Synergy of Data Assimilation and Inverse Problems Techniques

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February 28, 2017

Outline

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- The neurons can receive, process, and transmit signals through the transmission paths, which either excite or inhibit the others.
- In case of excitation, the neuron fires, when the excitation level reach a threshold.

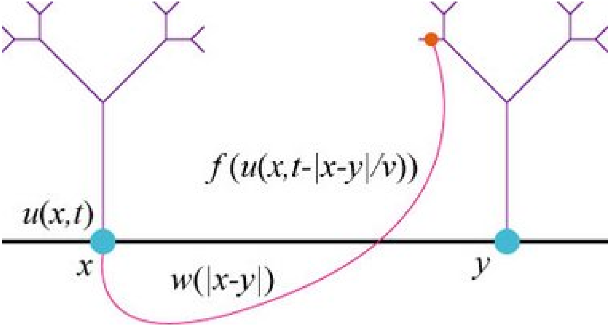

Figure: This figure is simplifying the connection between two neurons, in reality, they are part of a complex network

The Neural Field

In neural dynamics, neurons send electrical spikes to each other. The dynamics is described in its continuous version over the space Ω by the simplest form of the Amari neural field equation

$$\tau \frac{\partial u}{\partial t}(r,t) = -u(r,t) + \int_{\Omega} w(r,r') f(u(r',t))\, dr', \quad r \in \Omega. \tag{1}$$

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- The kernel $w(r,r')$ is the connectivity function.

The Delay Neural Field Equation

The Amari neural field equation ignores any delay embedded in the firing rate. In reality, the velocity and the time of transmission of the synaptic signals cause a delay. Taking it into account, the neural field equation with a delay term is

$$\tau \frac{\partial u}{\partial t}(r,t) = -u(r,t) + \int_{\Omega} w(r,r') f\left(u(r', t - \frac{D(r,r')}{v}) \right) dr'. \qquad (2)$$

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- $D(r,r')$ is the length of the fiber between r and r'.
- v is the finite propagation speed of signals. For simplicity, in some of our examples we will work with $D(r,r') = \|r - r'\|$ and $v = 1$.

The Direct Problem

Definition (Direct Neural Field Problem)

Given an initial state $u_0 \in C^1(\Omega)$, a delay function D and a neural kernel w, the direct neural field problem is to calculate $u(r, t)$ for $t \in [0, T]$ with some constant $T > 0$ and $x \in \Omega$ as a solution to the integro-differential equation.

This problem has a unique solution $u(r, t)$ on $r \in \Omega$ and $t \in [0, T]$.

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Theorem (The Existence of The Unique Solution)

If the kernel w is uniformly Hölder continuous and if the delay term D is bounded continuous, then for any initial field u_0 there exists a unique solution $u \in C^1(\Omega, L^1(\Omega))$ to the delay neural field equation on $[0, T]$.

The Kernel Assumptions

We assume that the kernel w satisfies

(C1) $w(r,.) \in L^1(\Omega) \quad \textit{for all } r \in \Omega \subset \mathbb{R}^m$

and

with constants $C_j > 0$, $j = 1, 2, 3$ and $\alpha \in (0, 1]$. For the function $f : R \to R^+$ we note that

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(C3) $\|w(r,.) - w(r^*,.)\|_{L^1(\Omega)} \leq C_2|r - r^*|^\alpha, \;\; r, r^* \in \Omega$

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(C4) $|w(r,r')| \leq C_3, \;\; r, r' \in \Omega$

with constants $C_j > 0$, $j = 1, 2, 3$ and $\alpha \in (0, 1]$. For the function $f : R \to R^+$ we note that

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The proof of The Existence of The Unique Solution

We will need to split the function $u(r, s - D(r, r'))$ into the part where the time variable $t = s - D(r, r')$ is in $(0, T]$ and where $t = s - D(r, r')$ is in $[-c_T, 0]$.

$$(A_1 u)(r, t) := \int_0^t -\frac{u(x, s)}{\tau}\, ds, \quad r \in \Omega, t \leq 0, \tag{3}$$

and

$$\begin{aligned}(A_2^{\pm} u)(r, t) \quad := \quad & \frac{1}{\tau} \int_0^t \int_\Omega w(r, r') \chi_{\pm}(r, s - D(r, r')) \cdot \\ & \qquad f\Big(u(r', s - D(r, r'))\Big)\, dr'\, ds\end{aligned} \tag{4}$$

for $r \in \Omega, t \in [0, T]$.

By integration with respect to time the solution of equation (**??**) can be reformulated as

$$u(r,t) - u(r,0) = \tag{5}$$
$$-\frac{1}{\tau}\int_0^t u(r,s)\,ds + \frac{1}{\tau}\int_0^t \int_\Omega w(r,r')f\Big(u(r', s - D(r,r'))\Big)\,dr'\,ds$$

for $r \in \Omega$ and $t \in [0,\rho]$ for some interval $[0,\rho]$ with an auxiliary parameter ρ. We then have

$$\begin{aligned}
&\frac{1}{\tau}\int_0^t \int_\Omega w(r,r')f\Big(u(r', s - D(r,r'))\Big)\,dr'\,ds \\
&= (A_2^+ u)(r,s) + (A_2^- u)(r,s) \\
&= (A_2^+ u)(r,s) + (A_2^- u_0)(r,s)
\end{aligned} \tag{6}$$

where we use that $u(r,t) = u_0(r,t)$ for $t \le 0$. With $A := A_1 + A_2^+$ the delay neural field equation is equivalent to the fixed point equation

$$u(r,t) = u(r,0) + (A_2^- u_0)(r,t) + (Au)(r,t), \quad r \in \Omega, t \in [0,\rho]. \tag{7}$$

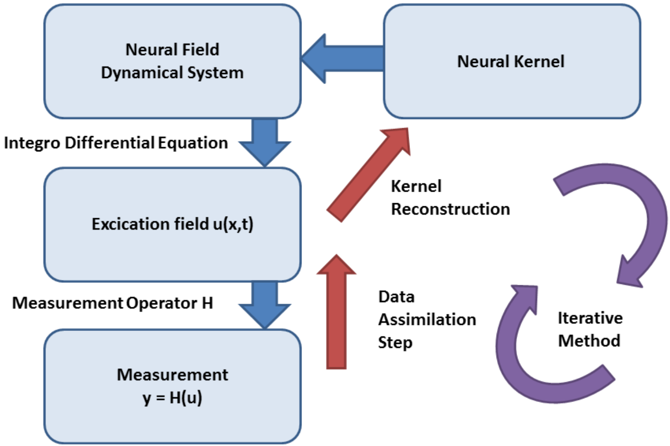
For the operator A we obtain the estimate

$$\begin{aligned} \|A(u_1) - A(u_2)\|_\rho &= \|A_1(u_1 - u_2) + (A_2(u_1) - A_2(u_2))\| \\ &\leq \frac{\rho}{\tau}\|u_1 - u_2\|_\rho + \frac{\rho L C_w}{\tau}\|u_1 - u_2\|_\rho \\ &\leq \frac{\rho}{\tau}(1 + LC_w)\|u_1 - u_2\|_\rho \end{aligned} \tag{8}$$

With the choice

$$q := \frac{\rho}{\tau}(1 + LC_w) \tag{9}$$

in the case where ρ is small enough to guarantee that $q < 1$, we have shown that A is a contraction on $BC(\Omega \times [0, \rho], ||\cdot||_\rho)$. According to the Banach fixed point theorem, there is one and only one fixed point u^* for the fixed-point equation (**??**). In the same way, this leads to the existence and uniqueness result on the interval $[0, T]$.


Neural Field
Dynamical System
Neural Kernel
Integro Differential Equation
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Kernel
Reconstruction
Measurement Operator H
Data
Assimilation
Step
Iterative
Method
Measurement
y = H(u)

Full Field Neural Inverse Problem

Definition (Full Field Neural Inverse Problem)

Given the time-dependent neural field $u(x, t)$ for $x \in \Omega$ and $t \in [0, T]$ the full field neural inverse problem is to determine the neural connectivity kernel $w(r, r')$ for $r, r' \in \Omega$ given the knowledge of u, such that u is a solution to the neural field equation with kernel w and delay D, where we assume that we know the delay D as a function of r, r'.

The Inverse Problem

Given N time-dependent neural activation patterns $u^{(\xi)}(r,t)$ for $(r,t) \in \Omega \times [0,T]$ corresponding to initial conditions $u_0^{(\xi)}(r,t)$ for $(r,t) \in \Omega \times [-c_T, 0]$ and $\xi = 1, ..., N$. We assume

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- the nonlinear activation function $f : \mathbb{R} \to \mathbb{R}^+$ to be known and
- the delay function $D : \Omega \times \Omega \to [0, c_T]$ to be given.
- The task is to find a kernel $w(r,r')$ for $(r,r') \in \Omega$ which generates the solutions $u^{(\xi)}$ with the initial conditions $u_0^{(\xi)}(r,t)$ for $\xi = 1, 2, ..., N$.

Let

$$w_r := w(r, r'), \quad r, r' \in \Omega. \tag{10}$$

$$\psi_r(t) = \int_\Omega \phi(r', t - D(r, r'))w_r(r')\, dr', \quad t \leq T \tag{11}$$

Here, we reformulate the inverse problem into a family of integral equations of the first kind and study their solution by regularization methods. As a first step, we define

$$\phi^{(\xi)}(r, s) := f\left(u^{(\xi)}(r, s - D(r, r'))\right), \quad (r, s) \in \Omega \times [0, T] \tag{12}$$

with $\xi = 1, 2, ..., N$, and

$$\psi^{(\xi)}(r, t) := \tau\left[\frac{\partial u^{(\xi)}}{\partial s}(r, s) + u^{(\xi)}(r, s)\right], \quad (s, t) \in \Omega \times [0, T] \tag{13}$$

for $\xi = 1, 2, ..., N$. With the integral operator W defined by

$$(W\phi)(r, t) := \int_\Omega w_r \phi(r', s - D(r, r'))\, dr'\ , \quad x \in \Omega, \tag{14}$$

The structure is given by the integral operator

$$(V_r g)(t) := \int_\Omega K_r(t, r')g(r')\, dr', \quad t \in [0, T], \tag{15}$$

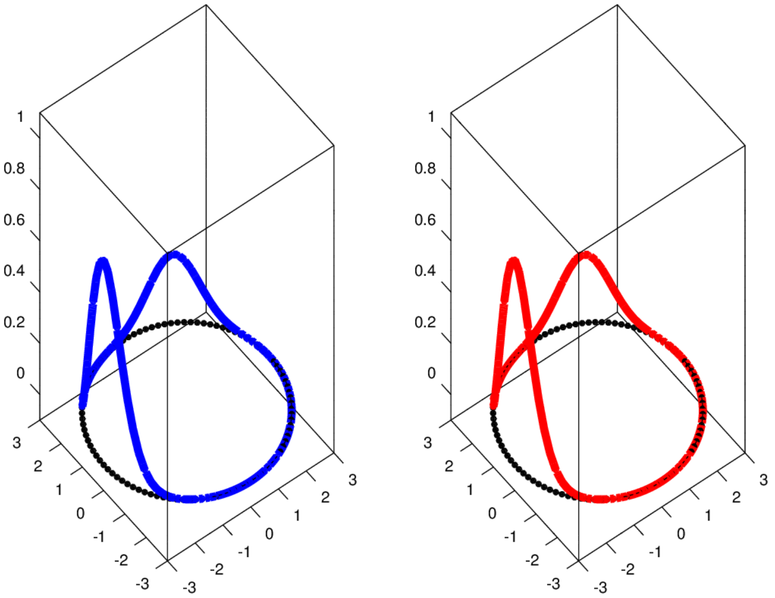
with kernel

$$K_r(t, r') := \phi(r', t - D(r, r')), \quad t \in [0, T],\ r' \in \Omega \tag{16}$$

for $r \in \Omega$. For $N > 1$ this kernel is vector valued in the sense that it is a vector of functions $\phi^{(\xi)}(r', t - D(r, r'))$, $\xi = 1, ..., N$. Now, our inverse problem is given by

$$\psi_r = V_r w_r \tag{17}$$

for $r \in \Omega$. For each $r \in \Omega$ equation is a *Fredholm integral equation of the first kind* with continuous kernel ϕ. The operator V_r is a compact operator on the spaces $C(\Omega)$ or $L^2(\Omega)$.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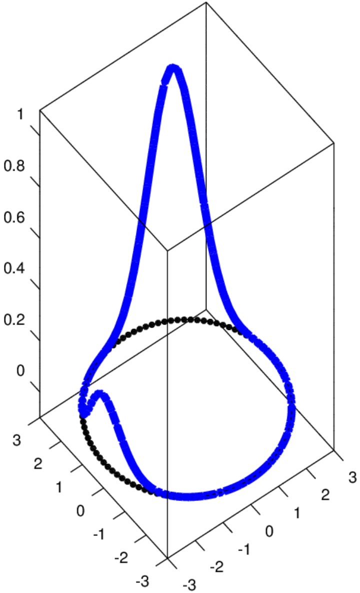

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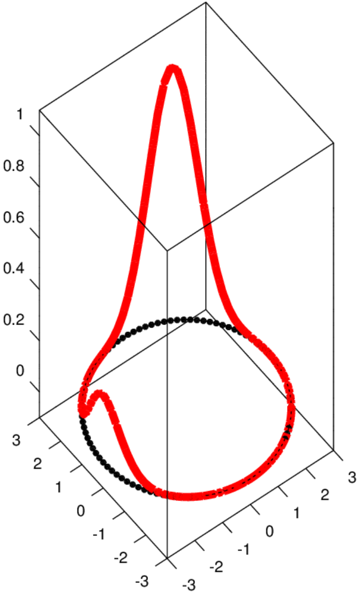

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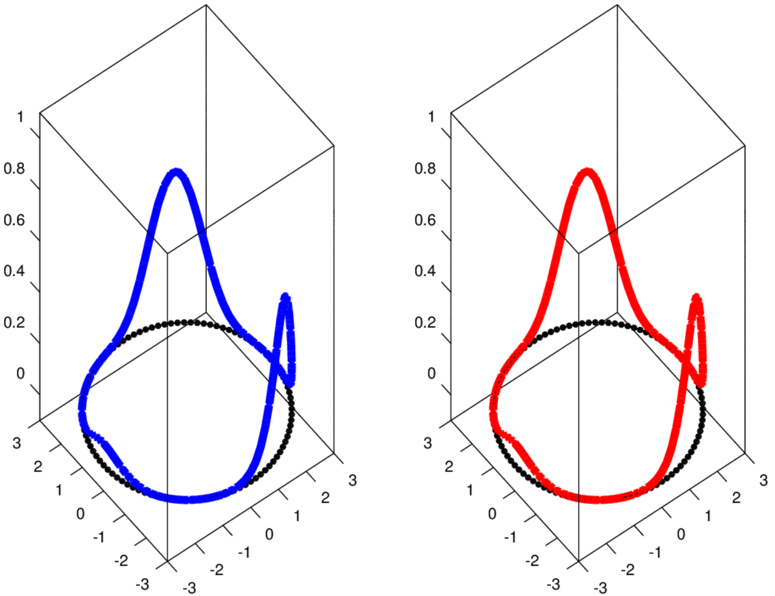

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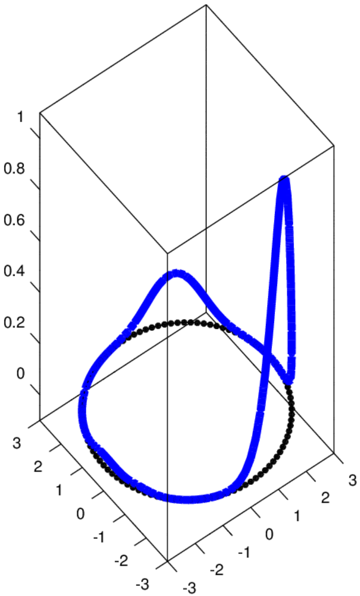
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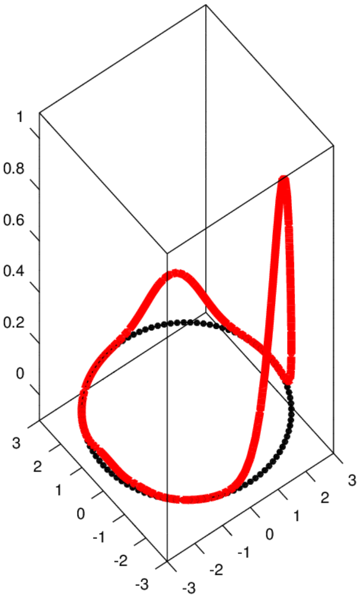
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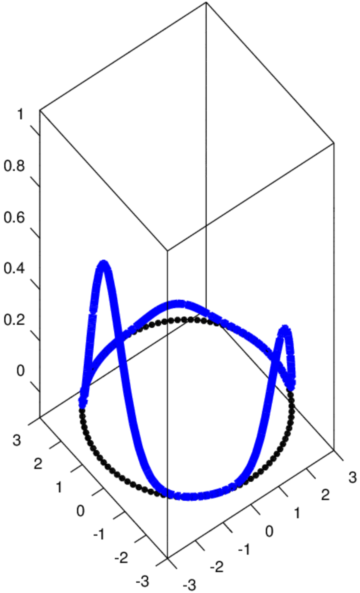
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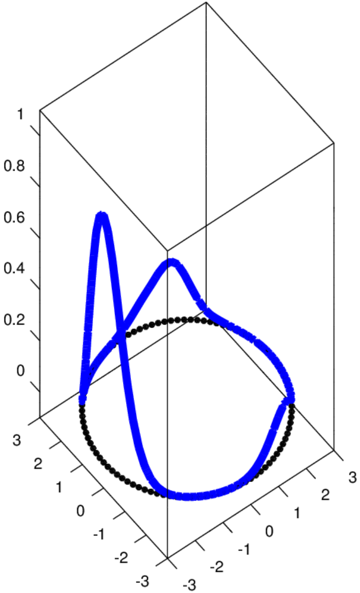
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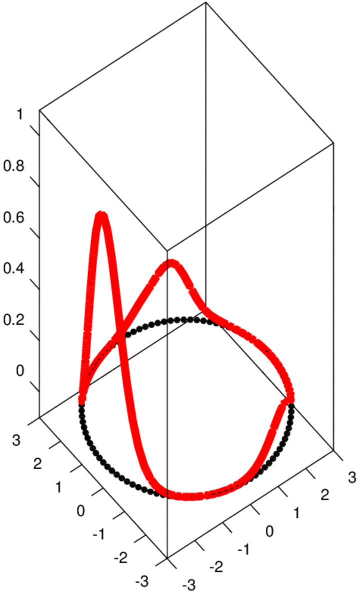
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- It is well-known that this equation is ill-posed, i.e. it does not need to have unique solutions.
- Its solution, if exist, does not depend continuously on the right-hand side.
- Ill-posed equations need some regularization method for their stable solution.

For each $r \in \Omega$ equation (**??**) corresponds to the operator equation

$$A[r]w(r,\cdot) = \psi(r,\cdot) \quad \text{on} \quad [0,T]. \tag{18}$$

The regularized solution with regularization parameter $\alpha > 0$ according to Tikhonov regularization is given by

$$w_\alpha(r,\cdot) := \Big(\alpha I + A^*[r]A[r]\Big)^{-1} A^*[r]\psi(r,\cdot) \tag{19}$$

for $r \in \Omega$.

Definition (3D-VAR for Neural State Estimation)

The three-dimensional variational method for neural state estimation from electrode measurements employs measurements

$$y_k = Hu^{(true)}(\cdot, t_k) + \epsilon \tag{20}$$

with error $\epsilon \in \mathbb{R}^m$ and a *first guess* or background

$$u^{(b)}(\cdot, t_k) := M_{k-1,k} u^{(a)}(\cdot, t_{k-1}) \tag{21}$$

to calculate an estimate $u^{(a)}(\cdot, t_k)$ of the neural activity at time t_k according to the equation:

$$u^{(a)} := u^{(b)} + BH^T(R + HBH^T)^{-1}(y - Hu^{(b)}), \tag{22}$$

For 3D-VAR, the background state covariance matrix B can be calculated from statistical evaluations of neural fields.

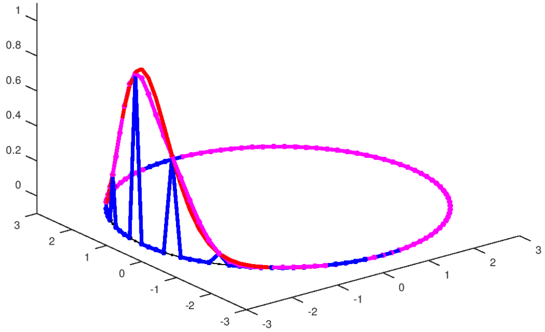

Figure: This figure shows the state estimation in one time slide

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Thank you