Visco-elastic Finite difference seismic modeling with time-dependent effective density

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Elastic equations of motion

Consider an elastic medium characterized by the density $\rho(\boldsymbol{x})$, and lamé parameters $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ at each spatial point $\boldsymbol{x}$. Now assume that $u(\boldsymbol{x})_i$ is the i'th component of the particle displacement and σ_{ij} is the stress tensor and c_{ijkl} is Hook's tensor. The elastic equations of motion and Hook's law in Cartesian coordinates are given as

$$\rho(\boldsymbol{x})\partial_t^2 u_i(\boldsymbol{x},t) = \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), \tag{1}$$

$$\sigma_{ij}(\boldsymbol{x},t) = c_{ijkl}e_{kl} + q_{ij}. \tag{2}$$

Here f_i is a driving force, and q_{ij} is a driving stress. In the isotropic case one has

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{3}$$

giving

$$\rho(\boldsymbol{x})\partial_t^2 u_i(\boldsymbol{x},t) = \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), \tag{4}$$

$$\sigma_{ij}(\boldsymbol{x},t) = \lambda(\boldsymbol{x})e_{kk}\delta_{ij} + 2\mu e_{ij} + q_{ij}. \tag{5}$$

Here f_i is the i'th component of the driving force, while the strain tensor e_{ij} is equal to

$$e_{ij} = \frac{1}{2}[\partial_i u_j(\boldsymbol{x},t) + \partial_j u_i(\boldsymbol{x},t)]. \tag{6}$$

Writing out the individual components of the equations above, one gets:

$$\rho\partial_t^2 u_x = \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} + f_x,$$
$$\rho\partial_t^2 u_y = \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + f_y,$$
$$\rho\partial_t^2 u_z = \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z,$$

$$\begin{aligned}
\sigma_{xx} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{xx} + q_{xx}, \\
\sigma_{yy} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{yy} + q_{yy}, \\
\sigma_{zz} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{zz} + q_{zz}, \\
\sigma_{xy} &= 2\mu e_{xy} + q_{xy}, \\
\sigma_{xz} &= 2\mu e_{xz} + q_{xz}, \\
\sigma_{yz} &= 2\mu e_{yz} + q_{yz}.
\end{aligned}$$

Viscoelastic equations of motion

Using the description in Appendix A, we can write down the viscoe-elastic equations of motion and the constitutive relation. In addition to visco-elastic stress-strain relation we have also introduced time-dependence for the density. The density then shows relaxation in the same way as a visco-elastic medium.

$$
\begin{aligned}
\partial_t^2 u_i(\boldsymbol{x},t) &= \rho_u^{-1}(\boldsymbol{x})\partial_j\sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), \\
&+ \chi(\boldsymbol{x},t) * \partial_j\sigma_{ij}(\boldsymbol{x},t), \\
\sigma_{ij}(\boldsymbol{x},t) &= \lambda_u e_{kk}\delta_{ij} + 2\mu_u e_{ij} + q_{ij} \\
&+ \delta_{ij}\phi_\lambda(t) * e_{mm} + 2\phi_\mu(t) * e_{ij}
\end{aligned}
$$

Writing out the individual components of the equations above, one gets:

$$
\begin{aligned}
\partial_t^2 u_x &= \rho_u^{-1}\left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right] + f_x, \\
&= +\chi * \left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right], \\
\partial_t^2 u_y &= \rho_u^{-1}\left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right] + f_y, \\
&= \chi * \left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right] \\
\partial_t^2 u_z &= \rho_u^{-1}\left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right] + f_z, \\
&= \chi * \left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right],
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xx} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{xx} + q_{xx} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{xx}, \\
\sigma_{yy} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{yy} + q_{yy} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{yy}, \\
\sigma_{zz} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{zz} + q_{zz} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{zz}, \\
\sigma_{xy} &= 2\mu_u e_{xy} + q_{xy} + 2\phi_\mu(t) * e_{xy}, \\
\sigma_{xz} &= 2\mu_u e_{xz} + q_{xz} + 2\phi_\mu(t) * e_{xz}, \\
\sigma_{yz} &= 2\mu_u e_{yz} + q_{yz} + 2\phi_\mu(t) * e_{yz}.
\end{aligned}
$$

Viscoelastic velocity-stress formulation

Using the velocity $v_i = \dot{u}_i$, one gets

$$
\partial_t v_x = \rho_u^{-1}\left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right] + f_x,
$$

$$\begin{aligned}
&= \quad +\chi * [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}], \\
\partial_t v_y &= \quad \rho_u^{-1} [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] + f_y, \\
&= \quad \chi * [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] \\
\partial_t v_z &= \quad \rho_u^{-1} [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}] + f_z, \\
&= \quad \chi * [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}],
\end{aligned}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \quad \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz} \right) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx} \\
&+ \quad \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{xx}, \\
\dot{\sigma}_{yy} &= \quad \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz} \right) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy} \\
&+ \quad \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{yy}, \\
\dot{\sigma}_{zz} &= \quad \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz} \right) + 2\mu_u \dot{e}_{zz} + q_{zz} \\
&+ \quad \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{zz}, \\
\dot{\sigma}_{xy} &= \quad 2\mu_u \dot{e}_{xy} + q_{xy} + 2\phi_\mu(t) * \dot{e}_{xy}, \\
\dot{\sigma}_{xz} &= \quad 2\mu_u \dot{e}_{xz} + q_{xz} + 2\phi_\mu(t) * \dot{e}_{xz}, \\
\dot{\sigma}_{yz} &= \quad 2\mu_u \dot{e}_{yz} + q_{yz} + 2\phi_\mu(t) * \dot{e}_{yz}.
\end{aligned}$$

Memory functions

We now define so-called memory variables by including the time convolution into one set of variables:

$$\begin{aligned}
\gamma_\lambda^l(t) &= \quad \frac{1}{\Delta \lambda_l} \phi_\lambda^l * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}], \\
\gamma_{ij}^l(t) &= \quad \frac{1}{\Delta \mu_l} \phi_\mu^l * [\dot{e}_{ij}],, \\
\theta_{kij}^l(t) &= \quad \frac{1}{\Delta \rho_l^{-1}} \chi^l * \partial_k [\sigma_{ij}],
\end{aligned}$$

Here

$$\Delta \rho_l^{-1} = \rho_u^{-1} \left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho} \right) \tag{7}$$

This gives the expressions for the γ functions as:

$$\gamma_\lambda^l = \left[\frac{\exp(-t/\tau_{\sigma l}^\lambda)}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \right] [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}]$$

$$\gamma_{ij}^l \;=\; \left[\frac{\exp(-t/\tau_{\sigma l}^{\mu})}{\tau_{\sigma l}^{\mu}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}}\right]\dot{e}_{ij}$$

$$\theta_{kij}^l \;=\; \left[\frac{\exp(-t/\tau_{\sigma l}^{\rho})}{\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}}}\right]\partial_k\sigma_{ij}$$

$$(8)$$

and also

$$\gamma_{\lambda}(t) \;=\; \sum_{l=1}^{N}\gamma_{\lambda}^l,$$

$$\gamma_{ij}(t) \;=\; \sum_{l=1}^{N}\gamma_{ij}^l.$$

$$\theta_{kij}(t) \;=\; \sum_{l=1}^{N}\theta_{ij}^l.$$

$$(9)$$

This gives the final form of the viscoelastic equations

$$\partial_t v_x \;=\; \rho_i^{-1}\left(\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right) + f_x \tag{10}$$

$$+ \;\sum_{l=0}^{N}\theta_{xxx}^l\Delta\rho_l^{-1} + \sum_{l=0}^{N}\theta_{yxy}^l\Delta\rho_l^{-1} + \sum_{l=0}^{N}\theta_{zxz}^l\Delta\rho_l^{-1}, \tag{11}$$

$$\partial_t v_y \;=\; \partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz} + f_y, \tag{12}$$

$$+ \;\sum_{l=0}^{N}\theta_{xyx}^l\Delta\rho_l^{-1} + \sum_{l=0}^{N}\theta_{yyy}^l\Delta\rho_{ll}^{-1} + \sum_{l=0}^{N}\theta_{zyz}^l\Delta\rho_{ll}^{-1}, \tag{13}$$

$$\partial_t v_z \;=\; \partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz} + f_z, \tag{14}$$

$$+ \;\sum_{l=0}^{N}\theta_{xzx}^l + \sum_{l=0}^{N}\theta_{yzy}^l + \sum_{l=0}^{N}\theta_{zzz}^l. \tag{15}$$

$$\dot{\sigma}_{xx} \;=\; \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u\dot{e}_{xx} + \dot{q}_{xx}$$

$$+ \;\sum_{l=1}^{N}\gamma_{\lambda}^l\Delta\lambda_l + 2\sum_{l=1}^{N}\gamma_{xx}^l\Delta\mu_l, \tag{16}$$

$$\dot{\sigma}_{yy} \;=\; \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u\dot{e}_{yy} + \dot{q}_{yy}$$

$$+ \quad \sum_{l=1}^{N} \gamma_\lambda^l \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma_{yy}^l \Delta\mu_l, \tag{17}$$

$$\dot{\sigma}_{zz} = \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz}$$

$$+ \quad \sum_{l=1}^{N} \gamma_\lambda^l \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma_{zz}\Delta\mu_l,$$

$$\dot{\sigma}_{xy} = 2\mu\dot{e}_{xy} + 2\sum_{l=1}^{N} \gamma_{xy}^l \Delta\mu_l + \dot{q}_{xy},$$

$$\dot{\sigma}_{xz} = 2\mu\dot{e}_{xz} + 2\sum_{l=1}^{N} \gamma_{xz}^l \Delta\mu_l + \dot{q}_{xz},$$

$$\dot{\sigma}_{yz} = 2\mu\dot{e}_{yz} + 2\sum_{l=1}^{N} \gamma_{yz}^l \Delta\mu_l + \dot{q}_{yz},$$

$$\tag{18}$$

$$\dot{e}_{xx} = \partial_x v_x, \tag{19}$$

$$\dot{e}_{yy} = \partial_y v_y, \tag{20}$$

$$\dot{e}_{zz} = \partial_z v_z, \tag{21}$$

$$\dot{e}_{xy} = \frac{1}{2}(\partial_x v_y + \partial_y v_x), \tag{22}$$

$$\dot{e}_{xz} = \frac{1}{2}(\partial_x v_z + \partial_z v_y), \tag{23}$$

$$\dot{e}_{yz} = \frac{1}{2}(\partial_y v_z + \partial_z v_y). \tag{24}$$

Integration of memory functions

The *mmeory* functions obeys approximately the relations

$$\gamma_\lambda^l(t) = \exp(-\frac{\Delta t}{\tau_\sigma^{\lambda l}})\gamma_\lambda^l(t - \Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^\lambda \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}}\right)(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}),$$

$$\gamma_{ij}^l(t) = \exp(-\frac{\Delta t}{N\tau_{\sigma l}^\mu})\gamma_{ij}^l(t - \Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^\mu \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}}\right)\dot{e}_{ij}.$$

$$\theta^l_{kij}(t) = \exp(-\frac{\Delta t}{N\tau^\rho_{\sigma l}})\theta^l_{kij}(t-\Delta t) + \left(\frac{\Delta t}{\tau^\rho_{\sigma l} \sum_{l=1}^N \frac{\tau^\rho_{\epsilon l}}{\tau^\mu_{\sigma l}}} \right) \partial_k \sigma_{ij}. \tag{25}$$

Defining the quantities

$$\alpha^l_1 = \exp(-\frac{\Delta t}{\tau^\lambda_{\sigma l}}), \tag{26}$$

$$\alpha^l_2 = \frac{\Delta t}{\tau^\lambda_{\sigma l} \sum_{l=1}^N \frac{\tau^\lambda_{\epsilon l}}{\tau^\lambda_{\sigma l}}}, \tag{27}$$

$$\beta^l_1 = \exp(-\frac{\Delta t}{\tau^\mu_{\sigma l}}), \tag{28}$$

$$\beta^l_2 = \frac{\Delta t}{\tau^\mu_{\sigma l} \sum_{l=1}^N \frac{\tau^\mu_{\epsilon l}}{\tau^\mu_{\sigma l}}}, \tag{29}$$

$$\eta^l_1 = \exp(-\frac{\Delta t}{\tau^\rho_{\sigma l}}), \tag{30}$$

$$\eta^l_2 = \frac{\Delta t}{\tau^\rho_{\sigma l} \sum_{l=1}^N \frac{\tau^\rho_{\epsilon l}}{\tau^\rho_{\sigma l}}} \tag{31}$$

we get

$$\begin{aligned}
\gamma^l_\lambda(t) &= \alpha^l_1 \gamma^l_\lambda(t-\Delta t) + \alpha^l_2 \left(\dot{e}_{xx} + \dot{e}_{y}y + \dot{e}_{z}z \right), \\
\gamma^l_{ij}(t) &= \beta^l_1 \gamma^l_{ij}(t-\Delta t) + \beta^l_2 \dot{e}_{ij}. \\
\theta^l_{kij}(t) &= \eta^l_1 \theta^l_{kij}(t-\Delta t) + \eta^l_2 \partial_k \sigma_{ij}.
\end{aligned} \tag{32}$$

Discretization of the three dimensional case

We now consider a regular grid with positions $\boldsymbol{x}$ defined as

$$\begin{aligned}
\boldsymbol{x} &= (x, y, z), \tag{33} \\
x &= p\Delta x, \tag{34} \\
y &= q\Delta y, \tag{35} \\
z &= r\Delta z, \tag{36} \\
t &= n\Delta t. \tag{37}
\end{aligned}$$

where $p = 0, 1, 2, \ldots, N_x$, $q = 0, 1, 2, \ldots, N_y$ and $r = 0, 1, 2, \ldots, N_z$ and N_x, N_y and N_z are the number of grid points in the x, y and z-directions. $n = 0, 1, 2, \ldots, N_t$ where N_t is the number of time steps.

We will also need a regular grid which is displaced, or staggered, relative to the the regular grid. Sometimes we will refer to the regular grid as the reference grid.

The particle velocities v_x, v_y and v_z are defined on staggered grids as follows

$$
\begin{align}
v_x(\boldsymbol{x}, t) &= v_x(x + \Delta x/2, y, z, t), \tag{38}\\
v_y(\boldsymbol{x}, t) &= v_y(x, y + \Delta y/2, z, t), \tag{39}\\
v_z(\boldsymbol{x}, t) &= v_z(x, y, z + \Delta z/2, t). \tag{40}
\end{align}
$$

The θ_{kij} are defined on the same staggered grid as the particle velocities:

$$
\begin{align}
\theta_{xxx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xxx}, \tag{41}\\
\theta_{yxy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yxy}, \tag{42}\\
\theta_{zxz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{zxz}, \tag{43}\\
\theta_{xyx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xyx}, \tag{44}\\
\theta_{yyy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yyy}, \tag{45}\\
\theta_{zyz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{zyz}, \tag{46}\\
\theta_{xzx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xzx}, \tag{47}\\
\theta_{yzy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yzy} \tag{48}\\
\theta_{zzz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{xxx}. \tag{49}
\end{align}
$$

The diagonal stresses and the gamma functions are defined on a regular grid:

$$
\begin{align}
\sigma_{xx}(\boldsymbol{x}, t) &= \sigma_{xx}(x, y, z, t), \tag{50}\\
\sigma_{yy}(\boldsymbol{x}, t) &= \sigma_{yy}(x, y, z, t), \tag{51}\\
\sigma_{zz}(\boldsymbol{x}, t) &= \sigma_{zz}(x, y, z, t), \tag{52}\\
\gamma_\lambda(\boldsymbol{x}, t) &= \gamma_\lambda(x, y, z, t), \tag{53}\\
\gamma_\mu(\boldsymbol{x}, t) &= \gamma_\mu(x, y, z, t), \tag{54}\\
\gamma_{xx}(\boldsymbol{x}, t) &= \gamma_{xx}(x, y, z, t), \tag{55}\\
\gamma_{yy}(\boldsymbol{x}, t) &= \gamma_{yy}(x, y, z, t), \tag{56}\\
\gamma_{zz}(\boldsymbol{x}, t) &= \gamma_{zz}(x, y, z, t). \tag{57}
\end{align}
$$

The off-diagonal stresses and the off-diagonal gamma functions are defined on several staggered grids

$$
\begin{aligned}
\sigma_{xy}(\boldsymbol{x}, t) &= \sigma_{xy}(x + \Delta x/2, y + \Delta y/2, z, t), & (58) \\
\sigma_{xz}(\boldsymbol{x}, t) &= \sigma_{xz}(x + \Delta x/2, y, z + \Delta z/2, t), & (59) \\
\sigma_{yz}(\boldsymbol{x}, t) &= \sigma_{yz}(x, y + \Delta y/2, z + \Delta z/2, t), & (60) \\
\gamma_{xy}(\boldsymbol{x}, t) &= \gamma_{xy}(x + \Delta x/2, y + \Delta y/2, z, t), & (61) \\
\gamma_{xz}(\boldsymbol{x}, t) &= \gamma_{xz}(x + \Delta x/2, y, z + \Delta z/2, t), & (62) \\
\gamma_{yz}(\boldsymbol{x}, t) &= \gamma_{yz}(x, y + \Delta y/2, z + \Delta z/2, t), & (63) \\
& & (64)
\end{aligned}
$$

The Lamé parameters are defined on a regular grid as follows: as follows:

$$
\begin{aligned}
\lambda_u(\boldsymbol{x}) &= \lambda_u(x, y, z), & (65) \\
\mu_u(\boldsymbol{x}) &= \mu_u(x, y, z). & (66)
\end{aligned}
$$

The inverse density and μ are defined on three different staggered grids as follows:

$$
\begin{aligned}
\rho_x^{-1}(\boldsymbol{x}) &= \rho^{-1}(x + \Delta x/2, y, z), & (67) \\
\rho_y^{-1}(\boldsymbol{x}) &= \rho^{-1}(x, y + \Delta y/2, z), & (68) \\
\rho_z^{-1}(\boldsymbol{x}) &= \rho^{-1}(x, y, z + \Delta z/2), & (69) \\
\mu_{uxy}(\boldsymbol{x}) &= \mu_u(x + \Delta x/2, y + \Delta y/2, z), & (70) \\
\mu_{uyz}(\boldsymbol{x}) &= \mu_u(x, y + \Delta y/2, z + \Delta z/2), & (71) \\
\mu_{uxz}(\boldsymbol{x}) &= \mu_u(x + \Delta x/2, y, z + \Delta z), & (72) \\
& & (73)
\end{aligned}
$$

The visco-elastic parameters β_1, β_2 and η_1 and η_2 are also defined on several staggered grids:

$$
\begin{aligned}
\beta_{1xy}\boldsymbol{x}) &= \beta_1(x + \Delta x, y + \Delta y, z), & (74) \\
\beta_{1yz}(\boldsymbol{x}) &= \beta_1(x, y + \Delta y, z + \Delta z), & (75) \\
\beta_{1xz}(\boldsymbol{x}) &= \beta_1(x + \Delta x, y, z + \Delta z), & (76) \\
\beta_{2xy}(\boldsymbol{x}) &= \beta_2(x + \Delta x, y + \Delta y, z), & (77) \\
\beta_{2yz}(\boldsymbol{x}) &= \beta_2(x, y + \Delta y, z + \Delta z), & (78) \\
\beta_{2xz}(\boldsymbol{x}) &= \beta_2(x + \Delta x, y, z + \Delta z), & (79) \\
\eta_{1x}(\boldsymbol{x}) &= \eta_1(x + \Delta x/2, y, z), & (80)
\end{aligned}
$$

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$$\eta_{2x}(\boldsymbol{x}) = \eta_2(x + \Delta x/2, y, z), \tag{81}$$
$$\eta_{1y}(\boldsymbol{x}) = \eta_1(x, y + \Delta y/2, z), \tag{82}$$
$$\eta_{2y}(\boldsymbol{x}) = \eta_1(x, y + \Delta y/2, z), \tag{83}$$
$$\eta_{1z}(\boldsymbol{x}) = \eta_1(x, y, z + \Delta z/2), \tag{84}$$
$$\eta_{2z}(\boldsymbol{x}) = \eta_1(x, y, z + \Delta z/2). \tag{85}$$

Differentiation is now replaced by numerical approximations so that ∂_x, ∂_y and ∂_z are replaced with numerical operators $d_x^+, d_x^-, d_y^+, d_y^-, d_z^+$ and d_z^-. These operators connects the staggered and reference grids, and we illustrate this with the differentiation in the x-direction. The derivative of a function $a(x)$ is approximately given at $a(x + \Delta x/2)$ and at $a(x - \Delta x/2)$ by

$$a'(x + \Delta x/2) = d_x^+ a(x), \tag{86}$$
$$a'(x - \Delta x/2) = d_x^- a(x). \tag{87}$$

The differentiators d^+ and d^- are given by

$$\partial^+ = \frac{1}{\Delta x} \sum_{l=1}^{L} \alpha_l \left[u(x + l\Delta x) - u(x - (l-1)\Delta x) \right]$$

$$\partial^- = \frac{1}{\Delta x} \sum_{l=1}^{L} \alpha_l \left[u(x + (l-1)\Delta x) - u(x - l\Delta x) \right] \tag{88}$$

where the coefficients α_l are found through an optimization procedure. Similar differentiators are defined for the y-direction and for the z-direction, with obvious names.

Using the numerical differentiators the equations of motion becomes:

$$\partial_t v_x = \rho_i^{-1} \left(d_x^+ \sigma_{xx} + d_y^+ \sigma_{xy} + d_z^+ \sigma_{xz} \right) + f_x \tag{89}$$

$$+ \sum_{l=0}^{N} \theta_{xxx}^l + \sum_{l=0}^{N} \theta_{yxy}^l + \sum_{l=0}^{N} \theta_{zxz}^l, \tag{90}$$

$$\partial_t v_y = d_x^+ \sigma_{yx} + d_y^+ \sigma_{yy} + d_z^+ \sigma_{yz} + f_y, \tag{91}$$

$$+ \sum_{l=0}^{N} \theta_{xyx}^l + \sum_{l=0}^{N} \theta_{yyy}^l + \sum_{l=0}^{N} \theta_{zyz}^l, \tag{92}$$

$$\partial_t v_z = d_x^+ \sigma_{zx} + d_y^+ \sigma_{zy} + d_z^+ \sigma_{zz} + f_z, \tag{93}$$

$$+ \sum_{l=0}^{N} \theta_{xzx}^l + \sum_{l=0}^{N} \theta_{yzy}^l + \sum_{l=0}^{N} \theta_{zzz}^l. \tag{94}$$

The computation of the strains becomes as follows:

$$
\begin{aligned}
\dot{e}_{xx} &= d_x^- v_x, \\
\dot{e}_{yy} &= d_y^- v_y, \\
\dot{e}_{zz} &= d_z^- v_z, \\
\dot{e}_{xy} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x), \\
\dot{e}_{xz} &= \frac{1}{2}(d_x^+ v_z + d_z^+ v_x), \\
\dot{e}_{yz} &= \frac{1}{2}(d_y^+ v_z + d_z^+ v_y).
\end{aligned}
\tag{95}
$$

The time derivatives is approximated by the central difference

$$
\dot{a}(t) = \frac{a(t + \Delta t/2) - a(t - \Delta t/2)}{\Delta t}
\tag{96}
$$

Solution algorithm for the three dimensional case

We are now in a position to formulate a complete numerical solution of the visco-elastic equations.

Computation of the particle velocity

We use the expression for the approximate time derivative given by equation (96) in equations (**??**) to obtain an expression for the components of the particle velocity

$$
\begin{aligned}
v_x(t + \Delta t/2) &= \Delta t \rho_x^{-1} \left[d_x^+ \sigma_{xx}(t) + d_y^+ \sigma_{xy}(t) + d_z^+ \sigma_{xz}(t) \right] + \Delta t \rho_x^{-1} f_x(t) + \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xxx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yxy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zxz}^l(t) + v_x(t - \Delta t/2), \\
v_y(t + \Delta t/2) &= \Delta t \rho_y^{-1} \left[d_x^+ \sigma_{yx}(t) + d_y^+ \sigma_{yy}(t) + d_z^+ \sigma_{yz}(t) \right] + \Delta t \rho_y^{-1} f_y(t) \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xyx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yyy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zyz}^l(t) + v_y(t - \Delta t/2), \\
v_z(t + \Delta t/2) &= \Delta t \rho_z^{-1} \left[d_x^+ \sigma_{zx}(t) + d_y^+ \sigma_{zy}(t) + d_z^+ \sigma_{zz}(t) \right] + \Delta t \rho_z^{-1} f_z(t) \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xzx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yzy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zzz}^l(t). + v_z(t - \Delta t/2)
\end{aligned}
\tag{97}
$$

The strains can now be computed from equation (95)

$$
\begin{aligned}
\dot{e}_{xx}(t + \Delta t/2) &= d_x^- v_x(t + \Delta t/2), \\
\dot{e}_{yy}(t + \Delta t/2) &= d_y^- v_y(t + \Delta t/2), \\
\dot{e}_{zz}(t + \Delta t/2) &= d_z^- v_z(t + \Delta t/2), \\
\dot{e}_{xy}(t + \Delta t/2) &= \frac{1}{2}\left[d_x^+ v_y(t + \Delta t/2) + d_y^+ v_x(t + \Delta t/2)\right], \\
\dot{e}_{xz}(t + \Delta t/2) &= \frac{1}{2}\left[d_x^+ v_z(t + \Delta t/2) + d_z^+ v_x(t + \Delta t/2)\right], \\
\dot{e}_{yz}(t + \Delta t/2) &= \frac{1}{2}\left[d_y^+ v_z(t + \Delta t/2) + d_z^+ v_y(t + \Delta t/2)\right].
\end{aligned}
\tag{98}
$$

Equations (106) can be solved for the stresses using the same approach as for the particle velocities:

$$
\begin{aligned}
\sigma_{xx}(t + \Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)\right] \\
&+ 2\mu_u \dot{e}_{xx}(t + \Delta t/2) + \Delta t \dot{q}_{xx} \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l(t + \Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{xx}^l(t + \Delta t/2)\Delta\mu_l(t + \Delta t/2) \\
&+ \sigma_{xx}(t), \hspace{7cm} (99)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{yy}(t + \Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)\right] \hspace{1cm} (100) \\
&+ 2\Delta t \mu_u \dot{e}_{yy}(t + \Delta t/2) + \Delta t \dot{q}_{yy}(t + \Delta t/2) \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l(t + \Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{yy}^l(t + \Delta t/2)\Delta\mu_l \hspace{1cm} (101) \\
&+ \sigma_{yy}(t) \hspace{7cm} (102)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{zz}(t + \Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)\right] \hspace{1cm} (103) \\
&+ 2\mu_u \dot{e}_{zz}(t + \Delta t/2) + \Delta t \dot{q}_{zz}(t + \Delta t/2) \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l(t + \Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{zz}\Delta\mu_l(t + \Delta t/2), \\
&+ \sigma_{zz}. \hspace{7cm} (104)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xy}(t + \Delta t) &= 2\Delta t \mu \dot{e}_{xy}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{xy}^l(t + \Delta t/2)\Delta\mu_l + \Delta t \dot{q}_{xy}(t + \Delta t/2) \\
&+ \sigma_{xy}(t),
\end{aligned}
$$

$$
\sigma_{xz}(t + \Delta t) = 2\Delta t \mu \dot{e}_{xz}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{xz}^l(t + \Delta t/2)\Delta\mu_l + \Delta t \dot{q}_{xz}(t + \Delta t/2)
$$

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$$+ \quad \sigma_{xz}(t), \tag{105}$$

$$\sigma_{yz}(t + \Delta t) \quad = \quad 2\Delta\mu\dot{e}_{yz}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{yz}^{l}(t + \Delta t/2)\Delta\mu_l + \dot{q}_{yz}(t + \Delta t/2)$$

$$+ \quad \sigma_{yz}(t). \tag{106}$$

The gamma functions are updated as follows

$$\gamma_{\lambda}^{l}(t + 3/2\Delta t) \quad = \quad \alpha_1^l \gamma_{\lambda}^{l}(t + \Delta t/2) + \alpha_2^l \left[\dot{e}_{xx}(t + \Delta t/) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2) \right],$$
$$\gamma_{\mu}^{l}(t + 3/2\Delta t) \quad = \quad \beta_1^l \gamma_{\mu}^{l}(t + \Delta t/2) + \beta_2^l \left[\dot{e}_{xx}(t + \Delta t/) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2) \right],$$
$$\gamma_{xy}^{l}(t + 3/2\Delta t) \quad = \quad \beta_{1xy}^l \gamma_{xy}^{l}(t + \Delta t/2) + \beta_{2xy}^l \dot{e}_{xy}(t + \Delta t/2).$$
$$\gamma_{xz}^{l}(t + 3/2\Delta t) \quad = \quad \beta_{1xz}^l \gamma_{xz}^{l}(t + \Delta t/2) + \beta_{2xz}^l \dot{e}_{xz}(t + \Delta t/2).$$
$$\gamma_{yz}^{l}(t + 3/2\Delta t) \quad = \quad \beta_{1xz}^l \gamma_{yz}^{l}(t + \Delta t/2) + \beta_{2xz}^l \dot{e}_{yz}(t + \Delta t/2). \tag{107}$$

The θ functions are updated as:

$$\theta_{xxx}(t + \Delta t) \quad = \quad \eta_1^l \theta_{xxx}^{l}(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{xx}(t + \Delta t/2).$$
$$\theta_{yxy}(t + \Delta t) \quad = \quad \eta_1^l \theta_{yxy}^{l}(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{xy}(t + \Delta t/2).$$
$$\theta_{zxz}(t + \Delta t) \quad = \quad \eta_1^l \theta_{zxz}^{l}(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{xz}(t + \Delta t/2).$$
$$\theta_{xyx}(t + \Delta t) \quad = \quad \eta_1^l \theta_{xyx}^{l}(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{yx}(t + \Delta t/2).$$
$$\theta_{yyy}(t + \Delta t) \quad = \quad \eta_1^l \theta_{yyy}^{l}(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{yy}(t + \Delta t/2).$$
$$\theta_{zyz}(t + \Delta t) \quad = \quad \eta_1^l \theta_{zyz}^{l}(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{yz}(t + \Delta t/2).$$
$$\theta_{xyx}(t + \Delta t) \quad = \quad \eta_1^l \theta_{xyx}^{l}(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{yx}(t + \Delta t/2).$$
$$\theta_{xzx}(t + \Delta t) \quad = \quad \eta_1^l \theta_{xzx}^{l}(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{zx}(t + \Delta t/2).$$
$$\theta_{yzy}(t + \Delta t) \quad = \quad \eta_1^l \theta_{yzy}^{l}(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{zy}(t + \Delta t/2).$$
$$\theta_{zzz}(t + \Delta t) \quad = \quad \eta_1^l \theta_{zzz}^{l}(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{zz}(t + \Delta t/2). \tag{108}$$

2D Acoustic case

For the acoustic 2D case we reduce the equations above by neglecting the y-axis terms and putting $\mu = 0$. We consider also the pseudo-stress σ defined by

$$\sigma = \frac{1}{2} \left(\sigma_{xx} + \sigma_{zz} \right)$$

We then get the acoustic 2D scheme as:

$$
\begin{aligned}
v_x(t + \Delta t/2) &= \Delta t \left[\rho_{ux}^{-1} d_x^+ \sigma_{xx}(t) + \rho_{ux}^{-1} f_x(t) \right] + \\
&+ \Delta t \sum_{l=0}^{N} \theta_x^l(t) \Delta \rho_x^{-1} + v_x(t - \Delta t/2), \\
v_z(t + \Delta t/2) &= \Delta t \left[\rho_{uz}^{-1} d_z^+ \sigma_{zz}(t) + \Delta t \rho_{uz}^{-1} f_z(t) \right] \\
&+ \Delta t \sum_{l=0}^{N} \theta_z^l(t) \Delta \rho_z^{-1} + v_z(t - \Delta t/2).
\end{aligned}
$$

The strains can now be computed from :

$$
\begin{aligned}
\dot{e}_{xx}(t + \Delta t/2) &= d_x^- v_x(t + \Delta t/2), \\
\dot{e}_{zz}(t + \Delta t/2) &= d_z^- v_z(t + \Delta t/2).
\end{aligned}
$$

Equations (106) can be solved for the stresses using the same approach as for the particle velocities:

$$
\begin{aligned}
\sigma(t + \Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2) \right] + \Delta t \dot{q}, \\
&+ \Delta t \sum_{l=1}^{N} \gamma^l(t + \Delta t/2) \Delta \lambda_l + \sigma(t).
\end{aligned}
$$

We now split the γ^l into two parts γ_x^l and γ_z^l as follows:

$$
\begin{aligned}
\sigma(t + \Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2) \right] + \Delta t \dot{q} \\
&+ \Delta t \sum_{l=1}^{N} \left[\gamma_x^l(t + \Delta t/2) \Delta \lambda_l + \gamma_z^l(t + \Delta t/2) \Delta \lambda_l \right] + \sigma(t).
\end{aligned}
$$

The θ functions are updated as:

$$
\begin{aligned}
\theta_x(t + \Delta t) &= \eta_{1x}^l \theta_x^l(t) + \eta_{2x}^l \partial_x \sigma(t), \\
\theta_z(t + \Delta t) &= \eta_{1z}^l \theta_x^l(t) + \eta_{2z}^l \partial_y \sigma(t).
\end{aligned}
$$

The γ functions are given by

$$
\begin{aligned}
\gamma_x^l(t + 3/2\Delta t) &= \alpha_{1x}^l \gamma_x^l(t + \Delta t/2) + \alpha_{2x}^l \dot{e}_{xx}(t + \Delta t/2), \\
\gamma_z^l(t + 3/2\Delta t) &= \alpha_{1z}^l \gamma_z^l(t + \Delta t/2) + \alpha_{2z}^l \dot{e}_{zz}(t + \Delta t/2).
\end{aligned}
$$

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Also the coeffients are

$$
\begin{aligned}
\alpha_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2x}^l &= \frac{-d_x(x)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\alpha_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2z}^l &= \frac{-d_z(z)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\eta_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}, \\
\eta_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}.
\end{aligned}
$$

The profile functions d_x and d_z are

$$
d_x(x) = (x/L), d_z(y) = (z/L),
$$

where L is the length of the absorbing layer and and we also have

$$
\Delta\lambda_l = \lambda_u\left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right) \Delta\rho^{-1} = \rho_u^{-1}\left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right) \tag{109}
$$

References

Hudson, J., 1985, The excitation and propagation of elastic waves: Cambridge University Press. Cambridge Monographs on Mechanics Series.

Komatitsch, D., and R. Martin, 2007, An unsplit convolutional perfectly matched layer improved at grazing incidence for the seismic wave equation: Geophysics, **72**, SM155–SM167.

APPENDIX A: The viscoelastic standard linear solid

Bolzman's generalization of Hook's law to the visco-elastic case is (Hudson, 1985):

$$\sigma_{ij} = \psi_{ijkl} * \dot{e}_{kl}, \tag{A-1}$$

where ψ_{ijkl} is known as the relaxation tensor. The $*$ denotes convolution defined by

$$a(t) * b(t) = \int_0^t a(t - \tau)b(\tau). \tag{A-2}$$

Integrating (A-1) by parts

$$\sigma_{ij}(t) = |_0^t \psi_{ijkl}(t - \tau)e_{kl}(\tau) + \int_0^t \dot{\psi}_{ijkl}(t - \tau)e_{kl}(\tau), \tag{A-3}$$

and using $e(t = 0) = 0$ I get

$$\sigma_{ij}(t) = \psi(0)_{ijkl}e_{kl}(t) + \int_{0+}^t \dot{\psi}_{ijkl}(t - \tau)e_{kl}(\tau) \tag{A-4}$$

For the Zener model the components of the ψ_{ijkl} tensor have the form

$$\psi(t) = K \left[1 - \frac{1}{N} \sum_{l=1}^N (1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}) \exp(-t/\tau_{\sigma l}) \right] H(t). \tag{A-5}$$

where K_r is a relaxed modulus, N is the number of Zener mechanisms, $\tau_{\sigma l}$ and $\tau_{\epsilon l}$ are relaxation times. $H(t)$ is the Heavy side function. The time derivative of ψ is equal to:

$$\dot{\psi} = \phi(t), \tag{A-6}$$

where ϕ is equal to:

$$\phi(t) = \frac{1}{N} \sum_{l=1}^N \left[\left(\frac{K_r}{\tau_{\sigma l}} \right) \left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \tag{A-7}$$

Using the above we have for the stress:

$$\sigma_{ij}(t) = c_{ijkl}e_{kl} + \int_{0+}^t \phi_{ijkl}(t - \tau)e_{kl}(\tau) \tag{A-8}$$

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This is most conveniently written as

$$\sigma_{ij}(t) = c_{ijkl}(t) * e_{kl}(t), \tag{A-9}$$

where

$$c_{ijkl}(t) = \psi(0)_{ijkl}\delta(t) + \phi_{ijkl}(t), \tag{A-10}$$

By definition $\psi(t = 0)$ corresponds to the unrelaxed modulus so that we have

$$K_u = \frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}K_r \tag{A-11}$$

or

$$K_r = \frac{K_u}{\frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \tag{A-12}$$

The ϕ function can then be expressed in terms of the unrelaxed moduli:

$$\phi(t) = \sum_{l=1}^{N}\left[\left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}}\right)K_u\left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\right] \tag{A-13}$$

Finally, we express ϕ as:

$$\phi(t) = \sum_{l=1}^{N}\phi^l(t) \tag{A-14}$$

where

$$\phi^l(t) = \left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}}\right)\Delta K_l \tag{A-15}$$

and ΔK_l is

$$\Delta K_l = K_u\left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right) \tag{A-16}$$

It is most practical to write the time-dependent visco-elastic constants as

$$\lambda(t) = \lambda_u\delta(t) + \phi_\lambda(t), \tag{A-17}$$
$$\mu(t) = \mu_u\delta(t) + \phi_\mu(t), \tag{A-18}$$

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where ϕ_λ is given as:

$$\phi_\lambda(t) = \sum_{l=1}^{N} \left(\frac{\exp(-t/\tau_{\sigma l}^\lambda)}{\tau_{\sigma l}^\lambda \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \right) \Delta\lambda_l \qquad \text{(A-19)}$$

and ϕ_μ is given as:

$$\phi_\mu(t) = \sum_{l=1}^{N} \left(\frac{\exp(-t/\tau_{\sigma l}^\mu)}{\tau_{\sigma l}^\mu \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}} \right) \Delta\mu_l \qquad \text{(A-20)}$$

Q-factors

The Q values are related to the Fourier transform of the λ modulus as

$$Q_\lambda^{-1}(\omega) = \frac{Im\lambda(\omega)}{Re\lambda(\omega)} \qquad \text{(A-21)}$$

Assuming λ is given as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t). \qquad \text{(A-22)}$$

The fourier transform of λ is given by

$$\lambda(\omega) = \lambda_u + \int_{-\infty}^{\infty} \phi_\lambda(t) \exp(-i\omega t). \qquad \text{(A-23)}$$

The Fourier transform of ϕ_λ is:

$$\phi_\lambda(\omega) = \frac{1}{N} \sum_{l=1}^{N} \left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \int_0^{+\infty} dt \exp(-i\omega) \exp(-t/\tau_{\sigma l}^\lambda). \quad \text{(A-24)}$$

The results is:

$$\phi_\lambda(\omega) = \frac{1}{N} \sum_{l=1}^{N} \left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \frac{1}{1 + i\omega\tau_{\sigma l}^\lambda}. \qquad \text{(A-25)}$$

The fourier transform of λ is then

$$\lambda(\omega) = \lambda_u + \frac{1}{N} \sum_{l=1}^{N} \left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \frac{1}{1 + i\omega\tau_{\sigma l}^\lambda}. \qquad \text{(A-26)}$$

After some (tedious) algebra one obtains

$$\lambda(\omega) = \lambda_r \frac{1}{N} \sum_{l=1}^{N} \frac{1 + i\omega\tau_{\epsilon l}^\lambda}{1 + i\omega\tau_{\sigma l}^\lambda} \tag{A-27}$$

Separating into real and imaginary parts, I get

$$\lambda(\omega) = \lambda_r \frac{1}{N} \sum_{l=1}^{N} \frac{1 + \omega^2 \tau_{\sigma l}^\lambda \tau_{\epsilon l}^\lambda}{1 + (\omega\tau_{\sigma l}^\lambda)^2} \tag{A-28}$$

$$+ \quad i\lambda_r \frac{1}{N} \sum_{l=1}^{N} \frac{\omega\tau_{\sigma l}^\lambda(\tau_{\epsilon}^\lambda/\tau_{\sigma l}^\lambda - 1)}{1 + (\omega\tau_{\sigma l}^\lambda)^2}$$

$$\tag{A-29}$$

We then have

$$Q_\lambda^{-1} = \frac{\sum_{l=1}^{N} \omega\tau_{\sigma l}^\lambda \left(\tau_{\epsilon l}^\lambda/\tau_{\sigma l}^\lambda - 1\right)/\left[1 + (\omega\tau_{\sigma l}^\lambda)^2\right]}{\sum_{l=1}^{N}(1 + \omega^2 \tau_{\sigma l}^\lambda \tau_{\epsilon l}^\lambda)/\left[1 + (\omega\tau_{\sigma l}^\lambda)^2\right]} \tag{A-30}$$

The results for the frequency dependence of μ is obtained in exactly the same manner as above:

$$\mu(\omega) = \mu_r \frac{1}{N} \sum_{l=1}^{N} \frac{1 + i\omega\tau_{\epsilon l}^\mu}{1 + i\omega\tau_{\sigma l}^\mu} \tag{A-31}$$

and the Q-factor for μ is

$$Q_\mu^{-1} = \frac{\sum_{l=1}^{N} \omega\tau_{\sigma l}^\lambda \left(\tau_{\epsilon l}^\mu/\tau_{\sigma l}^\lambda - 1\right)/\left[1 + (\omega\tau_{\sigma l}^\mu)^2\right]}{\sum_{l=1}^{N}(1 + \omega^2 \tau_{\sigma l}^\mu \tau_{\epsilon l}^\mu)/\left[1 + (\omega\tau_{\sigma l}^\mu)^2\right]} \tag{A-32}$$

In practice we need to relate Q_l and Q_μ to Q_κ. We use the relation

$$\kappa(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega). \tag{A-33}$$

Splitting into real and imaginary parts

$$\kappa_r(\omega) = \lambda_r(\omega) + \frac{2}{3}\mu_r(\omega).\kappa_i(\omega) = \lambda_i(\omega) + \frac{2}{3}\mu_i(\omega). \tag{A-34}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\lambda_i + (2/3)\mu_i}{\lambda_r + (2/3)\mu_r} \tag{A-35}$$

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$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\left(\frac{\lambda_i}{\lambda_r}\right)\lambda_r + (2/3)\left(\frac{\mu_i}{\mu_r}\right)\mu_r}{\lambda_r + (2/3)\mu_r} \tag{A-36}$$

$$Q_\kappa^{-1} = \frac{Q_\lambda^{-1}\lambda_r + (2/3)Q_\mu^{-1}}{\lambda_r + (2/3)\mu_r} \tag{A-37}$$

We can use the P-wave and S-wave velocities

$$\lambda = \rho V_p^2 - (2/3)\rho V_s^2$$
$$\mu = \rho V_s^2 \tag{A-38}$$

$$Q_\kappa^{-1} = Q_\lambda^{-1}\left[1 - \left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2\right] + Q_\mu^{-1}\left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2 \tag{A-39}$$

Effective density

We now assume that the effective density has the following form

$$\rho_{eff}^{-1}(t) = s(t). \tag{A-40}$$

We also assume that

$$s(t) = s(0)\delta(t) + \chi(t) \tag{A-41}$$

So that the inverse of the effective density reads

$$\rho_{eff}^{-1}(t) = s_u\delta(t) + \chi(t). \tag{A-42}$$

where $s_u = s(0) = \rho_u^{-1}$.

APPENDIX B: Recursive computation of γ and s.

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t + \Delta t) = \int_0^{t+\Delta t} d\tau \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{t + \Delta t - \tau}{\tau_\sigma^\lambda})\dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t + \Delta t) = \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t+\Delta t} d\tau \exp(-\frac{t - \tau}{\tau_\sigma^\lambda})\dot{e}_{xx}(\tau).$$

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$$\gamma_x^\lambda(t+\Delta t) \quad = \quad \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^t d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau)$$

$$+ \quad \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t + \Delta t$

$$\gamma_x^\lambda(t+\Delta t) \quad = \quad \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^t d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau)$$

$$+ \quad \frac{1}{\tau_\epsilon} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda})$$

Performing the integral we then get

$$\gamma_x^\lambda(t+\Delta t) \quad = \quad \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda})$$

$$+ \quad \frac{\tau_\sigma}{\tau_\epsilon} \left[1 - \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \right] \dot{e}_{xx}(t).$$

For small $\Delta t \ll 1$ the last equation is also

$$\gamma_x^\lambda(t+\Delta t) \quad = \quad \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda})$$

$$+ \quad \frac{1}{\tau_\epsilon} \dot{e}_{xx}(t).$$

APPENDIX C: Comparison with CPML

Komatitsch and Martin (2007) show that the implementation of the CPML method can be performed by replacing each spatial derivative with

$$\partial_x = \frac{1}{k_x} \partial_x + \psi_x$$

Where the memory variable ψ is computed as

$$\psi_x^n = b_x \psi^{n-1} + a_x (\partial_x))^{n+1/2}$$

Here the coefficients a_x and b_x are

$$b_x = \exp -(d_x/k_x + \alpha_x)\Delta t,$$

$$a_x = \frac{d_x}{k_x(d_x + k_x\alpha_x)}(b_x - 1)$$

In the numerical examples, $k_x = 1$, so one gets

$$\partial_x = \partial_x + \psi_x$$

$$b_x = \exp(-d_x \Delta t) \exp(-\alpha_x \Delta t),$$
$$a_x = \frac{d_x}{d_x + \alpha_x}(b_x - 1)$$

Here the profile function d_x is equal to

$$d_x(x) = d_0(x/L)^2, d_y(y) = d_0(y/L)^2,$$

where L is the length of the absorbing zone and $d_0 = 341.9$. Also we have

$$\alpha = \alpha_{max}(L - x)/L$$

where $\alpha_{max} = \pi f_0$.