Extending Bayesian analysis of circular data for comparison of multiple groups

Kees Mulder

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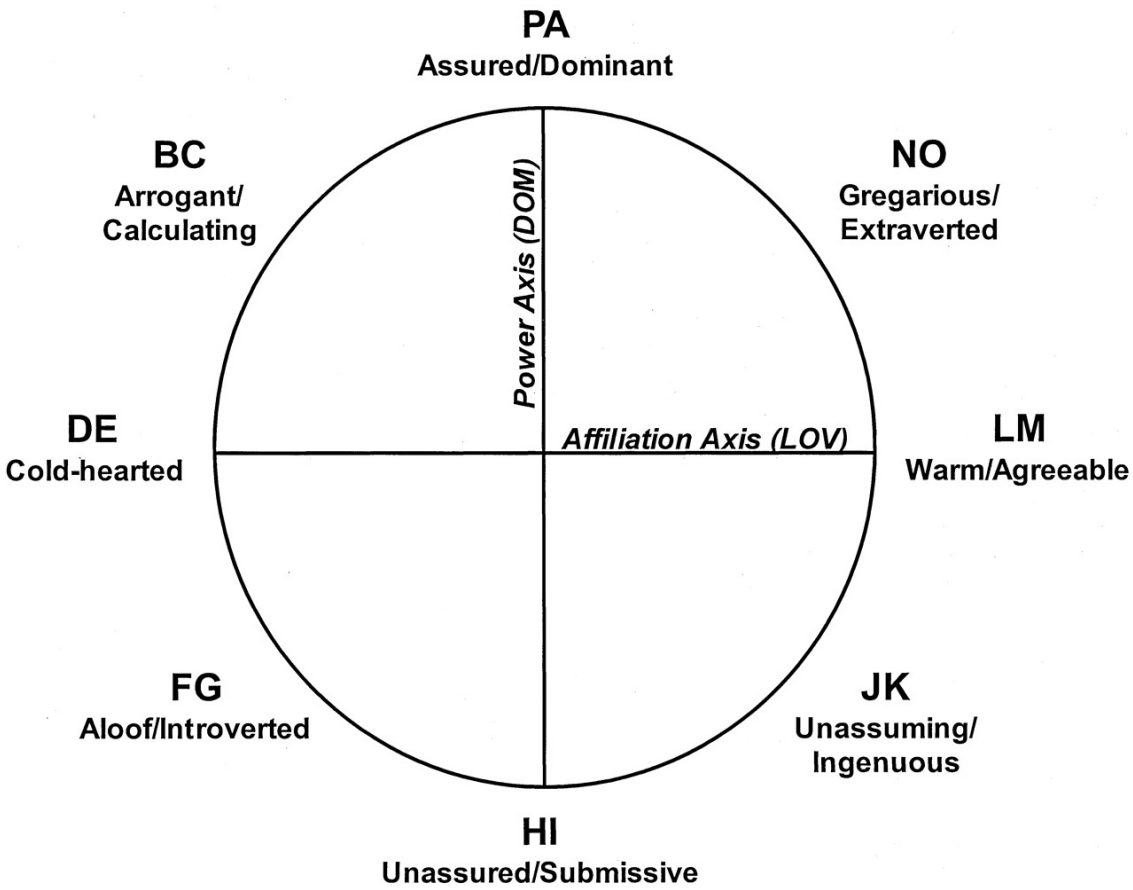

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	Score on Leary's rose			
Managers	46°	92°	102°	122°
Teachers	80°	47°	5°	355°

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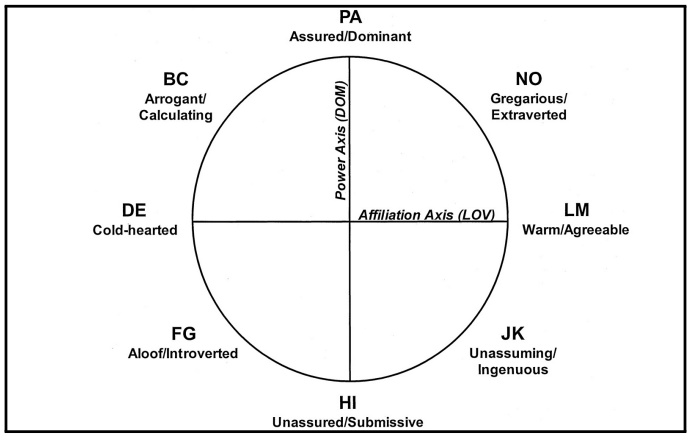

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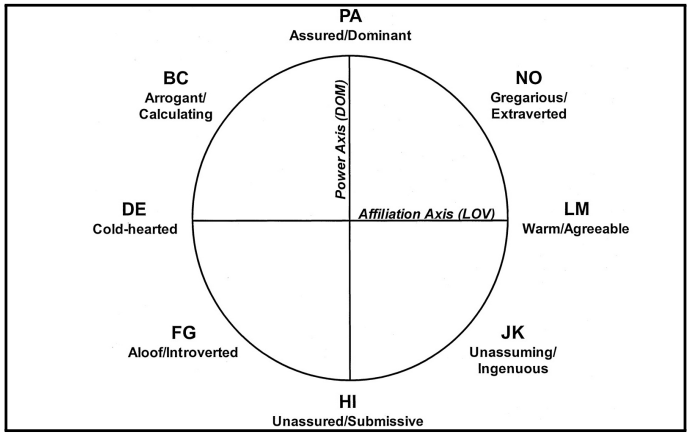

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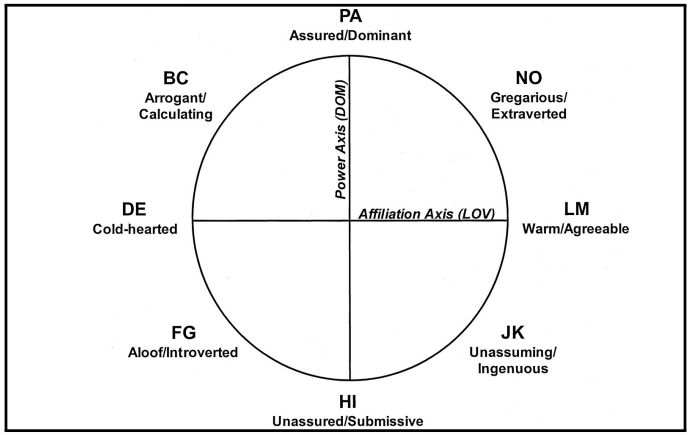

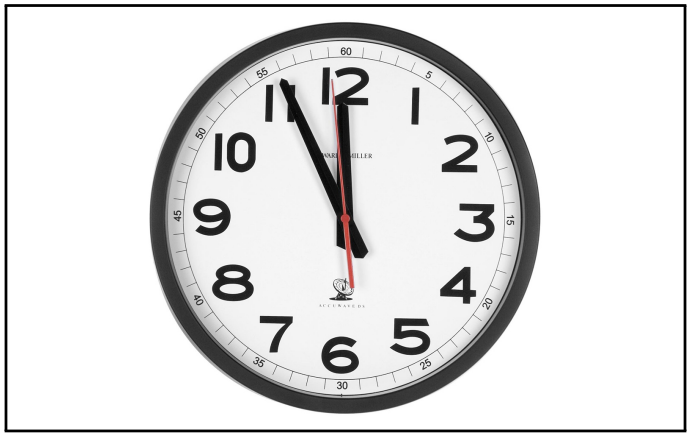

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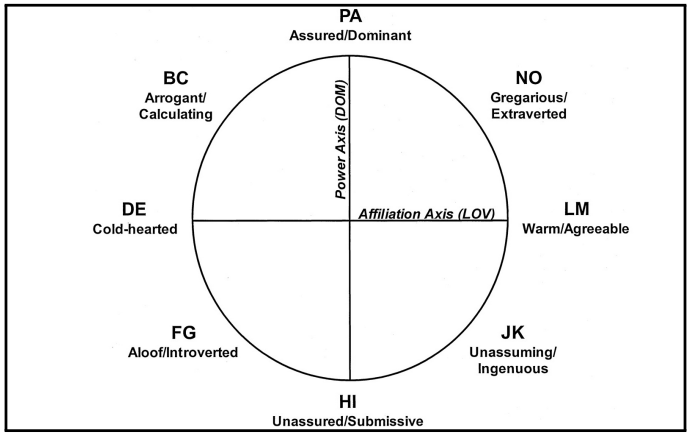

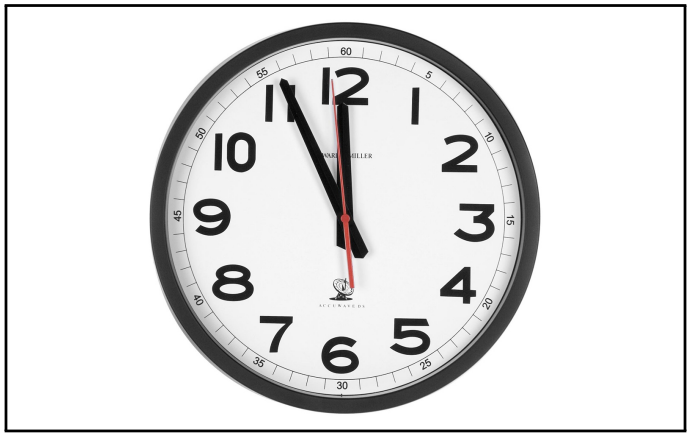

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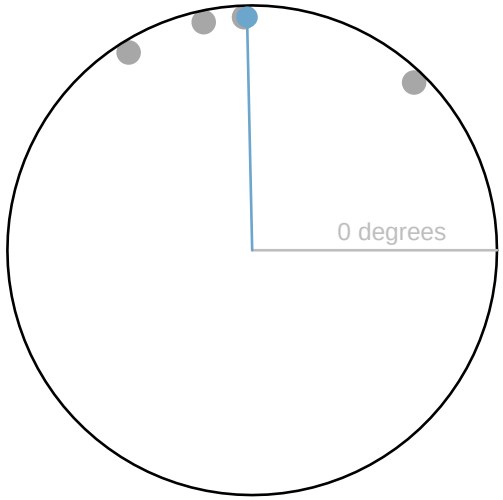
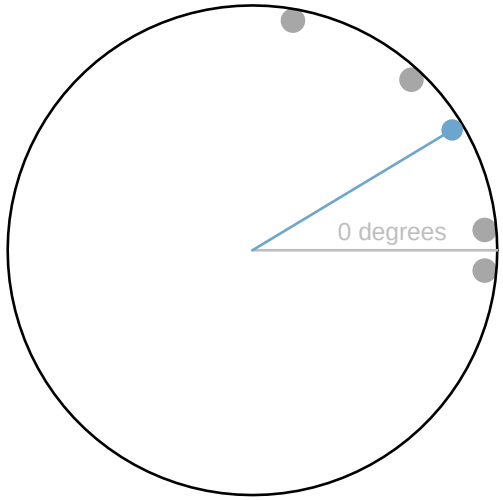
Managers



Mean direction = 91.262 degrees

Teachers



Mean direction = 31.072 degrees

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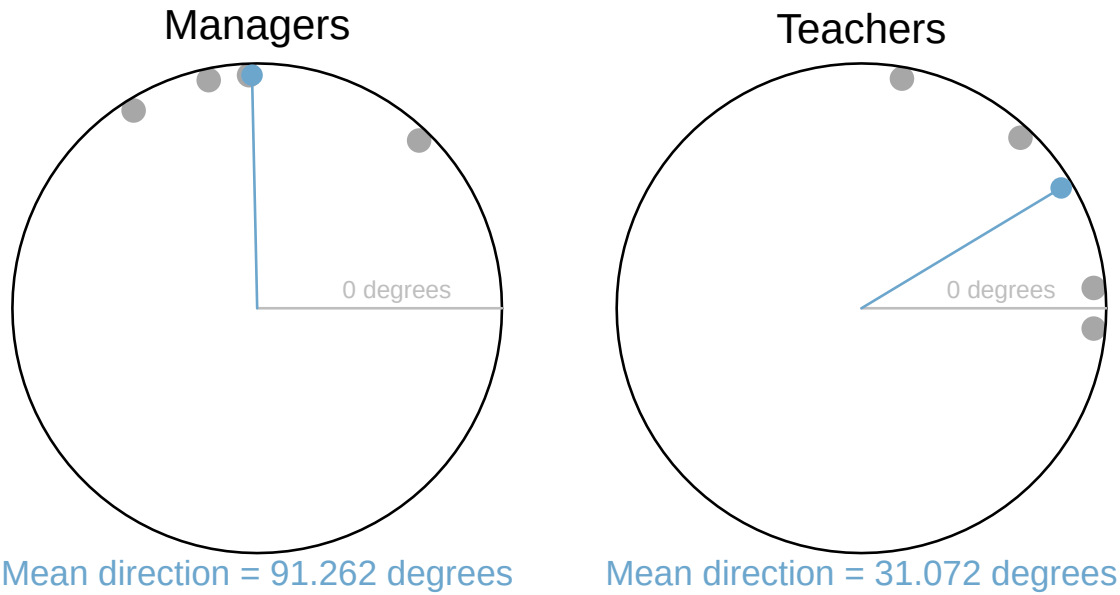

Difference with linear data

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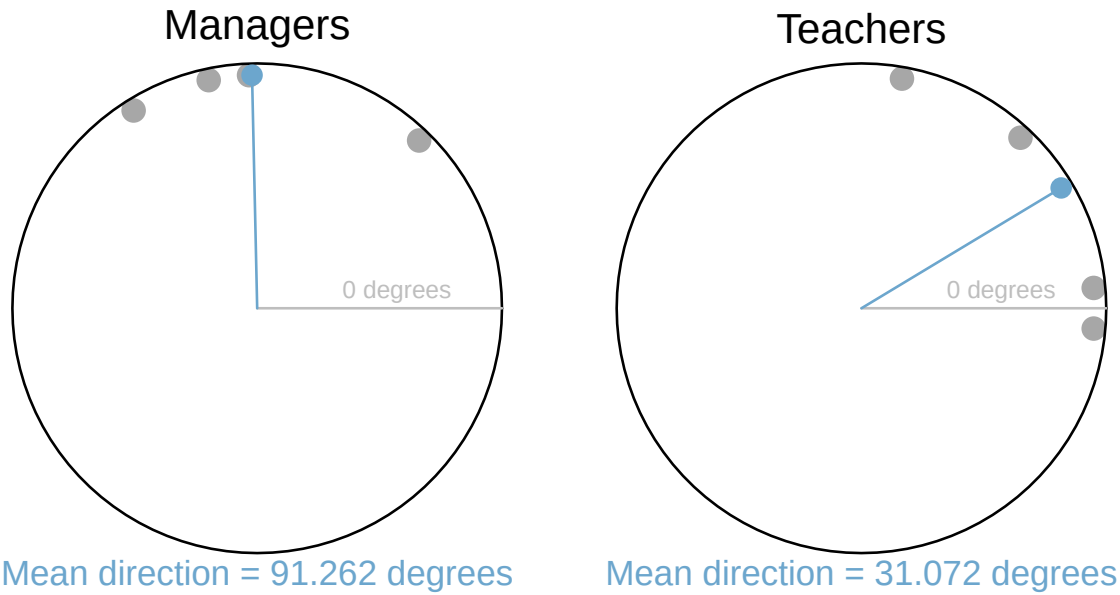

Difference with linear data

- Teacher with score 5°

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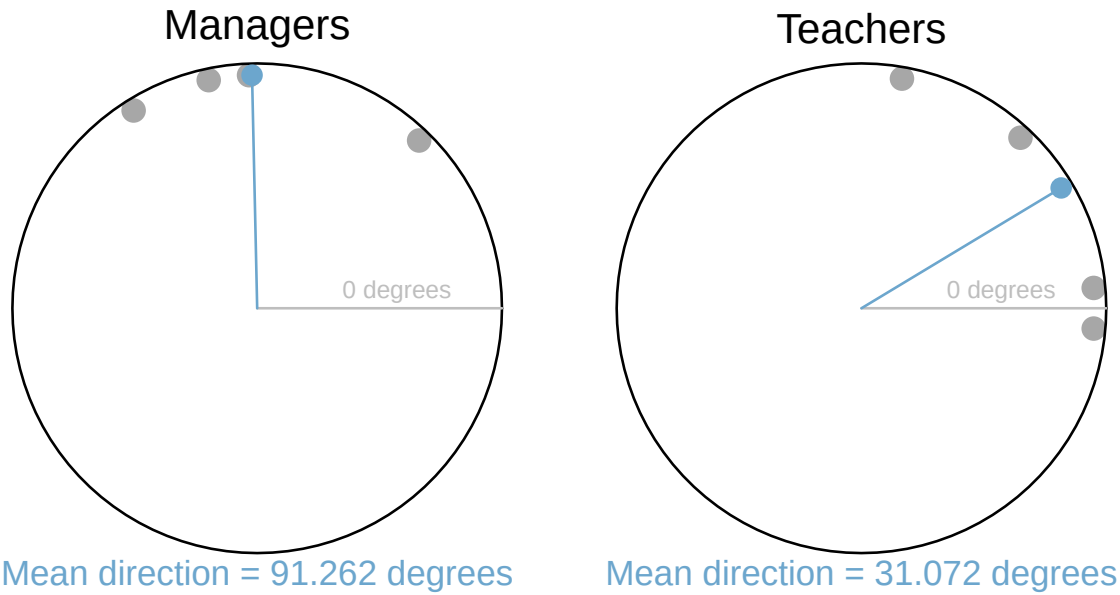

Difference with linear data

- Teacher with score 5°
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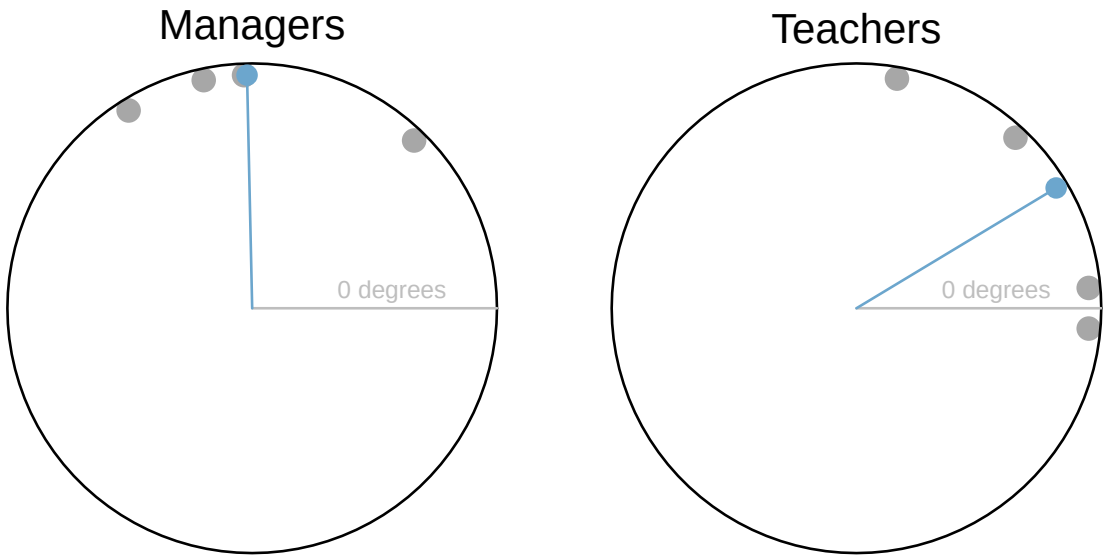

Difference with linear data

- Teacher with score 5°
- Teacher with score 355°
- Linear methods: difference of 350°

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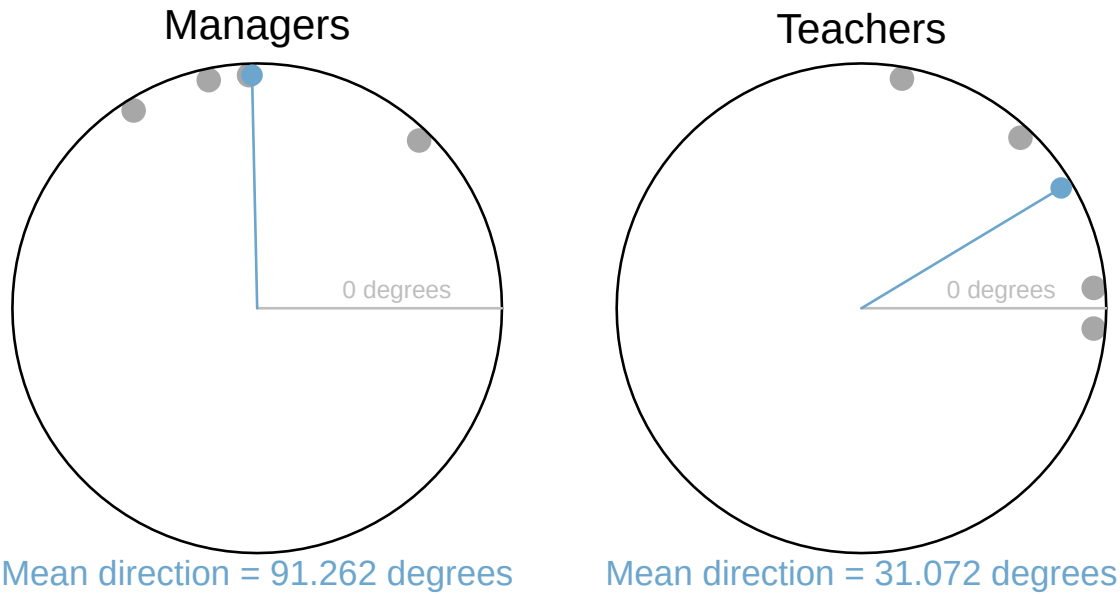

Difference with linear data

- Teacher with score 5°
- Teacher with score 355°
- Linear methods: difference of 350°
- Difference is really only 10°

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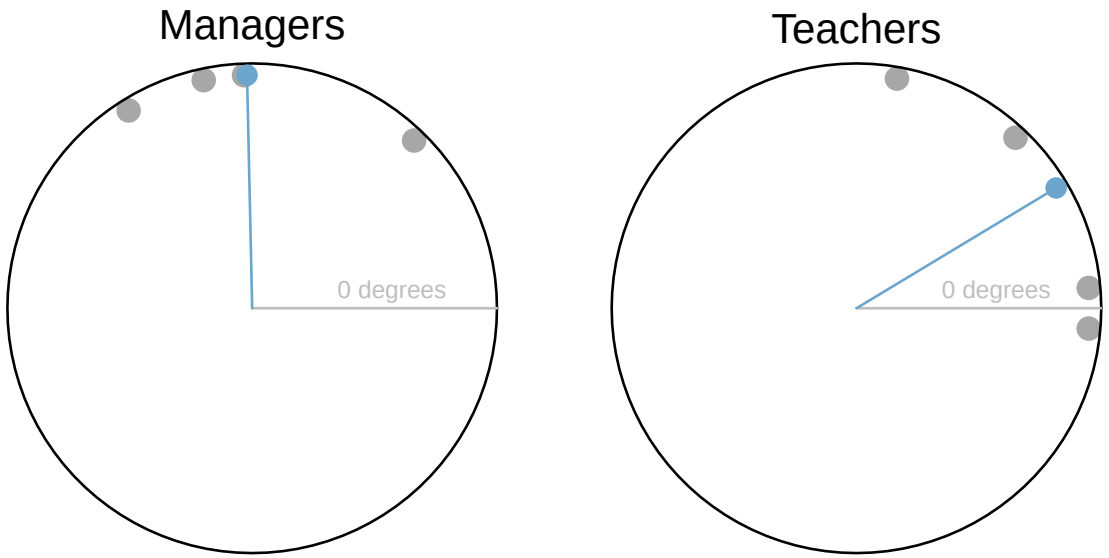

Difference with linear data

- Teacher with score 5°
- Teacher with score 355°
- Linear methods: difference of 350°
- Difference is really only 10°
- **Linear methods can not be used**

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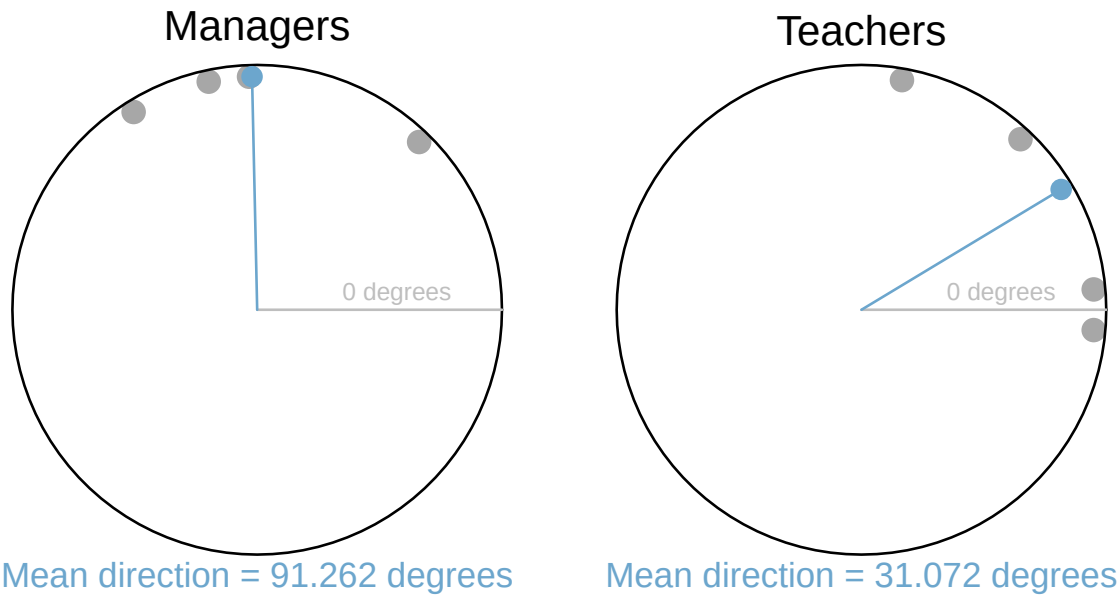

- Inherently difficult to analyse

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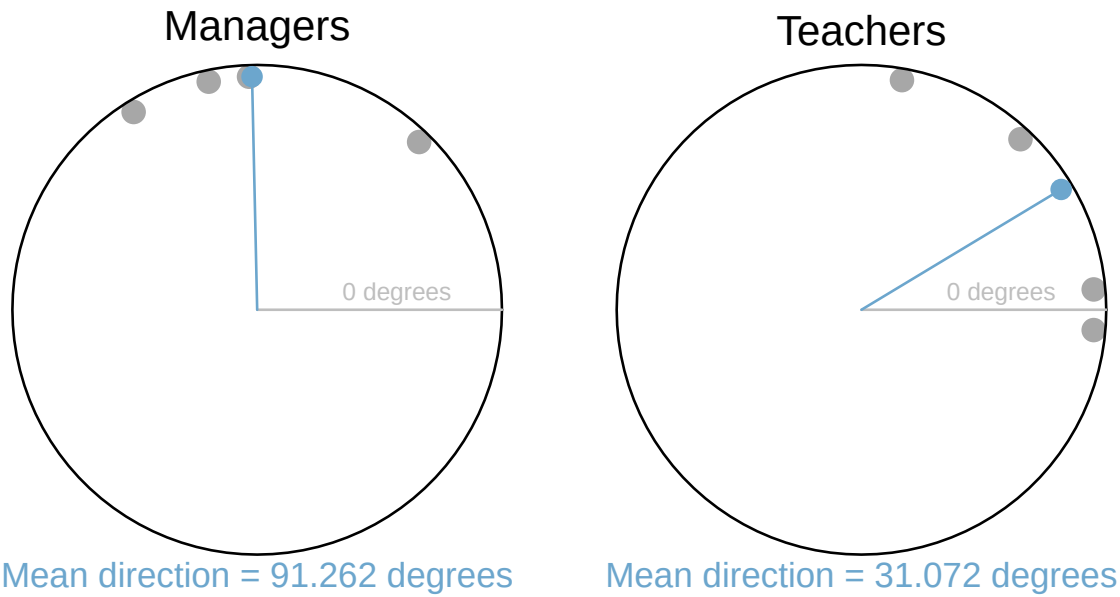

- Inherently difficult to analyse
- Bayesian methods may prove useful $\rightarrow$ flexible

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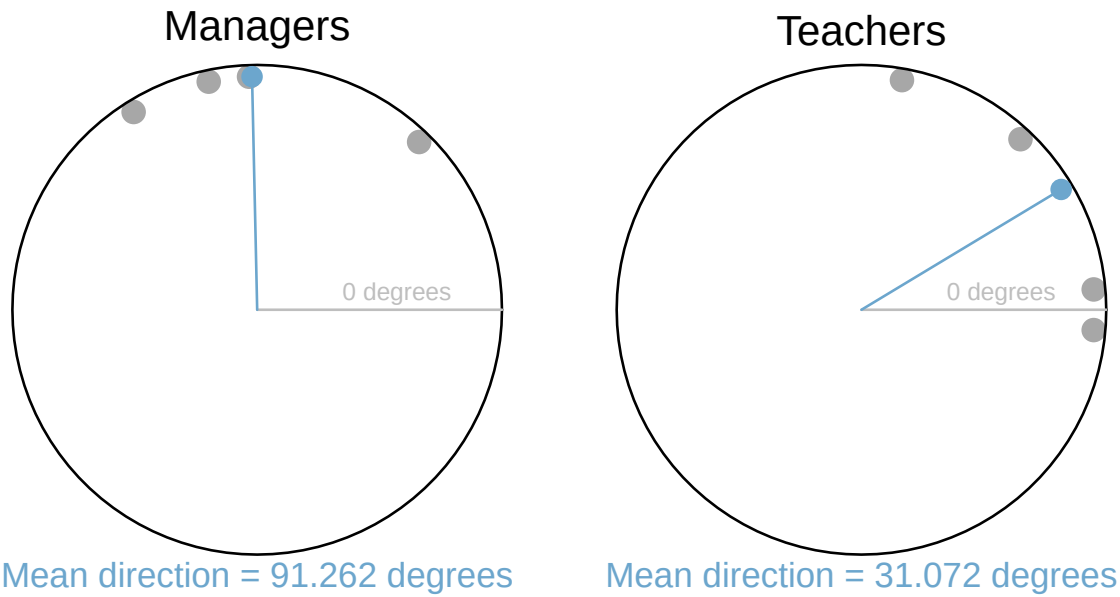

- Inherently difficult to analyse
- Bayesian methods may prove useful $\rightarrow$ flexible
- Three approaches are used

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Intrinsic Approach

- Define distributions on the circle

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- Sample space: $\mathbb{S}^1$

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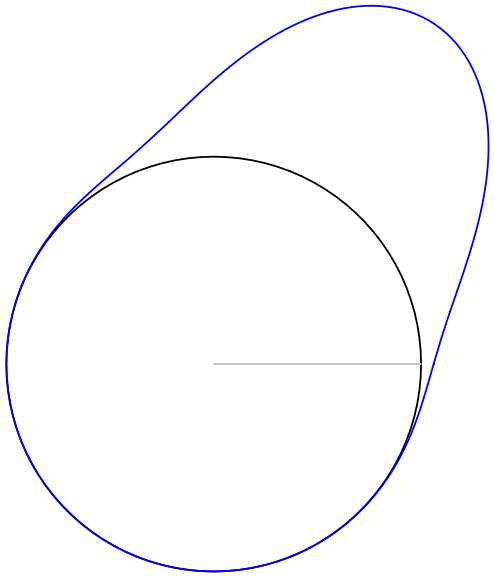

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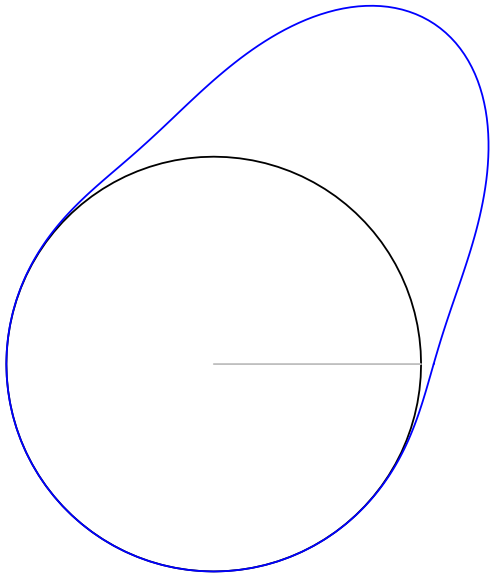

- Example: von Mises distribution

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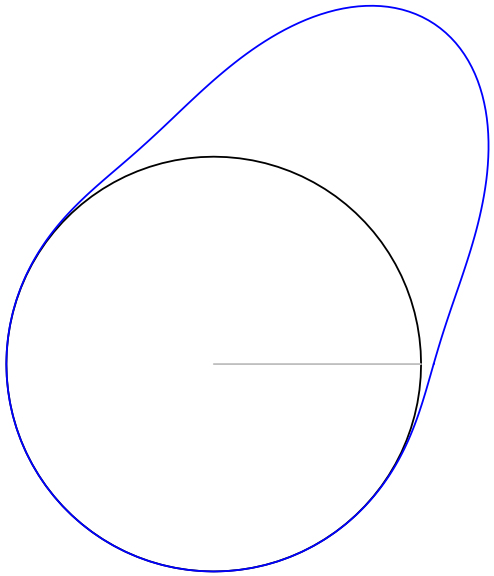

- Example: von Mises distribution

$$\text{VM}(\theta|\mu,\kappa) = \frac{\exp\{\kappa\cos(\theta-\mu)\}}{2\pi I_0(\kappa)}, 0 \leq \theta < 2\pi, \kappa \geq 0$$

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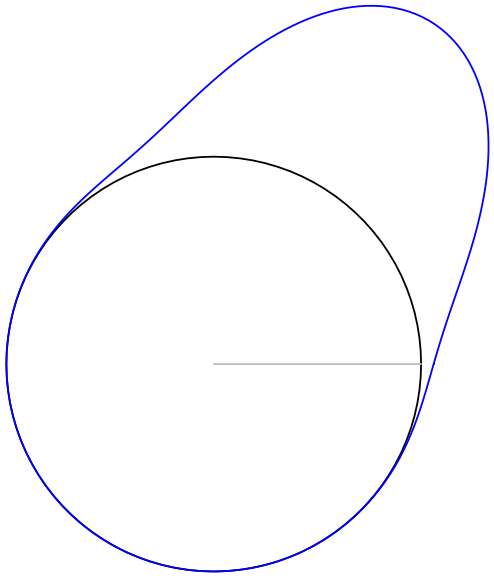

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Pro Straightforward

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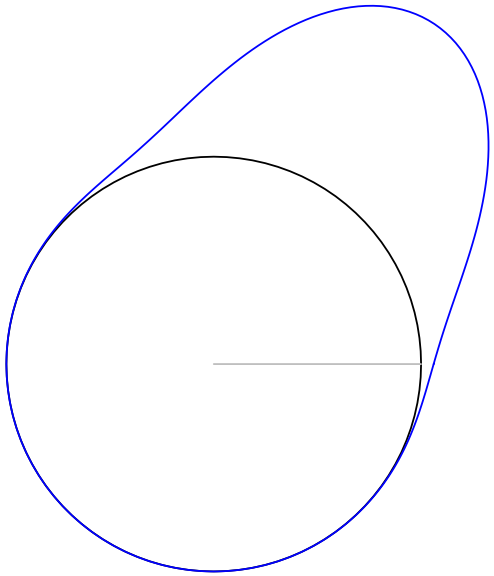

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Pro Straightforward

Con Bessel function

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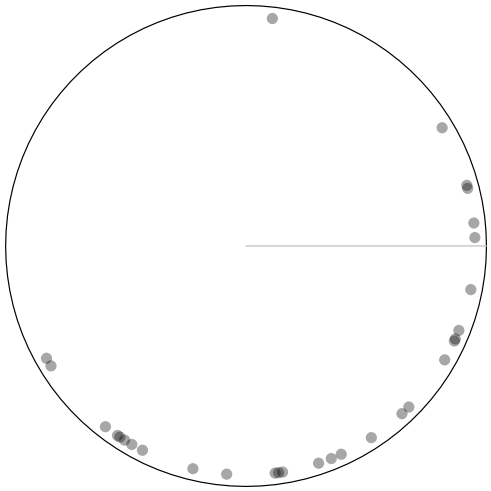
Wrapped Approach

- 'Wrap' distribution on the real line to the circle
- Map: $\mathbb{S}^1 \rightarrow \mathbb{R}^1$



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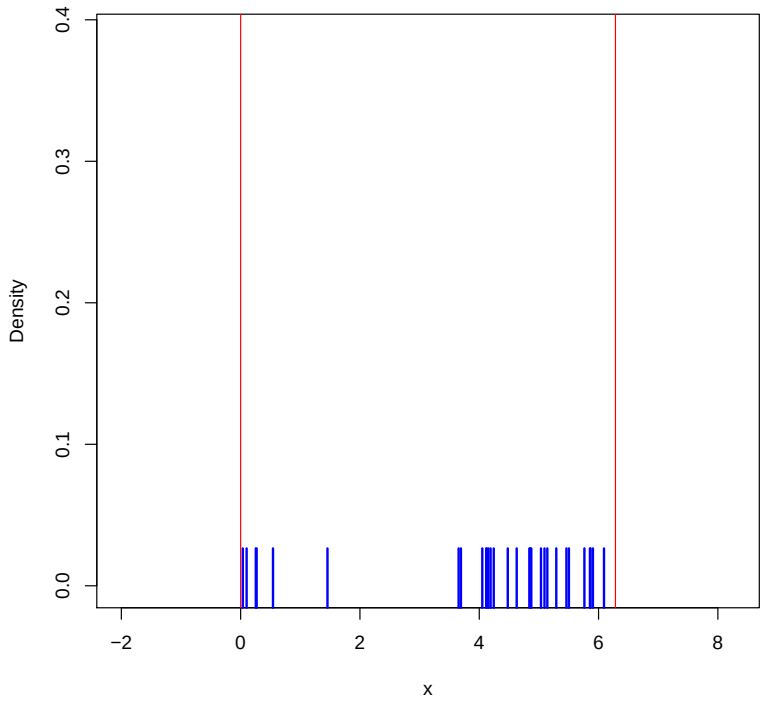

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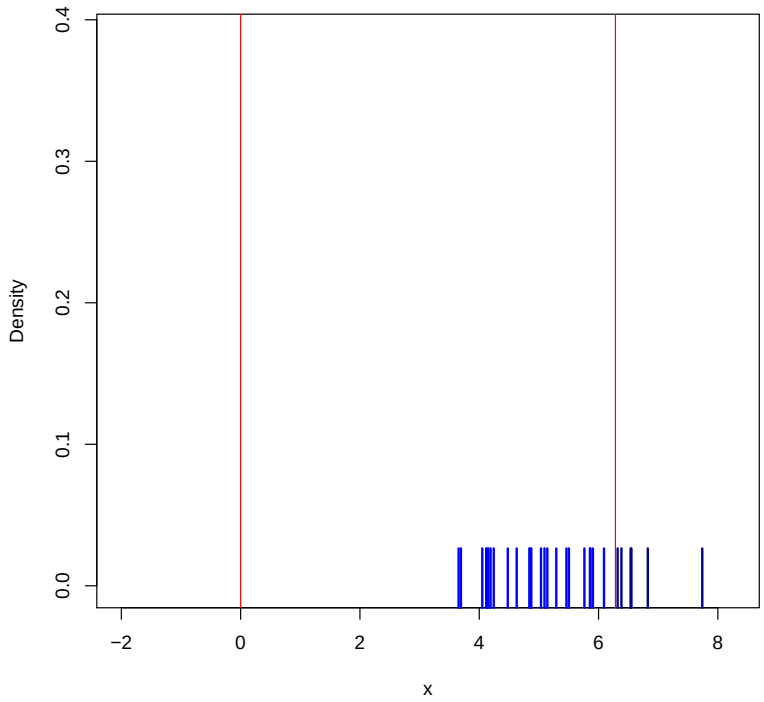

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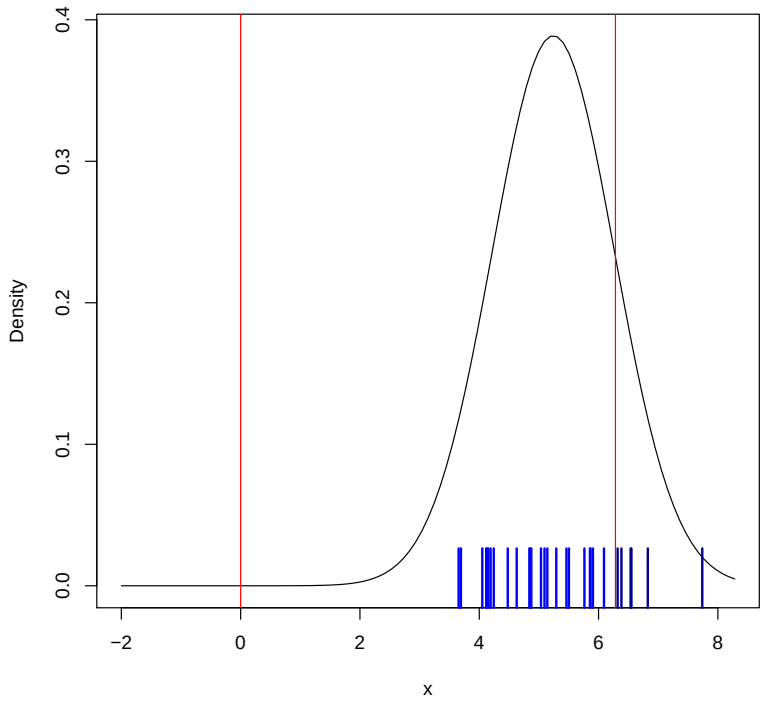

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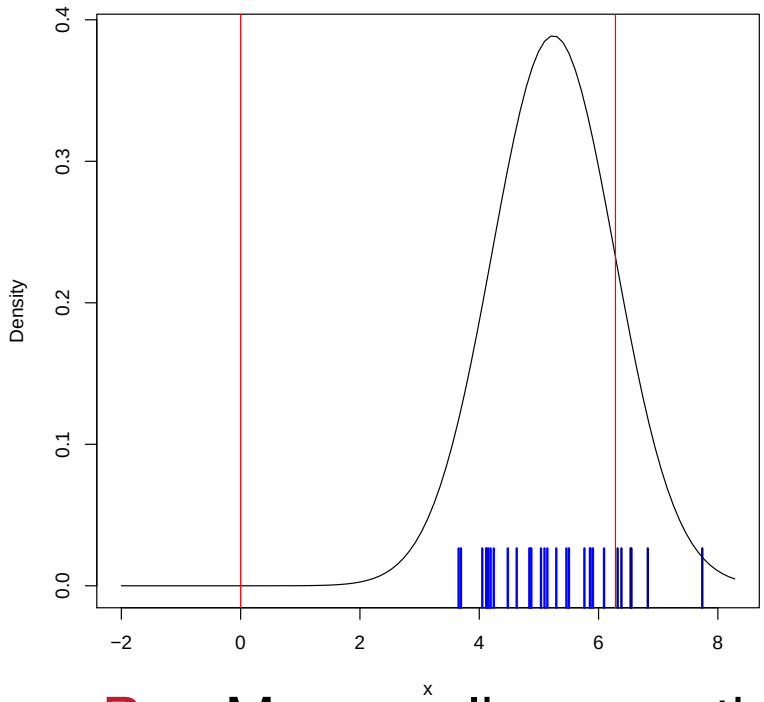

Pro May use linear methods

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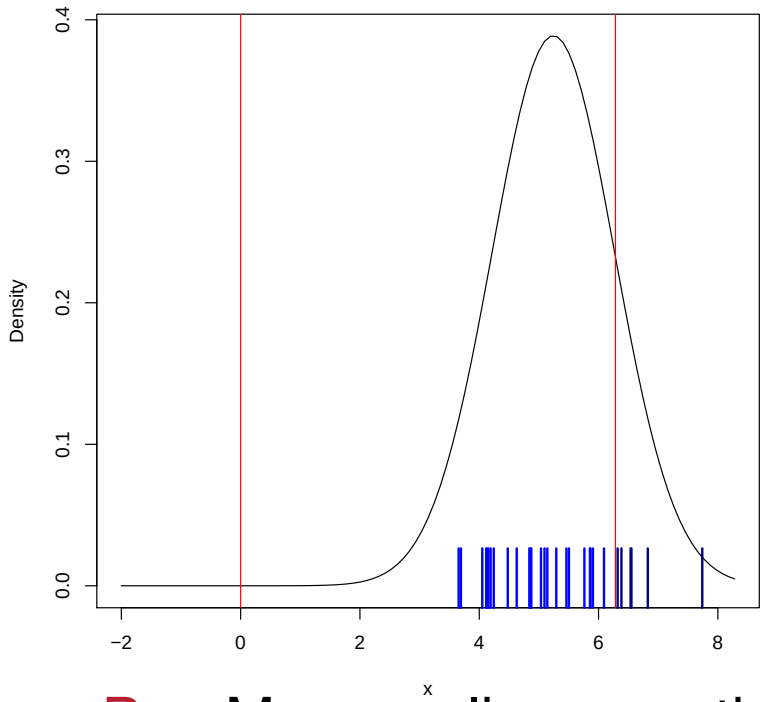

Pro May use linear methods

Con Wrapping process

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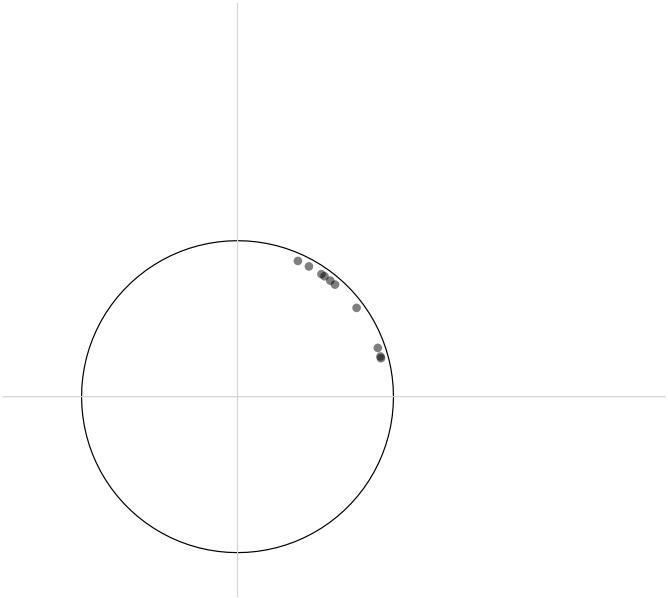
Embedding Approach

- 'Project' distribution in two-dimensional space to the circle
- Map: $\mathbb{S}^1 \rightarrow \mathbb{R}^2$



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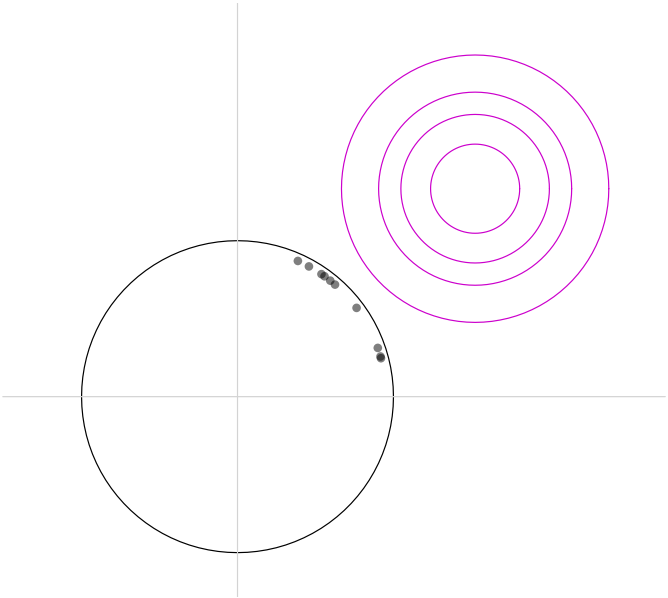

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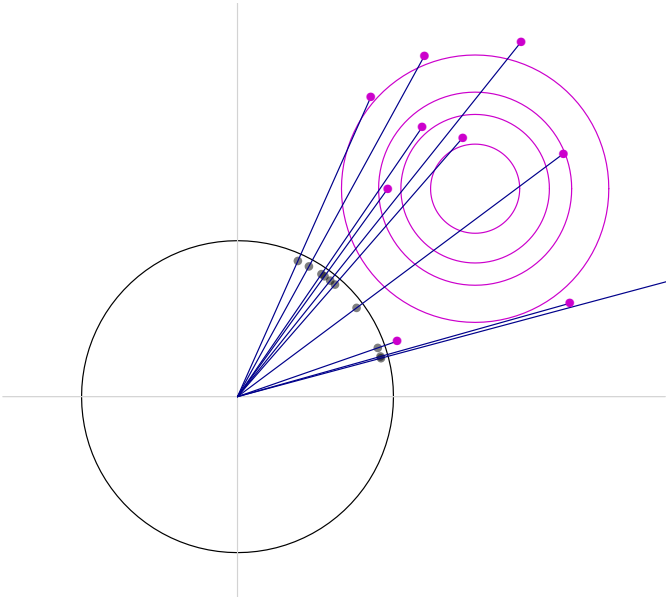

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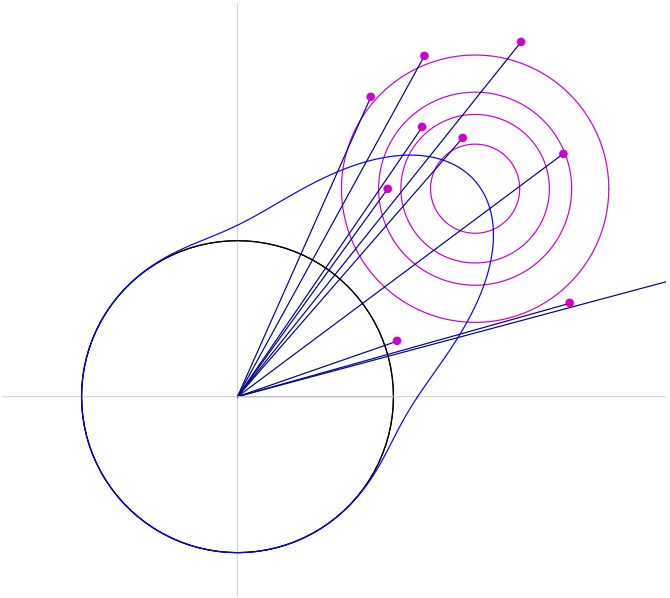

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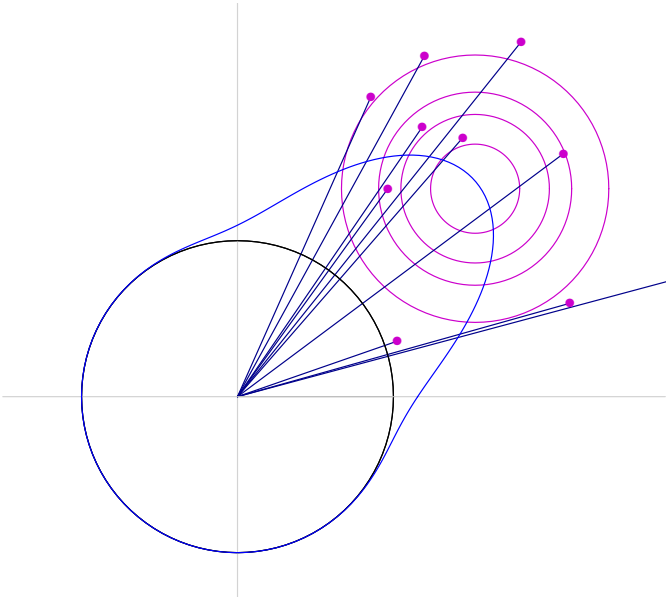

Pro May use bivariate linear methods

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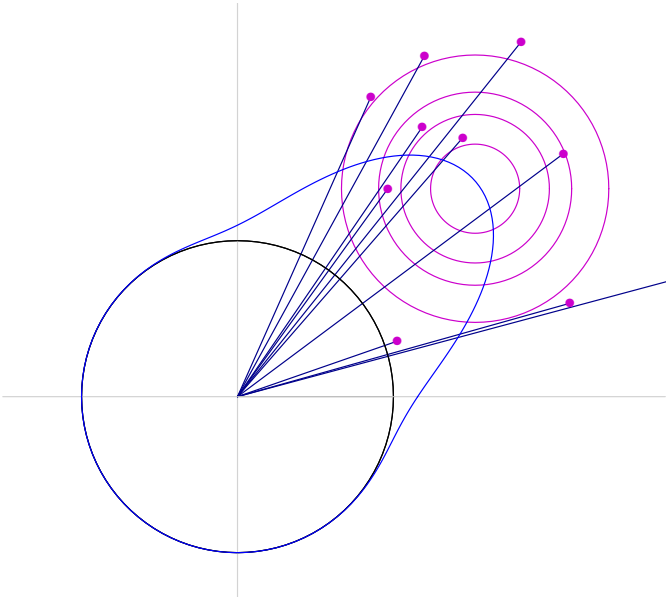

Pro May use bivariate linear methods

Con Complex, heteroscedasticity

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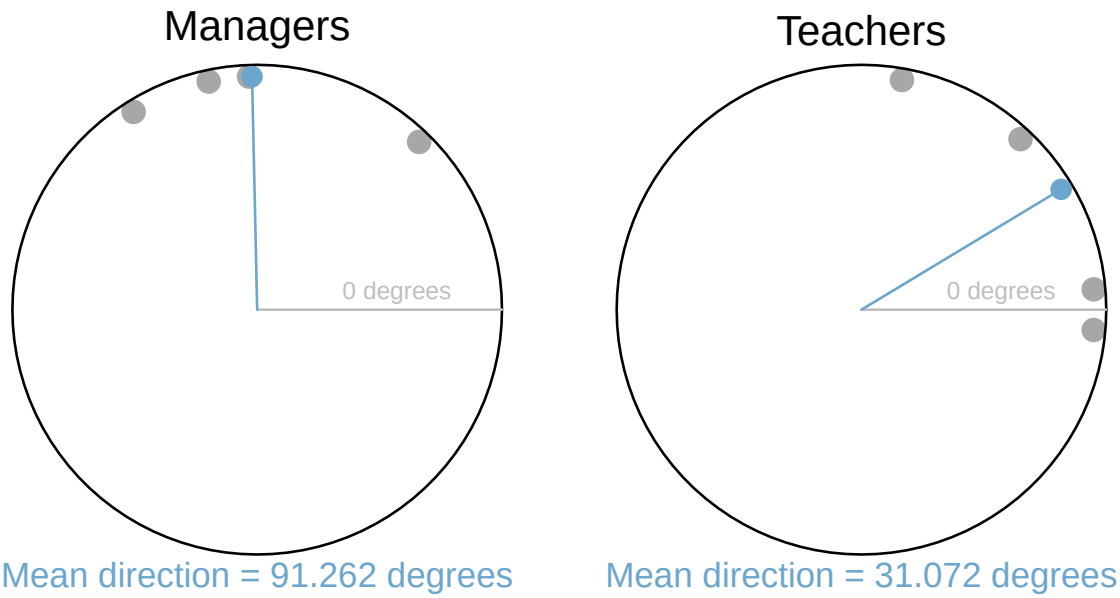
Which approach?

- Wrapping, embedding: practical solutions
- Intrinsic: most natural, direct



Intrinsic approach: what's missing?

- Methods analyze a single group of data

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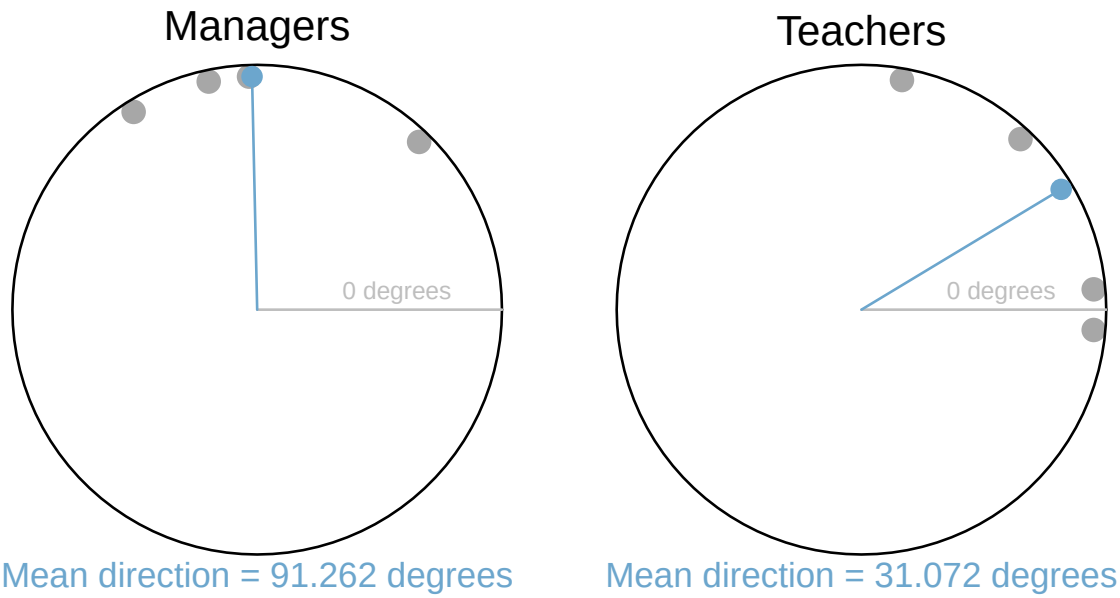

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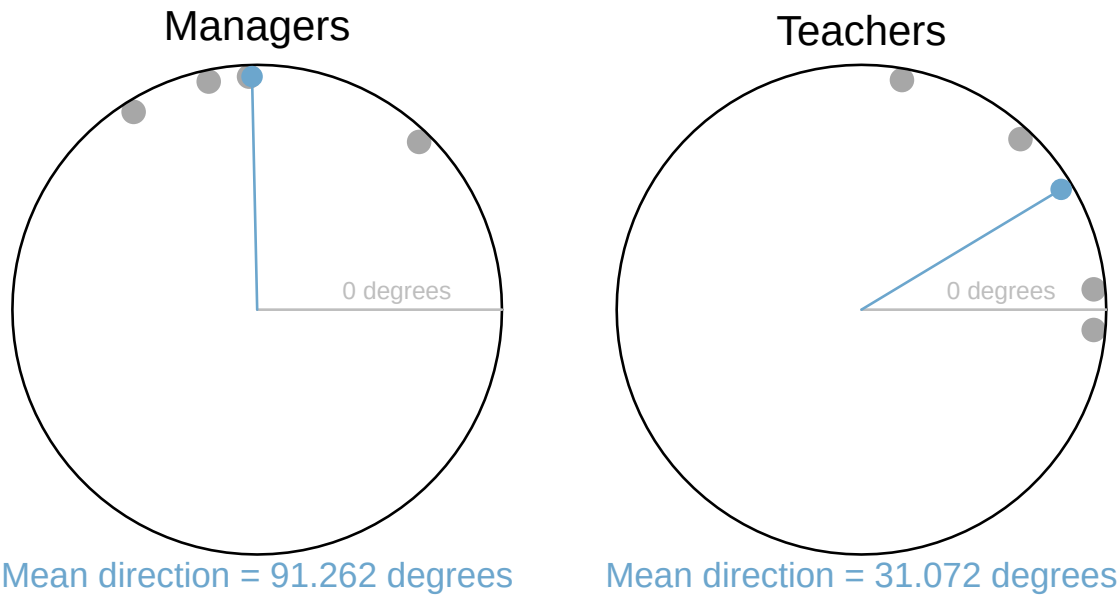

Intrinsic approach: what's missing?

- Methods analyze a single group of data
- We need **multiple groups** with **common variance**

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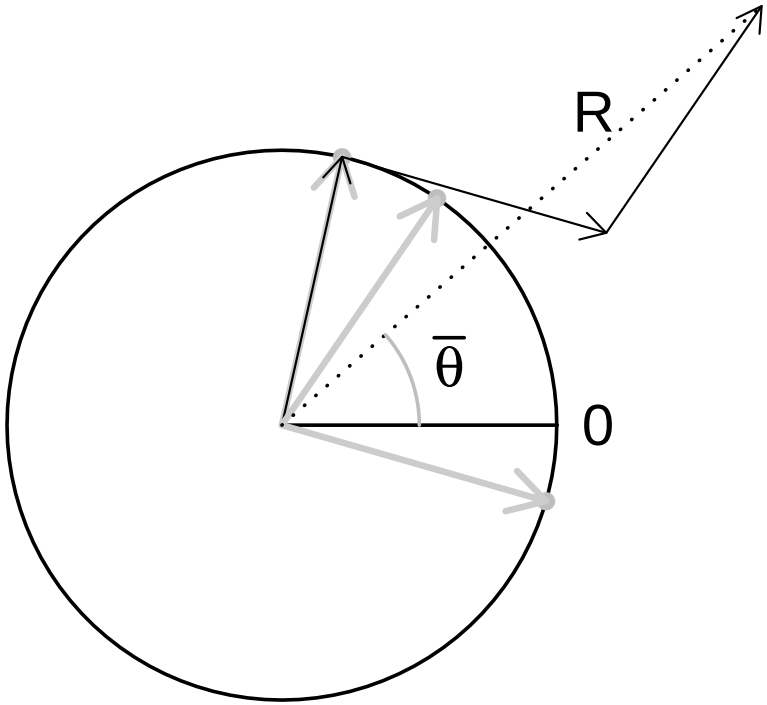
Previous methods

Method	Prior	Multiple groups
Gibbs sampler	✓	×
Rejection sampler	×	×

Extensions

Method	Prior	Multiple groups
Gibbs sampler	✓	✓
Rejection sampler	✓	✓
Metropolis-Hastings	✓	✓

Circular summary statistics



- $\bar{\theta}$: Unbiased estimate of μ
- R: Resultant length

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von Mises posterior

We can use a conjugate prior:

$$p(\mu, \kappa) \propto \frac{\exp\{R_0 \kappa \cos(\mu - \mu_0)\}}{I_0(\kappa)^c},$$

The posterior for multiple groups with a common κ is given by

$$f(\boldsymbol{\mu}, \kappa | \boldsymbol{\theta}) \propto \{I_0(\kappa)\}^{-m_t} \exp\left[\kappa \sum_{j=1}^{J} R_{nj} \cos(\mu_j - \mu_{nj})\right],$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_J)$ denotes the mean directions of the groups.

Gibbs sampler

- Damien & Walker, 2000
- Add latent variables w, v, x, and $u = (u_1, u_2, \dots)$ to the joint posterior density to obtain

$$f(\mu, \kappa, w, v, u, x|\theta) \propto$$
$$e^{-R_n\kappa} I(v < e^{R_n\kappa\{1+\cos(\mu-\mu_n)\}}, x < w^{m-1}) \times$$
$$\left(e^{-w} \prod_{k=1}^{\infty} I(u_k < e^{-w\lambda_k\kappa^{2k}}) \right),$$

- Fairly complex, contains sampling within Gibbs-step, which needs tuning
- Computationally intensive
- High autocorrelations

MH sampler

1. Draw each μ_j from $\text{VM}(\mu_j|\mu_{nj}, R_n\kappa_{cur})$.
2. Draw a candidate κ_{can} from $\chi^2(\kappa_{can}|\kappa_{cur})$.
3. Calculate the MH ratio as

$$\begin{aligned} a = & \ln f(\kappa_{can}|\boldsymbol{\mu}, \boldsymbol{\theta}) + \ln \chi^2(\kappa_{cur}|\kappa_{can}) \\ & - \ln f(\kappa_{cur}|\boldsymbol{\mu}, \boldsymbol{\theta}) - \ln \chi^2(\kappa_{can}|\kappa_{cur}). \end{aligned}$$

4. Draw a value u from $U(0, 1)$.
5. If $a > \ln u$, set $\kappa_{cur} = \kappa_{can}$.
6. Repeat

- Acceptance ratio mostly reasonable
- Computationally fast in most cases
- Some autocorrelation

Rejection sampler

- Forbes & Mardia, early 2014
  1. Draw each μ_j from $\text{VM}(\mu_j|\mu_{nj}, R_n\kappa_{cur})$.
  2. Calculate

$$\beta_t = -\frac{\sum_{j=1}^{J} R_{nj}\cos(\mu - \mu_{nj})}{m_t}.$$

  3. Tweak parameters of a Gamma proposal for κ using β_t
  4. Draw values from Gamma proposal until accepted
  5. Repeat
- Acceptance ratio great
- Computationally fast
- Low autocorrelation

Simulation study

- Simulation study comparing the methods
- Vary single or multiple groups, and spread

Results

All methods: some upwards bias in κ.

Method	**Performance**	**Ease of use**
Gibbs sampler	Bad	Complex
Metropolis-Hastings	Fairly good	Straightforward
Rejection sampler	Good	Complex

Conclusion

What's new?

- We can now compare groups of circular data using Bayesian analysis

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Up next: Extensions to more complex models

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