Dynamics of Computation in the Brain - Exercise 2

Due June 13, 2024

1 The Fourier series

1. Calculate the Fourier series of the function:

$$f(x) = \begin{cases} -1 & -\pi < x < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < x < \pi \end{cases}$$

   That is, find the coefficients a_n, b_n that will obey:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big)$$

   by using:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

2. We saw in class that for the function

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

   the Fourier series is given by:

$$f(x) = \sum_{n=1,3,5..}^{\infty} \frac{4}{\pi n} \sin(nx)$$

   What is the relation between the function we saw in class and the function in question (1)? Explain how this relation affects the relation between the Fourier coefficients of both functions. Relate to the odd/even nature of the functions, to which coefficients are zero and to the amplitude and phase of each frequency. Use the fact that $\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$.

2 Emergence of Oscillations

Consider the following model of an excitatory and an inhibitory neuron with firing rates r_E, r_I. The relation between the input $h_\alpha$and the firing rate r_α is defined as:

$$r_\alpha = g\left(h_\alpha\right) \equiv \lfloor h_\alpha \rfloor_+$$

for both $\alpha \in \{E, I\}$. The function$f(x) = \lfloor x \rfloor_+$ is the threshold non-linearity

$$\lfloor x \rfloor_+ = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

The dynamics of the inputs are as follows:

$$\tau\dot{h}_E = -h_E + J_{EE}r_E - J_{EI}r_I + h_E^0$$
$$\tau\dot{h}_I = -h_I + J_{IE}r_E - J_{II}r_I + h_I^0 \tag{2.1}$$

Each of the $J_{\alpha\beta}$ and h_α^0 parameters are positive, and for simplicity you may take $\tau = 1$. You can further assume $h_E^0 > h_I^0$.

2.1 Analytic Part

For simplicity, set $J_{II} = 0$ and $J_{IE} = 1$.

1. Split the dynamics into 4 regimes depending on the signs of h_α. For each regime, write the dynamical equations in vector notation (i.e. $\dot{\vec{h}} = M\vec{h} + \vec{h}^0$), as well as the inequality (constraint) associated with the regime.

2. Now we are going to find the fixed points of the regime and use self consistency arguments in order to explain which regimes are possible under our assumptions regarding the parameters:

   (a) Fixed points: find the values h_E^*, h_I^* for which there are fixed points in each one of the regimes.

   (b) Self consistency: for each one of the four regimes, we have different constraints (inequalities) on h_E, h_I. After you've found the fixed point in (a) use these constraints in order find a range of values for which the solution you found is viable. Be mindful, it might be that not all cases are possible given this set of constraints[1].

3. Calculate the eigenvalues of the dynamics near each fixed point and perform a stability analysis as a function of J_{EE} and J_{EI}. Summarize your results by plotting the parameter space of J_{EE} vs J_{EI} for each fixed point, and marking the regions with different qualitative dynamics (non-consistent regimes, stable and unstable dynamics). Mark the areas where the dynamics have complex eigenvalues - we will treat these as separate regions in the next sections. Are there parameters for which more than a single fixed point exists?

2.2 Numerical Part

1. Choose values for h_E^0, h_I^0. Using Euler's method, simulate this system of equations (with $J_{II} = 0$ and $J_{IE} = 1$), and verify your above analysis for at least two points per distinct region of parameter space – for each area in parameter space generate a single plot of the state space of r_E vs r_I and display the trajectories for 4 different initial conditions (Mark the fixed point in some distinct way) .What happens to the trajectories in regions of complex eigenvalues? How is this behavior different for stable vs. unstable fixed points?

2.3 Bonus (5 points)

1. For $J_{EI} = 4$, find the critical value of J_{EE} at which the fixed point loses stability. What are the eigenvalues at this critical point?

2. Write the solution to the dynamical equation (with a small perturbation from the fixed point) in the eigenbasis near this critical point, where it loses stability. Use the relation $e^{zt} = e^{(x+iy)t} = e^{xt}\Big(\cos(yt) + i\sin(yt)\Big)$ to claim that the solution has an oscilatory part and predict the period of these oscilations.

3. Simulate the dynamics for a value of J_{EE} just outside of the stability region, and estimate the period of the dynamics (i.e. how long does it take the trajectory to do a single rotation). Show that the period is very close to what you calculated above.

[1] The reason we say self-consistency is the following: When we divided the equations to four parts we added constraints on the dynamics. When we started looking for fixed points, we "ignored" these constraints. Therefore we must make sure that our solutions are consistent with the assumptions we started from. An example of inconsistency would be to allow a range of values of $h_E^* < 0$ while looking at the cases when $h_E^* > 0$

2.4 Qualitative Observation

1. This is the first time we see stability analysis in the context of the entire dynamics (away from the fixed points). What could we learn from the stability analysis? What did we miss?

2. We already saw several networks consist of multiple neurons. What qualitatively new dynamics appeared in this system and how does this relate to the connectivity structure? What is the importance of the non-linearity? What qualitative difference would it have made if we had simply used $g(h) = h$? What about $g(h) = (1 + \exp(\beta(\theta - h)))^{-1}$?

3 Fourier meets Oscillations

In this part we will look at the set of differential equations from Part 2 through the lens of the fourier series and what we learned in class about it. We also set $J_{II} = 0$ and $J_{IE} = 1$ here, as well as $\tau = 1[sec]$.

3.1 Numerical Part

Note that in part 3.2 you are required to repeat a similar analysis to what is done here, so it might be worth reading that part in advance to know what you are going to reuse and write your code accordingly.

1. Simulate the dynamics with $J_{EE} = J_{EI} = 2$ for 100 seconds. Plot r_E as function of time (in units of seconds) and estimate the duration of a single oscillation (a very rough estimation, with precision of a second, is enough).

2. Using the FFT algorithm, plot the power spectrum of the signal:

   (a) Run the FFT command on $r_E(t)$. In Python, this can be done by np.fft.fft(x).

   (b) Generate the right frequencies vector, that will relate every element of the FFT of x to its corresponding frequency. For python, this can be done by np.fft.fftfreq. For Matlab, you can implement the python's version of fftfreq by yourself (search for it online). Notice that your sampling space/period is the $\mathrm{d}t$ you've used for the simulation, while the sampling frequency is $1/\mathrm{d}t$. The highest frequency you should get is $k = \frac{1}{2 \cdot \mathrm{d}t}$.

   (c) **Understanding the meaning of the frequencies vector:** Each frequency k corresponds to the cosine wave $\cos(2\pi kt)$, which is equivalent to the n we saw in class via $n = 2\pi k$.

      i. How long does it take a wave with $k = 3\,[\frac{1}{sec}]$ to complete a single oscillation?

      ii. How long does it take a wave with $k = 6\,[\frac{1}{sec}]$ to complete a single oscillation?

      iii. In your frequencies vector, which k corresponds to the longest wave (i.e. the k for which single oscillation will take the longest time)? What is the corresponding duration of a single oscillation?

   (d) Plot the power spectrum of the signal: For every frequency, plot the power of that frequency. Remember: as we saw in class, the FFT algorithm returns complex values. To get the power spectrum we need to take the absolute value of the complex value of each frequency.

3. Zoom in to the range of frequencies $[-2, 2]$. Other than the $k = 0$ frequency (corresponding to the mean amplitude of the entire signal), what is the k with the highest power? Show that this k corresponds to your estimation of the duration of a single oscillation for r_E from the first question in this part (Question B1).

4. Repeat the simulation with different values of J_{EE}, J_{EI}. For each value, calculate the frequency of the oscillation using the same method described above[2]. Make sure you find the frequency in an efficient way and not manually, so that you can run many simulations. To get higher frequencies, you might need to use a large $\frac{h_E^0}{h_I^0}$ ratio. For $J_{EE} = 2$, $J_{EE} = 2.5$ and $J_{EE} = 3$, plot the frequency of oscillations as a function of J_{EI}.

3.2 Explaining the numerical results via the linear model

We will now try to explain the frequencies observed in part (B) with a linear model. For a linear system $\dot{\vec{r}} = M\vec{r}$, where M has an eigenvalue of $\lambda = a + ib$, the expected frequency of oscillation is $k = \frac{b}{2\pi}$[3].

[2] Notice that for the frequency to be relyable, you need to have at least a single oscillation in your data. In cases of fixed point without any oscillations, this is not the case.

[3] A detailed explanation of this statement can be found in the lecture notes.

1. Consider a 2d linear model with $M = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$.

   (a) What are the eigenvalues of the matrix? What is the expected frequency of this system?

   (b) Verify the above statement: Simulate these dynamics and use the same analysis as in 3.1 to show the predicted vs the observed frequency for this system, for a range of $J \in [0, 3]$. You might need to use a finer $\mathrm{d}t$ for this simulation.

2. Going back to the model from section 3.1, check how well the linear model describes the behavior of the non-linear network:

   Near the fixed point the model is approximately linear, and the eigenvalues of the dynamics can be extracted (This can be done by using the function np.linalg.eig() in Python or eig() in Matlab). For each of the three J_{EE} values of section 3.1 q.4, plot the predicted frequency of oscillations from the linear model vs the oscillations observed in the simulations.

3.3 Qualitative Observation

1. How does the frequency of oscillations depends on J_{EE} and J_{EI}? Can you explain this dependency intuitively?

2. How well does the linear model describe the non-linear dynamics? Is this approximation useful? In what sense?

3. How does the accuracy of the linear approximation depend on the value of J_{EE}? Try to explain this dependency.