Visco-elastic Finite difference seismic modeling with simple and effective boundary conditions

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PRELIMINARY VERSION UNDER DEVELOPMENT. CONTAINS ERRORS.

Elastic equations of motion

Consider an elastic medium characterized by the density $\rho(\boldsymbol{x})$, and lamé parameters $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ at each spatial point $\boldsymbol{x}$. Now assume that $u(\boldsymbol{x})_i$ is the i'th component of the particle displacement and σ_{ij} is the stress tensor and c_{ijkl} is Hook's tensor. The elastic equations of motion and Hook's law in Cartesian coordinates are given as

$$
\begin{aligned}
\rho(\boldsymbol{x})\partial_t^2 u_i(\boldsymbol{x},t) &= \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), &(1)\\
\sigma_{ij}(\boldsymbol{x},t) &= c_{ijkl} e_{kl} + q_{ij}. &(2)
\end{aligned}
$$

Here f_i is a driving force, and q_{ij} is a driving stress. In the isotropic case one has

$$
c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{3}
$$

giving

$$
\begin{aligned}
\rho(\boldsymbol{x})\partial_t^2 u_i(\boldsymbol{x},t) &= \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), &(4)\\
\sigma_{ij}(\boldsymbol{x},t) &= \lambda(\boldsymbol{x}) e_{kk}\delta_{ij} + 2\mu e_{ij} + q_{ij}. &(5)
\end{aligned}
$$

Here f_i is the i'th component of the driving force, while the strain tensor e_{ij} is equal to

$$
e_{ij} = \frac{1}{2}[\partial_i u_j(\boldsymbol{x},t) + \partial_j u_i(\boldsymbol{x},t)]. \tag{6}
$$

Writing out the individual components of the equations above, one gets:

$$
\begin{aligned}
\rho\partial_t^2 u_x &= \partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz} + f_x,\\
\rho\partial_t^2 u_y &= \partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz} + f_y,\\
\rho\partial_t^2 u_z &= \partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz} + f_z,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\sigma_{xx} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{xx} + q_{xx},\\
\sigma_{yy} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{yy} + q_{yy},\\
\sigma_{zz} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{zz} + q_{zz},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xy} &= 2\mu e_{xy} + q_{xy}, \\
\sigma_{xz} &= 2\mu e_{xz} + q_{xz}, \\
\sigma_{yz} &= 2\mu e_{yz} + q_{yz}.
\end{aligned}
\tag{8}
$$

Viscoelastic equations of motion

Using the description in Appendix A, we can write down the viscoe-elastic equations of motion and the constitutive relation. In addition to visco-elastic stress-strain relation we have also introduced time-dependence for the density. The density then shows relaxation in the same way as a visco-elastic medium.

$$
\begin{aligned}
\partial_t^2 u_i(\boldsymbol{x},t) &= \rho_u^{-1}(\boldsymbol{x})\partial_j\sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), \\
&+ \chi(\boldsymbol{x},t) * \partial_j\sigma_{ij}(\boldsymbol{x},t), \qquad (9) \\
\sigma_{ij}(\boldsymbol{x},t) &= \lambda_u e_{kk}\delta_{ij} + 2\mu_u e_{ij} + q_{ij} \\
&+ \delta_{ij}\phi_\lambda(t) * e_{mm} + 2\phi_\mu(t) * e_{ij} \qquad (10)
\end{aligned}
$$

Writing out the individual components of the equations above, one gets:

$$
\begin{aligned}
\partial_t^2 u_x &= \rho_u^{-1}\left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right] + f_x, \\
&= +\chi * \left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right], \\
\partial_t^2 u_y &= \rho_u^{-1}\left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right] + f_y, \\
&= \chi * \left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right] \\
\partial_t^2 u_z &= \rho_u^{-1}\left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right] + f_z, \\
&= \chi * \left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right].
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\sigma_{xx} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{xx} + q_{xx} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{xx}, \\
\sigma_{yy} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{yy} + q_{yy} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{yy}, \\
\sigma_{zz} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{zz} + q_{zz} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{zz}, \\
\sigma_{xy} &= 2\mu_u e_{xy} + q_{xy} + 2\phi_\mu(t) * e_{xy}, \\
\sigma_{xz} &= 2\mu_u e_{xz} + q_{xz} + 2\phi_\mu(t) * e_{xz},
\end{aligned}
$$

$$\sigma_{yz} = 2\mu_u e_{yz} + q_{yz} + 2\phi_\mu(t) * e_{yz}. \tag{12}$$

Viscoelastic velocity-stress formulation

Using the velocity $v_i = \dot{u}_i$, one gets

$$\begin{aligned}
\partial_t v_x &= \rho_u^{-1}\left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right] + f_x,\\
&= +\chi * \left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right],\\
\partial_t v_y &= \rho_u^{-1}\left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right] + f_y,\\
&= \chi * \left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right]\\
\partial_t v_z &= \rho_u^{-1}\left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right] + f_z,\\
&= \chi * \left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right].
\end{aligned} \tag{13}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u\dot{e}_{xx} + \dot{q}_{xx},\\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] + 2\phi_\mu(t) * \dot{e}_{xx},\\
\dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u\dot{e}_{yy} + \dot{q}_{yy}\\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] + 2\phi_\mu(t) * \dot{e}_{yy},\\
\dot{\sigma}_{zz} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u\dot{e}_{zz} + q_{zz},\\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] + 2\phi_\mu(t) * \dot{e}_{zz},\\
\dot{\sigma}_{xy} &= 2\mu_u\dot{e}_{xy} + q_{xy} + 2\phi_\mu(t) * \dot{e}_{xy},\\
\dot{\sigma}_{xz} &= 2\mu_u\dot{e}_{xz} + q_{xz} + 2\phi_\mu(t) * \dot{e}_{xz},\\
\dot{\sigma}_{yz} &= 2\mu_u\dot{e}_{yz} + q_{yz} + 2\phi_\mu(t) * \dot{e}_{yz}.
\end{aligned} \tag{14}$$

Memory functions

We now define so-called memory variables by including the time convolution into one set of variables:

$$\gamma_\lambda^l(t) = \frac{1}{\Delta\lambda_l}\phi_\lambda^l * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] \tag{15}$$

$$\gamma_{ij}^l(t) = \frac{1}{\Delta\mu_l}\phi_\mu^l * [\dot{e}_{ij}], \tag{16}$$

$$\theta_{kij}^l(t) = \frac{1}{\Delta\rho_l^{-1}}\chi^l * \partial_k[\sigma_{ij}], \tag{17}$$

Here

$$\Delta\rho_l^{-1} = \rho_u^{-1}\left(1-\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}}\right), \tag{18}$$

$$\Delta\lambda_l = \lambda_u\left(1-\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}\right), \tag{19}$$

$$\Delta\mu_l = \mu_u\left(1-\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\rho}}\right). \tag{20}$$

This gives the expressions for the γ functions as:

$$\gamma_\lambda^l = \left[\frac{\exp(-t/\tau_{\sigma l}^{\lambda})}{\tau_{\sigma l}^{\lambda}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}}\right][\dot{e}_{xx}+\dot{e}_{yy}+\dot{e}_{zz}], \tag{21}$$

$$\gamma_{ij}^l = \left[\frac{\exp(-t/\tau_{\sigma l}^{\mu})}{\tau_{\sigma l}^{\mu}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}}\right]\dot{e}_{ij}, \tag{22}$$

$$\theta_{kij}^l = \left[\frac{\exp(-t/\tau_{\sigma l}^{\rho})}{\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}}}\right]\partial_k\sigma_{ij}. \tag{23}$$

and also

$$\begin{aligned}
\gamma_\lambda(t) &= \sum_{l=1}^{N}\gamma_\lambda^l,\\
\gamma_{ij}(t) &= \sum_{l=1}^{N}\gamma_{ij}^l,\\
\theta_{kij}(t) &= \sum_{l=1}^{N}\theta_{ij}^l.
\end{aligned}$$

This gives the final form of the viscoelastic equations

$$\begin{aligned}
\partial_t v_x &= \rho_i^{-1}(\partial_x\sigma_{xx}+\partial_y\sigma_{xy}+\partial_z\sigma_{xz})+f_x,\\
&+ \sum_{l=0}^{N}\theta_{xxx}^l\Delta\rho_l^{-1}+\sum_{l=0}^{N}\theta_{yxy}^l\Delta\rho_l^{-1}+\sum_{l=0}^{N}\theta_{zxz}^l\Delta\rho_l^{-1},\\
\partial_t v_y &= \partial_x\sigma_{yx}+\partial_y\sigma_{yy}+\partial_z\sigma_{yz}+f_y,\\
&+ \sum_{l=0}^{N}\theta_{xyx}^l\Delta\rho_l^{-1}+\sum_{l=0}^{N}\theta_{yyy}^l\Delta\rho_{ll}^{-1}+\sum_{l=0}^{N}\theta_{zyz}^l\Delta\rho_{ll}^{-1}.
\end{aligned}$$

$$
\begin{aligned}
\partial_t v_z &= \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z, \\
&+ \sum_{l=0}^{N} \theta^l_{xzx} + \sum_{l=0}^{N} \theta^l_{yzy} + \sum_{l=0}^{N} \theta^l_{zzz}.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz} \right) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx} \\
&+ \sum_{l=1}^{N} \gamma^l_\lambda \Delta\lambda_l + 2 \sum_{l=1}^{N} \gamma^l_{xx} \Delta\mu_l, \\
\dot{\sigma}_{yy} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz} \right) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy}, \\
&+ \sum_{l=1}^{N} \gamma^l_\lambda \Delta\lambda_l + 2 \sum_{l=1}^{N} \gamma^l_{yy} \Delta\mu_l, \\
\dot{\sigma}_{zz} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz} \right) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz} \\
&+ \sum_{l=1}^{N} \gamma^l_\lambda \Delta\lambda_l + 2 \sum_{l=1}^{N} \gamma_{zz} \Delta\mu_l, \\
\dot{\sigma}_{xy} &= 2\mu \dot{e}_{xy} + 2 \sum_{l=1}^{N} \gamma^l_{xy} \Delta\mu_l + \dot{q}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu \dot{e}_{xz} + 2 \sum_{l=1}^{N} \gamma^l_{xz} \Delta\mu_l + \dot{q}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu \dot{e}_{yz} + 2 \sum_{l=1}^{N} \gamma^l_{yz} \Delta\mu_l + \dot{q}_{yz},
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\dot{e_{xx}} &= \partial_x v_x, \\
\dot{e_{yy}} &= \partial_y v_y, \\
\dot{e_{zz}} &= \partial_z v_z, \\
\dot{e_{xy}} &= \frac{1}{2}(\partial_x v_y + \partial_y v_x), \\
\dot{e_{xz}} &= \frac{1}{2}(\partial_x v_z + \partial_z v_y), \\
\dot{e_{yz}} &= \frac{1}{2}(\partial_y v_z + \partial_z v_y).
\end{aligned}
\tag{26}
$$

Integration of memory functions

The memory functions obeys approximately the relations

$$
\begin{aligned}
\gamma_\lambda^l(t) &= \exp(-\frac{\Delta t}{\tau_\sigma^{\lambda l}})\gamma_\lambda^l(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}}\right)(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}),\\
\gamma_{ij}^l(t) &= \exp(-\frac{\Delta t}{N\tau_{\sigma l}^\mu})\gamma_{ij}^l(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}}\right)\dot{e}_{ij}.\\
\theta_{kij}^l(t) &= \exp(-\frac{\Delta t}{N\tau_{\sigma l}^\rho})\theta_{kij}^l(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\mu}}\right)\partial_k\sigma_{ij}.
\end{aligned}
\tag{27}
$$

Defining the quantities

$$\alpha_1^l = \exp(-\frac{\Delta t}{\tau_{\sigma l}^\lambda}), \tag{28}$$

$$\alpha_2^l = \frac{\Delta t}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}}, \tag{29}$$

$$\beta_1^l = \exp(-\frac{\Delta t}{\tau_{\sigma l}^\mu}), \tag{30}$$

$$\beta_2^l = \frac{\Delta t}{\tau_{\sigma l}^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}}, \tag{31}$$

$$\eta_1^l = \exp(-\frac{\Delta t}{\tau_{\sigma l}^\rho}), \tag{32}$$

$$\eta_2^l = \frac{\Delta t}{\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \tag{33}$$

we get

$$
\begin{aligned}
\gamma_\lambda^l(t) &= \alpha_1^l\gamma_\lambda^l(t-\Delta t) + \alpha_2^l(\dot{e}_{xx} + \dot{e}_y y + \dot{e}_z z),\\
\gamma_{ij}^l(t) &= \beta_1^l\gamma_{ij}^l(t-\Delta t) + \beta_2^l\dot{e}_{ij}.\\
\theta_{kij}^l(t) &= \eta_1^l\theta_{kij}^l(t-\Delta t) + \eta_2^l\partial_k\sigma_{ij}.
\end{aligned}
\tag{34}
$$

Discretization of the three dimensional case

We now consider a regular grid with positions $\boldsymbol{x}$ defined as

$$\begin{aligned} \boldsymbol{x} &= (x, y, z), & (35)\\ x &= p\Delta x, & (36)\\ y &= q\Delta y, & (37)\\ z &= r\Delta z, & (38)\\ t &= n\Delta t. & (39) \end{aligned}$$

where $p = 0, 1, 2, \ldots, N_x,\ q = 0, 1, 2, \ldots, N_y$ and $r = 0, 1, 2, \ldots, N_z$ and N_x, N_y and N_z are the number of grid points in the x, y and z-directions. $n = 0, 1, 2, \ldots, N_t$ where N_t is the number of time steps.

We will also need a regular grid which is displaced, or staggered, relative to the the regular grid. Sometimes we will refer to the regular grid as the reference grid.

The particle velocities v_x,v_y and v_z are defined on staggered grids as follows

$$\begin{aligned} v_x(\boldsymbol{x}, t) &= v_x(x + \Delta x/2, y, z, t), & (40)\\ v_y(\boldsymbol{x}, t) &= v_y(x, y + \Delta y/2, z, t), & (41)\\ v_z(\boldsymbol{x}, t) &= v_z(x, y, z + \Delta z/2, t). & (42) \end{aligned}$$

The θ_{kij} are defined on the same staggered grid as the particle velocities:

$$\begin{aligned} \theta_{xxx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xxx}, & (43)\\ \theta_{yxy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yxy}, & (44)\\ \theta_{zxz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{zxz}, & (45)\\ \theta_{xyx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xyx}, & (46)\\ \theta_{yyy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yyy}, & (47)\\ \theta_{zyz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{zyz}, & (48)\\ \theta_{xzx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xzx}, & (49)\\ \theta_{yzy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yzy} & (50)\\ \theta_{zzz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{xxx}. & (51) \end{aligned}$$

The diagonal stresses and the gamma functions are defined on a regular grid:

$$\sigma_{xx}(\boldsymbol{x}, t) = \sigma_{xx}(x, y, z, t), \quad (52)$$

$$\begin{aligned}
\sigma_{yy}(\boldsymbol{x},t) &= \sigma_{yy}(x,y,z,t), && (53)\\
\sigma_{zz}(\boldsymbol{x},t) &= \sigma_{zz}(x,y,z,t), && (54)\\
\gamma_{\lambda}(\boldsymbol{x},t) &= \gamma_{\lambda}(x,y,z,t), && (55)\\
\gamma_{\mu}(\boldsymbol{x},t) &= \gamma_{\mu}(x,y,z,t), && (56)\\
\gamma_{xx}(\boldsymbol{x},t) &= \gamma_{xx}(x,y,z,t), && (57)\\
\gamma_{yy}(\boldsymbol{x},t) &= \gamma_{yy}(x,y,z,t), && (58)\\
\gamma_{zz}(\boldsymbol{x},t) &= \gamma_{zz}(x,y,z,t). && (59)
\end{aligned}$$

The off-diagonal stresses and the off-diagonal gamma functions are defined on several staggered grids

$$\begin{aligned}
\sigma_{xy}(\boldsymbol{x},t) &= \sigma_{xy}(x+\Delta x/2,y+\Delta y/2,z,t), && (60)\\
\sigma_{xz}(\boldsymbol{x},t) &= \sigma_{xz}(x+\Delta x/2,y,z+\Delta z/2,t), && (61)\\
\sigma_{yz}(\boldsymbol{x},t) &= \sigma_{yz}(x,y+\Delta y/2,z+\Delta z/2,t), && (62)\\
\gamma_{xy}(\boldsymbol{x},t) &= \gamma_{xy}(x+\Delta x/2,y+\Delta y/2,z,t), && (63)\\
\gamma_{xz}(\boldsymbol{x},t) &= \gamma_{xz}(x+\Delta x/2,y,z+\Delta z/2,t), && (64)\\
\gamma_{yz}(\boldsymbol{x},t) &= \gamma_{yz}(x,y+\Delta y/2,z+\Delta z/2,t), && (65)\\
& && (66)
\end{aligned}$$

The Lamé parameters are defined on a regular grid as follows: as follows:

$$\begin{aligned}
\lambda_u(\boldsymbol{x}) &= \lambda_u(x,y,z), && (67)\\
\mu_u(\boldsymbol{x}) &= \mu_u(x,y,z). && (68)
\end{aligned}$$

The inverse density and μ are defined on three different staggered grids as follows:

$$\begin{aligned}
\rho_x^{-1}(\boldsymbol{x}) &= \rho^{-1}(x+\Delta x/2,y,z), && (69)\\
\rho_y^{-1}(\boldsymbol{x}) &= \rho^{-1}(x,y+\Delta y/2,z), && (70)\\
\rho_z^{-1}(\boldsymbol{x}) &= \rho^{-1}(x,y,z+\Delta z/2), && (71)\\
\mu_{uxy}(\boldsymbol{x}) &= \mu_u(x+\Delta x/2,y+\Delta y/2,z), && (72)\\
\mu_{uyz}(\boldsymbol{x}) &= \mu_u(x,y+\Delta y/2,z+\Delta z/2), && (73)\\
\mu_{uxz}(\boldsymbol{x}) &= \mu_u(x+\Delta x/2,y,z+\Delta z), && (74)\\
& && (75)
\end{aligned}$$

The visco-elastic parameters β_1, β_2 and η_1 and η_2 are also defined on several staggered grids:

$$\beta_{1xy}\boldsymbol{x}) = \beta_1(x+\Delta x,y+\Delta y,z), \quad (76)$$

$$
\begin{aligned}
\beta_{1yz}(\boldsymbol{x}) &= \beta_1(x, y+\Delta y, z+\Delta z), && (77)\\
\beta_{1xz}(\boldsymbol{x}) &= \beta_1(x+\Delta x, y, z+\Delta z), && (78)\\
\beta_{2xy}(\boldsymbol{x}) &= \beta_2(x+\Delta x, y+\Delta y, z), && (79)\\
\beta_{2yz}(\boldsymbol{x}) &= \beta_2(x, y+\Delta y, z+\Delta z), && (80)\\
\beta_{2xz}(\boldsymbol{x}) &= \beta_2(x+\Delta x, y, z+\Delta z), && (81)\\
\eta_{1x}(\boldsymbol{x}) &= \eta_1(x+\Delta x/2, y, z), && (82)\\
\eta_{2x}(\boldsymbol{x}) &= \eta_2(x+\Delta x/2, y, z), && (83)\\
\eta_{1y}(\boldsymbol{x}) &= \eta_1(x, y+\Delta y/2, z), && (84)\\
\eta_{2y}(\boldsymbol{x}) &= \eta_1(x, y+\Delta y/2, z), && (85)\\
\eta_{1z}(\boldsymbol{x}) &= \eta_1(x, y, z+\Delta z/2), && (86)\\
\eta_{2z}(\boldsymbol{x}) &= \eta_1(x, y, z+\Delta z/2). && (87)
\end{aligned}
$$

Differentiation is now replaced by numerical approximations so that ∂_x, ∂_y and ∂_z are replaced with numerical operators $d_x^+, d_x^-, d_y^+, d_y^-, d_z^+$ and d_z^-. These operators connects the staggered and reference grids, and we illustrate this with the differentiation in the x-direction. The derivative of a function $a(x)$ is approximately given at $a(x+\Delta x/2)$ and at $a(x-\Delta x/2)$ by

$$
\begin{aligned}
a'(x+\Delta x/2) &= d_x^+ a(x), && (88)\\
a'(x-\Delta x/2) &= d_x^- a(x). && (89)
\end{aligned}
$$

The differentiators d^+ and d^- are given by (Holberg, 1987)

$$
\begin{aligned}
\partial^+ &= \frac{1}{\Delta x}\sum_{l=1}^{L} \alpha_l \left[u(x+l\Delta x) - u(x-(l-1)\Delta x\right] \\
\partial^- &= \frac{1}{\Delta x}\sum_{l=1}^{L} \alpha_l \left[u(x+(l-1)\Delta x) - u(x-l\Delta x)\right] && (90)
\end{aligned}
$$

where the coefficients α_l are found through an optimization procedure. Similar differentiators are defined for the y-direction and for the z-direction, with obvious names.

Using the numerical differentiators the equations of motion becomes:

$$
\begin{aligned}
\partial_t v_x &= \rho_i^{-1}\left(d_x^+\sigma_{xx} + d_y^+\sigma_{xy} + d_z^+\sigma_{xz}\right) + f_x && (91)\\
&+ \sum_{l=0}^{N}\theta_{xxx}^l + \sum_{l=0}^{N}\theta_{yxy}^l + \sum_{l=0}^{N}\theta_{zxz}^l, && (92)
\end{aligned}
$$

$$\partial_t v_y = d_x^+ \sigma_{yx} + d_y^+ \sigma_{yy} + d_z^+ \sigma_{yz} + f_y, \tag{93}$$

$$+ \sum_{l=0}^{N} \theta_{xyx}^l + \sum_{l=0}^{N} \theta_{yyy}^l + \sum_{l=0}^{N} \theta_{zyz}^l, \tag{94}$$

$$\partial_t v_z = d_x^+ \sigma_{zx} + d_y^+ \sigma_{zy} + d_z^+ \sigma_{zz} + f_z, \tag{95}$$

$$+ \sum_{l=0}^{N} \theta_{xzx}^l + \sum_{l=0}^{N} \theta_{yzy}^l + \sum_{l=0}^{N} \theta_{zzz}^l. \tag{96}$$

The computation of the strains becomes as follows:

$$\begin{aligned}
\dot{e}_{xx} &= d_x^- v_x, \\
\dot{e}_{yy} &= d_y^- v_y, \\
\dot{e}_{zz} &= d_z^- v_z, \\
\dot{e}_{xy} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x), \\
\dot{e}_{xz} &= \frac{1}{2}(d_x^+ v_z + d_z^+ v_x), \\
\dot{e}_{yz} &= \frac{1}{2}(d_y^+ v_z + d_z^+ v_y).
\end{aligned} \tag{97}$$

The time derivatives is approximated by the central difference

$$\dot{a}(t) = \frac{a(t + \Delta t/2) - a(t - \Delta t/2)}{\Delta t} \tag{98}$$

Solution algorithm for the three dimensional case

We are now in a position to formulate a complete numerical solution of the visco-elastic equations.

Computation of the particle velocity

We use the expression for the approximate time derivative given by equation (98) in equations (**??**) to obtain an expression for the components of the particle velocity

$$\begin{aligned}
v_x(t + \Delta t/2) &= \Delta t \rho_x^{-1} \left[d_x^+ \sigma_{xx}(t) + d_y^+ \sigma_{xy}(t) + d_z^+ \sigma_{xz}(t) \right] + \Delta t \rho_x^{-1} f_x(t) + \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xxx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yxy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zxz}^l(t) + v_x(t - \Delta t/2),
\end{aligned}$$

$$
\begin{aligned}
v_y(t+\Delta t/2) &= \Delta t \rho_y^{-1}\left[d_x^+\sigma_{yx}(t) + d_y^+\sigma_{yy}(t) + d_z^+\sigma_{yz}(t)\right] + \Delta t \rho_y^{-1} f_y(t) \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xyx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yyy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zyz}^l(t) + v_y(t-\Delta t/2), \\
v_z(t+\Delta t/2) &= \Delta t \rho_z^{-1}\left[d_x^+\sigma_{zx}(t) + d_y^+\sigma_{zy}(t) + d_z^+\sigma_{zz}(t)\right] + \Delta t \rho_z^{-1} f_z(t) \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xzx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yzy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zzz}^l(t). + v_z(t-\Delta t/2)
\end{aligned} \tag{99}
$$

The strains can now be computed from equation (97)

$$
\begin{aligned}
\dot{e}_{xx}(t+\Delta t/2) &= d_x^- v_x(t+\Delta t/2), \\
\dot{e}_{yy}(t+\Delta t/2) &= d_y^- v_y(t+\Delta t/2), \\
\dot{e}_{zz}(t+\Delta t/2) &= d_z^- v_z(t+\Delta t/2), \\
\dot{e}_{xy}(t+\Delta t/2) &= \frac{1}{2}\left[d_x^+ v_y(t+\Delta t/2) + d_y^+ v_x(t+\Delta t/2)\right], \\
\dot{e}_{xz}(t+\Delta t/2) &= \frac{1}{2}\left[d_x^+ v_z(t+\Delta t/2) + d_z^+ v_x(t+\Delta t/2)\right], \\
\dot{e}_{yz}(t+\Delta t/2) &= \frac{1}{2}\left[d_y^+ v_z(t+\Delta t/2) + d_z^+ v_y(t+\Delta t/2)\right].
\end{aligned} \tag{100}
$$

Equations (108) can be solved for the stresses using the same approach as for the particle velocities:

$$
\begin{aligned}
\sigma_{xx}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] \\
&+ 2\mu_u \dot{e}_{xx}(t+\Delta t/2) + \Delta t \dot{q}_{xx} \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l(t+\Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{xx}^l(t+\Delta t/2)\Delta\mu_l(t+\Delta t/2) \\
&+ \sigma_{xx}(t), && (101) \\
\sigma_{yy}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] && (102) \\
&+ 2\Delta t \mu_u \dot{e}_{yy}(t+\Delta t/2) + \Delta t \dot{q}_{yy}(t+\Delta t/2) \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l(t+\Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{yy}^l(t+\Delta t/2)\Delta\mu_l && (103) \\
&+ \sigma_{yy}(t) && (104) \\
\sigma_{zz}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] && (105) \\
&+ 2\mu_u \dot{e}_{zz}(t+\Delta t/2) + \Delta t \dot{q}_{zz}(t+\Delta t/2)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \Delta t \sum_{l=1}^{N} \gamma_\lambda^l (t+\Delta t/2) \Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{zz} \Delta\mu_l (t+\Delta t/2), \\
&+ \quad \sigma_{zz}. \qquad (106)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xy}(t+\Delta t) &= 2\Delta t \mu \dot{e}_{xy}(t+\Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{xy}^l (t+\Delta t/2)\Delta\mu_l + \Delta t \dot{q}_{xy}(t+\Delta t/2) \\
&+ \sigma_{xy}(t), \\
\sigma_{xz}(t+\Delta t) &= 2\Delta t \mu \dot{e}_{xz}(t+\Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{xz}^l (t+\Delta t/2)\Delta\mu_l + \Delta t \dot{q}_{xz}(t+\Delta t/2) \\
&+ \sigma_{xz}(t), \qquad (107) \\
\sigma_{yz}(t+\Delta t) &= 2\Delta \mu \dot{e}_{yz}(t+\Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{yz}^l (t+\Delta t/2)\Delta\mu_l + \dot{q}_{yz}(t+\Delta t/2) \\
&+ \sigma_{yz}(t).
\end{aligned} \qquad (108)
$$

The gamma functions are updated as follows

$$
\begin{aligned}
\gamma_\lambda^l(t+3/2\Delta t) &= \alpha_1^l \gamma_\lambda^l (t+\Delta t/2) + \alpha_2^l \left[\dot{e}_{xx}(t+\Delta t/) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right], \\
\gamma_\mu^l(t+3/2\Delta t) &= \beta_1^l \gamma_\mu^l (t+\Delta t/2) + \beta_2^l \left[\dot{e}_{xx}(t+\Delta t/) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right], \\
\gamma_{xy}^l(t+3/2\Delta t) &= \beta_{1xy}^l \gamma_{xy}^l (t+\Delta t/2) + \beta_{2xy}^l \dot{e}_{xy}(t+\Delta t/2). \\
\gamma_{xz}^l(t+3/2\Delta t) &= \beta_{1xz}^l \gamma_{xz}^l (t+\Delta t/2) + \beta_{2xz}^l \dot{e}_{xz}(t+\Delta t/2). \\
\gamma_{yz}^l(t+3/2\Delta t) &= \beta_{1xz}^l \gamma_{yz}^l (t+\Delta t/2) + \beta_{2xz}^l \dot{e}_{yz}(t+\Delta t/2).
\end{aligned} \qquad (109)
$$

The θ functions are updated as:

$$
\begin{aligned}
\theta_{xxx}(t+\Delta t) &= \eta_1^l \theta_{xxx}^l (t+\Delta t/2) + \eta_2^l \partial_x \sigma_{xx}(t+\Delta t/2). \\
\theta_{yxy}(t+\Delta t) &= \eta_1^l \theta_{yxy}^l (t+\Delta t/2) + \eta_2^l \partial_y \sigma_{xy}(t+\Delta t/2). \\
\theta_{zxz}(t+\Delta t) &= \eta_1^l \theta_{zxz}^l (t+\Delta t/2) + \eta_2^l \partial_z \sigma_{xz}(t+\Delta t/2). \\
\theta_{xyx}(t+\Delta t) &= \eta_1^l \theta_{xyx}^l (t+\Delta t/2) + \eta_2^l \partial_x \sigma_{yx}(t+\Delta t/2). \\
\theta_{yyy}(t+\Delta t) &= \eta_1^l \theta_{yyy}^l (t+\Delta t/2) + \eta_2^l \partial_y \sigma_{yy}(t+\Delta t/2). \\
\theta_{zyz}(t+\Delta t) &= \eta_1^l \theta_{zyz}^l (t+\Delta t/2) + \eta_2^l \partial_z \sigma_{yz}(t+\Delta t/2). \\
\theta_{xyx}(t+\Delta t) &= \eta_1^l \theta_{xyx}^l (t+\Delta t/2) + \eta_2^l \partial_x \sigma_{yx}(t+\Delta t/2). \\
\theta_{xzx}(t+\Delta t) &= \eta_1^l \theta_{xzx}^l (t+\Delta t/2) + \eta_2^l \partial_x \sigma_{zx}(t+\Delta t/2). \\
\theta_{yzy}(t+\Delta t) &= \eta_1^l \theta_{yzy}^l (t+\Delta t/2) + \eta_2^l \partial_y \sigma_{zy}(t+\Delta t/2).
\end{aligned}
$$

$$\theta_{zzz}(t+\Delta t) \quad = \quad \eta_1^l \theta_{zzz}^l(t+\Delta t/2) + \eta_2^l \partial_z \sigma_{zz}(t+\Delta t/2). \tag{110}$$

2D Acoustic case

For the acoustic 2D case we reduce the equations above by neglecting the y-axis terms and putting $\mu = 0$. We consider also the pseudo-stress σ defined by

$$\sigma = \frac{1}{2}\left(\sigma_{xx} + \sigma_{zz}\right)$$

We then get the acoustic 2D scheme as:

$$\begin{aligned}
v_x(t+\Delta t/2) &= \Delta t \left[\rho_{ux}^{-1} d_x^+ \sigma_{xx}(t) + \rho_{ux}^{-1} f_x(t)\right] + \\
&+ \Delta t \sum_{l=0}^{N} \theta_x^l(t) \Delta\rho_x^{-1} + v_x(t-\Delta t/2), \\
v_z(t+\Delta t/2) &= \Delta t \left[\rho_{uz}^{-1} d_z^+ \sigma_{zz}(t) + \Delta t \rho_{uz}^{-1} f_z(t)\right] \\
&+ \Delta t \sum_{l=0}^{N} \theta_z^l(t) \Delta\rho_z^{-1} + v_z(t-\Delta t/2).
\end{aligned}$$

The strains can now be computed from :

$$\begin{aligned}
\dot{e}_{xx}(t+\Delta t/2) &= d_x^- v_x(t+\Delta t/2), \\
\dot{e}_{zz}(t+\Delta t/2) &= d_z^- v_z(t+\Delta t/2).
\end{aligned}$$

Equations (108) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned}
\sigma(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] + \Delta t \dot{q}, \\
&+ \Delta t \sum_{l=1}^{N} \gamma^l(t+\Delta t/2) \Delta\lambda_l + \sigma(t).
\end{aligned}$$

We now split the γ^l into two parts γ_x^l and γ_z^l as follows:

$$\begin{aligned}
\sigma(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] + \Delta t \dot{q} \\
&+ \Delta t \sum_{l=1}^{N} \left[\gamma_x^l(t+\Delta t/2) \Delta\lambda_l + \gamma_z^l(t+\Delta t/2) \Delta\lambda_l\right] + \sigma(t).
\end{aligned}$$

The θ functions are updated as:

$$\begin{aligned}\theta_x(t+\Delta t) &= \eta_{1x}^l \theta_x^l(t) + \eta_{2x}^l \partial_x \sigma(t),\\ \theta_z(t+\Delta t) &= \eta_{1z}^l \theta_x^l(t) + \eta_{2z}^l \partial_y \sigma(t).\end{aligned}$$

The γ functions are given by

$$\begin{aligned}\gamma_x^l(t+3/2\Delta t) &= \alpha_{1x}^l \gamma_x^l(t+\Delta t/2) + \alpha_{2x}^l \dot{e}_{xx}(t+\Delta t/2),\\ \gamma_z^l(t+3/2\Delta t) &= \alpha_{1z}^l \gamma_z^l(t+\Delta t/2) + \alpha_{2z}^l \dot{e}_{zz}(t+\Delta t/2).\end{aligned}$$

Standard linear solid

The coefficents are

$$\begin{aligned}\alpha_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\lambda}\right),\\ \alpha_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})}\\ \alpha_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\lambda}\right),\\ \alpha_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})}\\ \eta_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\rho}\right),\\ \eta_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})},\\ \eta_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\rho}\right),\\ \eta_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}.\end{aligned}$$

The profile functions d_x and d_z are

$$d_x(x) = (x/L)^2, d_z(y) = (z/L)^2,$$

where L is the length of the absorbing layer and and we also have

$$\Delta\lambda_l = \lambda_u \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right) \Delta\rho^{-1} = \rho_u^{-1}\left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right) \tag{111}$$

Maxwell solid

The coefficents are

$$\begin{aligned}
\alpha_{1x} &= -\frac{1}{\tau_0^\lambda}\exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\lambda}\right), &(112)\\
\alpha_{2x} &= \frac{d_x(x)\Delta t}{\tau_0^\lambda}, &(113)\\
\alpha_{1z} &= -\frac{1}{\tau_0^\lambda}\exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\lambda}\right), &(114)\\
\alpha_{2z} &= \frac{d_z(z)\Delta t}{\tau_0^\lambda}. &(115)\\
\eta_{1x} &= -\frac{1}{\tau_0^\rho}\exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\rho}\right), &(116)\\
\eta_{2x} &= \frac{d_x(x)\Delta t}{\tau_0^\rho}, &(117)\\
\eta_{1z} &= -\frac{1}{\tau_0^\rho}\exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\rho}\right), &(118)\\
\eta_{2z} &= \frac{d_z(z)\Delta t}{\tau_0^\rho}. &(119)
\end{aligned}$$

We also have

$$\begin{aligned}
\Delta\lambda &= \lambda_u, &(120)\\
\Delta\rho^{-1} &= \rho_u^{-1}. &(121)
\end{aligned}$$

Plane Waves

The bulk modulus is considered to be visco-acoustic and is in the frequency domain given as

$$K(\omega) = K_0 G(\omega), \tag{122}$$

where K_0 is a constant to be specified. The density is also taken to be frequency dependent and complex

$$\rho(\omega) = \rho_0 G^{-1}(\omega), \tag{123}$$

Consider a horizontally layered medium with bulk-modulus and density given by equations 122 and 123. According to Ursin (1983), a downgoing wave in a homogeneous layer with unit amplitude is given by

$$\exp(-ik_z|z|), \tag{124}$$

where z is the depth of the layer and the vertical wavenumber k_z is given as:

$$k_z = \frac{\omega}{c}\cos(\theta). \tag{125}$$

Here θ is the angle between the z-axis and the propagation direction of the wave. The wave velocity c is given by

$$c = \sqrt{\frac{K(\omega)}{\rho(\omega)}} = \sqrt{\frac{K_0}{\rho_0}}G(\omega). \tag{126}$$

The reflection coefficient accross an interface between two layers (Ursin, 1983) is given by

$$R = \frac{Z_- - Z_+}{Z_- + Z_+}, \tag{127}$$

where the impedances Z_- and Z_+ above and below the interface, respectively, are given by

$$Z = \frac{\rho\omega}{k_z} = Z_0, \tag{128}$$

where $Z_0 = \rho_0 c_0 \cos(\theta)$. Consider the special case where the medium above the interface is purely acoustic, i.e $G = H = 1$, and the medium below the interface is visco-acoustic, and in addition choose Z_0 above the interface equal to Z_0 below the interface. Then R is exactly zero for all incidence angles.

Maxwell solid

A particular choice is the classical Maxwell visco-acoustic medium with G given by (Casula et al., 1992)

$$G(\omega) = \left(1 - \frac{i}{\omega\tau}\right)^{-1}, \tag{129}$$

We then get for the velocity c

$$c = \left(1 - \frac{i}{\omega\tau}\right)^{-1}\sqrt{\frac{K_0}{\rho_0}}, \tag{130}$$

and the vertical wavenumber k_z becomes

$$k_z = \frac{\omega}{c_0} - \frac{i}{c_0\tau}. \tag{131}$$

The wave are then described by

$$\exp\left(\frac{-i\omega z}{c_0}\right)\exp(\frac{-z}{c_0\tau}), \tag{132}$$

which is a dispersion-free wave with frequency independent attenuation.

Standard linear solid

A particular choice is the standard linear solid visco-acoustic medium with G given by (Casula et al., 1992)

$$G(\omega) = \frac{1+i\omega\tau_\epsilon}{1+i\omega\tau_\sigma} \tag{133}$$

We then get for the velocity c

$$c = \frac{1+i\omega\tau_\epsilon}{1+i\omega\tau_\sigma}\sqrt{\frac{K_0}{\rho_0}}, \tag{134}$$

and the vertical wavenumber k_z becomes

$$k_z = \frac{\omega}{c_0}\frac{1+i\omega\tau_\epsilon}{1+i\omega\tau_\sigma} \tag{135}$$

The wave are then described by

$$\exp\left[iz\left(\frac{\omega}{c_0}\right)\frac{1+i\omega\tau_\epsilon}{1+i\omega\tau_\sigma}\right]. \tag{136}$$

which is

$$\exp\left[iz\left(\frac{\omega}{c_0}\right)\frac{1+\omega^2\tau_\epsilon\tau_\sigma}{1+\omega^2\tau_\sigma^2}\right]\exp\left[\left(\frac{-z}{c_0}\right)\frac{\omega^2(\tau_\epsilon-\tau_\sigma)}{1+\omega^2\tau_\sigma^2}\right] \tag{137}$$

Time dependent density

Williams (2001) gives an expression for dynamic density used for acoustic waves in porous media:

$$\rho_{eff}(\omega) = \rho_f\left(\frac{\alpha(1-\beta)\rho_s+\beta(\alpha-1)\rho_f+\frac{i\beta\rho F\eta}{\rho_f\omega\kappa}}{\beta(1-\beta)\rho_s+(\alpha-2\beta+\beta^2)\rho_f+\frac{i\beta F\eta}{\omega\kappa}}\right) \tag{138}$$

The parameters are

β	porosity
ρ_s	Mass density of grains
η	Viscosity
κ	Permeability
α	Tortuosity
F	Frequency correction function

Multiply by $i\omega$ to get:

$$\rho_{eff}(\omega) = \rho_f \left(\frac{[\alpha(1-\beta)\rho_s + \beta(\alpha-1)\rho_f]i\omega - \frac{\beta\rho F\eta}{\rho_f\kappa}}{[\beta(1-\beta)\rho_s + (\alpha-2\beta+\beta^2)\rho_f]i\omega - \frac{\beta F\eta}{\kappa}} \right) \tag{139}$$

$$\rho_{eff}(\omega) = \frac{\beta\rho F\eta/\rho_f\kappa}{\beta F\eta/\kappa}\rho_f \left(\frac{\frac{[\alpha(1-\beta)\rho_s+\beta(\alpha-1)\rho_f]}{\beta\rho F\eta/\rho_f\kappa} i\omega - 1}{\frac{[\beta(1-\beta)\rho_s+(\alpha-2\beta+\beta^2)\rho_f]}{\beta F\eta/\kappa} i\omega - 1} \right) \tag{140}$$

$$\rho_{eff}(\omega) = \rho \left(\frac{1 - \frac{[\alpha(1-\beta)\rho_s+\beta(\alpha-1)\rho_f]}{\beta\rho F\eta/\rho_f\kappa} i\omega}{1 - \frac{[\beta(1-\beta)\rho_s+(\alpha-2\beta+\beta^2)\rho_f]}{\beta F\eta/\kappa} i\omega} \right) \tag{141}$$

$$\rho_{eff}^{-1}(\omega) = \rho^{-1} \left(\frac{1 - \frac{[\alpha(1-\beta)\rho_s+\beta(\alpha-1)\rho_f]}{\beta\rho F\eta/\rho_f\kappa} i\omega}{1 - \frac{[\beta(1-\beta)\rho_s+(\alpha-2\beta+\beta^2)\rho_f]}{\beta F\eta/\kappa} i\omega} \right)^{-1} \tag{142}$$

$$\rho_{eff}^{-1}(\omega) = \rho^{-1} \left(\frac{1 - \tau_\epsilon i\omega}{1 - \tau_\sigma 1\omega} \right)^{-1} \tag{143}$$

where

$$\begin{aligned} \tau_\sigma &= \frac{[\alpha(1-\beta)\rho_s + \beta(\alpha-1)\rho_f]}{\beta\rho F\eta/\rho_f\kappa}, \\ \tau_\epsilon &= \frac{[\beta(1-\beta)\rho_s + (\alpha-2\beta+\beta^2)\rho_f]}{\beta F\eta/\kappa}. \end{aligned} \tag{144}$$

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APPENDIX A: The viscoelastic standard linear solid

Bolzman's generalization of Hook's law to the visco-elastic case is (Hudson, 1985):

$$\sigma_{ij} = \psi_{ijkl} * \dot{e}_{kl}, \tag{A-1}$$

where ψ_{ijkl} is known as the relaxation tensor. The $*$ denotes convolution defined by

$$a(t) * b(t) = \int_0^t a(t-\tau)b(\tau). \tag{A-2}$$

Integrating (A-1) by parts

$$\sigma_{ij}(t) = |_0^t \psi_{ijkl}(t-\tau)e_{kl}(\tau) + \int_0^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau), \tag{A-3}$$

and using $e(t=0)=0$ I get

$$\sigma_{ij}(t) = \psi(0)_{ijkl}e_{kl}(t) + \int_{0+}^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau) \tag{A-4}$$

For the Zener model the components of the ψ_{ijkl} tensor have the form

$$\psi(t) = K\left[1 - \frac{1}{N}\sum_{l=1}^{N}(1-\frac{\tau_{\epsilon l}}{\tau_{\sigma l}})\exp(-t/\tau_{\sigma l})\right]H(t). \tag{A-5}$$

where K_r is a relaxed modulus, N is the number of Zener mechanisms,$\tau_{\sigma l}$ and $\tau_{\epsilon l}$ are relaxation times. $H(t)$ is the Heavy side function. The time derivative of ψ is equal to:

$$\dot{\psi} = \phi(t), \tag{A-6}$$

where ϕ is equal to:

$$\phi(t) = \frac{1}{N}\sum_{l=1}^{N}\left[\left(\frac{K_r}{\tau_{\sigma l}}\right)\left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\exp(-t/\tau_{\sigma l})\right]H(t). \tag{A-7}$$

Using the above we have for the stress:

$$\sigma_{ij}(t) = c_{ijkl}e_{kl} + \int_{0+}^{t}\phi_{ijkl}(t-\tau)e_{kl}(\tau) \tag{A-8}$$

This is most conveniently written as

$$\sigma_{ij}(t) = c_{ijkl}(t) * e_{kl}(t), \tag{A-9}$$

where

$$c_{ijkl}(t) = \psi(0)_{ijkl}\delta(t) + \phi_{ijkl}(t), \tag{A-10}$$

By definition $\psi(t=0)$ corresponds to the unrelaxed modulus so that we have

$$K_u = \frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}K_r \tag{A-11}$$

or

$$K_r = \frac{K_u}{\frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \tag{A-12}$$

The ϕ function can then be expressed in terms of the unrelaxed moduli:

$$\phi(t) = \sum_{l=1}^{N}\left[\left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}}\right)K_u\left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\right] \tag{A-13}$$

Finally, we express ϕ as:

$$\phi(t) = \sum_{l=1}^{N}\phi^{l}(t) \tag{A-14}$$

where

$$\phi^l(t) = \left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \right) \Delta K_l \tag{A-15}$$

and ΔK_l is

$$\Delta K_l = K_u \left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} \right) \tag{A-16}$$

It is most practical to write the time-dependent visco-elastic constants as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t), \tag{A-17}$$

$$\mu(t) = \mu_u \delta(t) + \phi_\mu(t), \tag{A-18}$$

where ϕ_λ is given as:

$$\phi_\lambda(t) = \sum_{l=1}^{N} \left(\frac{\exp(-t/\tau^\lambda_{\sigma l})}{\tau^\lambda_{\sigma l} \sum_{l=1}^{N} \frac{\tau^\lambda_{\epsilon l}}{\tau^\lambda_{\sigma l}}} \right) \Delta\lambda_l \tag{A-19}$$

and ϕ_μ is given as:

$$\phi_\mu(t) = \sum_{l=1}^{N} \left(\frac{\exp(-t/\tau^\mu_{\sigma l})}{\tau^\mu_{\sigma l} \sum_{l=1}^{N} \frac{\tau^\mu_{\epsilon l}}{\tau^\mu_{\sigma l}}} \right) \Delta\mu_l \tag{A-20}$$

Q-factors

See Casula et al. (1992) for further relations between Q and relaxation times. The Q values are related to the Fourier transform of the λ modulus as

$$Q^{-1}_\lambda(\omega) = \frac{Im\lambda(\omega)}{Re\lambda(\omega)} \tag{A-21}$$

Assuming λ is given as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t). \tag{A-22}$$

The fourier transform of λ is given by

$$\lambda(\omega) = \lambda_u + \int_{-\infty}^{\infty} \phi_\lambda(t) \exp(-i\omega t). \tag{A-23}$$

The Fourier transform of ϕ_λ is:

$$\phi_\lambda(\omega) = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1-\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\int_0^{+\infty} dt \exp(-i\omega)\exp(-t/\tau_{\sigma l}^\lambda). \quad \text{(A-24)}$$

The results is:

$$\phi_\lambda(\omega) = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1-\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\frac{1}{1+i\omega\tau_{\sigma l}^\lambda}. \quad \text{(A-25)}$$

The fourier transform of λ is then

$$\lambda(\omega) = \lambda_u + \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1-\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\frac{1}{1+i\omega\tau_{\sigma l}^\lambda}. \quad \text{(A-26)}$$

After some (tedious) algebra one obtains

$$\lambda(\omega) = \lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{1+i\omega\tau_{\epsilon l}^\lambda}{1+i\omega\tau_{\sigma l}^\lambda} \quad \text{(A-27)}$$

Separating into real and imaginary parts, I get

$$\begin{aligned}\lambda(\omega) &= \lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{1+\omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda}{1+(\omega\tau_{\sigma l}^\lambda)^2} && \text{(A-28)}\\ &+ i\lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{\omega\tau_{\sigma l}^\lambda(\tau_\epsilon^\lambda/\tau_{\sigma l}^\lambda - 1)}{1+(\omega\tau_{\sigma l}^\lambda)^2} && \\ & && \text{(A-29)}\end{aligned}$$

We then have

$$Q_\lambda^{-1} = \frac{\sum_{l=1}^{N}\omega\tau_{\sigma l}^\lambda\left(\tau_{\epsilon l}^\lambda/\tau_{\sigma l}^\lambda - 1\right)/\left[1+(\omega\tau_{\sigma l}^\lambda)^2\right]}{\sum_{l=1}^{N}(1+\omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda)/\left[1+(\omega\tau_{\sigma l}^\lambda)^2\right]} \quad \text{(A-30)}$$

The results for the frequency dependence of μ is obtained in exactly the same manner as above:

$$\mu(\omega) = \mu_r \frac{1}{N}\sum_{l=1}^{N}\frac{1+i\omega\tau_{\epsilon l}^\mu}{1+i\omega\tau_{\sigma l}^\mu} \quad \text{(A-31)}$$

and the Q-factor for μ is

$$Q_\mu^{-1} = \frac{\sum_{l=1}^{N} \omega\tau_{\sigma l}^{\lambda} \left(\tau_{\epsilon l}^{\mu}/\tau_{\sigma l}^{\lambda} - 1\right) / \left[1 + (\omega\tau_{\sigma l}^{\mu})^2\right]}{\sum_{l=1}^{N} (1 + \omega^2 \tau_{\sigma l}^{\mu}\tau_{\epsilon l}^{\mu}) / \left[1 + (\omega\tau_{\sigma l}^{\mu})^2\right]} \tag{A-32}$$

In practice we need to relate Q_l and Q_μ to Q_κ. We use the relation

$$\kappa(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega). \tag{A-33}$$

Splitting into real and imaginary parts

$$\kappa_r(\omega) = \lambda_r(\omega) + \frac{2}{3}\mu_r(\omega). \kappa_i(\omega) = \lambda_i(\omega) + \frac{2}{3}\mu_i(\omega). \tag{A-34}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\lambda_i + (2/3)\mu_i}{\lambda_r + (2/3)\mu_r} \tag{A-35}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\left(\frac{\lambda_i}{\lambda_r}\right)\lambda_r + (2/3)\left(\frac{\mu_i}{\mu_r}\right)\mu_r}{\lambda_r + (2/3)\mu_r} \tag{A-36}$$

$$Q_\kappa^{-1} = \frac{Q_\lambda^{-1}\lambda_r + (2/3)Q_\mu^{-1}}{\lambda_r + (2/3)\mu_r} \tag{A-37}$$

We can use the P-wave and S-wave velocities

$$\begin{aligned} \lambda &= \rho V_p^2 - (2/3)\rho V_s^2 \\ \mu &= \rho V_s^2 \end{aligned} \tag{A-38}$$

$$Q_\kappa^{-1} = Q_\lambda^{-1}\left[1 - \left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2\right] + Q_\mu^{-1}\left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2 \tag{A-39}$$

Q-model parametrization

Q-models can be described by the two relaxation times, τ_σ and τ_ϵ. However it is simpler to use the two parameters τ_0 and Q_0 to describe a model. According to Casula et al. (1992), Appendix B, we have

$$Q(\omega) = Q_0 \frac{1 + \omega^2\tau_0^2}{2\omega\tau_0}$$

where

$$Q_0 = \frac{2\tau_0}{\tau_\epsilon - \tau_\sigma},$$
$$\tau_0^2 = \tau_\epsilon \tau_\sigma.$$

$\omega = 1/\tau_0$ is the minimum value for $Q(\omega)$, i.e. the absorption top. We can now find τ_σ and τ_ϵ in terms of τ_0 and Q_0 as:

$$\tau_\epsilon = \frac{\tau_0}{Q_0}\left[\sqrt{Q_0^2+1}+1\right],$$
$$\tau_\sigma = \frac{\tau_0}{Q_0}\left[\sqrt{Q_0^2+1}-1\right].$$

Effective density

We now assume that the effective density has the following form

$$\rho_{eff}^{-1}(t) = s(t). \tag{A-40}$$

We also assume that

$$s(t) = s(0)\delta(t) + \chi(t) \tag{A-41}$$

So that the inverse of the effective density reads

$$\rho_{eff}^{-1}(t) = s_u\delta(t) + \chi(t). \tag{A-42}$$

where $s_u = s(0) = \rho_u^{-1}$.

APPENDIX C: The Maxwell visco-elastic solid

According to Casula et al. (1992) the Maxwell visco-elastic solid has a modulus given by

$$\lambda(t) = \lambda_u \exp(-t/\tau_0)H(t) \tag{A-1}$$

In the frequency domain one gets

$$\lambda(\omega) = \frac{\tau_0\omega}{\omega\tau_0 - i}. \tag{A-2}$$

or,

$$\lambda(\omega) = \frac{\tau_0 i\omega}{i\omega\tau_0 + 1}. \tag{A-3}$$

The function ϕ is now:

$$\phi(t) = -\Delta\lambda \frac{1}{\tau_0} \exp(-t/\tau_0), \tag{A-4}$$

where $\Delta\lambda = \lambda_u$. The Q-value are related to τ_0 by:

$$\tau_0 = Q(\omega)/\omega. \tag{A-5}$$

A plane wave

APPENDIX D: Recursive computation of γ and s.

Standard Linear Solid

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t+\Delta t) = \int_0^{t+\Delta t} d\tau \, \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{t+\Delta t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t+\Delta t) = \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\begin{aligned}
\gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\
&+ \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).
\end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t+\Delta t$

$$\begin{aligned}
\gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\
&+ \frac{1}{\tau_\epsilon} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda})
\end{aligned}$$

Performing the integral we then get

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\tau_\sigma}{\tau_\epsilon}\left[1-\exp(-\frac{\Delta t}{\tau_\sigma^\lambda})\right]\dot{e}_{xx}(t).\end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\Delta t}{\tau_\epsilon}\dot{e}_{xx}(t).\end{aligned}$$

Maxwell solid

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t+\Delta t) = -\int_0^{t+\Delta t} d\tau\,\frac{1}{\tau_0^\lambda}\exp(-\frac{t+\Delta t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t+\Delta t) = -\frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_0^{t+\Delta t} d\tau\,\exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).$$

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= -\frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_0^{t} d\tau\,\exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau) \\ &- \frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_t^{t+\Delta t} d\tau\,\exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).\end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t+\Delta t$

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= -\frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_0^{t} d\tau\,\exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau) \\ &- \frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\dot{e}_{xx}(t)\int_t^{t+\Delta t} d\tau\,\exp(-\frac{t-\tau}{\tau_0^\lambda})\end{aligned}$$

Performing the integral we then get

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \left[1-\exp(-\frac{\Delta t}{\tau_0^\lambda})\right]\dot{e}_{xx}(t).\end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\Delta t}{\tau_0}\dot{e}_{xx}(t).\end{aligned}$$

APPENDIX E: Comparison with CPML

Komatitsch and Martin (2007) show that the implementation of the CPML method can be performed by replacing each spatial derivative with (Their's equation (16) and (18)):

$$s_x = \delta(t) - d_x H(t)\exp\left[-(d_x+\alpha_x)t\right]. \tag{A-1}$$

We compare this with our equation

$$\lambda(t)/\lambda_u = \delta(t) + \frac{1}{\tau_\epsilon}\exp(-t/\tau_\sigma)H(t)\left(1-\frac{\tau_\epsilon}{\tau_\sigma}\right) \tag{A-2}$$

Comparing equation (A-1) with equation (A-2) one gets:

$$-d_x = \frac{1}{\tau_\epsilon}\left(1-\frac{\tau_\epsilon}{\tau_\sigma}\right), \tag{A-3}$$

$$d_x+\alpha = \frac{1}{\tau_\sigma} \tag{A-4}$$

Solving for τ_ϵ and τ_σ one gets

$$\tau_\sigma = \frac{1}{(d_x+\alpha)}, \tag{A-5}$$

$$\tau_\epsilon = \frac{1}{\alpha}. \tag{A-6}$$

. Here

$$d_x(x) = d_0\left(\frac{x}{L}\right)^2, \tag{A-7}$$

where d_0 is a constant and L is the length of the PML zone and x is the distance from the start (outer border) of the PML zone.