Bicausal optimal transport for SDEs with irregular coefficients

Benjamin A. Robinson (University of Vienna)

February 29, 2024 — Probability Seminar, University of Leeds

Supported by Austrian Science Fund (FWF) projects (Y782-N25), (P35519), (P34743).

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Adapted Wasserstein distance between the laws of SDEs
(with J. Backhoff-Veraguas and S. Källblad) — arXiv:2209.03243, 2022

Bicausal optimal transport for SDEs with irregular coefficients
(with M. Szölgyenyi) — Preprint, 2024

Comparing stochastic models

Aim: Compute a measure of model uncertainty

E.g.

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}}[\mathcal{J}(\omega, \alpha)]$$

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SDEs:

- Good computational methods available
- Rich class of models, beyond Lipschitz coefficients

Main result

$b, \bar{b} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon [0,T] \times \mathbb{R} \to [0,\infty)$,

$$\begin{aligned} \mathrm{d}X_t &= b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad X_0 = x_0, \\ \mathrm{d}\bar{X}_t &= \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t, \quad \bar{X}_0 = x_0. \end{aligned}$$

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$$d(\mu,\nu)^p = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\right], \quad \text{with } B = W.$$

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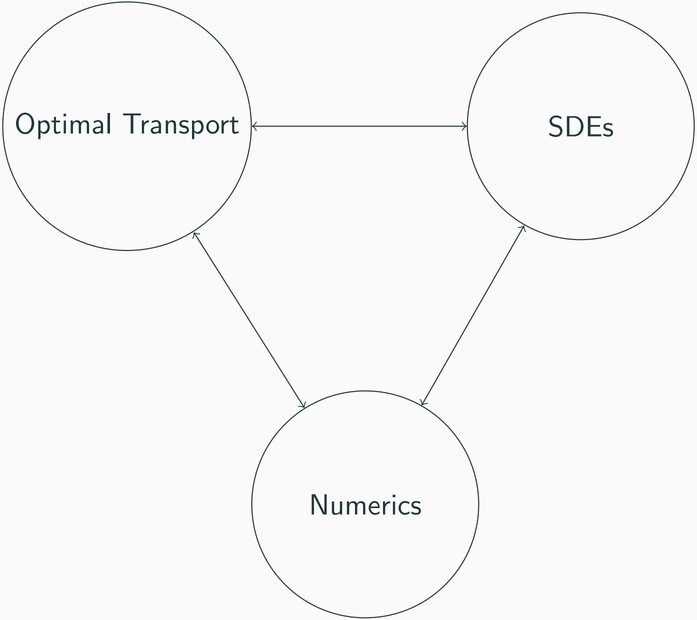
Strong existence, pathwise uniqueness, and moment bounds hold for (SDE) with coefficients satisfying (A). Moreover, for a transformation-based semi-implicit Euler scheme, we obtain strong convergence rates.

Ingredients



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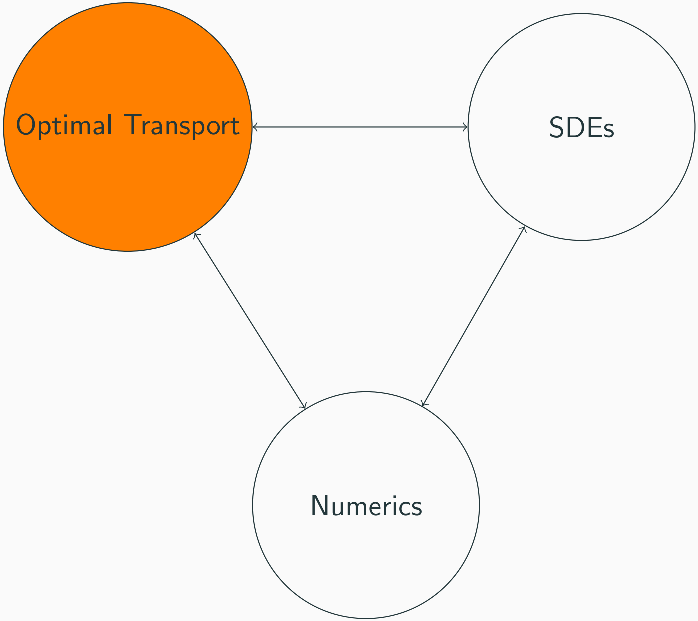

Optimal transport

Probability measures μ, ν on $\mathbb{R}^N$

Find

$$\inf_{T\colon T_{\#}\mu=\nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^p\right]$$

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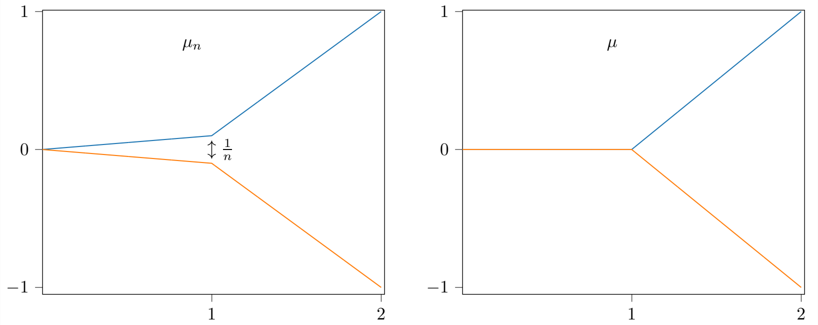
Wasserstein distance metrises usual weak topology

Example

[Backhoff-Veraguas, Bartl, Beiglböck, Eder '20], [Aldous '81]



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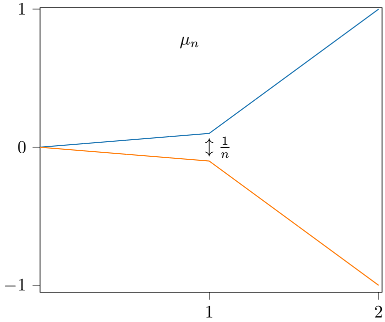
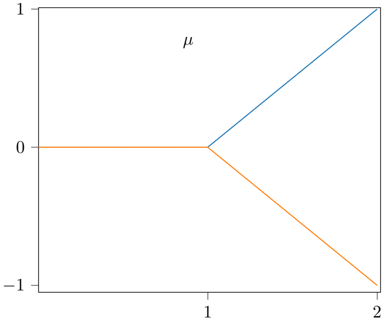


"Can get rich"



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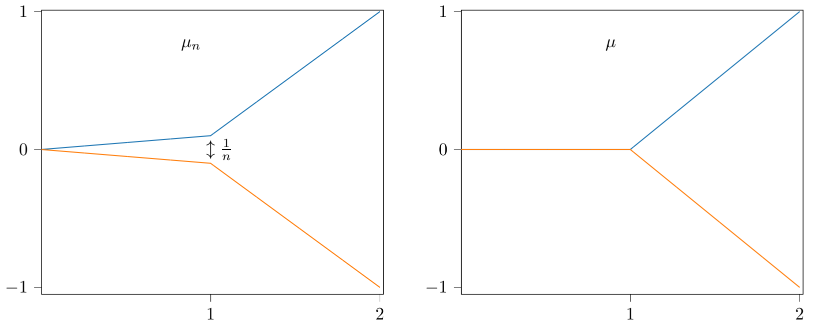

$$V_n := \sup_\tau \mathbb{E}^{\mu_n}[X_\tau] \approx \frac{1}{2} \qquad V := \sup_\tau \mathbb{E}^{\mu}[X_\tau] = 0$$

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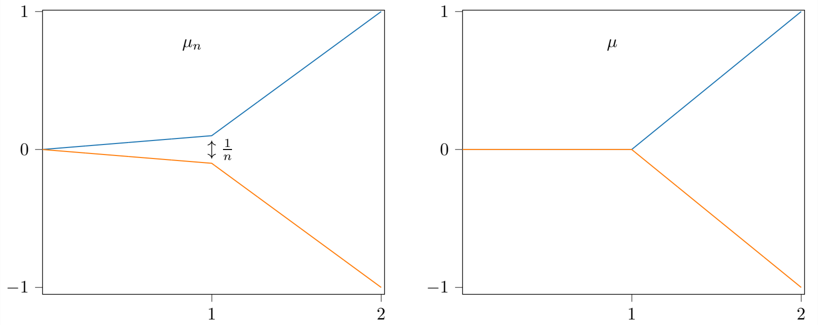

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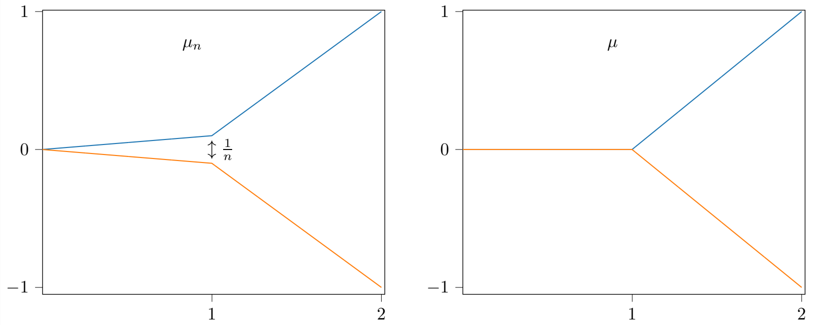

$$V_n \not\to V \quad \text{but} \quad \mu_n \rightharpoonup \mu$$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \inf_{T\colon T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^p\right]$$

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More general cost functions $\rightsquigarrow$ bicausal optimal transport

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c_n continuous, polynomial growth, quasi-monotone

$$c_n(x, y) + c_n(x', y') - c_n(x, y') - c_n(x', y) \geq 0, \quad \forall\, x \leq x', y \leq y'$$

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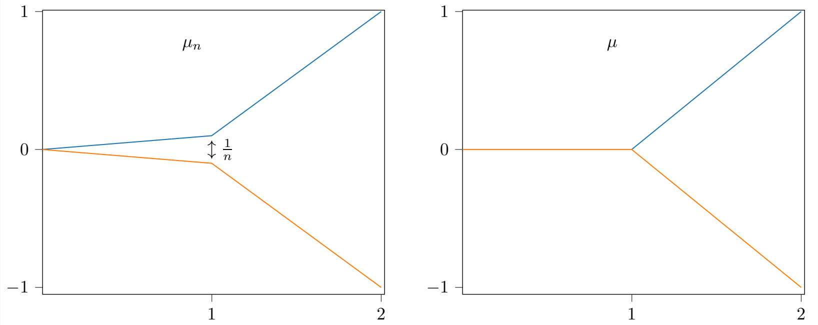
Acciaio, Aldous, Backhoff-Veraguas, Bartl, Beiglböck, Bion-Nadal, Eder, Hellwig, Källblad, Pammer, Pflug, Pichler, Talay, Zalaschko, ...

Example revisited

[Backhoff-Veraguas, Bartl, Beiglböck, Eder '20], [Aldous '81]



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$V_n \not\to V$ and $\mathcal{AW}_p(\mu_n, \mu) \not\to 0$

Continuous time

Similar definition of Wasserstein distance in continuous time w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$

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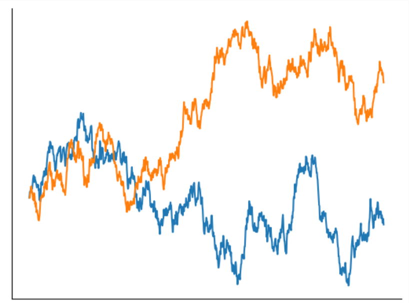

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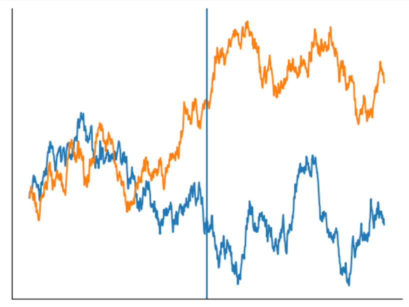

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Similar definition of adapted Wasserstein distance in continuous time w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$

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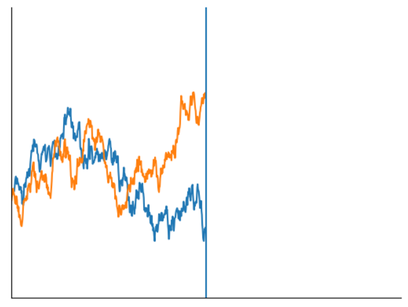

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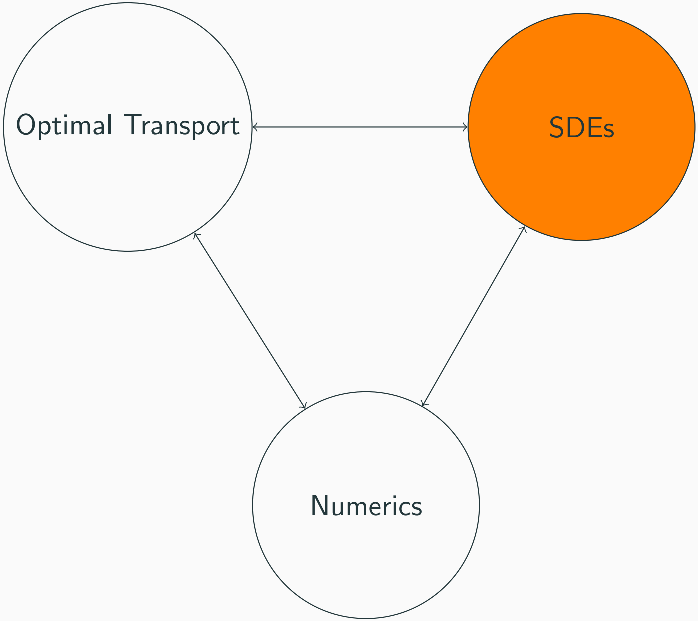

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$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \quad \text{with } B = W.$$

Coupling SDEs

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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Theorem [Backhoff-Veraguas, Källblad, R. '22]

Optimising over bicausal couplings $\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu, \nu)$
$\Leftrightarrow$
Optimising over correlations between B, W

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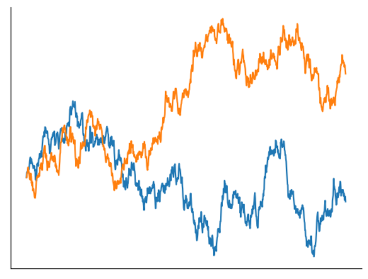
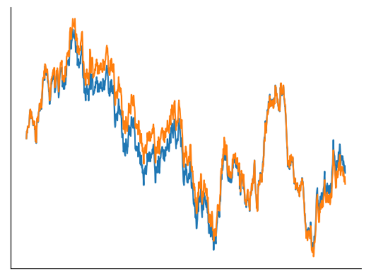
Product coupling

B, W independent

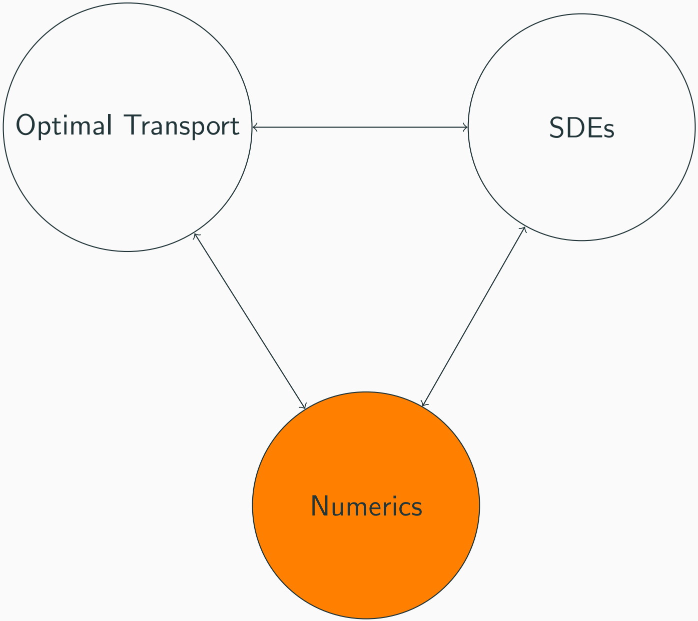


Synchronous coupling

Choose the same driving Brownian motion $B = W$.



Ingredients



Proof of main result

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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Under “weak assumptions” on the coefficients, we can compute the adapted Wasserstein distance by

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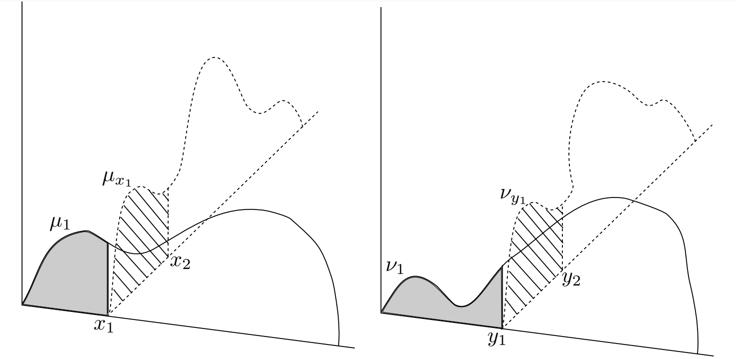
— generalisation of monotone rearrangement



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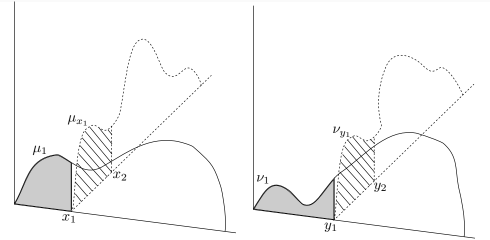

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Theorem [Rüschendorf '85] [Posch '23+]

For μ,ν stochastically co-monotone, the unique optimiser is the Knothe–Rosenblatt rearrangement.

This induces the adapted weak topology.

Proof of main result

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A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t$$

Euler scheme

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh), \quad t \in (kh, (k+1)h].$$

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$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t$$

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Lemma [Backhoff-Veraguas, Källblad, R. '22]

For b, σ Lipschitz, the monotone Euler–Maruyama scheme is stochastically increasing.

Hence the Knothe–Rosenblatt rearrangement is optimal for μ^h, ν^h.

Proof of main result

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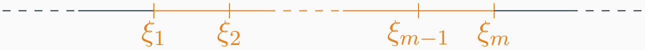
Transformation-based semi-implicit Euler scheme

Assumption (A)

Drift $b\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions piecewise:

- absolute continuity
- one-sided Lipschitz condition
- two-sided local Lipschitz condition
- exponential growth



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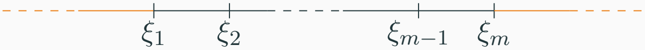

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Diffusion $\sigma\colon \mathbb{R} \to [0, \infty)$ satisfies

- global Lipschitz condition
- $\sigma(\xi_k) \neq 0$, for $k \in \{1, \dots, m\}$ — no uniform ellipticity

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Under Assumption (A) , the scheme is constructed as follows:

1. Apply the transformation G from [Leobacher, Szölgyenyi '17] to (SDE),

$$Z = G(X)$$

$$\mathrm{d}Z_t = \tilde{b}(Z_t)\mathrm{d}t + \tilde{\sigma}(Z_t)\mathrm{d}W_t$$

$\tilde{b}$ one-sided Lipschitz, exponential growth, locally Lipschitz, a.c.

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Theorem [R., Szölgyenyi '24]

Let (b, σ) satisfy Assumption (A). Then (SDE) admits a unique strong solution and, for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\Big[|X_T - X_T^h|^p\Big]^{\frac{1}{p}} \leq \begin{cases} C_p h^{\frac{1}{2}}, & p \in [1,2], \\ C_p h^{\frac{1}{p(p-1)}}, & p \geq 2. \end{cases}$$

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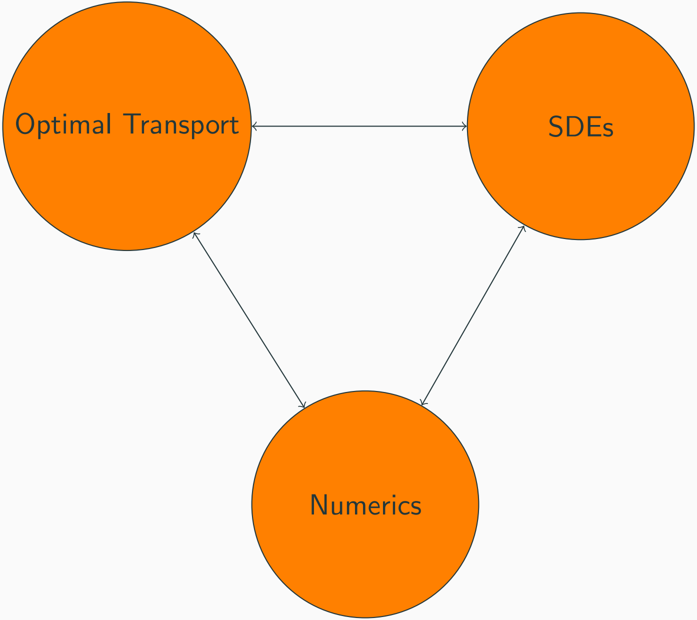

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Synchronous coupling solves general bicausal transport problem

Future research directions

- Extension to higher dimensions
  - Examples in [Backhoff-Veraguas, Källblad, R. '22] show that the synchronous coupling is not always optimal
- Extension to jump-diffusions
- Extension to neural SDEs, McKean–Vlasov SDEs
- Convergence of optimisers
  - Use density estimates for SDEs from [Backhoff-Veraguas, Unterberger '23]
- Application to uniqueness of mimicking martingales

Summary

- We compute adapted Wasserstein distance between SDEs with irregular coefficients
- We prove strong convergence rates for a numerical scheme for SDEs with discontinuous and exponentially growing drift

References: