Gaussian Beam-Mode Analysis and Phase-Centers of Corrugated Feed Horns

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Abstract—**The notion of phase-center for a feed-horn is critically examined and, using Gaussian beam-mode analysis, several variously-defined phase-centers, including a maximal-gain phase-center, are investigated in detail for corrugated feed-horns in balanced hybrid mode. Expressions for the locations of these phase centers are derived which are applicable for a wide range of corrugated horn dimensions and for near-field as well as far-field distances.**

I. Introduction

IN MOST multi-reflector antenna systems the individual reflectors have paraboloidal, hyperboloidal or ellipsoidal surfaces. The values of the geometrical parameters for the surface of such a reflector can be chosen so that the reflector would transform an incident beam which has a spherical phase-front with a given radius of curvature at the reflector, R_i say, into an emergent beam which would also have a spherical phase front at the reflector, with radius of curvature R_e, say: the focal length of such a reflector, f, can be defined in terms of the discrete change in phase-front curvature at the reflector, i.e.,

$$1/f = 1/R_i - 1/R_e$$

where, by convention, a positive value for the phase-front curvature is assigned for a diverging beam and a negative value for a converging beam.

Frequently, in designing or computing the performance of a reflector antenna system, the beam from the feed horn is represented as a spherical wave, in which all the phase-fronts are spherical, with a common center of curvature known as the *phase-center* of the beam. A hypothetical horn producing such a beam would be optically matched to the first reflector in an antenna system by placing it with its phase-center at a distance R_i from the reflector.

Real horns do not usually produce beams with spherical phase-fronts however. Only a horn specially shaped so as to give, over the aperture plane, uniform phase and an amplitude distribution with inversion symmetry will give spherical phase-fronts in the *far-field.* No horn gives spherical phase-fronts at near and intermediate distances. Optimized design of a dual or multi-reflector antenna system can require the first reflector to be in the near or intermediate field of the feed-horn. Determining the optimal distance for a horn from the reflector it feeds therefore requires careful consideration. The purpose of this paper is to show how this question can be approached using Gaussian beam-mode analysis [1], [2]. We deal in particular with corrugated feed-horns [3], [4] but the line we take can be extended to cover other types of feed-horn [5].

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Recently we introduced a beam-mode description of the field of a corrugated horn [4] and we shall make full use of that. Any paraxial free-space beam can be represented as a superposition of independently propagating orthogonal modes; the relative amplitudes of the constituent modes are determined by making the superposition fit the field over some cross-sectional plane where the form of the field is known—in the case of the beam of a feed-horn this is the aperture plane. The efficacy of this 'beam-mode' analysis springs from the fact that the phase-fronts in an individual mode *are* all spherical and, moreover, at a given down-beam distance the phase-fronts of all the constituent modes have the same radius of curvature. The transformation of a paraxial beam at a reflector can therefore be readily treated by considering each mode separately. The results we obtain will be applicable for a wide range of horn dimensions. For horns having very small apertures, or very large cone-angles, the results would be quantitatively in error to some extent because the highly divergent beams from such horns would not conform with the paraxial assumption on which beam-mode analysis is based (see the comment following equation 4 below). We shall assume the field amplitude distribution over the aperture of a corrugated horn is that of a balanced hybrid-mode, as appropriate for the frequency band over which such a horn has its optimal characteristics [3].

Analyzing the beam of a horn into beam-modes provides a clear basis for identifying a *beam-mode phase-center* for *each* given down-beam distance because, as we have noted above, the beam-modes' phase-fronts there are spherical with a common center. We shall show that referring to the beam-mode phase-center when placing a horn with respect to a reflector can come close to achieving optimal performance. We also show, however, using the beam-mode analysis, that further optimization might be possible, depending on the criterion of good performance; for this we use, as an example, the criterion of maximized gain for the horn-reflector combination, and we locate the *maximal-gain phase-center* of a corrugated feed-horn for any specified down-beam distance from the reflector. We shall also use beam-mode analysis to locate some of the 'apparent phase-centers' invoked, often uncritically, in the past.

Beam-mode analyses have been used recently to determine optimal locations for the feed-horns in specific systems [13]; the studies reported here provide a general context for understanding what is at stake.

II. Beam-Mode Analysis of the Fields of Corrugated Feed Horns

Any coherent linearly polarized paraxial beam can be decomposed [1] into a superposition of Gauss–Laguerre beam-modes. The beam-mode superposition representing the transverse electric field E_t of an *axially-symmetric* beam, in cylindrical polar co-ordinates (r, z) with the z-axis coincident with the beam-axis, is

$$E_t(r,z) = \sum_p A_p^c\{\exp(-r^2/w^2)\cdot L_p(2r^2/w^2)\cdot w^{-1}\} \cdot \{\exp(-ikr^2/2R)\}\{\exp -i(kz-\omega t-\theta_p)\} \quad (1)$$

where $k \equiv \omega/c$ is the wave-number and $p = 0, 1, 2, 3\cdots$ is the mode number.

A_p^c is the complex amplitude of the p^{th} mode and is independent of r and z. Its values are to be determined by fitting the superposition to the field over some transverse plane for which the form of the field is known. For a horn, this will be the aperture plane.

The first term in curly-brackets shows the form of the variation of the *modulus* of the p^{th} beam-mode over a cross-sectional plane. This is a Gaussian function of r^2 modulated by the p^{th} Laguerre polynomial, L_p (i.e., a power series in r^2 to the r^{2p} term). The *scale* of this variation changes with z through the z-dependence of the beam-width parameter w as given in equation 2 below. This expanding scale of the mode's field distribution as it propagates in the *diffractive spreading* of the beam.

The second term in curly-brackets shows the *variation of the phase* of the beam-mode field over a cross-sectional plane, relative to the on-axis value. The form of this term indicates (in paraxial approximation) a spherical phase-front with radius-of-curvature, R. The value of R varies with propagation distance, z, as shown in (2) below. The fact that R is not linearly dependent on z means that the location of the center-of-curvature of the beam-modes' equi-phase surfaces varies with down-beam distance (a second aspect of the diffractive spreading of the beam).

The third term in curly-brackets gives the *on-axis phase*. The phase-angle θ_p registers an on-axis phase slip of the p^{th} mode relative to a plane-wave phase $(kz - \omega t)$; it varies with down-beam distance as shown in equation 2. This phase-slippage is the third consequence of diffractive spreading.

The z-dependences of the beam-mode parameters w, R and θ_p are:

$$\begin{aligned} w^2 &= w_o^2 + \{2(z-z_o)/kw_o\}^2 \\ R &= (z-z_o) + \left\{(kw_o^2/2)^2 \Big/ (z-z_o)\right\} \\ \theta_p &= (2p+1)\arctan\{2(z-z_o)/kw_o^2\} \end{aligned} \quad (2)$$

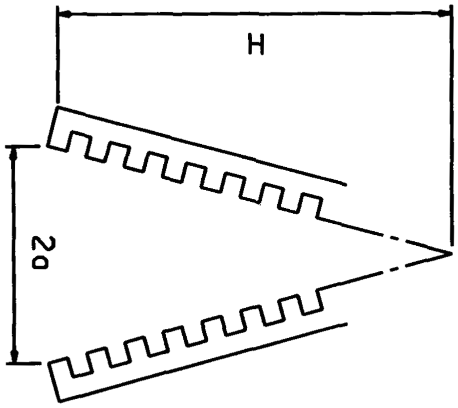

Fig. 1.

where w_o, z_o are constants (the 'beamwaist' width parameter and position) which can be expressed in terms of the values w_a, R_a assigned to w, R in the aperture plane at $z = z_a$ thus

$$\begin{aligned} w_o^2 &= w_a^2 \Big/ \left\{1 + (kw_a^2/2R_a)^2\right\} \\ z_o - z_a &= -R_a \Big/ \left\{1 + (2R_a/kw_a^2)^2\right\} \end{aligned} \quad (3)$$

The values of w_a and R_a are not uniquely determined because the Gauss–Laguerre (GL) functions, $\exp(-r/w^2)\cdot L_p(2r^2/w^2)\exp(-ikr^2/2R)$, for any choice of values for the parameters w and R, form a complete set. That is to say, an arbitrary axially-symmetric function can be fitted by a superposition of GL functions with any choice of values for the parameters w and R. The values which would have to be assigned to the complex amplitudes of the GL functions in order to achieve the fit will depend on the values assigned to w and R, of course. The optimum selection of values is a matter of computational economy because a good selection will minimize the number of GL functions which must be included in the superposition in order to obtain a good fit.

Now the field in the aperture of a corrugated feed-horn in balanced hybrid mode [3] has a spherical phase-front with a radius of curvature equal to the length of the horn, H, from aperture to apex (Fig. 1). The assignment $R_a = H$ is clearly indicated as optimum. The distribution of $|E_t|$ over the aperture plane of a corrugated feed-horn is given by a zero-order Bessel function truncated at its first zero, at $r = a$, the radius of the circular aperture. This approximates closely to a Gaussian function having a width parameter $w = 0.6435a$ [4]. For this reason, assigning the value $0.6435a$ to w_a minimizes the number of GL functions required to give a good fit to the field. The quantity $kw_a^2/2R_a$ appearing in (3) is thus equal to $(0.6435^2)ka^2/2H$ and can now be seen to be a parameter for the dimensions of the horn which we shall denote Δ; the dimensions of the horn are specified by H and Δ. (Δ is thus $(0.6435)^2 s$ where $s \equiv ka^2/2H$ is sometimes referred to as the 'phase error' across the aperture). From (3) we can now write

$$kw_0^2 = 2H\frac{\Delta}{1+\Delta^2} \quad (4)$$

$$z_0 = z_a - H\frac{\Delta^2}{1+\Delta^2} \qquad (5)$$

At this point it is possible to note the range of horn dimensions for which the beam-mode analysis will be subject to little error as a result of non-paraxiality, i.e., for which $kw_0 \leq 6$ (see [1], Fig. 1). From (4) above, this requirement is

$$a/H \leq 0.28 - \left(24.4/k^2a^2\right)$$

The fields of very wide-angle horns with $a/H > 0.28$ radians will therefore depart to some degree from the beam-mode predictions no matter how large the apertures; and those of small-aperture horns with $ka << 9.34$ will do so no matter how small the cone angle. Between these limits there is a wide range of horn dimensions for which the beam-mode analysis introduces very little error due non-paraxiality.

Having assigned the value H for the radius of curvature of the beam-modes' phase-fronts in the aperture plane, we must now see that the beam-modes' on-axis phase-angles are all the same at the aperture, modulo $\pm\pi$. (A superposition of beam-modes all of which have the same phase-front curvatures will give a resultant field with a spherical phase-front only in a cross-section at which the on-axis phases are the same for all the modes, modulo $\pm\pi$). The on-axis phase derives from both the complex amplitude A_p^c and the beam-mode phase-slip angle θ_p. The θ_p depends on p (2) and at the aperture has the value

$$\theta_p(z = z_a) = (2p+1)\cdot \arctan\left\{\frac{2(z_a - z_0)}{kw_0^2}\right\} \qquad (6)$$

This p-dependence must therefore be compensated for by assigning a complementary dependence of the phase-angle of the beam-modes' complex amplitudes, A_p^c. If we write $A_p^c \equiv A_p e^{i\theta_{A_p}}$ where A_p is a *real* amplitude, we may choose straightforwardly

$$\theta_{A_p} = -(2p+1)\cdot \arctan\left\{\frac{2(z_a - z_0)}{kw_0^2}\right\} \qquad (7)$$

The real amplitude, A_p, introduced in this way is not necessarily positive. Its magnitude and sign are determined in the fit to the aperture field.

The beam-mode on-axis phase at an arbitrary down-beam distance from the aperture is thus

$$\Theta_p \equiv \theta_p + \theta_{A_p} = (p + 1/2)\Theta \qquad (8)$$

where Θ is the *beam-mode phase-difference,* i.e., the common on-axis phase-difference between all successive pairs of modes, which is given by

$$\Theta \equiv \Theta_{p+1} - \Theta_p = 2\left\{\arctan\frac{2(z-z_0)}{kw_0^2} - \arctan\frac{2(z_a - z_0)}{kw_0^2}\right\}. \qquad (9)$$

(It should be noted that we are dealing here with axially symmetric modes only; inclusion of axially asymmetric modes would give a phase-difference between successive modes only one-half as large as this-see [1]).

TABLE I

$A_0 = 1.129890929909842$
$A_1 = -0.0001356877140080622$
$A_2 = -0.1374882595344276$
$A_3 = -0.04909626305813087$
$A_4 = 0.02238706418780562$
$A_5 = 0.03894626791308362$
$A_6 = 0.02280656850916452$
$A_7 = 0.0001986949746394339$
$A_8 = -0.0142902289947528$
$A_9 = -0.01731953253685816$
$A_{10} = -0.01197797055512092$
$A_{11} = 0.003309000367381503$
$A_{12} = 0.004435748450857954$
$A_{13} = 0.00888736454929259$
$A_{14} = 0.0095405941002746$
$A_{15} = 0.00718982193247121$
$A_{16} = 0.00323650627950287$
$A_{17} = -0.00089087534434813$
$A_{18} = -0.004106511863628689$
$A_{19} = -0.005823245165949175$
$A_{20} = -0.005941655866810328$
$A_{21} = -0.004736067571242572$
$A_{22} = -0.002697585732485339$
$A_{23} = -0.000380127790201935$
$A_{24} = 0.001721877574781014$
$A_{25} = 0.003251327526841281$
$A_{26} = 0.004021258673431002$
$A_{27} = 0.004010005168501314$
$A_{28} = 0.003330018034541174$
$A_{29} = 0.002182468621466627$

From (4) and (5) above for z_0 and w_0 this expression for Θ can be reduced to

$$\tan\frac{\Theta}{2} = \frac{\frac{z-z_a}{H}}{\Delta\left(1+\frac{z-z_a}{H}\right)} \qquad (10)$$

In this sense Θ can be regarded as a reduced down-beam distance.

As the beam propagates away from the aperture the modes thus lose the equality of on-axis phase and consequently the beam acquires non-spherical phase-fronts at all down-beam distances, in spite of the fact that the beam-modes' own phase fronts remain spherical. Both the amplitude and phase distributions over a beam cross-section develop considerable structure. To display this structure we require the values of the coefficients A_p. The assignment $w_a = 0.6435a$, $R_a = H$ leads [4] to the relative values for A_p for $p = 0$ to 29 given in Table I. We should remark that the same values of A_p apply for all corrugated feed-horns, i.e., there is no dependence of the A_p on the dimensions of the horn.

Writing the p^{th} beam-mode

$$E_{tp} = |E_{tp}|e^{i\phi_p}e^{-i(kz-\omega t)} \qquad (11)$$

we note that the mode's modulus $|E_{tp}|$, and phase ϕ_p, are functions both of r, z and of the horn dimensions H, Δ as follows. From equation 1

$$|E_{tp}| = (A_p/w)e^{\left(-r^2/w^2\right)}L_p\left(2r^2/w^2\right) \qquad (12)$$

the dependences on z, H, Δ being contained in $w(z, H, \Delta)$ ((2), (4), and (5)). And from (8)

$$\phi_p = -\frac{kr^2}{2R} + \Theta_p = -\frac{kr^2}{2R} + \left(p + \frac{1}{2}\right)\Theta \tag{13}$$

the dependences on z, H, Δ being contained in $R(z, H, \Delta)$ (equations (2), (4), and (5)) and in $\Theta(z, H, \Delta)$ (10).

Then writing the total field $E_t = \sum_p E_{tp}$ thus

$$E_t = |E_t| e^{i\Phi} e^{-i(kz-\omega t)} \tag{14}$$

we have

$$|E_t| = \left\{ \left(\sum_p |E_{tp}| \cos\phi_p \right)^2 + \left(\sum_p |E_{tp}| \sin\phi_p \right)^2 \right\}^{1/2} \tag{15}$$

and

$$\Phi = \arctan\left\{ \frac{\sum_p |E_{tp}| \sin\phi_p}{\sum_p |E_{tp}| \cos\phi_p} \right\} \tag{16}$$

Now changing the origin for the phase-angles ϕ_p, i.e., adding an arbitrary p-independent angle to ϕ_p in the right-hand sides of (15) and (16), would require no change to be made in the left-hand side of equation 15 and only the addition of that angle to the left-hand side of equation 16. If we choose $(kr^2/2R - \Theta/2)$ as the p-independent angle to be added, we have $\phi_p \to p\Theta$ from (13), and hence the weighted amplitude $|E_t|w$ is

$$|E_t|w = \left\{ \left(\sum_p |E_{tp}| w \cos p\Theta \right)^2 + \left(\sum_p |E_{tp}| w \sin p\Theta \right)^2 \right\}^{1/2} \tag{17}$$

and the phase-deviation from a spherical phase-front of curvature R is

$$(\Phi + kr^2/2R - \Theta/2) = \arctan\left(\frac{\sum_p |E_{tp}| w \sin p\Theta}{\sum_p |E_{tp}| w \cos p\Theta} \right) \tag{18}$$

The only r-dependence in the right-hand sides of these equations is in the $|E_{tp}|w$ which is a function of (r/w) only-see (12). It can be seen, therefore, that the field quantities, $|E_t|w$ and $(\Phi + kr^2/2R - \Theta/2)$, are functions of two variables only, namely (r/w) and Θ; the dependence of the field of a horn on the *four* variables, r, z and H, Δ, is *implicit* in the dependence of the field quantities on these two variables. The functions in equations 17 and 18 are universal in the sense that they are not specific to any particular horn dimensions, but the radiated field of a given horn can be readily determined from them since the dependences of w and Θ on r, z and H, Δ are simple (equations (2), (4), (5) and (10)). Figures 2(a) and (b) show the variation of $|E_t|w$ and of $(\Phi + kr^2/2R - \Theta/2)$ with (r/w) and Θ calculated using thirty modes, $p = 0$ to 29.

These computations are undemanding in computer time but the resulting fields in Fig. 2(a) and 2(b) span all the distributions that can be produced by corrugated feed-horns,

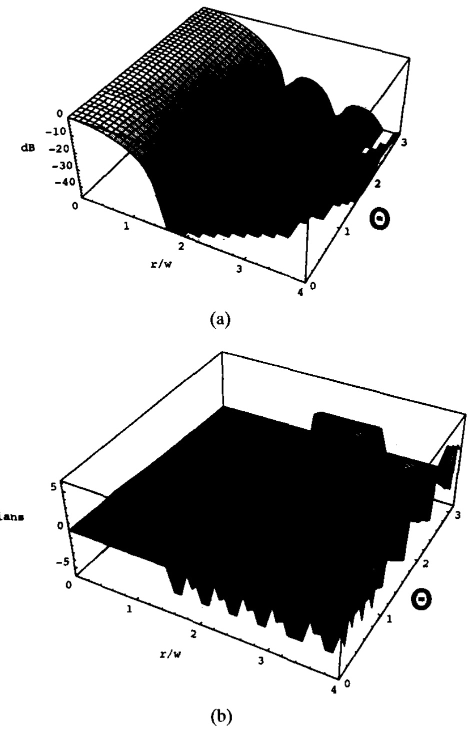

Fig. 2. (a) The weighted amplitude of the field in beams of corrugated feed horns as dependent upon the reduced off-axis and down-beam distances. (b) The deviation of the phase of the field in beams of corrugated feed horns from that given by the spherical beam-mode equiphase surface, as dependent upon the reduced off-axis and down-beam distances.

at all down-beam distances and for all horn-dimensions. The modulus and phase distributions in the field of a given feed horn, from the aperture plane to the far-field, are those for values of Θ lying between $\Theta = 0$ (corresponding to $z - z_a = 0$) and $\Theta = \arctan(1/\Delta)$ (corresponding to $(z - z_a) \to \infty$).

We show in Fig. 3 the field amplitude distribution in the aperture plane of the horn, $\Theta = 0$, as given by 30 modes. It can be seen that 30 modes give a good fit to the truncated Bessel function down to the -40 dB level. The amplitude and phase distributions at $\Theta = 1.97$ have a special significance-see later-and we show these distributions in Fig. 4.

We have found the software package MATHEMATICA [6], which allows algebraic symbolic analysis to be carried through prior to numerical evaluation, to be of great assistance in the studies we report here.

III. The Gain of a Reflector or Lens Antenna with a Corrugated Feed-Horn

We examine in this section how the gain of a reflector or lens antenna, with a corrugated feed-horn, can be treated by beam-mode analysis. We shall represent the antenna as an ideal thin lens or phase-transformer, i.e., a device which, when placed in a cross-sectional plane of the beam from the horn, produces a discrete phase-advance proportional to the square of the off-axis distance, with no modification of the amplitude. Bloomed thin lenses and ellipsoidal reflectors approximate to

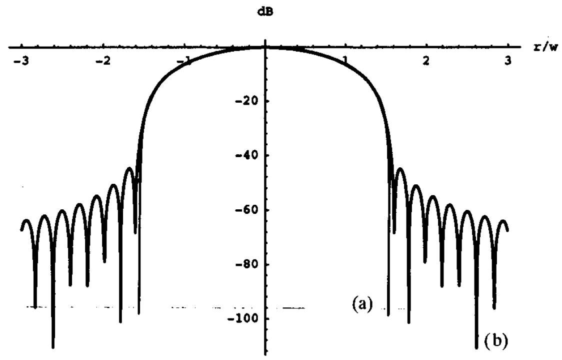

Fig. 3. Field amplitude distribution at the aperture plane of a corrugated feed-horn: (a) truncated Bessel function (b) representation by a superposition of 30 beam-modes.

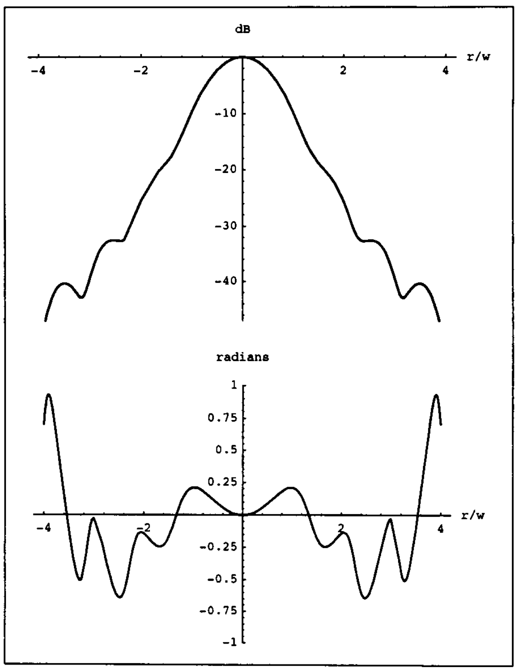

Fig. 4. The amplitude and phase distributions of the field of a corrugated feed-horn at reduced down-beam distance $\Theta = 1.97$.

such behavior but the ideal phase-transformer will exclude the small off-axis aberration in the case of an ellipsoidal reflector [7]. Our concern is with the influence of the horn dimensions on gain, including cases in which the antenna is not in the far-field of the horn.

The beam emerging from a phase-transformer of this kind (which we hereon call the antenna) is made up of beam modes having the same real amplitudes, A_p, as the modes of the incident beam since the field's amplitude distribution over the antenna is the same for the incident and emergent beams. The phase-fronts of each mode in the emergent beam will have a curvature $1/R_e$ equal to $(1/R_i - 1/f)$ where $1/R_i$ is the phase-front curvature of the incident modes at the antenna and f is the focal length of the antenna, i.e., the antenna's added phase is $kr^2/2f$. If $f = R_i$, $1/R_e$ is zero, i.e., the phase-fronts of the emergent modes are plane at the antenna. For $f < R_i$, $1/R_e$ is negative which means converging emergent modes. And for $f > R_i$, $1/R_e$ is positive, i.e., the beam-modes are diverging at the antenna.

The transverse field of the emergent beam at the antenna is thus

$$E_{tA} = \left\{ \sum_p E_{tpA} \right\} e^{-i(kz_A - \omega t)} \tag{19}$$

where the p^{th} mode's modulus is

$$|E_{tpA}| = \frac{A_p}{w_A} e^{(-r^2/w_A^2)} \cdot L_p(2r^2/w_A^2) \tag{20}$$

and its phase angle is

$$\phi_{pA} = \left(p + \frac{1}{2}\right)\Theta_A - \frac{kr^2}{2R_e} \tag{21}$$

in which w_A denotes the beam-width parameter at the antenna and Θ_A the beam-mode phase-difference there (both of which are the same for incident and emergent beams and are given by (2), (4), (5) and (10) with $z = z_A$, the location of the antenna).

The on-axis gain, G, of an antenna is a measure of the coupling between the antenna and a plane-wave propagating along the axis. It is given by [8]

$$G = \frac{k^2}{\pi} \frac{\left|\int_0^\infty E_{tA} 2\pi r \cdot dr\right|^2}{\int_0^\infty |E_{tA}|^2 2\pi r \cdot dr} \tag{22}$$

The integrals can be evaluated for each mode separately using the standard form [9]

$$\int_0^\infty e^{-bX} L_p(X)\, dX \equiv (b-1)^p b^{-p-1} \tag{23}$$

for $\mathcal{R}[b] > 0$. Using equation 20 and 21 in equation 22, and after some manipulation, we obtain

$$\frac{G}{G_F} = \frac{\cos^2\delta}{\sum_p A_p^2} \left\{ \left(\sum\nolimits_p (-1)^p A_p \cos\{p(\Theta_A - 2\delta)\}\right)^2 + \left(\sum\nolimits_p (-1)^p A_p \sin\{p(\Theta_A - 2\delta)\}\right)^2 \right\} \tag{24}$$

or

$$\frac{G}{G_F} = \frac{\cos^2\delta}{\sum_p A_p^2} \sum_{p,q} (-1)^{p-q} A_p A_q \cdot \cos\{(p-q)(\Theta_A - 2\delta)\} \tag{25}$$

where the angle δ is $\arctan\left(kw_A^2/2R_e\right)$, $|\delta| < \pi/2$. $G_F \equiv 2k^2 w_A^2$ is the gain for a beam made up of a fundamental (Gaussian) beam-mode only, with zero phase-front curvature at the antenna.

Evaluation of the summations in equation 25 is a relatively small computational task. Fig. 5 shows how the reduced gain G/G_F varies with the beam-mode phase-difference at the antenna Θ_A (i.e., with the dimensions of the horn and the distance between horn and antenna) and with the phase-front

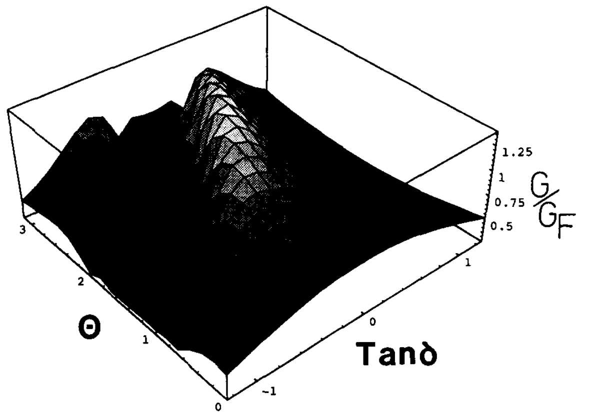

Fig. 5. Gain-ratio of the antenna as a function of Θ_A (the reduced down-beam distance of the antenna from the horn) and of $\tan\delta$ (the reduced phase-front curvature of the emergent beam-mode).

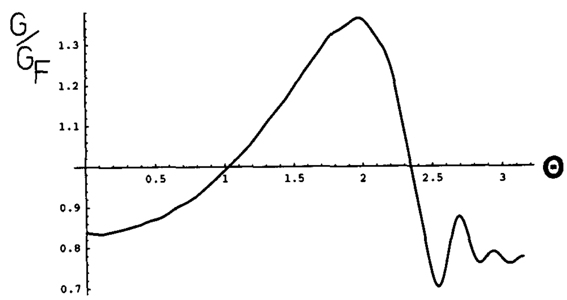

Fig. 6. Section through Fig. 5 at $\delta = 0$.

curvature of the emergent beam-modes, $\tan\delta = kw_A^2/2R_e$ (i.e., with the focal length of the antenna).

By plotting the ratio G/G_F, rather than G itself, attention is focussed on the range of gain that can be spanned by changing the dimensions of the horn, the separation of the horn and the antenna, and the focal length of the antenna, within the constraint of *fixed* beam-width at the antenna, w_A. The range can be seen to be substantial.

The absolute maximum of the gain ratio, G/G_F, occurs at $\Theta_A = 1.97$, $\delta = 0$ (see the section for $\delta = 0$ in Fig. 6). The reason for this can be seen in the forms of the amplitude and phase distributions in the beam from the feed-horn as shown in Fig. 2(a) and (b) above: it can be seen that, in the cross-section at $\Theta = 1.97$, the phase front remains close to spherical out to larger values of the off-axis distance than in other cross-sections and the amplitude falls away less rapidly. The amplitude and phase distributions for $\Theta = 1.97$ are shown in Fig. 4.

For a large gain the beam-mode width-parameter at the antenna, w_A, would have to be large, of course; conversely, a small antenna, i.e., small w_A, would be possible and desirable if a small gain were acceptable. Equations (2), (4), (5) and (10) can be used to establish the following relationship between w_A and the horn dimensions,

$$w_A = w_a\left(1 + b^2\right)^{1/2} \Big/ (1 - b\Delta) \tag{26}$$

and from (10)

$$z_A - z_a = H/(1 - b\Delta) \tag{27}$$

where $b \equiv \tan(\Theta_A/2)$. Hence for a horn specified for a maximum gain ratio, i.e., $\Theta_A = 1.97$, we have

$$\Delta = 0.662 - \frac{0.772}{w_A/a} \tag{28}$$

$$\frac{z_A - z_a}{H} = \frac{0.662}{0.662 - \Delta} \tag{29}$$

It is clear that, for maximum gain ratio, the value of Δ cannot exceed 0.662 because w_A and $(z_A - z_a)$ are positive. It is clear also that for a large antenna with $w_A >> a$, the required Δ will be close to the limiting value 0.622 and $(z_A - z_a) >> H$ so the antenna would be in the far-field of the feed-horn. For small w_A/a (and therefore more moderate gain) a smaller Δ is required and the antenna will be in the near-field of the feed-horn.

If other constraints prevent the selection of a horn having the dimensions required for the absolute maximum of gain-ratio, the reflector can be shaped to give δ the value which would maximize the gain-ratio at the given Θ_A (see Fig. 5), i.e., the focal length would be off-set from the value R_i; the phase fronts of the modes leaving the antenna would consequently *not* be plane there. The modes would converge, for negative δ, to a beam-waist at which their phase-fronts would be plane, or would diverge, for positive δ, from a virtual beam-waist; in either case the beam-width parameter at the beam-waist plane will be smaller than w_A and the gain-ratio will therefore be smaller than would be given by an optimally-dimensioned feed-horn for the same w_A.

We have implicitly assumed above that the antenna is sufficiently large that there is no significant truncation of the beam from the feed-horn at the rim of the antenna. When there is truncation there the gain can still be determined by evaluating the integrals given in the definition of gain in (22) but the limits of the integrals should be 0, r_A rather than 0, ∞ where r_A is the radius of the antenna. The result of doing this for $r_A = 1.27w_A$, which corresponds to truncation at the -14 dB level, are given in Fig. 7 (showing the gain relative to that for an untruncated pure Gaussian beam-mode). Truncation has resulted in a major reduction in the general level of gain (not surprisingly) and in a more simple surface *versus* Θ_A, δ. The maximum gain is now found at $\delta = 0$ for all values of Θ_A; i.e., for maximum gain the focal length of the antenna should be such as to give a *plane* phase-front in each emergent beam-mode at the antenna.

IV. The Phase-Centers of Corrugated Feed-Horns

An optical system incorporating thin lenses, or conic-section reflectors, is designed to convert an incident beam having a spherical phase-front of specified curvature, $1/R_i$, at the input port into an emergent beam having a spherical phase-front of specified curvature, $1/R_e$, at the output port. (We shall describe transmitting systems but there are, of course, reciprocal relationships to receiving systems).

The notion of apparent *phase-center of a feed horn* has frequently been invoked in deciding the precise location of a

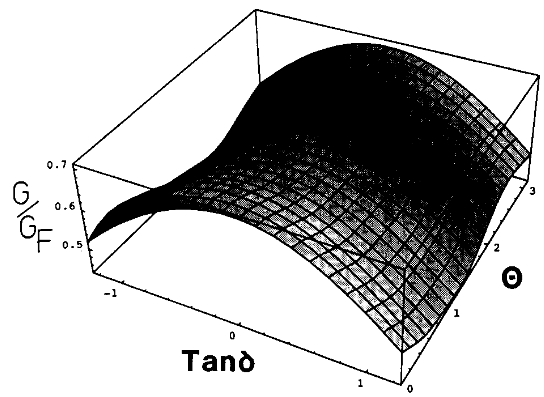

Fig. 7. As Fig. 5 but for an antenna which truncates the beam at the −14 dB level.

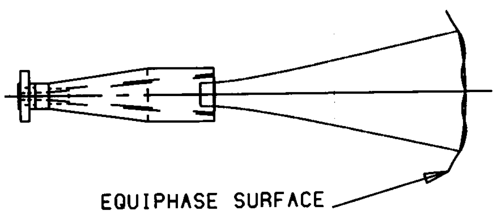

Fig. 8.

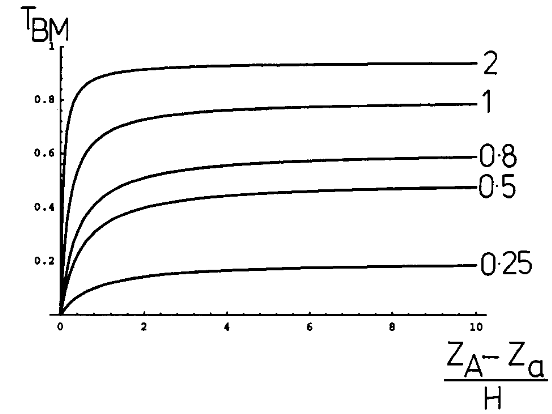

Fig. 9. Beam-mode phase-center as a function of reduced down-beam distances for selected values of Δ, from 0.25 to 2.0.

horn relative to the system it feeds. The apparent phase-center of a horn for a specified down-beam distance is the center of the spherical surface which bests fits the equi-phase surface of the beam of the horn at that down-beam distance (see Fig. 8). Its position in the horn will vary with the specified down-beam distance. A feed-horn would be judged to be properly placed with respect to an optical system when the best-fit spherical surface at the input port has radius R_i.

The best spherical fit to a non-spherical equi-phase surface is not uniquely defined, however. Several inequivalent criteria have been adopted in the past [10]. In each case the criterion is essentially a minimum for the weighted mean of the phase-deviation over the cross-section; they differ in the weights assigned to the deviation at different points in the cross-section. The adoption of one pattern of weighting rather than another has usually received little quantitative justification. We use the beam-mode analysis for a corrugated feed-horn below to examine several apparent phase-centers.

4.1 Beam-Mode Phase-Center

At any given down beam distance all the beam-modes in the beam of a feed-horn have spherical phase-fronts with a common center-of-curvature. This common center is the 'beam-mode phase-center' for the given down-beam distance. This center is thus at a distance $\{R_A - (z_A - z_a)\}$ behind the aperture plane of the horn, where R_A is the radius of curvature of the beam-mode phase-front at the down-beam distance $(z_A - z_a)$. Then using (2) and (3), we obtain the following expressions for T_{BM} (i.e., the distance of the beam-mode phase-center from the apex of the horn expressed as a fraction of the length of the horn, H).

$$T_{BM} = D/(1 + D) \tag{30}$$

where

$$D \equiv \frac{\frac{z_A - z_a}{H}}{\Delta^2\left(\frac{z_A - z_a}{H} + 1\right)} \tag{31}$$

This gives the location of the beam-mode phase-center for any down-beam distance and any horn dimensions. In Fig. 9, we show the location as a function of $(z_A - z_a)/H$ for selected values of the horn parameter, Δ. It is clear from this figure that the beam-mode phase-center has essentially reached its far-field location when $(z_A - z_a)/H$ exceeds about 5.

4.2 On-Axis Phase-Center

The fit for this case is simply that of matching the curvature of the equi-phase surface at the on-axis point. We can use the expression given in equation 16 for the phase Φ in the beam of a corrugated feed-horn to derive the following expression for the on-axis phase-front curvature, $1/R_o$ say:

$$\begin{aligned} 1/R_o &\equiv \frac{1}{k}\left(\frac{\partial^2 \Phi}{\partial r^2}\right)_{r=0} \\ &= 1/R - \frac{4}{kw^2}\frac{\sum_{p,q} pA_pA_q \sin\{(p-q)\Theta\}}{\sum_{p,q} pA_pA_q \cos\{(p-q)\Theta\}} \end{aligned} \tag{32}$$

where R, w, Φ take the values appropriate to the specified down-beam distance. The on-axis phase-center is at a distance $R_0 - (z_A - z_a)$ behind the horn aperture. Fig. 10 shows how the location of the *far-field* on-axis phase-center varies with the horn parameter Δ.

4.3 Least-Squares-Fit Phase-Center

The criterion for this case is minimization of the rms phase deviation of the true equi-phase surface from the fitted spherical surface over the main beam, within say, the −12 dB level. Equation 16 for Φ can be used to determine this phase-center for any down-beam distance but we present here, in Fig. 10, the location of this center only for *far-field* distances,

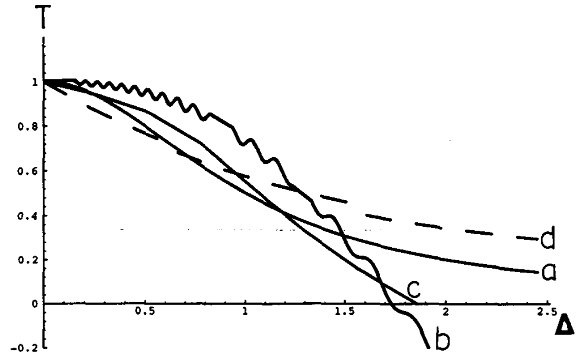

Fig. 10. The locations of far-field phase-centers of corrugated feed horns. T is the distance of the phase-center from the apex of the horn expressed as a fraction of the horn length, H. Δ is the dimensional parameter of the horn i.e., $\frac{(0.6435)^2 ka^2}{2H}$. (a) Beam-mode phase-center. (b) On-axis phase-center. This curve was calculated using 30 modes. Use of a smaller number of modes gives fewer but larger ripples. We believe that the ripples would disappear if a sufficiently large number of modes were used. (c) Least-squares-fit phase-center (to -12 dB level). (d) Maximal-gain phase-center.

as dependent on the horn parameter Δ. We have used, for this, the data given by Thomas [11] (note that our Δ is 2.6 times larger than the quantity for which Thomas uses the symbol Δ).

4.4 Best-Fit-Gaussian Phase-Center

A more graduated way of assigning greater weight to the phase at points where the field amplitude is larger is to find the best-fit Gaussian approximation to the field at the specified down-beam distance, i.e., a beam having a Gaussian amplitude distribution, and a spherical phase-front. The best-fit criterion would be the maximization of the fraction of beam-power in this Gaussian component, by appropriately choosing values for the Gaussian width-parameter and for the radius of curvature of the phase-front. This process is just that involved in an analysis of the beam into Gauss–Laguerre modes as treated in Section II, the best-fit Gaussian component being the fundamental mode produced in that analysis. The best fit Gaussian phase-center is therefore simply the beam-mode phase-center (Section 4.1) and we show in Fig. 10 the position of this phase-center for *far-field* distances, as dependent on the horn parameter Δ.

4.5 Maximal-Gain Phase-Center

The phase-center which is defined using the field-amplitude of the beam as a weighting function has a particular significance. The center is located by maximizing the modulus of the coupling integral between the horn's field at the specified down-beam distance and the spherical phase function, $e^{ikr^2/2R_s}$, that is to say, by finding the value of the radius of curvature R_s that maximizes

$$\left| \int_c |E_t| e^{i\Phi} e^{-ikr^2/2R_s} 2\pi r \, dr \right| \tag{33}$$

where the integral is over the beam cross-section, C, at the specified down-beam distance. (Expanding the exponential functions in (33) to the first order terms-i.e., taking the phase deviation to be small over C-shows that maximizing this overlap integral will be equivalent to minimizing the mean of the phase-deviation, $(\Phi - kr^2/2R_s)$, with $|E_t|$ as weighting function, over C.)

The significance of this phase-center becomes clear when the form of the expression above is compared with that for the gain of an antenna. If a lens or reflector antenna, of focal length f, were placed in the beam cross-section, C, the emergent beam would have the form $|E_t|e^{i\Phi}e^{-ikr^2/2f}$. The antenna gain is proportional to the overlap integral of this field with a plane wave, i.e., to

$$\left| \int_c |E_t| e^{i\Phi} e^{-ikr^2/2f} 2\pi r \, dr \right| \tag{34}$$

This has the same form as the expression in (33) above if f is identified with R_s. Finding the value of R_s which maximizes the expression in equation 33 is thus equivalent to finding the value of f which maximizes the gain of the antenna, as described in Section III. We can see, therefore, that the phase-center of the horn obtained by using the beam's field amplitude as the weighting function is the *maximal-gain phase-center.*

Given the above equivalence, the data in Fig. 5 can be used to determine the location of the maximal-gain phase-center of any corrugated feed-horn for any down-beam distance. The location of this phase-center for *far-field* distances, as dependent on the horn parameter, Δ, is included in Fig. 10.

A different situation exists when the antenna truncates the beam from the feed-horn. We have shown in Fig. 7 that truncation at the -14 dB level results in maximal gain being obtained when the focal length of the antenna is such as will give each beam-mode a plane phase-front as it emerges from the antenna. This means, of course, that the maximal-gain phase-center is at the *beam-mode phase-center* (or Gaussian best-fit phase-center) when the truncation is within the -14 dB level.

4.6 Empirical Phase-Centers

There are measured far-field power patterns for corrugated feed-horns in the literature which are in good agreement with what is expected if the field in the aperture-plane of a corrugated horn has the form assumed in the beam-mode analysis of Section III, i.e., a truncated zero-order Bessel function for the amplitude distribution and a spherical equiphase surface.

There are few reported measurements of phase however. Recent measurements of phase distributions for corrugated horns for 110 GHz and 183 GHz have been reported by Tuovinen et. al. [12], who determined the location of the far-field phase-center in each case using a method which gives essentially the *on-axis phase-center* discussed in Section 4.1 above. For the two horns the best estimate of the phase-center was 2.1 mm behind the aperture for the first case and 8.1 mm for the second. Using our results in Section 4.1 above for the particular dimensions of the horns in this work we would expect the on-axis phase-center to be at 2.2 mm and 7.6 mm behind the aperture respectively, in good accord with the empirical determination. Tuovinen et. al. compared their empirical phase-center locations with theoretical expectations previously published and found a lack of agreement. The phase-centers involved in the theoretical work they cited were, however, not the on-axis phase-center but what we here have called the Gaussian best-fit (or beam-mode) phase-center and the least-squares-fit phase-center. As we have noted above, the theoretically derived *on-axis* phase-center agrees well with the empirical result.

V. Other Types of Aperture Horn

A beam-mode analysis along the lines followed above for a corrugated feed-horn can be made of the beam of any feed-horn for which the form of the field in the aperture plane is known. The fields in the apertures of rectangular pyramidal horns and smooth-walled conical horns have been analyzed into Gauss–Hermite functions [5] and this would provide the basis for a beam-mode analysis of phase-centers along the lines set out here. If the field in the aperture plane of a horn is not plane-polarized it is necessary to resolve the field into orthogonally-polarized components and to develop a beam-mode analysis for each polarization.

VI. Conclusion

The idea of a phase-center having an identified location in a feed-horn is invoked when deciding the correct position for a horn in relation to the optical system it is to feed. The idea is compromised, however, by the fact that the equi-phase surfaces in the fields of feed-horn beams are never, or hardly ever, truly spherical. Phase-centers are consequently not uniquely definable. We have shown, for corrugated feed horns, how variously defined phase-centers span such a range of locations within a horn that significant differences in optical performance would result from adopting one rather than another when placing a horn is an optical system.

We conclude that a phase-center should be defined with reference to a selected performance criterion. We have used maximal gain as an example of a performance criterion to illustrate this question. Other criteria would be more appropriate in some applications, for example, maximum beam-efficiency. The beam-mode phase-center (which, as we have seen, can be interpreted as a best-fit-Gaussian phase-center) is probably the best choice if a detailed computation based on a specified performance criterion is excluded. We have shown that the location of the beam-mode phase-center is known for all horn dimensions and down-beam distances, from near-field to far.

Beam-mode analysis contains the detailed forms of the fields in the beams of feed-horns, not only for far-field distances, but also for near and intermediate distances. It is therefore an effective method for locating performance-related phase-centers, as we have illustrated here. Once a beam-mode analysis is undertaken, the optimum location of the horn can be directly determined without explicit use of the phase-center idea; nevertheless, giving a phase-center location for the selected performance criterion and down-beam distance can be a convenient way to express the results. Beam-mode analysis has previously been used to determine horn locations in specific systems [13]. We believe the studies we report here give a general context for understanding what is at stake.

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Derek H. Martin, for a photograph and biography, see this issue, p. 1690.