

# Visco-elastic Finite difference seismic modeling with time-dependent effective density

B.Arntsen

February 25, 2023

PRELIMINARY VERSION UNDER DEVELOPMENT. CONTAINS ERRORS.

## Elastic equations of motion

Consider an elastic medium characterized by the density  $\rho(\mathbf{x})$ , and lamé parameters  $\lambda(\mathbf{x})$  and  $\mu(\mathbf{x})$  at each spatial point  $\mathbf{x}$ . Now assume that  $u(\mathbf{x})_i$  is the  $i$ 'th component of the particle displacement and  $\sigma_{ij}$  is the stress tensor and  $c_{ijkl}$  is Hook's tensor. The elastic equations of motion and Hook's law in Cartesian coordinates are given as

$$\rho(\mathbf{x})\partial_t^2 u_i(\mathbf{x}, t) = \partial_j \sigma_{ij}(\mathbf{x}, t) + f_i(\mathbf{x}, t), \quad (1)$$

$$\sigma_{ij}(\mathbf{x}, t) = c_{ijkl} e_{kl} + q_{ij}. \quad (2)$$

Here  $f_i$  is a driving force, and  $q_{ij}$  is a driving stress. In the isotropic case one has

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (3)$$

giving

$$\rho(\mathbf{x})\partial_t^2 u_i(\mathbf{x}, t) = \partial_j \sigma_{ij}(\mathbf{x}, t) + f_i(\mathbf{x}, t), \quad (4)$$

$$\sigma_{ij}(\mathbf{x}, t) = \lambda(\mathbf{x}) e_{kk} \delta_{ij} + 2\mu e_{ij} + q_{ij}. \quad (5)$$

Here  $f_i$  is the  $i$ 'th component of the driving force, while the strain tensor  $e_{ij}$  is equal to

$$e_{ij} = \frac{1}{2} [\partial_i u_j(\mathbf{x}, t) + \partial_j u_i(\mathbf{x}, t)]. \quad (6)$$

Writing out the individual components of the equations above, one gets:

$$\rho \partial_t^2 u_x = \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} + f_x,$$

$$\rho \partial_t^2 u_y = \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + f_y,$$

$$\rho \partial_t^2 u_z = \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z,$$

$$\sigma_{xx} = \lambda (e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{xx} + q_{xx},$$

$$\sigma_{yy} = \lambda (e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{yy} + q_{yy},$$

$$\sigma_{zz} = \lambda (e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{zz} + q_{zz},$$

$$\sigma_{xy} = 2\mu e_{xy} + q_{xy},$$

$$\sigma_{xz} = 2\mu e_{xz} + q_{xz},$$

$$\sigma_{yz} = 2\mu e_{yz} + q_{yz}.$$

## Viscoelastic equations of motion

Using the description in Appendix A, we can write down the viscoelastic equations of motion and the constitutive relation. In addition to viscoelastic stress-strain relation we have also introduced time-dependence for the density. The density then shows relaxation in the same way as a viscoelastic medium.

$$\begin{aligned}\partial_t^2 u_i(\mathbf{x}, t) &= \rho_u^{-1}(\mathbf{x}) \partial_j \sigma_{ij}(\mathbf{x}, t) + f_i(\mathbf{x}, t), \\ &+ \chi(\mathbf{x}, t) * \partial_j \sigma_{ij}(\mathbf{x}, t), \\ \sigma_{ij}(\mathbf{x}, t) &= \lambda_u e_{kk} \delta_{ij} + 2\mu_u e_{ij} + q_{ij} \\ &+ \delta_{ij} \phi_\lambda(t) * e_{mm} + 2\phi_\mu(t) * e_{ij}\end{aligned}$$

Writing out the individual components of the equations above, one gets:

$$\begin{aligned}\partial_t^2 u_x &= \rho_u^{-1} [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}] + f_x, \\ &= +\chi * [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}], \\ \partial_t^2 u_y &= \rho_u^{-1} [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] + f_y, \\ &= \chi * [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] \\ \partial_t^2 u_z &= \rho_u^{-1} [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}] + f_z, \\ &= \chi * [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}],\end{aligned}$$

$$\begin{aligned}\sigma_{xx} &= \lambda_u (e_{xx} + e_{yy} + e_{zz}) + 2\mu_u e_{xx} + q_{xx} \\ &+ \phi_\lambda * [e_{xx} + e_{yy} + e_{zz}] + 2\phi_\mu(t) * e_{xx}, \\ \sigma_{yy} &= \lambda_u (e_{xx} + e_{yy} + e_{zz}) + 2\mu_u e_{yy} + q_{yy} \\ &+ \phi_\lambda * [e_{xx} + e_{yy} + e_{zz}] + 2\phi_\mu(t) * e_{yy}, \\ \sigma_{zz} &= \lambda_u (e_{xx} + e_{yy} + e_{zz}) + 2\mu_u e_{zz} + q_{zz} \\ &+ \phi_\lambda * [e_{xx} + e_{yy} + e_{zz}] + 2\phi_\mu(t) * e_{zz}, \\ \sigma_{xy} &= 2\mu_u e_{xy} + q_{xy} + 2\phi_\mu(t) * e_{xy}, \\ \sigma_{xz} &= 2\mu_u e_{xz} + q_{xz} + 2\phi_\mu(t) * e_{xz}, \\ \sigma_{yz} &= 2\mu_u e_{yz} + q_{yz} + 2\phi_\mu(t) * e_{yz}.\end{aligned}$$

## Viscoelastic velocity-stress formulation

Using the velocity  $v_i = \dot{u}_i$ , one gets

$$\partial_t v_x = \rho_u^{-1} [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}] + f_x,$$

$$\begin{aligned}
&= +\chi * [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}], \\
\partial_t v_y &= \rho_u^{-1} [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] + f_y, \\
&= \chi * [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] \\
\partial_t v_z &= \rho_u^{-1} [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}] + f_z, \\
&= \chi * [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}],
\end{aligned}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx} \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{xx}, \\
\dot{\sigma}_{yy} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy} \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{yy}, \\
\dot{\sigma}_{zz} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz} \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{zz}, \\
\dot{\sigma}_{xy} &= 2\mu_u \dot{e}_{xy} + q_{xy} + 2\phi_\mu(t) * \dot{e}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu_u \dot{e}_{xz} + q_{xz} + 2\phi_\mu(t) * \dot{e}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu_u \dot{e}_{yz} + q_{yz} + 2\phi_\mu(t) * \dot{e}_{yz}.
\end{aligned}$$

## Memory functions

We now define so-called memory variables by including the time convolution into one set of variables:

$$\begin{aligned}
\gamma_\lambda^l(t) &= \frac{1}{\Delta\lambda_l} \phi_\lambda^l * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}], \\
\gamma_{ij}^l(t) &= \frac{1}{\Delta\mu_l} \phi_\mu^l * [\dot{e}_{ij}], \\
\theta_{kij}^l(t) &= \frac{1}{\Delta\rho_l^{-1}} \chi^l * \partial_k[\sigma_{ij}],
\end{aligned}$$

Here

$$\Delta\rho_l^{-1} = \rho_u^{-1} \left( 1 - \frac{\tau_{el}^\rho}{\tau_{\sigma l}^\rho} \right) \quad (7)$$

This gives the expressions for the  $\gamma$  functions as:

$$\gamma_\lambda^l = \left[ \frac{\exp(-t/\tau_{\sigma l}^\lambda)}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{el}^\lambda}{\tau_{\sigma l}^\lambda}} \right] [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}]$$

$$\begin{aligned}
\gamma_{ij}^l &= \left[ \frac{\exp(-t/\tau_{\sigma l}^\mu)}{\tau_{\sigma l}^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}} \right] \dot{\epsilon}_{ij} \\
\theta_{kij}^l &= \left[ \frac{\exp(-t/\tau_{\sigma l}^\rho)}{\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \right] \partial_k \sigma_{ij}
\end{aligned} \tag{8}$$

and also

$$\begin{aligned}
\gamma_\lambda(t) &= \sum_{l=1}^N \gamma_\lambda^l, \\
\gamma_{ij}(t) &= \sum_{l=1}^N \gamma_{ij}^l. \\
\theta_{kij}(t) &= \sum_{l=1}^N \theta_{kij}^l.
\end{aligned} \tag{9}$$

This gives the final form of the viscoelastic equations

$$\partial_t v_x = \rho_i^{-1} (\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}) + f_x \tag{10}$$

$$+ \sum_{l=0}^N \theta_{xxx}^l \Delta \rho_l^{-1} + \sum_{l=0}^N \theta_{xyx}^l \Delta \rho_l^{-1} + \sum_{l=0}^N \theta_{xzx}^l \Delta \rho_l^{-1}, \tag{11}$$

$$\partial_t v_y = \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + f_y, \tag{12}$$

$$+ \sum_{l=0}^N \theta_{xyx}^l \Delta \rho_l^{-1} + \sum_{l=0}^N \theta_{yyy}^l \Delta \rho_{ll}^{-1} + \sum_{l=0}^N \theta_{zyz}^l \Delta \rho_{ll}^{-1}, \tag{13}$$

$$\partial_t v_z = \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z, \tag{14}$$

$$+ \sum_{l=0}^N \theta_{xzx}^l + \sum_{l=0}^N \theta_{yzy}^l + \sum_{l=0}^N \theta_{zzz}^l. \tag{15}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u (\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}) + 2\mu_u \dot{\epsilon}_{xx} + \dot{q}_{xx} \\
&+ \sum_{l=1}^N \gamma_\lambda^l \Delta \lambda_l + 2 \sum_{l=1}^N \gamma_{xx}^l \Delta \mu_l,
\end{aligned} \tag{16}$$

$$\dot{\sigma}_{yy} = \lambda_u (\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}) + 2\mu_u \dot{\epsilon}_{yy} + \dot{q}_{yy}$$

$$+ \sum_{l=1}^N \gamma_{\lambda}^l \Delta \lambda_l + 2 \sum_{l=1}^N \gamma_{yy}^l \Delta \mu_l, \quad (17)$$

$$\begin{aligned} \dot{\sigma}_{zz} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz} \\ &+ \sum_{l=1}^N \gamma_{\lambda}^l \Delta \lambda_l + 2 \sum_{l=1}^N \gamma_{zz}^l \Delta \mu_l, \\ \dot{\sigma}_{xy} &= 2\mu \dot{e}_{xy} + 2 \sum_{l=1}^N \gamma_{xy}^l \Delta \mu_l + \dot{q}_{xy}, \\ \dot{\sigma}_{xz} &= 2\mu \dot{e}_{xz} + 2 \sum_{l=1}^N \gamma_{xz}^l \Delta \mu_l + \dot{q}_{xz}, \\ \dot{\sigma}_{yz} &= 2\mu \dot{e}_{yz} + 2 \sum_{l=1}^N \gamma_{yz}^l \Delta \mu_l + \dot{q}_{yz}, \end{aligned} \quad (18)$$

$$\dot{e}_{xx} = \partial_x v_x, \quad (19)$$

$$\dot{e}_{yy} = \partial_y v_y, \quad (20)$$

$$\dot{e}_{zz} = \partial_z v_z, \quad (21)$$

$$\dot{e}_{xy} = \frac{1}{2}(\partial_x v_y + \partial_y v_x), \quad (22)$$

$$\dot{e}_{xz} = \frac{1}{2}(\partial_x v_z + \partial_z v_x), \quad (23)$$

$$\dot{e}_{yz} = \frac{1}{2}(\partial_y v_z + \partial_z v_y). \quad (24)$$

## Integration of memory functions

The *mmeory* functions obeys approximately the relations

$$\begin{aligned} \gamma_{\lambda}^l(t) &= \exp\left(-\frac{\Delta t}{\tau_{\sigma}^{\lambda l}}\right) \gamma_{\lambda}^l(t - \Delta t) + \left( \frac{\Delta t}{\tau_{\sigma l}^{\lambda} \sum_{l=1}^N \frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}} \right) (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}), \\ \gamma_{ij}^l(t) &= \exp\left(-\frac{\Delta t}{N \tau_{\sigma l}^{\mu}}\right) \gamma_{ij}^l(t - \Delta t) + \left( \frac{\Delta t}{\tau_{\sigma l}^{\mu} \sum_{l=1}^N \frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}} \right) \dot{e}_{ij}. \end{aligned}$$

$$\theta_{kij}^l(t) = \exp\left(-\frac{\Delta t}{N\tau_{\sigma l}^\rho}\right)\theta_{kij}^l(t - \Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\mu}}\right)\partial_k \sigma_{ij}. \quad (25)$$

Defining the quantities

$$\alpha_1^l = \exp\left(-\frac{\Delta t}{\tau_{\sigma l}^\lambda}\right), \quad (26)$$

$$\alpha_2^l = \frac{\Delta t}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}}, \quad (27)$$

$$\beta_1^l = \exp\left(-\frac{\Delta t}{\tau_{\sigma l}^\mu}\right), \quad (28)$$

$$\beta_2^l = \frac{\Delta t}{\tau_{\sigma l}^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}}, \quad (29)$$

$$\eta_1^l = \exp\left(-\frac{\Delta t}{\tau_{\sigma l}^\rho}\right), \quad (30)$$

$$\eta_2^l = \frac{\Delta t}{\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}}, \quad (31)$$

we get

$$\begin{aligned} \gamma_\lambda^l(t) &= \alpha_1^l \gamma_\lambda^l(t - \Delta t) + \alpha_2^l (\dot{e}_{xx} + \dot{e}_y y + \dot{e}_z z), \\ \gamma_{ij}^l(t) &= \beta_1^l \gamma_{ij}^l(t - \Delta t) + \beta_2^l \dot{e}_{ij}, \\ \theta_{kij}^l(t) &= \eta_1^l \theta_{kij}^l(t - \Delta t) + \eta_2^l \partial_k \sigma_{ij}. \end{aligned} \quad (32)$$

## Discretization of the three dimensional case

We now consider a regular grid with positions  $\mathbf{x}$  defined as

$$\mathbf{x} = (x, y, z), \quad (33)$$

$$x = p\Delta x, \quad (34)$$

$$y = q\Delta y, \quad (35)$$

$$z = r\Delta z, \quad (36)$$

$$t = n\Delta t. \quad (37)$$

where  $p = 0, 1, 2, \dots, N_x$ ,  $q = 0, 1, 2, \dots, N_y$  and  $r = 0, 1, 2, \dots, N_z$  and  $N_x, N_y$  and  $N_z$  are the number of grid points in the  $x, y$  and  $z$ -directions.  $n = 0, 1, 2, \dots, N_t$  where  $N_t$  is the number of time steps.

We will also need a regular grid which is displaced, or staggered, relative to the regular grid. Sometimes we will refer to the regular grid as the reference grid.

The particle velocities  $v_x, v_y$  and  $v_z$  are defined on staggered grids as follows

$$v_x(\mathbf{x}, t) = v_x(x + \Delta x/2, y, z, t), \quad (38)$$

$$v_y(\mathbf{x}, t) = v_y(x, y + \Delta y/2, z, t), \quad (39)$$

$$v_z(\mathbf{x}, t) = v_z(x, y, z + \Delta z/2, t). \quad (40)$$

The  $\theta_{kij}$  are defined on the same staggered grid as the particle velocities:

$$\theta_{xxx}(\mathbf{x}, t) = \theta(x + \Delta x/2, y, z)_{xxx}, \quad (41)$$

$$\theta_{yxy}(\mathbf{x}, t) = \theta(x, y + \Delta y/2, z)_{yxy}, \quad (42)$$

$$\theta_{zxx}(\mathbf{x}, t) = \theta(x, y, z + \Delta z/2)_{zxx}, \quad (43)$$

$$\theta_{xyx}(\mathbf{x}, t) = \theta(x + \Delta x/2, y, z)_{xyx}, \quad (44)$$

$$\theta_{yyy}(\mathbf{x}, t) = \theta(x, y + \Delta y/2, z)_{yyy}, \quad (45)$$

$$\theta_{zyz}(\mathbf{x}, t) = \theta(x, y, z + \Delta z/2)_{zyz}, \quad (46)$$

$$\theta_{xzx}(\mathbf{x}, t) = \theta(x + \Delta x/2, y, z)_{xzx}, \quad (47)$$

$$\theta_{yzy}(\mathbf{x}, t) = \theta(x, y + \Delta y/2, z)_{yzy}, \quad (48)$$

$$\theta_{zzz}(\mathbf{x}, t) = \theta(x, y, z + \Delta z/2)_{zzz}. \quad (49)$$

The diagonal stresses and the gamma functions are defined on a regular grid:

$$\sigma_{xx}(\mathbf{x}, t) = \sigma_{xx}(x, y, z, t), \quad (50)$$

$$\sigma_{yy}(\mathbf{x}, t) = \sigma_{yy}(x, y, z, t), \quad (51)$$

$$\sigma_{zz}(\mathbf{x}, t) = \sigma_{zz}(x, y, z, t), \quad (52)$$

$$\gamma_\lambda(\mathbf{x}, t) = \gamma_\lambda(x, y, z, t), \quad (53)$$

$$\gamma_\mu(\mathbf{x}, t) = \gamma_\mu(x, y, z, t), \quad (54)$$

$$\gamma_{xx}(\mathbf{x}, t) = \gamma_{xx}(x, y, z, t), \quad (55)$$

$$\gamma_{yy}(\mathbf{x}, t) = \gamma_{yy}(x, y, z, t), \quad (56)$$

$$\gamma_{zz}(\mathbf{x}, t) = \gamma_{zz}(x, y, z, t). \quad (57)$$



The off-diagonal stresses and the off-diagonal gamma functions are defined on several staggered grids

$$\sigma_{xy}(\mathbf{x}, t) = \sigma_{xy}(x + \Delta x/2, y + \Delta y/2, z, t), \quad (58)$$

$$\sigma_{xz}(\mathbf{x}, t) = \sigma_{xz}(x + \Delta x/2, y, z + \Delta z/2, t), \quad (59)$$

$$\sigma_{yz}(\mathbf{x}, t) = \sigma_{yz}(x, y + \Delta y/2, z + \Delta z/2, t), \quad (60)$$

$$\gamma_{xy}(\mathbf{x}, t) = \gamma_{xy}(x + \Delta x/2, y + \Delta y/2, z, t), \quad (61)$$

$$\gamma_{xz}(\mathbf{x}, t) = \gamma_{xz}(x + \Delta x/2, y, z + \Delta z/2, t), \quad (62)$$

$$\gamma_{yz}(\mathbf{x}, t) = \gamma_{yz}(x, y + \Delta y/2, z + \Delta z/2, t), \quad (63)$$

$$(64)$$

The Lamé parameters are defined on a regular grid as follows: as follows:

$$\lambda_u(\mathbf{x}) = \lambda_u(x, y, z), \quad (65)$$

$$\mu_u(\mathbf{x}) = \mu_u(x, y, z). \quad (66)$$

The inverse density and  $\mu$  are defined on three different staggered grids as follows:

$$\rho_x^{-1}(\mathbf{x}) = \rho^{-1}(x + \Delta x/2, y, z), \quad (67)$$

$$\rho_y^{-1}(\mathbf{x}) = \rho^{-1}(x, y + \Delta y/2, z), \quad (68)$$

$$\rho_z^{-1}(\mathbf{x}) = \rho^{-1}(x, y, z + \Delta z/2), \quad (69)$$

$$\mu_{uxy}(\mathbf{x}) = \mu_u(x + \Delta x/2, y + \Delta y/2, z), \quad (70)$$

$$\mu_{uyz}(\mathbf{x}) = \mu_u(x, y + \Delta y/2, z + \Delta z/2), \quad (71)$$

$$\mu_{uxz}(\mathbf{x}) = \mu_u(x + \Delta x/2, y, z + \Delta z/2), \quad (72)$$

$$(73)$$

The visco-elastic parameters  $\beta_1$ ,  $\beta_2$  and  $\eta_1$  and  $\eta_2$  are also defined on several staggered grids:

$$\beta_{1xy}(\mathbf{x}) = \beta_1(x + \Delta x, y + \Delta y, z), \quad (74)$$

$$\beta_{1yz}(\mathbf{x}) = \beta_1(x, y + \Delta y, z + \Delta z), \quad (75)$$

$$\beta_{1xz}(\mathbf{x}) = \beta_1(x + \Delta x, y, z + \Delta z), \quad (76)$$

$$\beta_{2xy}(\mathbf{x}) = \beta_2(x + \Delta x, y + \Delta y, z), \quad (77)$$

$$\beta_{2yz}(\mathbf{x}) = \beta_2(x, y + \Delta y, z + \Delta z), \quad (78)$$

$$\beta_{2xz}(\mathbf{x}) = \beta_2(x + \Delta x, y, z + \Delta z), \quad (79)$$

$$\eta_{1x}(\mathbf{x}) = \eta_1(x + \Delta x/2, y, z), \quad (80)$$

$$\eta_{2x}(\mathbf{x}) = \eta_2(x + \Delta x/2, y, z), \quad (81)$$

$$\eta_{1y}(\mathbf{x}) = \eta_1(x, y + \Delta y/2, z), \quad (82)$$

$$\eta_{2y}(\mathbf{x}) = \eta_1(x, y + \Delta y/2, z), \quad (83)$$

$$\eta_{1z}(\mathbf{x}) = \eta_1(x, y, z + \Delta z/2), \quad (84)$$

$$\eta_{2z}(\mathbf{x}) = \eta_1(x, y, z + \Delta z/2). \quad (85)$$

Differentiation is now replaced by numerical approximations so that  $\partial_x, \partial_y$  and  $\partial_z$  are replaced with numerical operators  $d_x^+, d_x^-, d_y^+, d_y^-, d_z^+$  and  $d_z^-$ . These operators connects the staggered and reference grids, and we illustrate this with the differentiation in the x-direction. The derivative of a function  $a(x)$  is approximately given at  $a(x + \Delta x/2)$  and at  $a(x - \Delta x/2)$  by

$$a'(x + \Delta x/2) = d_x^+ a(x), \quad (86)$$

$$a'(x - \Delta x/2) = d_x^- a(x). \quad (87)$$

The differentiators  $d^+$  and  $d^-$  are given by (Holberg, 1987)

$$\begin{aligned} \partial^+ &= \frac{1}{\Delta x} \sum_{l=1}^L \alpha_l [u(x + l\Delta x) - u(x - (l-1)\Delta x)] \\ \partial^- &= \frac{1}{\Delta x} \sum_{l=1}^L \alpha_l [u(x + (l-1)\Delta x) - u(x - l\Delta x)] \end{aligned} \quad (88)$$

where the coefficients  $\alpha_l$  are found through an optimization procedure. Similar differentiators are defined for the  $y$ -direction and for the  $z$ -direction, with obvious names.

Using the numerical differentiators the equations of motion becomes:

$$\partial_t v_x = \rho_i^{-1} (d_x^+ \sigma_{xx} + d_y^+ \sigma_{xy} + d_z^+ \sigma_{xz}) + f_x \quad (89)$$

$$+ \sum_{l=0}^N \theta_{xxx}^l + \sum_{l=0}^N \theta_{xyx}^l + \sum_{l=0}^N \theta_{xzx}^l, \quad (90)$$

$$\partial_t v_y = d_x^+ \sigma_{yx} + d_y^+ \sigma_{yy} + d_z^+ \sigma_{yz} + f_y, \quad (91)$$

$$+ \sum_{l=0}^N \theta_{xyy}^l + \sum_{l=0}^N \theta_{yyy}^l + \sum_{l=0}^N \theta_{yyz}^l, \quad (92)$$

$$\partial_t v_z = d_x^+ \sigma_{zx} + d_y^+ \sigma_{zy} + d_z^+ \sigma_{zz} + f_z, \quad (93)$$

$$+ \sum_{l=0}^N \theta_{xzx}^l + \sum_{l=0}^N \theta_{zyx}^l + \sum_{l=0}^N \theta_{zzx}^l. \quad (94)$$

The computation of the strains becomes as follows:

$$\begin{aligned}
\dot{e}_{xx} &= d_x^- v_x, \\
\dot{e}_{yy} &= d_y^- v_y, \\
\dot{e}_{zz} &= d_z^- v_z, \\
\dot{e}_{xy} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x), \\
\dot{e}_{xz} &= \frac{1}{2}(d_x^+ v_z + d_z^+ v_x), \\
\dot{e}_{yz} &= \frac{1}{2}(d_y^+ v_z + d_z^+ v_y).
\end{aligned} \tag{95}$$

The time derivatives is approximated by the central difference

$$\dot{a}(t) = \frac{a(t + \Delta t/2) - a(t - \Delta t/2)}{\Delta t} \tag{96}$$

## Solution algorithm for the three dimensional case

We are now in a position to formulate a complete numerical solution of the visco-elastic equations.

### Computation of the particle velocity

We use the expression for the approximate time derivative given by equation (96) in equations (??) to obtain an expression for the components of the particle velocity

$$\begin{aligned}
v_x(t + \Delta t/2) &= \Delta t \rho_x^{-1} \left[ d_x^+ \sigma_{xx}(t) + d_y^+ \sigma_{xy}(t) + d_z^+ \sigma_{xz}(t) \right] + \Delta t \rho_x^{-1} f_x(t) + \\
&+ \Delta t \sum_{l=0}^N \theta_{xxx}^l(t) + \Delta t \sum_{l=0}^N \theta_{xyx}^l(t) + \Delta t \sum_{l=0}^N \theta_{zxx}^l(t) + v_x(t - \Delta t/2), \\
v_y(t + \Delta t/2) &= \Delta t \rho_y^{-1} \left[ d_x^+ \sigma_{yx}(t) + d_y^+ \sigma_{yy}(t) + d_z^+ \sigma_{yz}(t) \right] + \Delta t \rho_y^{-1} f_y(t) \\
&+ \Delta t \sum_{l=0}^N \theta_{xyx}^l(t) + \Delta t \sum_{l=0}^N \theta_{yyy}^l(t) + \Delta t \sum_{l=0}^N \theta_{zyx}^l(t) + v_y(t - \Delta t/2), \\
v_z(t + \Delta t/2) &= \Delta t \rho_z^{-1} \left[ d_x^+ \sigma_{zx}(t) + d_y^+ \sigma_{zy}(t) + d_z^+ \sigma_{zz}(t) \right] + \Delta t \rho_z^{-1} f_z(t) \\
&+ \Delta t \sum_{l=0}^N \theta_{xzx}^l(t) + \Delta t \sum_{l=0}^N \theta_{yzy}^l(t) + \Delta t \sum_{l=0}^N \theta_{zzz}^l(t) + v_z(t - \Delta t/2)
\end{aligned} \tag{97}$$

The strains can now be computed from equation (95)

$$\begin{aligned}
\dot{e}_{xx}(t + \Delta t/2) &= d_x^- v_x(t + \Delta t/2), \\
\dot{e}_{yy}(t + \Delta t/2) &= d_y^- v_y(t + \Delta t/2), \\
\dot{e}_{zz}(t + \Delta t/2) &= d_z^- v_z(t + \Delta t/2), \\
\dot{e}_{xy}(t + \Delta t/2) &= \frac{1}{2} \left[ d_x^+ v_y(t + \Delta t/2) + d_y^+ v_x(t + \Delta t/2) \right], \\
\dot{e}_{xz}(t + \Delta t/2) &= \frac{1}{2} \left[ d_x^+ v_z(t + \Delta t/2) + d_z^+ v_x(t + \Delta t/2) \right], \\
\dot{e}_{yz}(t + \Delta t/2) &= \frac{1}{2} \left[ d_y^+ v_z(t + \Delta t/2) + d_z^+ v_y(t + \Delta t/2) \right].
\end{aligned} \tag{98}$$

Equations (106) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned}
\sigma_{xx}(t + \Delta t) &= \Delta t \lambda_u [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)] \\
&+ 2\mu_u \dot{e}_{xx}(t + \Delta t/2) + \Delta t \dot{q}_{xx} \\
&+ \Delta t \sum_{l=1}^N \gamma_\lambda^l(t + \Delta t/2) \Delta \lambda_l + 2\Delta t \sum_{l=1}^N \gamma_{xx}^l(t + \Delta t/2) \Delta \mu_l(t + \Delta t/2) \\
&+ \sigma_{xx}(t),
\end{aligned} \tag{99}$$

$$\begin{aligned}
\sigma_{yy}(t + \Delta t) &= \Delta t \lambda_u [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)] \\
&+ 2\Delta t \mu_u \dot{e}_{yy}(t + \Delta t/2) + \Delta t \dot{q}_{yy}(t + \Delta t/2)
\end{aligned} \tag{100}$$

$$\begin{aligned}
&+ \Delta t \sum_{l=1}^N \gamma_\lambda^l(t + \Delta t/2) \Delta \lambda_l + 2\Delta t \sum_{l=1}^N \gamma_{yy}^l(t + \Delta t/2) \Delta \mu_l \\
&+ \sigma_{yy}(t)
\end{aligned} \tag{101}$$

$$+ \sigma_{yy}(t) \tag{102}$$

$$\begin{aligned}
\sigma_{zz}(t + \Delta t) &= \Delta t \lambda_u [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)] \\
&+ 2\mu_u \dot{e}_{zz}(t + \Delta t/2) + \Delta t \dot{q}_{zz}(t + \Delta t/2)
\end{aligned} \tag{103}$$

$$\begin{aligned}
&+ \Delta t \sum_{l=1}^N \gamma_\lambda^l(t + \Delta t/2) \Delta \lambda_l + 2\Delta t \sum_{l=1}^N \gamma_{zz}^l(t + \Delta t/2) \Delta \mu_l(t + \Delta t/2), \\
&+ \sigma_{zz}.
\end{aligned} \tag{104}$$

$$\begin{aligned}
\sigma_{xy}(t + \Delta t) &= 2\Delta t \mu \dot{e}_{xy}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^N \gamma_{xy}^l(t + \Delta t/2) \Delta \mu_l + \Delta t \dot{q}_{xy}(t + \Delta t/2) \\
&+ \sigma_{xy}(t),
\end{aligned}$$

$$\begin{aligned}
\sigma_{xz}(t + \Delta t) &= 2\Delta t \mu \dot{e}_{xz}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^N \gamma_{xz}^l(t + \Delta t/2) \Delta \mu_l + \Delta t \dot{q}_{xz}(t + \Delta t/2)
\end{aligned}$$

$$+ \sigma_{xz}(t), \quad (105)$$

$$\begin{aligned} \sigma_{yz}(t + \Delta t) &= 2\Delta\mu\dot{e}_{yz}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^N \gamma_{yz}^l(t + \Delta t/2)\Delta\mu_l + \dot{q}_{yz}(t + \Delta t/2) \\ &+ \sigma_{yz}(t). \end{aligned} \quad (106)$$

The gamma functions are updated as follows

$$\begin{aligned} \gamma_\lambda^l(t + 3/2\Delta t) &= \alpha_1^l \gamma_\lambda^l(t + \Delta t/2) + \alpha_2^l [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)], \\ \gamma_\mu^l(t + 3/2\Delta t) &= \beta_1^l \gamma_\mu^l(t + \Delta t/2) + \beta_2^l [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)], \\ \gamma_{xy}^l(t + 3/2\Delta t) &= \beta_{1xy}^l \gamma_{xy}^l(t + \Delta t/2) + \beta_{2xy}^l \dot{e}_{xy}(t + \Delta t/2). \\ \gamma_{xz}^l(t + 3/2\Delta t) &= \beta_{1xz}^l \gamma_{xz}^l(t + \Delta t/2) + \beta_{2xz}^l \dot{e}_{xz}(t + \Delta t/2). \\ \gamma_{yz}^l(t + 3/2\Delta t) &= \beta_{1yz}^l \gamma_{yz}^l(t + \Delta t/2) + \beta_{2yz}^l \dot{e}_{yz}(t + \Delta t/2). \end{aligned} \quad (107)$$

The  $\theta$  functions are updated as:

$$\begin{aligned} \theta_{xxx}(t + \Delta t) &= \eta_1^l \theta_{xxx}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{xx}(t + \Delta t/2). \\ \theta_{xyx}(t + \Delta t) &= \eta_1^l \theta_{xyx}^l(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{xy}(t + \Delta t/2). \\ \theta_{xzx}(t + \Delta t) &= \eta_1^l \theta_{xzx}^l(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{xz}(t + \Delta t/2). \\ \theta_{xyy}(t + \Delta t) &= \eta_1^l \theta_{xyy}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{yy}(t + \Delta t/2). \\ \theta_{yyx}(t + \Delta t) &= \eta_1^l \theta_{yyx}^l(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{yx}(t + \Delta t/2). \\ \theta_{yzy}(t + \Delta t) &= \eta_1^l \theta_{yzy}^l(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{yz}(t + \Delta t/2). \\ \theta_{xyx}(t + \Delta t) &= \eta_1^l \theta_{xyx}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{yx}(t + \Delta t/2). \\ \theta_{xzx}(t + \Delta t) &= \eta_1^l \theta_{xzx}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{zx}(t + \Delta t/2). \\ \theta_{yzy}(t + \Delta t) &= \eta_1^l \theta_{yzy}^l(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{zy}(t + \Delta t/2). \\ \theta_{zzz}(t + \Delta t) &= \eta_1^l \theta_{zzz}^l(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{zz}(t + \Delta t/2). \end{aligned} \quad (108)$$

## 2D Acoustic case

For the acoustic 2D case we reduce the equations above by neglecting the  $y$ -axis terms and putting  $\mu = 0$ . We consider also the pseudo-stress  $\sigma$  defined by

$$\sigma = \frac{1}{2} (\sigma_{xx} + \sigma_{zz})$$

We then get the acoustic 2D scheme as:

$$\begin{aligned}
v_x(t + \Delta t/2) &= \Delta t \left[ \rho_{ux}^{-1} d_x^+ \sigma_{xx}(t) + \rho_{ux}^{-1} f_x(t) \right] + \\
&+ \Delta t \sum_{l=0}^N \theta_x^l(t) \Delta \rho_x^{-1} + v_x(t - \Delta t/2), \\
v_z(t + \Delta t/2) &= \Delta t \left[ \rho_{uz}^{-1} d_z^+ \sigma_{zz}(t) + \Delta t \rho_{uz}^{-1} f_z(t) \right] \\
&+ \Delta t \sum_{l=0}^N \theta_z^l(t) \Delta \rho_z^{-1} + v_z(t - \Delta t/2).
\end{aligned}$$

The strains can now be computed from :

$$\begin{aligned}
\dot{e}_{xx}(t + \Delta t/2) &= d_x^- v_x(t + \Delta t/2), \\
\dot{e}_{zz}(t + \Delta t/2) &= d_z^- v_z(t + \Delta t/2).
\end{aligned}$$

Equations (106) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned}
\sigma(t + \Delta t) &= \Delta t \lambda_u [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)] + \Delta t \dot{q}, \\
&+ \Delta t \sum_{l=1}^N \gamma^l(t + \Delta t/2) \Delta \lambda_l + \sigma(t).
\end{aligned}$$

We now split the  $\gamma^l$  into two parts  $\gamma_x^l$  and  $\gamma_z^l$  as follows:

$$\begin{aligned}
\sigma(t + \Delta t) &= \Delta t \lambda_u [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)] + \Delta t \dot{q} \\
&+ \Delta t \sum_{l=1}^N \left[ \gamma_x^l(t + \Delta t/2) \Delta \lambda_l + \gamma_z^l(t + \Delta t/2) \Delta \lambda_l \right] + \sigma(t).
\end{aligned}$$

The  $\theta$  functions are updated as:

$$\begin{aligned}
\theta_x(t + \Delta t) &= \eta_{1x}^l \theta_x^l(t) + \eta_{2x}^l \partial_x \sigma(t), \\
\theta_z(t + \Delta t) &= \eta_{1z}^l \theta_x^l(t) + \eta_{2z}^l \partial_y \sigma(t).
\end{aligned}$$

The  $\gamma$  functions are given by

$$\begin{aligned}
\gamma_x^l(t + 3/2 \Delta t) &= \alpha_{1x}^l \gamma_x^l(t + \Delta t/2) + \alpha_{2x}^l \dot{e}_{xx}(t + \Delta t/2), \\
\gamma_z^l(t + 3/2 \Delta t) &= \alpha_{1z}^l \gamma_z^l(t + \Delta t/2) + \alpha_{2z}^l \dot{e}_{zz}(t + \Delta t/2).
\end{aligned}$$

Also the coefficients are

$$\begin{aligned}
\alpha_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2x}^l &= \frac{-d_x(x)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\alpha_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2z}^l &= \frac{-d_z(z)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\eta_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}, \\
\eta_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}.
\end{aligned}$$

The profile functions  $d_x$  and  $d_z$  are

$$d_x(x) = (x/L), d_z(y) = (z/L),$$

where  $L$  is the length of the absorbing layer and we also have

$$\Delta\lambda_l = \lambda_u \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right) \Delta\rho^{-1} = \rho_u^{-1} \left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right) \quad (109)$$

## References

- Casula, G., J. Carcione, et al., 1992, Generalized mechanical model analogies of linear viscoelastic behaviour: *Bollettino di geofisica teorica ed applicata*, **34**, 235–256.
- Holberg, O., 1987, Computational aspects of the choice of operator and sampling interval for numerical differentiation in large-scale simulation of wave phenomena: *Geophysical P*, **35**, 629–655.

Hudson, J., 1985, The excitation and propagation of elastic waves: Cambridge University Press. Cambridge Monographs on Mechanics Series.  
 Komatitsch, D., and R. Martin, 2007, An unsplit convolutional perfectly matched layer improved at grazing incidence for the seismic wave equation: Geophysics, **72**, SM155–SM167.

## APPENDIX A: The viscoelastic standard linear solid

Bolzmann's generalization of Hook's law to the visco-elastic case is (Hudson, 1985):

$$\sigma_{ij} = \psi_{ijkl} * \dot{e}_{kl}, \quad (\text{A-1})$$

where  $\psi_{ijkl}$  is known as the relaxation tensor. The  $*$  denotes convolution defined by

$$a(t) * b(t) = \int_0^t a(t - \tau)b(\tau). \quad (\text{A-2})$$

Integrating (A-1) by parts

$$\sigma_{ij}(t) = \int_0^t \dot{\psi}_{ijkl}(t - \tau)e_{kl}(\tau) + \psi_{ijkl}(t)e_{kl}(t), \quad (\text{A-3})$$

and using  $e(t = 0) = 0$  I get

$$\sigma_{ij}(t) = \psi(0)_{ijkl}e_{kl}(t) + \int_0^t \dot{\psi}_{ijkl}(t - \tau)e_{kl}(\tau) \quad (\text{A-4})$$

For the Zener model the components of the  $\psi_{ijkl}$  tensor have the form

$$\psi(t) = K \left[ 1 - \frac{1}{N} \sum_{l=1}^N \left( 1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \quad (\text{A-5})$$

where  $K_r$  is a relaxed modulus,  $N$  is the number of Zener mechanisms,  $\tau_{\sigma l}$  and  $\tau_{el}$  are relaxation times.  $H(t)$  is the Heavy side function. The time derivative of  $\psi$  is equal to:

$$\dot{\psi} = \phi(t), \quad (\text{A-6})$$

where  $\phi$  is equal to:

$$\phi(t) = \frac{1}{N} \sum_{l=1}^N \left[ \left( \frac{K_r}{\tau_{\sigma l}} \right) \left( 1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \quad (\text{A-7})$$



Using the above we have for the stress:

$$\sigma_{ij}(t) = c_{ijkl}e_{kl} + \int_{0+}^t \phi_{ijkl}(t-\tau)e_{kl}(\tau) \quad (\text{A-8})$$

This is most conveniently written as

$$\sigma_{ij}(t) = c_{ijkl}(t) * e_{kl}(t), \quad (\text{A-9})$$

where

$$c_{ijkl}(t) = \psi(0)_{ijkl}\delta(t) + \phi_{ijkl}(t), \quad (\text{A-10})$$

By definition  $\psi(t=0)$  corresponds to the unrelaxed modulus so that we have

$$K_u = \frac{1}{N} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}} K_r \quad (\text{A-11})$$

or

$$K_r = \frac{K_u}{\frac{1}{N} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}}} \quad (\text{A-12})$$

The  $\phi$  function can then be expressed in terms of the unrelaxed moduli:

$$\phi(t) = \sum_{l=1}^N \left[ \left( \frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}}} \right) K_u \left( 1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \right] \quad (\text{A-13})$$

Finally, we express  $\phi$  as:

$$\phi(t) = \sum_{l=1}^N \phi^l(t) \quad (\text{A-14})$$

where

$$\phi^l(t) = \left( \frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}}} \right) \Delta K_l \quad (\text{A-15})$$

and  $\Delta K_l$  is

$$\Delta K_l = K_u \left( 1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \quad (\text{A-16})$$

It is most practical to write the time-dependent visco-elastic constants as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t), \quad (\text{A-17})$$

$$\mu(t) = \mu_u \delta(t) + \phi_\mu(t), \quad (\text{A-18})$$

where  $\phi_\lambda$  is given as:

$$\phi_\lambda(t) = \sum_{l=1}^N \left( \frac{\exp(-t/\tau_{\sigma l}^\lambda)}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \right) \Delta \lambda_l \quad (\text{A-19})$$

and  $\phi_\mu$  is given as:

$$\phi_\mu(t) = \sum_{l=1}^N \left( \frac{\exp(-t/\tau_{\sigma l}^\mu)}{\tau_{\sigma l}^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}} \right) \Delta \mu_l \quad (\text{A-20})$$

### Q-factors

See Casula et al. (1992) for further relations between Q and relaxation times. The  $Q$  values are related to the Fourier transform of the  $\lambda$  modulus as

$$Q_\lambda^{-1}(\omega) = \frac{\text{Im} \lambda(\omega)}{\text{Re} \lambda(\omega)} \quad (\text{A-21})$$

Assuming  $\lambda$  is given as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t). \quad (\text{A-22})$$

The fourier transform of  $\lambda$  is given by

$$\lambda(\omega) = \lambda_u + \int_{-\infty}^{\infty} \phi_\lambda(t) \exp(-i\omega t) dt. \quad (\text{A-23})$$

The Fourier transform of  $\phi_\lambda$  is:

$$\phi_\lambda(\omega) = \frac{1}{N} \sum_{l=1}^N \left( \frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left( 1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \int_0^{+\infty} dt \exp(-i\omega t) \exp(-t/\tau_{\sigma l}^\lambda). \quad (\text{A-24})$$

The results is:

$$\phi_\lambda(\omega) = \frac{1}{N} \sum_{l=1}^N \left( \frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left( 1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \frac{1}{1 + i\omega \tau_{\sigma l}^\lambda}. \quad (\text{A-25})$$

The fourier transform of  $\lambda$  is then

$$\lambda(\omega) = \lambda_u + \frac{1}{N} \sum_{l=1}^N \left( \frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left( 1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \frac{1}{1 + i\omega\tau_{\sigma l}^\lambda}. \quad (\text{A-26})$$

After some (tedious) algebra one obtains

$$\lambda(\omega) = \lambda_r \frac{1}{N} \sum_{l=1}^N \frac{1 + i\omega\tau_{\epsilon l}^\lambda}{1 + i\omega\tau_{\sigma l}^\lambda} \quad (\text{A-27})$$

Separating into real and imaginary parts, I get

$$\lambda(\omega) = \lambda_r \frac{1}{N} \sum_{l=1}^N \frac{1 + \omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda}{1 + (\omega\tau_{\sigma l}^\lambda)^2} \quad (\text{A-28})$$

$$+ i\lambda_r \frac{1}{N} \sum_{l=1}^N \frac{\omega\tau_{\sigma l}^\lambda(\tau_{\epsilon l}^\lambda/\tau_{\sigma l}^\lambda - 1)}{1 + (\omega\tau_{\sigma l}^\lambda)^2} \quad (\text{A-29})$$

We then have

$$Q_\lambda^{-1} = \frac{\sum_{l=1}^N \omega\tau_{\sigma l}^\lambda (\tau_{\epsilon l}^\lambda/\tau_{\sigma l}^\lambda - 1) / [1 + (\omega\tau_{\sigma l}^\lambda)^2]}{\sum_{l=1}^N (1 + \omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda) / [1 + (\omega\tau_{\sigma l}^\lambda)^2]} \quad (\text{A-30})$$

The results for the frequency dependence of  $\mu$  is obtained in exactly the same manner as above:

$$\mu(\omega) = \mu_r \frac{1}{N} \sum_{l=1}^N \frac{1 + i\omega\tau_{\epsilon l}^\mu}{1 + i\omega\tau_{\sigma l}^\mu} \quad (\text{A-31})$$

and the Q-factor for  $\mu$  is

$$Q_\mu^{-1} = \frac{\sum_{l=1}^N \omega\tau_{\sigma l}^\mu (\tau_{\epsilon l}^\mu/\tau_{\sigma l}^\mu - 1) / [1 + (\omega\tau_{\sigma l}^\mu)^2]}{\sum_{l=1}^N (1 + \omega^2\tau_{\sigma l}^\mu\tau_{\epsilon l}^\mu) / [1 + (\omega\tau_{\sigma l}^\mu)^2]} \quad (\text{A-32})$$

In practice we need to relate  $Q_l$  and  $Q_\mu$  to  $Q_\kappa$ . We use the relation

$$\kappa(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega). \quad (\text{A-33})$$

Splitting into real and imaginary parts

$$\kappa_r(\omega) = \lambda_r(\omega) + \frac{2}{3}\mu_r(\omega), \quad \kappa_i(\omega) = \lambda_i(\omega) + \frac{2}{3}\mu_i(\omega). \quad (\text{A-34})$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\lambda_i + (2/3)\mu_i}{\lambda_r + (2/3)\mu_r} \quad (\text{A-35})$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\left(\frac{\lambda_i}{\lambda_r}\right) \lambda_r + (2/3) \left(\frac{\mu_i}{\mu_r}\right) \mu_r}{\lambda_r + (2/3)\mu_r} \quad (\text{A-36})$$

$$Q_\kappa^{-1} = \frac{Q_\lambda^{-1} \lambda_r + (2/3) Q_\mu^{-1}}{\lambda_r + (2/3)\mu_r} \quad (\text{A-37})$$

We can use the P-wave and S-wave velocities

$$\begin{aligned} \lambda &= \rho V_p^2 - (2/3) \rho V_s^2 \\ \mu &= \rho V_s^2 \end{aligned} \quad (\text{A-38})$$

$$Q_\kappa^{-1} = Q_\lambda^{-1} \left[ 1 - \left(\frac{2}{3}\right) \left(\frac{V_s}{V_p}\right)^2 \right] + Q_\mu^{-1} \left(\frac{2}{3}\right) \left(\frac{V_s}{V_p}\right)^2 \quad (\text{A-39})$$

### Q-model parametrization

Q-models can be described by the two relaxation times,  $\tau_\sigma$  and  $\tau_\epsilon$ . However it is simpler to use the two parameters  $\tau_0$  and  $Q_0$  to describe a model. According to Casula et al. (1992), Appendix B, we have

$$Q(\omega) = Q_0 \frac{1 + \omega^2 \tau_0^2}{2\omega \tau_0}$$

where

$$\begin{aligned} Q_0 &= \frac{2\tau_0}{\tau_\epsilon - \tau_\sigma}, \\ \tau_0^2 &= \tau_\epsilon \tau_\sigma. \end{aligned}$$

$\omega = 1/\tau_0$  is the minimum value for  $Q(\omega)$ , i.e. the absorption top. We can now find  $\tau_\sigma$  and  $\tau_\epsilon$  in terms of  $\tau_0$  and  $Q_0$  as:

$$\begin{aligned} \tau_\epsilon &= \frac{\tau_0}{Q_0} \left[ \sqrt{Q_0^2 + 1} + 1 \right], \\ \tau_\sigma &= \frac{\tau_0}{Q_0} \left[ \sqrt{Q_0^2 + 1} - 1 \right]. \end{aligned}$$

## Effective density

We now assume that the effective density has the following form

$$\rho_{eff}^{-1}(t) = s(t). \quad (\text{A-40})$$

We also assume that

$$s(t) = s(0)\delta(t) + \chi(t) \quad (\text{A-41})$$

So that the inverse of the effective density reads

$$\rho_{eff}^{-1}(t) = s_u\delta(t) + \chi(t). \quad (\text{A-42})$$

where  $s_u = s(0) = \rho_u^{-1}$ .

## APPENDIX B: Recursive computation of $\gamma$ and $s$ .

The  $\gamma$ 's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\begin{aligned} \gamma_x^\lambda(t + \Delta t) &= \int_0^{t+\Delta t} d\tau \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp\left(-\frac{t + \Delta t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau). \\ \gamma_x^\lambda(t + \Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_0^{t+\Delta t} d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau). \\ \gamma_x^\lambda(t + \Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_0^t d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_t^{t+\Delta t} d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau). \end{aligned}$$

The second intergral is approximated by assuming that  $\dot{e}_{xx}(t)$  is constant in the interval  $t$  to  $t + \Delta t$

$$\begin{aligned} \gamma_x^\lambda(t + \Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_0^t d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_\epsilon} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \end{aligned}$$

Performing the integral we then get

$$\begin{aligned}\gamma_x^\lambda(t + \Delta t) &= \gamma_x^\lambda(t) \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \\ &+ \frac{\tau_\sigma}{\tau_\epsilon} \left[1 - \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right)\right] \dot{e}_{xx}(t).\end{aligned}$$

For small  $\Delta t \ll 1$  the last equation is also

$$\begin{aligned}\gamma_x^\lambda(t + \Delta t) &= \gamma_x^\lambda(t) \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \\ &+ \frac{1}{\tau_\epsilon} \dot{e}_{xx}(t).\end{aligned}$$

## APPENDIX C: Comparison with CPML

Komatitsch and Martin (2007) show that the implementation of the CPML method can be performed by replacing each spatial derivative with (Their's equation (16) and (18)):

$$s_x = \delta(t) - d_x H(t) \exp[-(d_x + \alpha_x)t]. \quad (\text{A-1})$$

We compare this with our equation

$$\lambda(t)/\lambda_u = \delta(t) + \frac{1}{\tau_\epsilon} \exp(-t/\tau_\sigma) H(t) \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right) \quad (\text{A-2})$$

Comparing equation (A-1) with equation (A-2) one gets:

$$-d_x = \frac{1}{\tau_\epsilon} \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right), \quad (\text{A-3})$$

$$d_x + \alpha = \frac{1}{\tau_\sigma} \quad (\text{A-4})$$

Solving for  $\tau_\epsilon$  and  $\tau_\sigma$  one gets

$$\tau_\sigma = \frac{1}{(d_x + \alpha)}, \quad (\text{A-5})$$

$$\tau_\epsilon = \frac{1}{\alpha}. \quad (\text{A-6})$$

. Here

$$d_x(x) = d_0 \left(\frac{x}{L}\right)^2, \quad (\text{A-7})$$

where  $d_0$  is a constant and  $L$  is the length of the PML zone and  $x$  is the distance from the start (outer border) of the PML zone.