just like univariate, but longer

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Regularized Modeling

- Pushes towards sparsity in solution
- Remember: LASSO ("L_1 *norm*") returns some coefficients = 0
  - Coefficients = 0 removes terms (e.g., creates a "sparse" solution)
- Make the library large
  - Within reason...round-off errors accumulate...

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Example: Linear Oscillations

- 2nd order equation (simulate data)

$$m\ddot{x} + b\dot{x} + kx = 0$$

- Let $y = \dot{x}$

$$\dot{x} = y = f_x$$

$$\dot{y} = \frac{-by - kx}{m} = f_y$$

- Notice: solution should have only linear terms

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Example: Linear Oscillations

- Try a 3rd order polynomial library for both equations

1	x	x^2	x^3	y^3
	y	y^2	x^2y	xy^2
		xy		

- Can investigate how well this works with noise, too.

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Fixed Points

- We have equations, so we can find fixed points
  - Set all the derivatives equal to zero
- Stability needs a *Jacobian*
  - Calculus
  - Numerically
- Not a calculus course, but for polynomials, it is doable
  - Remember: A polynomial library will often model things very well

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Derivatives of Polynomials

$$\frac{\partial(Ax^n)}{\partial x} = Anx^{n-1}$$

- Multiply by the exponent
- Drop the exponent by 1
- Leave everything else the same
- The ∂ is a “d” but it means “only use the variable in the numerator”

- Examples
  - $\frac{\partial(5x^6)}{\partial x} = 5 * 6x^{6-1} = 30x^5$
  - $\frac{\partial(2y^2x^3)}{\partial x} = 2y^2 3x^{3-1} = 6y^2x^2$

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Jacobian

- 1-d: Jacobian is a 1 x 1 matrix

$$J = \frac{\partial f}{\partial x}$$

- 2-d:

$$J = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{bmatrix}$$

- After calculus, evaluate the terms at the fixed-point values
  - Eigenvectors and eigenvalues after that

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Linear Oscillator

- Fixed point: (0,0)
  - Pick values of k, b, and m for the demonstration
- Calculus Jacobian: $J = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}$
  - Find the eigenvectors and eigenvalues
  - eigen(J)
- Numerical Jacobian
  - pracma::jacobian(f = c(f_x, f_y), x0 = c(x*, y*))

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Example: Lorenz SINDy

- Original Equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

- Try a polynomial library
  - We'll do 3rd order and 4th order

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Mayport SINDy

- Now here's one we have no idea about
  - Ran all the tests, however, in the first seminar of this series
- Again, try a polynomial library

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Non-polynomial Terms

- When should we include non-polynomial terms in the library?

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Latent SINDy

- Little work done so far
- Possible approaches
  - Multilevel (a la GMM)
  - K-means of coefficients
  - Genetic algorithms

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Questions

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STOP

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Next session @ UTC 1900

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