Dynamics of Computation in the Brain
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Solution EX #1

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1 Euler's Method for Numerical Integration

$$\dot{x} = f(x)$$

$$\hat{x}(t + \Delta t) = \hat{x}(t) + f(\hat{x}(t))\,\Delta t$$

$$\dot{x} = -\lambda x$$

1.1 Analytical Part

$$\epsilon(t) = |x(t) - \hat{x}(t)|$$

1.1.1 Single-step Simulation Error

<u>(a)</u> The approximated solution after a single-step is:

$$\hat{x}(0 + \Delta t) = \hat{x}(0) + f(\hat{x}(0))\,\Delta t = \hat{x}(0) - \lambda\hat{x}(0)\,\Delta t$$

Note $\hat{x}(0) = x_0$ the starting condition for the solution and we get:

$$\boxed{\hat{x}(\Delta t) = x_0 - \lambda x_0\,\Delta t}$$

<u>(b)</u> The analytical solution as we know is $x(t) = x_0 e^{-\lambda t}$. The single-step error is:

$$\begin{aligned}\epsilon(\Delta t) &= |x(\Delta t) - \hat{x}(\Delta t)| = |x_0 e^{-\lambda\Delta t} - x_0 + \lambda x_0\,\Delta t| \\ &= |x_0|\,|e^{-\lambda\Delta t} - 1 + \lambda\Delta t|\end{aligned}$$

Using the fact that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ (Tylor expansion of e^x at 0), we get:

$$\epsilon(\Delta t) = |x_0|\,\left|1 - \lambda\Delta t + \frac{\lambda^2\Delta t^2}{2} + O(\Delta t^3) - 1 + \lambda\Delta t\right| = |x_0|\,\left|\frac{\lambda^2\Delta t^2}{2} + O(\Delta t^3)\right|$$

$$\boxed{\approx |x_0|\,\frac{\lambda^2\Delta t^2}{2}}$$

1.1.2 Fixed-time Simulation Error

<u>(a)</u> Given t_1 and Δt, let n be the maximal integer such that: $n < \frac{t_1}{\Delta t}$ (less formally, this means $t_1 = n\Delta t$ up to a small residual). We then get:

$$\hat{x}(0) = x_0;\ \hat{x}(\Delta t) = x_0 - \lambda x_0\,\Delta t = x_1;\ \ldots;\ \hat{x}(n\Delta t) = x_{n-1} - \lambda x_{n-1}\,\Delta t = x_n = \hat{x}(t_1)$$

So, for $1 \le i \le n$:

$$\frac{x_i}{x_{i-1}} = \frac{x_{i-1} - \lambda x_{i-1}\,\Delta t}{x_{i-1}} = 1 - \lambda\Delta t$$

And now we can get a closed formula:

$$\hat{x}(i\Delta t) = x_i = x_0(1-\lambda\Delta t)^i$$

Specifically:

$$\boxed{\hat{x}(t_1) = x_0(1-\lambda\Delta t)^n}$$

(b) First, note that $0<\lambda;\Delta t$, so we get $0<\lambda\Delta t$ and thus $-\lambda\Delta t<0$ and $1-\lambda\Delta t<1$. Ergo, we only need to consider what happens to $q := 1-\lambda\Delta t$ in the ray $(-\infty;1)$. Given a fixed λ, this will singularly determine the different qualitative outcomes. Moreover, note $t_1 \to \infty$ iff $n \to \infty$. Thus, it suffices to consider the asymptotic behavior of q^n. We'll look at 3 regimes: $(-\infty;-1];(-1;0);[0;1)$:

$$\begin{cases} 0\le q<1 & q^n \searrow_{n\to\infty} 0 \\ -1<q<0 & q^n \to_{n\to\infty} 0 \\ q\le -1 & q^n \text{ diverges} \end{cases}$$

I.e., for $0\le q<1$, q^n will monotonically decrease to 0. For $-1<q<0$, q^n will still converge to 0 but will do it in an alternating fashion where the sign of q^n will socialite between + and $-$. Similar oscillations will occurs for $q\le -1$ however, in this case there will be no convergence. In the spacial case of $q=-1$, q^n will oscillate between ± 1, otherwise, between $\pm\infty$. Now:

$$0\le 1-\lambda\Delta t<1 \iff -1\le -\lambda\Delta t<0 \iff 1\ge \lambda\Delta t>0 \iff \frac{1}{\lambda}\ge \Delta t>0$$

$$-1<1-\lambda\Delta t<0 \iff -2<-\lambda\Delta t<-1 \iff 2>\lambda\Delta t>1 \iff \frac{2}{\lambda}>\Delta t>\frac{1}{\lambda}$$

$$1-\lambda\Delta t\le -1 \iff -\lambda\Delta t\le -2 \iff \lambda\Delta t\ge 2 \iff \Delta t\ge \frac{2}{\lambda}$$

Thus we're left with:

$$\boxed{\begin{cases} \frac{1}{\lambda}\ge\Delta t>0 & (0<\lambda\Delta t\le 1)\ \textit{monotonical convergence} \\ \frac{2}{\lambda}>\Delta t>\frac{1}{\lambda} & (1<\lambda\Delta t<2)\ \textit{alternating convergence} \\ \Delta t\ge\frac{2}{\lambda} & (2\le\lambda\Delta t)\ \textit{divergence} \end{cases}}$$

(c) The error is:

$$\epsilon(t_1) = |x(t_1)-\hat{x}(t_1)| = |x_0e^{-\lambda t_1}-x_0(1-\lambda\Delta t)^n| = |x_0||e^{-\lambda t_1}-(1-\lambda\Delta t)^n|$$

Assume for simplicity $|x_0|=1$:

$$\epsilon(t_1) = |e^{-\lambda t_1}-e^{n\ln(2-\lambda\Delta t)}| = |e^{-\lambda t_1}-e^{n\ln(1-\lambda\Delta t)}|$$

We use Tylor's expansion around 0 of $ln(1+x) = x-\frac{x^2}{2}+o(x^3)$. For this to work, we'll add the condition that $(-\lambda\Delta t=)x\ll 1$ such that $ln(1-\lambda\Delta t)$ is defined. Also, remember $n\approx\frac{t_1}{\Delta t}$ so:

$$\epsilon(t_1)\approx|e^{-\lambda t_1}-e^{\frac{t_1}{\Delta t}(-\lambda\Delta t-\frac{(-\lambda\Delta t)^2}{2})}| = |e^{-\lambda t_1}-e^{-\lambda t_1-\frac{1}{2}\lambda^2 t_1\Delta t}| = e$$