PyAc2d - python library for visco-elastic modeling

B.Arntsen

May 15, 2024

Contents

1 Introduction

PyAc2d is a small and simple python library for Finit-Difference propagation of viscoelastic waves. The core of the library is capable of running on multi-core cpu and also on gpu hardware.

In the following sections we outline the basic equations for visco-elastic waves in two and three dimensions. It is followed by sections on the numerical implementation of the Finite-Difference solution of viscoelastic wave propagation. The PyAc2d python library is then described in detail, using several examples. A separate section on the implementation details of the PyAc2d library is also given.

2 The Viscoelastic equations of motion

2.1 The three dimensional case

The fundamental equation for viscoelastic wave propagation are written as **?**

$$
\begin{aligned}
\partial_t^2 u_i(\boldsymbol{x},t) &= \rho^{-1}(\boldsymbol{x},t) * \partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), \\
\sigma_{ij}(\boldsymbol{x},t) &= \lambda(\boldsymbol{x},t) * e_{kk}\delta_{ij} + 2\mu(\boldsymbol{x},t) * e_{ij} + q_{ij}
\end{aligned} \tag{1}
$$

$\boldsymbol{x} = x, y, z$ denotes a spatial position with cartesioan components x, y, z and t is the time. u_i is the i'th component of the particle displacement and $i = x, y, z$. $\sigma_{i,j}$ is the stress tensor, while f_i is the i'th component of an external (source) body force. $q_{i,j}$ is an external (source) stress tensor. ρ is the density and λ and μ are the Lamé parameters. Note that both density and Lameé parameters are time dependent in order to describe the effect of visco-elasticity.

Using the derivations in Appendix A the viscoelastic equations given above in equations (1) and (1) can be formulated as

$$
\begin{aligned}
\partial_t^2 u_i(\boldsymbol{x},t) &= \rho_u^{-1}(\boldsymbol{x})\partial_j \sigma_{ij}(\boldsymbol{x},t) + f_i(\boldsymbol{x},t), \\
&+ \delta_{i,j}\chi_\lambda(\boldsymbol{x},t) * \partial_j\sigma_{ij}(\boldsymbol{x},t) + (1-\delta_{i,j})\chi_\mu(\boldsymbol{x},t) * \partial_j\sigma_{ij}(\boldsymbol{x},t) \qquad (2) \\
\sigma_{ij}(\boldsymbol{x},t) &= \lambda_u e_{kk}\delta_{ij} + 2\mu_u e_{ij} + q_{ij} \\
&+ \delta_{ij}\phi_\lambda(t) * e_{mm} + 2\phi_\mu(t) * e_{ij}. \qquad (3)
\end{aligned}
$$

Here ρ_u is the elastic unrelaxed part of the density ρ, while λ_u and μ_u are the corresponding unrelaxed part of the Lamé parameters. The relaxation functions χ and ϕ contains the effects of visco-elasticity and is equal to zero for a pure elastic medium.

Using the velocity $v_i = \dot{u}_i$, and writing out individual components one gets

$$\begin{aligned}
\partial_t v_x &= \rho_u^{-1}\left[\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz}\right] + f_x, \\
&= +\chi_\lambda * \partial_x \sigma_{xx} + \chi_\mu * \left[(\partial_y \sigma_{xy} + \partial_z \sigma_{xz}\right], \\
\partial_t v_y &= \rho_u^{-1}\left[p_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz}\right] + f_y, \\
&+ \chi_\lambda * \partial_x \sigma_{yx} + \chi_\mu * \left[\partial_y \sigma_{yy} + \partial_z \sigma_{yz}\right] \\
\partial_t v_z &= \rho_u^{-1}\left[\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}\right] + f_z, \\
&+ \chi * \left[\partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz}\right].
\end{aligned} \tag{4}$$

$$\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx}, \\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] + 2\phi_\mu(t) * \dot{e}_{xx}, \\
\dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy} \\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] + 2\phi_\mu(t) * \dot{e}_{yy}, \\
\dot{\sigma}_{zz} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right) + 2\mu_u \dot{e}_{zz} + q_{zz}, \\
&+ \phi_\lambda * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] + 2\phi_\mu(t) * \dot{e}_{zz}, \\
\dot{\sigma}_{xy} &= 2\mu_u \dot{e}_{xy} + q_{xy} + 2\phi_\mu(t) * \dot{e}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu_u \dot{e}_{xz} + q_{xz} + 2\phi_\mu(t) * \dot{e}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu_u \dot{e}_{yz} + q_{yz} + 2\phi_\mu(t) * \dot{e}_{yz}.
\end{aligned} \tag{5}$$

2.1.1 Memory functions

We now define so-called memory variables by including the time convolution into one set of variables:

$$\begin{aligned}
\gamma_\lambda^l(t) &= \frac{1}{\Delta\lambda_l}\phi_\lambda^l * \left[\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}\right] \\
\gamma_{ij}^l(t) &= \frac{1}{\Delta\mu_l}\phi_\mu^l * [\dot{e}_{ij}], \\
\theta_{kij}^l(t) &= \frac{1}{\Delta\rho_l^{-1}}\chi^l * \partial_k[\sigma_{ij}],
\end{aligned} \tag{6}$$

Here

$$\Delta\rho_l^{-1} = \rho_u^{-1}\left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right),$$

$$
\begin{aligned}
\Delta\lambda_l &= \lambda_u \left(1 - \frac{\tau^{\lambda}_{\epsilon l}}{\tau^{\lambda}_{\sigma l}}\right), \\
\Delta\mu_l &= \mu_u \left(1 - \frac{\tau^{\mu}_{\epsilon l}}{\tau^{\rho}_{\sigma l}}\right).
\end{aligned} \tag{7}
$$

This gives the expressions for the γ functions as:

$$
\begin{aligned}
\gamma^l_{\lambda} &= \left[\frac{\exp(-t/\tau^{\lambda}_{\sigma l})}{\tau^{\lambda}_{\sigma l}\sum_{l=1}^{N}\frac{\tau^{\lambda}_{\epsilon l}}{\tau^{\lambda}_{\sigma l}}}\right][\dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}], \\
\gamma^l_{ij} &= \left[\frac{\exp(-t/\tau^{\mu}_{\sigma l})}{\tau^{\mu}_{\sigma l}\sum_{l=1}^{N}\frac{\tau^{\mu}_{\epsilon l}}{\tau^{\mu}_{\sigma l}}}\right]\dot{e}_{ij}, \\
\theta^l_{kij} &= \left[\frac{\exp(-t/\tau^{\rho}_{\sigma l})}{\tau^{\rho}_{\sigma l}\sum_{l=1}^{N}\frac{\tau^{\rho}_{\epsilon l}}{\tau^{\rho}_{\sigma l}}}\right]\partial_k\sigma_{ij}.
\end{aligned} \tag{8}
$$

and also

$$
\begin{aligned}
\gamma_{\lambda}(t) &= \sum_{l=1}^{N}\gamma^l_{\lambda}, \\
\gamma_{ij}(t) &= \sum_{l=1}^{N}\gamma^l_{ij}, \\
\theta_{kij}(t) &= \sum_{l=1}^{N}\theta^l_{ij}.
\end{aligned}
$$

This gives the final form of the viscoelastic equations

$$
\begin{aligned}
\partial_t v_x &= \rho_i^{-1}\left(\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}\right) + f_x, \\
&+ \sum_{l=0}^{N}\theta^l_{xxx}\Delta\rho_l^{-1} + \sum_{l=0}^{N}\theta^l_{yxy}\Delta\rho_l^{-1} + \sum_{l=0}^{N}\theta^l_{zxz}\Delta\rho_l^{-1}, \\
\partial_t v_y &= \partial_x\sigma_{yx} + \partial_y\sigma_{yy} + \partial_z\sigma_{yz} + f_y, \\
&+ \sum_{l=0}^{N}\theta^l_{xyx}\Delta\rho_l^{-1} + \sum_{l=0}^{N}\theta^l_{yyy}\Delta\rho_{ll}^{-1} + \sum_{l=0}^{N}\theta^l_{zyz}\Delta\rho_{ll}^{-1}. \\
\partial_t v_z &= \partial_x\sigma_{zx} + \partial_y\sigma_{zy} + \partial_z\sigma_{zz} + f_z, \\
&+ \sum_{l=0}^{N}\theta^l_{xzx} + \sum_{l=0}^{N}\theta^l_{yzy} + \sum_{l=0}^{N}\theta^l_{zzz}.
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\dot{\sigma}_{xx} &= \lambda_u\left(\dot{e}_{xx}+\dot{e}_{yy}+\dot{e}_{zz}\right)+2\mu_u\dot{e}_{xx}+\dot{q}_{xx} \\
&+ \sum_{l=1}^{N}\gamma_\lambda^l\Delta\lambda_l+2\sum_{l=1}^{N}\gamma_{xx}^l\Delta\mu_l, \\
\dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx}+\dot{e}_{yy}+\dot{e}_{zz}\right)+2\mu_u\dot{e}_{yy}+\dot{q}_{yy}, \\
&+ \sum_{l=1}^{N}\gamma_\lambda^l\Delta\lambda_l+2\sum_{l=1}^{N}\gamma_{yy}^l\Delta\mu_l, \\
\dot{\sigma}_{zz} &= \lambda_u\left(\dot{e}_{xx}+\dot{e}_{yy}+\dot{e}_{zz}\right)+2\mu_u\dot{e}_{zz}+\dot{q}_{zz} \\
&+ \sum_{l=1}^{N}\gamma_\lambda^l\Delta\lambda_l+2\sum_{l=1}^{N}\gamma_{zz}\Delta\mu_l, \\
\dot{\sigma}_{xy} &= 2\mu\dot{e}_{xy}+2\sum_{l=1}^{N}\gamma_{xy}^l\Delta\mu_l+\dot{q}_{xy}, \\
\dot{\sigma}_{xz} &= 2\mu\dot{e}_{xz}+2\sum_{l=1}^{N}\gamma_{xz}^l\Delta\mu_l+\dot{q}_{xz}, \\
\dot{\sigma}_{yz} &= 2\mu\dot{e}_{yz}+2\sum_{l=1}^{N}\gamma_{yz}^l\Delta\mu_l+\dot{q}_{yz},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\dot{e_{xx}} &= \partial_x v_x, \\
\dot{e_{yy}} &= \partial_y v_y, \\
\dot{e_{zz}} &= \partial_z v_z, \\
\dot{e_{xy}} &= \frac{1}{2}(\partial_x v_y+\partial_y v_x), \\
\dot{e_{xz}} &= \frac{1}{2}(\partial_x v_z+\partial_z v_y), \\
\dot{e_{yz}} &= \frac{1}{2}(\partial_y v_z+\partial_z v_y).
\end{aligned}
\tag{11}
$$

2.1.2 Integration of memory functions

The memory functions obeys approximately the relations

$$
\gamma_\lambda^l(t) = \exp(-\frac{\Delta t}{\tau_\sigma^{\lambda l}})\gamma_\lambda^l(t-\Delta t)+\left(\frac{\Delta t}{\tau_{\sigma l}^\lambda\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}}\right)\left(\dot{e}_{xx}+\dot{e}_{yy}+\dot{e}_{zz}\right),
$$

$$
\begin{aligned}
\gamma_{ij}^l(t) &= \exp(-\frac{\Delta t}{N\tau_{\sigma l}^{\mu}})\gamma_{ij}^l(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^{\mu}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}}\right)\dot{e}_{ij}. \\
\theta_{kij}^l(t) &= \exp(-\frac{\Delta t}{N\tau_{\sigma l}^{\rho}})i\theta_{kij}^l(t-\Delta t) + \left(\frac{\Delta t}{\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}}}\right)\partial_k\sigma_{ij}.
\end{aligned}
\tag{12}
$$

Defining the quantities

$$\alpha_1^l = \exp(-\frac{\Delta t}{\tau_{\sigma l}^{\lambda}}), \tag{13}$$

$$\alpha_2^l = \frac{\Delta t}{\tau_{\sigma l}^{\lambda}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}}, \tag{14}$$

$$\beta_1^l = \exp(-\frac{\Delta t}{\tau_{\sigma l}^{\mu}}), \tag{15}$$

$$\beta_2^l = \frac{\Delta t}{\tau_{\sigma l}^{\mu}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\mu}}{\tau_{\sigma l}^{\mu}}}, \tag{16}$$

$$\eta_1^l = \exp(-\frac{\Delta t}{\tau_{\sigma l}^{\rho}}), \tag{17}$$

$$\eta_2^l = \frac{\Delta t}{\tau_{\sigma l}^{\rho}\sum_{l=1}^{N}\frac{\tau_{\epsilon l}^{\rho}}{\tau_{\sigma l}^{\rho}}} \tag{18}$$

we get

$$
\begin{aligned}
\gamma_{\lambda}^l(t) &= \alpha_1^l\gamma_{\lambda}^l(t-\Delta t) + \alpha_2^l\left(\dot{e}_{xx} + \dot{e}_y y + \dot{e}_z z\right), \\
\gamma_{ij}^l(t) &= \beta_1^l\gamma_{ij}^l(t-\Delta t) + \beta_2^l\dot{e}_{ij}. \\
\theta_{kij}^l(t) &= \eta_1^l\theta_{kij}^l(t-\Delta t) + \eta_2^l\partial_k\sigma_{ij}.
\end{aligned}
\tag{19}
$$

2.2 The two dimensional case

$$
\begin{aligned}
\dot{v}_x &= \rho_u^{-1}\left[\partial_x\sigma_{xx} + \partial_y\sigma_{xy}\right] + \dot{f}_x, \\
\dot{v}_y &= \rho_u^{-1}\left[\partial_x\sigma_{xy} + \partial_y\sigma_{yy}\right] + \dot{f}_y.
\end{aligned}
$$

$$\dot{e}_{xx} = \partial_x v_x,$$

$$\begin{aligned}\dot{e}_{yy} &= \partial_y v_x, \\ \dot{e}_{xy} &= \frac{1}{2}\left[\partial_x v_y + \partial_y v_x\right].\end{aligned}$$

$$\begin{aligned}\dot{\sigma}_{xx} &= \lambda_u\left[\dot{e}_{xx} + \dot{e}_{yy}\right] + 2\mu_u \dot{e}_{xx} + \dot{q}_{xx}, \\ \dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy} +\right) + 2\mu_u \dot{e}_{yy} + \dot{q}_{yy}, \\ \dot{\sigma}_{xy} &= 2\mu_u \dot{e}_{xy} + q_{xy}.\end{aligned}$$

3 Numerical grids

The equations given above can be solved numerically by using the Finite-Difference method.

3.1 Two dimensional case

We now consider a regular grid with positions $\boldsymbol{x}$ defined as

$$\boldsymbol{x} = (x, y), \quad (20)$$

$$x = p\Delta x, \quad (21)$$

$$y = q\Delta y, \quad (22)$$

$$t = n\Delta t. \quad (23)$$

where $p = 0, 1, 2, \ldots, N_x$, $q = 0, 1, 2, \ldots, N_y$ and N_x, N_y are the number of grid points in the x and y-directions. $n = 0, 1, 2, \ldots, N_t$ where N_t is the number of time steps.

We will also need a regular grid which is displaced, or staggered, relative to the the regular grid. Sometimes we will refer to the regular grid as the reference grid.

The particle velocities v_x and v_y are defined on grids as follows

$$\begin{aligned}v_x(\boldsymbol{x}, t) &= v_x(x + \Delta x/2, y, t), \\ v_y(\boldsymbol{x}, t) &= v_y(x, y + \Delta y/2, t),\end{aligned} \quad (24)$$

The diagonal stresses are defined on the grids

$$\begin{aligned}\sigma_{xx}(\boldsymbol{x}, t) &= \sigma_{xx}(x, y, t), \\ \sigma_{yy}(\boldsymbol{x}, t) &= \sigma_{yy}(x, y, t),\end{aligned} \quad (25)$$

The off-diagonal stress

$$\sigma_{xy}(\boldsymbol{x}, t) = \sigma_{xy}(x + \Delta x/2, y + \Delta y/2t) \tag{26}$$

The Lamé parameters are defined on grids as follows: as follows:

$$\begin{aligned} \lambda_u(\boldsymbol{x}) &= \lambda_u(x, y), \quad (27) \\ \mu_u(\boldsymbol{x}) &= \mu_u(x + \Delta x/2, y + \Delta y/2), \\ \mu_u(\boldsymbol{x}) &= \mu_u(x, y). \quad (28) \end{aligned}$$

The inverse density and μ are defined on three different staggered grids as follows:

$$\begin{aligned} \rho_x^{-1}(\boldsymbol{x}) &= \rho^{-1}(x + \Delta x/2, y, z), \\ \rho_y^{-1}(\boldsymbol{x}) &= \rho^{-1}(x, y + \Delta y/2, z). \end{aligned}$$

4 Finite-Difference Solution algorithms

We are now in a position to formulate a complete numerical solution of the visco-elastic equations. Below we give full expressions for solution algorithms in two dimensions, as well as specialization to the visco-acoustic case.

4.1 The two dimensional case

Differentiation is now replaced by numerical approximations so that ∂_x and ∂_y are replaced with numerical operators $d_x^+, d_x^-, d_y^+, d_y^-$. These operators connects the staggered and reference grids, and we illustrate this with the differentiation in the x-direction. The derivative of a function $a(x)$ is approximately given at $a(x + \Delta x/2)$and at $a(x - \Delta x/2)$ by

$$\begin{aligned} a'(x + \Delta x/2) &= d_x^+ a(x), \\ a'(x - \Delta x/2) &= d_x^- a(x). \end{aligned} \tag{29}$$

The differentiators d^+ and d^- are given by (**?**)

$$\begin{aligned} \partial^+ &= \frac{1}{\Delta x} \sum_{l=1}^{L} \alpha_l \left[u(x + l\Delta x) - u(x - (l-1)\Delta x \right] \\ \partial^- &= \frac{1}{\Delta x} \sum_{l=1}^{L} \alpha_l \left[u(x + (l-1)\Delta x) - u(x - l\Delta x) \right] \end{aligned} \tag{30}$$

where the coefficients α_l are found through an optimization procedure. Similar differentiators are defined for the y-direction and for the z-direction, with obvious names.

Using the numerical differentiators the equations of motion becomes:

$$\begin{aligned}\dot{v}_x &= \rho_x^{-1}\left(d_x^+\sigma_{xx} + d_y^-\sigma_{xy}\right) + \rho_x^{-1}\dot{f}_x \qquad (31)\\ \dot{v}_y &= \rho_y^{-1}\left(d_x^-\sigma_{xy} + d_y^+\sigma_{yy}\right) + \rho_y^{-1}\dot{f}_y. \qquad (32)\end{aligned}$$

$$\begin{aligned}\dot{e}_{xx} &= d_x^- v_x,\\ \dot{e}_{yy} &= d_y^- v_y,\\ \dot{e}_{xy} &= \frac{1}{2}(d_x^+ v_y + d_y^+ v_x),\end{aligned} \qquad (33)$$

$$\begin{aligned}\dot{\sigma}_{xx} &= \lambda_u\left[\dot{e}_{xx} + \dot{e}_{yy}\right] + 2\mu_u\dot{e}_{xx} + \dot{q}_{xx},\\ \dot{\sigma}_{yy} &= \lambda_u\left(\dot{e}_{xx} + \dot{e}_{yy}+\right) + 2\mu_u\dot{e}_{yy} + \dot{q}_{yy},\\ \dot{\sigma}_{xy} &= 2\mu_u\dot{e}_{xy} + q_{xy}.\end{aligned}$$

The time derivatives is approximated by the central difference

$$\dot{a}(t) = \frac{a(t+\Delta t/2) - a(t-\Delta t/2)}{\Delta t} \qquad (34)$$

We use the expression for the approximate time derivative given by equation (34) in equations (10) to obtain an expression for the components of the particle velocity

$$\begin{aligned}v_x(t+\Delta t/2) &= \Delta t\rho_x^{-1}\left[d_x^+\sigma_{xx}(t) + d_y^+\sigma_{xy}(t)\right] + \Delta t\rho_x^{-1}f_x(t) +\\ &+v_x(t-\Delta t/2),\\ v_y(t+\Delta t/2) &= \Delta t\rho_z^{-1}\left[d_x^+\sigma_{zx}(t) + d_y^+\sigma_{zz}(t)\right] + \Delta t\rho_z^{-1}f_z(t)\\ &+v_y(t-\Delta t/2)\end{aligned} \qquad (35)$$

The strains can now be computed from equation (33)

$$\begin{aligned}\dot{e}_{xx}(t+\Delta t/2) &= d_x^- v_x(t+\Delta t/2),\\ \dot{e}_{zz}(t+\Delta t/2) &= d_z^- v_z(t+\Delta t/2),\\ \dot{e}_{xz}(t+\Delta t/2) &= \frac{1}{2}\left[d_x^+ v_z(t+\Delta t/2) + d_z^+ v_x(t+\Delta t/2)\right],\end{aligned} \qquad (36)$$

Equations (**??**) can be solved for the stresses using the same approach as for the particle velocities:

$$\begin{aligned}
\sigma_{xx}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] \\
&+ 2\mu_u \dot{e}_{xx}(t+\Delta t/2) + \Delta t \dot{q}_{xx} \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l (t+\Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{xx}^l (t+\Delta t/2)\Delta\mu_l(t+\Delta t/2) \\
&+ \sigma_{xx}(t), \qquad (37) \\
\sigma_{zz}(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] \qquad (38) \\
&+ 2\mu_u \dot{e}_{zz}(t+\Delta t/2) + \Delta t \dot{q}_{zz}(t+\Delta t/2) \\
&+ \Delta t \sum_{l=1}^{N} \gamma_\lambda^l (t+\Delta t/2)\Delta\lambda_l + 2\Delta t \sum_{l=1}^{N} \gamma_{zz} \Delta\mu_l(t+\Delta t/2), \\
&+ \sigma_{zz}(t). \qquad (39) \\
\sigma_{xz}(t+\Delta t) &= 2\Delta t \mu \dot{e}_{xz}(t+\Delta t/2) + 2\Delta t \sum_{l=1}^{N} \gamma_{xz}^l (t+\Delta t/2)\Delta\mu_l + \Delta t \dot{q}_{xz}(t+\Delta t/2) \\
&+ \sigma_{xz}(t), \qquad (40) \\
& \qquad (41)
\end{aligned}$$

4.2 The two dimensional Acoustic case

For the acoustic 2D case we reduce the equations above by neglecting the y-axis terms and putting $\mu = 0$. We consider also the pseudo-stress σ defined by

$$\sigma = \frac{1}{2}\left(\sigma_{xx} + \sigma_{zz}\right)$$

We then get the acoustic 2D scheme as:

$$\begin{aligned}
v_x(t+\Delta t/2) &= \Delta t \left[\rho_{ux}^{-1} d_x^+ \sigma_{xx}(t) + \rho_{ux}^{-1} f_x(t)\right] + \\
&+ \Delta t \sum_{l=0}^{N} \theta_x^l (t)\Delta\rho_x^{-1} + v_x(t-\Delta t/2), \\
v_z(t+\Delta t/2) &= \Delta t \left[\rho_{uz}^{-1} d_z^+ \sigma_{zz}(t) + \Delta t \rho_{uz}^{-1} f_z(t)\right] \\
&+ \Delta t \sum_{l=0}^{N} \theta_z^l (t)\Delta\rho_z^{-1} + v_z(t-\Delta t/2).
\end{aligned}$$

The strains can now be computed from :

$$
\begin{aligned}
\dot{e}_{xx}(t+\Delta t/2) &= d_x^- v_x(t+\Delta t/2), \\
\dot{e}_{zz}(t+\Delta t/2) &= d_z^- v_z(t+\Delta t/2).
\end{aligned}
$$

Equations (**??**) can be solved for the stresses using the same approach as for the particle velocities:

$$
\begin{aligned}
\sigma(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] + \Delta t \dot{q}, \\
&+ \Delta t \sum_{l=1}^{N} \gamma^l (t+\Delta t/2) \Delta\lambda_l + \sigma(t).
\end{aligned}
$$

We now split the γ^l into two parts γ_x^l and γ_z^l as follows:

$$
\begin{aligned}
\sigma(t+\Delta t) &= \Delta t \lambda_u \left[\dot{e}_{xx}(t+\Delta t/2) + \dot{e}_{zz}(t+\Delta t/2)\right] + \Delta t \dot{q} \\
&+ \Delta t \sum_{l=1}^{N} \left[\gamma_x^l (t+\Delta t/2) \Delta\lambda_l + \gamma_z^l (t+\Delta t/2) \Delta\lambda_l\right] + \sigma(t).
\end{aligned}
$$

The θ functions are updated as:

$$
\begin{aligned}
\theta_x(t+\Delta t) &= \eta_{1x}^l \theta_x^l(t) + \eta_{2x}^l \partial_x \sigma(t), \\
\theta_z(t+\Delta t) &= \eta_{1z}^l \theta_x^l(t) + \eta_{2z}^l \partial_y \sigma(t).
\end{aligned}
$$

The γ functions are given by

$$
\begin{aligned}
\gamma_x^l(t+3/2\Delta t) &= \alpha_{1x}^l \gamma_x^l (t+\Delta t/2) + \alpha_{2x}^l \dot{e}_{xx}(t+\Delta t/2), \\
\gamma_z^l(t+3/2\Delta t) &= \alpha_{1z}^l \gamma_z^l (t+\Delta t/2) + \alpha_{2z}^l \dot{e}_{zz}(t+\Delta t/2).
\end{aligned}
$$

Standard linear solid

The coefficents are

$$
\begin{aligned}
\alpha_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^{\lambda}}\right), \\
\alpha_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^{\lambda} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}})} \\
\alpha_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^{\lambda}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda})} \\
\eta_{1x}^l &= \exp\left(-\frac{d_x(x)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2x}^l &= \frac{d_x(x)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}, \\
\eta_{1z}^l &= \exp\left(-\frac{d_z(z)\Delta t}{\tau_{\sigma l}^\rho}\right), \\
\eta_{2z}^l &= \frac{d_z(z)\Delta t}{(\tau_{\sigma l}^\rho \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho})}.
\end{aligned}
$$

The profile functions d_x and d_z are

$$d_x(x) = (x/L)^2, d_z(y) = (z/L)^2,$$

where L is the length of the absorbing layer and and we also have

$$\Delta\lambda_l = \lambda_u \left(1 - \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}\right) \Delta\rho^{-1} = \rho_u^{-1} \left(1 - \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}\right) \tag{42}$$

Maxwell solid

The coefficents are

$$
\begin{aligned}
\alpha_{1x} &= -\frac{1}{\tau_0^\lambda} \exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\lambda}\right), && (43) \\
\alpha_{2x} &= \frac{d_x(x)\Delta t}{\tau_0^\lambda}, && (44) \\
\alpha_{1z} &= -\frac{1}{\tau_0^\lambda} \exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\lambda}\right), && (45) \\
\alpha_{2z} &= \frac{d_z(z)\Delta t}{\tau_0^\lambda}. && (46) \\
\eta_{1x} &= -\frac{1}{\tau_0^\rho} \exp\left(-\frac{d_x(x)\Delta t}{\tau_0^\rho}\right), && (47) \\
\eta_{2x} &= \frac{d_x(x)\Delta t}{\tau_0^\rho}, && (48)
\end{aligned}
$$

$$\begin{aligned}
\eta_{1z} &= -\frac{1}{\tau_0^\rho}\exp\left(-\frac{d_z(z)\Delta t}{\tau_0^\rho}\right), && (49)\\
\eta_{2z} &= \frac{d_z(z)\Delta t}{\tau_0^\rho}. && (50)
\end{aligned}$$

We also have

$$\begin{aligned}
\Delta\lambda &= \lambda_u, && (51)\\
\Delta\rho^{-1} &= \rho_u^{-1}. && (52)
\end{aligned}$$

5 The PyAc2d python library

APPENDIX A: The viscoelastic standard linear solid

Bolzman's generalization of Hook's law to the visco-elastic case is (**?**):

$$\sigma_{ij} = \psi_{ijkl} * \dot{e}_{kl}, \tag{A-1}$$

where ψ_{ijkl} is known as the relaxation tensor. The $*$ denotes convolution defined by

$$a(t) * b(t) = \int_0^t a(t-\tau)b(\tau). \tag{A-2}$$

Integrating (A-1) by parts

$$\sigma_{ij}(t) = |_0^t \psi_{ijkl}(t-\tau)e_{kl}(\tau) + \int_0^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau), \tag{A-3}$$

and using $e(t=0) = 0$ I get

$$\sigma_{ij}(t) = \psi(0)_{ijkl}e_{kl}(t) + \int_{0+}^t \dot{\psi}_{ijkl}(t-\tau)e_{kl}(\tau) \tag{A-4}$$

For the Zener model the components of the ψ_{ijkl} tensor have the form

$$\psi(t) = K\left[1 - \frac{1}{N}\sum_{l=1}^{N}(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}})\exp(-t/\tau_{\sigma l})\right]H(t). \tag{A-5}$$

where K_r is a relaxed modulus, N is the number of Zener mechanisms,$\tau_{\sigma l}$ and $\tau_{\epsilon l}$ are relaxation times. $H(t)$ is the Heavy side function. The time derivative of ψ is equal to:

$$\dot{\psi} = \phi(t), \tag{A-6}$$

where ϕ is equal to:

$$\phi(t) = \frac{1}{N} \sum_{l=1}^{N} \left[\left(\frac{K_r}{\tau_{\sigma l}} \right) \left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \tag{A-7}$$

Using the above we have for the stress:

$$\sigma_{ij}(t) = c_{ijkl} e_{kl} + \int_{0+}^{t} \phi_{ijkl}(t-\tau) e_{kl}(\tau) \tag{A-8}$$

This is most conveniently written as

$$\sigma_{ij}(t) = c_{ijkl}(t) * e_{kl}(t), \tag{A-9}$$

where

$$c_{ijkl}(t) = \psi(0)_{ijkl} \delta(t) + \phi_{ijkl}(t), \tag{A-10}$$

By definition $\psi(t=0)$ corresponds to the unrelaxed modulus so that we have

$$K_u = \frac{1}{N} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} K_r \tag{A-11}$$

or

$$K_r = \frac{K_u}{\frac{1}{N} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \tag{A-12}$$

The ϕ function can then be expressed in terms of the unrelaxed moduli:

$$\phi(t) = \sum_{l=1}^{N} \left[\left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \right) K_u \left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} \right) \right] \tag{A-13}$$

Finally, we express ϕ as:

$$\phi(t) = \sum_{l=1}^{N} \phi^l(t) \tag{A-14}$$

where

$$\phi^l(t) = \left(\frac{\exp(-t/\tau_{\sigma l})}{\tau_{\sigma l} \sum_{l=1}^{N} \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}} \right) \Delta K_l \tag{A-15}$$

and ΔK_l is

$$\Delta K_l = K_u \left(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}\right) \tag{A-16}$$

It is most practical to write the time-dependent visco-elastic constants as

$$\begin{aligned} \lambda(t) &= \lambda_u \delta(t) + \phi_\lambda(t), &\text{(A-17)} \\ \mu(t) &= \mu_u \delta(t) + \phi_\mu(t), &\text{(A-18)} \end{aligned}$$

where ϕ_λ is given as:

$$\phi_\lambda(t) = \sum_{l=1}^{N} \left(\frac{\exp(-t/\tau^\lambda_{\sigma l})}{\tau^\lambda_{\sigma l} \sum_{l=1}^{N} \frac{\tau^\lambda_{\epsilon l}}{\tau^\lambda_{\sigma l}}} \right) \Delta \lambda_l \tag{A-19}$$

and ϕ_μ is given as:

$$\phi_\mu(t) = \sum_{l=1}^{N} \left(\frac{\exp(-t/\tau^\mu_{\sigma l})}{\tau^\mu_{\sigma l} \sum_{l=1}^{N} \frac{\tau^\mu_{\epsilon l}}{\tau^\mu_{\sigma l}}} \right) \Delta \mu_l \tag{A-20}$$

Q-factors

See **?** for further relations between Q and relaxation times. The Q values are related to the Fourier transform of the λ modulus as

$$Q^{-1}_\lambda(\omega) = \frac{Im\lambda(\omega)}{Re\lambda(\omega)} \tag{A-21}$$

Assuming λ is given as

$$\lambda(t) = \lambda_u \delta(t) + \phi_\lambda(t). \tag{A-22}$$

The fourier transform of λ is given by

$$\lambda(\omega) = \lambda_u + \int_{-\infty}^{\infty} \phi_\lambda(t) \exp(-i\omega t). \tag{A-23}$$

The Fourier transform of ϕ_λ is:

$$\phi_\lambda(\omega) = \frac{1}{N} \sum_{l=1}^{N} \left(\frac{\lambda_r}{\tau^\lambda_{\sigma l}}\right) \left(1 - \frac{\tau^\lambda_{\epsilon l}}{\tau^\lambda_{\sigma l}}\right) \int_0^{+\infty} dt \exp(-i\omega) \exp(-t/\tau^\lambda_{\sigma l}). \tag{A-24}$$

The results is:

$$\phi_\lambda(\omega) = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^{\lambda}}\right)\left(1-\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}\right)\frac{1}{1+i\omega\tau_{\sigma l}^{\lambda}}. \tag{A-25}$$

The fourier transform of λ is then

$$\lambda(\omega) = \lambda_u + \frac{1}{N}\sum_{l=1}^{N}\left(\frac{\lambda_r}{\tau_{\sigma l}^{\lambda}}\right)\left(1-\frac{\tau_{\epsilon l}^{\lambda}}{\tau_{\sigma l}^{\lambda}}\right)\frac{1}{1+i\omega\tau_{\sigma l}^{\lambda}}. \tag{A-26}$$

After some (tedious) algebra one obtains

$$\lambda(\omega) = \lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{1+i\omega\tau_{\epsilon l}^{\lambda}}{1+i\omega\tau_{\sigma l}^{\lambda}} \tag{A-27}$$

Separating into real and imaginary parts, I get

$$\begin{aligned}\lambda(\omega) &= \lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{1+\omega^2\tau_{\sigma l}^{\lambda}\tau_{\epsilon l}^{\lambda}}{1+(\omega\tau_{\sigma l}^{\lambda})^2} && \text{(A-28)}\\ &+ i\lambda_r \frac{1}{N}\sum_{l=1}^{N}\frac{\omega\tau_{\sigma l}^{\lambda}(\tau_{\epsilon}^{\lambda}/\tau_{\sigma l}^{\lambda}-1)}{1+(\omega\tau_{\sigma l}^{\lambda})^2} && \\ & && \text{(A-29)}\end{aligned}$$

We then have

$$Q_\lambda^{-1} = \frac{\sum_{l=1}^{N}\omega\tau_{\sigma l}^{\lambda}\left(\tau_{\epsilon l}^{\lambda}/\tau_{\sigma l}^{\lambda}-1\right)/\left[1+(\omega\tau_{\sigma l}^{\lambda})^2\right]}{\sum_{l=1}^{N}(1+\omega^2\tau_{\sigma l}^{\lambda}\tau_{\epsilon l}^{\lambda})/\left[1+(\omega\tau_{\sigma l}^{\lambda})^2\right]} \tag{A-30}$$

The results for the frequency dependence of μ is obtained in exactly the same manner as above:

$$\mu(\omega) = \mu_r \frac{1}{N}\sum_{l=1}^{N}\frac{1+i\omega\tau_{\epsilon l}^{\mu}}{1+i\omega\tau_{\sigma l}^{\mu}} \tag{A-31}$$

and the Q-factor for μ is

$$Q_\mu^{-1} = \frac{\sum_{l=1}^{N}\omega\tau_{\sigma l}^{\lambda}\left(\tau_{\epsilon l}^{\mu}/\tau_{\sigma l}^{\lambda}-1\right)/\left[1+(\omega\tau_{\sigma l}^{\mu})^2\right]}{\sum_{l=1}^{N}(1+\omega^2\tau_{\sigma l}^{\mu}\tau_{\epsilon l}^{\mu})/\left[1+(\omega\tau_{\sigma l}^{\mu})^2\right]} \tag{A-32}$$

In practice we need to relate Q_l and Q_μ to Q_κ. We use the relation

$$\kappa(\omega) = \lambda(\omega) + \frac{2}{3}\mu(\omega). \tag{A-33}$$

Splitting into real and imaginary parts

$$\kappa_r(\omega) = \lambda_r(\omega) + \frac{2}{3}\mu_r(\omega).\kappa_i(\omega) = \lambda_i(\omega) + \frac{2}{3}\mu_i(\omega). \tag{A-34}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\lambda_i + (2/3)\mu_i}{\lambda_r + (2/3)\mu_r} \tag{A-35}$$

$$\frac{1}{Q_\kappa} = \frac{\kappa_i}{\kappa_r} = \frac{\left(\frac{\lambda_i}{\lambda_r}\right)\lambda_r + (2/3)\left(\frac{\mu_i}{\mu_r}\right)\mu_r}{\lambda_r + (2/3)\mu_r} \tag{A-36}$$

$$Q_\kappa^{-1} = \frac{Q_\lambda^{-1}\lambda_r + (2/3)Q_\mu^{-1}}{\lambda_r + (2/3)\mu_r} \tag{A-37}$$

We can use the P-wave and S-wave velocities

$$\begin{aligned} \lambda &= \rho V_p^2 - (2/3)\rho V_s^2 \\ \mu &= \rho V_s^2 \end{aligned} \tag{A-38}$$

$$Q_\kappa^{-1} = Q_\lambda^{-1}\left[1 - \left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2\right] + Q_\mu^{-1}\left(\frac{2}{3}\right)\left(\frac{V_s}{V_p}\right)^2 \tag{A-39}$$

Q-model parametrization

Q-models can be described by the two relaxation times, τ_σ and τ_ϵ. However it is simpler to use the two parameters τ_0 and Q_0 to describe a model. According to **?**, Appendix B, we have

$$Q(\omega) = Q_0\frac{1 + \omega^2\tau_0^2}{2\omega\tau_0}$$

where

$$\begin{aligned} Q_0 &= \frac{2\tau_0}{\tau_\epsilon - \tau_\sigma}, \\ \tau_0^2 &= \tau_\epsilon\tau_\sigma. \end{aligned}$$

$\omega = 1/\tau_0$ is the minimum value for $Q(\omega)$, i.e. the absorption top. We can now find τ_σ and τ_ϵ in terms of τ_0 and Q_0 as:

$$\tau_\epsilon = \frac{\tau_0}{Q_0}\left[\sqrt{Q_0^2+1}+1\right],$$
$$\tau_\sigma = \frac{\tau_0}{Q_0}\left[\sqrt{Q_0^2+1}-1\right].$$

Effective density

We now assume that the effective density has the following form

$$\rho_{eff}^{-1}(t) = s(t). \tag{A-40}$$

We also assume that

$$s(t) = s(0)\delta(t) + \chi(t) \tag{A-41}$$

So that the inverse of the effective density reads

$$\rho_{eff}^{-1}(t) = s_u\delta(t) + \chi(t). \tag{A-42}$$

where $s_u = s(0) = \rho_u^{-1}$.

APPENDIX B: The Maxwell visco-elastic solid

According to **?** the Maxwell visco-elastic solid has a modulus given by

$$\lambda(t) = \lambda_u \exp(-t/\tau_0)H(t) \tag{A-1}$$

In the frequency domain one gets

$$\lambda(\omega) = \frac{\tau_0\omega}{\omega\tau_0 - i}. \tag{A-2}$$

or,

$$\lambda(\omega) = \frac{\tau_0 i\omega}{i\omega\tau_0 + 1}. \tag{A-3}$$

The function ϕ is now:

$$\phi(t) = -\Delta\lambda\frac{1}{\tau_0}\exp(-t/\tau_0), \tag{A-4}$$

where $\Delta\lambda = \lambda_u$. The Q-value are related to τ_0 by:

$$\tau_0 = Q(\omega)/\omega. \tag{A-5}$$

A plane wave

APPENDIX C: Recursive computation of γ and s.

Standard Linear Solid

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t+\Delta t) = \int_0^{t+\Delta t} d\tau \, \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{t+\Delta t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t+\Delta t) = \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\lambda}{\tau_{\sigma l}^\lambda}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau).$$

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau). \end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t+\Delta t$

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \frac{1}{\tau_{\sigma l}^\lambda \sum_{l=1}^N \frac{\tau_{\epsilon l}^\rho}{\tau_{\sigma l}^\rho}} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \dot{e}_{xx}(\tau) \\ &+ \frac{1}{\tau_\epsilon} \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_\sigma^\lambda}) \end{aligned}$$

Performing the integral we then get

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\tau_\sigma}{\tau_\epsilon} \left[1 - \exp(-\frac{\Delta t}{\tau_\sigma^\lambda})\right] \dot{e}_{xx}(t). \end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\Delta t}{\tau_\epsilon} \dot{e}_{xx}(t). \end{aligned}$$

Maxwell solid

The γ's involve a convolution, which is difficult to compute directly. However, a recursive relation can be found by considering

$$\gamma_x^\lambda(t+\Delta t) = -\int_0^{t+\Delta t} d\tau \, \frac{1}{\tau_0^\lambda} \exp(-\frac{t+\Delta t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).$$

$$\gamma_x^\lambda(t+\Delta t) = -\frac{1}{\tau_0^\lambda} \exp(-\frac{\Delta t}{\tau_0^\lambda}) \int_0^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau).$$

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= -\frac{1}{\tau_0^\lambda} \exp(-\frac{\Delta t}{\tau_0^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau) \\ &- \frac{1}{\tau_0^\lambda} \exp(-\frac{\Delta t}{\tau_0^\lambda}) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau). \end{aligned}$$

The second intergral is approximated by assuming that $\dot{e}_{xx}(t)$ is constant in the interval t to $t+\Delta t$

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= -\frac{1}{\tau_0^\lambda} \exp(-\frac{\Delta t}{\tau_0^\lambda}) \int_0^{t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda})\dot{e}_{xx}(\tau) \\ &- \frac{1}{\tau_0^\lambda} \exp(-\frac{\Delta t}{\tau_0^\lambda})\dot{e}_{xx}(t) \int_t^{t+\Delta t} d\tau \, \exp(-\frac{t-\tau}{\tau_0^\lambda}) \end{aligned}$$

Performing the integral we then get

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \left[1-\exp(-\frac{\Delta t}{\tau_0^\lambda})\right] \dot{e}_{xx}(t). \end{aligned}$$

For small $\Delta t \ll 1$ the last equation is also

$$\begin{aligned} \gamma_x^\lambda(t+\Delta t) &= \gamma_x^\lambda(t) \exp(-\frac{\Delta t}{\tau_\sigma^\lambda}) \\ &+ \frac{\Delta t}{\tau_0}\dot{e}_{xx}(t). \end{aligned}$$

APPENDIX E: Comparison with CPML

? show that the implementation of the CPML method can be performed by replacing each spatial derivative with (Their's equation (16) and (18)):

$$s_x = \delta(t) - d_x H(t) \exp\left[-(d_x + \alpha_x)t\right]. \tag{A-1}$$

We compare this with our equation

$$\lambda(t)/\lambda_u = \delta(t) + \frac{1}{\tau_\epsilon} \exp(-t/\tau_\sigma) H(t) \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right) \tag{A-2}$$

Comparing equation (A-1) with equation (A-2) one gets:

$$-d_x = \frac{1}{\tau_\epsilon}\left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right), \tag{A-3}$$

$$d_x + \alpha = \frac{1}{\tau_\sigma} \tag{A-4}$$

Solving for τ_ϵ and τ_σ one gets

$$\tau_\sigma = \frac{1}{(d_x + \alpha)}, \tag{A-5}$$

$$\tau_\epsilon = \frac{1}{\alpha}. \tag{A-6}$$

. Here

$$d_x(x) = d_0 \left(\frac{x}{L}\right)^2, \tag{A-7}$$

where d_0 is a constant and L is the length of the PML zone and x is the distance from the start (outer border) of the PML zone.