Discontinuous Galerkin Methods for Nonlinear Wave Interactions

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Motivation and examples

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- Conservative vectorial modified Korteweg-de Vres: $\vec{u}_t + \frac{1}{2}\left(|\vec{u}|^2 \vec{u}\right)_x + \vec{u}_{xxx}$

http://createinfife.co.uk/wp-content/uploads/2016/01/Wave-new.jpg

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and can be generalised to the

- Elastodynamics equations: $u_{tt} - \left(W'(u_x)\right)_x = 0$
- Incompressible Euler equations $\vec{u}_t + \vec{u}\cdot\nabla\vec{u} = -\nabla p, \quad \nabla\cdot\vec{u} = 0$ (Olver, 1982, Math. Ana. Comp.)

Existing schemes

KdV has been thoroughly investigated numerically, see (Sanz-Serna, 1982, J. Comput. Phys.) for a finite difference scheme, and (Karakashian and Makridakis, 2015, Math. Comp.) for a discontinuous finite element scheme.
The physical behaviour of solutions is also well understood for this problem, so in addition performing numerical analysis we can verify solution behaviour in the eyeball norm.

http://www.douglasbaldwin.com/Figures/A12_large_120502_DSC0213.jpg

Main example

Throughout this talk we will focus on the Korteweg-de Vres (KdV) equation,

$$u_t + 6uu_x + u_{xxx} = 0,$$

where $u = u(t, x)$ for $t \in [0, T]$ and $x \in \Omega := [0, 1)$ periodic.

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where $u = u(t,x)$ for $t \in [0, T]$ and $x \in \Omega := [0,1)$ periodic. This equation admits soliton solutions in the form of linear combination (asymptotically) of

$$u(t,x) = \frac{1}{2}\text{csech}^2\left(\frac{\sqrt{c}}{2}(x - ct - p)\right),$$

where c denotes wavespeed, and p denotes the initial position of the soliton.

Hamiltonian structure

Let $u = u(t, x)$, where $t \in [0, T]$ and $x \in \Omega := [0, 1)$ is periodic. Further let the Hamiltonian $H[u] = H(u, u_x, u_{xx}, ...)$, then a Hamiltonian PDE can be written in the form

$$u_t = -B \frac{\delta}{\delta u} H(u)$$

where B is a differential operator and $\frac{\delta}{\delta u}$ denotes the variational derivative.

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where B is a differential operator and $\frac{\delta}{\delta u}$ denotes the variational derivative. We introduce the auxiliary variable

$$v := \frac{\delta}{\delta u} H[u],$$

allowing us to rewrite our Hamiltonian PDE as the system

$$u_t + Bv = 0, \qquad v - \frac{\delta}{\delta u} H[u] = 0.$$

Hamiltonian conservation

Recall the system form of a general Hamiltonian PDE

$$u_t + Bv = 0, \qquad v - \frac{\delta}{\delta u} H[u] = 0.$$

Multiplying the system with v and u_t respectively and integrating over the domain

$$\int_\Omega v u_t + vBv \,\mathrm{d}x = 0, \qquad \int_\Omega u_t v - u_t \frac{\delta}{\delta u} H[u] \,\mathrm{d}x = 0.$$

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which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega H[u] \, \mathrm{d}x = 0,$$

so the Hamiltonian is conserved by the continuous system.

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KdV as a (bi-)Hamiltonian PDE

Recall a Hamiltonian PDE has the form

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We can write KdV

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Note that KdV has two Hamiltonian formulations, the second is

$$B_2 = \frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x, \qquad H_2 = \frac{u^2}{2}.$$

Is our assumption true?

To show conservation of the Hamiltonian we assumed

$$\int_\Omega vBv \, \mathrm{d}x = 0.$$

Recall KdV can be written in the Hamiltonian forms

$$H = u^3 - \frac{u_x^2}{2}, \qquad B = \frac{\partial}{\partial x},$$
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Yes!

Discretisation for KdV

In time we use method of lines with time step τ, and in space we employ a discontinuous finite element method. For simplicity we consider uniform step sizes in time and space.

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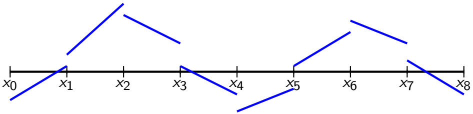

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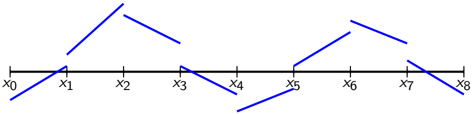

We will denote spatially discretisation by capitalisation, and temporal discretisation by superscripts, i.e.,

- $u(t, x)$ is a continuous function
- $U(t, x)$ is a spatially discrete function
- $u^n(x)$ is a temporally discrete function
- $U^n(x)$ is a fully discrete function

Some definitions

Notation for interactions between elements

We define the jump over the node m to be

$$[\![W_m]\!] := \lim_{x \nearrow x_m} W(x) - \lim_{x \searrow x_m} W(x)$$

and the average to be

$$\{W_m\} := \frac{1}{2}\left(\lim_{x \nearrow x_m} W(x) + \lim_{x \searrow x_m} W(x)\right).$$

We will write the sum of the product of these operators over all nodes as $\int_{\mathcal{E}} [\![W]\!]\{W\}\ \mathrm{d}S$ for consistency with our finite element notation.

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Operator for discontinuous first derivative (Di Pietro and Ern, 2010, Math. Comput.)

Define $G : \mathbb{V}_q \to \mathbb{V}_q$ such that for $W \in \mathbb{V}_q$

$$\int_\Omega G(W)\phi \, \mathrm{d}x = \int_\Omega W_x \phi \, \mathrm{d}x - \int_{\mathcal{E}} [\![W]\!]\{\phi\} \, \mathrm{d}S \qquad \forall \phi \in \mathbb{V}_q.$$

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Operator for discontinuous second derivative (Wheeler, 1978, SIAM J. Comput. Phys.)

Define $a_h : \mathbb{V}_q \times \mathbb{V}_q \to \mathbb{V}_q$ such that for $W \in \mathbb{V}_q$
$$\begin{aligned} a_h(W, \psi) = \int_\Omega W_x \psi_x\ \mathrm{d}x &- \int_{\mathcal{E}} [\![W]\!]\{\psi_x\} + [\![\psi]\!]\{W_x\}\ \mathrm{d}S \\ &+ \int_{\mathcal{E}} \frac{\sigma}{h}[\![W]\!][\![\psi]\!]\ \mathrm{d}S, \end{aligned}$$

$\forall \psi \in \mathbb{V}_q$ for $\sigma \in \mathbb{R}$ sufficiently large. This bilinear form is known as the symmetric interior penalty method.

Semi-discrete numerical scheme

Given initial data at $t = 0$, we seek $U(t,x), V(t,x) \in \mathbb{V}_q$ such that

$$\int_\Omega (U_t + G(V))\phi \,\mathrm{d}x = 0 \qquad \forall \phi \in \mathbb{V}_q$$

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This scheme conserves a discrete mass

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega U \, \mathrm{d}x = 0,$$

and energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(U^3 - \frac{1}{2} a_h(U, U) \right) \mathrm{d}x = 0.$$

A semi-discrete a priori bound

The semi-discrete scheme satisfies the bound

$$||U - u||_{H_h^1} \leq Ch^q.$$

Fully discrete numerical scheme

Let U^n, V^n be given, then the fully discrete scheme is given by seeking $U^{n+1}, V^{n+1} \in \mathbb{V}_q$ such that

$$\int_\Omega \left(\frac{U^{n+1} - U^n}{\tau} + G\left(V^{n+\frac{1}{2}}\right)\right) \phi \, \mathrm{d}x = 0 \qquad \forall \phi \in \mathbb{V}_q$$

$$\int_\Omega \left(V^{n+\frac{1}{2}} - ((U^{n+1})^2 + U^{n+1}U^n + (U^n)^2)\right) \psi \, \mathrm{d}x + a_h(U^{n+\frac{1}{2}}, \psi) = 0 \qquad \forall \psi \in \mathbb{V}_q,$$

where $U^{n+\frac{1}{2}} := \frac{U^{n+1}+U^n}{2}$.

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where $U^{n+\frac{1}{2}} := \frac{U^{n+1}+U^n}{2}$.
The fully discrete scheme conserves the discrete mass

$$M^n := \int_\Omega U^n \, \mathrm{d}x$$

and energy

$$H^n := \int_\Omega (U^n)^3 \, \mathrm{d}x - \frac{1}{2} a_h(U^n, U^n),$$

i.e.,

$$M^{n+1} = M^n \qquad H^{n+1} = H^n.$$

Fully discrete a priori bound

The fully discrete scheme satisfies the a priori bound

$$||U^{n+1} - u(t_{n+1})||_{H_h^1} \leq C\left(h^q + \tau^2\right).$$

Conserved quantities

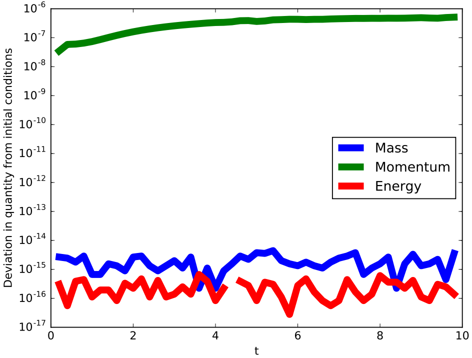

Figure: The deviation in discrete mass, momentum and energy from their value at $t = 0$. We chose $x_l = 0, x_r = 40$, $h = 0.8$ and $\tau = 0.005$ for the 1-soliton solution to KdV $\frac{1}{2}\mathrm{sech}^2\left(\frac{1}{2}(x - 20 - t)\right)$.

Convergence rates

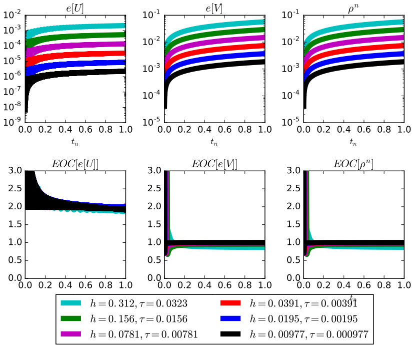

Figure: $h = \tau$

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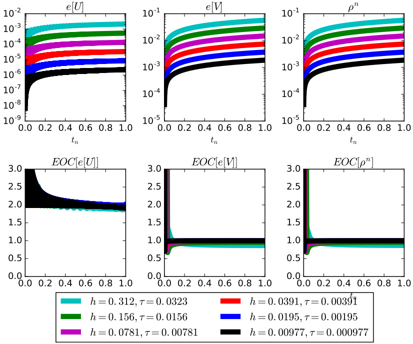

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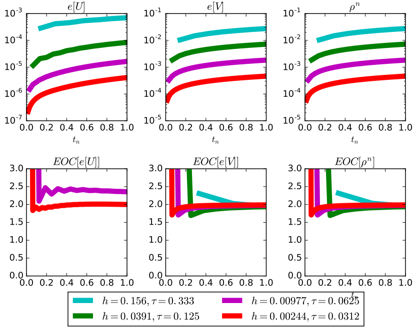

Figure: $h = \tau^{\frac{1}{2}}$

KdV soliton simulations

cvmKdV

Let $\vec{u}(t,x) \in \mathbb{R}^D$ be sufficiently smooth for $t \in [0, T]$ and $x \in \Omega := [x_l, x_t)$, then we define conservative vectorial modified KdV *(vmKdV)* by

$$\vec{u}_t + \frac{1}{2}\left(|\vec{u}|^2\,\vec{u}\right)_x + \vec{u}_{xxx} = 0.$$

There are many different forms of cvmKdV but we choose this one because of the simple Hamiltonian structure.

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There are many different forms of cvmKdV but we choose this one because of the simple Hamiltonian structure. Note that cvmKdV conserves the Hamiltonians

$$H_1 = \frac{1}{4}\,|\vec{u}|^4 - |\vec{u}_x|^2$$
$$H_2 = |\vec{u}|^2\,.$$

cvmKdV as a system

With a view to conserve the energy of our problem we introduce the first variation of the energy $\frac{1}{4}\left|\vec{u}\right|^4 - \left|\vec{u}_x\right|^2$ as the auxiliary variable

$$\vec{w} = \frac{1}{2}\left|\vec{u}\right|^2 \vec{u} + \vec{u}_{xx},$$

leading us to write vmKdV as

$$\begin{aligned} \vec{u}_t + \vec{w}_x &= 0 \\ \vec{w} - \frac{1}{2}\left|\vec{u}\right|^2 \vec{u} - \vec{u}_{xx} &= 0. \end{aligned}$$

Spatially discrete scheme for cvmKdV

Seek $\vec{U}, \vec{W} \in (\mathbb{V}_q)^D$ such that

$$\int_\Omega \left(\vec{U}_t + G\left(\vec{W}\right)\right) \cdot \vec{\phi} \,\mathrm{d}x = 0$$
$$\int_\Omega \left(\vec{W} - \frac{1}{2}\left|\vec{U}\right|^2 \vec{U}\right) \cdot \vec{\psi} \,\mathrm{d}x + a_h\left(\vec{U}, \vec{\psi}\right) = 0,$$

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for all $\vec{\phi}, \vec{\psi} \in (\mathbb{V}_q)^D$. This scheme preserves the energy of vmKdV, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{4}\left|\vec{U}\right|^4 - \left|\vec{U}_x\right|^2 \,\mathrm{d}x = 0.$$

Numerical simulations

In these numerical experiments we are fix $D = 2$.

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$$u_1(t,x) = \frac{2a_1\mu}{\cosh\xi + a_0 \sinh\xi}$$
$$u_2(t,x) = \frac{2a_2\mu}{\cosh\xi + a_0 \sinh\xi}$$

where $\vec{u} = (u_1, u_2)^T$ solves our spatially discrete scheme for vmKdV.

Convergence rates

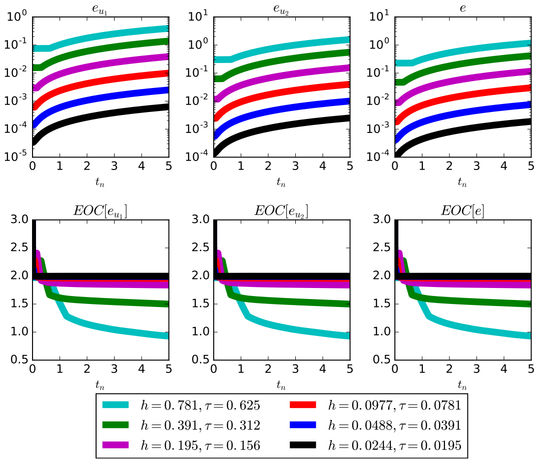

Figure: $h = \tau$

cvmKdV soliton simulations

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