PyEl2d - python library for visco-elastic modeling

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Contents

1 Introduction

PyAc2d is a small and simple python library for Finit-Difference propagation of viscoelastic waves. The core of the library is capable of running on multi-core cpu and also on gpu hardware.

In the following sections we outline the basic equations for visco-elastic waves in two and three dimensions. It is followed by sections on the numerical implementation of the Finite-Difference solution of viscoelastic wave propagation. The PyAc2d python library is then described in detail, using several examples. A separate section on the implementation details of the PyAc2d library is also given.

2 The Viscoelastic equations of motion

In the following sections the viscoelastic equations are presented. The only change from the standard case is that both the density and the stress include relaxation.

2.1 The three dimensional case

The fundamental equation for viscoelastic wave propagation are written as

$$\begin{aligned} \partial_t^2 u_i(\boldsymbol{x},t) &= \rho^{-1}(\boldsymbol{x},t) * \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), && (1) \\ \sigma_{ij}(\boldsymbol{x},t) &= \lambda(\boldsymbol{x},t) * e_{kk}\delta_{ij} + 2\mu(\boldsymbol{x},t) * e_{ij} + q_{ij}. && (2) \end{aligned}$$

$\boldsymbol{x} = x,y,z$ denotes a spatial position with cartesioan components x,y,z and t is the time. u_i is the i'th component of the particle displacement and $i = x,y,z$. $\sigma_{i,j}$ is the stress tensor, while f_i is the i'th component of an external (source) body force. $q_{i,j}$ is an external (source) stress tensor. ρ is the density and λ and μ are the Lamé parameters. Note that both density and Lameé parameters are time dependent in order to describe the effect of visco-elasticity.

Using the derivations in Appendix A the viscoelastic equations given above in equations (1) and (2) can be formulated as

$$\begin{aligned} \partial_t^2 u_i(\boldsymbol{x},t) &= \rho_u^{-1}(\boldsymbol{x})\partial_j\sigma_{ij} + \delta_{i,j}\chi_p(\boldsymbol{x},t) * \partial_j\sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t) \\ &+ (1-\delta_{i,j})\chi_s(\boldsymbol{x},t) * \partial_j\sigma_{ij}(\boldsymbol{x},t), && (3) \\ \sigma_{ij}(\boldsymbol{x},t) &= \lambda_u e_{kk}\delta_{ij} + 2\mu_u e_{ij} + q_{ij} \\ &+ \delta_{ij}\phi_\lambda(t) * e_{ij} + 2\phi_\mu(t) * e_{ij}. && (4) \end{aligned}$$

Here ρ_u is the elastic unrelaxed part of the density ρ, while λ_u and μ_u are the corresponding unrelaxed part of the Lamé parameters. The relaxation functions χ and ϕ contains the effects of visco-elasticity and is equal to zero for a pure elastic medium.

Using the velocity $v_i = \dot{u}_i$, and writing out individual components one gets

$$\begin{aligned}
\partial_t v_x &= \rho_u^{-1}(\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}) + f_x, \\
&= +\chi_p * \partial_x \sigma_{xx} + \chi_s * (\partial_y \sigma_{xy} + \partial_z \sigma_{xz}), \\
\partial_t v_y &= \rho_u^{-1}(p_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}) + f_y, \\
&+ \chi_p * \partial_x \sigma_{yx} + \chi_s * (\partial_y \sigma_{yy} + \partial_z \sigma_{yz}) \\
\partial_t v_z &= \rho_u^{-1} \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}) + f_z, \\
&+ \chi_p * \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \chi_s * \partial_y \sigma_{zz} + \partial_z \sigma_{zz}).
\end{aligned} \tag{5}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx}, \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{xx}, \\
\dot{\sigma}_{yy} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy} \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{yy}, \\
\dot{\sigma}_{zz} &= \lambda_u (\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}) + 2\mu_u \dot{e}_{zz} + q_{zz}, \\
&+ \phi_\lambda * [\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}] + 2\phi_\mu(t) * \dot{e}_{zz}, \\
\dot{\sigma}_{xy} &= 2\mu_u \dot{e}_{xy} + q_{xy} + 2\phi_\mu(t) * \dot{e}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu_u \dot{e}_{xz} + q_{xz} + 2\phi_\mu(t) * \dot{e}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu_u \dot{e}_{yz} + q_{yz} + 2\phi_\mu(t) * \dot{e}_{yz}.
\end{aligned} \tag{6}$$

2.1.1 Memory functions

We now define so-called memory variables by including the time convolution into one set of variables:

$$\gamma_{ij}^l(t) = \delta_{i,j} \frac{1}{\Delta\lambda_l} \phi_\lambda^l * \dot{e}_{ij} + (1 - \delta_{i,j}) \frac{1}{\Delta\mu_l} \phi_\mu^l * \dot{e}_{ij} \tag{7}$$

$$\theta_{kij}^l(t) = \delta_{i,j} \frac{1}{\Delta\rho_l^p} \chi_p^l * \partial_k \sigma_{ij} + (1 - \delta_{i,j}) \frac{1}{\Delta\rho_l^p} \chi_s^l * \partial_k \sigma_{ij}. \tag{8}$$

Here

$$\Delta\rho_l^{p-1} = \rho_u^{-1} \left(1 - \frac{\upsilon_{\epsilon l}^p}{\upsilon_{\sigma l}^p}\right),$$

$$\begin{aligned}
\Delta\rho_l^{s-1} &= \rho_u^{-1}\left(1-\frac{\upsilon_{\epsilon l}^s}{\upsilon_{\sigma l}^s}\right),\\
\Delta\lambda_l &= \lambda_u\left(1-\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right),\\
\Delta\mu_l &= \mu_u\left(1-\frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\rho}\right). \qquad (9)
\end{aligned}$$

Using equation (A-15) for the ϕ and χ functions give the the γ and θ functions as:

$$\gamma_{ij}^l = \delta_{i,j}\left[\frac{\exp(-t/\tau_{\sigma l}^\lambda)}{\tau_{\sigma l}^\mu\sum_{l=1}^N\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\mu}}\right]*\dot{e}_{ij}+(1-\delta_{i,j})\left[\frac{\exp(-t/\tau_{\sigma l}^\mu)}{\tau_{\sigma l}^\mu\sum_{l=1}^N\frac{\tau_{\epsilon l}^\mu}{\tau_{\sigma l}^\mu}}\right]*\dot{e}_{ij}, \quad (10)$$

$$\theta_{kij}^l = \delta_{i,j}\left[\frac{\exp(-t/\upsilon_{\sigma l}^\lambda)}{\upsilon_{\sigma l}^\lambda\sum_{l=1}^N\frac{\upsilon_{\epsilon l}^\rho}{\upsilon_{\sigma l}^\rho}}\right]*\partial_k\sigma_{ij}+(1-\delta_{i,j})\left[\frac{\exp(-t/\upsilon_{\sigma l}^\mu)}{\upsilon_{\sigma l}^\rho\sum_{l=1}^N\frac{\upsilon_{\epsilon l}^\rho}{\upsilon_{\sigma l}^\rho}}\right]*\partial_k\sigma_{ij} \quad (11)$$

This gives the final form of the viscoelastic equations

$$\begin{aligned}
\partial_t v_x &= \rho_i^{-1}\left(\partial_x\sigma_{xx}+\partial_y\sigma_{xy}+\partial_z\sigma_{xz}\right)+f_x,\\
&+ \sum_{l=0}^N\theta_{xxx}^l\Delta\rho_l^{-1}+\sum_{l=0}^N\theta_{yxy}^l\Delta\rho_l^{-1}+\sum_{l=0}^N\theta_{zxz}^l\Delta\rho_l^{-1},\\
\partial_t v_y &= \partial_x\sigma_{yx}+\partial_y\sigma_{yy}+\partial_z\sigma_{yz}+f_y,\\
&+ \sum_{l=0}^N\theta_{xyx}^l\Delta\rho_l^{-1}+\sum_{l=0}^N\theta_{yyy}^l\Delta\rho_{ll}^{-1}+\sum_{l=0}^N\theta_{zyz}^l\Delta\rho_{ll}^{-1}.\\
\partial_t v_z &= \partial_x\sigma_{zx}+\partial_y\sigma_{zy}+\partial_z\sigma_{zz}+f_z,\\
&+ \sum_{l=0}^N\theta_{xzx}^l+\sum_{l=0}^N\theta_{yzy}^l+\sum_{l=0}^N\theta_{zzz}^l.
\end{aligned} \quad (12)$$

$$\begin{aligned}
\dot{e_{xx}} &= \partial_x v_x,\\
\dot{e_{yy}} &= \partial_y v_y,\\
\dot{e_{zz}} &= \partial_z v_z,\\
\dot{e_{xy}} &= \frac{1}{2}(\partial_x v_y+\partial_y v_x),
\end{aligned}$$

$$
\begin{aligned}
\dot{e_{xz}} &= \frac{1}{2}(\partial_x v_z + \partial_z v_y), \\
\dot{e_{yz}} &= \frac{1}{2}(\partial_y v_z + \partial_z v_y).
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx} \\
&+ \sum_{l=1}^{N} \left[\gamma_{xx}^l + \gamma_{yy}^l + \gamma_{zz}^l\right] \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma_{xx}^l \Delta\mu_l, \\
\dot{\sigma}_{yy} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy}, \\
&+ \sum_{l=1}^{N} \left[\gamma_{xx}^l + \gamma_{yy}^l + \gamma_{zz}^l\right] \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma_{yy}^l \Delta\mu_l, \\
\dot{\sigma}_{zz} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{zz} + \dot{q}_{zz} \\
&+ \sum_{l=1}^{N} \left[\gamma_{xx}^l + \gamma_{yy}^l + \gamma_{zz}^l\right] \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma_{zz} \Delta\mu_l, \\
\dot{\sigma}_{xy} &= 2\mu \dot{e}_{xy} + 2\sum_{l=1}^{N} \gamma_{xy}^l \Delta\mu_l + \dot{q}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu \dot{e}_{xz} + 2\sum_{l=1}^{N} \gamma_{xz}^l \Delta\mu_l + \dot{q}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu \dot{e}_{yz} + 2\sum_{l=1}^{N} \gamma_{yz}^l \Delta\mu_l + \dot{q}_{yz},
\end{aligned}
\tag{14}
$$

2.1.2 Integration of memory functions

The memory functions obeys approximately a recursive relation (See the Appendix). As an example we consider γ_{xx}^l

$$
\gamma_{xx}^l(t) = \exp(-\frac{\Delta t}{\tau_\sigma^{\lambda l}})\gamma_\lambda^l(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^\lambda \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}}\right) \dot{e}_{xx}(t). \tag{15}
$$

The other components of γ and θ obeys similar recursive relations. Defining the quantities

$$
\alpha_1^l = \exp(-\frac{\Delta t}{\tau_{\sigma l}^\lambda}),
$$

$$\alpha_2^l = \frac{\Delta t}{\tau_{\sigma l}^{\lambda} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}}, \tag{16}$$

we get

$$\gamma_{xx}^l(t) = \alpha_1^l \gamma_{xx}^l(t - \Delta t) + \alpha_2^l \dot{e}_{xx}(t). \tag{17}$$

For the other components we can write in a similar way:

$$\begin{aligned}
\gamma_{yy}^l(t) &= \alpha_1^l \gamma_{yy}^l(t - \Delta t) + \alpha_2^l \dot{e}_{yy}(t - \Delta t), \\
\gamma_{zz}^l(t) &= \alpha_1^l \gamma_{zz}^l(t - \Delta t) + \alpha_2^l \dot{e}_{zz}(t - \Delta t), \\
\gamma_{xy}^l(t) &= \beta_1^l \gamma_{xy}^l(t - \Delta t) + \beta_2^l \dot{e}_{xy}(t - \Delta t), \\
\gamma_{xz}^l(t) &= \beta_1^l \gamma_{xz}^l(t - \Delta t) + \beta_2^l \dot{e}_{xz}(t - \Delta t), \\
\gamma_{yz}^l(t) &= \beta_1^l \gamma_{yz}^l(t - \Delta t) + \beta_2^l \dot{e}_{yz}(t - \Delta t)
\end{aligned} \tag{18}$$

Here the β coefficients are given by

$$\begin{aligned}
\beta_1^l &= \exp(-\frac{\Delta t}{\tau_{\sigma l}^{\mu}}), \\
\beta_2^l &= \frac{\Delta t}{\tau_{\sigma l}^{\mu} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}}.
\end{aligned} \tag{19}$$

The theta relations are

$$\begin{aligned}
\theta_{xxx}^l(t) &= \eta_1^l \theta_{xxx}^l(t - \Delta t) + \eta_2^l \partial_x \sigma_{xx}(t - \Delta t), \\
\theta_{yyy}^l(t) &= \eta_1^l \theta_{yyy}^l(t - \Delta t) + \eta_2^l \partial_y \sigma_{yy}(t - \Delta t), \\
\theta_{zzz}^l(t) &= \eta_1^l \theta_{zzz}^l(t - \Delta t) + \eta_2^l \partial_z \sigma_{zz}(t - \Delta t), \\
\theta_{yxy}^l(t) &= \nu_1^l \theta_{yxy}^l(t - \Delta t) + \nu_2^l \partial_y \sigma_{xy}(t - \Delta t), \\
\theta_{zxz}^l(t) &= \nu_1^l \theta_{zxz}^l(t - \Delta t) + \nu_2^l \partial_z \sigma_{xz}(t - \Delta t), \\
\theta_{xyx}^l(t) &= \nu_1^l \theta_{xyx}^l(t - \Delta t) + \nu_2^l \partial_x \sigma_{yx}(t - \Delta t), \\
\theta_{zyz}^l(t) &= \nu_1^l \theta_{zyz}^l(t - \Delta t) + \nu_2^l \partial_z \sigma_{yz}(t - \Delta t), \\
\theta_{xzx}^l(t) &= \nu_1^l \theta_{xzx}^l(t - \Delta t) + \nu_2^l \partial_x \sigma_{zx}(t - \Delta t), \\
\theta_{yzy}^l(t) &= \nu_1^l \theta_{yzy}^l(t - \Delta t) + \nu_2^l \partial_y \sigma_{zy}(t - \Delta t).
\end{aligned} \tag{20}$$

The η and ν coefficients are given by

$$\begin{aligned}
\eta_1^l &= \exp(-\frac{\Delta t}{\tau_{\sigma l}^p}), \\
\eta_2^l &= \frac{\Delta t}{\tau_{\sigma l}^p \sum_{l=1}^N \frac{\tau_{\epsilon l}^p}{\tau_{\sigma l}^p}}, \\
\nu_1^l &= \exp(-\frac{\Delta t}{\tau_{\sigma l}^s}), \\
\nu_2^l &= \frac{\Delta t}{\tau_{\sigma l}^s \sum_{l=1}^N \frac{\tau_{\epsilon l}^s}{\tau_{\sigma l}^s}},
\end{aligned} \tag{21}$$

2.2 The two dimensional case

In two dimensions equation 6 reduces to:

$$\begin{aligned}
\dot{v}_x &= \rho_u^{-1}\left[\partial_x \sigma_{xx} + \partial_y \sigma_{xy}\right] + \dot{f}_x \\
&+ \chi_p * \partial_x \sigma_{xx} + \chi_s * \partial_y \sigma_{xy}, \\
\dot{v}_y &= \rho_u^{-1}\left[\partial_x \sigma_{xy} + \partial_y \sigma_{yy}\right] + \dot{f}_y \\
&+ \chi_p * \partial_y \sigma_{yy} + \chi_s * \partial_x \sigma_{xy}.
\end{aligned} \tag{22}$$

$$\begin{aligned}
\dot{e}_{xx} &= \partial_x v_x, \\
\dot{e}_{yy} &= \partial_y v_x, \\
\dot{e}_{xy} &= \frac{1}{2}\left[\partial_x v_y + \partial_y v_x\right].
\end{aligned} \tag{23}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u\left[\dot{e}_{xx} + \dot{e}_{yy}\right] + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx} \\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy}\right] + 2\phi_\mu(t) * \dot{e}_{xy} \\
\dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \right) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy}, \\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy}\right] + 2\phi_\mu(t) * \dot{e}_{xy} \\
\dot{\sigma}_{xy} &= 2\mu_u \dot{e}_{xy} + q_{xy}.
\end{aligned} \tag{24}$$

The final form of the visco-elastic equations in two dimensions is then

$$\begin{aligned}
\partial_t v_x &= \rho^{-1}\left(\partial_x \sigma_{xx} + \partial_y \sigma_{xy}\right) + \rho^{-1} f_x, \\
&+ \sum_{l=0}^{N} \theta^l_{xxx} \Delta\rho_l^{-1} + \sum_{l=0}^{N} \theta^l_{yxy} \Delta\rho_l^{-1} \\
\partial_t v_y &= \rho^{-1}\left(\partial_x \sigma_{yx} + \partial_y \sigma_{yy}\right) + \rho^{-1} f_y, \\
&+ \sum_{l=0}^{N} \theta^l_{xyx} \Delta\rho_l^{-1} + \sum_{l=0}^{N} \theta^l_{yyy} \Delta\rho_l^{-1}.
\end{aligned} \tag{25}$$

$$\begin{aligned}
\dot{e_{xx}} &= \partial_x v_x, \\
\dot{e_{yy}} &= \partial_y v_y, \\
\dot{e_{zz}} &= \partial_z v_z, \\
\dot{e_{xy}} &= \frac{1}{2}(\partial_x v_y + \partial_y v_x),
\end{aligned} \tag{26}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy}\right) + 2\mu_u \dot{e}_{xx} \\
&+ \sum_{l=1}^{N} \left[\gamma^l_{xx} + \gamma^l_{yy}\right] \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma^l_{xx} \Delta\mu_l, \\
\dot{\sigma}_{yy} &= \lambda_u \left(\dot{e}_{xx} + \dot{e}_{yy}\right) + 2\mu_u \dot{e}_{yy}, \\
&+ \sum_{l=1}^{N} \left[\gamma^l_{xx} + \gamma^l_{yy} +\right] \Delta\lambda_l + 2\sum_{l=1}^{N} \gamma^l_{yy} \Delta\mu_l, \\
\dot{\sigma}_{xy} &= 2\mu \dot{e}_{xy} + 2\sum_{l=1}^{N} \gamma^l_{xy} \Delta\mu_l + \dot{q}_{xy},
\end{aligned} \tag{27}$$

3 Numerical grids

The equations given above can be solved numerically by using the Finite-Difference method. We first need to define the numerical grids involved.

3.1 Two dimensional case

We now consider a regular grid with positions $\boldsymbol{x}$ defined as

$$\begin{aligned} \boldsymbol{x} &= (x, y), && (28) \\ x &= p\Delta x, && (29) \\ y &= q\Delta y, && (30) \\ t &= n\Delta t. && (31) \end{aligned}$$

where $p = 0, 1, 2, \ldots, N_x$, $q = 0, 1, 2, \ldots, N_y$ and N_x, N_y are the number of grid points in the x and y-directions. $n = 0, 1, 2, \ldots, N_t$ where N_t is the number of time steps.

We will also need a regular grid which is displaced, or staggered, relative to the the regular grid. Sometimes we will refer to the regular grid as the reference grid.

The particle velocities v_x and v_y are defined on grids as follows

$$\begin{aligned} v_x(\boldsymbol{x}, t) &= v_x(x + \Delta x/2, y, t), \\ v_y(\boldsymbol{x}, t) &= v_y(x, y + \Delta y/2, t), \end{aligned} \qquad (32)$$

The diagonal stresses are defined on the grids

$$\begin{aligned} \sigma_{xx}(\boldsymbol{x}, t) &= \sigma_{xx}(x, y, t), \\ \sigma_{yy}(\boldsymbol{x}, t) &= \sigma_{yy}(x, y, t), \end{aligned} \qquad (33)$$

The off-diagonal stress

$$\sigma_{xy}(\boldsymbol{x}, t) = \sigma_{xy}(x + \Delta x/2, y + \Delta y/2t) \qquad (34)$$

The Lamé parameters are defined on grids as follows: as follows:

$$\begin{aligned} \lambda_u(\boldsymbol{x}) &= \lambda_u(x, y), && (35) \\ \mu_u(\boldsymbol{x}) &= \mu_u(x + \Delta x/2, y + \Delta y/2), \\ \mu_u(\boldsymbol{x}) &= \mu_u(x, y). && (36) \end{aligned}$$

The inverse density and μ are defined on three different staggered grids as follows:

$$\begin{aligned} \rho_x^{-1}(\boldsymbol{x}) &= \rho^{-1}(x + \Delta x/2, y), \\ \rho_y^{-1}(\boldsymbol{x}) &= \rho^{-1}(x, y + \Delta y/2). \end{aligned}$$

The θ functions are defined on two different grids:

$$\begin{aligned}
\theta_{xxx} &= \theta(x+\Delta x/2), && (37)\\
\theta_{yxy} &= \theta(x+\Delta x/2), && (38)\\
\theta_{xyx} &= \theta(x,y+\Delta y/2), && (39)\\
\theta_{yyy} &= \theta(x,y+\Delta y/2). && (40)
\end{aligned}$$

The *gamma* functions are defined on the grids

$$\begin{aligned}
\gamma_{xx} &= \gamma(x,y) && (41)\\
\gamma_{yy} &= \gamma(x,y) && (42)\\
\gamma_{xy} &= \gamma(x+\Delta x/2,y+\Delta y/2). && (43)
\end{aligned}$$

4 Finite-Difference Solution algorithms

We are now in a position to formulate a complete numerical solution of the visco-elastic equations. Below we give full expressions for solution algorithms in two dimensions, as well as specialization to the visco-acoustic case.

Differentiation is now replaced by numerical approximations so that ∂_x and ∂_y are replaced with numerical operators $d_x^+, d_x^-, d_y^+, d_y^-$. These operators connects the staggered and reference grids, and we illustrate this with the differentiation in the x-direction. The derivative of a function $a(x)$ is approximately given at $a(x+\Delta x/2)$and at $a(x-\Delta x/2)$ by

$$\begin{aligned}
a'(x+\Delta x/2) &= d_x^+ a(x),\\
a'(x-\Delta x/2) &= d_x^- a(x). && (44)
\end{aligned}$$

The differentiators d^+ and d^- are given by (**?**)

$$\begin{aligned}
\partial^+ &= \frac{1}{\Delta x}\sum_{l=1}^{L}\alpha_l\left[u(x+l\Delta x)-u(x-(l-1)\Delta x\right]\\
\partial^- &= \frac{1}{\Delta x}\sum_{l=1}^{L}\alpha_l\left[u(x+(l-1)\Delta x)-u(x-l\Delta x)\right] && (45)
\end{aligned}$$

where the coefficients α_l are found through an optimization procedure. Similar differentiators are defined for the y-direction and for the z-direction, with obvious names.

4.1 The two dimensional case

Using the numerical differentiators the equations of motion becomes:

$$
\begin{aligned}
\dot{v}_x &= \rho_x^{-1}\left(d_x^+\sigma_{xx} + d_y^-\sigma_{xy}\right) + \rho_x^{-1}\dot{f}_x \\
&+ \sum_{l=0}^{N}\theta_{xxx}^l\Delta\rho_{lx}^{-1} + \sum_{l=0}^{N}\theta_{yxy}^l\Delta\rho_{lix}^{-1} \\
\dot{v}_y &= \rho_y^{-1}\left(d_x^-\sigma_{xy} + d_y^+\sigma_{yy}\right) + \rho_y^{-1}\dot{f}_y. \\
&+ \sum_{l=0}^{N}\theta_{xyx}^l\Delta\rho_{ly}^{-1} + \sum_{l=0}^{N}\theta_{yyy}^l\Delta\rho_{ly}^{-1}.
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\dot{e}_{xx} &= d_x^- v_x, \\
\dot{e}_{yy} &= d_y^- v_y, \\
\dot{e}_{xy} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x), \\
\dot{e}_{yx} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x),
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u\left[\dot{e}_{xx} + \dot{e}_{yy}\right] + 2\mu_u\dot{e}_{xx} + \dot{q}_{xx}, \\
&+ \sum_{l=1}^{N}\left[\gamma_{xx}^l + \gamma_{yy}^l\right]\Delta\lambda_l + 2\sum_{l=1}^{N}\gamma_{xx}^l\Delta\mu_l, \\
\dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} +\right) + 2\mu_u\dot{e}_{yy} + \dot{q}_{yy}, \\
&+ \sum_{l=1}^{N}\left[\gamma_{xx}^l + \gamma_{yy}^l\right]\Delta\lambda_l + 2\sum_{l=1}^{N}\gamma_{yy}^l\Delta\mu_l, \\
\dot{\sigma}_{xy} &= 2\mu\dot{e}_{xy} + 2\sum_{l=1}^{N}\gamma_{xy}^l\Delta\mu_l + \dot{q}_{xy}. \\
\dot{\sigma}_{yx} &= 2\mu\dot{e}_{yx} + 2\sum_{l=1}^{N}\gamma_{xy}^l\Delta\mu_l + \dot{q}_{xy}.
\end{aligned}
\tag{48}
$$

The time derivatives is approximated by the central difference

$$
\dot{a}(t) = \frac{a(t+\Delta t/2) - a(t-\Delta t/2)}{\Delta t} \tag{49}
$$

We use the expression for the approximate time derivative given by equation (49) in equations (14) to obtain an expression for the components of the particle velocity

$$\begin{aligned}
v_x(t+\Delta t/2) &= \Delta t \rho_x^{-1}\left[d_x^+\sigma_{xx}(t) + d_y^+\sigma_{xy}(t)\right] + \Delta t \rho_x^{-1} f_x(t) + \\
&+ \Delta t \sum_{l=0}^{N} \theta_{xxx}^l(t)\Delta\rho_{lx}^{-1} + \Delta t \sum_{l=0}^{N} \theta_{yxy}^l(t)\Delta\rho_{lx}^{-1} + v_x(t-\Delta t/2), \\
v_y(t+\Delta t/2) &= \Delta t \rho_z^{-1}\left[d_x^+\sigma_{yx}(t) + d_y^+\sigma_{yy}(t)\right] + \Delta t \rho_y^{-1} f_y(t) \\
&+ \Delta t \sum_{l=0}^{N} \theta_{yyy}^l(t)\Delta\rho_{ly}^{-1} + \Delta t \sum_{l=0}^{N} \theta_{yxy}^l(t)\Delta\rho_{ly}^{-1} + v_y(t-\Delta t/2),
\end{aligned} \tag{50}$$

The strains can now be computed from equation (47)

$$\begin{aligned}
\dot{e}_{xx}(t+\Delta t/2) &= d_x^- v_x(t+\Delta t/2), \\
\dot{e}_{zz}(t+\Delta t/2) &= d_z^- v_z(t+\Delta t/2), \\
\dot{e}_{xz}(t+\Delta t/2) &= \frac{1}{2}\left[d_x^+ v_z(t+\Delta t/2) + d_z^+ v_x(t+\Delta t/2)\right],
\end{aligned} \tag{51}$$

Equations (**??**) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned}
\sigma_{xx}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] \\
&+ 2\Delta t \mu_u \dot{e}_{xx}(t+\Delta t/2) + \Delta t \dot{q}_{xx} \\
&+ \Delta t \sum_{l=1}^{N}\left[\gamma_{xx}^l(t+\Delta t/2) + \gamma_{yy}^l(t+\Delta t/2)\right]\Delta\lambda_l \\
&+ 2\Delta t \sum_{l=1}^{N} \gamma_{xx}^l(t+\Delta t/2)\Delta\mu_l \\
&+ \sigma_{xx}(t),
\end{aligned} \tag{52}$$

$$\begin{aligned}
\sigma_{yy}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{yy}(t+\Delta t/2)\right] \qquad (53) \\
&+ 2\Delta t \mu_u \dot{e}_{yy}(t+\Delta t/2) + \Delta t \dot{q}_{yy}(t+\Delta t/2) \\
&+ \Delta t \sum_{l=1}^{N}\left[\gamma_{xx}^l(t+\Delta t/2) + \gamma_{yy}^l(t+\Delta t/2)\right]\Delta\lambda_l \\
&+ 2\Delta t \sum_{l=1}^{N} \gamma_{yy}(t+\Delta t/2)\Delta\mu_l,
\end{aligned}$$

$$
\begin{aligned}
& + \quad \sigma_{yy}(t). \qquad (54)\\
\sigma_{xy}(t+\Delta t) &= 2\Delta t\mu\dot{e}_{xy}(t+\Delta t/2) + 2\Delta t\sum_{l=1}^{N}\gamma_{xy}^{l}(t+\Delta t/2)\Delta\mu_l + \Delta t\dot{q}_{xy}(t+\Delta t/2)\\
&+ \quad \sigma_{xy}(t), \qquad (55)
\end{aligned}
$$

The θ functions are updated as:

$$
\begin{aligned}
\theta_{xxx}(t+\Delta t) &= \eta_{1x}^{l}\theta_{x}^{l}(t) + \eta_{2x}^{l}\partial_x\sigma(t),\\
\theta_{yyy}(t+\Delta t) &= \eta_{1z}^{l}\theta_{x}^{l}(t) + \eta_{2y}^{l}\partial_y\sigma(t).\\
\theta_{yxy}(t+\Delta t) &= \nu_{1x}^{l}\theta_{x}^{l}(t) + \nu_{2x}^{l}\partial_x\sigma(t),\\
\theta_{xyx}(t+\Delta t) &= \nu_{1z}^{l}\theta_{x}^{l}(t) + \nu_{2y}^{l}\partial_y\sigma(t).
\end{aligned}
$$

The γ functions are given by

$$
\begin{aligned}
\gamma_{xx}^{l}(t+3/2\Delta t) &= \alpha_{1x}^{l}\gamma_{xx}^{l}(t+\Delta t/2) + \alpha_{2x}^{l}\dot{e}_{xx}(t+\Delta t/2),\\
\gamma_{yy}^{l}(t+3/2\Delta t) &= \alpha_{1y}^{l}\gamma_{yy}^{l}(t+\Delta t/2) + \alpha_{2y}^{l}\dot{e}_{yy}(t+\Delta t/2).\\
\gamma_{xy}^{l}(t+3/2\Delta t) &= \beta_{1y}^{l}\gamma_{xy}^{l}(t+\Delta t/2) + \beta_{2y}^{l}\dot{e}_{xy}(t+\Delta t/2).
\end{aligned}
$$

The coefficents for the γ and θ functions are

$$
\begin{aligned}
\alpha_{1x}^{l} &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^{\lambda}}\right),\\
\alpha_{2x}^{l} &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^{\lambda}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}})}\\
\alpha_{1y}^{l} &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^{\lambda}}\right),\\
\alpha_{2y}^{l} &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^{\lambda}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}})}\\
\beta_{1x}^{l} &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^{\mu}}\right),\\
\beta_{2x}^{l} &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^{\mu}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}})}\\
\beta_{1y}^{l} &= \exp\left(-\frac{d_y(y)\Delta t}{\tau_{\sigma l}^{\mu}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\beta_{2y}^{l} &= \frac{d_y(y)\Delta t}{(\tau_{\sigma l}^{\mu}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}})} \\
\eta_{1x}^{l} &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^{\rho}}\right), \\
\eta_{2x}^{l} &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}})}, \\
\eta_{1y}^{l} &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^{\rho}}\right), \\
\eta_{2y}^{l} &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}})}. \\
\nu_{1x}^{l} &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^{\rho}}\right), \\
\nu_{2x}^{l} &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}})}. \\
\nu_{1y}^{l} &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^{\rho}}\right), \\
\nu_{2y}^{l} &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}})}.
\end{aligned}
$$

4.2 The two dimensional Acoustic case

For the acoustic 2D case we reduce the equations above by neglecting the y-axis terms and putting $\mu = 0$. We consider also the pseudo-stress σ defined by

$$\sigma = \frac{1}{2}(\sigma_{xx} + \sigma_{zz})$$

We then get the acoustic 2D scheme as:

$$
\begin{aligned}
v_x(t + \Delta t/2) &= \Delta t\left[\rho_{ux}^{-1} d_x^{+} \sigma_{xx}(t) + \rho_{ux}^{-1} f_x(t)\right] + \\
&+ \Delta t \sum_{l=0}^{N} \theta_x^l(t) \Delta \rho_x^{-1} + v_x(t - \Delta t/2), \\
v_z(t + \Delta t/2) &= \Delta t\left[\rho_{uz}^{-1} d_z^{+} \sigma_{zz}(t) + \Delta t \rho_{uz}^{-1} f_z(t)\right]
\end{aligned}
$$

$$+ \quad \Delta t \sum_{l=0}^{N} \theta_z^l(t) \Delta\rho_z^{-1} + v_z(t - \Delta t/2).$$

The strains can now be computed from :

$$\begin{aligned}
\dot{e}_{xx}(t + \Delta t/2) &= d_x^- v_x(t + \Delta t/2), \\
\dot{e}_{zz}(t + \Delta t/2) &= d_z^- v_z(t + \Delta t/2).
\end{aligned}$$

Equations (**??**) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned}
\sigma(t + \Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)\right] + \Delta t \dot{q}, \\
&+ \Delta t \sum_{l=1}^{N} \gamma^l(t + \Delta t/2) \Delta\lambda_l + \sigma(t).
\end{aligned}$$

We now split the γ^l into two parts γ_x^l and γ_z^l as follows:

$$\begin{aligned}
\sigma(t + \Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t + \Delta t/2) + \dot{e}_{zz}(t + \Delta t/2)\right] + \Delta t \dot{q} \\
&+ \Delta t \sum_{l=1}^{N} \left[\gamma_x^l(t + \Delta t/2) \Delta\lambda_l + \gamma_z^l(t + \Delta t/2) \Delta\lambda_l\right] + \sigma(t).
\end{aligned}$$

The θ functions are updated as:

$$\begin{aligned}
\theta_x(t + \Delta t) &= \eta_{1x}^l \theta_x^l(t) + \eta_{2x}^l \partial_x \sigma(t), \\
\theta_z(t + \Delta t) &= \eta_{1z}^l \theta_x^l(t) + \eta_{2z}^l \partial_y \sigma(t).
\end{aligned}$$

The γ functions are given by

$$\begin{aligned}
\gamma_x^l(t + 3/2\Delta t) &= \alpha_{1x}^l \gamma_x^l(t + \Delta t/2) + \alpha_{2x}^l \dot{e}_{xx}(t + \Delta t/2), \\
\gamma_z^l(t + 3/2\Delta t) &= \alpha_{1z}^l \gamma_z^l(t + \Delta t/2) + \alpha_{2z}^l \dot{e}_{zz}(t + \Delta t/2).
\end{aligned}$$

Standard linear solid

The coefficents are

$$\begin{aligned}
\alpha_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})}
\end{aligned}$$

$$
\begin{aligned}
\alpha_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\lambda}\right), \\
\alpha_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\eta_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}, \\
\eta_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}.
\end{aligned}
$$

The profile functions d_x and d_z are

$$
d_x(x) = (x/L)^2, d_z(y) = (z/L)^2,
$$

where L is the length of the absorbing layer and and we also have

$$
\Delta\lambda_l = \lambda_u \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right) \Delta\rho^{-1} = \rho_u^{-1} \left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right) \tag{56}
$$

Maxwell solid

The coefficents are

$$
\begin{aligned}
\alpha_{1x} &= -\frac{1}{\tau_0^\lambda} \exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\lambda}\right), && (57) \\
\alpha_{2x} &= \frac{d_x(x)\Delta t}{\tau_0^\lambda}, && (58) \\
\alpha_{1z} &= -\frac{1}{\tau_0^\lambda} \exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\lambda}\right), && (59) \\
\alpha_{2z} &= \frac{d_z(z)\Delta t}{\tau_0^\lambda}. && (60) \\
\eta_{1x} &= -\frac{1}{\tau_0^\rho} \exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\rho}\right), && (61)
\end{aligned}
$$

$$\eta_{2x} = \frac{d_x(x)\Delta t}{\tau_0^\rho}, \tag{62}$$

$$\eta_{1z} = -\frac{1}{\tau_0^\rho}\exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\rho}\right), \tag{63}$$

$$\eta_{2z} = \frac{d_z(z)\Delta t}{\tau_0^\rho}. \tag{64}$$

We also have

$$\Delta\lambda = \lambda_u, \tag{65}$$

$$\Delta\rho^{-1} = \rho_u^{-1}. \tag{66}$$

5 The PyAc2d python library

APPENDIX A: The viscoelastic standard linear solid

Bolzman's generalization of Hook's law to the visco-elastic case is (**?**):

$$\sigma_{ij} = \psi_{ijkl} * \dot{e}_{kl}, \tag{A-1}$$

where ψ_{ijkl} is known as the relaxation tensor. The $*$ denotes convolution defined by

$$a(t) * b(t) = \int_0^t a(t-\tau)b(\tau). \tag{A-2}$$

Integrating (A-1) by parts

$$\sigma_{ij}(t) = |_0^t \psi_{ijkl}(t-\tau)e_{kl}(\tau) + \int_0^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau), \tag{A-3}$$

and using $e(t=0) = 0$ I get

$$\sigma_{ij}(t) = \psi(0)_{ijkl}e_{kl}(t) + \int_{0+}^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau) \tag{A-4}$$

For the Zener model the components of the ψ_{ijkl} tensor have the form

$$\psi(t) = K\left[1 - \frac{1}{N}\sum_{l=1}^{N}(1-\frac{\tau_{\epsilon l}}{\tau_{\sigma l}})\exp(-t/\tau_{\sigma l})\right]H(t). \tag{A-5}$$

where K_r is a relaxed modulus, N is the number of Zener mechanisms,$\tau_{\sigma l}$ and $\tau_{\epsilon l}$ are relaxation times. $H(t)$ is the Heavy side function. The time derivative of ψ is equal to:

$$\dot{\psi} = \phi(t), \tag{A-6}$$

where ϕ is equal to:

$$\phi(t) = \frac{1}{N}\sum_{l=1}^{N}\left[\left(\frac{K_r}{\tau_{\sigma l}}\right)\left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\exp(-t/\tau_{\sigma l})\right]H(t). \tag{A-7}$$

Using the above we have for the stress:

$$\sigma_{ij}(t) = c_{ijkl}e_{kl} + \int_{0+}^{t}\phi_{ijkl}(t-\tau)e_{kl}(\tau) \tag{A-8}$$

This is most conveniently written as

$$\sigma_{ij}(t) = c_{ijkl}(t) * e_{kl}(t), \tag{A-9}$$

where

$$c_{ijkl}(t) = \psi(0)_{ijkl}\delta(t) + \phi_{ijkl}(t), \tag{A-10}$$

By definition $\psi(t=0)$ corresponds to the unrelaxed modulus so that we have

$$K_u = \frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}K_r \tag{A-11}$$

or

$$K_r = \frac{K_u}{\frac{1}{N}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \tag{A-12}$$

The ϕ function can then be expressed in terms of the unrelaxed moduli:

$$\phi(t) = \sum_{l=1}^{N}\left[\left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}}\right)K_u\left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right)\right] \tag{A-13}$$

Finally, we express ϕ as:

$$\phi(t) = \sum_{l=1}^{N}\phi^l(t) \tag{A-14}$$

where

$$\phi^l(t) = \left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}}\right)\Delta K_l \tag{A-15}$$

and ΔK_l is

$$\Delta K_l = K_u\left(1-\frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right) \tag{A-16}$$

It is most practical to write the time-dependent visco-elastic constants as

$$\lambda(t) = \lambda_u\delta(t) + \phi_\lambda(t), \tag{A-17}$$
$$\mu(t) = \mu_u\delta(t) + \phi_\mu(t), \tag{A-18}$$

where ϕ_λ is given as:

$$\phi_\lambda(t) = \sum_{l=1}^{N}\left(\frac{\exp(-t/\tau^\lambda_{\sigma l})}{\tau^\lambda_{\sigma l}\sum_{l=1}^{N}\frac{\tau^\lambda_{\epsilon l}}{\tau^\lambda_{\sigma l}}}\right)\Delta\lambda_l \tag{A-19}$$

and ϕ_μ is given as:

$$\phi_\mu(t) = \sum_{l=1}^{N}\left(\frac{\exp(-t/\tau^\mu_{\sigma l})}{\tau^\mu_{\sigma l}\sum_{l=1}^{N}\frac{\tau^\mu_{\epsilon l}}{\tau^\mu_{\sigma l}}}\right)\Delta\mu_l \tag{A-20}$$

Q-factors

See **?** for further relations between Q and relaxation times. The Q values are related to the Fourier transform of the λ modulus as

$$Q_\lambda^{-1}(\omega) = \frac{Im\lambda(\omega)}{Re\lambda(\omega)} \tag{A-21}$$

Assuming λ is given as

$$\lambda(t) = \lambda_u\delta(t) + \phi_\lambda(t). \tag{A-22}$$

The fourier transform of λ is given by

$$\lambda(\omega) = \lambda_u + \int_{-\infty}^{\infty}\phi_\lambda(t)\exp(-i\omega t). \tag{A-23}$$

The Fourier transform of ϕ_λ is:

$$\phi_\lambda(\omega) = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1-\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\int_0^{+\infty} dt\,\exp(-i\omega)\exp(-t/\tau_{\sigma l}^\lambda). \quad \text{(A-24)}$$

The results is:

$$\phi_\lambda(\omega) = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1-\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\frac{1}{1+i\omega\tau_{\sigma l}^\lambda}. \quad \text{(A-25)}$$

The fourier transform of λ is then

$$\lambda(\omega) = \lambda_u + \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^\lambda}\right)\left(1-\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right)\frac{1}{1+i\omega\tau_{\sigma l}^\lambda}. \quad \text{(A-26)}$$

After some (tedious) algebra one obtains

$$\lambda(\omega) = \lambda_r\frac{1}{N}\sum_{l=1}^{N}\frac{1+i\omega\tau_{\epsilon l}^\lambda}{1+i\omega\tau_{\sigma l}^\lambda} \quad \text{(A-27)}$$

Separating into real and imaginary parts, I get

$$\begin{aligned}\lambda(\omega) &= \lambda_r\frac{1}{N}\sum_{l=1}^{N}\frac{1+\omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda}{1+(\omega\tau_{\sigma l}^\lambda)^2} && \text{(A-28)}\\ &+ i\lambda_r\frac{1}{N}\sum_{l=1}^{N}\frac{\omega\tau_{\sigma l}^\lambda(\tau_\epsilon^\lambda/\tau_{\sigma l}^\lambda-1)}{1+(\omega\tau_{\sigma l}^\lambda)^2} && \\ & && \text{(A-29)}\end{aligned}$$

We then have

$$Q_\lambda^{-1} = \frac{\sum_{l=1}^{N}\omega\tau_{\sigma l}^\lambda\left(\tau_{\epsilon l}^\lambda/\tau_{\sigma l}^\lambda-1\right)/\left[1+(\omega\tau_{\sigma l}^\lambda)^2\right]}{\sum_{l=1}^{N}(1+\omega^2\tau_{\sigma l}^\lambda\tau_{\epsilon l}^\lambda)/\left[1+(\omega\tau_{\sigma l}^\lambda)^2\right]} \quad \text{(A-30)}$$

The results for the frequency dependence of μ is obtained in exactly the same manner as above:

$$\mu(\omega) = \mu_r\frac{1}{N}\sum_{l=1}^{N}\frac{1+i\omega\tau_{\epsilon l}^\mu}{1+i\omega\tau_{\sigma l}^\mu} \quad \text{(A-31)}$$

and the Q-factor for μ is

$$Q_\mu^{-1} = \frac{\sum_{l=1}^{N} \omega\tau_{\sigma l}^{\lambda}\left(\tau_{\epsilon l}^{\mu}/\tau_{\sigma l}^{\lambda} - 1\right)/\left[1 + (\omega\tau_{\sigma l}^{\mu})^2\right]}{\sum_{l=1}^{N}(1 + \omega^2\tau_{\sigma l}^{\mu}\tau_{\epsilon l}^{\mu})/\left[1 + (\omega\tau_{\sigma l}^{\mu})^2\right]} \tag{A-32}$$

In practice we need to relate Q_l and Q_μ to Q_κ. We use the relation

$$\kappa(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega). \tag{A-33}$$

Splitting into real and imaginary parts

$$\kappa_r(\omega) = \lambda_r(\omega) + \frac{2}{3}\mu_r(\omega). \kappa_i(\omega) = \lambda_i(\omega) + \frac{2}{3}\mu_i(\omega). \tag{A-34}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\lambda_i + (2/3)\mu_i}{\lambda_r + (2/3)\mu_r} \tag{A-35}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\left(\frac{\lambda_i}{\lambda_r}\right)\lambda_r + (2/3)\left(\frac{\mu_i}{\mu_r}\right)\mu_r}{\lambda_r + (2/3)\mu_r} \tag{A-36}$$

$$Q_\kappa^{-1} = \frac{Q_\lambda^{-1}\lambda_r + (2/3)Q_\mu^{-1}}{\lambda_r + (2/3)\mu_r} \tag{A-37}$$

We can use the P-wave and S-wave velocities

$$\begin{aligned} \lambda &= \rho V_p^2 - (2/3)\rho V_s^2 \\ \mu &= \rho V_s^2 \end{aligned} \tag{A-38}$$

$$Q_\kappa^{-1} = Q_\lambda^{-1}\left[1 - \left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2\right] + Q_\mu^{-1}\left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2 \tag{A-39}$$

Q-model parametrization

Q-models can be described by the two relaxation times, τ_σ and τ_ϵ. However it is simpler to use the two parameters τ_0 and Q_0 to describe a model. According to **?**, Appendix B, we have

$$Q(\omega) = Q_0\frac{1 + \omega^2\tau_0^2}{2\omega\tau_0}$$

where

$$\begin{aligned} Q_0 &= \frac{2\tau_0}{\tau_\epsilon - \tau_\sigma}, \\ \tau_0^2 &= \tau_\epsilon \tau_\sigma. \end{aligned}$$

$\omega = 1/\tau_0$ is the minimum value for $Q(\omega)$, i.e. the absorption top. We can now find τ_σ and τ_ϵ in terms of τ_0 and Q_0 as:

$$\begin{aligned} \tau_\epsilon &= \frac{\tau_0}{Q_0}\left[\sqrt{Q_0^2+1}+1\right], \\ \tau_\sigma &= \frac{\tau_0}{Q_0}\left[\sqrt{Q_0^2+1}-1\right]. \end{aligned}$$

Effective density

We now assume that the effective density has the following form

$$\rho_{eff}^{-1}(t) = s(t). \tag{A-40}$$

We also assume that

$$s(t) = s(0)\delta(t) + \chi(t) \tag{A-41}$$

So that the inverse of the effective density reads

$$\rho_{eff}^{-1}(t) = s_u\delta(t) + \chi(t). \tag{A-42}$$

where $s_u = s(0) = \rho_u^{-1}$.

APPENDIX B: The Maxwell visco-elastic solid

According to **?** the Maxwell visco-elastic solid has a modulus given by

$$\lambda(t) = \lambda_u \exp(-t/\tau_0)H(t) \tag{A-1}$$

In the frequency domain one gets

$$\lambda(\omega) = \frac{\tau_0\omega}{\omega\tau_0 - i}. \tag{A-2}$$

or,

$$\lambda(\omega) = \frac{\tau_0 i\omega}{i\omega\tau_0 + 1}. \tag{A-3}$$

The function ϕ is now:

$$\phi(t) = -\Delta\lambda \frac{1}{\tau_0} \exp(-t/\tau_0), \tag{A-4}$$

where $\Delta\lambda = \lambda_u$. The Q-value are related to τ_0 by:

$$\tau_0 = Q(\omega)/\omega. \tag{A-5}$$

A plane wave

APPENDIX C: Recursive computation of γ and s.

Standard Linear Solid

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t+\Delta t) = \int_0^{t+\Delta t} d\tau \, \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{t+\Delta t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t+\Delta t) = \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau). \end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t+\Delta t$

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_\epsilon} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \end{aligned}$$

Performing the integral we then get

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\tau_\sigma}{\tau_\epsilon}\left[1-\exp(-\frac{\Delta t}{\tau_\sigma^\lambda})\right]\dot{e}_{xx}(t).\end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\Delta t}{\tau_\epsilon}\dot{e}_{xx}(t).\end{aligned}$$

Maxwell solid

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t+\Delta t) = -\int_0^{t+\Delta t} d\tau \, \frac{1}{\tau_0^\lambda}\exp(-\frac{t+\Delta t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t+\Delta t) = -\frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_0^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).$$

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= -\frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau) \\ &- \frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).\end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t+\Delta t$

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= -\frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau) \\ &- \frac{1}{\tau_0^\lambda}\exp(-\frac{\Delta t}{\tau_0^\lambda})\dot{e}_{xx}(t)\int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\end{aligned}$$

Performing the integral we then get

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \left[1-\exp(-\frac{\Delta t}{\tau_0^\lambda})\right]\dot{e}_{xx}(t).\end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned}\gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t)\exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\Delta t}{\tau_0}\dot{e}_{xx}(t).\end{aligned}$$

APPENDIX E: Comparison with CPML

? show that the implementation of the CPML method can be performed by replacing each spatial derivative with (Their's equation (16) and (18)):

$$s_x = \delta(t) - d_x H(t)\exp\left[-(d_x+\alpha_x)t\right]. \tag{A-1}$$

We compare this with our equation

$$\lambda(t)/\lambda_u = \delta(t) + \frac{1}{\tau_\epsilon}\exp(-t/\tau_\sigma)H(t)\left(1-\frac{\tau_\epsilon}{\tau_\sigma}\right) \tag{A-2}$$

Comparing equation (A-1) with equation (A-2) one gets:

$$-d_x = \frac{1}{\tau_\epsilon}\left(1-\frac{\tau_\epsilon}{\tau_\sigma}\right), \tag{A-3}$$

$$d_x+\alpha = \frac{1}{\tau_\sigma} \tag{A-4}$$

Solving for τ_ϵ and τ_σ one gets

$$\tau_\sigma = \frac{1}{(d_x+\alpha)}, \tag{A-5}$$

$$\tau_\epsilon = \frac{1}{\alpha}. \tag{A-6}$$

. Here

$$d_x(x) = d_0\left(\frac{x}{L}\right)^2, \tag{A-7}$$

where d_0 is a constant and L is the length of the PML zone and x is the distance from the start (outer border) of the PML zone.