Bicausal optimal transport between the laws of SDEs

Benjamin A. Robinson (University of Klagenfurt)

NAASDE, Będlewo — September 23, 2024

Supported by Austrian Science Fund (FWF) projects (Y782-N25), (P35519), (P34743).

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Joint work with

Julio Backhoff-Veraguas
University of Vienna

Sigrid Källblad
KTH Stockholm

Comparing stochastic models

Aim: Compute a measure of model uncertainty

E.g. optimal stopping:

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[L(\tau, \omega)]$$

$\mathbb{P}$ law of solution of SDE

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$\mathbb{P}$ law of solution of SDE

Want:

- Appropriate topology on laws of stochastic processes
- Distance we can actually compute

Main result

$b, \bar{b} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon [0,T] \times \mathbb{R} \to [0,\infty)$,

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad X_0 = x_0,$$
$$\mathrm{d}\bar{X}_t = \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t, \quad \bar{X}_0 = x_0.$$

$\mu = \mathrm{Law}(X)$, $\nu = \mathrm{Law}(\bar{X})$

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Theorem [Backhoff-Veraguas, Källblad, R. '24]

For "sufficiently nice" coefficients, we can compute an "appropriate distance" d_p, $p \geq 1$,

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Theorem [Backhoff-Veraguas, Källblad, R. '24]

For "sufficiently nice" coefficients, we can compute an "appropriate distance" d_p, $p \geq 1$, by

$$d_p(\mu,\nu)^p = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\right], \quad \text{with } B = W.$$

Application

E.g. optimal stopping:

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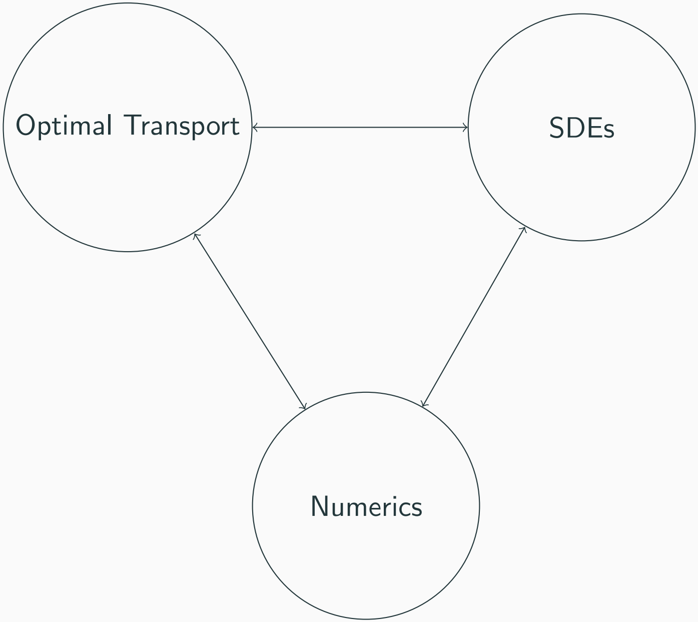
Theorem
[Acciaio, Backhoff-Veraguas, Zalashko '19], [R. Szölgyenyi '24]

$\omega \mapsto L(t, \omega)$ Lipschitz on $(\Omega, \|\cdot\|_{L^p})$ unif. in $t \in [0, T]$

$\Rightarrow$

$\mathbb{P} \mapsto v(\mathbb{P})$ Lipschitz on $(\mathcal{P}_p(\Omega), d_p)$

Ingredients



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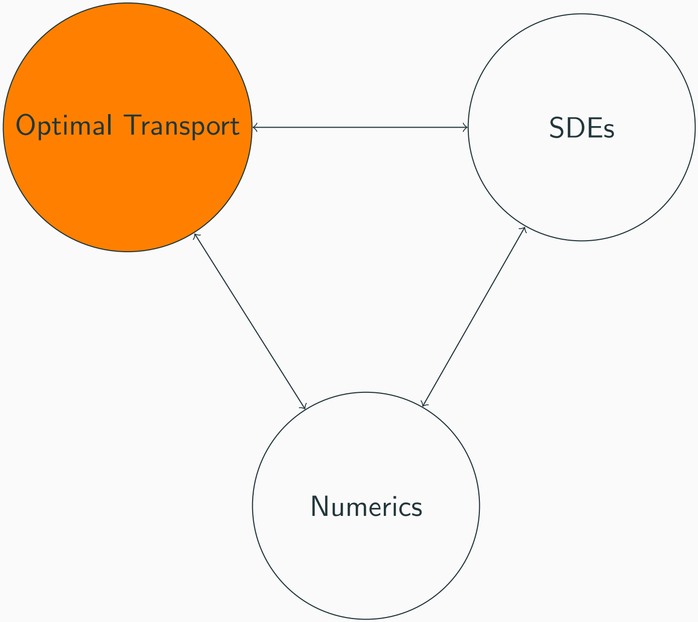


Optimal transport

Probability measures $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^N)$, $p \in [1, \infty)$.

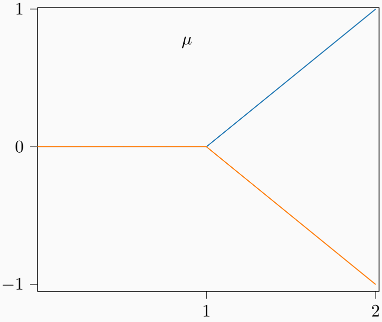
Find Wasserstein distance

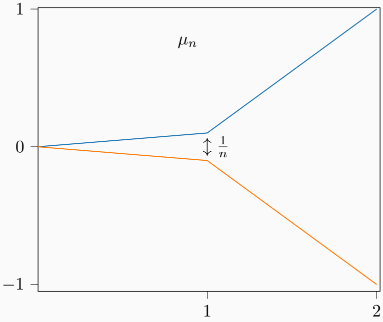
$$\mathcal{W}_p^p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|^p]$$

Metrises weak topology on $\mathcal{P}_p(\mathbb{R}^N)$

Example

[Aldous '81], [Backhoff-Veraguas, Bartl, Beiglböck, Eder '20]





$$V := \sup_{\tau} \mathbb{E}^{\mu}[X_\tau] \qquad V_n := \sup_{\tau} \mathbb{E}^{\mu_n}[X_\tau]$$

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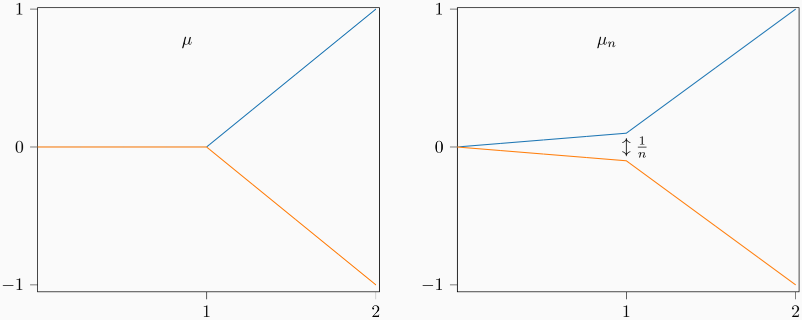

$$V := \sup_\tau \mathbb{E}^\mu[X_\tau] = 0 \qquad V_n := \sup_\tau \mathbb{E}^{\mu_n}[X_\tau] \approx \frac{1}{2}$$

$V_n \not\to V$ but $\mu_n \rightharpoonup \mu$

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$$T(X) = (T_1(X_1,\dots,X_N),\dots,T_N(X_1,\dots,X_N))$$

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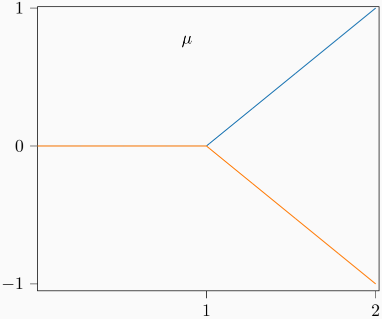

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and symmetric condition.

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Metrises adapted weak topology on $\mathcal{P}_p(\mathbb{R}^N)$

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Acciaio, Aldous, Backhoff-Veraguas, Bartl, Beiglböck, Bion-Nadal, Eder, Hellwig, Källblad, Pammer, Pflug, Pichler, Talay, Zalaschko, ...

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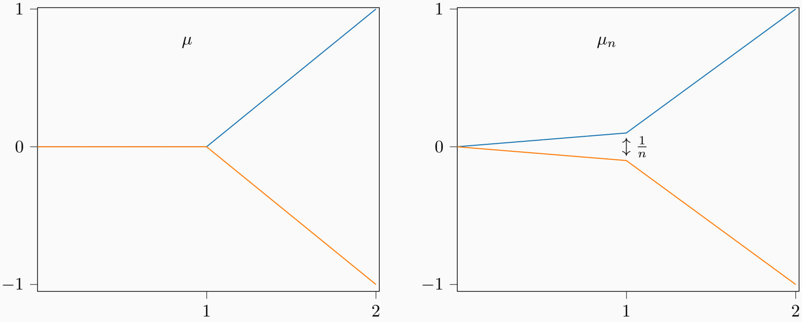

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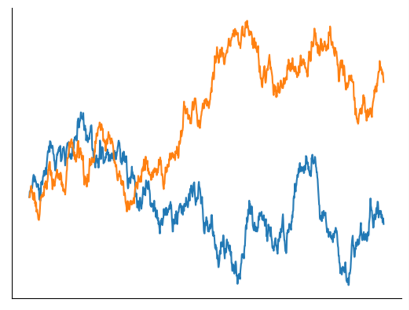
$$V_n \not\to V \quad \text{and} \quad \mathcal{AW}_p^p(\mu_n, \mu) \not\to 0$$

Continuous time

Similar definition of Wasserstein distance in continuous time w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$

$$\mu,\nu \in \mathcal{P}(\Omega) \quad \rightsquigarrow \quad \mathcal{W}_p^p(\mu,\nu) := \inf_{\pi\in \mathrm{Cpl}(\mu,\nu)} \mathbb{E}^\pi\left[\int_0^T |\omega_t - \bar{\omega}_t|^p \mathrm{d}t\right]$$

$$\mathrm{Cpl}(\mu,\nu) = \{\pi = \mathrm{Law}(X,Y)\colon\ X\sim\mu, Y\sim\nu\}$$



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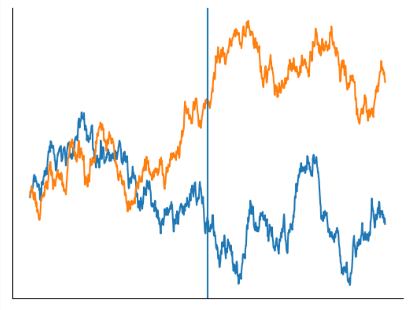

Continuous time

Similar definition of adapted Wasserstein distance in continuous time w.r.t. L^p norm on $\Omega := C([0,T],\mathbb{R})$

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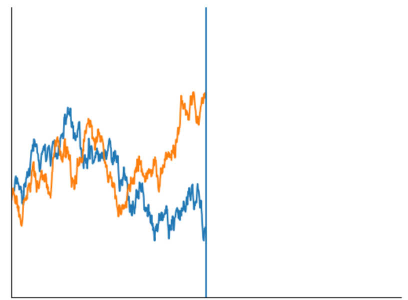
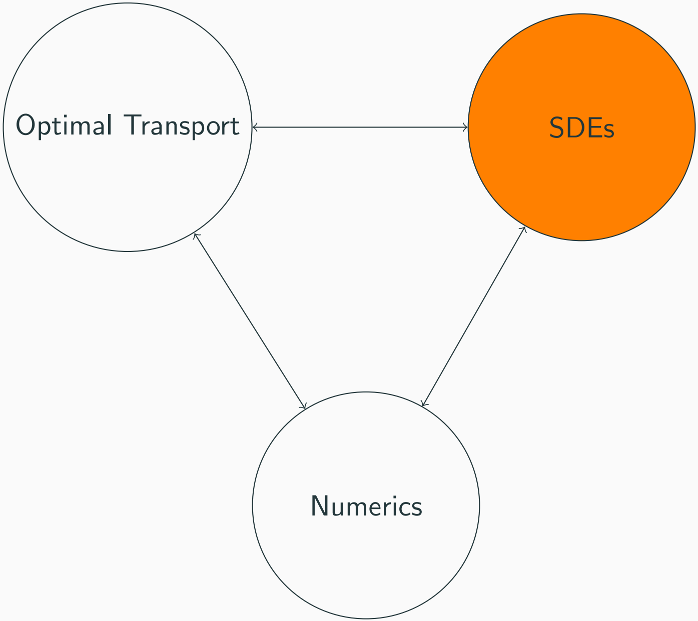
$$\mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu) = \{\pi \in \mathrm{Cpl}(\mu,\nu)\colon\ \pi \text{ bicausal}\}$$

"$\mathcal{F}_t^X$ independent of $\mathcal{F}_T^Y$ conditional on F_t^Y" and vice versa



Ingredients



Main result

$b, \bar{b} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon [0,T] \times \mathbb{R} \to [0,\infty)$,

$$\begin{aligned} \mathrm{d}X_t &= b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t, \quad X_0 = x_0, \\ \mathrm{d}\bar{X}_t &= \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t, \quad \bar{X}_0 = x_0. \end{aligned}$$

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Theorem [Backhoff-Veraguas, Källblad, R. '24]

For "sufficiently nice" coefficients, we can compute the adapted Wasserstein distance by

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Coupling SDEs

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
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Theorem [Backhoff-Veraguas, Källblad, R. '24]

Optimising over bicausal couplings $\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu, \nu)$

$\Leftrightarrow$

Optimising over correlations between B, W

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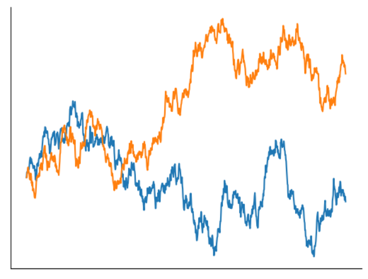
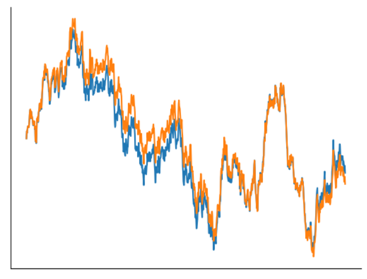
Product coupling

B, W independent



Synchronous coupling

Choose the same driving Brownian motion $B = W$.



Proof of main result

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1. Discretise SDEs;
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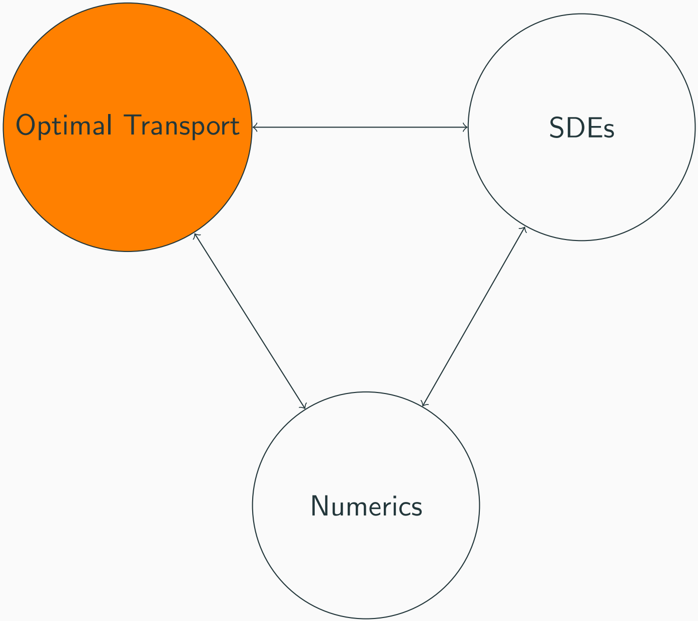


Classical optimal transport on $\mathbb{R}$

Probability measures $\mu, \nu \in \mathcal{P}_p(\mathbb{R})$, $p \in [1, \infty)$.

Wasserstein distance

$$\mathcal{W}_p^p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|^p]$$

is attained by monotone rearrangement

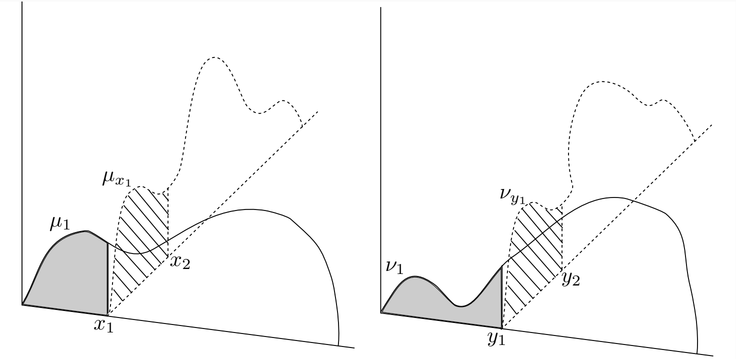
$$X = F_X^{-1}(U), \quad Y = F_Y^{-1}(U), \quad U \text{ uniform}$$

Key result in discrete time

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \mathrm{Cpl}_{\mathrm{bc}}(\mu,\nu)} \mathbb{E}^\pi \left[\sum_{n=1}^N |X_n - Y_n|^p \right]$$

Knothe–Rosenblatt rearrangement

— generalisation of monotone rearrangement to N time steps



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Knothe–Rosenblatt rearrangement

$X_1 = F_{\mu_1}^{-1}(U_1)$, $Y_1 = F_{\nu_1}^{-1}(U_1)$,

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Knothe–Rosenblatt rearrangement

$X_1 = F_{\mu_1}^{-1}(U_1)$, $Y_1 = F_{\nu_1}^{-1}(U_1)$, and for $k \in \{2, \ldots, N\}$

$$X_k = F^{-1}_{\mu_{X_1,\ldots,X_{k-1}}}(U_k), \quad Y_k = F^{-1}_{\nu_{Y_1,\ldots,Y_{k-1}}}(U_k)$$

$U_1, \ldots, U_N$ independent uniform

$$\pi_{\mathrm{KR}}(\mu, \nu) := \mathrm{Law}(X, Y)$$

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$U_1, \dots, U_N$ independent uniform

$$\pi_{\mathrm{KR}}(\mu, \nu) := \mathrm{Law}(X, Y)$$

Theorem [Rüschendorf ’85]

For μ, ν Markov and stochastically comonotone, the Knothe–Rosenblatt rearrangement is optimal.

Proof of main result

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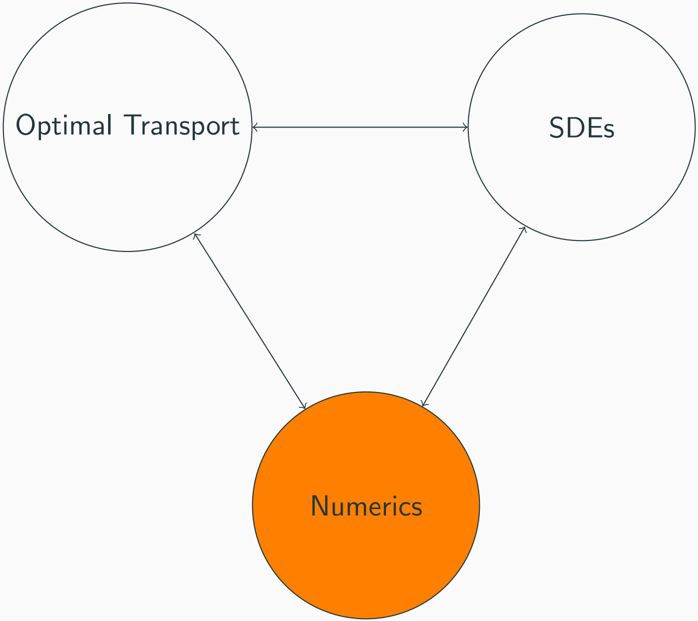

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t$$

Euler scheme

$$\begin{aligned} X_0^h &= X_0, \\ X_t^h &= X_{kh}^h + b(X_{kh})(t - kh), \quad t \in (kh, (k+1)h]. \end{aligned}$$

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t$$

Euler–Maruyama scheme

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh) + W_t - W_{kh}, \quad t \in (kh, (k+1)h].$$

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Write $X_k^h := X_{kh}^h$ and $\mu^h = \mathrm{Law}((X_k^h)_k)$.

Remark

$X_k^h \mapsto X_{(k+1)}^h$ is increasing if b is Lipschitz, $h \ll 1$

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

Monotone Euler–Maruyama scheme

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh})(t-kh) + \sigma(X_{kh})(W_t^h - W_{kh}^h),\ t \in (kh,(k+1)h].$$

$$W_t^h - W_{kh}^h = W_{t\wedge\tau_k^h} - W_{kh}, \quad \tau_k^h \coloneqq \inf\{t > kh\colon\ |W_t - W_{kh}| > A_h|\}$$

Cf. [Milstein, Repin, Tretyakov ‘02], [Liu, Pagès ‘22], [Jourdain, Pagès ’23]

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Lemma [Backhoff-Veraguas, Källblad, R. '24]

For b, σ Lipschitz, the monotone Euler–Maruyama scheme is stochastically increasing.

Hence the Knothe–Rosenblatt rearrangement is optimal for μ^h, ν^h.

Moreover, $\pi_{\mathrm{KR}}(\mu^h, \nu^h) = \mathrm{Law}(X^h, \bar{X}^h)$, $B = W$.

Proof of main result

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
$$\mathrm{d}\bar{X}_t = \bar{b}_t(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_t(\bar{X}_t)\mathrm{d}W_t,\ \bar{X}_0 = x \ \rightsquigarrow\ \mathrm{Law}(\bar{X}) = \nu$$

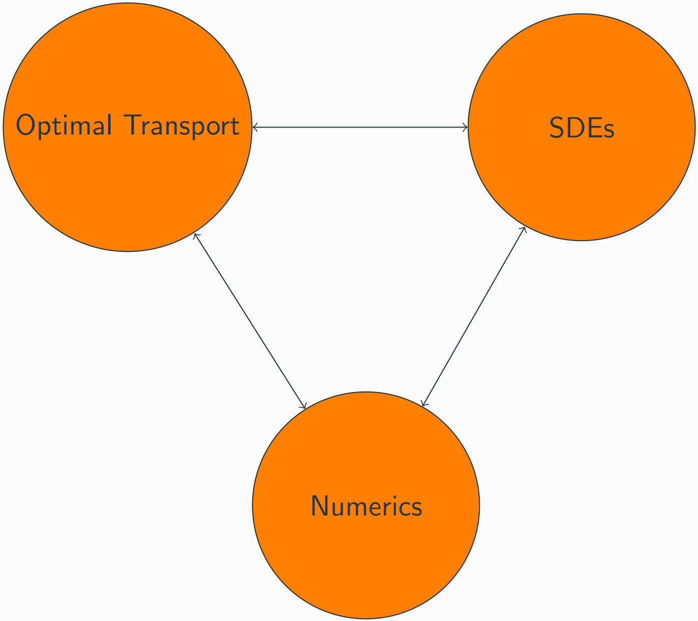
Theorem [Backhoff-Veraguas, Källblad, R. 24]

For "sufficiently nice" coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \quad \text{with } B = W.$$

1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

Ingredients



Main result

Assumptions

- Continuous coefficients with linear growth
- Strong existence and uniqueness

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Main Theorem [Backhoff-Veraguas, Källblad, R. 24]

The adapted Wasserstein distance is given by

$$\mathcal{AW}_p^p(\mu,\nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \quad \text{with } B = W.$$

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Synchronous coupling solves general bicausal transport problem

Extensions

- **Irregular coefficients** [R. Szölgyenyi '24]
  - discontinuous drift with exponential growth
  - bounded measurable drift
- **Higher dimensions**
  - counterexamples [Backhoff-Veraguas, Källblad, R. '24]
  - different techniques needed
- **More general processes** (work in progress...)
  - jump-diffusions, McKean–Vlasov equations, ...

Summary

- Study distance between stochastic processes
- Identify optimal bicausal coupling of SDEs
- Exploit properties of numerical approximations of SDEs

References: