

Visco-elastic Finite difference seismic modeling with simple and effective boundary conditions

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PRELIMINARY VERSION UNDER DEVELOPMENT. CONTAINS ERRORS.

Elastic equations of motion

Consider an elastic medium characterized by the density $\rho(\mathbf{x})$, and lamé parameters $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ at each spatial point \mathbf{x} . Now assume that $u(\mathbf{x})_i$ is the i 'th component of the particle displacement and σ_{ij} is the stress tensor and c_{ijkl} is Hook's tensor. The elastic equations of motion and Hook's law in Cartesian coordinates are given as

$$\rho(\mathbf{x})\partial_t^2 u_i(\mathbf{x}, t) = \partial_j \sigma_{ij}(\mathbf{x}, t) + f_i(\mathbf{x}, t), \quad (1)$$

$$\sigma_{ij}(\mathbf{x}, t) = c_{ijkl} e_{kl} + q_{ij}. \quad (2)$$

Here f_i is a driving force, and q_{ij} is a driving stress. In the isotropic case one has

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (3)$$

giving

$$\rho(\mathbf{x})\partial_t^2 u_i(\mathbf{x}, t) = \partial_j \sigma_{ij}(\mathbf{x}, t) + f_i(\mathbf{x}, t), \quad (4)$$

$$\sigma_{ij}(\mathbf{x}, t) = \lambda(\mathbf{x}) e_{kk} \delta_{ij} + 2\mu e_{ij} + q_{ij}. \quad (5)$$

Here f_i is the i 'th component of the driving force, while the strain tensor e_{ij} is equal to

$$e_{ij} = \frac{1}{2} [\partial_i u_j(\mathbf{x}, t) + \partial_j u_i(\mathbf{x}, t)]. \quad (6)$$

Writing out the individual components of the equations above, one gets:

$$\begin{aligned} \rho \partial_t^2 u_x &= \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} + f_x, \\ \rho \partial_t^2 u_y &= \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + f_y, \\ \rho \partial_t^2 u_z &= \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z, \end{aligned} \quad (7)$$

$$\begin{aligned} \sigma_{xx} &= \lambda (e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{xx} + q_{xx}, \\ \sigma_{yy} &= \lambda (e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{yy} + q_{yy}, \\ \sigma_{zz} &= \lambda (e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{zz} + q_{zz}, \end{aligned}$$

$$\begin{aligned}
\sigma_{xy} &= 2\mu e_{xy} + q_{xy}, \\
\sigma_{xz} &= 2\mu e_{xz} + q_{xz}, \\
\sigma_{yz} &= 2\mu e_{yz} + q_{yz}.
\end{aligned} \tag{8}$$

Viscoelastic equations of motion

Using the description in Appendix A, we can write down the visco-elastic equations of motion and the constitutive relation. In addition to visco-elastic stress-strain relation we have also introduced time-dependence for the density. The density then shows relaxation in the same way as a visco-elastic medium.

$$\begin{aligned}
\partial_t^2 u_i(\mathbf{x}, t) &= \rho_u^{-1}(\mathbf{x}) \partial_j \sigma_{ij}(\mathbf{x}, t) + f_i(\mathbf{x}, t), \\
&+ \chi(\mathbf{x}, t) * \partial_j \sigma_{ij}(\mathbf{x}, t),
\end{aligned} \tag{9}$$

$$\begin{aligned}
\sigma_{ij}(\mathbf{x}, t) &= \lambda_u e_{kk} \delta_{ij} + 2\mu_u e_{ij} + q_{ij} \\
&+ \delta_{ij} \phi_\lambda(t) * e_{mm} + 2\phi_\mu(t) * e_{ij}
\end{aligned} \tag{10}$$

Writing out the individual components of the equations above, one gets:

$$\begin{aligned}
\partial_t^2 u_x &= \rho_u^{-1} [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}] + f_x, \\
&= +\chi * [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}], \\
\partial_t^2 u_y &= \rho_u^{-1} [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] + f_y, \\
&= \chi * [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] \\
\partial_t^2 u_z &= \rho_u^{-1} [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}] + f_z, \\
&= \chi * [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}].
\end{aligned} \tag{11}$$

$$\begin{aligned}
\sigma_{xx} &= \lambda_u (e_{xx} + e_{yy} + e_{zz}) + 2\mu_u e_{xx} + q_{xx} \\
&+ \phi_\lambda * [e_{xx} + e_{yy} + e_{zz}] + 2\phi_\mu(t) * e_{xx}, \\
\sigma_{yy} &= \lambda_u (e_{xx} + e_{yy} + e_{zz}) + 2\mu_u e_{yy} + q_{yy} \\
&+ \phi_\lambda * [e_{xx} + e_{yy} + e_{zz}] + 2\phi_\mu(t) * e_{yy}, \\
\sigma_{zz} &= \lambda_u (e_{xx} + e_{yy} + e_{zz}) + 2\mu_u e_{zz} + q_{zz} \\
&+ \phi_\lambda * [e_{xx} + e_{yy} + e_{zz}] + 2\phi_\mu(t) * e_{zz}, \\
\sigma_{xy} &= 2\mu_u e_{xy} + q_{xy} + 2\phi_\mu(t) * e_{xy}, \\
\sigma_{xz} &= 2\mu_u e_{xz} + q_{xz} + 2\phi_\mu(t) * e_{xz},
\end{aligned}$$

$$\sigma_{yz} = 2\mu_u e_{yz} + q_{yz} + 2\phi_\mu(t) * e_{yz}. \quad (12)$$

Viscoelastic velocity-stress formulation

Using the velocity $v_i = \dot{u}_i$, one gets

$$\begin{aligned} \partial_t v_x &= \rho_u^{-1} [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}] + f_x, \\ &= +\chi * [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}], \\ \partial_t v_y &= \rho_u^{-1} [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] + f_y, \\ &= \chi * [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] \\ \partial_t v_z &= \rho_u^{-1} [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}] + f_z, \\ &= \chi * [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}]. \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{\sigma}_{xx} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx}, \\ &+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{xx}, \\ \dot{\sigma}_{yy} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy}, \\ &+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{yy}, \\ \dot{\sigma}_{zz} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz}, \\ &+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{zz}, \\ \dot{\sigma}_{xy} &= 2\mu_u \dot{e}_{xy} + q_{xy} + 2\phi_\mu(t) * \dot{e}_{xy}, \\ \dot{\sigma}_{xz} &= 2\mu_u \dot{e}_{xz} + q_{xz} + 2\phi_\mu(t) * \dot{e}_{xz}, \\ \dot{\sigma}_{yz} &= 2\mu_u \dot{e}_{yz} + q_{yz} + 2\phi_\mu(t) * \dot{e}_{yz}. \end{aligned} \quad (14)$$

Memory functions

We now define so-called memory variables by including the time convolution into one set of variables:

$$\gamma_\lambda^l(t) = \frac{1}{\Delta\lambda_l} \phi_\lambda^l * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] \quad (15)$$

$$\gamma_{ij}^l(t) = \frac{1}{\Delta\mu_l} \phi_\mu^l * [\dot{e}_{ij}], \quad (16)$$

$$\theta_{kij}^l(t) = \frac{1}{\Delta\rho_l^{-1}} \chi^l * \partial_k [\sigma_{ij}], \quad (17)$$

Here

$$\Delta\rho_l^{-1} = \rho_u^{-1} \left(1 - \frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}} \right), \quad (18)$$

$$\Delta\lambda_l = \lambda_u \left(1 - \frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}} \right), \quad (19)$$

$$\Delta\mu_l = \mu_u \left(1 - \frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}} \right). \quad (20)$$

This gives the expressions for the γ functions as:

$$\gamma_{\lambda}^l = \left[\frac{\exp(-t/\tau_{\sigma l}^{\lambda})}{\tau_{\sigma l}^{\lambda} \sum_{l=1}^N \frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}} \right] [\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz}], \quad (21)$$

$$\gamma_{ij}^l = \left[\frac{\exp(-t/\tau_{\sigma l}^{\mu})}{\tau_{\sigma l}^{\mu} \sum_{l=1}^N \frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}} \right] \dot{\epsilon}_{ij}, \quad (22)$$

$$\theta_{kij}^l = \left[\frac{\exp(-t/\tau_{\sigma l}^{\rho})}{\tau_{\sigma l}^{\rho} \sum_{l=1}^N \frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}}} \right] \partial_k \sigma_{ij}. \quad (23)$$

and also

$$\begin{aligned} \gamma_{\lambda}(t) &= \sum_{l=1}^N \gamma_{\lambda}^l, \\ \gamma_{ij}(t) &= \sum_{l=1}^N \gamma_{ij}^l, \\ \theta_{kij}(t) &= \sum_{l=1}^N \theta_{kij}^l. \end{aligned}$$

This gives the final form of the viscoelastic equations

$$\begin{aligned} \partial_t v_x &= \rho_i^{-1} (\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}) + f_x, \\ &+ \sum_{l=0}^N \theta_{xxx}^l \Delta \rho_l^{-1} + \sum_{l=0}^N \theta_{xyx}^l \Delta \rho_l^{-1} + \sum_{l=0}^N \theta_{xxz}^l \Delta \rho_l^{-1}, \\ \partial_t v_y &= \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + f_y, \\ &+ \sum_{l=0}^N \theta_{xyx}^l \Delta \rho_l^{-1} + \sum_{l=0}^N \theta_{yyy}^l \Delta \rho_{ll}^{-1} + \sum_{l=0}^N \theta_{zyz}^l \Delta \rho_{ll}^{-1}. \end{aligned}$$

$$\begin{aligned}
\partial_t v_z &= \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z, \\
&+ \sum_{l=0}^N \theta_{xzx}^l + \sum_{l=0}^N \theta_{yzy}^l + \sum_{l=0}^N \theta_{zzz}^l.
\end{aligned} \tag{24}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx} \\
&+ \sum_{l=1}^N \gamma_{\lambda}^l \Delta \lambda_l + 2 \sum_{l=1}^N \gamma_{xx}^l \Delta \mu_l, \\
\dot{\sigma}_{yy} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy}, \\
&+ \sum_{l=1}^N \gamma_{\lambda}^l \Delta \lambda_l + 2 \sum_{l=1}^N \gamma_{yy}^l \Delta \mu_l, \\
\dot{\sigma}_{zz} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz} \\
&+ \sum_{l=1}^N \gamma_{\lambda}^l \Delta \lambda_l + 2 \sum_{l=1}^N \gamma_{zz}^l \Delta \mu_l, \\
\dot{\sigma}_{xy} &= 2\mu \dot{e}_{xy} + 2 \sum_{l=1}^N \gamma_{xy}^l \Delta \mu_l + \dot{q}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu \dot{e}_{xz} + 2 \sum_{l=1}^N \gamma_{xz}^l \Delta \mu_l + \dot{q}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu \dot{e}_{yz} + 2 \sum_{l=1}^N \gamma_{yz}^l \Delta \mu_l + \dot{q}_{yz},
\end{aligned} \tag{25}$$

$$\begin{aligned}
\dot{e}_{xx} &= \partial_x v_x, \\
\dot{e}_{yy} &= \partial_y v_y, \\
\dot{e}_{zz} &= \partial_z v_z, \\
\dot{e}_{xy} &= \frac{1}{2}(\partial_x v_y + \partial_y v_x), \\
\dot{e}_{xz} &= \frac{1}{2}(\partial_x v_z + \partial_z v_x), \\
\dot{e}_{yz} &= \frac{1}{2}(\partial_y v_z + \partial_z v_y).
\end{aligned} \tag{26}$$

Integration of memory functions

The memory functions obeys approximately the relations

$$\begin{aligned}
\gamma_\lambda^l(t) &= \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \gamma_\lambda^l(t - \Delta t) + \left(\frac{\Delta t}{\tau_\sigma^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_\sigma^\lambda}} \right) (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}), \\
\gamma_{ij}^l(t) &= \exp\left(-\frac{\Delta t}{N \tau_\sigma^\mu}\right) \gamma_{ij}^l(t - \Delta t) + \left(\frac{\Delta t}{\tau_\sigma^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_\sigma^\mu}} \right) \dot{e}_{ij}. \\
\theta_{kij}^l(t) &= \exp\left(-\frac{\Delta t}{N \tau_\sigma^\rho}\right) \theta_{kij}^l(t - \Delta t) + \left(\frac{\Delta t}{\tau_\sigma^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_\sigma^\rho}} \right) \partial_k \sigma_{ij}.
\end{aligned} \tag{27}$$

Defining the quantities

$$\alpha_1^l = \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right), \tag{28}$$

$$\alpha_2^l = \frac{\Delta t}{\tau_\sigma^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_\sigma^\lambda}}, \tag{29}$$

$$\beta_1^l = \exp\left(-\frac{\Delta t}{\tau_\sigma^\mu}\right), \tag{30}$$

$$\beta_2^l = \frac{\Delta t}{\tau_\sigma^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_\sigma^\mu}}, \tag{31}$$

$$\eta_1^l = \exp\left(-\frac{\Delta t}{\tau_\sigma^\rho}\right), \tag{32}$$

$$\eta_2^l = \frac{\Delta t}{\tau_\sigma^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_\sigma^\rho}}, \tag{33}$$

we get

$$\begin{aligned}
\gamma_\lambda^l(t) &= \alpha_1^l \gamma_\lambda^l(t - \Delta t) + \alpha_2^l (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}), \\
\gamma_{ij}^l(t) &= \beta_1^l \gamma_{ij}^l(t - \Delta t) + \beta_2^l \dot{e}_{ij}. \\
\theta_{kij}^l(t) &= \eta_1^l \theta_{kij}^l(t - \Delta t) + \eta_2^l \partial_k \sigma_{ij}.
\end{aligned} \tag{34}$$

Discretization of the three dimensional case

We now consider a regular grid with positions \mathbf{x} defined as

$$\mathbf{x} = (x, y, z), \quad (35)$$

$$x = p\Delta x, \quad (36)$$

$$y = q\Delta y, \quad (37)$$

$$z = r\Delta z, \quad (38)$$

$$t = n\Delta t. \quad (39)$$

where $p = 0, 1, 2, \dots, N_x$, $q = 0, 1, 2, \dots, N_y$ and $r = 0, 1, 2, \dots, N_z$ and N_x, N_y and N_z are the number of grid points in the x, y and z -directions. $n = 0, 1, 2, \dots, N_t$ where N_t is the number of time steps.

We will also need a regular grid which is displaced, or staggered, relative to the regular grid. Sometimes we will refer to the regular grid as the reference grid.

The particle velocities v_x, v_y and v_z are defined on staggered grids as follows

$$v_x(\mathbf{x}, t) = v_x(x + \Delta x/2, y, z, t), \quad (40)$$

$$v_y(\mathbf{x}, t) = v_y(x, y + \Delta y/2, z, t), \quad (41)$$

$$v_z(\mathbf{x}, t) = v_z(x, y, z + \Delta z/2, t). \quad (42)$$

The θ_{kij} are defined on the same staggered grid as the particle velocities:

$$\theta_{xxx}(\mathbf{x}, t) = \theta(x + \Delta x/2, y, z)_{xxx}, \quad (43)$$

$$\theta_{yxy}(\mathbf{x}, t) = \theta(x, y + \Delta y/2, z)_{yxy}, \quad (44)$$

$$\theta_{zxx}(\mathbf{x}, t) = \theta(x, y, z + \Delta z/2)_{zxx}, \quad (45)$$

$$\theta_{xyx}(\mathbf{x}, t) = \theta(x + \Delta x/2, y, z)_{xyx}, \quad (46)$$

$$\theta_{yyy}(\mathbf{x}, t) = \theta(x, y + \Delta y/2, z)_{yyy}, \quad (47)$$

$$\theta_{zyz}(\mathbf{x}, t) = \theta(x, y, z + \Delta z/2)_{zyz}, \quad (48)$$

$$\theta_{xzx}(\mathbf{x}, t) = \theta(x + \Delta x/2, y, z)_{xzx}, \quad (49)$$

$$\theta_{yzy}(\mathbf{x}, t) = \theta(x, y + \Delta y/2, z)_{yzy}, \quad (50)$$

$$\theta_{zzz}(\mathbf{x}, t) = \theta(x, y, z + \Delta z/2)_{zzz}. \quad (51)$$

The diagonal stresses and the gamma functions are defined on a regular grid:

$$\sigma_{xx}(\mathbf{x}, t) = \sigma_{xx}(x, y, z, t), \quad (52)$$

$$\sigma_{yy}(\mathbf{x}, t) = \sigma_{yy}(x, y, z, t), \quad (53)$$

$$\sigma_{zz}(\mathbf{x}, t) = \sigma_{zz}(x, y, z, t), \quad (54)$$

$$\gamma_\lambda(\mathbf{x}, t) = \gamma_\lambda(x, y, z, t), \quad (55)$$

$$\gamma_\mu(\mathbf{x}, t) = \gamma_\mu(x, y, z, t), \quad (56)$$

$$\gamma_{xx}(\mathbf{x}, t) = \gamma_{xx}(x, y, z, t), \quad (57)$$

$$\gamma_{yy}(\mathbf{x}, t) = \gamma_{yy}(x, y, z, t), \quad (58)$$

$$\gamma_{zz}(\mathbf{x}, t) = \gamma_{zz}(x, y, z, t). \quad (59)$$

The off-diagonal stresses and the off-diagonal gamma functions are defined on several staggered grids

$$\sigma_{xy}(\mathbf{x}, t) = \sigma_{xy}(x + \Delta x/2, y + \Delta y/2, z, t), \quad (60)$$

$$\sigma_{xz}(\mathbf{x}, t) = \sigma_{xz}(x + \Delta x/2, y, z + \Delta z/2, t), \quad (61)$$

$$\sigma_{yz}(\mathbf{x}, t) = \sigma_{yz}(x, y + \Delta y/2, z + \Delta z/2, t), \quad (62)$$

$$\gamma_{xy}(\mathbf{x}, t) = \gamma_{xy}(x + \Delta x/2, y + \Delta y/2, z, t), \quad (63)$$

$$\gamma_{xz}(\mathbf{x}, t) = \gamma_{xz}(x + \Delta x/2, y, z + \Delta z/2, t), \quad (64)$$

$$\gamma_{yz}(\mathbf{x}, t) = \gamma_{yz}(x, y + \Delta y/2, z + \Delta z/2, t), \quad (65)$$

$$(66)$$

The Lamé parameters are defined on a regular grid as follows: as follows:

$$\lambda_u(\mathbf{x}) = \lambda_u(x, y, z), \quad (67)$$

$$\mu_u(\mathbf{x}) = \mu_u(x, y, z). \quad (68)$$

The inverse density and μ are defined on three different staggered grids as follows:

$$\rho_x^{-1}(\mathbf{x}) = \rho^{-1}(x + \Delta x/2, y, z), \quad (69)$$

$$\rho_y^{-1}(\mathbf{x}) = \rho^{-1}(x, y + \Delta y/2, z), \quad (70)$$

$$\rho_z^{-1}(\mathbf{x}) = \rho^{-1}(x, y, z + \Delta z/2), \quad (71)$$

$$\mu_{uxy}(\mathbf{x}) = \mu_u(x + \Delta x/2, y + \Delta y/2, z), \quad (72)$$

$$\mu_{uyz}(\mathbf{x}) = \mu_u(x, y + \Delta y/2, z + \Delta z/2), \quad (73)$$

$$\mu_{uxz}(\mathbf{x}) = \mu_u(x + \Delta x/2, y, z + \Delta z/2), \quad (74)$$

$$(75)$$

The visco-elastic parameters β_1 , β_2 and η_1 and η_2 are also defined on several staggered grids:

$$\beta_{1xy}(\mathbf{x}) = \beta_1(x + \Delta x, y + \Delta y, z), \quad (76)$$

$$\beta_{1yz}(\mathbf{x}) = \beta_1(x, y + \Delta y, z + \Delta z), \quad (77)$$

$$\beta_{1xz}(\mathbf{x}) = \beta_1(x + \Delta x, y, z + \Delta z), \quad (78)$$

$$\beta_{2xy}(\mathbf{x}) = \beta_2(x + \Delta x, y + \Delta y, z), \quad (79)$$

$$\beta_{2yz}(\mathbf{x}) = \beta_2(x, y + \Delta y, z + \Delta z), \quad (80)$$

$$\beta_{2xz}(\mathbf{x}) = \beta_2(x + \Delta x, y, z + \Delta z), \quad (81)$$

$$\eta_{1x}(\mathbf{x}) = \eta_1(x + \Delta x/2, y, z), \quad (82)$$

$$\eta_{2x}(\mathbf{x}) = \eta_2(x + \Delta x/2, y, z), \quad (83)$$

$$\eta_{1y}(\mathbf{x}) = \eta_1(x, y + \Delta y/2, z), \quad (84)$$

$$\eta_{2y}(\mathbf{x}) = \eta_1(x, y + \Delta y/2, z), \quad (85)$$

$$\eta_{1z}(\mathbf{x}) = \eta_1(x, y, z + \Delta z/2), \quad (86)$$

$$\eta_{2z}(\mathbf{x}) = \eta_1(x, y, z + \Delta z/2). \quad (87)$$

Differentiation is now replaced by numerical approximations so that ∂_x, ∂_y and ∂_z are replaced with numerical operators $d_x^+, d_x^-, d_y^+, d_y^-, d_z^+$ and d_z^- . These operators connects the staggered and reference grids, and we illustrate this with the differentiation in the x-direction. The derivative of a function $a(x)$ is approximately given at $a(x + \Delta x/2)$ and at $a(x - \Delta x/2)$ by

$$a'(x + \Delta x/2) = d_x^+ a(x), \quad (88)$$

$$a'(x - \Delta x/2) = d_x^- a(x). \quad (89)$$

The differentiators d^+ and d^- are given by (Holberg, 1987)

$$\begin{aligned} \partial^+ &= \frac{1}{\Delta x} \sum_{l=1}^L \alpha_l [u(x + l\Delta x) - u(x - (l-1)\Delta x)] \\ \partial^- &= \frac{1}{\Delta x} \sum_{l=1}^L \alpha_l [u(x + (l-1)\Delta x) - u(x - l\Delta x)] \end{aligned} \quad (90)$$

where the coefficients α_l are found through an optimization procedure. Similar differentiators are defined for the y -direction and for the z -direction, with obvious names.

Using the numerical differentiators the equations of motion becomes:

$$\partial_t v_x = \rho_i^{-1} (d_x^+ \sigma_{xx} + d_y^+ \sigma_{xy} + d_z^+ \sigma_{xz}) + f_x \quad (91)$$

$$+ \sum_{l=0}^N \theta_{xxx}^l + \sum_{l=0}^N \theta_{xyy}^l + \sum_{l=0}^N \theta_{zzz}^l, \quad (92)$$

$$\partial_t v_y = d_x^+ \sigma_{yx} + d_y^+ \sigma_{yy} + d_z^+ \sigma_{yz} + f_y, \quad (93)$$

$$+ \sum_{l=0}^N \theta_{xyx}^l + \sum_{l=0}^N \theta_{yyy}^l + \sum_{l=0}^N \theta_{zyz}^l, \quad (94)$$

$$\partial_t v_z = d_x^+ \sigma_{zx} + d_y^+ \sigma_{zy} + d_z^+ \sigma_{zz} + f_z, \quad (95)$$

$$+ \sum_{l=0}^N \theta_{xxz}^l + \sum_{l=0}^N \theta_{yzy}^l + \sum_{l=0}^N \theta_{zzz}^l. \quad (96)$$

The computation of the strains becomes as follows:

$$\begin{aligned} \dot{e}_{xx} &= d_x^- v_x, \\ \dot{e}_{yy} &= d_y^- v_y, \\ \dot{e}_{zz} &= d_z^- v_z, \\ \dot{e}_{xy} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x), \\ \dot{e}_{xz} &= \frac{1}{2}(d_x^+ v_z + d_z^+ v_x), \\ \dot{e}_{yz} &= \frac{1}{2}(d_y^+ v_z + d_z^+ v_y). \end{aligned} \quad (97)$$

The time derivatives is approximated by the central difference

$$\dot{a}(t) = \frac{a(t + \Delta t/2) - a(t - \Delta t/2)}{\Delta t} \quad (98)$$

Solution algorithm for the three dimensional case

We are now in a position to formulate a complete numerical solution of the visco-elastic equations.

Computation of the particle velocity

We use the expression for the approximate time derivative given by equation (98) in equations (??) to obtain an expression for the components of the particle velocity

$$\begin{aligned} v_x(t + \Delta t/2) &= \Delta t \rho_x^{-1} \left[d_x^+ \sigma_{xx}(t) + d_y^+ \sigma_{xy}(t) + d_z^+ \sigma_{xz}(t) \right] + \Delta t \rho_x^{-1} f_x(t) + \\ &+ \Delta t \sum_{l=0}^N \theta_{xxx}^l(t) + \Delta t \sum_{l=0}^N \theta_{yxy}^l(t) + \Delta t \sum_{l=0}^N \theta_{zxx}^l(t) + v_x(t - \Delta t/2), \end{aligned}$$

$$\begin{aligned}
v_y(t + \Delta t/2) &= \Delta t \rho_y^{-1} \left[d_x^+ \sigma_{yx}(t) + d_y^+ \sigma_{yy}(t) + d_z^+ \sigma_{yz}(t) \right] + \Delta t \rho_y^{-1} f_y(t) \\
&+ \Delta t \sum_{l=0}^N \theta_{xyx}^l(t) + \Delta t \sum_{l=0}^N \theta_{yyy}^l(t) + \Delta t \sum_{l=0}^N \theta_{zyz}^l(t) + v_y(t - \Delta t/2), \\
v_z(t + \Delta t/2) &= \Delta t \rho_z^{-1} \left[d_x^+ \sigma_{zx}(t) + d_y^+ \sigma_{zy}(t) + d_z^+ \sigma_{zz}(t) \right] + \Delta t \rho_z^{-1} f_z(t) \\
&+ \Delta t \sum_{l=0}^N \theta_{xzx}^l(t) + \Delta t \sum_{l=0}^N \theta_{yzy}^l(t) + \Delta t \sum_{l=0}^N \theta_{zzz}^l(t) + v_z(t - \Delta t/2)
\end{aligned} \tag{99}$$

The strains can now be computed from equation (97)

$$\begin{aligned}
\dot{\epsilon}_{xx}(t + \Delta t/2) &= d_x^- v_x(t + \Delta t/2), \\
\dot{\epsilon}_{yy}(t + \Delta t/2) &= d_y^- v_y(t + \Delta t/2), \\
\dot{\epsilon}_{zz}(t + \Delta t/2) &= d_z^- v_z(t + \Delta t/2), \\
\dot{\epsilon}_{xy}(t + \Delta t/2) &= \frac{1}{2} \left[d_x^+ v_y(t + \Delta t/2) + d_y^+ v_x(t + \Delta t/2) \right], \\
\dot{\epsilon}_{xz}(t + \Delta t/2) &= \frac{1}{2} \left[d_x^+ v_z(t + \Delta t/2) + d_z^+ v_x(t + \Delta t/2) \right], \\
\dot{\epsilon}_{yz}(t + \Delta t/2) &= \frac{1}{2} \left[d_y^+ v_z(t + \Delta t/2) + d_z^+ v_y(t + \Delta t/2) \right].
\end{aligned} \tag{100}$$

Equations (108) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned}
\sigma_{xx}(t + \Delta t) &= \Delta t \lambda_u [\dot{\epsilon}_{xx}(t + \Delta t/2) + \dot{\epsilon}_{yy}(t + \Delta t/2) + \dot{\epsilon}_{zz}(t + \Delta t/2)] \\
&+ 2\mu_u \dot{\epsilon}_{xx}(t + \Delta t/2) + \Delta t \dot{q}_{xx} \\
&+ \Delta t \sum_{l=1}^N \gamma_{\lambda}^l(t + \Delta t/2) \Delta \lambda_l + 2\Delta t \sum_{l=1}^N \gamma_{xx}^l(t + \Delta t/2) \Delta \mu_l(t + \Delta t/2) \\
&+ \sigma_{xx}(t),
\end{aligned} \tag{101}$$

$$\begin{aligned}
\sigma_{yy}(t + \Delta t) &= \Delta t \lambda_u [\dot{\epsilon}_{xx}(t + \Delta t/2) + \dot{\epsilon}_{yy}(t + \Delta t/2) + \dot{\epsilon}_{zz}(t + \Delta t/2)] \\
&+ 2\Delta t \mu_u \dot{\epsilon}_{yy}(t + \Delta t/2) + \Delta t \dot{q}_{yy}(t + \Delta t/2)
\end{aligned} \tag{102}$$

$$\begin{aligned}
&+ \Delta t \sum_{l=1}^N \gamma_{\lambda}^l(t + \Delta t/2) \Delta \lambda_l + 2\Delta t \sum_{l=1}^N \gamma_{yy}^l(t + \Delta t/2) \Delta \mu_l \\
&+ \sigma_{yy}(t)
\end{aligned} \tag{103}$$

$$\tag{104}$$

$$\begin{aligned}
\sigma_{zz}(t + \Delta t) &= \Delta t \lambda_u [\dot{\epsilon}_{xx}(t + \Delta t/2) + \dot{\epsilon}_{yy}(t + \Delta t/2) + \dot{\epsilon}_{zz}(t + \Delta t/2)] \\
&+ 2\mu_u \dot{\epsilon}_{zz}(t + \Delta t/2) + \Delta t \dot{q}_{zz}(t + \Delta t/2)
\end{aligned} \tag{105}$$

$$\begin{aligned}
& + \Delta t \sum_{l=1}^N \gamma_{\lambda}^l(t + \Delta t/2) \Delta \lambda_l + 2\Delta t \sum_{l=1}^N \gamma_{zz} \Delta \mu_l(t + \Delta t/2), \\
& + \sigma_{zz}.
\end{aligned} \tag{106}$$

$$\begin{aligned}
\sigma_{xy}(t + \Delta t) &= 2\Delta t \mu \dot{e}_{xy}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^N \gamma_{xy}^l(t + \Delta t/2) \Delta \mu_l + \Delta t \dot{q}_{xy}(t + \Delta t/2) \\
&+ \sigma_{xy}(t), \\
\sigma_{xz}(t + \Delta t) &= 2\Delta t \mu \dot{e}_{xz}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^N \gamma_{xz}^l(t + \Delta t/2) \Delta \mu_l + \Delta t \dot{q}_{xz}(t + \Delta t/2) \\
&+ \sigma_{xz}(t),
\end{aligned} \tag{107}$$

$$\begin{aligned}
\sigma_{yz}(t + \Delta t) &= 2\Delta \mu \dot{e}_{yz}(t + \Delta t/2) + 2\Delta t \sum_{l=1}^N \gamma_{yz}^l(t + \Delta t/2) \Delta \mu_l + \dot{q}_{yz}(t + \Delta t/2) \\
&+ \sigma_{yz}(t).
\end{aligned} \tag{108}$$

The gamma functions are updated as follows

$$\begin{aligned}
\gamma_{\lambda}^l(t + 3/2\Delta t) &= \alpha_1^l \gamma_{\lambda}^l(t + \Delta t/2) + \alpha_2^l [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)], \\
\gamma_{\mu}^l(t + 3/2\Delta t) &= \beta_1^l \gamma_{\mu}^l(t + \Delta t/2) + \beta_2^l [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{yy}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)], \\
\gamma_{xy}^l(t + 3/2\Delta t) &= \beta_{1xy}^l \gamma_{xy}^l(t + \Delta t/2) + \beta_{2xy}^l \dot{e}_{xy}(t + \Delta t/2). \\
\gamma_{xz}^l(t + 3/2\Delta t) &= \beta_{1xz}^l \gamma_{xz}^l(t + \Delta t/2) + \beta_{2xz}^l \dot{e}_{xz}(t + \Delta t/2). \\
\gamma_{yz}^l(t + 3/2\Delta t) &= \beta_{1yz}^l \gamma_{yz}^l(t + \Delta t/2) + \beta_{2yz}^l \dot{e}_{yz}(t + \Delta t/2).
\end{aligned} \tag{109}$$

The θ functions are updated as:

$$\begin{aligned}
\theta_{xxx}(t + \Delta t) &= \eta_1^l \theta_{xxx}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{xx}(t + \Delta t/2). \\
\theta_{yxy}(t + \Delta t) &= \eta_1^l \theta_{yxy}^l(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{xy}(t + \Delta t/2). \\
\theta_{zxx}(t + \Delta t) &= \eta_1^l \theta_{zxx}^l(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{xx}(t + \Delta t/2). \\
\theta_{xyx}(t + \Delta t) &= \eta_1^l \theta_{xyx}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{yx}(t + \Delta t/2). \\
\theta_{yyy}(t + \Delta t) &= \eta_1^l \theta_{yyy}^l(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{yy}(t + \Delta t/2). \\
\theta_{zyz}(t + \Delta t) &= \eta_1^l \theta_{zyz}^l(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{yz}(t + \Delta t/2). \\
\theta_{xyx}(t + \Delta t) &= \eta_1^l \theta_{xyx}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{yx}(t + \Delta t/2). \\
\theta_{xzx}(t + \Delta t) &= \eta_1^l \theta_{xzx}^l(t + \Delta t/2) + \eta_2^l \partial_x \sigma_{zx}(t + \Delta t/2). \\
\theta_{yzy}(t + \Delta t) &= \eta_1^l \theta_{yzy}^l(t + \Delta t/2) + \eta_2^l \partial_y \sigma_{zy}(t + \Delta t/2).
\end{aligned}$$

$$\theta_{zzz}(t + \Delta t) = \eta_1^l \theta_{zzz}^l(t + \Delta t/2) + \eta_2^l \partial_z \sigma_{zz}(t + \Delta t/2). \quad (110)$$

2D Acoustic case

For the acoustic 2D case we reduce the equations above by neglecting the y-axis terms and putting $\mu = 0$. We consider also the pseudo-stress σ defined by

$$\sigma = \frac{1}{2} (\sigma_{xx} + \sigma_{zz})$$

We then get the acoustic 2D scheme as:

$$\begin{aligned} v_x(t + \Delta t/2) &= \Delta t \left[\rho_{ux}^{-1} d_x^+ \sigma_{xx}(t) + \rho_{ux}^{-1} f_x(t) \right] + \\ &+ \Delta t \sum_{l=0}^N \theta_x^l(t) \Delta \rho_x^{-1} + v_x(t - \Delta t/2), \\ v_z(t + \Delta t/2) &= \Delta t \left[\rho_{uz}^{-1} d_z^+ \sigma_{zz}(t) + \Delta t \rho_{uz}^{-1} f_z(t) \right] \\ &+ \Delta t \sum_{l=0}^N \theta_z^l(t) \Delta \rho_z^{-1} + v_z(t - \Delta t/2). \end{aligned}$$

The strains can now be computed from :

$$\begin{aligned} \dot{e}_{xx}(t + \Delta t/2) &= d_x^- v_x(t + \Delta t/2), \\ \dot{e}_{zz}(t + \Delta t/2) &= d_z^- v_z(t + \Delta t/2). \end{aligned}$$

Equations (108) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned} \sigma(t + \Delta t) &= \Delta t \lambda_u [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)] + \Delta t \dot{q}, \\ &+ \Delta t \sum_{l=1}^N \gamma^l(t + \Delta t/2) \Delta \lambda_l + \sigma(t). \end{aligned}$$

We now split the γ^l into two parts γ_x^l and γ_z^l as follows:

$$\begin{aligned} \sigma(t + \Delta t) &= \Delta t \lambda_u [\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)] + \Delta t \dot{q} \\ &+ \Delta t \sum_{l=1}^N \left[\gamma_x^l(t + \Delta t/2) \Delta \lambda_l + \gamma_z^l(t + \Delta t/2) \Delta \lambda_l \right] + \sigma(t). \end{aligned}$$

The θ functions are updated as:

$$\begin{aligned}\theta_x(t + \Delta t) &= \eta_{1x}^l \theta_x^l(t) + \eta_{2x}^l \partial_x \sigma(t), \\ \theta_z(t + \Delta t) &= \eta_{1z}^l \theta_z^l(t) + \eta_{2z}^l \partial_z \sigma(t).\end{aligned}$$

The γ functions are given by

$$\begin{aligned}\gamma_x^l(t + 3/2\Delta t) &= \alpha_{1x}^l \gamma_x^l(t + \Delta t/2) + \alpha_{2x}^l \dot{\epsilon}_{xx}(t + \Delta t/2), \\ \gamma_z^l(t + 3/2\Delta t) &= \alpha_{1z}^l \gamma_z^l(t + \Delta t/2) + \alpha_{2z}^l \dot{\epsilon}_{zz}(t + \Delta t/2).\end{aligned}$$

Standard linear solid

The coefficients are

$$\begin{aligned}\alpha_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\ \alpha_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})}, \\ \alpha_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\ \alpha_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})}, \\ \eta_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\rho}\right), \\ \eta_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}, \\ \eta_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\rho}\right), \\ \eta_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}.\end{aligned}$$

The profile functions d_x and d_z are

$$d_x(x) = (x/L)^2, d_z(z) = (z/L)^2,$$

where L is the length of the absorbing layer and we also have

$$\Delta\lambda_l = \lambda_u \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right) \Delta\rho^{-1} = \rho_u^{-1} \left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right) \quad (111)$$

Maxwell solid

The coefficients are

$$\alpha_{1x} = \exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\lambda}\right), \quad (112)$$

$$\alpha_{2x} = -\frac{d_x(x)\Delta t}{\tau_0^\lambda}, \quad (113)$$

$$\alpha_{1z} = \exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\lambda}\right), \quad (114)$$

$$\alpha_{2z} = -\frac{d_z(z)\Delta t}{\tau_0^\lambda}. \quad (115)$$

$$\eta_{1x} = \exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\rho}\right), \quad (116)$$

$$\eta_{2x} = -\frac{d_x(x)\Delta t}{\tau_0^\rho}, \quad (117)$$

$$\eta_{1z} = \exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\rho}\right), \quad (118)$$

$$\eta_{2z} = -\frac{d_z(z)\Delta t}{\tau_0^\rho}. \quad (119)$$

We also have

$$\Delta\lambda = \lambda_u, \quad (120)$$

$$\Delta\rho^{-1} = \rho_u^{-1}. \quad (121)$$

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APPENDIX A: The viscoelastic standard linear solid

Bolzmann's generalization of Hook's law to the visco-elastic case is (Hudson, 1985):

$$\sigma_{ij} = \psi_{ijkl} * \dot{e}_{kl}, \quad (\text{A-1})$$

where ψ_{ijkl} is known as the relaxation tensor. The $*$ denotes convolution defined by

$$a(t) * b(t) = \int_0^t a(t - \tau)b(\tau). \quad (\text{A-2})$$

Integrating (A-1) by parts

$$\sigma_{ij}(t) = \int_0^t \dot{\psi}_{ijkl}(t - \tau)e_{kl}(\tau) + \psi_{ijkl}(t)e_{kl}(0), \quad (\text{A-3})$$

and using $e(t = 0) = 0$ I get

$$\sigma_{ij}(t) = \psi(0)_{ijkl}e_{kl}(t) + \int_0^t \dot{\psi}_{ijkl}(t - \tau)e_{kl}(\tau) \quad (\text{A-4})$$

For the Zener model the components of the ψ_{ijkl} tensor have the form

$$\psi(t) = K \left[1 - \frac{1}{N} \sum_{l=1}^N \left(1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \quad (\text{A-5})$$

where K_r is a relaxed modulus, N is the number of Zener mechanisms, $\tau_{\sigma l}$ and τ_{el} are relaxation times. $H(t)$ is the Heavy side function. The time derivative of ψ is equal to:

$$\dot{\psi} = \phi(t), \quad (\text{A-6})$$

where ϕ is equal to:

$$\phi(t) = \frac{1}{N} \sum_{l=1}^N \left[\left(\frac{K_r}{\tau_{\sigma l}} \right) \left(1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \quad (\text{A-7})$$

Using the above we have for the stress:

$$\sigma_{ij}(t) = c_{ijkl}e_{kl} + \int_{0+}^t \phi_{ijkl}(t-\tau)e_{kl}(\tau) \quad (\text{A-8})$$

This is most conveniently written as

$$\sigma_{ij}(t) = c_{ijkl}(t) * e_{kl}(t), \quad (\text{A-9})$$

where

$$c_{ijkl}(t) = \psi(0)_{ijkl}\delta(t) + \phi_{ijkl}(t), \quad (\text{A-10})$$

By definition $\psi(t=0)$ corresponds to the unrelaxed modulus so that we have

$$K_u = \frac{1}{N} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}} K_r \quad (\text{A-11})$$

or

$$K_r = \frac{K_u}{\frac{1}{N} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}}} \quad (\text{A-12})$$

The ϕ function can then be expressed in terms of the unrelaxed moduli:

$$\phi(t) = \sum_{l=1}^N \left[\left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}}} \right) K_u \left(1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \right] \quad (\text{A-13})$$

Finally, we express ϕ as:

$$\phi(t) = \sum_{l=1}^N \phi^l(t) \quad (\text{A-14})$$

where

$$\phi^l(t) = \left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l} \sum_{l=1}^N \frac{\tau_{el}}{\tau_{\sigma l}}} \right) \Delta K_l \quad (\text{A-15})$$

and ΔK_l is

$$\Delta K_l = K_u \left(1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \quad (\text{A-16})$$

It is most practical to write the time-dependent visco-elastic constants as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t), \quad (\text{A-17})$$

$$\mu(t) = \mu_u \delta(t) + \phi_\mu(t), \quad (\text{A-18})$$

where ϕ_λ is given as:

$$\phi_\lambda(t) = \sum_{l=1}^N \left(\frac{\exp(-t/\tau_{\sigma l}^\lambda)}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \right) \Delta \lambda_l \quad (\text{A-19})$$

and ϕ_μ is given as:

$$\phi_\mu(t) = \sum_{l=1}^N \left(\frac{\exp(-t/\tau_{\sigma l}^\mu)}{\tau_{\sigma l}^\mu \sum_{l=1}^N \frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}} \right) \Delta \mu_l \quad (\text{A-20})$$

Q-factors

See Casula et al. (1992) for further relations between Q and relaxation times. The Q values are related to the Fourier transform of the λ modulus as

$$Q_\lambda^{-1}(\omega) = \frac{\text{Im}\lambda(\omega)}{\text{Re}\lambda(\omega)} \quad (\text{A-21})$$

Assuming λ is given as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t). \quad (\text{A-22})$$

The fourier transform of λ is given by

$$\lambda(\omega) = \lambda_u + \int_{-\infty}^{\infty} \phi_\lambda(t) \exp(-i\omega t) dt. \quad (\text{A-23})$$

The Fourier transform of ϕ_λ is:

$$\phi_\lambda(\omega) = \frac{1}{N} \sum_{l=1}^N \left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \int_0^{+\infty} dt \exp(-i\omega t) \exp(-t/\tau_{\sigma l}^\lambda). \quad (\text{A-24})$$

The results is:

$$\phi_\lambda(\omega) = \frac{1}{N} \sum_{l=1}^N \left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \frac{1}{1 + i\omega \tau_{\sigma l}^\lambda}. \quad (\text{A-25})$$

The fourier transform of λ is then

$$\lambda(\omega) = \lambda_u + \frac{1}{N} \sum_{l=1}^N \left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda} \right) \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda} \right) \frac{1}{1 + i\omega\tau_{\sigma l}^\lambda}. \quad (\text{A-26})$$

After some (tedious) algebra one obtains

$$\lambda(\omega) = \lambda_r \frac{1}{N} \sum_{l=1}^N \frac{1 + i\omega\tau_{\epsilon l}^\lambda}{1 + i\omega\tau_{\sigma l}^\lambda} \quad (\text{A-27})$$

Separating into real and imaginary parts, I get

$$\lambda(\omega) = \lambda_r \frac{1}{N} \sum_{l=1}^N \frac{1 + \omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda}{1 + (\omega\tau_{\sigma l}^\lambda)^2} \quad (\text{A-28})$$

$$+ i\lambda_r \frac{1}{N} \sum_{l=1}^N \frac{\omega\tau_{\sigma l}^\lambda(\tau_{\epsilon l}^\lambda/\tau_{\sigma l}^\lambda - 1)}{1 + (\omega\tau_{\sigma l}^\lambda)^2} \quad (\text{A-29})$$

We then have

$$Q_\lambda^{-1} = \frac{\sum_{l=1}^N \omega\tau_{\sigma l}^\lambda (\tau_{\epsilon l}^\lambda/\tau_{\sigma l}^\lambda - 1) / [1 + (\omega\tau_{\sigma l}^\lambda)^2]}{\sum_{l=1}^N (1 + \omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda) / [1 + (\omega\tau_{\sigma l}^\lambda)^2]} \quad (\text{A-30})$$

The results for the frequency dependence of μ is obtained in exactly the same manner as above:

$$\mu(\omega) = \mu_r \frac{1}{N} \sum_{l=1}^N \frac{1 + i\omega\tau_{\epsilon l}^\mu}{1 + i\omega\tau_{\sigma l}^\mu} \quad (\text{A-31})$$

and the Q-factor for μ is

$$Q_\mu^{-1} = \frac{\sum_{l=1}^N \omega\tau_{\sigma l}^\mu (\tau_{\epsilon l}^\mu/\tau_{\sigma l}^\mu - 1) / [1 + (\omega\tau_{\sigma l}^\mu)^2]}{\sum_{l=1}^N (1 + \omega^2\tau_{\sigma l}^\mu\tau_{\epsilon l}^\mu) / [1 + (\omega\tau_{\sigma l}^\mu)^2]} \quad (\text{A-32})$$

In practice we need to relate Q_l and Q_μ to Q_κ . We use the relation

$$\kappa(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega). \quad (\text{A-33})$$

Splitting into real and imaginary parts

$$\kappa_r(\omega) = \lambda_r(\omega) + \frac{2}{3}\mu_r(\omega), \quad \kappa_i(\omega) = \lambda_i(\omega) + \frac{2}{3}\mu_i(\omega). \quad (\text{A-34})$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\lambda_i + (2/3)\mu_i}{\lambda_r + (2/3)\mu_r} \quad (\text{A-35})$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\left(\frac{\lambda_i}{\lambda_r}\right) \lambda_r + (2/3) \left(\frac{\mu_i}{\mu_r}\right) \mu_r}{\lambda_r + (2/3)\mu_r} \quad (\text{A-36})$$

$$Q_\kappa^{-1} = \frac{Q_\lambda^{-1} \lambda_r + (2/3) Q_\mu^{-1}}{\lambda_r + (2/3)\mu_r} \quad (\text{A-37})$$

We can use the P-wave and S-wave velocities

$$\begin{aligned} \lambda &= \rho V_p^2 - (2/3) \rho V_s^2 \\ \mu &= \rho V_s^2 \end{aligned} \quad (\text{A-38})$$

$$Q_\kappa^{-1} = Q_\lambda^{-1} \left[1 - \left(\frac{2}{3}\right) \left(\frac{V_s}{V_p}\right)^2 \right] + Q_\mu^{-1} \left(\frac{2}{3}\right) \left(\frac{V_s}{V_p}\right)^2 \quad (\text{A-39})$$

Q-model parametrization

Q-models can be described by the two relaxation times, τ_σ and τ_ϵ . However it is simpler to use the two parameters τ_0 and Q_0 to describe a model. According to Casula et al. (1992), Appendix B, we have

$$Q(\omega) = Q_0 \frac{1 + \omega^2 \tau_0^2}{2\omega \tau_0}$$

where

$$\begin{aligned} Q_0 &= \frac{2\tau_0}{\tau_\epsilon - \tau_\sigma}, \\ \tau_0^2 &= \tau_\epsilon \tau_\sigma. \end{aligned}$$

$\omega = 1/\tau_0$ is the minimum value for $Q(\omega)$, i.e. the absorption top. We can now find τ_σ and τ_ϵ in terms of τ_0 and Q_0 as:

$$\begin{aligned} \tau_\epsilon &= \frac{\tau_0}{Q_0} \left[\sqrt{Q_0^2 + 1} + 1 \right], \\ \tau_\sigma &= \frac{\tau_0}{Q_0} \left[\sqrt{Q_0^2 + 1} - 1 \right]. \end{aligned}$$

Effective density

We now assume that the effective density has the following form

$$\rho_{eff}^{-1}(t) = s(t). \quad (\text{A-40})$$

We also assume that

$$s(t) = s(0)\delta(t) + \chi(t) \quad (\text{A-41})$$

So that the inverse of the effective density reads

$$\rho_{eff}^{-1}(t) = s_u\delta(t) + \chi(t). \quad (\text{A-42})$$

where $s_u = s(0) = \rho_u^{-1}$.

APPENDIX C: The Maxwell visco-elastic solid

According to Casula et al. (1992) the Maxwell visco-elastic solid has a modulus given by

$$\lambda(t) = \lambda_u \left[\delta(t) - \frac{1}{\tau_0} \exp(-t/\tau_0) \right] \quad (\text{A-1})$$

In the frequency domain one gets

$$\lambda(\omega) = \frac{\tau_0 \omega}{\omega \tau_0 - i}. \quad (\text{A-2})$$

or,

$$\lambda(\omega) = \frac{\tau_0 i \omega}{i \omega \tau_0 + 1}. \quad (\text{A-3})$$

The function ϕ is now:

$$\phi(t) = -\Delta\lambda \frac{1}{\tau_0} \exp(-t/\tau_0), \quad (\text{A-4})$$

where $\Delta\lambda = \lambda_u$. The Q-value are related to τ_0 by:

$$\tau_0 = Q(\omega)/\omega. \quad (\text{A-5})$$

APPENDIX D: Recursive computation of γ and s .

The γ 's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\begin{aligned}\gamma_x^\lambda(t + \Delta t) &= \int_0^{t+\Delta t} d\tau \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp\left(-\frac{t + \Delta t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau). \\ \gamma_x^\lambda(t + \Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_0^{t+\Delta t} d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau). \\ \gamma_x^\lambda(t + \Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_0^t d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau) \\ &\quad + \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_t^{t+\Delta t} d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau).\end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t + \Delta t$

$$\begin{aligned}\gamma_x^\lambda(t + \Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \int_0^t d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(\tau) \\ &\quad + \frac{1}{\tau_\epsilon} \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \exp\left(-\frac{t - \tau}{\tau_\sigma^\lambda}\right)\end{aligned}$$

Performing the integral we then get

$$\begin{aligned}\gamma_x^\lambda(t + \Delta t) &= \gamma_x^\lambda(t) \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \\ &\quad + \frac{\tau_\sigma}{\tau_\epsilon} \left[1 - \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right)\right] \dot{e}_{xx}(t).\end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned}\gamma_x^\lambda(t + \Delta t) &= \gamma_x^\lambda(t) \exp\left(-\frac{\Delta t}{\tau_\sigma^\lambda}\right) \\ &\quad + \frac{\Delta t}{\tau_\epsilon} \dot{e}_{xx}(t).\end{aligned}$$

APPENDIX E: Comparison with CPML

Komatitsch and Martin (2007) show that the implementation of the CPML method can be performed by replacing each spatial derivative with (Their's equation (16) and (18)):

$$s_x = \delta(t) - d_x H(t) \exp[-(d_x + \alpha_x)t]. \quad (\text{A-1})$$

We compare this with our equation

$$\lambda(t)/\lambda_u = \delta(t) + \frac{1}{\tau_\epsilon} \exp(-t/\tau_\sigma) H(t) \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right) \quad (\text{A-2})$$

Comparing equation (A-1) with equation (A-2) one gets:

$$-d_x = \frac{1}{\tau_\epsilon} \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right), \quad (\text{A-3})$$

$$d_x + \alpha = \frac{1}{\tau_\sigma} \quad (\text{A-4})$$

Solving for τ_ϵ and τ_σ one gets

$$\tau_\sigma = \frac{1}{(d_x + \alpha)}, \quad (\text{A-5})$$

$$\tau_\epsilon = \frac{1}{\alpha}. \quad (\text{A-6})$$

. Here

$$d_x(x) = d_0 \left(\frac{x}{L}\right)^2, \quad (\text{A-7})$$

where d_0 is a constant and L is the length of the PML zone and x is the distance from the start (outer border) of the PML zone.