A regularized Kellerer theorem in arbitrary dimension

Benjamin A. Robinson

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November 20, 2023 — Vienna Probability Seminar

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Joint work with

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ETH Zürich

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University of Vienna

Motivating problem

Given probability measures μ, ν on $\mathbb{R}^d$ do there exist random variables M_0, M_1 such that

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For any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

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For any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

$$\int v \mathrm{d}\mu \leq \int v \mathrm{d}\nu.$$

… also sufficient [Strassen '65]

Problem statement

Given a family of probability measures $(\mu_t)_{t\in I}$ on $\mathbb{R}^d$, does there exist a mimicking martingale M such that

$$\mathrm{Law}(M_t) = \mu_t, \quad \forall t \in I?$$

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Peacocks

Assume that μ is a peacock; i.e. for any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

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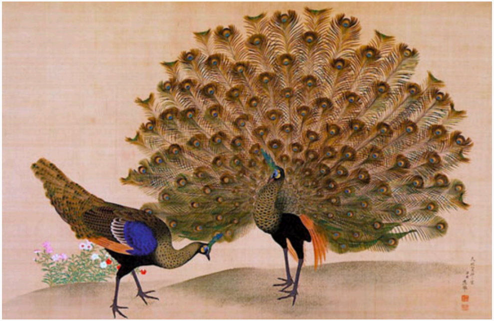
Processus Croissant pour l'Ordre Convexe



[Hirsch, Profetta, Roynette, Yor '11]

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Yes – [Strassen '65], [Doob '68], [Hirsch, Roynette '13]

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Desirable properties

- strong Markovianity
- continuity of paths
- uniqueness

Discrete time

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Volker Strassen *The Existence of Probability Measures with Given Marginals*, Ann. Math. Stat. 1965

Continuous time, $d = 1$

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Subsequent contributions (incomplete!)

Albin, Baker, Beiglböck, Brückerhoff, Boubel, Donati-Martin, Hamza, Hirsch, Huesmann, Juillet, Källblad, Klebaner, Lowther, Profetta, Roynette, Stebegg, Tan, Touzi, Yor, ...

Continuous time, $d = 1$

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Given a continuous-time peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}$, there exists a mimicking strong Markov martingale M.

[Lowther '08–10]

- M is the unique strong Markov mimicking martingale

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... without the Markov property

[Doob '68] (compact support)

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[Doob '68] (compact support), [Hirsch, Roynette '13] ($\mathbb{R}^d$)

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no known results

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Theorem 1 [Pammer, R., Schachermayer '23+]

There exists a strong Markov martingale diffusion mimicking a *regularized* continuous-time peacock on $\mathbb{R}^d$, $d \in \mathbb{N}$.

Main result

Weakly continuous $\mathbb{R}^d$-valued square-integrable peacock $(\mu_t)_{t\in[0,1]}$.
Regularize with a Gaussian $\mu_t^{\mathrm{r}} := \mu_t * \gamma^{\varepsilon(t+\delta)}$

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In dimension one, compactness w.r.t. f.d.d. convergence holds for the sets of processes that are

- càdlàg and Lipschitz Markov [Kellerer '72], [Lowther '09],
- almost-continuous diffusions [Lowther '09].

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In $\mathbb{R}^2$, these notions no longer help us [Lowther '09], [Juillet '16]!

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Discretize and take Bass martingales from μ_{t_k} to $\mu_{t_{k+1}}$ to get a diffusion process

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Optimal Transport

$$\inf_{\substack{\pi \in \mathrm{Cpl}(\mu_0, \mu_1) \\ (X,Y) \sim \pi}} \mathbb{E}[c(X,Y)]$$

$$\mathrm{Cpl}(\mu_0, \mu_1) = \{\pi \colon\ \pi(A \times \mathbb{R}^d) = \mu_0(A), \pi(\mathbb{R}^d \times B) = \mu_1(B)\}$$

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Martingale Optimal Transport

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$$\mathcal{M}(\mu_0,\mu_1) = \{\pi \in \mathrm{Cpl}(\mu_0,\mu_1) \colon\ (U,V) \sim \pi \Rightarrow \mathbb{E}[V \mid U] = U\}$$

[Hobson, Neuberger '12], [Beiglböck, Juillet '16]

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Martingale Optimal Transport

$$\inf_{\substack{\pi \in \mathcal{M}(\mu_0,\mu_1) \\ (X,Y)\sim\pi}} \mathbb{E}[c(X,Y)] \text{``}=\text{''}\, \mathbb{E}[c(M_0,M_1)]$$

$$(M_t)_{t\in[0,1]} \quad \text{Bass martingale}$$

[Backhoff, Beiglböck, Huesmann, Källblad '19]

[Backhoff, Beiglböck, Schachermayer, Tschiderer '23]

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Bass martingales

M_t

μ_0 μ_1

$$M_t = \mathbb{E}[\nabla v(B_1) \mid \mathcal{F}_t], \quad v\colon \mathbb{R}^d \to \mathbb{R} \text{ convex}, \quad B_0 \sim \alpha.$$

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Proof idea II

Regularize with a Gaussian and make a Markovian projection

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Mimicking Itô processes [Krylov '85], [Gyöngy '85], [Brunick, Shreve '13]

$$\mathrm{d}X_t = \sigma_t \mathrm{d}W_t$$

$$\mathrm{d}\hat{X}_t = \hat{\sigma}_t(\hat{X}_t)\mathrm{d}W_t, \quad \mathrm{Law}(X_t) = \mathrm{Law}(\hat{X}_t),\ t \in [0,1]$$

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$\mathrm{Law}(\hat{X}_{t_k}) = \mu^{\mathrm{r}}_{t_k}$ and $\hat{\sigma}$ "nice"

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For $\mathrm{d}X_t^k = \sigma_t^k(X_t^k)\mathrm{d}B_t$ for "nice" σ^k, suppose for each (t,x)

$$\int_0^t \sigma_s^k(x)^2 \mathrm{d}s \to \int_0^t \sigma_s(x)^2 \mathrm{d}s.$$

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Then $X^k \to X$ in f.d.d., $\mathrm{d}X_t = \sigma_t(X_t)\mathrm{d}B_t$ and σ "nice".

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Counterexamples

Theorem 3 [Pammer, R., Schachermayer '23+]

There exists a weakly continuous square-integrable peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}^4$ such that, for the peacock $(\mu_t * \gamma^t)_{t\in[0,1]}$, there exists no mimicking Markov martingale.

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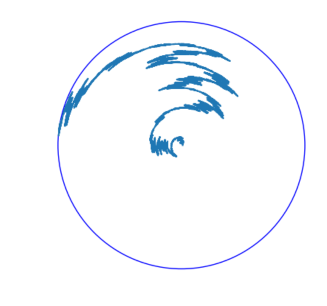
Circular Brownian Motion

[Émery, Schachermayer '99]

[Fernholz, Karatzas, Ruf '18]

[Larsson, Ruf '20]



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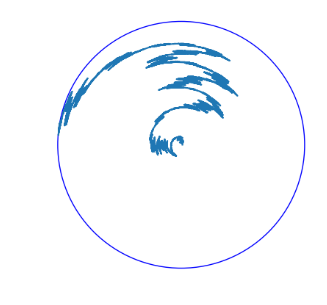
Theorem [Cox, R. '23]
There is a unique weak solution but no strong solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}\begin{bmatrix} -X_t^2 \\ X_t^1 \end{bmatrix}\mathrm{d}W_t, \quad X_0 = 0.$$



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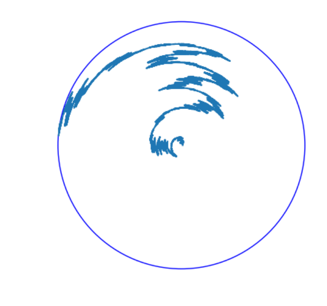
1. No continuous Markov martingale mimicking μ;



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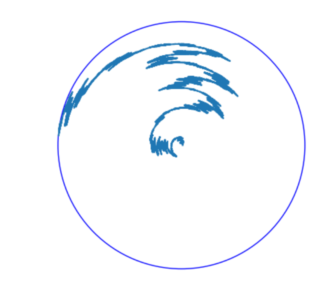
1. No continuous Markov martingale mimicking μ;
2. No Markov martingale mimicking μ;



[Cox, R. '23]

Counterexamples

Theorem 3 [Pammer, R., Schachermayer '23+]

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1. No continuous Markov martingale mimicking μ;
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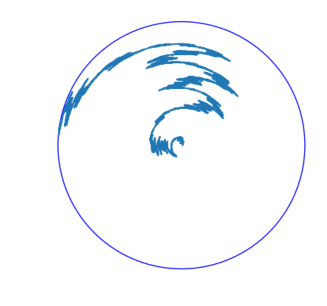

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Main result

Weakly continuous $\mathbb{R}^d$-valued square-integrable peacock $(\mu_t)_{t\in[0,1]}$.
Regularize with a Gaussian $\mu_t^{\mathrm{r}} := \mu_t * \gamma^{\varepsilon(t+\delta)}$

Theorem 1 [Pammer, R., Schachermayer '23+]

There exists a measurable $(t,x) \mapsto \sigma_t^{\mathrm{r}}(x)$ that is locally Lipschitz in x and non-degenerate, uniformly in $t \in [0,1]$, and a Brownian motion B such that $\mathrm{Law}(M_t^{\mathrm{r}}) = \mu_t^{\mathrm{r}}$, for all $t \in [0,1]$, where

$$\mathrm{d}M_t^{\mathrm{r}} = \sigma_t^{\mathrm{r}}(M_t^{\mathrm{r}})\mathrm{d}B_t.$$

Moreover:

- M^{r} is a strong Markov martingale with continuous paths;
- The result does not hold without regularization;
- There is no uniqueness for $d \geq 2$.

Faking Brownian motion

Do there exist stochastic processes with Brownian marginals that are not Brownian motion?

[Hamza, Klebaner '07]

There exists some fake Brownian motion.

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There exists a "very fake" Brownian motion in dimension $d = 1$.

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There exists a Markov process with continuous paths that mimics Brownian marginals in dimension $d = 1$.

Faking Brownian motion

Do there exist stochastic processes with Brownian marginals that are not Brownian motion?

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There exists some fake Brownian motion.

[Beiglböck, Lowther, Pammer, Schachermayer '21]

There exists a in dimension $d = 1$.

Recall: for $d = 1$, a strong Markov process is uniquely determined by its marginals [Lowther '09].

Non-uniqueness

Theorem 4 [Pammer, R., Schachermayer '23+]

There exists an $\mathbb{R}^2$-valued strong Markov martingale diffusion with Brownian marginals, which is not a Brownian motion.

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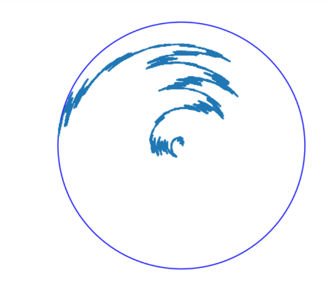
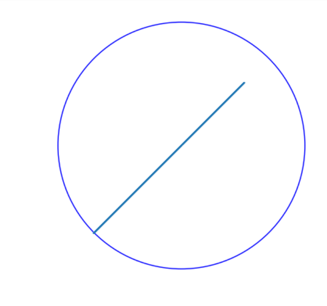


$$\mathrm{d}X_t = |X_t|^{-1} X_t^\top \mathrm{d}W_t$$



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Solutions of a stochastic control problem for martingales [Cox, R. '23]

Non-uniqueness

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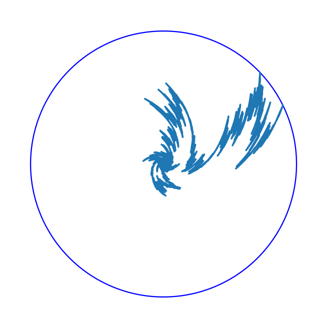
Theorem [Cox, R. '23]
Let X be a weak solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}(X_t + X_t^{\perp})\mathrm{d}W_t,\ X_0 \sim \eta.$$

Then X is a continuous strong Markov fake Brownian motion.



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References

- Alexander M. G. Cox and Benjamin A. Robinson, *Optimal control of martingales in a radially symmetric environment*, Stochastic Processes and their Applications **159** (2023), 149–198.
- Alexander M. G. Cox and Benjamin A. Robinson, *SDEs with no strong solution arising from a problem of stochastic control*, Electronic Journal of Probability **28** (2023), 1–24.
- Gudmund Pammer, Benjamin A. Robinson, and Walter Schachermayer, *A regularized Kellerer theorem in arbitrary dimension*, Annals of Applied Probability, to appear.

Summary

- We prove the first known Kellerer-type result in arbitrary dimension;
- In general, the result can fail without some regularization;
- In dimension $d \geq 2$, uniqueness fails.

G. Pammer, B. R., W. Schachermayer, *A regularized Kellerer theorem in arbitrary dimension*