Dynamics of Computation in the Brain - Exercise 1

Due May 23, 2024

1 Euler's Method for Numerical Integration

Euler's Method for numerical integration is the simplest way to solve differential equations on a computer, i.e. simulate them. It will most likely be sufficient for the numerical exercises throughout this course. In particular, for a system of differential equations given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Euler's method yields an approximate solution $\hat{\mathbf{x}}(t)$ by fixing a time step Δt and iterating:

$$\hat{\mathbf{x}}(t + \Delta t) = \hat{\mathbf{x}}(t) + \mathbf{f}(\hat{\mathbf{x}}(t))\,\Delta t$$

In this exercise you will gain an understanding for the effectiveness of this method, in particular in relation to the step-size, Δt, by applying it to solve a differential equation which we are able to solve analytically:

$$\dot{x} = -\lambda x$$

Throughout the exercise we will assume $\lambda > 0$.

1.1 Analytical Part

Assume an initial condition $x(0) = x_0$.
Define the error:

$$\epsilon(t) = |x(t) - \hat{x}(t)|$$

1. **Single-step Simulation Error**
   (a) Calculate the Euler's Method approximate solution after a single-step, $\hat{x}(\Delta t)$.
   (b) Using the analytical solution, estimate the single-step error, $\epsilon(\Delta t)$, up to the leading order[1] of Δt. What power of Δt is the leading order? Hint: Use the power series expansion of e^x.
2. **Fixed-time Simulation Error**
   (a) Solve for Euler's Method approximate solution $\hat{x}(t_1)$ for arbitrary time t_1.
   (b) As $t_1 \to \infty$ find the three different ranges of values of Δt that will lead to different qualitative outcomes.
   (c) Estimate $\epsilon(t_1)$ for small Δt up to leading order in Δt for a fixed t_1. What determines the scale for how small Δt needs to be in order for this estimate to be a good one? **Hint:** notice that $(1+x)^y = e^{y\ln(1+x)}$.
   (d) What power of Δt is the leading order? How does the error depend on t_1?
   (e) Show that if we take the step-size Δt to be small enough the error goes to zero for all orders of Δt, i.e. $\epsilon(t_1) \to 0$ as $\Delta t \to 0$. You can use the identity: $\lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N = e^x$.

[1] The leading-order term of an expression is the term with the largest order of magnitude. Since other terms in the expression are smaller (and regarded as negligible), the leading order term gives the main behaviour of the expression (the true behaviour is only small deviations away from it). In our case, since $\Delta t < 1$, the leading order is the first power of Δt for which the error is not zero.

1.2 Numerical Part

Use python/matlab to solve the differential equation numerically using Euler's method.

1. Show numerically the three qualitative outcomes you found in question (2b) in the previous section. Don't forget to mention the parameters you used for each of the three graphs.
2. Simulate the system for $\lambda = 2, t_1 = 20$ and plot the error $\epsilon(t)$ as a function of time. The error plot should match the expression in question (2c):
   (a) What is the leading order of the expression in (2c) for small t_1? Does is match the behavior of the graph?
   (b) How does the expression in (2c) behave as $t_1 \to \infty$?

1.3 Qualitative Observation

How does the quality of the approximation obtained depend on the step-size and the length of the simulation? How does it depend on the dynamical equations themselves (that is, on the function $f(x)$)? Can you suggest a rule of thumb for how to select Δt in order to make this method useful?

2 Differential equation with time-dependent input

Show that the solution:

$$x(t) = x_0 e^{at} + \int\limits_0^t I(\tilde{t}) e^{a(t-\tilde{t})} d\tilde{t}$$

solves the differential equation:

$$\dot{x}(t) = ax(t) + I(t)$$