Constrained optimal transport problems and stochastic differential equations

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Overview

Current research

1. Computing distances between stochastic processes;
2. Fitting processes to given marginals.

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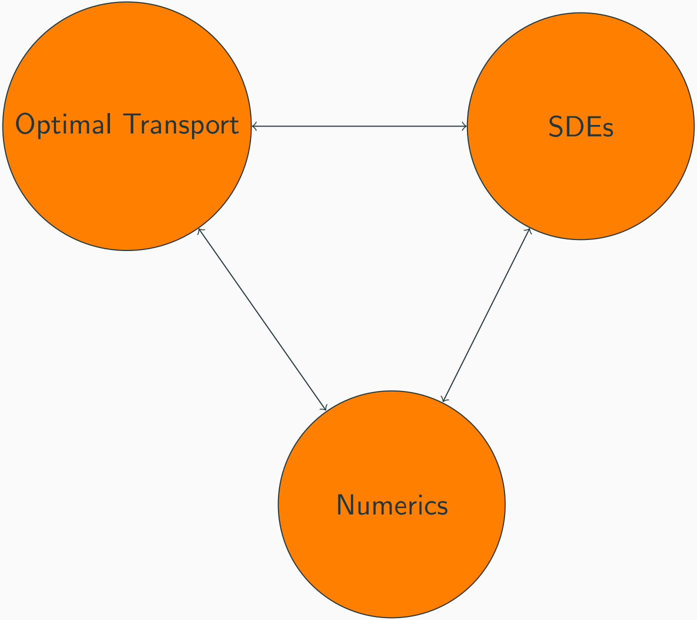

Comparing stochastic models

Aim: Compute a measure of model uncertainty

E.g.

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}}[\mathcal{J}(\omega, \alpha)]$$

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SDEs:

- Good computational methods available
- Rich class of models, beyond Lipschitz coefficients

Distances between stochastic processes

Adapted Wasserstein distance between the laws of SDEs

arXiv:2209.03243, 2022

Julio Backhoff-Veraguas

Universität Wien

Sigrid Källblad

KTH Stockholm

Distances between stochastic processes

Adapted Wasserstein distance between the laws of SDEs
with Julio Backhoff-Veraguas (Universität Wien) and
Sigrid Källblad (KTH Stockholm), arXiv:2209.03243, 2022

Bicausal optimal transport for SDEs with irregular coefficients
Preprint, 2024

Michaela Szölgyenyi
Universität Klagenfurt

Distances between stochastic processes

$b, \bar{b} \colon \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon \mathbb{R} \to [0, \infty)$, $X_0 = \bar{X}_0 = x \in \mathbb{R}$,

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$
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Main result

Under "weak assumptions" on the coefficients, we can compute an "appropriate distance"

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Main result

Under "weak assumptions" on the coefficients, we can compute an "appropriate distance" by

$$d(\mu,\nu)^2 = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^2 \mathrm{d}t\right], \quad \text{with } B = W.$$

Optimal transport

Probability measures μ, ν on $\mathbb{R}^N$

Find

$$\mathcal{W}_2^2(\mu,\nu) \coloneqq \inf_{T\colon T_\#\mu=\nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2\right].$$

$$T(X) = (T_1(X_1,\dots,X_N),\dots,T_N(X_1,\dots,X_N))$$

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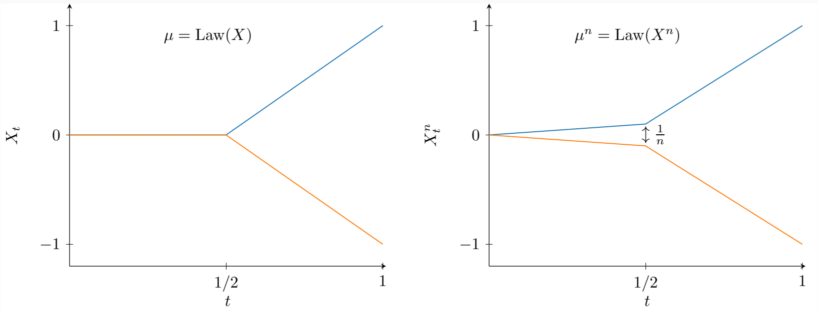
Wasserstein distance metrises usual weak topology

Example



$\mu = \mathrm{Law}(X)$
X_t
1
0
-1
1/2
1
t
$\mu^n = \mathrm{Law}(X^n)$
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0
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$\frac{1}{n}$
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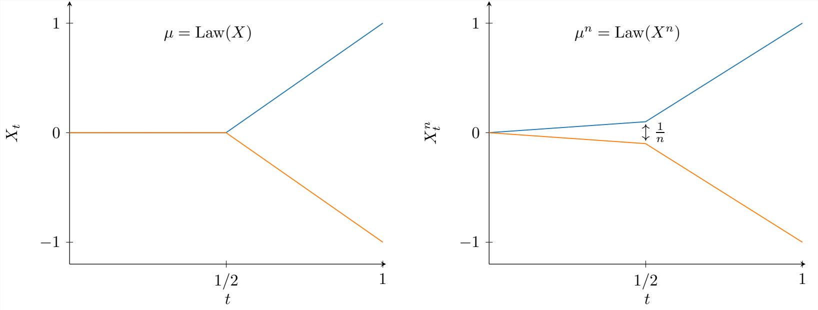

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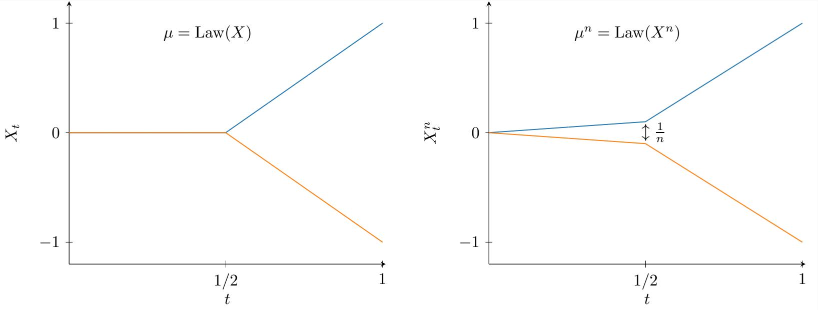

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But

$$\mathcal{W}_2^2(\mu, \mu_n) = \inf_{T\colon T_\#\mu = \mu_n} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2\right] \to 0$$

$$\mu_n \rightharpoonup \mu$$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{W}_2^2(\mu,\nu) := \inf_{T:\, T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^N |T_n(X) - X_n|^2\right].$$

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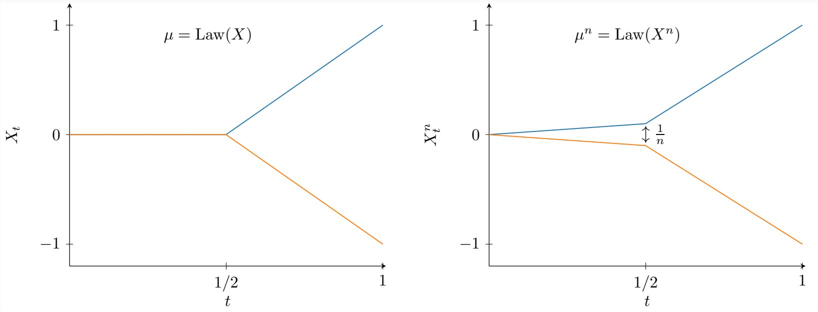

$$\mathcal{AW}_2^2(\mu, \mu_n) > \frac{1}{2}$$

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Similar definition of adapted Wasserstein distance in continuous time w.r.t. L^p norm on $C([0,T], \mathbb{R})$

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More general cost functions $\rightsquigarrow$ bicausal optimal transport

Adapted topology

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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Theorem [Backhoff-Veraguas, Källblad, R. '22]

Optimising over biadapted maps $T_\# \mu = \nu$

$\Leftrightarrow$

Optimising over correlations between B, W

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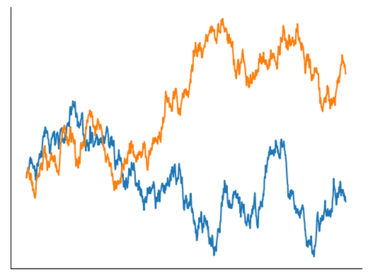
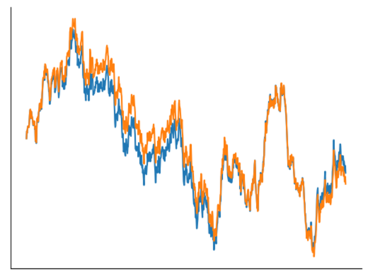
Product coupling

B, W independent



Synchronous coupling

Choose the same driving Brownian motion $B = W$.



Regular SDEs

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow \ \mathrm{Law}(X) = \mu$$
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Theorem [Backhoff-Veraguas, Källblad, R. '22]

For Lipschitz coefficients, the adapted Wasserstein distance is

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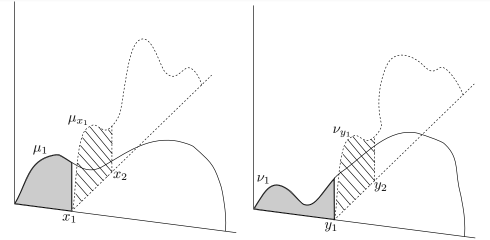
1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

Discretisation

$$\mu_N, \nu_N \in \mathcal{P}(\mathbb{R}^N) \ \rightsquigarrow \ \mathcal{AW}_p^p(\mu_N, \nu_N) := \inf_{\substack{T:\, T_\# \mu_N = \nu_N \\ \text{biadapted}}} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^p\right]$$

Theorem [Rüschendorf '85]

For μ_N, ν_N stochastically co-monotone, the unique optimiser is the Knothe–Rosenblatt rearrangement



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Theorem [Backhoff-Veraguas, Källblad, R. '22]

In the case of Lipschitz coefficients, the Euler–Maruyama scheme is stochastically increasing when the Brownian increments are truncated

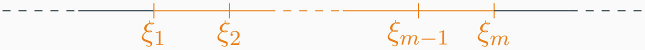
$$X_{k+1}^h = X_k^h + b(X_k^h) \cdot h + \sigma(X_k^h) \Delta W_{k+1}^h$$

Irregular SDEs

Assumption (A)

Drift $b\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions piecewise:

- absolute continuity
- one-sided Lipschitz condition
- two-sided local Lipschitz condition
- exponential growth



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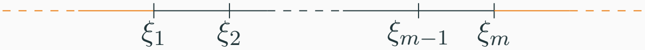

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Diffusion $\sigma\colon \mathbb{R} \to [0, \infty)$ satisfies

– global Lipschitz condition

– $\sigma(\xi_k) \neq 0$, for $k \in \{1, \dots, m\}$ — no uniform ellipticity

Irregular SDEs

Assumption (A)

Drift b piecewise regular with at most exponential growth,
σ Lipschitz and non-zero at discontinuity points of b.

Transformation-based semi-implicit Euler scheme

1. Transform the SDE $Z = G(X)$, with G increasing Lipschitz,
2. Apply a semi-implicit Euler scheme with truncated Brownian increments to Z,
3. Transform back $X^h = G^{-1}(Z^h)$.

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Theorem [R., Szölgyenyi '24+]

Let (b, σ) satisfy Assumption (A). Then for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\Big[|X_T - X_T^h|^p\Big]^{\frac{1}{p}} \leq \begin{cases} C_p h^{\frac{1}{2}}, & p \in [1,2], \\ C_p h^{\frac{1}{p(p-1)}}, & p \geq 2. \end{cases}$$

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Then, for $p \in [1, \infty)$, the adapted Wasserstein distance is given by

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Synchronous coupling solves general bicausal transport problem

Future research directions

- Extension to higher dimensions
  - Examples in [Backhoff-Veraguas, Källblad, R. '22] show that the synchronous coupling is not always optimal
- Extension to jump-diffusions
- Extension to neural SDEs, McKean–Vlasov SDEs
- Convergence of optimisers
  - Use density estimates for SDEs from [Backhoff-Veraguas, Unterberger 2023]
- Application to uniqueness of mimicking martingales

Mimicking martingales

Given probability measures μ, ν on $\mathbb{R}^d$ do there exist random variables M_0, M_1 such that

- $\mathbb{E}[M_1 | M_0] = M_0$ (martingale property)

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Necessary condition $\mu \preceq \nu$ in convex order

For any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

$$\int v \mathrm{d}\mu \leq \int v \mathrm{d}\nu.$$

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... also sufficient [Strassen '65]

Mimicking martingales

Given a family of probability measures $(\mu_t)_{t\in I}$ on $\mathbb{R}^d$, does there exist a mimicking martingale M such that

$$\mathrm{Law}(M_t) = \mu_t, \quad \forall t \in I?$$

Mimicking martingales

A regularized Kellerer theorem in arbitrary dimension
Annals of Applied Probability, to appear, with

Gudmund Pammer
ETH Zürich

Walter Schachermayer
Universität Wien

Peacocks

Assume that μ is a peacock; i.e. for any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

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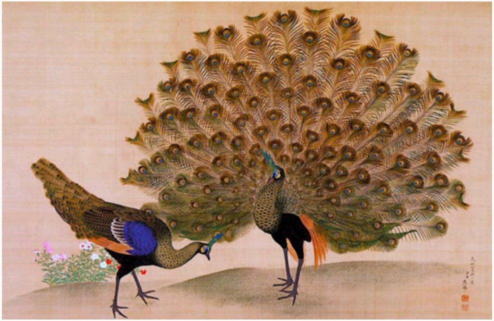
Processus Croissant pour l'Ordre Convexe



[Hirsch, Profetta, Roynette, Yor '11]

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Yes – [Strassen '65], [Doob '68], [Hirsch, Roynette '13]

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Continuous time, $d \geq 2$

No previous results with Markov property

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Weakly continuous $\mathbb{R}^d$-valued square-integrable peacock $(\mu_t)_{t\in[0,1]}$.

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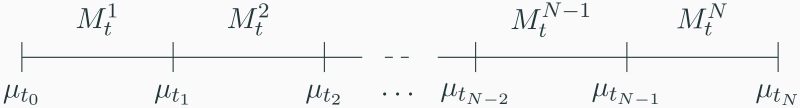
- The result does not hold without regularization;
- There is no uniqueness for $d \geq 2$.

Discretize and take Bass martingales from μ_{t_k} to $\mu_{t_{k+1}}$ to get a "nice" diffusion process
[Backhoff-Veraguas, Beiglböck, Huesmann, Källblad 2020]
[Backhoff-Veraguas, Beiglböck, Schachermayer, Tschiderer 2023]

Bass martingales (Martingale Optimal Transport)



$$M_t = \mathbb{E}[\nabla v(B_1) \mid \mathcal{F}_t] = \nabla v_t(B_t), \quad v_t \colon \mathbb{R}^d \to \mathbb{R} \text{ convex}, \quad B_0 \sim \alpha.$$

Prove compactness of such "nice" diffusions.

Mimicking martingales

Theorem [Pammer, R., Schachermayer '24]

There exists a weakly continuous square-integrable peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}^4$ such that, for the peacock $(\mu_t * \gamma^t)_{t\in[0,1]}$, there exists no mimicking Markov martingale.

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Optimal control of martingales in a radially symmetric environment
Alexander Cox, B.R., *Stochastic Processes and their Applications* 2023

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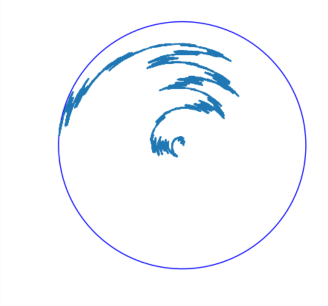

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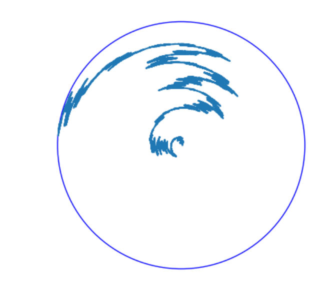
Theorem [Cox, R. '23]

There is a unique weak solution but no strong solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}\begin{bmatrix} -X_t^2 \\ X_t^1 \end{bmatrix}\mathrm{d}W_t, \quad X_0 = 0.$$



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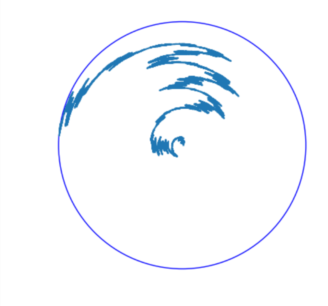

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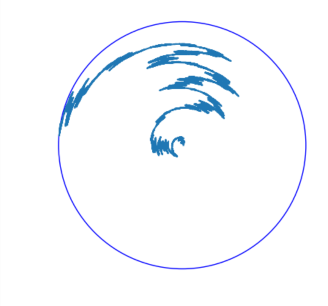

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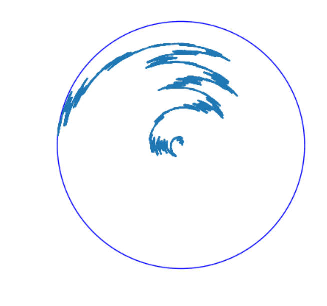
3. No Markov martingale mimicking $(\mu * \gamma^t)_{t\in[0,1]}$.



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Fake Brownian motion in 1D

There exists a *continuous Markov martingale with Brownian marginals*, which is *not* a Brownian motion.
[Albin 2008, Beiglböck, Lowther, Pammer, Schachermayer 2023, Hamza, Klebaner 2007, Oleszkiewicz 2008]

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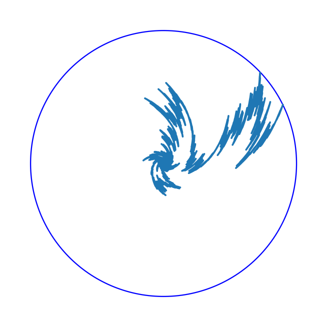
Let X be a weak solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}(X_t + X_t^\perp)\mathrm{d}W_t,\ X_0 \sim \eta.$$

Then X is a continuous strong Markov fake Brownian motion.



[Pammer, R., Schachermayer '22]

Future research directions

- Find minimal conditions for the existence of Markov mimicking martingales
- Find the appropriate sense in which a mimicking martingale is canonical

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$$\{M \text{ martingale}: \ M_0 \sim \mu_0, M_1 \sim \mu_1\}$$

with respect to adapted Wasserstein distance

[Backhoff-Veraguas, Beiglböck, Huesmann, Källblad 2020]
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[Conforti, Lacker, Pal 2023, Lacker 2023] — Hessian projection

Summary

Main contributions:

- Computation of adapted distance between laws of SDEs
- Strong approximation of SDEs with discontinuous and exponentially growing drift
- Proved first multidimensional Kellerer theorem

Future directions:

- Understand adapted distance for more general processes
- Prove full Kellerer theorem with uniqueness
- Applications in robust optimisation

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