Bicausal optimal transport for SDEs with irregular coefficients

Benjamin A. Robinson (University of Klagenfurt)

Austrian Stochastic Days — September 5, 2024

Supported by Austrian Science Fund (FWF) projects (Y782-N25), (P35519), (P34743).

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Adapted Wasserstein distance between the laws of SDEs
(with J. Backhoff-Veraguas and S. Källblad) — arXiv:2209.03243, 2024

Bicausal optimal transport for SDEs with irregular coefficients
(with M. Szölgyenyi) — arXiv:2403.09941, 2024

Comparing stochastic models

Aim: Compute a measure of model uncertainty

E.g. optimal stopping:

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[L(\tau, \omega)]$$

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SDEs:

- Good computational methods available
- Rich class of models, beyond Lipschitz coefficients

Main result

$b, \bar{b} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon [0,T] \times \mathbb{R} \to [0,\infty)$,

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad X_0 = x_0,$$
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Theorem [R., Szölgyenyi '24]

Under "weak assumptions" on the coefficients, we can compute an "appropriate distance"

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$$d_p(\mu,\nu)^p = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\right], \quad \text{with } B = W.$$

Application

E.g. optimal stopping:

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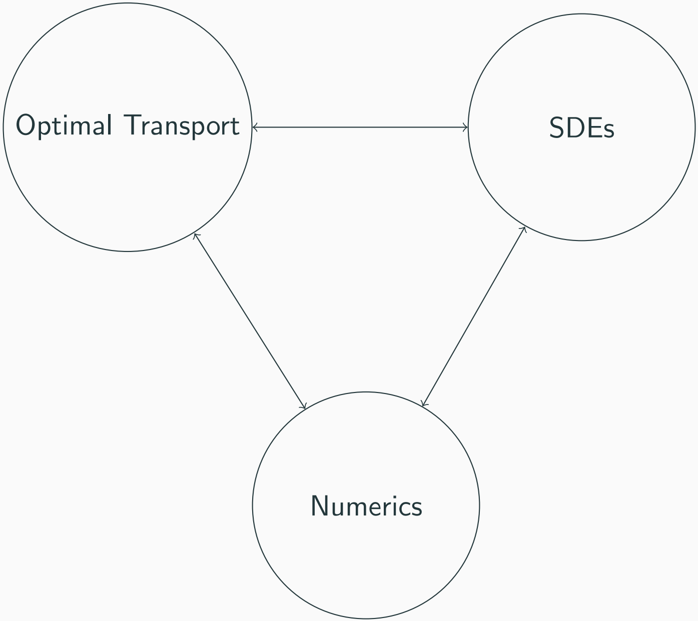
Theorem
[R., Szölgyenyi '24], [Acciaio, Backhoff-Veraguas, Zalashko '19]

$$\omega \mapsto L(t, \omega) \quad \text{Lipschitz on } (\Omega, \| \cdot \|_{L^p})$$

$$\Rightarrow$$

$$\mathbb{P} \mapsto v(\mathbb{P}) \quad \text{Lipschitz on } (\mathcal{P}_p(\Omega), d_p)$$

Ingredients



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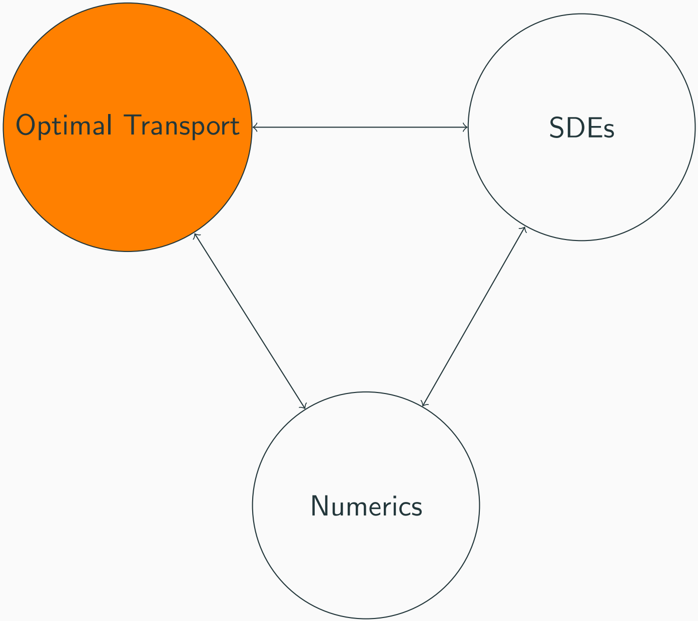

Adapted Wasserstein distance

Define the Wasserstein distance w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$ by

$$\mu,\nu \in \mathcal{P}_p(\Omega) \quad \rightsquigarrow \quad \mathcal{W}_p^p(\mu,\nu) := \inf_{\pi \in \mathrm{Cpl}(\mu,\nu)} \mathbb{E}^\pi \left[\int_0^T |\omega_t - \bar{\omega}_t|^p \mathrm{d}t \right]$$

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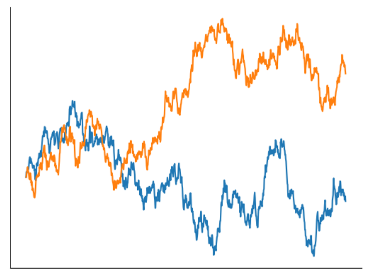

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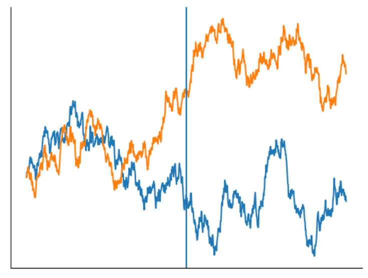

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Define the adapted Wasserstein distance w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$ by

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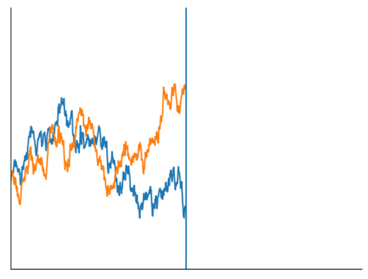
$$\mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu) = \{\pi \in \mathrm{Cpl}(\mu,\nu) \colon \ \pi \text{ bicausal}\}$$

"$\mathcal{F}_t^Y$ independent of $\mathcal{F}_T^X$, conditional on $\mathcal{F}_t^X$" and vice-versa



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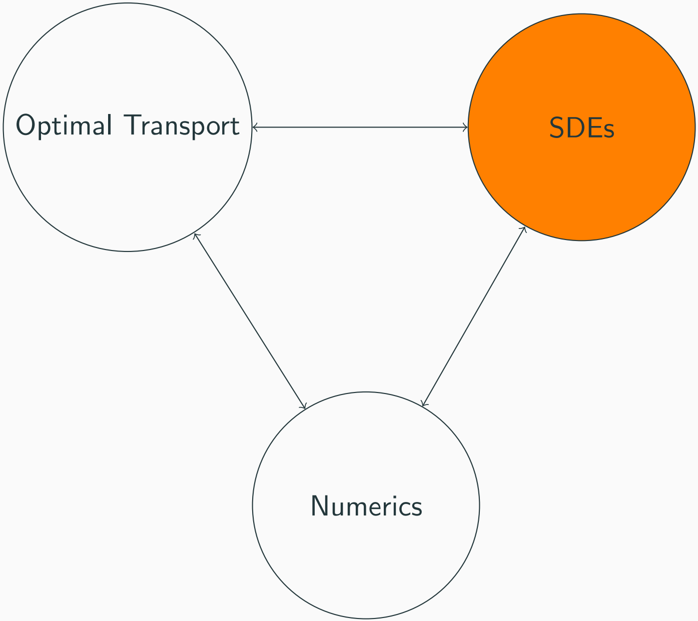

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Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \quad \text{with } B = W.$$

Coupling SDEs

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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Lemma [Backhoff-Veraguas, Källblad, R. '24]

Optimising over bicausal couplings $\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu, \nu)$

$\Leftrightarrow$

Optimising over correlations between B, W

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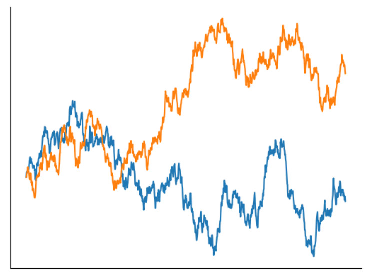
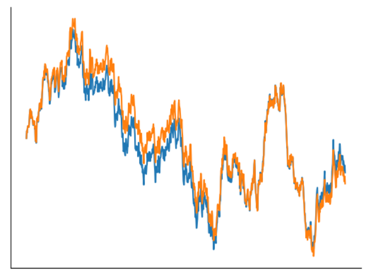
Product coupling

B, W independent



Synchronous coupling

Choose the same driving Brownian motion $B = W$.



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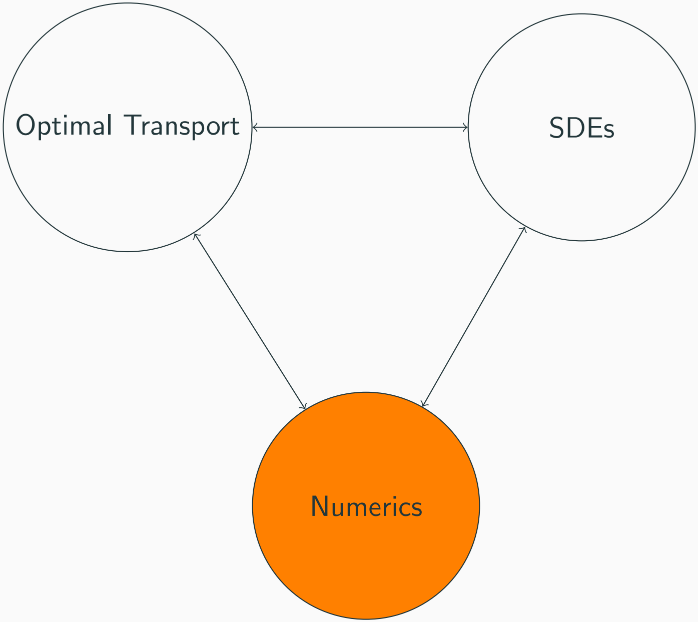

Proof of main result

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Key result in discrete time

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Knothe–Rosenblatt rearrangement

— generalisation of monotone rearrangement

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Knothe–Rosenblatt rearrangement

— generalisation of monotone rearrangement

Theorem **[Rüschendorf '85] [Beiglböck, Pammer, Posch '23]**

For μ, ν stochastically co-monotone, the unique optimiser is the Knothe–Rosenblatt rearrangement.

This induces the adapted weak topology.

Proof of main result

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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A monotone numerical scheme

$$dX_t = b(X_t)dt \tag{ODE}$$

Euler scheme

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh), \quad t \in (kh, (k+1)h].$$

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t \qquad \text{(SDE)}$$

Euler–Maruyama scheme

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$X_k^h \mapsto X_{(k+1)}^h$ is increasing if b is Lipschitz, $h \ll 1$

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Lemma [Backhoff-Veraguas, Källblad, R. '24]

For b, σ Lipschitz, the monotone Euler–Maruyama scheme is stochastically increasing.

Hence the Knothe–Rosenblatt rearrangement is optimal for μ^h, ν^h.

Proof of main result

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Theorem [R., Szölgyenyi '24]

Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\right], \quad \text{with } B = W.$$

1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

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Assumptions

(A) discontinuous drift with exponential growth (time-homog.);

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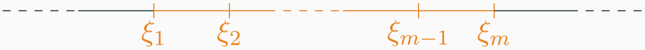
Transformation-based semi-implicit Euler scheme

Assumption (A)

Drift $b\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions piecewise:

- absolute continuity
- one-sided Lipschitz condition
- two-sided local Lipschitz condition
- exponential growth



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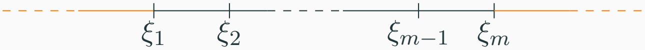

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Diffusion $\sigma\colon \mathbb{R} \to [0, \infty)$ satisfies

- global Lipschitz condition
- $\sigma(\xi_k) \neq 0$, for $k \in \{1, \ldots, m\}$ — no uniform ellipticity

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Under Assumption (A) , the scheme is constructed as follows:

1. Apply the transformation G from [Leobacher, Szölgyenyi '17] to (SDE),

$$Z = G(X)$$

$$\mathrm{d}Z_t = \tilde{b}(Z_t)\mathrm{d}t + \tilde{\sigma}(Z_t)\mathrm{d}W_t$$

- $\tilde{b}$ one-sided Lipschitz, locally Lip., a.c., exponential growth,
- $\tilde{\sigma}$ Lipschitz

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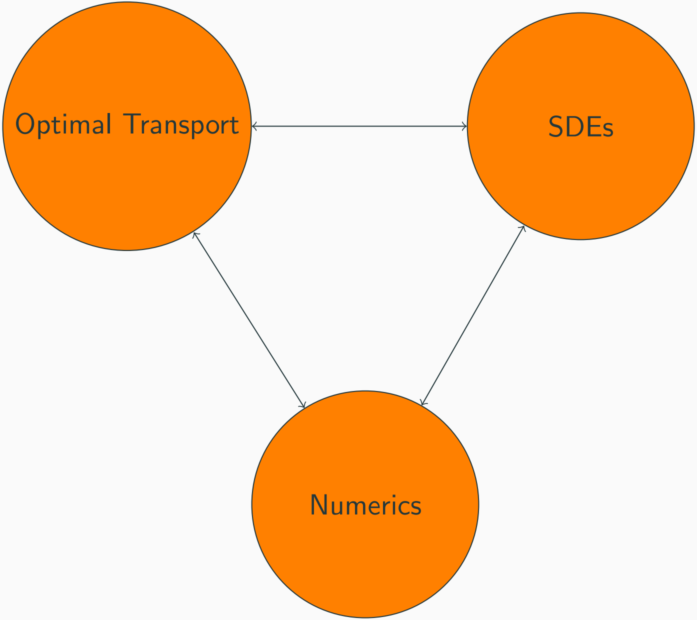
Theorem [R., Szölgyenyi '24]

Let (b, σ) satisfy Assumption (A). Then (SDE) admits a unique strong solution and, for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\Big[|X_T - X_T^h|^p\Big]^{\frac{1}{p}} \leq \begin{cases} C_p h^{\frac{1}{2}}, & p \in [1,2], \\ C_p h^{\frac{1}{p(p-1)}}, & p \geq 2. \end{cases}$$

Ingredients



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Let (b, σ) and $(\bar{b}, \bar{\sigma})$ each satisfy one of assumptions (A), (B), (C).
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Synchronous coupling solves general bicausal transport problem

Summary

- We compute adapted Wasserstein distance between SDEs with irregular coefficients
- We prove strong convergence rates for a numerical scheme for SDEs with discontinuous and exponentially growing drift

References: