TPG4190 Seismic data acquisition and processing
Lecture 4: Fourier Transforms

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Overview

- The Fourier Transform

The Continous Fourier Transform

The Fourier transform $A(f)$ of a time signal $a(t)$ is defined as

$$A(f) = \int_{-\infty}^{+\infty} dt\, a(t) \exp(-2\pi i f t), \tag{1}$$

with the inverse transform given as

$$a(t) = \int_{-\infty}^{+\infty} df\, A(f) \exp(2\pi i f t). \tag{2}$$

Here t is time and f is frequency.

The Continous Fourier Transform

Angular frequency $\omega = 2\pi f$ is often used instead of the frequency, so that equations (1) and (2) becomes

$$a(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, A(\omega)\exp(i\omega t), \tag{3}$$

$$A(\omega) = \int_{-\infty}^{+\infty} dt\, a(t)\exp(-i\omega t). \tag{4}$$

Amplitude and phase

We now introduce the concepts *Amplitude spectrum* and *Phase spectrum*.

$$A(f) = |A(f)| \exp[i\phi(f)], \tag{5}$$

where

$$|A(f)| = \sqrt{A^*(f)A(f)}, \tag{6}$$

and

$$\tan[\phi(f)] = \mathrm{Im}[A(f)]/\mathrm{Re}[A(f)]. \tag{7}$$

$|A(f)| = \sqrt{A^*(f)A(f)}$ is called the amplitude spectrum, while the phase spectrum (or just the phase for short) is defined by $\phi(f) = \tan^{-1}[Im(A(f)]/Re[A(f)]$.

The Fourier transforms of a constant

$$a(t) = c, \tag{8}$$

where c is a constant. Inserting the equation above into (2) we get

$$A(f) = \int_{-\infty}^{+\infty} dt\, c \exp(-2\pi ift), \tag{9}$$

and because c is a constant we get

$$A(f) = c \int_{-\infty}^{+\infty} dt\; \exp(-2\pi ift). \tag{10}$$

The δ - function

Before we can proceed we need to introduce the delta function, defined by

$$f(0) = \int_{-\infty}^{+\infty} dt\ f(t)\delta(t). \tag{11}$$

The Fourier transform of a δ-function

$$a(t) = 1 = \int_{-\infty}^{+\infty} df\ \delta(f) \exp(2\pi i f t), \tag{12}$$

and hence

$$\delta(f) = \int_{-\infty}^{+\infty} dt\ \exp(-2\pi i f t), \tag{13}$$

The Fourier transform of a constant

By using the delta function we see that that equation (10) is equal to

$$A(f) = c\delta(f). \tag{14}$$

The Fourier transform of a constant is a delta function at zero frequency.

The Fourier transforms of a boxcar

Another Fourier transform which will turn out to be usefull, is the case of a boxcar function defined by:

$$h(t) = \begin{cases} 1 & \text{if } |t| < L/2 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Inserting equation (15) into (2) we get

$$A(f) = \int_{-L/2}^{+L/2} dt \, \exp(-2\pi i f t), \tag{16}$$

which can be integrated to give

$$A(f) = \Big|_{-L/2}^{+L/2} dt \, \frac{\exp(-2\pi i f t)}{-2\pi i f}. \tag{17}$$

The Fourier transforms of a boxcar

Inserting the upper and lower limits this is equal to

$$A(f) = \frac{\exp(\pi i f L) - \exp(-\pi i f L)}{2\pi i f}. \tag{18}$$

By using the identity

$$\sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i}, \tag{19}$$

equation (18) becomes

$$A(f) = \frac{\sin(\pi f L)}{\pi f}, \tag{20}$$

which is finally rewritten to

$$A(f) = L\frac{\sin(\pi f L)}{\pi f L}. \tag{21}$$

The Fourier transforms of a time-delayed function

Consider the function $a(t-\tau)$, where τ is a constant time delay. Inserting this function into (2) we get

$$A(f) = \int_{-\infty}^{+\infty} dt\, a(t-\tau)\exp(-2\pi i f t). \tag{22}$$

Change of variable $u = t - \tau$ and observing $t = u + \tau$

$$A(f) = \int_{-\infty}^{+\infty} du\, a(u)\exp[-2\pi i f(u+\tau)], \tag{23}$$

which is equal to

$$A(f) = \exp(-2\pi i f\tau)\int_{-\infty}^{+\infty} du\, a(u)\exp(-2\pi i f u). \tag{24}$$

Which implies that a delay of τ leads to a change in the phase of the Fourier transform by $-2\pi i f\tau$.

The Fourier transforms of a derivative

The Fourier transform of the derivative of a function can be calculated by taking the derivative of both sides of equation (3)

$$\frac{da(t)}{dt} = a'(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \, i\omega A(\omega) \exp(i\omega t) \tag{25}$$

Compar with the definition of the inverse Fourier transform,(3), we conclude that the Fourier transform of $a'(t)$ has to be equal to $i\omega A(\omega)$.

The Fourier transforms of the convolution integral

The convolution integral is defined by

$$a(t) = \int_{-\infty}^{+\infty} d\tau s(t-\tau)h(\tau), \tag{26}$$

where s and h are arbitrary functions.
A very important property of convolution is that it can be expressed as an operation on the Fourier transforms of $s(t)$ and $h(t)$.
Assume that the Fourier transforms of $s(t)$ and $h(t)$ are given by

$$S(f) = \int_{-\infty}^{+\infty} dt\, s(t) \exp(-2\pi i f t), \tag{27}$$

$$H(f) = \int_{-\infty}^{+\infty} dt\, h(t) \exp(-2\pi i f t). \tag{28}$$

The Fourier transforms of the convolution integral

The Fourier transform of the convolution of $h(t)$ and $s(t)$ is

$$\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\tau \, s(\tau) h(t-\tau) \exp(-2\pi i f t), \tag{29}$$

with the change of variable $u = t - \tau$, one gets

$$\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} d\tau \, s(\tau) h(u) \exp[-2\pi i f (u+\tau)], \tag{30}$$

The Fourier transforms of the convolution integral

which is

$$\int_{-\infty}^{+\infty} du\, h(u) \exp(-2\pi i f u) \int_{-\infty}^{+\infty} d\tau\, s(\tau) \exp(-2\pi i f \tau). \tag{31}$$

We recognize equation (31) as

$$H(f)S(f). \tag{32}$$

The Fourier transforms of the convolution integral

We have established that the convolution of two functions

$$g(t) = h(t) * s(t), \tag{33}$$

corresponds to the product of the two functions after a Fourier transform

$$G(f) = H(f)S(f). \tag{34}$$

The Discrete Fourier Transform

In acquisition of seismic data we deal with sampled time functions instead of continuous time functions.
A sampled time function is known only at a finite number of time instances. If $a(t)$ is a continuous time function defined on the interval $(0, T)$, a corresponding sampled time function a_k is given by

$$a_k = a(k\Delta t), k = 0, ..., N \tag{35}$$

where

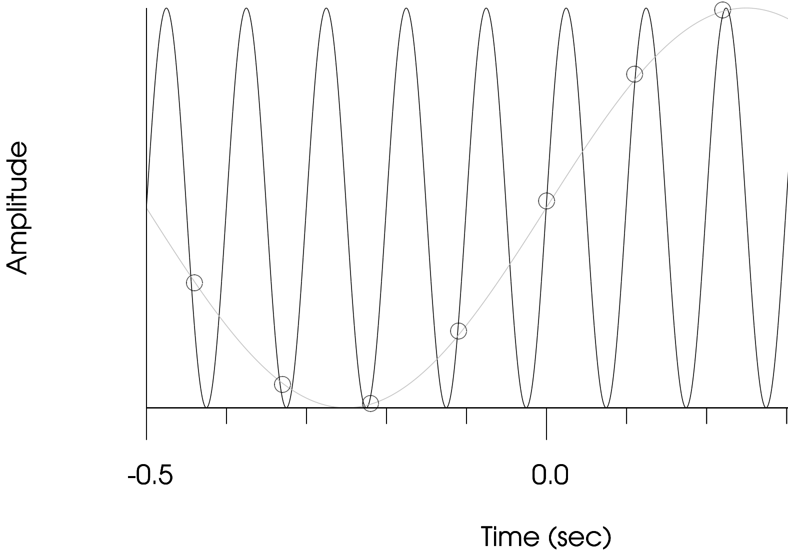
- Δt: time sampling interval
- N: $N = T/\Delta t$.

The Discrete Fourier Transform

Key questions:

- Is it possible to recover the continuous function $a(t)$ from the samples given by a_k?
- What (if any) are the requirements on $a(t)$ and how should the sampling interval Δt be chosen?

The Discrete Fourier Transform



The Discrete Fourier Transform

The discrete analogue of the continuous Fourier transform is defined by the equations

$$A_n = \frac{1}{N}\sum_{k=0}^{N-1} a_k \exp(-2\pi i n k/N), \qquad (36)$$

and the inverse transform

$$a_k = \sum_{n=0}^{N-1} A_n \exp(2\pi i n k/N). \qquad (37)$$

The Discrete Fourier Transform

Here A_n correspond to $A(f)$. The frequency, f, is now discrete and related to the time period T through

$$f_n = n/T \tag{38}$$

. We also have that the frequency sampling interval $\Delta f = f_{n+1} - f_n$ is given by

$$\Delta f = 1/T. \tag{39}$$

Equations (36) and (37) are the standard formulation of the Discrete Fourier Transform (DFT).

The Discrete Fourier Transform

The DFT is periodic with period N,

$$A_{N+l} = A_l, \tag{40}$$

which can be seen by setting $n = N + l$ in equation (36):

$$\begin{aligned} A_{N+l} &= \frac{1}{N} \sum_{k=0}^{N-1} a_k \exp[-2\pi i (N + l) k/N], \\ A_{N+l} &= \frac{1}{N} \sum_{k=0}^{N-1} a_k \exp(-2\pi i k) \exp[(-2\pi i l k)/N] \\ , &= A_l \end{aligned} \tag{41}$$

because $\exp(-2\pi i k) = cos(2\pi k) + i \sin(2\pi k) = 1$ for integer values of k.

The Discrete Fourier Transform

Another important property follows from the fact that $a(t)$ must always be real. Taking the complex conjugate of equation (36) one gets

$$
\begin{aligned}
A_n^* &= \frac{1}{N}\sum_{k=0}^{N-1} a_k \exp(2\pi i n k/N), \\
A_n^* &= \frac{1}{N}\sum_{k=0}^{N-1} a_k \exp[-2\pi i(-n)k/N], \\
&= A_{-n}
\end{aligned}
\tag{42}
$$

which implies

$$
A_{-n} = A^*. \tag{43}
$$

An immediate implication is that

$$
A_0^* = A_0, \tag{44}
$$

hence the zero-frequency value is always real.

The Fast Fourier Transform

A particular efficient implementation of the DFT is known as the Fast Fourier Transform (FFT). The FFT is in practice used instead of the DFT for numerical work.

- Compute time for DFT is proportional to N^2
- Compute time for FFT is proportional to $N \log N$

Aliasing

Equation (43) implies that the spectrum is symmetric around $n = 0$

$$|A_n| = |A_{-n}|, \tag{45}$$

and since A_n is periodic with peeriod N it implies that A_n is unique only for values of $|n| \leq N/2$. In other words, if the Discrete Fourier Transform is going to be used to yield a unique transform we must demand tha

$$A_n = 0 \quad \text{for} |n| > N/2. \tag{46}$$

This implies that there exist a largest frequency, the Nyquist frequency, given by

$$f_{Nyq} = \Delta f N/2 = \frac{N}{2T}, \tag{47}$$

which is equal to

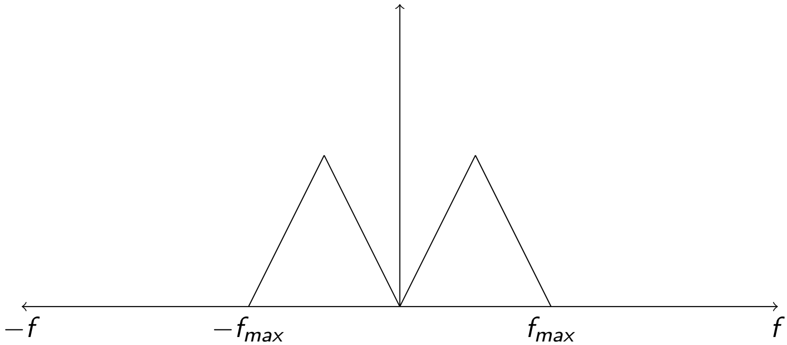
$$f_{Nyq} = \frac{1}{2\Delta t}. \tag{48}$$

The Nyquist limit

$$f \leq \frac{1}{2\Delta t}. \tag{49}$$

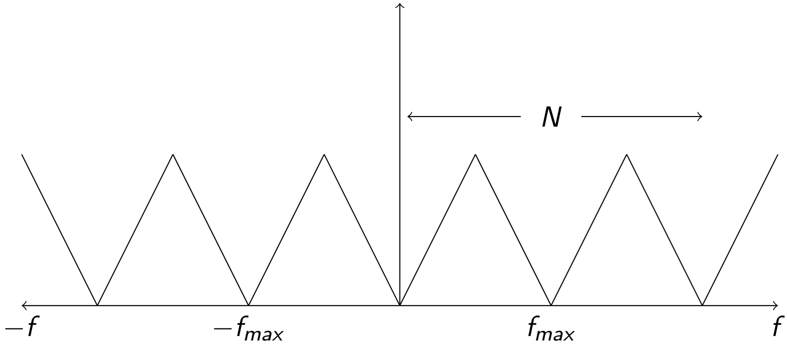
Aliasing

True (continous) amplitude spectrum



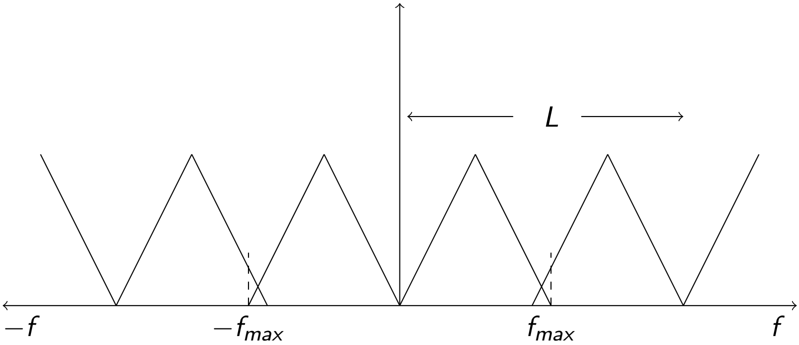
Aliasing

Spectrum of sampled function



Aliasing

Spectrum of incorrectly sampled function



Discrete Fourier Transform