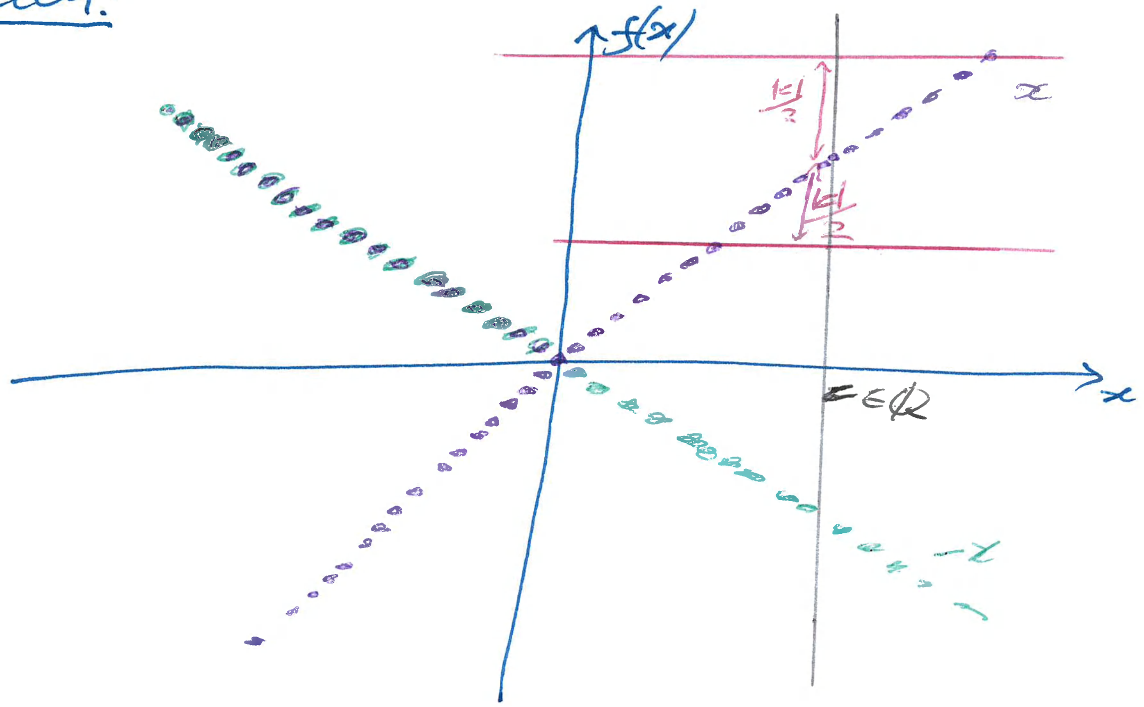
$f:\mathbb{R}\to\mathbb{R}$, $f(x) = \begin{cases} x, & x\in\mathbb{Q}, \\ -x, & x\in\mathbb{R}\setminus\mathbb{Q}. \end{cases}$

Show that, for any $c\in\mathbb{R}\setminus\{0\}$, f is <u>not</u> continuous at c.

Sketch:



In any interval $(c-\delta, c+\delta)$, there is an irrational $x\in\mathbb{R}\setminus\mathbb{Q}$, so $f(x) = -x$.
With $\varepsilon = \frac{|c|}{2}$, $f(x) = -x \notin (f(c)-\varepsilon, f(c)+\varepsilon)$.

KEY FACT: $\mathbb{Q}$ is dense in $\mathbb{R}$.

This means: $\forall x\in\mathbb{R}, \forall\delta>0, \exists q\in\mathbb{Q}$ s.t. $q\in(x-\delta, x+\delta)$.

Equivalently: $\forall x\in\mathbb{R}$, $\exists$ sequence $(x_n)_{n\in\mathbb{N}}\subseteq\mathbb{Q}$ s.t. $x_n \xrightarrow[n\to\infty]{} x$.

[Similarly, $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$.]

Proof 1: (use sequential definition of continuity)

Let $c \in \mathbb{R} \setminus \{0\}$.

Assume for contradiction that f is continuous at c.

Case 1: Suppose that $c \in \mathbb{R} \setminus \mathbb{Q}$.

Then $f(c) = -c$.

By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ s.t. $x_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$.

By continuity of f,

$$\lim_{n\to\infty} f(x_n) = f(c) = -c.$$

But $x_n \in \mathbb{Q}$, $\forall n \in \mathbb{N} \Rightarrow f(x_n) = x_n$, $\forall n \in \mathbb{N}$.

So

$$\lim_{n\to\infty} x_n = -c \neq c.$$

This contradicts uniqueness of the limit.

Case 2: Suppose that $c \in \mathbb{Q}$. Then $f(c) = c$.

By density of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$, $\exists (y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}\setminus\mathbb{Q}$ s.t. $\lim_{n\to\infty} y_n = c$.

By continuity of f,

$$\lim_{n\to\infty} f(x_n) = f(c) = c.$$

But $y_n \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow f(y_n) = -y_n$, $\forall n \in \mathbb{N}$.

So $\lim_{n\to\infty} (-y_n) = c \Rightarrow \lim_{n\to\infty} y_n = -c \neq c$. Contradiction $\square$

Proof 2: (use ε-δ definition - see sketch)

Let $c \in \mathbb{R} \setminus \{0\}$.

Assume for contradiction that f is continuous at c.

Fix $\varepsilon = \frac{|c|}{2}$.

Then $\exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \frac{|c|}{2}$.

NOT PART OF THE PROOF

Idea: Take $x_1 \in \mathbb{Q}$, $x_2 \in \mathbb{R} \setminus \mathbb{Q}$, "close" to c.

[can do this because of density]

① Continuity of f $\Rightarrow$ $f(x_1)$ "close" to $f(c)$

& $f(x_2)$ "close" to $f(c)$

$\Rightarrow$ $f(x_1)$ "close" to $f(x_2)$.

② Definition of f $\Rightarrow$ $f(x_1) = x_1 \simeq c$

& $f(x_2) = -x_2 \simeq -c$

$\Rightarrow |f(x_1) - f(x_2)| \simeq 2|c|$ ("not close")

Defining "close" carefully, ① contradicts ②

Let $\delta_0 := \min\{\delta, \frac{|c|}{2}\}$ ← [Making δ smaller to get x_1 "close enough" to x_2.]

By $\boxed{\text{density of } \mathbb{Q} \text{ in } \mathbb{R}}$, can choose $x_1 \in \mathbb{Q}$ s.t. $|x_1 - c| < \delta_0$.

By $\boxed{\text{density of } \mathbb{R} \setminus \mathbb{Q} \text{ in } \mathbb{R}}$, can choose $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $|x_2 - c| < \delta_0$.

By continuity of f at c,

$$|f(x_1) - f(c)| < \frac{|c|}{2}$$

$$\& \ |f(x_2) - f(c)| < \frac{|c|}{2}.$$

Using the triangle inequality,

$$(*) \quad \begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - f(c) - (f(x_2) - f(c))| \\ &\leq |f(x_1) - f(c)| + |f(x_2) - f(c)| \\ &< \frac{|c|}{2} + \frac{|c|}{2} = |c|. \end{aligned}$$

[$f(x_1)$ is "close" to $f(x_2)$]

Now to get a contradiction, look at

$$|f(x_1) - f(x_2)| = |x_1 - (-x_2)| \quad \text{by definition of } f$$
$$= |x_1 + x_2|.$$

Case 1: If $c > 0$, then $|c| = c$ and

$c - \delta_0 < x_1 < c + \delta_0 \Rightarrow \frac{c}{2} < x_1 < \frac{3c}{2}$.

Similarly, $x_2 > \frac{c}{2} > 0$.

So $|x_1 + x_2| = x_1 + x_2 > \frac{c}{2} + \frac{c}{2} = c = |c|$.

Case 2: If $c < 0$, then $|c| = -c$ and

$c - \delta_0 < x_1 < c + \delta_0 \Rightarrow \frac{3c}{2} < x_1 < \frac{c}{2} < 0$.

Similarly, $x_2 < \frac{c}{2} < 0$.

So $|x_1 + x_2| = -(x_1 + x_2) > -\left(\frac{c}{2} + \frac{c}{2}\right) = -c = |c|$.

In either case,

$$(**) \quad \boxed{|f(x_1) - f(x_2)| = |x_1 + x_2| > |c|.}$$

[$f(x_1)$ "not close" to $f(x_2)$]

Contradiction between $(*)$ & $(**)$ $\blacksquare$