

# Bicausal optimal transport between the laws of SDEs

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**Benjamin A. Robinson** (University of Klagenfurt)

NAASDE, Będlewo — September 23, 2024

*Supported by Austrian Science Fund (FWF) projects (Y782-N25), (P35519), (P34743).*

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Joint work with*

**Julio Backhoff-Veraguas**

University of Vienna



**Sigrid Källblad**

KTH Stockholm



# Comparing stochastic models

**Aim:** Compute a measure of model uncertainty

E.g. optimal stopping:

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[L(\tau, \omega)]$$

$\mathbb{P}$  law of solution of SDE

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$\mathbb{P}$  law of solution of SDE

**Want:**

- Appropriate topology on laws of stochastic processes
- Distance we can actually compute

## Main result

$$b, \bar{b}: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \sigma, \bar{\sigma}: [0, T] \times \mathbb{R} \rightarrow [0, \infty),$$

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 = x_0,$$

$$d\bar{X}_t = \bar{b}_t(\bar{X}_t)dt + \bar{\sigma}_t(\bar{X}_t)dW_t, \quad \bar{X}_0 = x_0.$$

$$\mu = \text{Law}(X), \nu = \text{Law}(\bar{X})$$

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**Theorem [Backhoff-Veraguas, Källblad, R. '24]**

For “sufficiently nice” coefficients, we can compute an  
“appropriate distance”  $d_p, p \geq 1$ ,

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## Theorem [Backhoff-Veraguas, Källblad, R. '24]

For “sufficiently nice” coefficients, we can compute an “appropriate distance”  $d_p$ ,  $p \geq 1$ , by

$$d_p(\mu, \nu)^p = \mathbb{E} \left[ \int_0^T |X_t - \bar{X}_t|^p dt \right], \quad \text{with } B = W.$$

# Application

E.g. optimal stopping:

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[L(\tau, \omega)]$$



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## Theorem

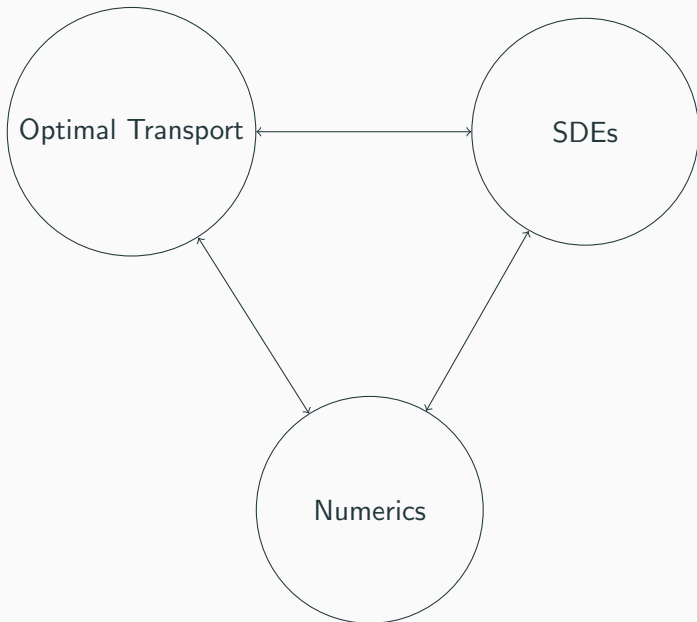
[Acciaio, Backhoff-Veraguas, Zalashko '19], [R. Szölgényi '24]

$\omega \mapsto L(t, \omega)$  Lipschitz on  $(\Omega, \|\cdot\|_{L^p})$  unif. in  $t \in [0, T]$

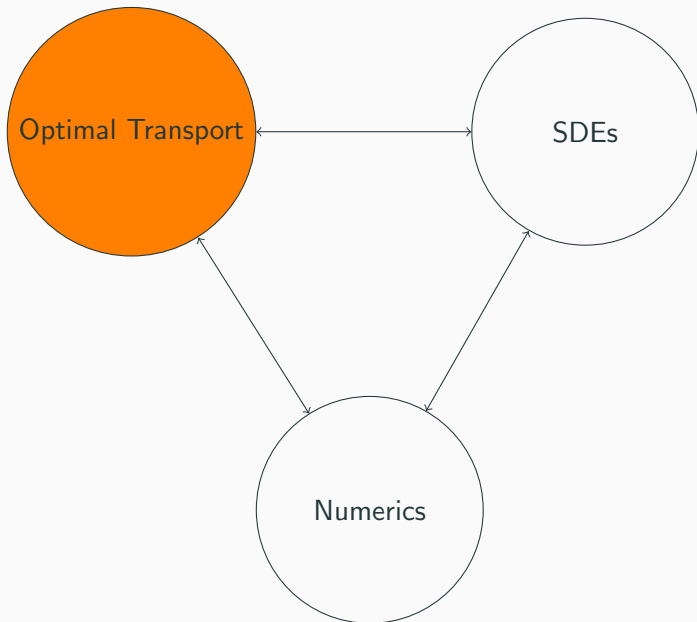
$\Rightarrow$

$\mathbb{P} \mapsto v(\mathbb{P})$  Lipschitz on  $(\mathcal{P}_p(\Omega), d_p)$

# Ingredients



# Ingredients



# Optimal transport

Probability measures  $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^N)$ ,  $p \in [1, \infty)$ .

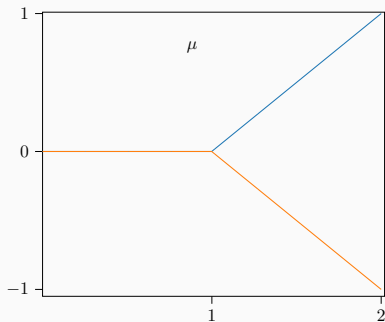
Find Wasserstein distance

$$\mathcal{W}_p^p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|^p]$$

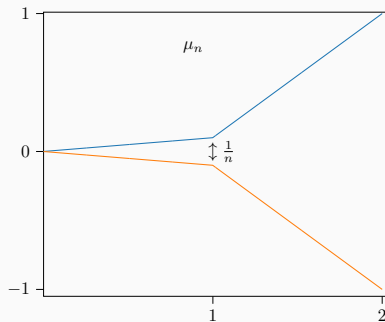
Metrises weak topology on  $\mathcal{P}_p(\mathbb{R}^N)$

# Example

[Aldous '81], [Backhoff-Veraguas, Bartl, Beiglböck, Eder '20]



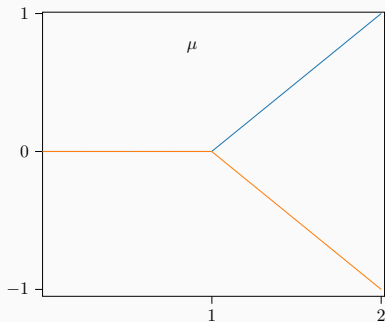
$$V := \sup_{\tau} \mathbb{E}^{\mu}[X_{\tau}]$$



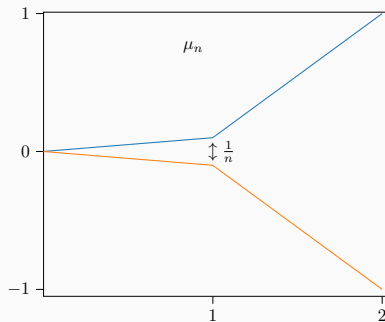
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$$V := \sup_{\tau} \mathbb{E}^{\mu}[X_{\tau}] = 0$$



$$V_n := \sup_{\tau} \mathbb{E}^{\mu_n}[X_{\tau}] \approx \frac{1}{2}$$

$$V_n \not\rightarrow V \quad \text{but} \quad \mu_n \rightarrow \mu$$

# Optimal transport

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$$\begin{aligned}\mathcal{W}_p^p(\mu, \nu) &:= \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|^p] \\ &= \inf_{\substack{T: \mathbb{R}^d \rightarrow \mathbb{R}^d \\ T_{\#}\mu = \nu}} \mathbb{E} \left[ \sum_{n=1}^N |T_n(X) - X_n|^p \right]\end{aligned}$$



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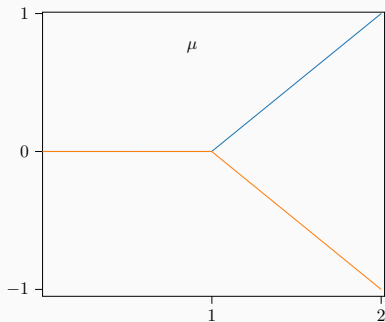
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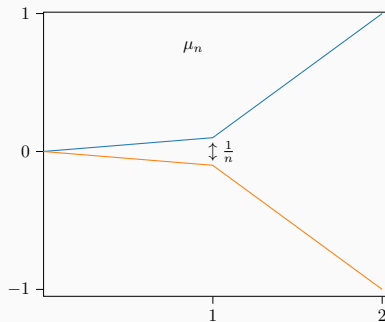
$$T(X) = (T_1(X_1, \dots, X_N), \dots, T_N(X_1, \dots, X_N))$$

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# Adapted topology

Probability measures  $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^N)$ ,  $p \in [1, \infty)$ .

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Probability measures  $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^N)$ ,  $p \in [1, \infty)$ .

Find **adapted Wasserstein distance**

$$\begin{aligned} \mathcal{AW}_p^p(\mu, \nu) &:= \inf_{\substack{X \sim \mu, Y \sim \nu \\ \text{bicausal}}} \mathbb{E}[|X - Y|^p] \\ &= \inf_{\substack{T: \mathbb{R}^d \rightarrow \mathbb{R}^d \\ T_{\#}\mu = \nu \\ T \text{ biadapted}}} \mathbb{E} \left[ \sum_{n=1}^N |T_n(X) - X_n|^p \right] \end{aligned}$$

$$T(X) = (T_1(X_1), T_2(X_1, X_2), \dots, T_N(X_1, \dots, X_N))$$

and symmetric condition.

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Metrises adapted weak topology on  $\mathcal{P}_p(\mathbb{R}^N)$

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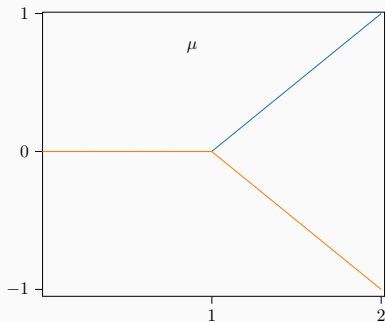
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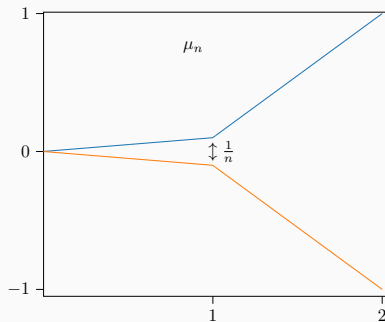
Acciaio, Aldous, Backhoff-Veraguas, Bartl, Beiglböck, Bion-Nadal, Eder, Hellwig, Källblad, Pammer, Pflug, Pichler, Talay, Zalaschko,

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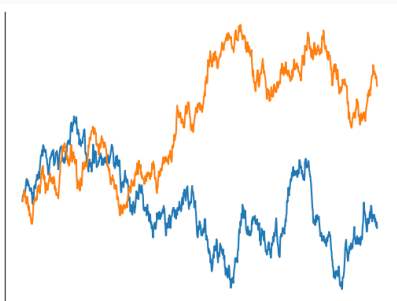
$$V_n \not\rightarrow V \quad \text{and} \quad \mathcal{AW}_p^p(\mu_n, \mu) \not\rightarrow 0$$

# Continuous time

Similar definition of Wasserstein distance in **continuous time** w.r.t.  $L^p$  norm on  $\Omega := C([0, T], \mathbb{R})$

$$\mu, \nu \in \mathcal{P}(\Omega) \quad \rightsquigarrow \quad \mathcal{W}_p^p(\mu, \nu) := \inf_{\pi \in \text{Cpl}(\mu, \nu)} \mathbb{E}^\pi \left[ \int_0^T |\omega_t - \bar{\omega}_t|^p dt \right]$$

$$\text{Cpl}(\mu, \nu) = \{ \pi = \text{Law}(X, Y) : X \sim \mu, Y \sim \nu \}$$



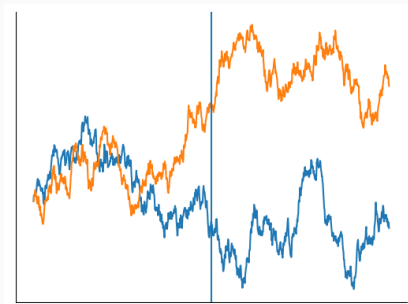


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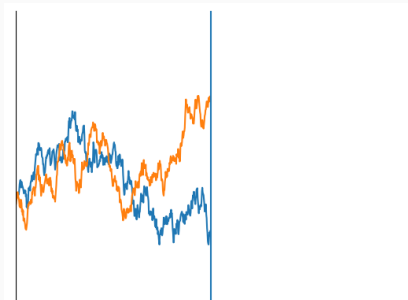
# Continuous time

Similar definition of **adapted** Wasserstein distance in continuous time w.r.t.  $L^p$  norm on  $\Omega := C([0, T], \mathbb{R})$

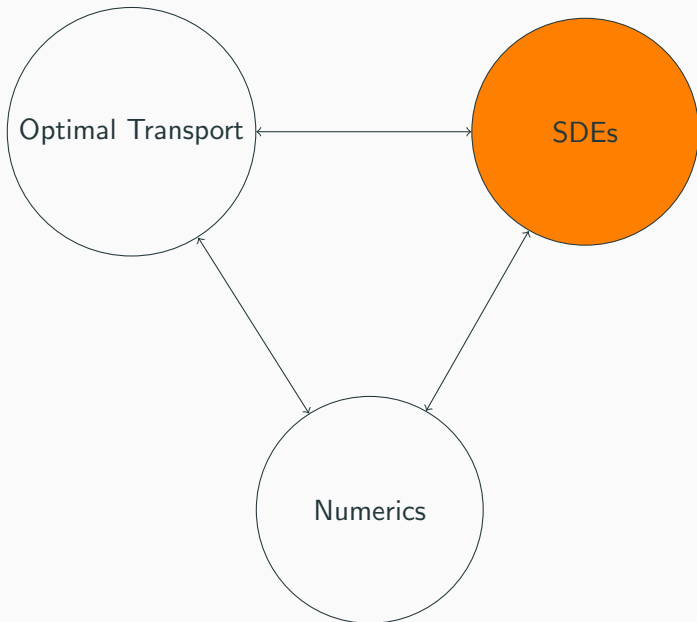
$$\mu, \nu \in \mathcal{P}(\Omega) \quad \rightsquigarrow \quad \mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \mathbf{Cpl}_{bc}(\mu, \nu)} \mathbb{E}^\pi \left[ \int_0^T |\omega_t - \bar{\omega}_t|^p dt \right]$$

$$\mathbf{Cpl}_{bc}(\mu, \nu) = \{ \pi \in \mathbf{Cpl}(\mu, \nu) : \pi \text{ **bicausal** } \}$$

“ $\mathcal{F}_t^X$  independent of  $\mathcal{F}_T^Y$  conditional on  $F_t^Y$ ” and vice versa



# Ingredients



# Main result

$$b, \bar{b}: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \sigma, \bar{\sigma}: [0, T] \times \mathbb{R} \rightarrow [0, \infty),$$

$$\begin{aligned} dX_t &= b_t(X_t)dt + \sigma_t(X_t)dB_t, & X_0 &= x_0, \\ d\bar{X}_t &= \bar{b}_t(\bar{X}_t)dt + \bar{\sigma}_t(\bar{X}_t)dW_t, & \bar{X}_0 &= x_0. \end{aligned}$$

$$\mu = \text{Law}(X), \nu = \text{Law}(\bar{X})$$

**Theorem [Backhoff-Veraguas, Källblad, R. '24]**

For “sufficiently nice” coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E} \left[ \int_0^T |X_t - \bar{X}_t|^p dt \right], \quad \text{with } B = W.$$

# Coupling SDEs

$$\begin{aligned}dX_t &= b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 = x \rightsquigarrow \text{Law}(X) = \mu \\d\bar{X}_t &= \bar{b}_t(\bar{X}_t)dt + \bar{\sigma}_t(\bar{X}_t)dW_t, \quad \bar{X}_0 = x \rightsquigarrow \text{Law}(\bar{X}) = \nu\end{aligned}$$

**Theorem [Backhoff-Veraguas, Källblad, R. '24]**

Optimising over **bicausal couplings**  $\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)$

$\Leftrightarrow$

Optimising over **correlations** between  $B, W$

# Coupling SDEs

## Theorem [Backhoff-Veraguas, Källblad, R. '24]

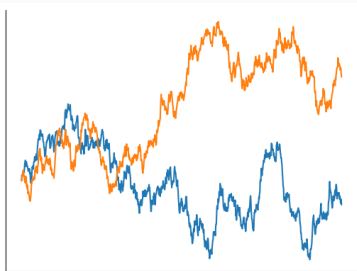
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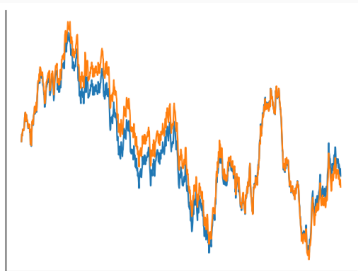
### Product coupling

$B, W$  independent



### Synchronous coupling

Choose the same driving Brownian motion  $B = W$ .



## Proof of main result

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 = x \rightsquigarrow \text{Law}(X) = \mu$$

$$d\bar{X}_t = \bar{b}_t(\bar{X}_t)dt + \bar{\sigma}_t(\bar{X}_t)dW_t, \quad \bar{X}_0 = x \rightsquigarrow \text{Law}(\bar{X}) = \nu$$

### Theorem [Backhoff-Veraguas, Källblad, R. 24]

For “sufficiently nice” coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E} \left[ \int_0^T |X_t - \bar{X}_t|^p dt \right], \quad \text{with } B = W.$$

1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

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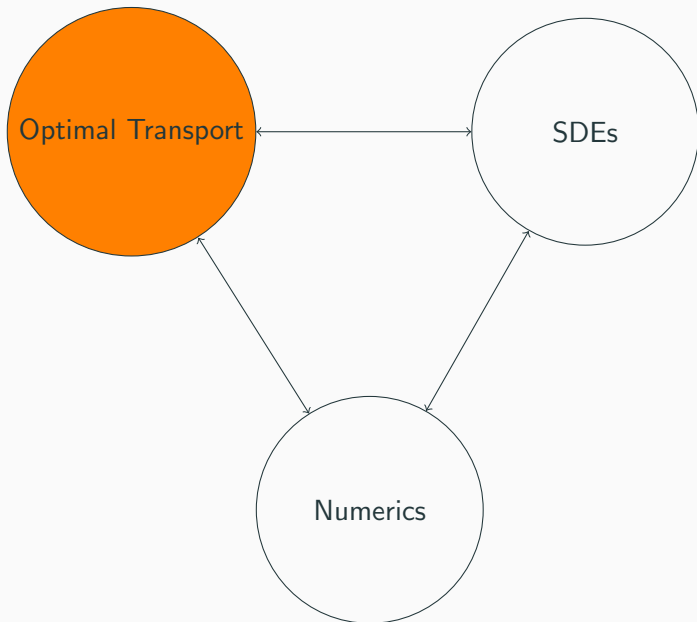
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# Ingredients



# Classical optimal transport on $\mathbb{R}$

Probability measures  $\mu, \nu \in \mathcal{P}_p(\mathbb{R})$ ,  $p \in [1, \infty)$ .

Wasserstein distance

$$\mathcal{W}_p^p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|^p]$$

is attained by **monotone rearrangement**

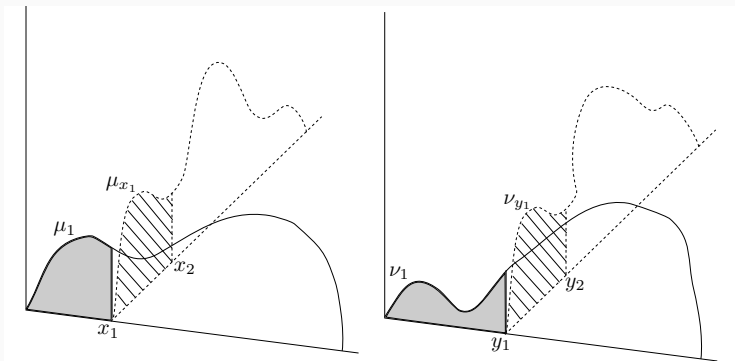
$$X = F_X^{-1}(U), \quad Y = F_Y^{-1}(U), \quad U \text{ uniform}$$

# Key result in discrete time

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)} \mathbb{E}^\pi \left[ \sum_{n=1}^N |X_n - Y_n|^p \right]$$

## Knothe–Rosenblatt rearrangement

— generalisation of **monotone rearrangement** to  $N$  time steps



## Key result in discrete time

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### Knothe–Rosenblatt rearrangement

$$X_1 = F_{\mu_1}^{-1}(U_1), \quad Y_1 = F_{\nu_1}^{-1}(U_1),$$

## Key result in discrete time

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### Knothe–Rosenblatt rearrangement

$X_1 = F_{\mu_1}^{-1}(U_1)$ ,  $Y_1 = F_{\nu_1}^{-1}(U_1)$ , and for  $k \in \{2, \dots, N\}$

$$X_k = F_{\mu_{X_1, \dots, X_{k-1}}}^{-1}(U_k), \quad Y_k = F_{\nu_{Y_1, \dots, Y_{k-1}}}^{-1}(U_k)$$

$U_1, \dots, U_N$  independent uniform

$$\pi_{\text{KR}}(\mu, \nu) := \text{Law}(X, Y)$$

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$U_1, \dots, U_N$  independent uniform

$$\pi_{\text{KR}}(\mu, \nu) := \text{Law}(X, Y)$$

### Theorem [Rüschendorf '85]

For  $\mu, \nu$  Markov and **stochastically comonotone**, the **Knothe–Rosenblatt rearrangement** is optimal.

## Proof of main result

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 = x \rightsquigarrow \text{Law}(X) = \mu$$

$$d\bar{X}_t = \bar{b}_t(\bar{X}_t)dt + \bar{\sigma}_t(\bar{X}_t)dW_t, \quad \bar{X}_0 = x \rightsquigarrow \text{Law}(\bar{X}) = \nu$$

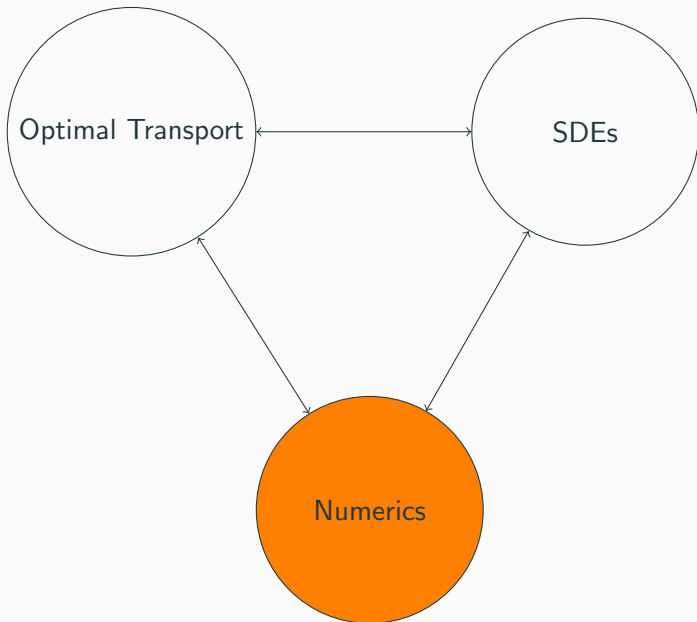
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1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

# Ingredients





## A monotone numerical scheme

$$dX_t = b(X_t)dt$$

### Euler scheme

$$X_0^h = X_0,$$

$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh), \quad t \in (kh, (k+1)h].$$

## A monotone numerical scheme

$$dX_t = b(X_t)dt + dW_t$$

### Euler–Maruyama scheme

$$X_0^h = X_0,$$

$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh) + W_t - W_{kh}, \quad t \in (kh, (k+1)h].$$

# A monotone numerical scheme

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$$X_0^h = X_0,$$

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Write  $X_k^h := X_{kh}^h$  and  $\mu^h = \text{Law}((X_k^h)_k)$ .

## Remark

$X_k^h \mapsto X_{(k+1)}^h$  is **increasing** if  $b$  is Lipschitz,  $h \ll 1$

# A monotone numerical scheme

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

## Monotone Euler–Maruyama scheme

$$X_0^h = X_0,$$

$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh) + \sigma(X_{kh})(W_t^h - W_{kh}^h), \quad t \in (kh, (k+1)h].$$

$$W_t^h - W_{kh}^h = W_{t \wedge \tau_k^h} - W_{kh}, \quad \tau_k^h := \inf\{t > kh : |W_t - W_{kh}| > A_h\}$$

Cf. [Milstein, Repin, Tretyakov '02], [Liu, Pagès '22], [Jourdain, Pagès '23]

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## Lemma [Backhoff-Veraguas, Källblad, R. '24]

For  $b, \sigma$  Lipschitz, the monotone Euler–Maruyama scheme is stochastically increasing.

Hence the Knothe–Rosenblatt rearrangement is optimal for  $\mu^h, \nu^h$ .

Moreover,  $\pi_{\text{KR}}(\mu^h, \nu^h) = \text{Law}(X^h, \bar{X}^h)$ ,  $B = W$ .

## Proof of main result

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 = x \rightsquigarrow \text{Law}(X) = \mu$$

$$d\bar{X}_t = \bar{b}_t(\bar{X}_t)dt + \bar{\sigma}_t(\bar{X}_t)dW_t, \quad \bar{X}_0 = x \rightsquigarrow \text{Law}(\bar{X}) = \nu$$

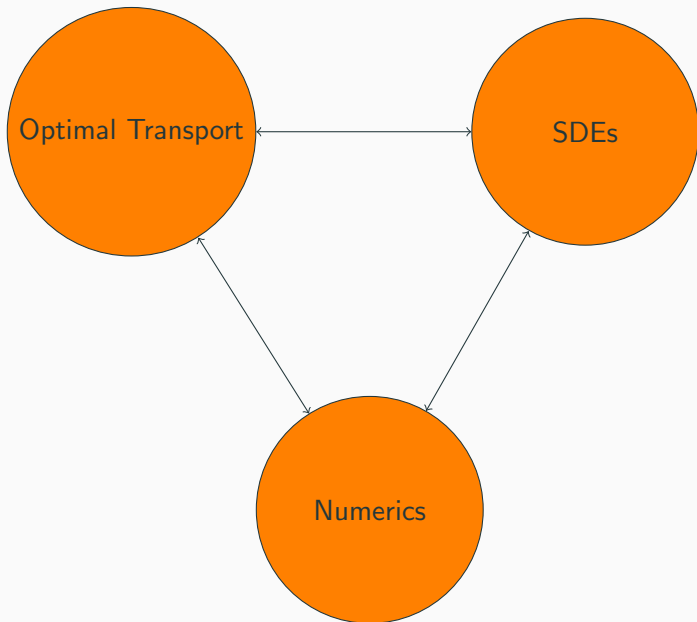
### Theorem [Backhoff-Veraguas, Källblad, R. 24]

For “sufficiently nice” coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E} \left[ \int_0^T |X_t - \bar{X}_t|^p dt \right], \quad \text{with } B = W.$$

1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

# Ingredients





# Main result

## Assumptions

- Continuous coefficients with linear growth
- Strong existence and uniqueness

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 = x \rightsquigarrow \text{Law}(X) = \mu$$

$$d\bar{X}_t = \bar{b}_t(\bar{X}_t)dt + \bar{\sigma}_t(\bar{X}_t)dW_t, \quad \bar{X}_0 = x \rightsquigarrow \text{Law}(\bar{X}) = \nu$$

## Main Theorem [Backhoff-Veraguas, Källblad, R. 24]

The adapted Wasserstein distance is given by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E} \left[ \int_0^T |X_t - \bar{X}_t|^p dt \right], \quad \text{with } B = W.$$

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## Main Theorem [Backhoff-Veraguas, Källblad, R. 24]

The adapted Wasserstein distance is given by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E} \left[ \int_0^T |X_t - \bar{X}_t|^p dt \right], \quad \text{with } B = W.$$

**Synchronous coupling** solves general bicausal transport problem

- **Irregular coefficients** [R. Szölgyenyi '24]
  - discontinuous drift with exponential growth
  - bounded measurable drift
- **Higher dimensions**
  - counterexamples [Backhoff-Veraguas, Källblad, R. '24]
  - different techniques needed
- **More general processes** (work in progress...)
  - jump-diffusions, McKean–Vlasov equations, ...

# Summary

- Study distance between stochastic processes
- Identify optimal bicausal coupling of SDEs
- Exploit properties of numerical approximations of SDEs

References:

