Dynamics of Computation in the Brain
76908
Solution EX #1

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Contents

1 Euler's Method for Numerical Integration

$$\dot{x} = f(x)$$
$$\hat{x}(t + \Delta t) = \hat{x}(t) + f(\hat{x}(t))\Delta t$$
$$\dot{x} = -\lambda x$$

1.1 Analytical Part

$$\epsilon(t) = |x(t) - \hat{x}(t)|$$

1.1.1 Single-step Simulation Error

(a) The approximated solution after a single-step is:

$$\hat{x}(0 + \Delta t) = \hat{x}(0) + f(\hat{x}(0))\Delta t = \hat{x}(0) - \lambda\hat{x}(0)\Delta t$$

Note $\hat{x}(0) = x_0$ the starting condition for the solution and we get:

$$\boxed{\hat{x}(\Delta t) = x_0 - \lambda x_0 \Delta t}$$

(b) The analytical solution as we know is $x(t) = x_0 e^{-\lambda t}$. The single-step error is:

$$\begin{aligned}\epsilon(\Delta t) &= |x(\Delta t) - \hat{x}(\Delta t)| = |x_0 e^{-\lambda\Delta t} - x_0 + \lambda x_0 \Delta t| \\ &= |x_0| \cdot |e^{-\lambda\Delta t} - 1 + \lambda\Delta t|\end{aligned}$$

Using the fact that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ (Tylor expansion of e^xat 0), we get:

$$\epsilon(\Delta t) = |x_0| \cdot |1 - \lambda\Delta t + \frac{\lambda^2 \Delta t^2}{2} + O(t^3) - 1 + \lambda\Delta t| = |x_0| \cdot |\frac{\lambda^2\Delta t^2}{2} + O(t^3)|$$

$$\boxed{\approx |x_0| \cdot \frac{\lambda^2 \Delta t^2}{2}}$$

1.1.2 Fixed-time Simulation Error

(a) Given t_1 and Δt , let n be the maximal integer such that: $n < \frac{t_1}{\Delta t}$ (less formally, this means $t_1 = n \cdot \Delta t$ up to a small residual). We then get:

$$\hat{x}(0) = x_0,\ \hat{x}(\Delta t) = x_0 - \lambda x_0 \Delta t = x_1,\ ...,\ \hat{x}(n\Delta t) = x_{n-1} - \lambda x_{n-1}\Delta t = x_n = \hat{x}(t_1)$$

So, for $1 \leq i \leq n$:

$$\frac{x_i}{x_{i-1}} = \frac{x_{i-1} - \lambda x_{i-1}\Delta t}{x_{i-1}} = 1 - \lambda\Delta t$$

And now we can get a closed formula:

$$\hat{x}(i\cdot\Delta t)=x_i=x_0(1-\lambda\Delta t)^i$$

Specifically:

$$\boxed{\hat{x}(t_1)=x_0(1-\lambda\Delta t)^n}$$

(b) First, note that $0<\lambda,\Delta t$, so we get $0<\lambda\cdot\Delta t$ and thus $-\lambda\Delta t<0$ and $1-\lambda\Delta t<1$. Ergo, we only need to consider what happens to $q\coloneqq 1-\lambda\Delta t$ in the ray $(-\infty,1)$. Given a fixed λ , this will singularly determine the different qualitative outcomes. Moreover, note $t_1\to\infty$ iff $n\to\infty$. Thus, it suffices to consider the asymptotic behavior of q^n. We'll look at 3 regimes: $(-\infty,-1],(-1,0),[0,1)$:

$$\begin{cases}0\le q<1 & q^n \underset{n\to\infty}{\searrow} 0\\ -1<q<0 & q^n \underset{n\to\infty}{\to} 0\\ q\le -1 & q^n\ diverges\end{cases}$$

I.e. , for $0\le q<1$, q^n will monotonically decrease to 0. For $-1<q<0$, q^n will still converge to 0 but will do it in an alternating fashion where the sign of q^nwill socialite between $+$ and $-$. Similar oscillations will occurs for $q\le-1$ however, in this case there will be no convergence. In the spacial case of $q=-1$, q^n will oscillate between ±1, otherwise, between $\pm\infty$. Now:

$$0\le 1-\lambda\Delta t<1\Leftrightarrow -1\le-\lambda\Delta t<0\Leftrightarrow 1\ge\lambda\Delta t>0\Leftrightarrow \frac{1}{\lambda}\ge\Delta t>0$$

$$-1<1-\lambda\Delta t<0\Leftrightarrow -2<-\lambda\Delta t<-1\Leftrightarrow 2>\lambda\Delta t>1\Leftrightarrow \frac{2}{\lambda}>\Delta t>\frac{1}{\lambda}$$

$$1-\lambda\Delta t\le-1\Leftrightarrow -\lambda\Delta t\le-2\Leftrightarrow \lambda\Delta t\ge2\Leftrightarrow \Delta t\ge\frac{2}{\lambda}$$

Thus we're left with:

$$\boxed{\begin{cases}\frac{1}{\lambda}\ge\Delta t>0 & (0<\lambda\Delta t\le1)\ monotonical\ \ convergence\\ \frac{2}{\lambda}>\Delta t>\frac{1}{\lambda} & (1<\lambda\Delta t<2)\ alternating\ \ convergence\\ \Delta t\ge\frac{2}{\lambda} & (2<\lambda\Delta t)\ divergence\end{cases}}$$

(c) The error is:

$$\epsilon(t_1)=|x(t_1)-\hat{x}(t_1)|=|x_0e^{-\lambda t_1}-x_0(1-\lambda\Delta t)^n|=|x_0||e^{-\lambda t_!}-(1-\lambda\Delta t)^n|$$

Assume for simplicity $|x_0|=1$:

$$\epsilon(t_1)=|e^{-\lambda t_1}-e^{n\ln(2-\lambda\Delta t)}|=|e^{-\lambda t_1}-e^{n\ln(1-\lambda\Delta t)}|$$

We use Tylor's expansion around 0 of $ln(1+x)=x-\frac{x^2}{2}+o(x^3)$. For this to work, we'll add the condition that $(\lambda\Delta t=)x\ll1$ such that $ln(1-\lambda\Delta t)$ is defined.Also, remember $n\approx\frac{t_1}{\Delta t}$ so:

$$\epsilon(t_1)\approx|e^{-\lambda t_1}-e^{\frac{t_1}{\Delta t}(-\lambda\Delta t-\frac{(\lambda\Delta t)^2}{2})}|=|e^{-\lambda t_1}-e^{-\lambda t_1-\frac{1}{2}\lambda^2t_1\Delta t}|=e^{-\lambda t_1}\cdot|1-e^{-\frac{1}{2}\lambda^2t_1\Delta t}|$$

Once more we'll use Tylor series expansion of e^x around 0 up too the first order for $x = -\frac{1}{2}\lambda^2 t_1 \Delta t$. Note that the firs order is enough as we can have Δt as small as we'd like, specifically small enough such that $\left(-\frac{1}{2}\lambda^2 t_1 \Delta t\right)^2$ is a negligible term. Moreover, from the conclusions of article (b), we'll add the condition that $\lambda \Delta t \ll 1$.We then get:

$$e^{-\frac{1}{2}\lambda^2 t_1 \Delta t} \approx 1 - \frac{1}{2}\lambda^2 t_1 \Delta t$$

And:

$$e^{-\lambda t_1} \cdot |1 - e^{-\frac{1}{2}\lambda^2 t_1 \Delta t}| \approx e^{-\lambda t_1} \cdot |1 - 1 + \frac{1}{2}\lambda^2 t_1 \Delta t| = e^{-\lambda t_1} \cdot \frac{1}{2}\lambda^2 t_1 \Delta t$$

Remember to account for $|x_0|$ and we get:

$$\boxed{\epsilon(t_1) \approx \frac{1}{2}|x_0|\lambda^2 t_1 \Delta t e^{-\lambda t_1}}$$

(d)

$$\boxed{\Delta t \quad \& \quad t_1 e^{-\lambda t_1}}$$

The leading order of Δt is its linear term, i.e. the power of 1. Also, the term in the error affected by t_1 is $t_1 e^{-\lambda t_1}$. This term has a big impact for small $t_1's$ and $t_1 e^{-\lambda t_1} \approx t_1$, however, for large enough $t_1's$, $t_1 e^{-\lambda t_1}$ goes quickly to 0.

(e) In article (a) we showed (for $\frac{1}{\lambda} \geq \Delta t > 0$):

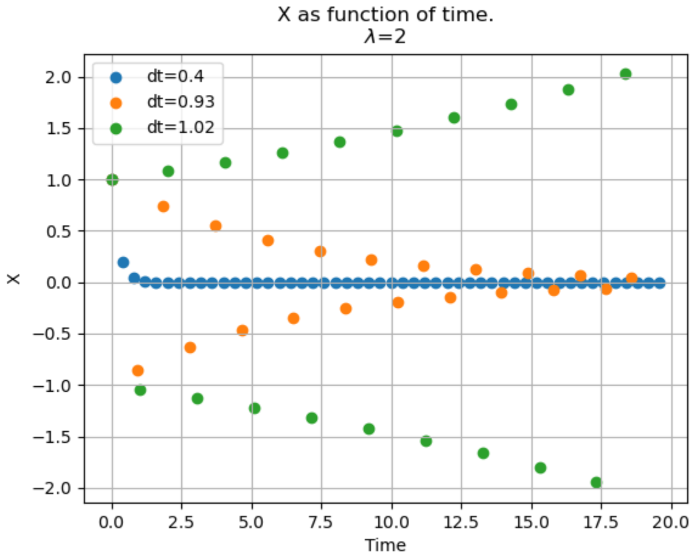
$$\hat{x}(t_1) = x_0(1 - \lambda \Delta t)^n = x_0 \left(1 - \frac{\lambda t_1}{n}\right)^n \underset{n \to \infty}{\rightarrow} x_0 e^{\lambda t_1} = x(t_1)$$

Thus, from arithmetics of limits:

$$\boxed{\epsilon(t_1) = |x(t_1) - \hat{x}(t_1)| \underset{n \to \infty}{\rightarrow} |x(t_1) - x(t_1)| = 0}$$

1.2 Numerical Part

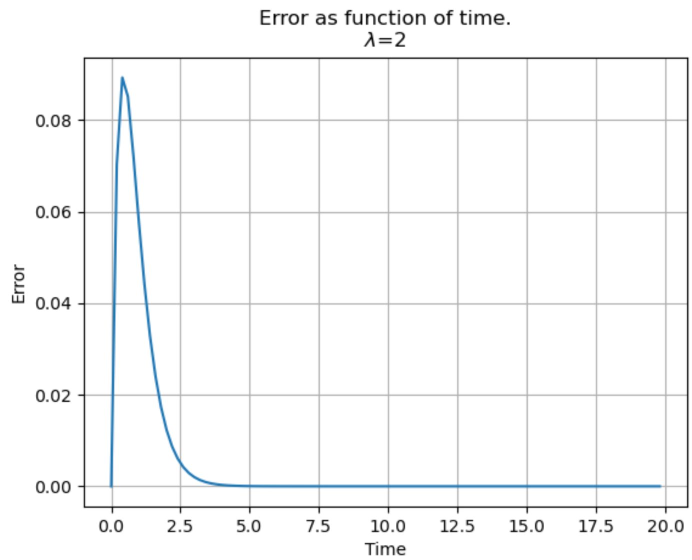
1.2.1 Numerical Simulations of $\hat{x}(t)$



Here $x_0 = 1, t_1 = 20, \lambda = 2$**.**

1.2.2 Error Function

Here $x_0 = 1, t_1 = 20, \lambda = 2$ **and** $dt = 0.2$**.**



(a+b) As explained in 1.1.2 **we can see that for small** t **the error function increases with** t **but** $\epsilon(t) \underset{t\to\infty}{\to} 0$**.**

1.3 Qualitative Observation

In the previous sections, we saw numerically and analytically that size of Δt does matter. First, as we saw in 1.1.2, it important to have $\Delta t < \frac{1}{\lambda}$ and keep $\left(-\frac{1}{2}\lambda^2 t_1 \Delta t\right)^2$ small. On the other hand, as $\Delta t \to 0$, the number of iterations needed in order to approximate $x(t_1)$ increases. Moreover, our analysis was limited to a very specific case where $\dot{x} = -\lambda x$. This case practically calls for a first order approximation like Euler's Method, however, for other functions, this may not show such a "nice" behavior. When trying to determine Δt, as a rule of thumb, I'd first consider the timescales of the system we're trying to simulate. E.g. when dealing with dynamics of an action potential we're talking in milliseconds but weather changes in minuets if not hours. <u>I suggest choosing the largest Δt possible such that $\Delta t < \frac{1}{\lambda}$ and is also minimal with respect to the dynamical system.</u>

2 Differential equation with time-dependent input

$$\dot{x}(t) = ax(t) + I(t)$$

$$x(t) = x_0 e^{at} + \int_0^t I(\tilde{t}) e^{a(t-\tilde{t})} d\tilde{t}$$

we want to show

$$\frac{d}{dt}\left(x_0 e^{at} + \int_0^t I(\tilde{t}) e^{a(t-\tilde{t})} d\tilde{t}\right) = ax(t) + I(t)$$

Using the linearity of the derivation operator to break the sum:

$$\frac{d}{dt}\left(x_0 e^{at}\right) = a x_0 e^{at}$$

And:

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^t I(\tilde{t}) e^{a(t-\tilde{t})} d\tilde{t}\right) &= \frac{d}{dt}\left(e^{at}\int_0^t I(\tilde{t}) e^{-a\tilde{t}} d\tilde{t}\right) \\
&= \frac{d}{dt}\left(e^{at}\right) \cdot \int_0^t I(\tilde{t}) e^{-a\tilde{t}} d\tilde{t} + e^{at} \cdot \frac{d}{dt}\left(\int_0^t I(\tilde{t}) e^{-a\tilde{t}} d\tilde{t}\right) \\
&=_1 a e^{at} \int_0^t I(\tilde{t}) e^{-a\tilde{t}} d\tilde{t} + e^{at} I(t) e^{-at} \\
&= a \int_0^t I(\tilde{t}) e^{a(t-\tilde{t})} d\tilde{t} + I(t)
\end{aligned}$$

Where 1 **is due to the fundamental theorem of calculus. Together we have:**

$$\begin{aligned}\frac{d}{dt}\left(x_0e^{at}+\int\limits_0^t I(\tilde{t})e^{a(t-\tilde{t})}d\tilde{t}\right) &= ax_0e^{at}+a\int\limits_0^t I(\tilde{t})e^{a(t-\tilde{t})}d\tilde{t}+I(t)\\ &=a\left(x_0e^{at}+\int\limits_0^t I(\tilde{t})e^{a(t-\tilde{t})}d\tilde{t}\right)+I(t)\\ &=ax(t)+I(t)\end{aligned}$$

QED.