A regularized Kellerer theorem in arbitrary dimension

Benjamin A. Robinson (Universität Klagenfurt)

December 19, 2024 — *Rough Analysis and Stochastic Dynamics* Seminar, TU Berlin

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Joint work with

Gudmund Pammer
ETH Zürich

Walter Schachermayer
Universität Wien

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Pammer, G., Robinson, B. A. and Schachermayer, W. (2025) A regularized Kellerer theorem in arbitrary dimension. *Annals of Applied Probability.* arXiv:2210.13847

Simple example

Given probability measures μ, ν on $\mathbb{R}^d$ do there exist random variables M_0, M_1 such that

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$$\int v \mathrm{d}\mu \leq \int v \mathrm{d}\nu.$$

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… also sufficient [Strassen '65]

Problem statement

Given a family of probability measures $(\mu_t)_{t \in I}$ on $\mathbb{R}^d$, does there exist a mimicking martingale M such that

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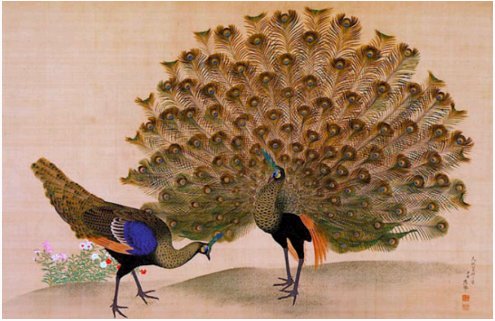
We call μ a peacock if it satisfies this condition.

Problem statement

Assume that μ is a peacock; i.e. for any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

$$\int v \mathrm{d}\mu_s \leq \int v \mathrm{d}\mu_t, \quad s \leq t.$$

Processus Croissant pour l'Ordre Convexe



[Hirsch, Profetta, Roynette, Yor '11]

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Yes – [Strassen '65], [Doob '68], [Hirsch, Roynette '13]

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Desirable properties

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- continuity of paths
- uniqueness

Discrete time

[Strassen '65]

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Volker Strassen *The Existence of Probability Measures with Given Marginals*, Ann. Math. Stat. 1965

[Kellerer '72]

Given a continuous-time peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}$,

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Subsequent contributions (incomplete!)

Albin, Baker, Beiglböck, Brückerhoff, Boubel, Donati-Martin, Hamza, Hirsch, Huesmann, Juillet, Källblad, Klebaner, Lowther, Profetta, Roynette, Stebegg, Tan, Touzi, Yor, ...

Continuous time, $d = 1$

[Kellerer '72]

Given a continuous-time peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}$, there exists a mimicking strong Markov martingale M.

[Lowther '08–10]

- M is the unique strong Markov mimicking martingale

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... without the Markov property

[Doob '68] (compact support)

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[Doob '68] (compact support), [Hirsch, Roynette '13] ($\mathbb{R}^d$)

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no known results

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Theorem 1 [Pammer, R., Schachermayer '25]

There exists a strong Markov martingale diffusion mimicking a *regularized* continuous-time peacock on $\mathbb{R}^d$, $d \in \mathbb{N}$.

Main result

Weakly continuous $\mathbb{R}^d$-valued square-integrable peacock $(\mu_t)_{t\in[0,1]}$.

Regularize with a Gaussian $\mu_t^{\mathrm{r}} := \mu_t * \gamma^{\varepsilon(t+\delta)}$

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There exists a measurable $(t,x) \mapsto \sigma_t^{\mathrm{r}}(x)^2$ that is locally Lipschitz in x and non-degenerate, uniformly in $t \in [0,1]$, and a Brownian motion B such that $\mathrm{Law}(M_t^{\mathrm{r}}) = \mu_t^{\mathrm{r}}$, for all $t \in [0,1]$, where

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- M^{r} is a strong Markov martingale with continuous paths;

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Moreover:

- M^{r} is a strong Markov martingale with continuous paths;
- The result does not hold without regularization;
- There is no uniqueness for $d \geq 2$.

Proof idea

- Discretize and solve discrete-time problem;
- Pass to the continuous-time limit.

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In dimension one, compactness w.r.t. f.d.d. convergence holds for the sets of processes that are

- càdlàg and Lipschitz Markov [Kellerer '72], [Lowther '09],
- almost-continuous diffusions [Lowther '09].

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In $\mathbb{R}^2$, these notions no longer help us [Lowther '09], [Juillet '16]!

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Discretize and take stretched Brownian motion from μ_{t_k} to $\mu_{t_{k+1}}$ to get a diffusion process

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Optimal Transport

$$\inf_{\substack{\pi \in \mathrm{Cpl}(\mu_0,\mu_1) \\ (X,Y)\sim\pi}} \mathbb{E}[c(X,Y)]$$

$$\mathrm{Cpl}(\mu_0,\mu_1) = \{\pi\colon\ \pi(A\times\mathbb{R}^d) = \mu_0(A), \pi(\mathbb{R}^d\times B) = \mu_1(B)\}$$

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Martingale Optimal Transport

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$$\mathcal{M}(\mu_0,\mu_1) = \{\pi \in \mathrm{Cpl}(\mu_0,\mu_1) \colon\ (U,V)\sim\pi \Rightarrow \mathbb{E}[V \mid U] = U\}$$

[Hobson, Neuberger '12], [Beiglböck, Juillet '16]

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Martingale Optimal Transport

$$\inf_{\substack{\pi \in \mathcal{M}(\mu_0,\mu_1) \\ (X,Y)\sim\pi}} \mathbb{E}[c(X,Y)] \text{“}=\text{”}\, \mathbb{E}[c(M_0,M_1)]$$

$(M_t)_{t\in[0,1]}$ stretched Brownian motion

[Backhoff, Beiglböck, Huesmann, Källblad ’19]
[Backhoff, Beiglböck, Schachermayer, Tschiderer ’23]
[Schachermayer, Tschiderer ’24]

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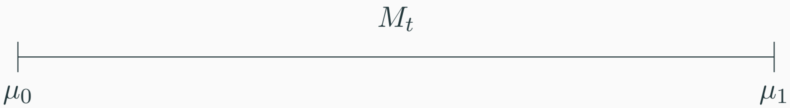
Stretched Brownian motion



$$M_t = \mathbb{E}[\nabla v(B_1) \mid \mathcal{F}_t], \quad v\colon \mathbb{R}^d \to \mathbb{R} \text{ convex}, \quad B_0 \sim \alpha.$$

Bass martingale — under an irreducibility condition

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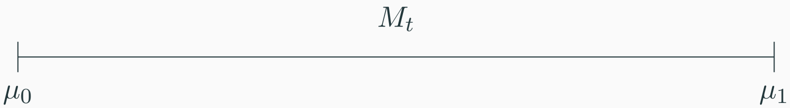

$M_t = \mathbb{E}[\nabla v(B_1) \mid \mathcal{F}_t] = \nabla v_t(B_t), \quad v_t \colon \mathbb{R}^d \to \mathbb{R}$ convex, $\quad B_0 \sim \alpha.$

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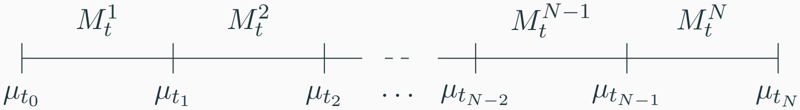

$$\mathrm{d}M_t^k = \sigma_t^k(M_t^k)\mathrm{d}W_t$$

Do not have uniform-in-time bounds on σ^k

Proof idea II

Regularize with a Gaussian and make a Markovian projection

$$\mu_t \mapsto \hat{\mu}_t = \mu_t * \gamma^{\varepsilon(t_k+\delta)}, \quad t \in [t_k, t_{k+1})$$

$$\hat{\mu}_{t_0} \qquad \hat{\mu}_{t_1} \qquad \hat{\mu}_{t_2} \quad \cdots \quad \hat{\mu}_{t_{N-2}} \qquad \hat{\mu}_{t_{N-1}} \qquad \hat{\mu}_{t_N}$$

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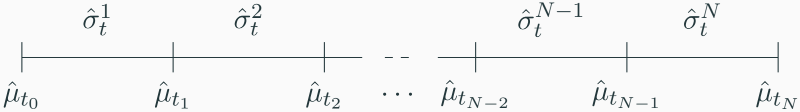
$\mathrm{Law}(\hat{X}_{t_k}) = \mu^{\mathrm{r}}_{t_k}$ and $(\hat{\sigma})^2$ locally Lipschitz and bounded, unif. in t



Proof idea III

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But $(\hat{\sigma})^2$ may be degenerate

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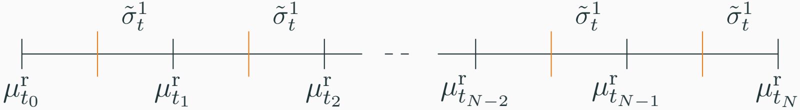
But $(\hat{\sigma})^2$ may be degenerate ... make a further regularization

$$\hat{\mu}_{t_{k+1}} \mapsto \hat{\mu}_{t_{k+1}} * \gamma^{\varepsilon 2^{-n}} = \mu^{\mathrm{r}}_{t_{k+1}}$$



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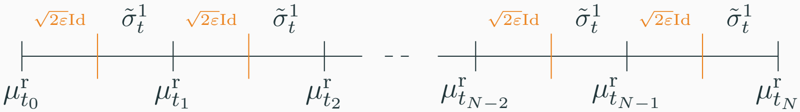

Proof idea IV

Use stability of diffusions to pass to a limit

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Assumption I
Suppose that $(t, x) \mapsto \sigma_t^k(x)^2$ are

- locally Lipschitz, uniformly in k and t;
- bounded locally in x, uniformly in k and t;

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Suppose that $(t,x) \mapsto \sigma_t^k(x)^2$ are

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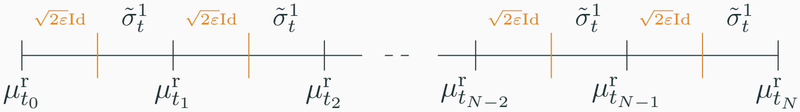

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Theorem 2 [Pammer, R., Schachermayer '25]

For $\mathrm{d}X_t^k = \sigma_t^k(X_t^k)\mathrm{d}B_t$ and σ^k satisfying Assumption I, suppose for each (t, x)

$$\int_0^t \sigma_s^k(x)^2 \mathrm{d}s \to \int_0^t \sigma_s(x)^2 \mathrm{d}s.$$

Proof idea IV

Use stability of diffusions to pass to a limit

Assumption I
Suppose that $(t,x) \mapsto \sigma_t^k(x)^2$ are "nice enough".

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Then $X^k \to X$ in f.d.d., $\mathrm{d}X_t = \sigma_t(X_t)\mathrm{d}B_t$ and σ "nice".

Main result

Weakly continuous $\mathbb{R}^d$-valued square-integrable peacock $(\mu_t)_{t\in[0,1]}$.
Regularize with a Gaussian $\mu_t^{\mathrm{r}} := \mu_t * \gamma^{\varepsilon(t+\delta)}$

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Theorem 3 [Pammer, R., Schachermayer '25]

There exists a weakly continuous square-integrable peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}^4$ such that, for the peacock $(\mu_t * \gamma^t)_{t\in[0,1]}$, there exists no mimicking Markov martingale.

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Circular Brownian Motion
[Émery, Schachermayer '99]
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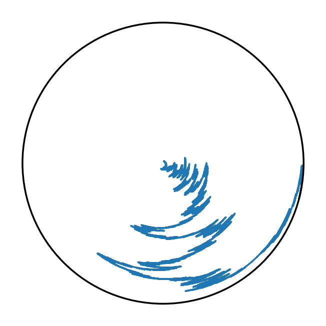

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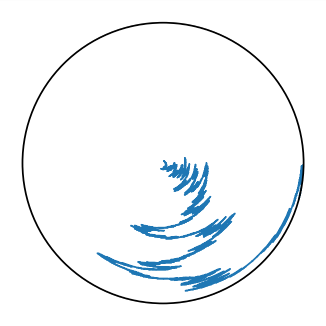
There is a unique weak solution but no strong solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}\begin{bmatrix} -X_t^2 \\ X_t^1 \end{bmatrix}\mathrm{d}W_t, \quad X_0 = 0.$$



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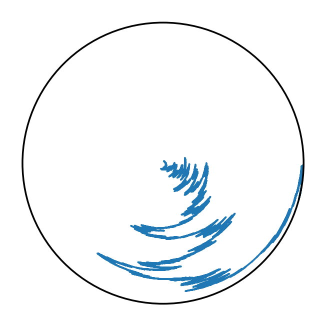

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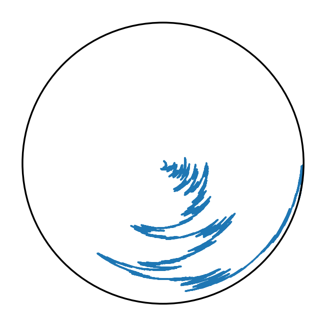

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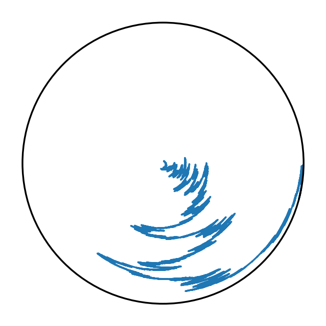
2. No Markov martingale mimicking μ;
3. No Markov martingale mimicking $(\mu * \gamma^t)_{t\in[0,1]}$.



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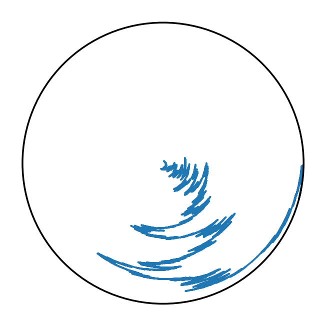
Open questions

- Minimal conditions for existence?
- Markov property with respect to raw filtration?



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Faking Brownian motion

Do there exist stochastic processes with Brownian marginals that are not Brownian motion?

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There exists a "very fake" Brownian motion in dimension $d = 1$.

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Recall: for $d = 1$, a strong Markov process is uniquely determined by its marginals [Lowther '09].

Non-uniqueness

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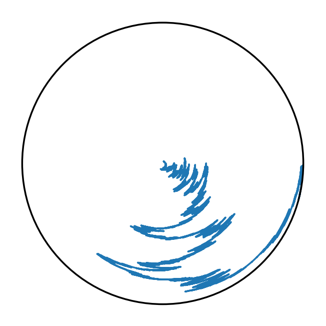
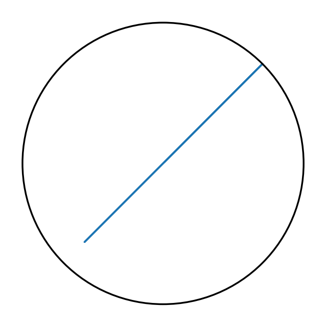


$$\mathrm{d}X_t = |X_t|^{-1} X_t^\top \mathrm{d}W_t$$



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Solutions of a stochastic control problem for martingales [Cox, R. '23]

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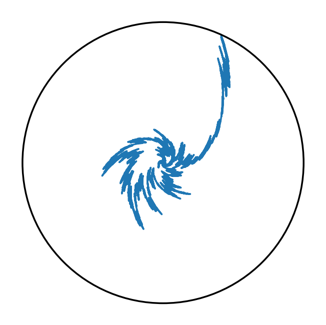
Theorem [Cox, R. '23]
Let X be a weak solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}(X_t + X_t^{\perp})\mathrm{d}W_t,\ X_0 \sim \eta.$$

Then X is a continuous strong Markov fake Brownian motion.



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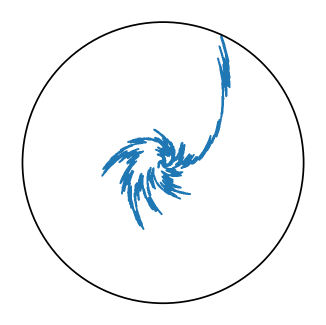
Open questions:

- Can we find mimicking martingales with additional structure?
- Is there a subclass of mimicking martingales in which uniqueness does hold?



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References

Alexander M. G. Cox and Benjamin A. Robinson, *Optimal control of martingales in a radially symmetric environment*, Stoch. Proc. Appl. **159** (2023), 149–198.

———, *SDEs with no strong solution arising from a problem of stochastic control*, Electron. J. Probab. **28** (2023), 1–24.

Gudmund Pammer, Benjamin A. Robinson, and Walter Schachermayer, *A regularized Kellerer theorem in arbitrary dimension*, Ann. Appl. Probab. (2025), arXiv:2210.13847.

Summary

- We prove the first known Kellerer-type result in arbitrary dimension;
- In general, the result can fail without some regularization;
- In dimension $d \geq 2$, uniqueness fails.

Pammer, G., Robinson, B. A. and Schachermayer, W. (2025)
A regularized Kellerer theorem in arbitrary dimension. *Annals of Applied Probability.* arXiv:2210.13847


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