Visco-elastic Finite difference seismic modeling with time-dependent effective density

B.Arntsen

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PRELIMINARY VERSION UNDER DEVELOPMENT. CONTAINS ERRORS.

Elastic equations of motion

Consider an elastic medium characterized by the density $\rho(\boldsymbol{x})$, and lamé parameters $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ at each spatial point $\boldsymbol{x}$. Now assume that $u(\boldsymbol{x})_i$ is the i'th component of the particle displacement and σ_{ij} is the stress tensor and c_{ijkl} is Hook's tensor. The elastic equations of motion and Hook's law in Cartesian coordinates are given as

$$\begin{aligned} \rho(\boldsymbol{x})\partial_t^2 u_i(\boldsymbol{x},t) &= \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), && (1)\\ \sigma_{ij}(\boldsymbol{x},t) &= c_{ijkl}e_{kl} + q_{ij}. && (2) \end{aligned}$$

Here f_i is a driving force, and q_{ij} is a driving stress. In the isotropic case one has

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{3}$$

giving

$$\begin{aligned} \rho(\boldsymbol{x})\partial_t^2 u_i(\boldsymbol{x},t) &= \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), && (4)\\ \sigma_{ij}(\boldsymbol{x},t) &= \lambda(\boldsymbol{x})e_{kk}\delta_{ij} + 2\mu e_{ij} + q_{ij}. && (5) \end{aligned}$$

Here f_i is the i'th component of the driving force, while the strain tensor e_{ij} is equal to

$$e_{ij} = \frac{1}{2}[\partial_i u_j(\boldsymbol{x},t) + \partial_j u_i(\boldsymbol{x},t)]. \tag{6}$$

Writing out the individual components of the equations above, one gets:

$$\begin{aligned} \rho\partial_t^2 u_x &= \partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz} + f_x,\\ \rho\partial_t^2 u_y &= \partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz} + f_y,\\ \rho\partial_t^2 u_z &= \partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz} + f_z, \end{aligned}$$

$$\begin{aligned} \sigma_{xx} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{xx} + q_{xx},\\ \sigma_{yy} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{yy} + q_{yy},\\ \sigma_{zz} &= \lambda\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu e_{zz} + q_{zz},\\ \sigma_{xy} &= 2\mu e_{xy} + q_{xy},\\ \sigma_{xz} &= 2\mu e_{xz} + q_{xz},\\ \sigma_{yz} &= 2\mu e_{yz} + q_{yz}. \end{aligned}$$

Viscoelastic equations of motion

Using the description in Appendix A, we can write down the viscoe-elastic equations of motion and the constitutive relation. In addition to visco-elastic stress-strain relation we have also introduced time-dependence for the density. The density then shows relaxation in the same way as a visco-elastic medium.

$$
\begin{aligned}
\partial_t^2 u_i(\boldsymbol{x},t) &= \rho_u^{-1}(\boldsymbol{x})\partial_j\sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), \\
&+ \chi(\boldsymbol{x},t) * \partial_j\sigma_{ij}(\boldsymbol{x},t), \\
\sigma_{ij}(\boldsymbol{x},t) &= \lambda_u e_{kk}\delta_{ij} + 2\mu_u e_{ij} + q_{ij} \\
&+ \delta_{ij}\phi_\lambda(t) * e_{mm} + 2\phi_\mu(t) * e_{ij}
\end{aligned}
$$

Writing out the individual components of the equations above, one gets:

$$
\begin{aligned}
\partial_t^2 u_x &= \rho_u^{-1}\left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right] + f_x, \\
&= +\chi * \left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right], \\
\partial_t^2 u_y &= \rho_u^{-1}\left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right] + f_y, \\
&= \chi * \left[\partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz}\right] \\
\partial_t^2 u_z &= \rho_u^{-1}\left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right] + f_z, \\
&= \chi * \left[\partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz}\right],
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xx} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{xx} + q_{xx} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{xx}, \\
\sigma_{yy} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{yy} + q_{yy} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{yy}, \\
\sigma_{zz} &= \lambda_u\left(e_{xx} + e_{yy} + e_{zz}\right) + 2\mu_u e_{zz} + q_{zz} \\
&+ \phi_\lambda * \left[e_{xx} + e_{yy} + e_{zz}\right] + 2\phi_\mu(t) * e_{zz}, \\
\sigma_{xy} &= 2\mu_u e_{xy} + q_{xy} + 2\phi_\mu(t) * e_{xy}, \\
\sigma_{xz} &= 2\mu_u e_{xz} + q_{xz} + 2\phi_\mu(t) * e_{xz}, \\
\sigma_{yz} &= 2\mu_u e_{yz} + q_{yz} + 2\phi_\mu(t) * e_{yz}.
\end{aligned}
$$

Viscoelastic velocity-stress formulation

Using the velocity $v_i = \dot{u}_i$, one gets

$$
\partial_t v_x = \rho_u^{-1}\left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right] + f_x,
$$

$$
\begin{aligned}
&= +\chi * [\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}]\,, \\
\partial_t v_y &= \rho_u^{-1} [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] + f_y, \\
&= \chi * [\partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}] \\
\partial_t v_z &= \rho_u^{-1} [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}] + f_z, \\
&= \chi * [\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}]\,,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx} \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{xx}, \\
\dot{\sigma}_{yy} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy} \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{yy}, \\
\dot{\sigma}_{zz} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{zz} + q_{zz} \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{zz}, \\
\dot{\sigma}_{xy} &= 2\mu_u \dot{e}_{xy} + q_{xy} + 2\phi_\mu(t) * \dot{e}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu_u \dot{e}_{xz} + q_{xz} + 2\phi_\mu(t) * \dot{e}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu_u \dot{e}_{yz} + q_{yz} + 2\phi_\mu(t) * \dot{e}_{yz}.
\end{aligned}
$$

Memory functions

We now define so-called memory variables by including the time convolution into one set of variables:

$$
\begin{aligned}
\gamma_\lambda^l(t) &= \frac{1}{\Delta\lambda_l} \phi_\lambda^l * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}], \\
\gamma_{ij}^l(t) &= \frac{1}{\Delta\mu_l} \phi_\mu^l * [\dot{e}_{ij}],\,, \\
\theta_{kij}^l(t) &= \frac{1}{\Delta\rho_l^{-1}} \chi^l * \partial_k [\sigma_{ij}],
\end{aligned}
$$

Here

$$
\Delta\rho_l^{-1} = \rho_u^{-1} \left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho} \right) \tag{7}
$$

This gives the expressions for the γ functions as:

$$
\gamma_\lambda^l = \left[\frac{\exp(-t/\tau_{\sigma l}^\lambda)}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \right] [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}]
$$

$$
\begin{aligned}
\gamma^l_{ij} &= \left[\frac{\exp(-t/\tau^{\mu}_{\sigma l})}{\tau^{\mu}_{\sigma l}\sum_{l=1}^{N}\frac{\tau^{\mu}_{\epsilon l}}{\tau^{\mu}_{\sigma l}}}\right]\dot{e}_{ij} \\
\theta^l_{kij} &= \left[\frac{\exp(-t/\tau^{\rho}_{\sigma l})}{\tau^{\rho}_{\sigma l}\sum_{l=1}^{N}\frac{\tau^{\rho}_{\epsilon l}}{\tau^{\rho}_{\sigma l}}}\right]\partial_k\sigma_{ij}
\end{aligned}
\tag{8}
$$

and also

$$
\begin{aligned}
\gamma_\lambda(t) &= \sum_{l=1}^{N}\gamma^l_\lambda, \\
\gamma_{ij}(t) &= \sum_{l=1}^{N}\gamma^l_{ij}. \\
\theta_{kij}(t) &= \sum_{l=1}^{N}\theta^l_{ij}.
\end{aligned}
\tag{9}
$$

This gives the final form of the viscoelastic equations

$$
\begin{aligned}
\partial_t v_x &= \rho_i^{-1}\left(\partial_x\sigma_{xx}+\partial_y\sigma_{xy}+\partial_z\sigma_{xz}\right)+f_x && (10)\\
&+ \sum_{l=0}^{N}\theta^l_{xxx}\Delta\rho_l^{-1}+\sum_{l=0}^{N}\theta^l_{yxy}\Delta\rho_l^{-1}+\sum_{l=0}^{N}\theta^l_{zxz}\Delta\rho_l^{-1}, && (11)\\
\partial_t v_y &= \partial_x\sigma_{yx}+\partial_y\sigma_{yy}+\partial_z\sigma_{yz}+f_y, && (12)\\
&+ \sum_{l=0}^{N}\theta^l_{xyx}\Delta\rho_l^{-1}+\sum_{l=0}^{N}\theta^l_{yyy}\Delta\rho_{ll}^{-1}+\sum_{l=0}^{N}\theta^l_{zyz}\Delta\rho_{ll}^{-1}, && (13)\\
\partial_t v_z &= \partial_x\sigma_{zx}+\partial_y\sigma_{zy}+\partial_z\sigma_{zz}+f_z, && (14)\\
&+ \sum_{l=0}^{N}\theta^l_{xzx}+\sum_{l=0}^{N}\theta^l_{yzy}+\sum_{l=0}^{N}\theta^l_{zzz}. && (15)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u\left(\dot{e}_{xx}+\dot{e}_{yy}+\dot{e}_{zz}\right)+2\mu_u\dot{e}_{xx}+\dot{q}_{xx} \\
&+ \sum_{l=1}^{N}\gamma^l_\lambda\Delta\lambda_l+2\sum_{l=1}^{N}\gamma^l_{xx}\Delta\mu_l, && (16)\\
\dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx}+\dot{e}_{yy}+\dot{e}_{zz}\right)+2\mu_u\dot{e}_{yy}+\dot{q}_{yy}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sum_{l=1}^{N} \gamma_{\lambda}^{l} \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma_{yy}^{l} \Delta\mu_l, \qquad (17)\\
\dot{\sigma}_{zz} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz}\\
&+ \quad \sum_{l=1}^{N} \gamma_{\lambda}^{l} \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma_{zz} \Delta\mu_l,\\
\dot{\sigma}_{xy} &= 2\mu \dot{e}_{xy} + 2\sum_{l=1}^{N} \gamma_{xy}^{l} \Delta\mu_l + \dot{q}_{xy},\\
\dot{\sigma}_{xz} &= 2\mu \dot{e}_{xz} + 2\sum_{l=1}^{N} \gamma_{xz}^{l} \Delta\mu_l + \dot{q}_{xz},\\
\dot{\sigma}_{yz} &= 2\mu \dot{e}_{yz} + 2\sum_{l=1}^{N} \gamma_{yz}^{l} \Delta\mu_l + \dot{q}_{yz},
\end{aligned}
\qquad (18)
$$

$$
\begin{aligned}
\dot{e_{xx}} &= \partial_x v_x, & (19)\\
\dot{e_{yy}} &= \partial_y v_y, & (20)\\
\dot{e_{zz}} &= \partial_z v_z, & (21)\\
\dot{e_{xy}} &= \frac{1}{2}(\partial_x v_y + \partial_y v_x), & (22)\\
\dot{e_{xz}} &= \frac{1}{2}(\partial_x v_z + \partial_z v_y), & (23)\\
\dot{e_{yz}} &= \frac{1}{2}(\partial_y v_z + \partial_z v_y). & (24)
\end{aligned}
$$

Integration of memory functions

The *mmeory* functions obeys approximately the relations

$$
\begin{aligned}
\gamma_{\lambda}^{l}(t) &= \exp(-\frac{\Delta t}{\tau_{\sigma}^{\lambda l}})\gamma_{\lambda}^{l}(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^{\lambda} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}}\right)\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right),\\
\gamma_{ij}^{l}(t) &= \exp(-\frac{\Delta t}{N\tau_{\sigma l}^{\mu}})\gamma_{ij}^{l}(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^{\mu} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}}\right)\dot{e}_{ij}.
\end{aligned}
$$

$$\theta^l_{kij}(t) = \exp(-\frac{\Delta t}{N\tau^\rho_{\sigma l}})\theta^l_{kij}(t-\Delta t) + \left(\frac{\Delta t}{\tau^\rho_{\sigma l}\sum_{l=1}^N \frac{\tau^\rho_{\epsilon l}}{\tau^\mu_{\sigma l}}}\right)\partial_k\sigma_{ij}. \tag{25}$$

Defining the quantities

$$\alpha^l_1 = \exp(-\frac{\Delta t}{\tau^\lambda_{\sigma l}}), \tag{26}$$

$$\alpha^l_2 = \frac{\Delta t}{\tau^\lambda_{\sigma l}\sum_{l=1}^N \frac{\tau^\lambda_{\epsilon l}}{\tau^\lambda_{\sigma l}}}, \tag{27}$$

$$\beta^l_1 = \exp(-\frac{\Delta t}{\tau^\mu_{\sigma l}}), \tag{28}$$

$$\beta^l_2 = \frac{\Delta t}{\tau^\mu_{\sigma l}\sum_{l=1}^N \frac{\tau^\mu_{\epsilon l}}{\tau^\mu_{\sigma l}}}, \tag{29}$$

$$\eta^l_1 = \exp(-\frac{\Delta t}{\tau^\rho_{\sigma l}}), \tag{30}$$

$$\eta^l_2 = \frac{\Delta t}{\tau^\rho_{\sigma l}\sum_{l=1}^N \frac{\tau^\rho_{\epsilon l}}{\tau^\rho_{\sigma l}}} \tag{31}$$

we get

$$\begin{aligned}
\gamma^l_\lambda(t) &= \alpha^l_1\gamma^l_\lambda(t-\Delta t) + \alpha^l_2\left(\dot{e}_{xx} + \dot{e}_y y + \dot{e}_z z\right), \\
\gamma^l_{ij}(t) &= \beta^l_1\gamma^l_{ij}(t-\Delta t) + \beta^l_2\dot{e}_{ij}. \\
\theta^l_{kij}(t) &= \eta^l_1\theta^l_{kij}(t-\Delta t) + \eta^l_2\partial_k\sigma_{ij}.
\end{aligned} \tag{32}$$

Discretization of the three dimensional case

We now consider a regular grid with positions $\boldsymbol{x}$ defined as

$$\boldsymbol{x} = (x, y, z), \tag{33}$$

$$x = p\Delta x, \tag{34}$$

$$y = q\Delta y, \tag{35}$$

$$z = r\Delta z, \tag{36}$$

$$t = n\Delta t. \tag{37}$$

where $p = 0, 1, 2, \ldots, N_x$, $q = 0, 1, 2, \ldots, N_y$ and $r = 0, 1, 2, \ldots, N_z$ and N_x, N_y and N_z are the number of grid points in the x, y and z-directions. $n = 0, 1, 2, \ldots, N_t$ where N_t is the number of time steps.

We will also need a regular grid which is displaced, or staggered, relative to the the regular grid. Sometimes we will refer to the regular grid as the reference grid.

The particle velocities v_x,v_y and v_z are defined on staggered grids as follows

$$
\begin{aligned}
v_x(\boldsymbol{x}, t) &= v_x(x + \Delta x/2, y, z, t), &(38)\\
v_y(\boldsymbol{x}, t) &= v_y(x, y + \Delta y/2, z, t), &(39)\\
v_z(\boldsymbol{x}, t) &= v_z(x, y, z + \Delta z/2, t). &(40)
\end{aligned}
$$

The θ_{kij} are defined on the same staggered grid as the particle velocities:

$$
\begin{aligned}
\theta_{xxx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xxx}, &(41)\\
\theta_{yxy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yxy}, &(42)\\
\theta_{zxz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{zxz}, &(43)\\
\theta_{xyx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xyx}, &(44)\\
\theta_{yyy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yyy}, &(45)\\
\theta_{zyz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{zyz}, &(46)\\
\theta_{xzx}(\boldsymbol{x}, t) &= \theta(x + \Delta x/2, y, z)_{xzx}, &(47)\\
\theta_{yzy}(\boldsymbol{x}, t) &= \theta(x, y + \Delta y/2, z)_{yzy} &(48)\\
\theta_{zzz}(\boldsymbol{x}, t) &= \theta(x, y, z + \Delta z/2)_{xxx}. &(49)
\end{aligned}
$$

The diagonal stresses and the gamma functions are defined on a regular grid:

$$
\begin{aligned}
\sigma_{xx}(\boldsymbol{x}, t) &= \sigma_{xx}(x, y, z, t), &(50)\\
\sigma_{yy}(\boldsymbol{x}, t) &= \sigma_{yy}(x, y, z, t), &(51)\\
\sigma_{zz}(\boldsymbol{x}, t) &= \sigma_{zz}(x, y, z, t), &(52)\\
\gamma_{\lambda}(\boldsymbol{x}, t) &= \gamma_{\lambda}(x, y, z, t), &(53)\\
\gamma_{\mu}(\boldsymbol{x}, t) &= \gamma_{\mu}(x, y, z, t), &(54)\\
\gamma_{xx}(\boldsymbol{x}, t) &= \gamma_{xx}(x, y, z, t), &(55)\\
\gamma_{yy}(\boldsymbol{x}, t) &= \gamma_{yy}(x, y, z, t), &(56)\\
\gamma_{zz}(\boldsymbol{x}, t) &= \gamma_{zz}(x, y, z, t). &(57)
\end{aligned}
$$

The off-diagonal stresses and the off-diagonal gamma functions are defined on several staggered grids

$$
\begin{aligned}
\sigma_{xy}(\boldsymbol{x},t) &= \sigma_{xy}(x+\Delta x/2, y+\Delta y/2, z, t), &(58)\\
\sigma_{xz}(\boldsymbol{x},t) &= \sigma_{xz}(x+\Delta x/2, y, z+\Delta z/2, t), &(59)\\
\sigma_{yz}(\boldsymbol{x},t) &= \sigma_{yz}(x, y+\Delta y/2, z+\Delta z/2, t), &(60)\\
\gamma_{xy}(\boldsymbol{x},t) &= \gamma_{xy}(x+\Delta x/2, y+\Delta y/2, z, t), &(61)\\
\gamma_{xz}(\boldsymbol{x},t) &= \gamma_{xz}(x+\Delta x/2, y, z+\Delta z/2, t), &(62)\\
\gamma_{yz}(\boldsymbol{x},t) &= \gamma_{yz}(x, y+\Delta y/2, z+\Delta z/2, t), &(63)\\
& &(64)
\end{aligned}
$$

The Lamé parameters are defined on a regular grid as follows: as follows:

$$
\begin{aligned}
\lambda_u(\boldsymbol{x}) &= \lambda_u(x,y,z), &(65)\\
\mu_u(\boldsymbol{x}) &= \mu_u(x,y,z). &(66)
\end{aligned}
$$

The inverse density and μ are defined on three different staggered grids as follows:

$$
\begin{aligned}
\rho_x^{-1}(\boldsymbol{x}) &= \rho^{-1}(x+\Delta x/2, y, z), &(67)\\
\rho_y^{-1}(\boldsymbol{x}) &= \rho^{-1}(x, y+\Delta y/2, z), &(68)\\
\rho_z^{-1}(\boldsymbol{x}) &= \rho^{-1}(x, y, z+\Delta z/2), &(69)\\
\mu_{uxy}(\boldsymbol{x}) &= \mu_u(x+\Delta x/2, y+\Delta y/2, z), &(70)\\
\mu_{uyz}(\boldsymbol{x}) &= \mu_u(x, y+\Delta y/2, z+\Delta z/2), &(71)\\
\mu_{uxz}(\boldsymbol{x}) &= \mu_u(x+\Delta x/2, y, z+\Delta z), &(72)\\
& &(73)
\end{aligned}
$$

The visco-elastic parameters β_1, β_2 and η_1 and η_2 are also defined on several staggered grids:

$$
\begin{aligned}
\beta_{1xy}\boldsymbol{x}) &= \beta_1(x+\Delta x, y+\Delta y, z), &(74)\\
\beta_{1yz}(\boldsymbol{x}) &= \beta_1(x, y+\Delta y, z+\Delta z), &(75)\\
\beta_{1xz}(\boldsymbol{x}) &= \beta_1(x+\Delta x, y, z+\Delta z), &(76)\\
\beta_{2xy}(\boldsymbol{x}) &= \beta_2(x+\Delta x, y+\Delta y, z), &(77)\\
\beta_{2yz}(\boldsymbol{x}) &= \beta_2(x, y+\Delta y, z+\Delta z), &(78)\\
\beta_{2xz}(\boldsymbol{x}) &= \beta_2(x+\Delta x, y, z+\Delta z), &(79)\\
\eta_{1x}(\boldsymbol{x}) &= \eta_1(x+\Delta x/2, y, z), &(80)
\end{aligned}
$$

$$
\begin{aligned}
\eta_{2x}(\boldsymbol{x}) &= \eta_2(x+\Delta x/2, y, z), && (81)\\
\eta_{1y}(\boldsymbol{x}) &= \eta_1(x, y+\Delta y/2, z), && (82)\\
\eta_{2y}(\boldsymbol{x}) &= \eta_1(x, y+\Delta y/2, z), && (83)\\
\eta_{1z}(\boldsymbol{x}) &= \eta_1(x, y, z+\Delta z/2), && (84)\\
\eta_{2z}(\boldsymbol{x}) &= \eta_1(x, y, z+\Delta z/2). && (85)
\end{aligned}
$$

Differentiation is now replaced by numerical approximations so that ∂_x, ∂_y and ∂_z are replaced with numerical operators $d_x^+, d_x^-, d_y^+, d_y^-, d_z^+$ and d_z^-. These operators connects the staggered and reference grids, and we illustrate this with the differentiation in the x-direction. The derivative of a function $a(x)$ is approximately given at $a(x+\Delta x/2)$ and at $a(x-\Delta x/2)$ by

$$
\begin{aligned}
a'(x+\Delta x/2) &= d_x^+ a(x), && (86)\\
a'(x-\Delta x/2) &= d_x^- a(x). && (87)
\end{aligned}
$$

The differentiators d^+ and d^- are given by (?)

$$
\begin{aligned}
\partial^+ &= \frac{1}{\Delta x}\sum_{l=1}^{L}\alpha_l\,[u(x+l\Delta x)-u(x-(l-1)\Delta x]\\
\partial^- &= \frac{1}{\Delta x}\sum_{l=1}^{L}\alpha_l\,[u(x+(l-1)\Delta x)-u(x-l\Delta x)] && (88)
\end{aligned}
$$

where the coefficients α_l are found through an optimization procedure. Similar differentiators are defined for the y-direction and for the z-direction, with obvious names.

Using the numerical differentiators the equations of motion becomes:

$$
\begin{aligned}
\partial_t v_x &= \rho_i^{-1}\left(d_x^+\sigma_{xx}+d_y^+\sigma_{xy}+d_z^+\sigma_{xz}\right)+f_x && (89)\\
&+ \sum_{l=0}^{N}\theta_{xxx}^l+\sum_{l=0}^{N}\theta_{yxy}^l+\sum_{l=0}^{N}\theta_{zxz}^l, && (90)\\
\partial_t v_y &= d_x^+\sigma_{yx}+d_y^+\sigma_{yy}+d_z^+\sigma_{yz}+f_y, && (91)\\
&+ \sum_{l=0}^{N}\theta_{xyx}^l+\sum_{l=0}^{N}\theta_{yyy}^l+\sum_{l=0}^{N}\theta_{zyz}^l, && (92)\\
\partial_t v_z &= d_x^+\sigma_{zx}+d_y^+\sigma_{zy}+d_z^+\sigma_{zz}+f_z, && (93)\\
&+ \sum_{l=0}^{N}\theta_{xzx}^l+\sum_{l=0}^{N}\theta_{yzy}^l+\sum_{l=0}^{N}\theta_{zzz}^l. && (94)
\end{aligned}
$$

The computation of the strains becomes as follows:

$$\begin{aligned}
\dot{e}_{xx} &= d_x^- v_x, \\
\dot{e}_{yy} &= d_y^- v_y, \\
\dot{e}_{zz} &= d_z^- v_z, \\
\dot{e}_{xy} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x), \\
\dot{e}_{xz} &= \frac{1}{2}(d_x^+ v_z + d_z^+ v_x), \\
\dot{e}_{yz} &= \frac{1}{2}(d_y^+ v_z + d_z^+ v_y).
\end{aligned} \tag{95}$$

The time derivatives is approximated by the central difference

$$\dot{a}(t) = \frac{a(t + \Delta t/2) - a(t - \Delta t/2)}{\Delta t} \tag{96}$$

Solution algorithm for the three dimensional case

We are now in a position to formulate a complete numerical solution of the visco-elastic equations.

Computation of the particle velocity

We use the expression for the approximate time derivative given by equation (96) in equations (**??**) to obtain an expression for the components of the particle velocity

$$\begin{aligned}
v_x(t + \Delta t/2) &= \Delta t \rho_x^{-1} \left[d_x^+ \sigma_{xx}(t) + d_y^+ \sigma_{xy}(t) + d_z^+ \sigma_{xz}(t) \right] + \Delta t \rho_x^{-1} f_x(t) + \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xxx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yxy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zxz}^l(t) + v_x(t - \Delta t/2), \\
v_y(t + \Delta t/2) &= \Delta t \rho_y^{-1} \left[d_x^+ \sigma_{yx}(t) + d_y^+ \sigma_{yy}(t) + d_z^+ \sigma_{yz}(t) \right] + \Delta t \rho_y^{-1} f_y(t) \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xyx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yyy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zyz}^l(t) + v_y(t - \Delta t/2), \\
v_z(t + \Delta t/2) &= \Delta t \rho_z^{-1} \left[d_x^+ \sigma_{zx}(t) + d_y^+ \sigma_{zy}(t) + d_z^+ \sigma_{zz}(t) \right] + \Delta t \rho_z^{-1} f_z(t) \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xzx}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{yzy}^l(t) + \Delta t \sum_{l=0}^{N} \theta_{zzz}^l(t). + v_z(t - \Delta t/2)
\end{aligned} \tag{97}$$

The strains can now be computed from equation (95)

$$
\begin{aligned}
\dot{e}_{xx}(t+\Delta t/2) &= d_x^- v_x(t+\Delta t/2), \\
\dot{e}_{yy}(t+\Delta t/2) &= d_y^- v_y(t+\Delta t/2), \\
\dot{e}_{zz}(t+\Delta t/2) &= d_z^- v_z(t+\Delta t/2), \\
\dot{e}_{xy}(t+\Delta t/2) &= \frac{1}{2}\left[d_x^+ v_y(t+\Delta t/2) + d_y^+ v_x(t+\Delta t/2)\right], \\
\dot{e}_{xz}(t+\Delta t/2) &= \frac{1}{2}\left[d_x^+ v_z(t+\Delta t/2) + d_z^+ v_x(t+\Delta t/2)\right], \\
\dot{e}_{yz}(t+\Delta t/2) &= \frac{1}{2}\left[d_y^+ v_z(t+\Delta t/2) + d_z^+ v_y(t+\Delta t/2)\right].
\end{aligned}
\tag{98}
$$

Equations (106) can be solved for the stresses using the same approach as for the particle velocities:

$$
\begin{aligned}
\sigma_{xx}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] \\
&+ 2\mu_u \dot{e}_{xx}(t+\Delta t/2) + \Delta t \dot{q}_{xx} \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l (t+\Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{xx}^l (t+\Delta t/2)\Delta\mu_l(t+\Delta t/2) \\
&+ \sigma_{xx}(t), && (99) \\
\sigma_{yy}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] && (100) \\
&+ 2\Delta t \mu_u \dot{e}_{yy}(t+\Delta t/2) + \Delta t \dot{q}_{yy}(t+\Delta t/2) \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l (t+\Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{yy}^l (t+\Delta t/2)\Delta\mu_l && (101) \\
&+ \sigma_{yy}(t) && (102) \\
\sigma_{zz}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] && (103) \\
&+ 2\mu_u \dot{e}_{zz}(t+\Delta t/2) + \Delta t \dot{q}_{zz}(t+\Delta t/2) \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l (t+\Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{zz}\Delta\mu_l(t+\Delta t/2), \\
&+ \sigma_{zz}. && (104)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xy}(t+\Delta t) &= 2\Delta t \mu \dot{e}_{xy}(t+\Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{xy}^l (t+\Delta t/2)\Delta\mu_l + \Delta t \dot{q}_{xy}(t+\Delta t/2) \\
&+ \sigma_{xy}(t), \\
\sigma_{xz}(t+\Delta t) &= 2\Delta t \mu \dot{e}_{xz}(t+\Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{xz}^l (t+\Delta t/2)\Delta\mu_l + \Delta t \dot{q}_{xz}(t+\Delta t/2)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sigma_{xz}(t), && (105)\\
\sigma_{yz}(t+\Delta t) \quad &= \quad 2\Delta\mu\dot{e}_{yz}(t+\Delta t/2) + 2\Delta t\sum_{l=1}^{N}\gamma_{yz}^{l}(t+\Delta t/2)\Delta\mu_l + \dot{q}_{yz}(t+\Delta t/2)\\
&+ \quad \sigma_{yz}(t).
\end{aligned}
\tag{106}
$$

The gamma functions are updated as follows

$$
\begin{aligned}
\gamma_{\lambda}^{l}(t+3/2\Delta t) &= \alpha_1^l\gamma_{\lambda}^{l}(t+\Delta t/2) + \alpha_2^l\left[\dot{e}_{xx}(t+\Delta t/) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right],\\
\gamma_{\mu}^{l}(t+3/2\Delta t) &= \beta_1^l\gamma_{\mu}^{l}(t+\Delta t/2) + \beta_2^l\left[\dot{e}_{xx}(t+\Delta t/) + \dot{e}_{yy}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right],\\
\gamma_{xy}^{l}(t+3/2\Delta t) &= \beta_{1xy}^l\gamma_{xy}^{l}(t+\Delta t/2) + \beta_{2xy}^l\dot{e}_{xy}(t+\Delta t/2).\\
\gamma_{xz}^{l}(t+3/2\Delta t) &= \beta_{1xz}^l\gamma_{xz}^{l}(t+\Delta t/2) + \beta_{2xz}^l\dot{e}_{xz}(t+\Delta t/2).\\
\gamma_{yz}^{l}(t+3/2\Delta t) &= \beta_{1xz}^l\gamma_{yz}^{l}(t+\Delta t/2) + \beta_{2xz}^l\dot{e}_{yz}(t+\Delta t/2).
\end{aligned}
\tag{107}
$$

The θ functions are updated as:

$$
\begin{aligned}
\theta_{xxx}(t+\Delta t) &= \eta_1^l\theta_{xxx}^l(t+\Delta t/2) + \eta_2^l\partial_x\sigma_{xx}(t+\Delta t/2).\\
\theta_{yxy}(t+\Delta t) &= \eta_1^l\theta_{yxy}^l(t+\Delta t/2) + \eta_2^l\partial_y\sigma_{xy}(t+\Delta t/2).\\
\theta_{zxz}(t+\Delta t) &= \eta_1^l\theta_{zxz}^l(t+\Delta t/2) + \eta_2^l\partial_z\sigma_{xz}(t+\Delta t/2).\\
\theta_{xyx}(t+\Delta t) &= \eta_1^l\theta_{xyx}^l(t+\Delta t/2) + \eta_2^l\partial_x\sigma_{yx}(t+\Delta t/2).\\
\theta_{yyy}(t+\Delta t) &= \eta_1^l\theta_{yyy}^l(t+\Delta t/2) + \eta_2^l\partial_y\sigma_{yy}(t+\Delta t/2).\\
\theta_{zyz}(t+\Delta t) &= \eta_1^l\theta_{zyz}^l(t+\Delta t/2) + \eta_2^l\partial_z\sigma_{yz}(t+\Delta t/2).\\
\theta_{xyx}(t+\Delta t) &= \eta_1^l\theta_{xyx}^l(t+\Delta t/2) + \eta_2^l\partial_x\sigma_{yx}(t+\Delta t/2).\\
\theta_{xzx}(t+\Delta t) &= \eta_1^l\theta_{xzx}^l(t+\Delta t/2) + \eta_2^l\partial_x\sigma_{zx}(t+\Delta t/2).\\
\theta_{yzy}(t+\Delta t) &= \eta_1^l\theta_{yzy}^l(t+\Delta t/2) + \eta_2^l\partial_y\sigma_{zy}(t+\Delta t/2).\\
\theta_{zzz}(t+\Delta t) &= \eta_1^l\theta_{zzz}^l(t+\Delta t/2) + \eta_2^l\partial_z\sigma_{zz}(t+\Delta t/2).
\end{aligned}
\tag{108}
$$

2D Acoustic case

For the acoustic 2D case we reduce the equations above by neglecting the y-axis terms and putting $\mu = 0$. We consider also the pseudo-stress σ defined by

$$
\sigma = \frac{1}{2}\left(\sigma_{xx} + \sigma_{zz}\right)
$$

We then get the acoustic 2D scheme as:

$$
\begin{aligned}
v_x(t+\Delta t/2) &= \Delta t\left[\rho_{ux}^{-1}d_x^{+}\sigma_{xx}(t)+\rho_{ux}^{-1}f_x(t)\right]+\\
&+ \Delta t\sum_{l=0}^{N}\theta_x^l(t)\Delta\rho_x^{-1}+v_x(t-\Delta t/2),\\
v_z(t+\Delta t/2) &= \Delta t\left[\rho_{uz}^{-1}d_z^{+}\sigma_{zz}(t)+\Delta t\rho_{uz}^{-1}f_z(t)\right]\\
&+ \Delta t\sum_{l=0}^{N}\theta_z^l(t)\Delta\rho_z^{-1}+v_z(t-\Delta t/2).
\end{aligned}
$$

The strains can now be computed from :

$$
\begin{aligned}
\dot{e}_{xx}(t+\Delta t/2) &= d_x^{-}v_x(t+\Delta t/2),\\
\dot{e}_{zz}(t+\Delta t/2) &= d_z^{-}v_z(t+\Delta t/2).
\end{aligned}
$$

Equations (106) can be solved for the stresses using the same approach as for the particle velocities:

$$
\begin{aligned}
\sigma(t+\Delta t) &= \Delta t\lambda_u\left[\dot{e}_{xx}(t+\Delta t/2)+\dot{e}_{zz}(t+\Delta t/2)\right]+\Delta t\dot{q},\\
&+ \Delta t\sum_{l=1}^{N}\gamma^l(t+\Delta t/2)\Delta\lambda_l+\sigma(t).
\end{aligned}
$$

We now split the γ^l into two parts γ_x^l and γ_z^l as follows:

$$
\begin{aligned}
\sigma(t+\Delta t) &= \Delta t\lambda_u\left[\dot{e}_{xx}(t+\Delta t/2)+\dot{e}_{zz}(t+\Delta t/2)\right]+\Delta t\dot{q}\\
&+ \Delta t\sum_{l=1}^{N}\left[\gamma_x^l(t+\Delta t/2)\Delta\lambda_l+\gamma_z^l(t+\Delta t/2)\Delta\lambda_l\right]+\sigma(t).
\end{aligned}
$$

The θ functions are updated as:

$$
\begin{aligned}
\theta_x(t+\Delta t) &= \eta_{1x}^l\theta_x^l(t)+\eta_{2x}^l\partial_x\sigma(t),\\
\theta_z(t+\Delta t) &= \eta_{1z}^l\theta_x^l(t)+\eta_{2z}^l\partial_y\sigma(t).
\end{aligned}
$$

The γ functions are given by

$$
\begin{aligned}
\gamma_x^l(t+3/2\Delta t) &= \alpha_{1x}^l\gamma_x^l(t+\Delta t/2)+\alpha_{2x}^l\dot{e}_{xx}(t+\Delta t/2),\\
\gamma_z^l(t+3/2\Delta t) &= \alpha_{1z}^l\gamma_z^l(t+\Delta t/2)+\alpha_{2z}^l\dot{e}_{zz}(t+\Delta t/2).
\end{aligned}
$$

Also the coefficents are

$$\begin{aligned}
\alpha_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\alpha_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\eta_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}, \\
\eta_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}.
\end{aligned}$$

The profile functions d_x and d_z are

$$d_x(x) = (x/L)^2, d_z(y) = (z/L)^2,$$

where L is the length of the absorbing layer and and we also have

$$\Delta\lambda_l = \lambda_u \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right) \Delta\rho^{-1} = \rho_u^{-1}\left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right) \tag{109}$$

APPENDIX A: The viscoelastic standard linear solid

Bolzman's generalization of Hook's law to the visco-elastic case is (**?**):

$$\sigma_{ij} = \psi_{ijkl} * \dot{e}_{kl}, \tag{A-1}$$

where ψ_{ijkl} is known as the relaxation tensor. The $*$ denotes convolution defined by

$$a(t) * b(t) = \int_0^t a(t-\tau)b(\tau). \tag{A-2}$$

Integrating (A-1) by parts

$$\sigma_{ij}(t) = |_0^t \psi_{ijkl}(t-\tau)e_{kl}(\tau) + \int_0^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau), \tag{A-3}$$

and using $e(t=0)=0$ I get

$$\sigma_{ij}(t) = \psi(0)_{ijkl}e_{kl}(t) + \int_{0+}^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau) \tag{A-4}$$

For the Zener model the components of the ψ_{ijkl} tensor have the form

$$\psi(t) = K\left[1 - \frac{1}{N}\sum_{l=1}^{N}(1-\frac{\tau_{\epsilon l}}{\tau_{\sigma l}})\exp(-t/\tau_{\sigma l})\right]H(t). \tag{A-5}$$

where K_r is a relaxed modulus, N is the number of Zener mechanisms,$\tau_{\sigma l}$ and $\tau_{\epsilon l}$ are relaxation times. $H(t)$ is the Heavy side function. The time derivative of ψ is equal to:

$$\dot{\psi} = \phi(t), \tag{A-6}$$

where ϕ is equal to:

$$\phi(t) = \frac{1}{N}\sum_{l=1}^{N}\left[\left(\frac{K_r}{\tau_{\sigma l}}\right)\left(1-\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\exp(-t/\tau_{\sigma l})\right]H(t). \tag{A-7}$$

Using the above we have for the stress:

$$\sigma_{ij}(t) = c_{ijkl}e_{kl} + \int_{0+}^t \phi_{ijkl}(t-\tau)e_{kl}(\tau) \tag{A-8}$$

This is most conveniently written as

$$\sigma_{ij}(t) = c_{ijkl}(t) * e_{kl}(t), \tag{A-9}$$

where

$$c_{ijkl}(t) = \psi(0)_{ijkl}\delta(t) + \phi_{ijkl}(t), \tag{A-10}$$

By definition $\psi(t=0)$ corresponds to the unrelaxed modulus so that we have

$$K_u = \frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}K_r \tag{A-11}$$

or

$$K_r = \frac{K_u}{\frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \tag{A-12}$$

The ϕ function can then be expressed in terms of the unrelaxed moduli:

$$\phi(t) = \sum_{l=1}^{N}\left[\left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}}\right)K_u\left(1-\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\right] \tag{A-13}$$

Finally, we express ϕ as:

$$\phi(t) = \sum_{l=1}^{N}\phi^l(t) \tag{A-14}$$

where

$$\phi^l(t) = \left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}}\right)\Delta K_l \tag{A-15}$$

and ΔK_l is

$$\Delta K_l = K_u\left(1-\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right) \tag{A-16}$$

It is most practical to write the time-dependent visco-elastic constants as

$$\lambda(t) = \lambda_u\delta(t) + \phi_\lambda(t), \tag{A-17}$$
$$\mu(t) = \mu_u\delta(t) + \phi_\mu(t), \tag{A-18}$$

where ϕ_λ is given as:

$$\phi_\lambda(t) = \sum_{l=1}^{N}\left(\frac{\exp(-t/\tau_{\sigma l}^{\lambda})}{\tau_{\sigma l}^{\lambda}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}}\right)\Delta\lambda_l \tag{A-19}$$

and ϕ_μ is given as:

$$\phi_\mu(t) = \sum_{l=1}^{N}\left(\frac{\exp(-t/\tau_{\sigma l}^{\mu})}{\tau_{\sigma l}^{\mu}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}}\right)\Delta\mu_l \tag{A-20}$$

Q-factors

See **?** for further relations between Q and relaxation times. The Q values are related to the Fourier transform of the λ modulus as

$$Q_\lambda^{-1}(\omega) = \frac{Im\lambda(\omega)}{Re\lambda(\omega)} \tag{A-21}$$

Assuming λ is given as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t). \tag{A-22}$$

The fourier transform of λ is given by

$$\lambda(\omega) = \lambda_u + \int_{-\infty}^{\infty} \phi_\lambda(t) \exp(-i\omega t). \tag{A-23}$$

The Fourier transform of ϕ_λ is:

$$\phi_\lambda(\omega) = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\int_0^{+\infty} dt \exp(-i\omega)\exp(-t/\tau_{\sigma l}^\lambda). \tag{A-24}$$

The results is:

$$\phi_\lambda(\omega) = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\frac{1}{1 + i\omega\tau_{\sigma l}^\lambda}. \tag{A-25}$$

The fourier transform of λ is then

$$\lambda(\omega) = \lambda_u + \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\frac{1}{1 + i\omega\tau_{\sigma l}^\lambda}. \tag{A-26}$$

After some (tedious) algebra one obtains

$$\lambda(\omega) = \lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{1 + i\omega\tau_{\epsilon l}^\lambda}{1 + i\omega\tau_{\sigma l}^\lambda} \tag{A-27}$$

Separating into real and imaginary parts, I get

$$\begin{aligned}\lambda(\omega) &= \lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{1 + \omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda}{1 + (\omega\tau_{\sigma l}^\lambda)^2} &\text{(A-28)}\\ &+ i\lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{\omega\tau_{\sigma l}^\lambda(\tau_\epsilon^\lambda/\tau_{\sigma l}^\lambda - 1)}{1 + (\omega\tau_{\sigma l}^\lambda)^2} & \\ & &\text{(A-29)}\end{aligned}$$

We then have

$$Q_\lambda^{-1} = \frac{\sum_{l=1}^{N} \omega\tau_{\sigma l}^{\lambda} \left(\tau_{\epsilon l}^{\lambda}/\tau_{\sigma l}^{\lambda} - 1\right) / \left[1 + (\omega\tau_{\sigma l}^{\lambda})^2\right]}{\sum_{l=1}^{N} (1 + \omega^2 \tau_{\sigma l}^{\lambda} \tau_{\epsilon l}^{\lambda}) / \left[1 + (\omega\tau_{\sigma l}^{\lambda})^2\right]} \tag{A-30}$$

The results for the frequency dependence of μ is obtained in exactly the same manner as above:

$$\mu(\omega) = \mu_r \frac{1}{N} \sum_{l=1}^{N} \frac{1 + i\omega\tau_{\epsilon l}^{\mu}}{1 + i\omega\tau_{\sigma l}^{\mu}} \tag{A-31}$$

and the Q-factor for μ is

$$Q_\mu^{-1} = \frac{\sum_{l=1}^{N} \omega\tau_{\sigma l}^{\lambda} \left(\tau_{\epsilon l}^{\mu}/\tau_{\sigma l}^{\lambda} - 1\right) / \left[1 + (\omega\tau_{\sigma l}^{\mu})^2\right]}{\sum_{l=1}^{N} (1 + \omega^2 \tau_{\sigma l}^{\mu} \tau_{\epsilon l}^{\mu}) / \left[1 + (\omega\tau_{\sigma l}^{\mu})^2\right]} \tag{A-32}$$

In practice we need to relate Q_l and Q_μ to Q_κ. We use the relation

$$\kappa(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega). \tag{A-33}$$

Splitting into real and imaginary parts

$$\kappa_r(\omega) = \lambda_r(\omega) + \frac{2}{3}\mu_r(\omega). \kappa_i(\omega) = \lambda_i(\omega) + \frac{2}{3}\mu_i(\omega). \tag{A-34}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\lambda_i + (2/3)\mu_i}{\lambda_r + (2/3)\mu_r} \tag{A-35}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\left(\frac{\lambda_i}{\lambda_r}\right)\lambda_r + (2/3)\left(\frac{\mu_i}{\mu_r}\right)\mu_r}{\lambda_r + (2/3)\mu_r} \tag{A-36}$$

$$Q_\kappa^{-1} = \frac{Q_\lambda^{-1}\lambda_r + (2/3)Q_\mu^{-1}}{\lambda_r + (2/3)\mu_r} \tag{A-37}$$

We can use the P-wave and S-wave velocities

$$\begin{aligned} \lambda &= \rho V_p^2 - (2/3)\rho V_s^2 \\ \mu &= \rho V_s^2 \end{aligned} \tag{A-38}$$

$$Q_\kappa^{-1} = Q_\lambda^{-1} \left[1 - \left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2\right] + Q_\mu^{-1}\left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2 \tag{A-39}$$

Q-model parametrization

Q-models can be described by the two relaxation times, τ_σ and τ_ϵ. However it is simpler to use the two parameters τ_0 and Q_0 to describe a model. According to **?**, Appendix B, we have

$$Q(\omega) = Q_0 \frac{1 + \omega^2 \tau_0^2}{2\omega\tau_0}$$

where

$$\begin{aligned} Q_0 &= \frac{2\tau_0}{\tau_\epsilon - \tau_\sigma}, \\ \tau_0^2 &= \tau_\epsilon \tau_\sigma. \end{aligned}$$

$\omega = 1/\tau_0$ is the minimum value for $Q(\omega)$, i.e. the absorption top. We can now find τ_σ and τ_ϵ in terms of τ_0 and Q_0 as:

$$\begin{aligned} \tau_\epsilon &= \frac{\tau_0}{Q_0} \left[\sqrt{Q_0^2 + 1} + 1 \right], \\ \tau_\sigma &= \frac{\tau_0}{Q_0} \left[\sqrt{Q_0^2 + 1} - 1 \right]. \end{aligned}$$

Effective density

We now assume that the effective density has the following form

$$\rho_{eff}^{-1}(t) = s(t). \tag{A-40}$$

We also assume that

$$s(t) = s(0)\delta(t) + \chi(t) \tag{A-41}$$

So that the inverse of the effective density reads

$$\rho_{eff}^{-1}(t) = s_u \delta(t) + \chi(t). \tag{A-42}$$

where $s_u = s(0) = \rho_u^{-1}$.

APPENDIX B: Recursive computation of γ and s.

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t + \Delta t) \quad = \quad \int_0^{t+\Delta t} d\tau \, \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{t + \Delta t - \tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t+\Delta t) = \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).\end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t+\Delta t$

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_\epsilon} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda})\end{aligned}$$

Performing the integral we then get

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\tau_\sigma}{\tau_\epsilon}\left[1-\exp(-\frac{\Delta t}{\tau_\sigma^\lambda})\right] \dot{e}_{xx}(t).\end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\Delta t}{\tau_\epsilon} \dot{e}_{xx}(t).\end{aligned}$$

APPENDIX C: Comparison with CPML

? show that the implementation of the CPML method can be performed by replacing each spatial derivative with (Their's equation (16) and (18)):

$$s_x = \delta(t) - d_x H(t) \exp\left[-(d_x+\alpha_x)t\right]. \tag{A-1}$$

We compare this with our equation

$$\lambda(t)/\lambda_u = \delta(t) + \frac{1}{\tau_\epsilon} \exp(-t/\tau_\sigma) H(t) \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right) \tag{A-2}$$

Comparing equation (A-1) with equation (A-2) one gets:

$$-d_x = \frac{1}{\tau_\epsilon}\left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right), \tag{A-3}$$

$$d_x + \alpha = \frac{1}{\tau_\sigma} \tag{A-4}$$

Solving for τ_ϵ and τ_σ one gets

$$\tau_\sigma = \frac{1}{(d_x + \alpha)}, \tag{A-5}$$

$$\tau_\epsilon = \frac{1}{\alpha}. \tag{A-6}$$

. Here

$$d_x(x) = d_0 \left(\frac{x}{L}\right)^2, \tag{A-7}$$

where d_0 is a constant and L is the length of the PML zone and x is the distance from the start (outer border) of the PML zone.