Explanation of rejection sampler for multiple groups based on Forbes & Mardia (2014)

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November 17, 2014

Introduction

In Forbes & Mardia (2014), a fast algorithm is applied to sample the concentration parameter κ from the posterior of a von Mises distribution. Forbes & Mardia (2014) employ a parameter, β_0, given by

$$\beta_0 = -\frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i - \mu).$$

Here, their methods will adapted, first to using summary statistics instead of using the data $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_n\}$ directly, and secondly to allowing for comparison of multiple groups.

The first adaptation will allow us to employ the conjugate prior by Guttorp & Lockhart (1988), which uses the summary statistics. The second will allow us to apply the algorithm in more typical research scenarios.

Summary statistics

Some common circular summary statistics of dataset $\boldsymbol{\theta} = \{\theta_1, \cdots, \theta_n\}$, given by Fisher (1995), are

$$C = \sum_{i=1}^{n}\cos(\theta_i), \quad S = \sum_{i=1}^{n}\sin(\theta_i)$$

$$R = \sqrt{C^2 + S^2}, \quad \bar{R} = R/n$$

$$\bar{\theta} = \begin{cases} \tan^{-1}(S/C) & \text{if } C > 0, S > 0 \\ \tan^{-1}(S/C) + \pi & \text{if } C < 0 \\ \tan^{-1}(S/C) + 2\pi & \text{if } C > 0, S < 0 \end{cases}$$

$$\cos(\bar{\theta}) = C/R, \quad \sin(\bar{\theta}) = S/R$$

Note that when modeling using a von Mises distribution, either $\{C, S\}$ or $\bar{\theta}$ are sufficient statistics for μ, while $\bar{R}$ is sufficient for κ.

Using summary statistics

In Forbes & Mardia (2014), β_0 is a discrepancy measure that captures all information for the current sample of κ to be drawn. In terms of $\bar{R}$ and $\bar{\theta}$, it is given by

$$\begin{aligned}
\beta_0 &= -\frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i - \mu) \\
&= -\frac{1}{n}\left(\sum_{i=1}^{n}\cos\theta_i\cos\mu + \sin\theta_i\sin\mu\right) \\
&= -\frac{1}{n}\left(C\cos\mu + S\sin\mu\right) \\
&= -\frac{R}{n}\left(\frac{C}{R}\cos\mu + \frac{S}{R}\sin\mu\right) \\
&= -\bar{R}\left(\cos\bar{\theta}\cos\mu + \sin\bar{\theta}\sin\mu\right) \\
&= -\bar{R}\cos\left(\mu - \bar{\theta}\right).
\end{aligned}$$

Incorporating prior information

The previous result can be used to create a "posterior" version of β_0, say β_n, by combining this information with the conjugate prior given by Guttorp & Lockhart (1988). They denote the prior mean by μ_0, the prior resultant length by R_0, and a 'prior sample size' by c. A method for combination of the data is suggested by Damien & Walker (1999), and results in computing

$$C_n = R_0\cos\mu_0 + R\cos\bar{\theta}, \quad S_n = R_0\sin\mu_0 + R\sin\bar{\theta}$$

$$m = n + c$$

$$R_n = \sqrt{C_n^2 + S_n^2}, \quad \bar{R_n} = R_n/m$$

$$\mu_n = \begin{cases} \tan^{-1}(S_n/C_n) & \text{if } C_n > 0, S_n > 0 \\ \tan^{-1}(S_n/C_n) + \pi & \text{if } C_n < 0 \\ \tan^{-1}(S_n/C_n) + 2\pi & \text{if } C_n > 0, S_n < 0 \end{cases}$$

These can simply be plugged into the result of the previous section to obtain

$$\beta_n = -\bar{R_n} \cos\left(\mu - \mu_n\right).$$

Then, the algorithm as provided by Forbes & Mardia (2014) can be applied, where β_n is used in place of β_0, and $\eta = m$.

Incorporating prior information and multiple groups

Note that β_0 is a discrepancy measure, which finds the distance between some current mean μ and the information about the location and spread in the posterior, as captured in μ_n, R_n, and m. This discrepancy is then averaged over the total number of observations. For multiple groups, we can calculate this discrepancy measure for each group separately, and average over the total number of observations. Denote the final, combined β_0 by β_t. Adding subscripts j to denote the respective parameters μ, μ_n, R_n and m for each group, it is found by

$$m_t = \sum_{j=1}^{J} m_j,$$

$$\beta_t = -\frac{1}{m_t} \sum_{j=1}^{J} R_{nj} \cos(\mu_j - \mu_{nj}).$$

Then, again, the algorithm as provided by Forbes & Mardia (2014) can be applied, where β_t is used in place of β_0, and $\eta = m_t$.

References

Damien, P., & Walker, S. (1999). A full Bayesian analysis of circular data using the von mises distribution. *Canadian Journal of Statistics*, *27*(2), 291–298.

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