Dynamics of Computation in the Brain - Exercise 3

Due June 27, 2024

1 Analytical part

1.1 Coupled oscillators

We revisit the case in which we analyze two neuronal oscillators interacting with one another. We observed 2 neuronal oscillators and defined the phase shift on the limit cycle[1] of the postsynaptic cell as $\delta\psi$ and the phase shift on the presynaptic cell as $\delta\psi'$. The change in phase over time for $\delta\psi$ is given by the interaction of the two oscillators over an entire cycle, which we defined as $\Gamma\left(\delta\psi, \delta\psi'\right)$:

$$\frac{\mathrm{d}\delta\psi}{\mathrm{d}t} = \Gamma\left(\delta\psi, \delta\psi'\right)$$

We looked at a pair of neurons when the interaction is a function of the phase difference between them. We thus wrote:

$$\Gamma\left(\delta\psi, \delta\psi'\right) = \Gamma\left(\delta\psi' - \delta\psi\right) = \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \vec{R}\left(\theta - \left(\delta\psi' - \delta\psi\right)\right) \cdot \vec{S}\left(\theta\right)$$

Where R is the postsynaptic response and S is the presynaptic activation. Instead of looking at ωt we looked directly at the angle θ for the case when:

$$R\left(\theta\right) = \begin{cases} 0 & \theta < 0 \\ \frac{g_{\mathrm{syn}}}{c_m}\left(\frac{\theta}{\omega\tau}\right)e^{-\frac{\theta}{\omega\tau}} & \theta \geq 0 \end{cases}$$
$$S\left(\theta\right) = \sin\theta$$

For g_{syn} the conductance of the synapse and c_m the capacitance. ω is the frequency and τ is the time constant. We would like to analyze this case and see how the interactions in oscillations affect the system.

1.1.1 Approximate integral solution (Optional)

First we will solve the integral, in order to reach a solution we will use the following approximation[2]:

$$\Gamma\left(\delta\psi, \delta\psi'\right) = \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \vec{R}\left(\theta - \left(\delta\psi' - \delta\psi\right)\right) \cdot \vec{S}\left(\theta\right) \approx \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta \vec{R}\left(\theta - \left(\delta\psi' - \delta\psi\right)\right) \cdot \vec{S}\left(\theta\right)$$

Solve the integral by doing the following:

1. Change variables so that $\theta \mapsto \tilde{\theta} - \left(\delta\psi - \delta\psi'\right)$.

[1] After moving from the position on the limit cycle $\vec{X}\left(t\right)$ to the constant orbit given by $\psi\left(t\right) = \delta\psi + \omega t$

[2] For a justification of the approximation, see David's lecture on 30/5/24

2. Plug the definition of $R(\theta), S(\theta)$. Note how this changes the boundaries of the integral.
3. Multiply and divide by $\omega\tau$ so that $\tilde{\theta}$ appears as $\frac{\tilde{\theta}}{\omega\tau}$, then change variables again with $\frac{\tilde{\theta}}{\omega\tau} \mapsto x$.
4. Use the identity $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ in order to simplify the integral so that it only contains exponents of x or products of x.
5. Use integration by parts to finally solve the integral.
6. Using the identity for $\sin(x)$ written above and $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$, simplify the integral so that it does not contain any complex exponents or factors with i.
7. We now define the difference $\tilde{\Gamma}(\delta\psi' - \delta\psi) = \Gamma(\delta\psi' - \delta\psi) - \Gamma(\delta\psi - \delta\psi')$ which is the phase difference of the two oscillators. Using your answer in step (6) show that:

$$\boxed{\tilde{\Gamma}(\delta\psi' - \delta\psi) = \frac{g_{\text{syn}}}{c_m}\frac{\epsilon}{\pi}\frac{\omega\tau\left[1 - (\omega\tau)^2\right]}{\left[1 + (\omega\tau)^2\right]^2}\sin(\delta\psi - \delta\psi')}$$

1.1.2 Fixed point analysis of $\tilde{\Gamma}$

The term $\tilde{\Gamma}(\delta\psi' - \delta\psi)$ refers to the phase difference (see section 1.1.1) which we wrote as $\tilde{\Gamma}(\delta\psi' - \delta\psi) = \Gamma(\delta\psi' - \delta\psi) - \Gamma(\delta\psi - \delta\psi') = \frac{\mathrm{d}\left(\delta\psi - \delta\psi'\right)}{\mathrm{d}t}$, since we know how to explicitly write this derivative, we can examine it as a dynamical system:

$$\tilde{\Gamma}(\delta\psi' - \delta\psi) = \frac{\mathrm{d}(\delta\psi - \delta\psi')}{\mathrm{d}t} = \frac{g_{\text{syn}}}{c_m}\frac{\epsilon}{\pi}\frac{\omega\tau\left[1 - (\omega\tau)^2\right]}{\left[1 + (\omega\tau)^2\right]^2}\sin(\delta\psi - \delta\psi')$$

Find what the fixed points of this system. What happens to the coupled oscillating neurons when we are at a fixed point of $\tilde{\Gamma}$, when $\tilde{\Gamma}(\delta\psi' - \delta\psi) = \frac{\mathrm{d}\left(\delta\psi - \delta\psi'\right)}{\mathrm{d}t} = 0$? Explain the dynamics of the oscillating neurons at the fixed points.

1.1.3 Stability analysis

Perform stability anlaysis on $\tilde{\Gamma}$ at the fixed points, consider the cases when the synapse is excitatory and inhibitory ($g_{\text{syn}} < 0$ and $g_{\text{syn}} > 0$) as well as what happens in high frequencies ($\omega > \frac{1}{\tau}$) and low frequencies ($\omega < \frac{1}{\tau}$)

2 Numerical Part

We've seen in class how we can simulate a neuron as an oscillator using the FitzHugh Nagumo model. In this section we will start by plotting various configurations and parameters of the FHN model. In order to simplify the model so that it contains less parameters, we write the following equations for the FHN model:

$$\begin{aligned}\dot{v} &= f(v) - w + I_{\text{ext}} \\ \dot{w} &= \epsilon(v + a - bw)\end{aligned}$$

$$\text{with } f(v) = v - \frac{v^3}{3}$$

The values for the simulation are the following: $\epsilon = 0.08, a = 0.7, b = 0.8, I_{\text{ext}} = 0.8$. Choose $\Delta t = 0.1$ and run the simulation for at least $t_{\text{end}} > 2000$ steps. If you are instructed to change one of the variables for a particular section, the rest should remain as written in here.

2.1 Plotting the FHN model

2.1.1 Plotting activity

Using the euler method, plot (v, w) for three or more points (initial conditions) in space, describe the behavior, is the system dependent on your choice of points? If it is, how? If not, explain.

2.1.2 Effect of external input I_{ext} on the system

Run the simulation for the values $I_{\text{ext}} = 0, I_{\text{ext}} \geq 1.5$ and 3 additional values of I_{ext} of your choice. Describe what happens to the system as I_{ext} is increased.

2.1.3 Effect of b on the system

Choose at least 5 values of b between $[0.8, 2]$, plot them all together on one graph. Describe the results as we increase b.

2.2 Oscillations in FHN

2.2.1 Cycles of v

Instead of (v, w), plot v as a function of time: (t, v)[3]. How long does the system take to finish one cycle?

2.2.2 Perturbation of initial conditions

Denote the initial condition from the previous part (v_0, w_0). We would like to see what happens to the phase when we start near (v_0, w_0). For your choice of values of δ_w, δ_v[4], plot (t, v) on the same figure for the following initial conditions: (v_0, w_0) and $(v_0 + \delta_v, w_0 + \delta_w)$. Describe the results. Can you explain why this happens?

2.3 Sensitivity of phase to perturbations

2.3.1 System reaction to a single push

Change your euler simulation so that a particular time step $t = 300 < t_1 < 1000$, $w_{t_1} \mapsto w_{t_1} + \delta_w, v_{t_1} \mapsto v_{t_1} + \delta_v$. What we are essentially doing is simulating what would happen if our system was perturbed by some $\vec{\delta}$ at time t_1. Plot (t, v) for this case as well as for the normal case for some (v_0, w_0) on the same figure. Explain what effects did the perturbation have on the system.

2.3.2 Sensitivity to perturbations in different phases

For the initial condition (v_0, w_0) from the previous section and for some $\vec{\delta}$ of the same magnitude from the previous section, Change t_1 to be other values around $[300, 680]$ and repeat the previous section using the new t_1. Describe the results for different values of t_1. Why does it behave the way it does? When is the phase most affected by perturbations?

2.4 Coupled oscillators

We now look at the case when two oscillators are coupled by having two sets of FitzHugh Nagumo along with an interaction term between them:

$$\begin{aligned}
\dot{v}_1 &= f(v_1) - w_1 + I_{\text{ext}} + \gamma (v_1 - v_2) \\
\dot{v}_2 &= f(v_2) - w_2 + I_{\text{ext}} + \gamma (v_2 - v_1)
\end{aligned}$$

$$\begin{aligned}
\dot{w}_1 &= \epsilon (v_1 + a - b w_1) \\
\dot{w}_2 &= \epsilon (v_2 + a - b w_2)
\end{aligned}$$

Here $0 \leq \gamma < 0.4$, the rest of the values are as written above.

[3] This means you still go through all of the simulation, only now you only plot v as a function of the steps

[4] Make sure it's not 0, and not larger than your initial condition though.

2.4.1 Coupled system behavior after a long time

Simulate the coupled FitzHugh Nagumo model written above, with initial conditions for v_1, v_2, w_1, w_2. Plot both trajectories of (v_1, w_1) and (v_2, w_2) on the same graph, on a different graph, plot both (t, v_1) and (t, v_2). What happens after a long period of time?

2.4.2 Perturbation from same initial condition

What happens if we choose the same initial condition for both (v_1, w_1) and (v_2, w_2) for long periods of time? What happens if $(v_2, w_2) = (v_1 + \delta, w_1 + \delta)$ for some small $\delta > 0$ for long periods of time? Find the frequency of oscillations of each neuron for this case. What is the phase difference between the neurons after a long period of time, and what can you conclude on $\tilde{\Gamma}$ in this case?