# PHYS 512: Assignment 5

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## 1 Problem 5

### (a)

$$\begin{aligned}
&\sum_{x=0}^{N-1} \exp(-2\pi i k x/N) \\
&= \sum_{x=0}^{N-1} a^x,
\end{aligned} \tag{1}$$

where $a = \exp(-2\pi i k/N)$. We know this has a closed form (from WolframAlpha):

$$\begin{aligned}
\sum_{x=0}^{N-1} a^x &= \frac{1-a^N}{1-a} \\
&= \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}
\end{aligned} \tag{2}$$

As required.

### (b)

We know $e^{-iy} = \cos(y) - i\sin(y)$. Therefore:

$$\frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)} = \frac{1-\cos(2\pi k)+i\sin(2\pi k)}{1-\cos(2\pi k/N)+i\sin(2\pi k/N)} \tag{3}$$

If we plug in $k=0$, we see that we will get an indeterminate fraction of the form $\frac{0}{0}$. To avoid this, we must use l'Hopital's rule:

$$\begin{aligned}
\lim_{k\to 0} \frac{1-\cos(2\pi k)+i\sin(2\pi k)}{1-\cos(2\pi k/N)+i\sin(2\pi k/N)} &= \lim_{k\to 0} \frac{2\pi\sin(2\pi k)+i2\pi\cos(2\pi k)}{\frac{2\pi}{N}\sin(2\pi k/N)+i\frac{2\pi}{N}\cos(2\pi k/N)} \\
&= \frac{i2\pi}{i\frac{2\pi}{N}} \\
&= N
\end{aligned} \tag{4}$$

where in the first line I take the derivative of the function with respect to $k$, and in the second line I plug in $k=0$ and simplify to obtain $N$ as required.

Next, consider once more the function in its sinusoidal form:

$$\frac{1-\cos(2\pi k)+i\sin(2\pi k)}{1-\cos(2\pi k/N)+i\sin(2\pi k/N)}$$

In order for this to be zero, we need the numerator to be equal to zero, i.e. both the real *and* imaginary parts. Let's start with the real part:

$$\begin{aligned} 1 - \cos(2\pi k) &= 0 \\ \cos(2\pi k) &= 1 \\ 2\pi k &= 2\pi n \\ k &= n \end{aligned} \tag{5}$$

Next, consider the imaginary part:

$$\begin{aligned} \sin(2\pi k) &= 0 \\ 2\pi k &= \pi n \\ k &= \frac{1}{2}n \end{aligned} \tag{6}$$

Since we want both the real and imaginary parts to be zero, we see that we require $k = n$, i.e. $k$ equal to *any* integer.

However, we know that a problem arises when the denominator is equal to zero. Let's investigate that, starting with the real part:

$$\begin{aligned} 1 - \cos(2\pi k/N) &= 0 \\ \cos(2\pi k/N) &= 1 \\ 2\pi k/N &= 2\pi n \\ k &= nN \end{aligned} \tag{7}$$

Now the imaginary part:

$$\begin{aligned} \sin(2\pi k/N) &= 0 \\ 2\pi k/N &= \pi n \\ k &= \frac{1}{2}nN \end{aligned} \tag{8}$$

Problems will only arise if both the real and imaginary parts are equal to zero, which occurs whenever $k = nN$, i.e. whenever $k$ is an integer multiple of $N$.

So, if $k$ is an integer multiple of $N$, the fraction is *not* equal to zero; it is simply undefined. If we want it to be equal to zero, $k$ must be an integer (from Eq. 5) that is not an integer multiple of $N$ (from Eq. 7).