Dynamics of Computation in the Brain
76908
Solution EX #3

Barak Haim

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Contents

1 Analytical part

1.1 Coupled oscillators

$$\frac{d\delta\psi}{dt} = \Gamma(\delta\psi, \delta\psi') = \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} d\theta \overrightarrow{R}(\theta - (\delta\psi - \delta\psi'))\overrightarrow{S}(\theta)$$

$$R(\theta) = \begin{cases} 0 & \theta < 0 \\ \frac{g_{syn}}{c_m}\left(\frac{\theta}{\omega\tau}\right)e^{-\frac{\theta}{\omega\tau}} & \theta \geq 0 \end{cases}$$

1.1.1 Approximate integral solution

$$\begin{aligned}
\Gamma(\delta\psi, \delta\psi') = & \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} d\theta \overrightarrow{R}(\theta - (\delta\psi - \delta\psi'))\overrightarrow{S}(\theta) \\
= & \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} d\theta \overrightarrow{R}(\theta)\overrightarrow{S}(\theta - (\delta\psi' - \delta\psi)) \\
= & \frac{\epsilon}{2\pi}\int_{0}^{\infty} d\theta \frac{g_{syn}}{c_m}\left(\frac{\theta}{\omega\tau}\right)e^{-\frac{\theta}{\omega\tau}} sin(\theta - (\delta\psi' - \delta\psi)) \\
= & \frac{\epsilon}{2\pi}\omega\tau\frac{g_{syn}}{c_m}\int_{0}^{\infty} d\left(\frac{\theta}{\omega\tau}\right) sin\left(\omega\tau\left(\frac{\theta}{\omega\tau}\right) - (\delta\psi' - \delta\psi)\right)\left(\frac{\theta}{\omega\tau}\right)e^{-\frac{\theta}{\omega\tau}} \\
= & \frac{\epsilon}{2\pi}\omega\tau\frac{g_{syn}}{c_m}\int_{0}^{\infty} d\left(\frac{\theta}{\omega\tau}\right) sin\left(\omega\tau\left(\frac{\theta}{\omega\tau}\right) - (\delta\psi' - \delta\psi)\right)\left(\frac{\theta}{\omega\tau}\right)e^{-\frac{\theta}{\omega\tau}} \\
= & \frac{\epsilon}{2\pi}\omega\tau\frac{g_{syn}}{c_m}\int_{0}^{\infty} d\left(\frac{\theta}{\omega\tau}\right) \frac{e^{i\left(\omega\tau\left(\frac{\theta}{\omega\tau}\right)-(\delta\psi'-\delta\psi)\right)} - e^{-i\left(\omega\tau\left(\frac{\theta}{\omega\tau}\right)-(\delta\psi'-\delta\psi)\right)}}{2i}\left(\frac{\theta}{\omega\tau}\right)e^{-\frac{\theta}{\omega\tau}} \\
= & \frac{\epsilon}{2\pi}\omega\tau\frac{g_{syn}}{c_m}\frac{1}{2i}\int_{0}^{\infty} d\left(\frac{\theta}{\omega\tau}\right)\left[\left(e^{-(\delta\psi'-\delta\psi)i}e^{\left(\omega\tau\left(\frac{\theta}{\omega\tau}\right)i\right)}\right) - \left(e^{(\delta\psi'-\delta\psi)i}e^{\left(-\omega\tau\left(\frac{\theta}{\omega\tau}\right)i\right)}\right)\right]\left(\frac{\theta}{\omega\tau}\right)e^{-\frac{\theta}{\omega\tau}} \\
= & \frac{\epsilon}{2\pi}\omega\tau\frac{g_{syn}}{c_m}\frac{1}{2i}\int_{0}^{\infty} dx\left[\left(e^{-(\delta\psi'-\delta\psi)i}e^{(\omega\tau xi)}xe^{-x}\right) - \left(e^{(\delta\psi'-\delta\psi)i}e^{(-\omega\tau xi)}xe^{-x}\right)\right] \\
= & \frac{\epsilon}{2\pi}\omega\tau\frac{g_{syn}}{c_m}\frac{1}{2i}\left[\left(e^{-(\delta\psi'-\delta\psi)i}\int_{0}^{\infty} dxe^{(\omega\tau xi)}xe^{-x}\right) - \left(e^{(\delta\psi'-\delta\psi)i}\int_{0}^{\infty} dxe^{(-\omega\tau xi)}xe^{-x}\right)\right] \\
= & \frac{\epsilon}{2\pi}\omega\tau\frac{g_{syn}}{c_m}\frac{1}{2i}\left[\frac{e^{-i(\delta\psi'-\delta\psi)}}{(1-i\omega\tau)^2} - \frac{e^{i(\delta\psi'-\delta\psi)}}{(1+i\omega\tau)^2}\right]\int_{0}^{\infty} xdxe^{-x} \\
= & \frac{g_{syn}}{c_m}\frac{\epsilon}{2\pi}\frac{\omega\tau}{[1+(\omega\tau)^2]^2}\left[\left((\omega\tau)^2 - 1\right)sin(\delta\psi' - \delta\psi) + 2\omega\tau cos(\delta\psi' - \delta\psi)\right]
\end{aligned}$$

Now:

$$
\begin{aligned}
\tilde{\Gamma}(\delta\psi' - \delta\psi) = \Gamma(\delta\psi, \delta\psi') - \Gamma(\delta\psi, \delta\psi')& \\
=& \frac{g_{syn}}{c_m}\frac{\epsilon}{2\pi}\frac{\omega\tau}{\left[1+(\omega\tau)^2\right]^2}\left[\left((\omega\tau)^2-1\right)sin(\delta\psi'-\delta\psi)+2\omega\tau cos(\delta\psi'-\delta\psi)\right] \\
&- \\
&\frac{g_{syn}}{c_m}\frac{\epsilon}{2\pi}\frac{\omega\tau}{\left[1+(\omega\tau)^2\right]^2}\left[\left((\omega\tau)^2-1\right)sin(\delta\psi-\delta\psi')+2\omega\tau cos(\delta\psi-\delta\psi')\right] \\
=& \frac{g_{syn}}{c_m}\frac{\epsilon}{2\pi}\frac{\omega\tau}{\left[1+(\omega\tau)^2\right]^2} \\
&\times \\
&\left[\left((\omega\tau)^2-1\right)sin(\delta\psi'-\delta\psi)+2\omega\tau cos(\delta\psi'-\delta\psi)-\left((\omega\tau)^2-1\right)sin(\delta\psi-\delta\psi')-2\omega\tau cos(\delta\psi-\delta\psi')\right]
\end{aligned}
$$

Note that *cosine* **is an even function and thus:**

$$cos(\delta\psi'-\delta\psi) = cos(-1(\delta\psi'-\delta\psi)) = cos(\delta\psi-\delta\psi')$$

and so we get:

$$2\omega\tau cos(\delta\psi'-\delta\psi) - 2\omega\tau cos(\delta\psi-\delta\psi') = 0$$

Also, as *sine* **is an odd function:**

$$
\begin{aligned}
\left((\omega\tau)^2-1\right)sin(\delta\psi'-\delta\psi)-\left((\omega\tau)^2-1\right)sin(\delta\psi-\delta\psi') &= \left((\omega\tau)^2-1\right)\left[sin(\delta\psi'-\delta\psi)-sin(\delta\psi-\delta\psi')\right] \\
&= \left((\omega\tau)^2-1\right)\left[sin(\delta\psi'-\delta\psi)-\left(-sin(-\left(\delta\psi-\delta\psi'\right))\right)\right] \\
&=2\left((\omega\tau)^2-1\right)sin(\delta\psi'-\delta\psi)
\end{aligned}
$$

Altogether we get:

$$
\begin{aligned}
\tilde{\Gamma}(\delta\psi'-\delta\psi) &= \frac{g_{syn}}{c_m}\frac{\epsilon}{2\pi}\frac{\omega\tau}{\left[1+(\omega\tau)^2\right]^2}2\left((\omega\tau)^2-1\right)sin(\delta\psi'-\delta\psi) \\
&=\frac{g_{syn}}{c_m}\frac{\epsilon}{2\pi}\frac{\omega\tau}{\left[1+(\omega\tau)^2\right]^2}2\left((\omega\tau)^2-1\right)\cdot -1\cdot sin(-1\left(\delta\psi'-\delta\psi\right)) \\
&=\frac{g_{syn}}{c_m}\frac{\epsilon}{2\pi}\frac{\omega\tau}{\left[1+(\omega\tau)^2\right]^2}2\left(1-(\omega\tau)^2\right)sin(\delta\psi-\delta\psi') \\
&=\frac{g_{syn}}{c_m}\frac{\epsilon}{\pi}\frac{\omega\tau\left(1-(\omega\tau)^2\right)}{\left[1+(\omega\tau)^2\right]^2}sin(\delta\psi-\delta\psi')
\end{aligned}
$$

QED.

1.1.2 Fixed point analysis of $\tilde{\Gamma}$

$$\tilde{\Gamma}(\delta\psi'-\delta\psi) = \frac{d(\delta\psi-\delta\psi')}{dt} = \frac{g_{syn}}{c_m}\cdot\frac{\epsilon}{\pi}\cdot\frac{\omega\tau\left[1-(\omega\tau)^2\right]}{\left[1+(\omega\tau)^2\right]^2}\cdot sin(\delta\psi-\delta\psi')$$

Let $=0$**, then:**

$$\frac{g_{syn}}{c_m}\cdot\frac{\epsilon}{\pi}\cdot\frac{\omega\tau\left[1-(\omega\tau)^2\right]}{\left[1+(\omega\tau)^2\right]^2}\cdot sin(\delta\psi-\delta\psi') = 0$$

If and only if #1. $sin(\delta\psi - \delta\psi') = 0$ **or #2.** $1 - (\omega\tau)^2 = 0$**. As the cell's time constant** (τ) **and frequency** (ω) **should be strictly positive (there is no sense in it otherwise), #2 is not possible and we're left with #1, i.e.:**

$$sin(\delta\psi - \delta\psi') = 0$$

iff $(\delta\psi - \delta\psi') = 0$ **or** $(\delta\psi - \delta\psi') = \pi$**. Note there's no need to look outside of** $[0, 2\pi)$ **as we're looking at the difference in phase. and the sine function is periodical. We get than, 2 possible situations:**

1. $(\delta\psi - \delta\psi') = 0$: The neurons fire together.

2. $(\delta\psi - \delta\psi') = \pi$: The neurons are out of phase, i.e., fire alternatingly.

1.1.3 Stability analysis

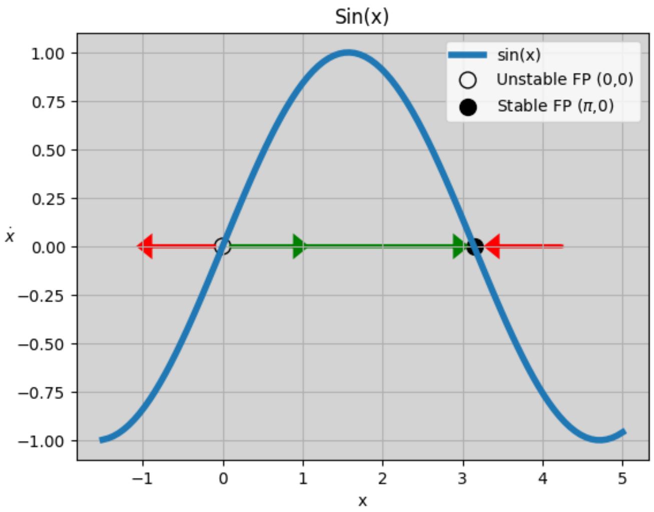
1. We'll use Strogatz's geometric way of analyzing flow on the line (see section 2.1 in his book "Nonlinear Dynamics And Chaos", specifically, figure 2.1.1). Note that is basically the sine function multiplied by a constant, strictly negative or positive. The constant itself doesn't make any qualitative difference, the only thing that matters is the sign of the constant. Now, Note there are only 2 terms which may be negative: $g_{syn}$and $\left[1 - (\omega\tau)^2\right]$. We get then 4 possible states in which all in all give 2 possible outcomes - $sin(x)$ or $-sin(x)$ up too multiplication by a (positive) constant. The case of excitatory or inhibitory synapse determines the sign of g_{syn}. Also:

$$\omega < \frac{1}{\tau} \Longrightarrow 1 - (\omega\tau)^2 < 0$$
$$\frac{1}{\tau} < \omega \Longrightarrow 0 < 1 - (\omega\tau)^2$$

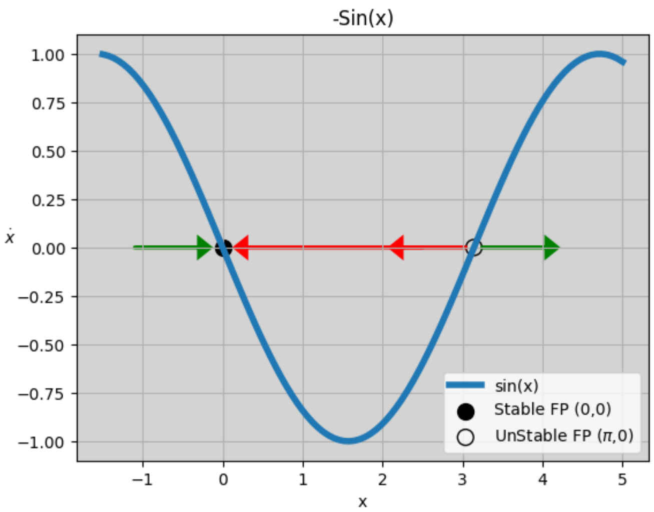
Together we get:

$$1.\ \left[\omega < \frac{1}{\tau} \& g_{syn} < 0\right] \quad OR \quad \left[\frac{1}{\tau} < \omega \& 0 < g_{syn}\right] \quad \Longrightarrow \quad 0 < g_{syn}\left[1 - (\omega\tau)^2\right]$$
$$2.\ \left[\omega < \frac{1}{\tau} \& 0 < g_{syn}\right] \quad OR \quad \left[\omega < \frac{1}{\tau} \& 0 < g_{syn}\right] \quad \Longrightarrow \quad g_{syn}\left[1 - (\omega\tau)^2\right] < 0$$

Now we get:
1. $0 < g_{syn}\left[1 - (\omega\tau)^2\right] \Rightarrow$ **0 is an Unstable** FP and π **is a Stable** FP as $\tilde{\Gamma}(\delta\psi' - \delta\psi)$ behaves like $sin(x)$. Graphical it looks like this:



2. $g_{syn}\left[1-(\omega\tau)^2\right] < 0 \Rightarrow$ **0 is a Stable** FP and π **is an Unstable** FP as $\tilde{\Gamma}(\delta\psi' - \delta\psi)$ behaves like $-sin(x)$. Graphical it looks like this:



Here, the green arrows mark regimes when $0 < \dot{x}$ and so x will increase . Red arrows are the opposite, they mark regimes where x tends to decrease. A stable fixed point is one such that to its left, x tends to increase (thus pushing the dynamics back to the FP if it perturbs below it) and to its right, x tends to decrease (also, back towards the FP). Note that this is equivalent to looking at the eigenvalues of system at the FP's.

2 Numerical Part

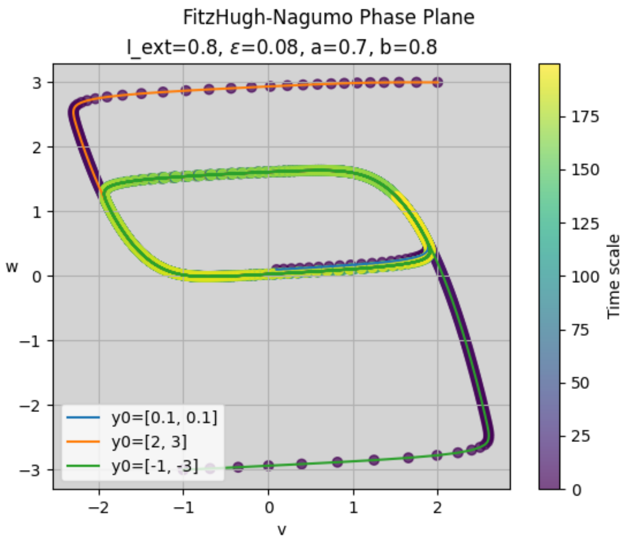
$$\dot{v} = f(v) - w + I_{ext}$$

$$\dot{w} = \epsilon \cdot (v + a - bw)$$

$$f(v) = v - \frac{v^3}{3}$$

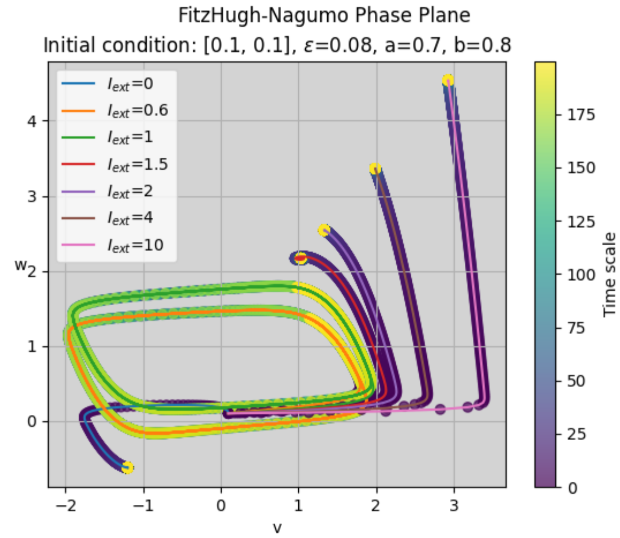
2.1 Plotting the FHN model

2.1.1 Plotting activity of different initial conditions



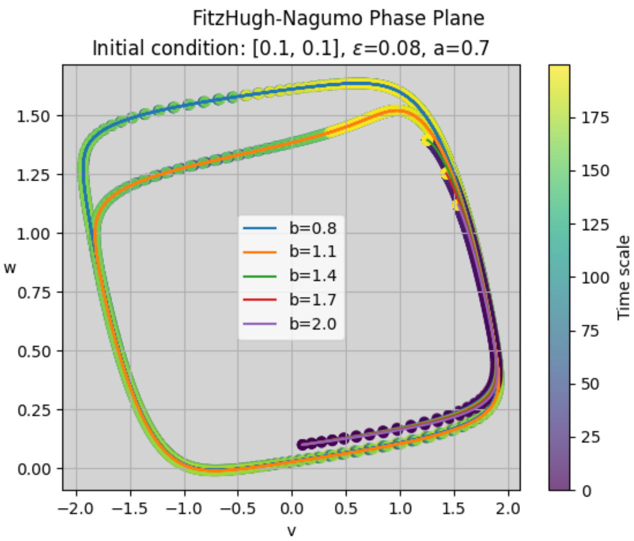
We see here that although the starting points are different, the dynamics converges to the same cycle as it is a limit cycle. I.e. the dynamics is attracted to a stable trajectory in the phase space and keeps at it. Thus, the behavior of the system is not dependent on your choice of points.

2.1.2 Effect of external input I_{ext} on the system



We see here that the value of I_{ext} changes the behavior of the system from convergence to a fixed point to a limit cycle. For $0 \leq b < 0.6$ the system converges to a fixed point. For $b's$ around 0.8, a limit cycle emerge's and past 1, again we go back to a fixed point. We can also see the location of the fixed point changes and as it seems, also the time it takes to converge. Note this is based solely on observation of the graphs and not an analytical analysis.

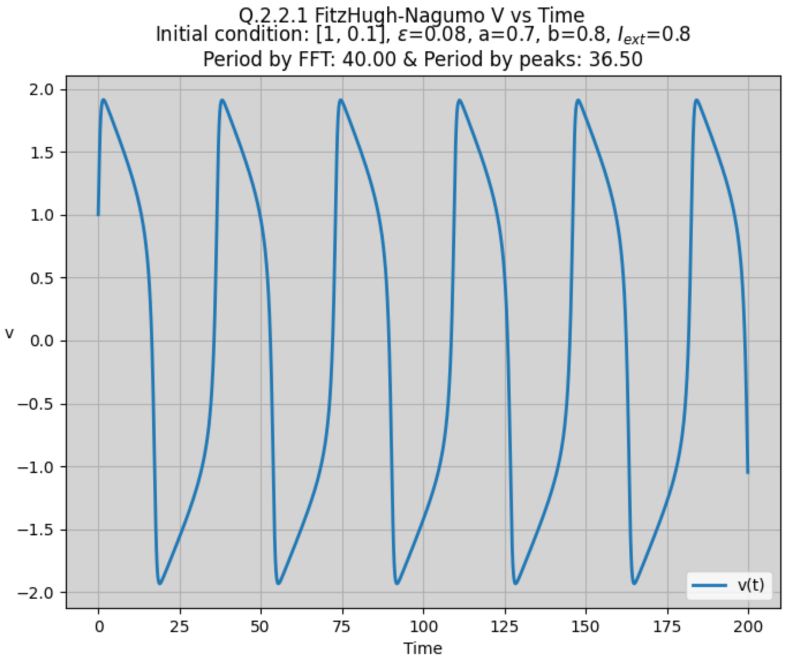
2.1.3 Effect of b on the system



Again, we see a limit cycle emerging only for small enough values of b. Otherwise, theres a fixed point. It's like increasing the value of b "slows down" the transition to phase #3 (by David's numbering system of the cycle) until it stop altogether at a fixed point.

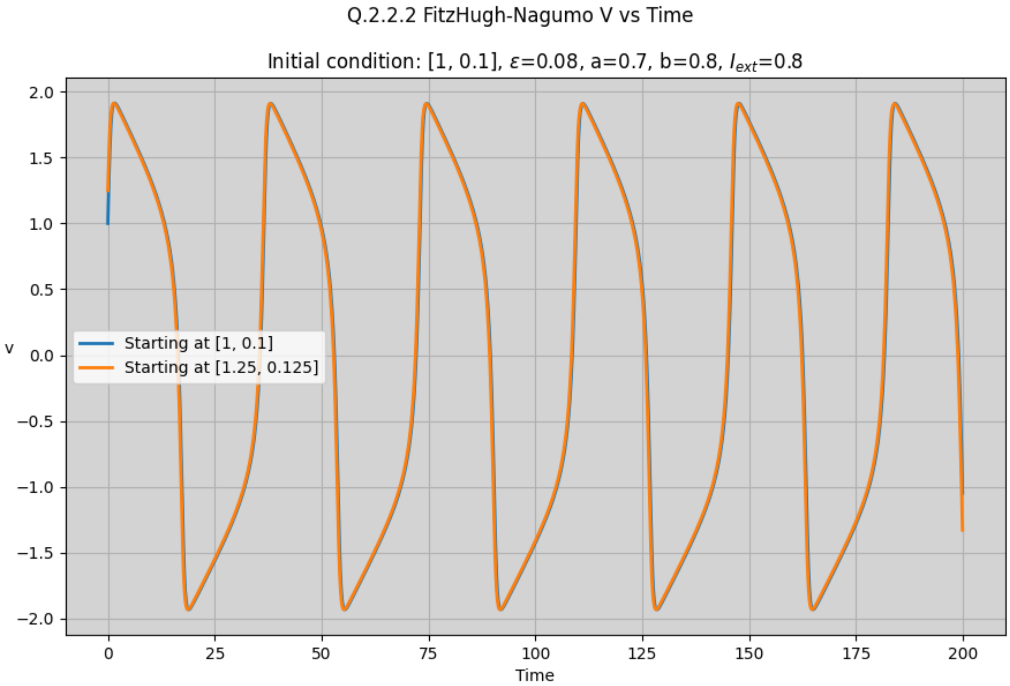
2.2 Oscillations in FHN

2.2.1 Cycles of v



The system take about 36-40 units of time (Milliseconds in the real world but...) to complete a cycle. I used two methods to verify that. One is by FFT like in exercise #2 and the the other is finding the peaks and taking the mean of the distances between them. Also, by looking at the graph, we can see it's just about right.

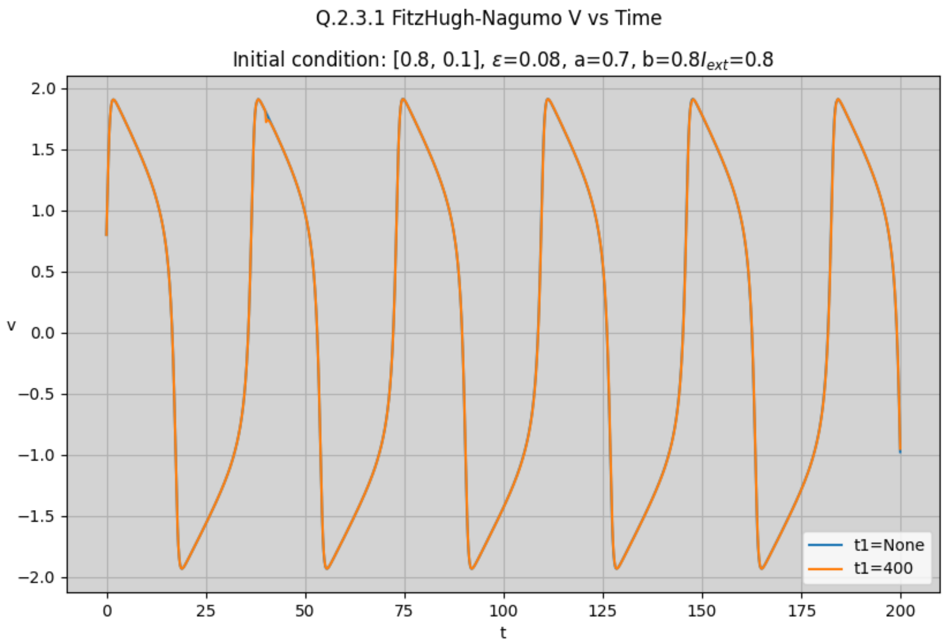
2.2.2 Perturbation of initial conditions



Here, the perturbation of initial conditions change the starting values slightly but before too long everything becomes the same. This is the meaning of converging to a limit cycle - you may start different but everything converges to the same trajectory eventually (similar to fixed point).

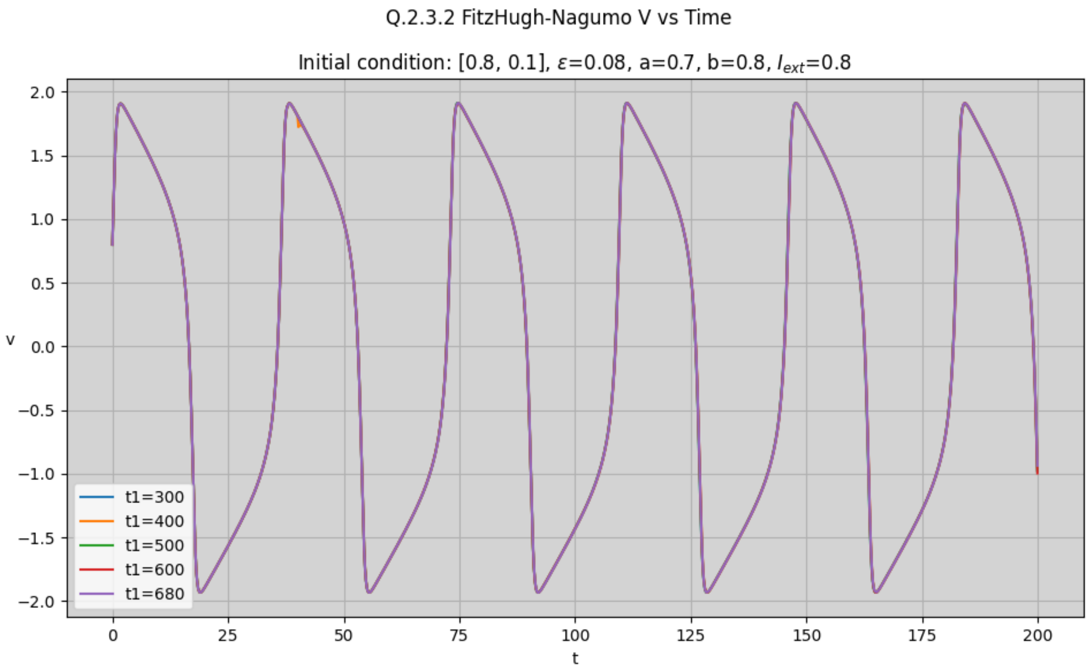
2.3 Sensitivity of phase to perturbations

2.3.1 System reaction to a single push



Much like in article 2.2.2, the perturbation may change the trajectory slightly in one moment in time but changes nothing in the grand scheme of things. Here, the situation is even more dire as the perturbation occurs at a much later time (at time-step $t_1 = 400$, i.e at $t = 40$.), much closer to the limit cycle and thus much less powerful. To the effect it's not even visible on the plot.

2.3.2 Sensitivity to perturbations in different phases



Again, we see the effects of the limit cycle - nothing really changes. The most sensitive time for a perturbation are near the extreme points where v changes direction. In the plot we can see that at time-step $t_1 = 400$ (i.e. $t = 40$) where we can see a slight change in values. Also, at time-step $t_1 = 600$ (i.e. $t = 60$) where we can see it causes a slight overshoot in the end.

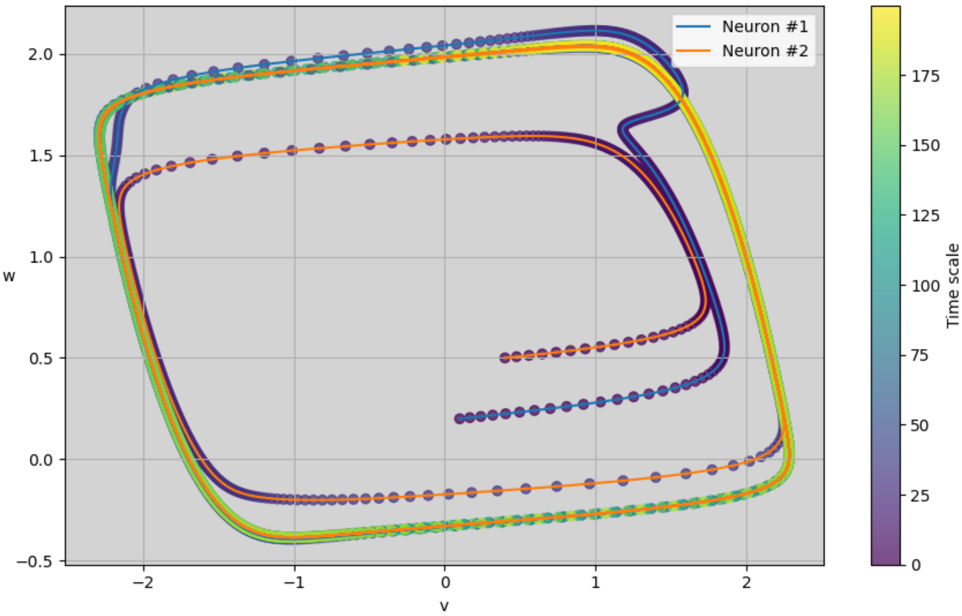
2.4 Coupled oscillators

$$\begin{gathered}
\dot{v}_1 = f(v_1) - w_1 + I_{ext} + \gamma(v_1 - v_2) \\
\dot{v}_2 = f(v_2) - w_2 + I_{ext} + \gamma(v_2 - v_1) \\
\dot{w}_1 = \epsilon \cdot (v_1 + a - bw_1) \\
\dot{w}_2 = \epsilon \cdot (v_2 + a - bw_2)
\end{gathered}$$

2.4.1 Coupled system behavior after a long time

Here we see the coupled neurons at different initial conditions get looked out of phase (when one fires the other refractors) after some time, although they converge to the same limit cycle.

Q.2.4: Coupled oscillators - FitzHugh-Nagumo Phase Plane
I_{ext}=0.8, ε=0.08, a=0.7, b=0.8, γ=0.2
v1=0.1, w1=0.2 & v2=0.4, w2=0.5



Q.2.4: Coupled oscillators - FitzHugh-Nagumo Phase Plane
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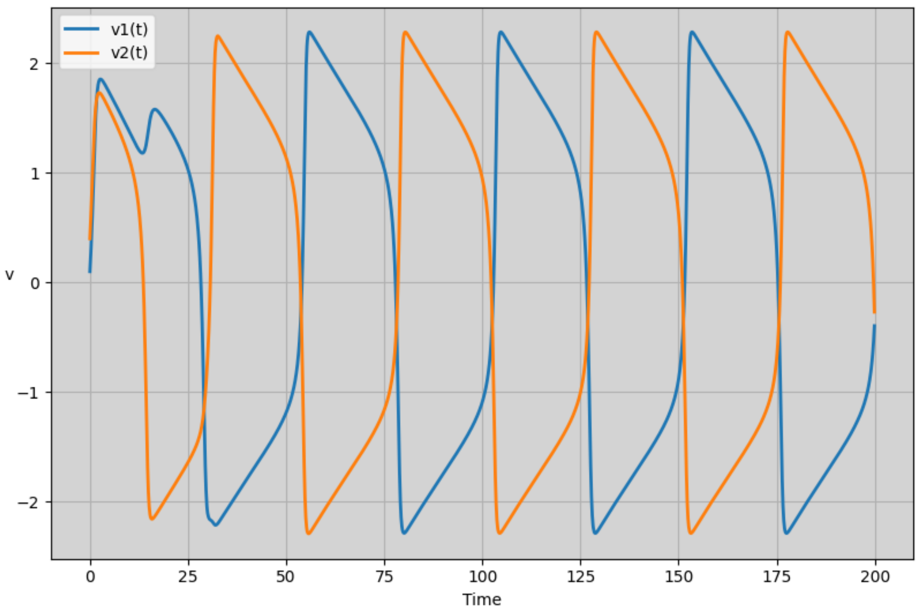
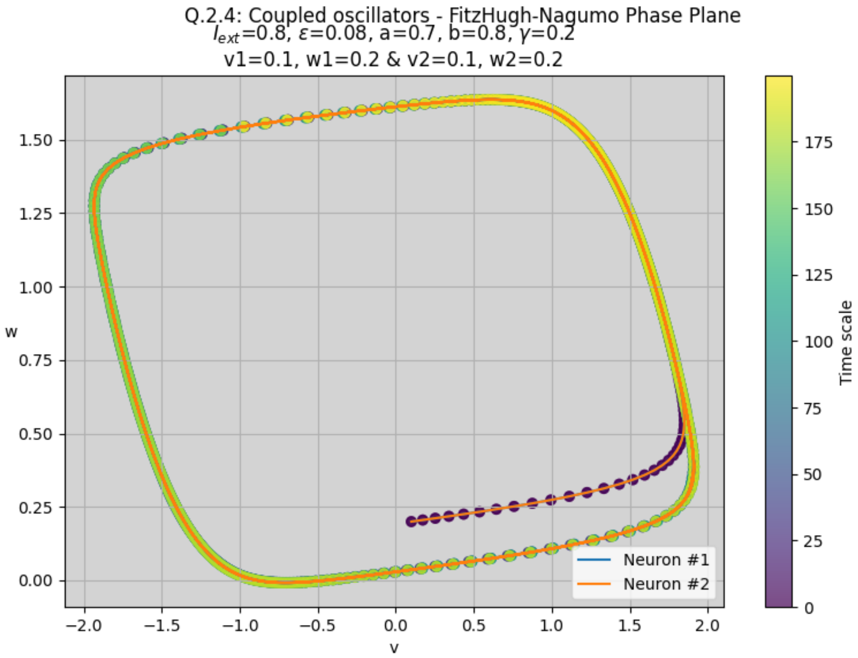
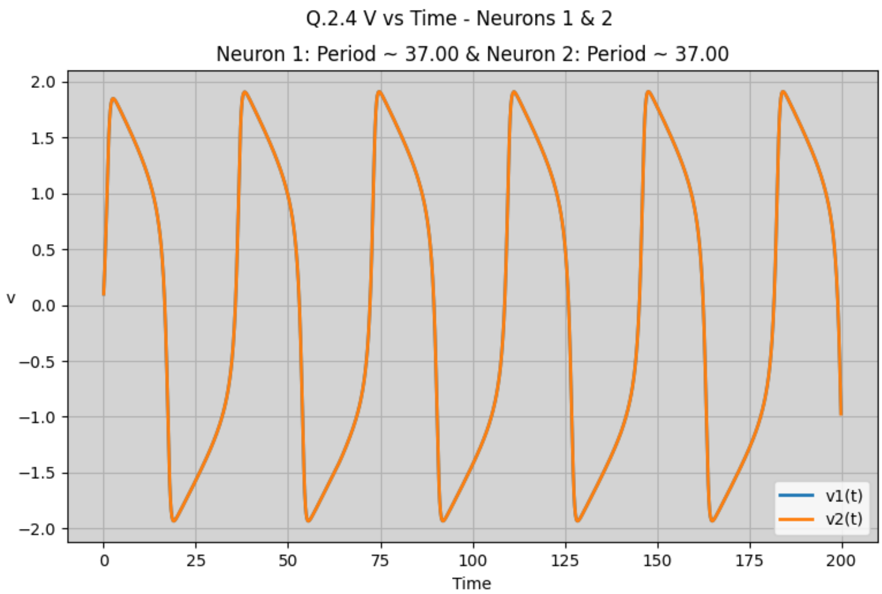


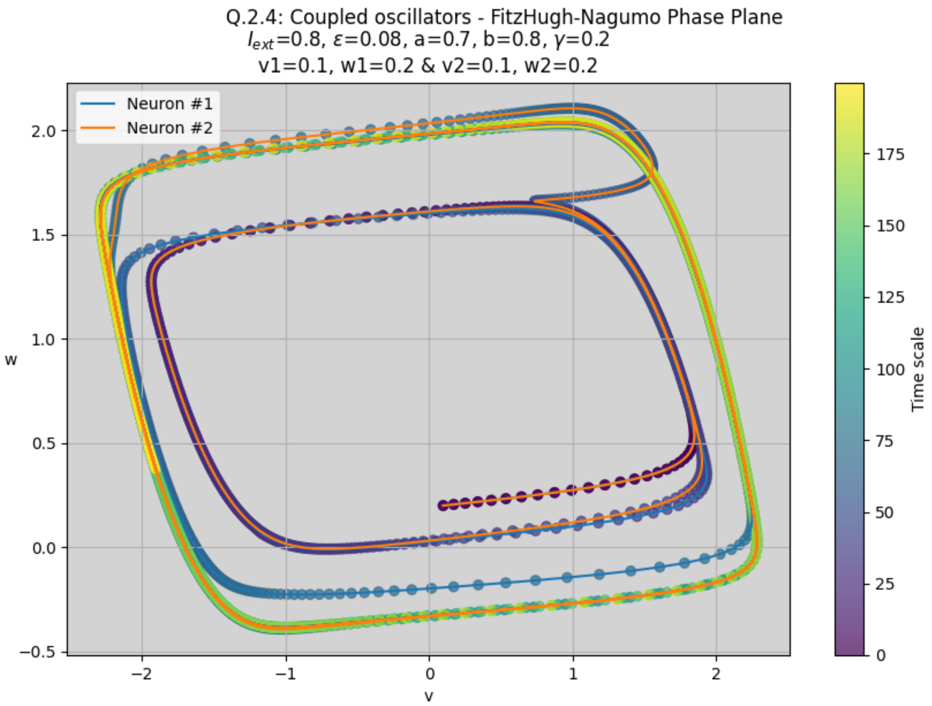
2.4.2 Perturbation from same initial condition

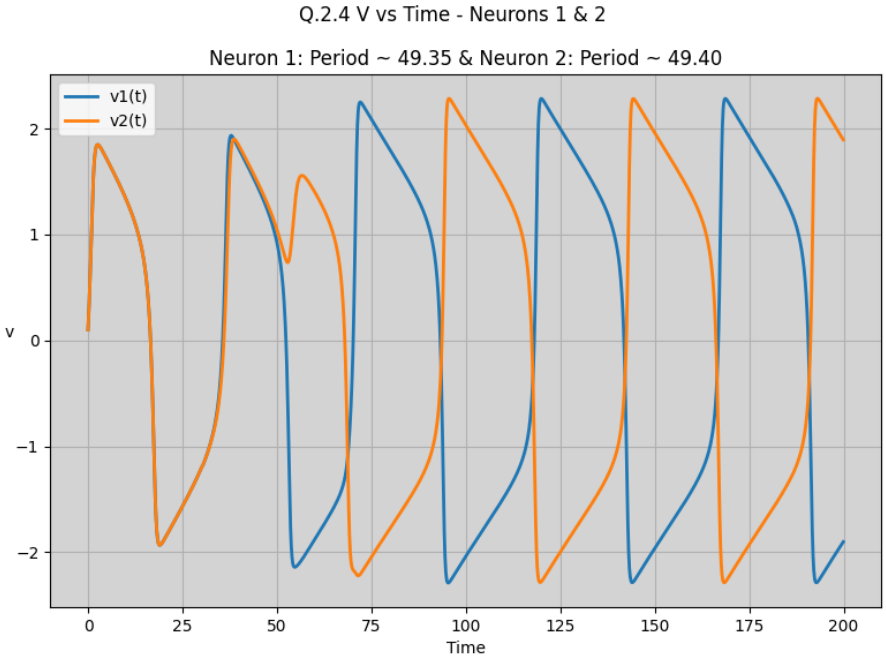
Here we see the neurons at the same initial condition without perturbation behave exactly the same.





When we add a perturbation at time-step 300 (i.e. $t = 30$), we get the neurons get looked out of phase. They still converge to a the same limit cycle after some time. Here, the perturbation has some noticeable effect on the the trajectory. The neurons take about 49.4 time units to finish one cycle and they do it out of phase, i.e. when one fires, the other is in its refractory period and vise versa.





Q.2.4 V vs Time - Neurons 1 & 2
Neuron 1: Period ~ 49.35 & Neuron 2: Period ~ 49.40
v1(t)
v2(t)
v
Time