Dynamics of Computation in the Brain
76908
Solution EX #2

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Contents

1 The Fourier Series

1.1 Fourier Series Of A Step Function

$$f(x) = \begin{cases} -1 & -\pi < x < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < x < \pi \end{cases}$$

First we note the function is symmetric around 0, i.e. an even function ($f(x) = f(-x)$). Thus for all n we get that $b_n = 0$. Now let's deal with a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx = \frac{1}{\pi}\left(\left[\int_{-\pi}^{-\frac{\pi}{2}} -1\cdot cos(nx)dx\right] + \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\cdot cos(nx)dx\right] + \left[\int_{\frac{\pi}{2}}^{\pi} -1\cdot cos(nx)dx\right]\right)$$
$$=\frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(nx)dx - \int_{-\pi}^{-\frac{\pi}{2}} cos(nx)dx - \int_{\frac{\pi}{2}}^{\pi} cos(nx)dx\right)$$

Because cos is an even function ($cos(x) = cos(-x)$) we get:$\int_{-\pi}^{-\frac{\pi}{2}} cos(nx)dx = \int_{\frac{\pi}{2}}^{\pi} cos(-nx)dx = \int_{\frac{\pi}{2}}^{\pi} cos(nx)dx$ **and so:**

$$a_n = \frac{1}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(nx)dx - 2\int_{\frac{\pi}{2}}^{\pi} cos(nx)dx\right)$$

Using the same argument we get: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(nx)dx = \int_{-\frac{\pi}{2}}^{0} cos(nx)dx + \int_{0}^{\frac{\pi}{2}} cos(nx)dx = 2\int_{0}^{\frac{\pi}{2}} cos(nx)dx$ **and so:**

$$a_n = \frac{1}{\pi}\left(2\int_{0}^{\frac{\pi}{2}} cos(nx)dx - 2\int_{\frac{\pi}{2}}^{\pi} cos(nx)dx\right) = \frac{2}{\pi}\left(\int_{0}^{\frac{\pi}{2}} cos(nx)dx - \int_{\frac{\pi}{2}}^{\pi} cos(nx)dx\right)$$

Using the fact that $\int cos(nx)dx = \frac{1}{n}sin(nx)$ **we get:**

$$a_n = \frac{2}{\pi}\left(\int_{0}^{\frac{\pi}{2}} cos(nx)dx - \int_{\frac{\pi}{2}}^{\pi} cos(nx)dx\right) = \frac{2}{\pi n}\left(sin(nx)\mid_0^{\frac{\pi}{2}} - sin(nx)\mid_{\frac{\pi}{2}}^{\pi}\right)$$
$$=\frac{2}{\pi n}\left(sin(n\frac{\pi}{2}) - \underbrace{sin(0)}_{=0} - sin(n\pi) + sin(n\frac{\pi}{2})\right)$$
$$=\frac{2}{\pi n}\left(2sin(n\frac{\pi}{2})\right) = \frac{4}{\pi n}sin(\frac{n\pi}{2})$$

Now note that for all $n \in \mathbb{N}$: $sin(\frac{2\pi n}{2}) = 0$, $sin(\frac{2n\pi}{2} + \frac{\pi}{2}) = 1$ and $sin(\frac{2n\pi}{2} + \frac{3\pi}{2}) = -1$ and so:

$$a_n = \frac{4}{\pi n}\begin{cases} 0 & n = 2m \\ 1 & n = 2m+1 \\ -1 & n = 2m+3 \end{cases}$$

For the spacial case of a_0:

$$a_0 = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{-\frac{\pi}{2}} -1dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1dx + \int_{\frac{\pi}{2}}^{\pi} -1dx = -1\cdot\frac{\pi}{2} + 1\cdot\pi - 1\cdot\frac{\pi}{2} = 0$$

At last, we get the series expansion:

$$f(x) = \frac{4}{\pi n}\left(\sum_{n=1,5,9,...} cos(nx) - \sum_{n=3,7,11,...} cos(nx)\right)$$

1.2 Compare $f(x)$ To The Function We Saw In Class

In class we saw a function also named $f(x)$, here I'll refer to it as g:

$$g(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases} = \sum_{n=1,3,5,...} \frac{4}{\pi n} sin(nx)$$

Denote $\bar{f}, \bar{g} : \mathbb{R} \to \{-1, 1\}$ be the expansion of the functions f and g to all of the real numbers in the following way:

For all $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$such that $|x| \in [(2n-1)\pi, (2n+1)\pi]$. Let $\tilde{x} = |x| - (2n-1)$. Note that $\tilde{x} \in [-\pi, \pi]$. We then define:

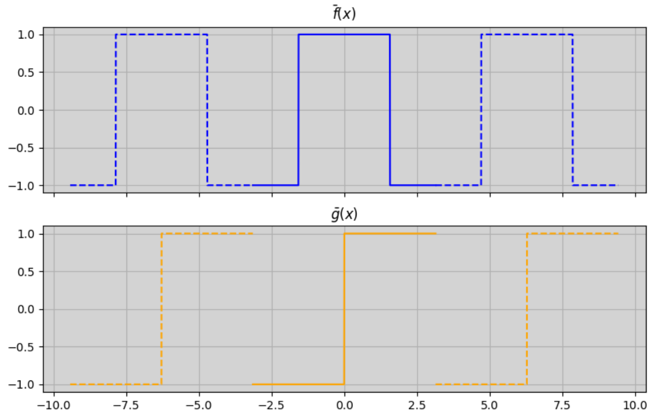
$$\bar{f}(x) = \begin{cases} f(\tilde{x}) & if\ |x| \in ((2n-1)\pi, (2n+1)\pi) \\ \underset{t \nearrow \pi}{lim}(f(t)) & \tilde{x} = \pi \\ \underset{t \searrow \pi}{lim}(f(t)) & \tilde{x} = -\pi \end{cases}$$

and $\bar{g}$ is defined in the same fashion. Note that when $n = \pi, -\pi$ we take the limit from the left or the right sides respectively, so it would make sense from the way f & g are defined. This is basically a concatenation of of the functions across the real line in a periodic manner. Just copy and paste the function infinitely many times and complete an $\aleph_0$ amount of undefined concatenation points using the defined limits. We get here bound periodic functions with at most $\aleph_0$ points of discontinuity. From reasons of functional analysis that I would not get into here, there is a uniform convergence of $\bar{f}'s$ & $\bar{g}'s$ Fourier series to them in all of the real line (note that this is not true for any function but ours are good enough). Moreover, all the arguments we've needed in order to calculate f & $g's$ Fourier decomposition apply here. Thus:

$$\bar{f}(x) = \frac{4}{\pi n}\left(\sum_{n=1,5,9,...} cos(nx) - \sum_{n=3,7,11,...} cos(nx)\right)$$

$$\bar{g}(x) = \sum_{n=1,3,5,...} \frac{4}{\pi n} sin(nx) = \frac{4}{\pi n}\left(\sum_{n=1,5,9,...} sin(nx) + \sum_{n=3,7,11,...} sin(nx)\right)$$

It looks like this:



Now, note that $\bar{f}$ is merely a shift of $\bar{g}$ by $\frac{\pi}{2}$ just like the cosine functions is a shift of sine function. I.e.:

$$\begin{aligned}\bar{g}(x+\frac{\pi}{2}) = \frac{4}{\pi n}&\left(\sum_{n=1,5,9,\ldots} sin(n(x+\frac{\pi}{2})) + \sum_{n=3,7,11,\ldots} sin(n(x+\frac{\pi}{2}))\right)\\ =&\frac{4}{\pi n}\left(\sum_{n=1,5,9,\ldots} sin(nx+\frac{n\pi}{2}) + \sum_{n=3,7,11,\ldots} sin(nx+\frac{n\pi}{2})\right)\\ \underset{\bigstar}{=}&\frac{4}{\pi n}\left(\sum_{n=1,5,9,\ldots} cos(nx) - \sum_{n=3,7,11,\ldots} cos(nx)\right) = \bar{f}(x)\end{aligned}$$

Where $\bigstar$ is from the fact that:

$$\sum_{n=1,5,9,\ldots} sin(nx+\frac{n\pi}{2}) = \sum_{n=1,5,9,\ldots}\left[sin(nx)cos\left(\frac{n\pi}{2}\right) + cos(nx)sin\left(\frac{n\pi}{2}\right)\right] = \sum_{n=1,5,9,\ldots} cos(nx)$$

and:

$$\sum_{n=3,7,11,\ldots} sin(nx+\frac{n\pi}{2}) = \sum_{n=3,7,11,\ldots}\left[sin(nx)cos\left(\frac{n\pi}{2}\right) + cos(nx)sin\left(\frac{n\pi}{2}\right)\right] = \sum_{n=3,7,11,\ldots} -cos(nx)$$

Lastly, we just need to remember $\bar{f}|_{(-\pi,\pi)}, \bar{g}|_{(-\pi,\pi)} = f, g$.

2 Emergence of Oscillations

$$\begin{aligned}\tau\dot{h}_E &= -h_E + J_{EE}r_E - J_{EI}r_I + h_E^0\\ \tau\dot{h}_I &= -h_I + J_{IE}r_E - J_{II}r_I + h_I^0\end{aligned}$$

$$0 < J_{\alpha\beta}, h_\alpha^0 \ \&\ \tau = 1 \ \&\ 0 < h_I^0 < h_E^0$$

2.1 Analytic Part

$$J_{II}=0\ ,\ J_{IE}=1,\ \tau=1$$

$$\begin{aligned}\dot{h_E}&=-\,h_E+J_{EE}\left\lfloor h_E\right\rfloor_+-J_{EI}\left\lfloor h_I\right\rfloor_++h_E^0\\ \dot{h_I}&=-\,h_I+\left\lfloor h_E\right\rfloor_++h_I^0\end{aligned}$$

2.1.1 Dynamical Equations In Vector Notation By Regime - $\dot{h}=Mh+h_0$

We can write: $\dot{h}=Mh+h_0$for $h_0=\left(\begin{array}{cc}h_E^0, & h_I^0\end{array}\right)^T$, $h=\left(\begin{array}{cc}h_E, & h_I\end{array}\right)^T$and a 2×2 matrix M which depends on the values of $\left\lfloor h_\alpha\right\rfloor_+$. For each $\alpha\in\{E,I\}$, either $h_\alpha\leq 0$ or $0<h_\alpha$. Note, here h is used as a vector notation instead of $\vec{h}$for connivance. Altogether we have 4 conditions:

1. $0<h_E$ And $0<h_I$ (i.e. $\left\lfloor h_\alpha\right\rfloor_+=h_\alpha$):

$$M=\left[\begin{array}{cc}-1+J_{EE} & -J_{EI}\\ 1 & -1\end{array}\right]$$

2. $0<h_E$ But $h_I\leq 0$ (i.e. $\left\lfloor h_E\right\rfloor_+=h_E$ and $\left\lfloor h_I\right\rfloor_+=0$):

$$M=\left[\begin{array}{cc}-1+J_{EE} & 0\\ 1 & -1\end{array}\right]$$

3. $h_E\leq 0$ But $0<h_I$ (i.e. $\left\lfloor h_E\right\rfloor_+=0$ and $\left\lfloor h_I\right\rfloor_+=h_I$):

$$M=\left[\begin{array}{cc}-1 & -J_{EI}\\ 0 & -1\end{array}\right]$$

4. $h_E\leq 0$ And $h_I\leq 0$ (i.e. $\left\lfloor h_\alpha\right\rfloor_+=0$):

$$M=\left[\begin{array}{cc}-1 & 0\\ 0 & -1\end{array}\right]$$

2.1.2 Fixed Points By Regimes

Let h^*be a fixed point (FP) of the system, then - $0=Mh^*+h_0$:

1. The equations defined by are ($0=Mh^*+h_0$):

$$M=\left[\begin{array}{cc}-1+J_{EE} & -J_{EI}\\ 1 & -1\end{array}\right]$$

$$\begin{cases}1) & 0=(-1+J_{EE})h_E^*-J_{EI}h_I^*+h_E^0\\ 2) & 0=h_E^*-h_I^*+h_I^0\end{cases}$$

From #2 we get: $h_I^*=h_E^*+h_I^0$. Plug it to #1:

$$0=(-1+J_{EE})h_E^*-J_{EI}(h_E^*+h_I^0)+h_E^0=-h_E^*+J_{EE}h_E^*-J_{EI}h_E^*-J_{EI}h_I^0+h_E^0$$

Iff:

$$(1 - J_{EE} + J_{EI})h_E^* = h_E^* - J_{EE}h_E^* + J_{EI}h_E^* = -J_{EI}h_I^0 + h_E^0$$

So:

$$h_E^* = \frac{h_E^0 - J_{EI}h_I^0}{1 - J_{EE} + J_{EI}}$$

Plug it in to #2 and we get:

$$h_I^* = \frac{h_E^0 - J_{EI}h_I^0}{1 - J_{EE} + J_{EI}} + h_I^0$$

As we assumed here that $0 < h_E$ And $0 < h_I$ and that $0 \neq 1 - J_{EE} + J_{EI}$ (iff $J_{EI} \neq J_{EE} - 1$) such that the FP is defined.
This is possible with the conditions that:

$$0 < 1 - J_{EE} + J_{EI} \quad AND \quad 0 < h_E^0 - J_{EI}h_I^0$$

Or:

$$1 - J_{EE} + J_{EI} < 0 \quad AND \quad h_E^0 - J_{EI}h_I^0 \ < 0$$

2. $det(M) = ((-1 + J_{EE}) \cdot (-1)) - (1 \cdot 0) = 1 - J_{EE}$ and so:

$$M^{-1} = \frac{1}{1 - J_{EE}} \begin{bmatrix} -1 & 0 \\ -1 & -1 + J_{EE} \end{bmatrix}$$

And $h^* = M^{-1}(-1 \cdot h_0)$. Note this is only possible in case that $1 - J_{EE} \neq 0$, iff $1 \neq J_{EE}$. Furthermore, we get:

$$\begin{bmatrix} h_E^* \\ h_I^* \end{bmatrix} = \frac{1}{1 - J_{EE}} \begin{bmatrix} -1 & 0 \\ -1 & -1 + J_{EE} \end{bmatrix} \begin{bmatrix} -h_E^0 \\ -h_I^0 \end{bmatrix} = \frac{1}{1 - J_{EE}} \begin{bmatrix} h_E^0 \\ h_E^0 + h_I^0(1 - J_{EE}) \end{bmatrix}$$

So:

$$\begin{cases} h_E^* = & \frac{h_E^0}{1 - J_{EE}} \\ h_i^* = & \frac{h_E^0}{1 - J_{EE}} + h_I^0 = h_E^* + h_I^0 \end{cases}$$

Remember, we assumed $0 < h_E$ But $h_I \leq 0$.
This is not possible as both $0 < h_E^*, h_I^0$ and thus $0 < h_i^* = h_E^* + h_I^0$ and so $0 < h_I$ in contradiction to our assumption.

3. $det(M) = ((-1) \cdot (-1)) - ((-J_{EI}) \cdot 0) = 1$ and so:

$$M^{-1} = \begin{bmatrix} -1 & J_{EI} \\ 0 & -1 \end{bmatrix}$$

And $h^* = M^{-1}(-1 \cdot h_0)$. We get:

$$\begin{bmatrix} h_E^* \\ h_I^* \end{bmatrix} = \begin{bmatrix} -1 & J_{EI} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -h_E^0 \\ -h_I^0 \end{bmatrix} = \begin{bmatrix} h_E^0 - h_I^0 J_{EI} \\ h_I^0 \end{bmatrix}$$

So:

$$\begin{cases} h_E^* = & h_E^0 - h_I^0 J_{EI} \\ h_i^* = & h_I^0 \end{cases}$$

Remember, here we demand $h_E \leq 0$ But $0 < h_I$. This means we must have $h_E^0 - h_I^0 J_{EI} \ll 0$ which **is possible** for large enough $h_I^0 J_{EI}$.

4. $det(M) = ((-1) \cdot (-1)) - (0 \cdot 0) = 1$ and so:

$$M^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = M$$

And $h^* = M^{-1}(-1 \cdot h_0)$. We get:

$$\begin{bmatrix} h_E^* \\ h_I^* \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -h_E^0 \\ -h_I^0 \end{bmatrix} = \begin{bmatrix} h_E^0 \\ h_I^0 \end{bmatrix}$$

So:

$$\begin{cases} h_E^* = & h_E^0 \\ h_i^* = & h_I^0 \end{cases}$$

Remember we assumed $h_E \leq 0$ And $h_I \leq 0$, but, also, $0 < h_\alpha^0$. so: $0 < h_\alpha^*$ and thus $0 < h_\alpha$ in contradiction to the assumption. Thus: **This is not possible.**

2.1.3 Stability Analysis

We'll us the fact that for a 2D linear system defined by a matrix M, its eigenvalues are given by:

$$\mu_{1,2} = \frac{tr(M) \pm \sqrt{tr(M)^2 - 4det(m)}}{2}$$

1. $0 < h_E$ And $0 < h_I$: $det(M) = 1 - J_{EE} + J_{EI}$ and $tr(M) = -2 + J_{EE}$

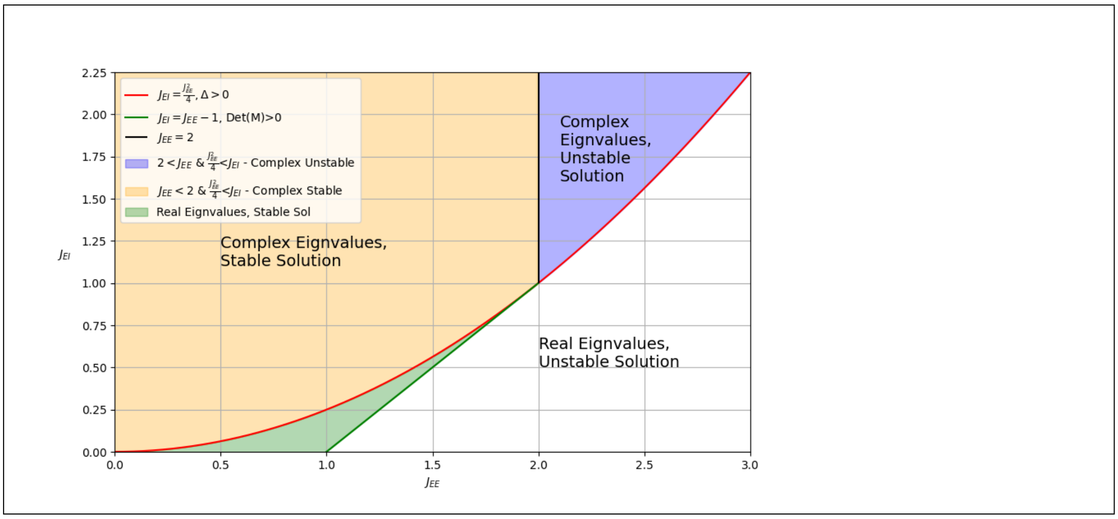
$$M = \begin{bmatrix} -1 + J_{EE} & -J_{EI} \\ 1 & -1 \end{bmatrix}$$

$$\mu_{1,2} = \frac{-2 + J_{EE} \pm \sqrt{4 - 4J_{EE} + J_{EE}^2 - 4 + 4J_{EE} - 4J_{EI})}}{2} = \frac{-2 + J_{EE} \pm \sqrt{J_{EE}^2 - 4J_{EI}}}{2}$$

First, we consider the case when the eigenvalues are real. That is $0 < J_{EE}^2 - 4J_{EI}$ (iff $J_{EI} < \frac{J_{EE}^2}{4}$). Then, the dynamics would be stable iff $tr(M) < 0$ (iff $J_{EE} < 2$) and $0 < det(M) = 1 - J_{EE} + J_{EI}$ (iff $J_{EE} - 1 < J_{EI}$), i.e. :

$$J_{EE} - 1 < J_{EI} \;\; \&\& \;\; J_{EE} < 2 \;\; \&\& \;\; J_{EI} < \frac{J_{EE}^2}{4}$$

Are the conditions for real eigenvalues and a stable solution. For $J_{EI} \leq J_{EE} - 1$ the eigenvalues are real and the solution is not stable.
Now, in the case when the eigenvalues are complex, i.e. $\frac{J_{EE}^2}{4} < J_{EI}$. In this case the sign of the real part of the eigenvalues would be determined by $tr(M)$, when it's negative (iff $J_{EE} < 2$), the dynamics converges and otherwise diverges. As we've already calculated, the is separated by $J_{EE} = 2$. Plotting it altogether we get:



2. $0 < h_E$ But $h_I \leq 0$: Not Possible.
3. $h_E \leq 0$ But $0 < h_I$: $det(M) = 1$ and $tr(M) = -2$

$$M = \begin{bmatrix} -1 & -J_{EI} \\ 0 & -1 \end{bmatrix}$$

$$\mu_{1,2} = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2}{2} \pm \frac{0}{2} = -1$$

I.e. , the FP is stable for any values of J_{EE} (which is not even a factor here) and J_{EI}

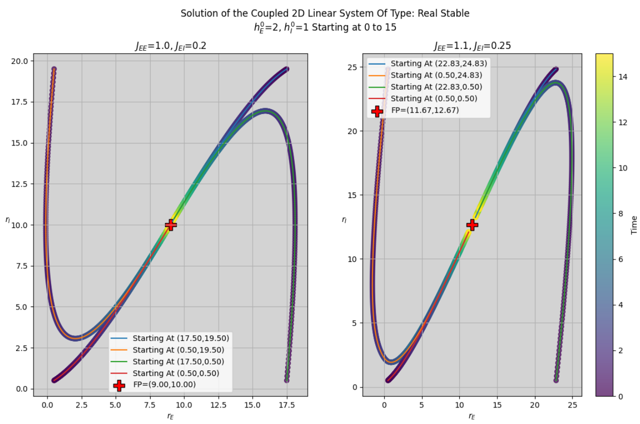
4. $h_E \leq 0$ And $h_I \leq 0$: Not Possible.

2.2 Numerical Part

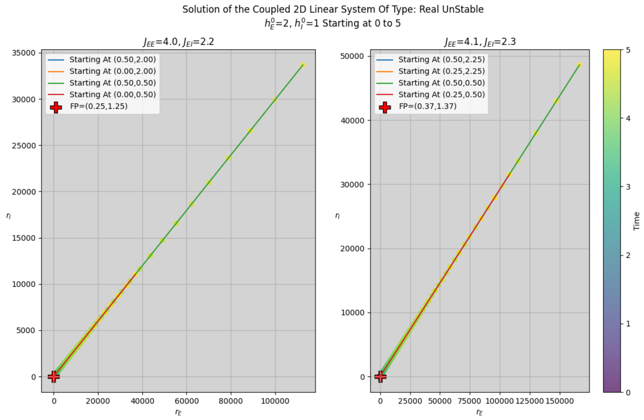
2.2.1 Simulating The System

For $h_E^0 = 2$ and $h_I^0 = 1$ Here are the plots:

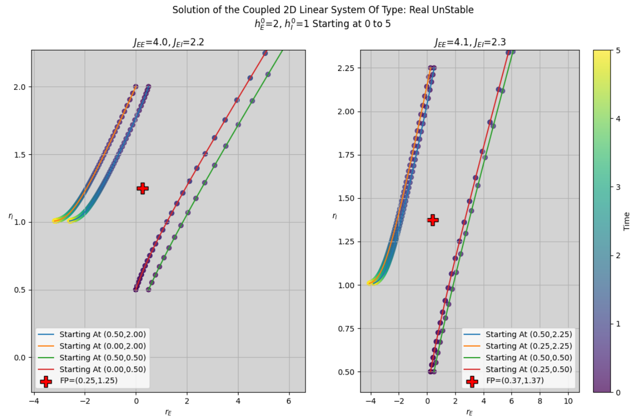
1. Real Stable:



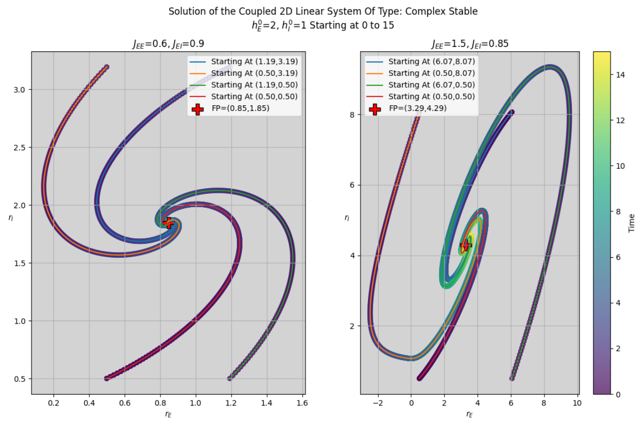
2. Real Unstable:



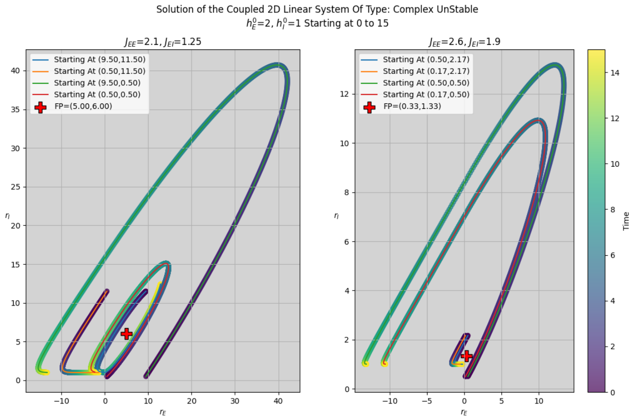
And zoomed in to show 4 unique staring points (the legend rounds it up...):



3. Complex Stable:



4. Complex Unstable:



2.3 Bonus

2.3.1 $J_{EI} = 4$, $J_{EE} = 2$

The value past which the system loses stability is $J_{EE} = 2$ as was explained in the stability analysis in section 2.1.3.1. We get the matrix and eigenvalues at that point:

$$M = \begin{bmatrix} -1 + J_{EE} & -J_{EI} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -1 \end{bmatrix}$$

$$\mu_{1,2} = \frac{-2 + J_{EE} \pm \sqrt{J_{EE}^2 - 4J_{EI}}}{2} = \frac{-2 + 2 \pm \sqrt{2^2 - 4 \cdot 4}}{2} = \pm\sqrt{3}i$$

2.3.2 Analytical Assent Of The Period Of The Oscillations

As we've seen thus far, If we look at the system in a small neighborhood of the FP, it can be approximated by the linear system. Moreover, if we look at the linear system in its complex eigenbase, it would look like this:

$$S = \begin{bmatrix} \sqrt{3}i & 0 \\ 0 & -\sqrt{3}i \end{bmatrix}$$

Specifically, in this base, we would get the solutions:

$$h_E = h_E^0 e^{\sqrt{3}it} = ae^t(cos(\sqrt{3}t) + isin(\sqrt{3}t))$$

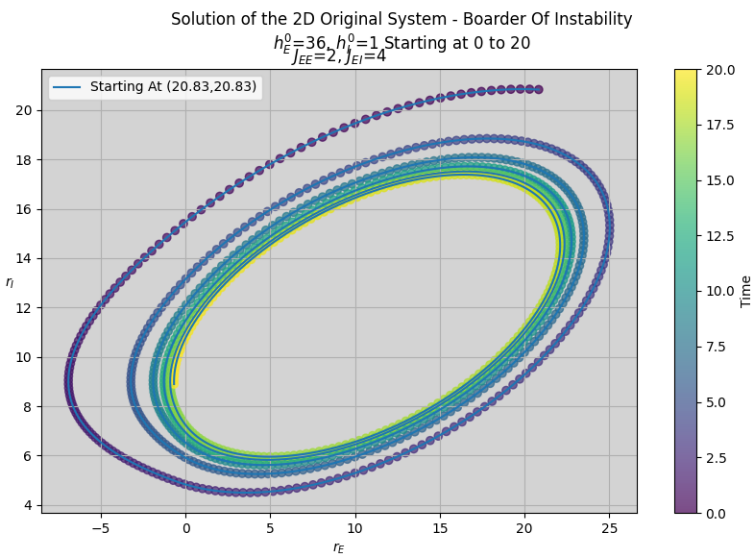
And the same goes for the other dimension. Using what we've learned in Q.3, we know the expected frequency is

$$k = \frac{im(\mu_1)}{2\pi} = \frac{\sqrt{3}}{2\pi} \approx 0.275[Hz]$$

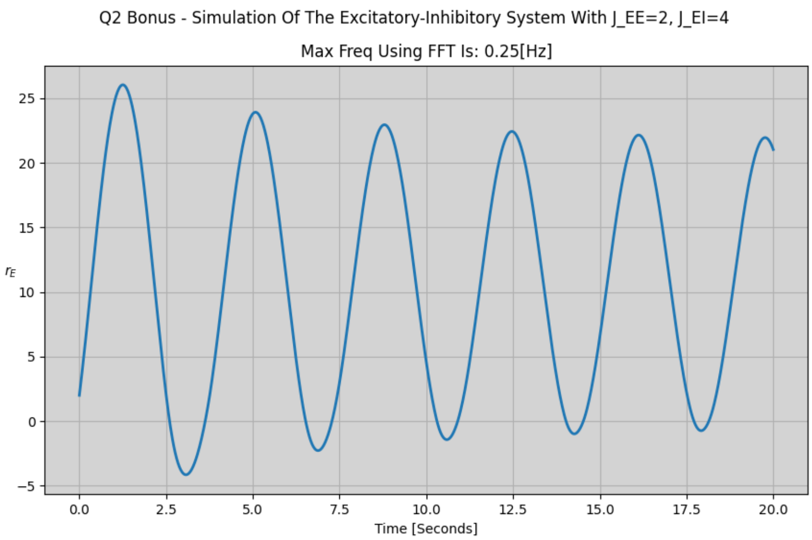
Now, sure, you can say we didn't need the Euler's identity for that, just the eigenvalue and you'll be right. As I'm trying to please the crowd - it's included anyway. Now, for the magic...

2.3.3 Simulation Gives the same results $\pm$

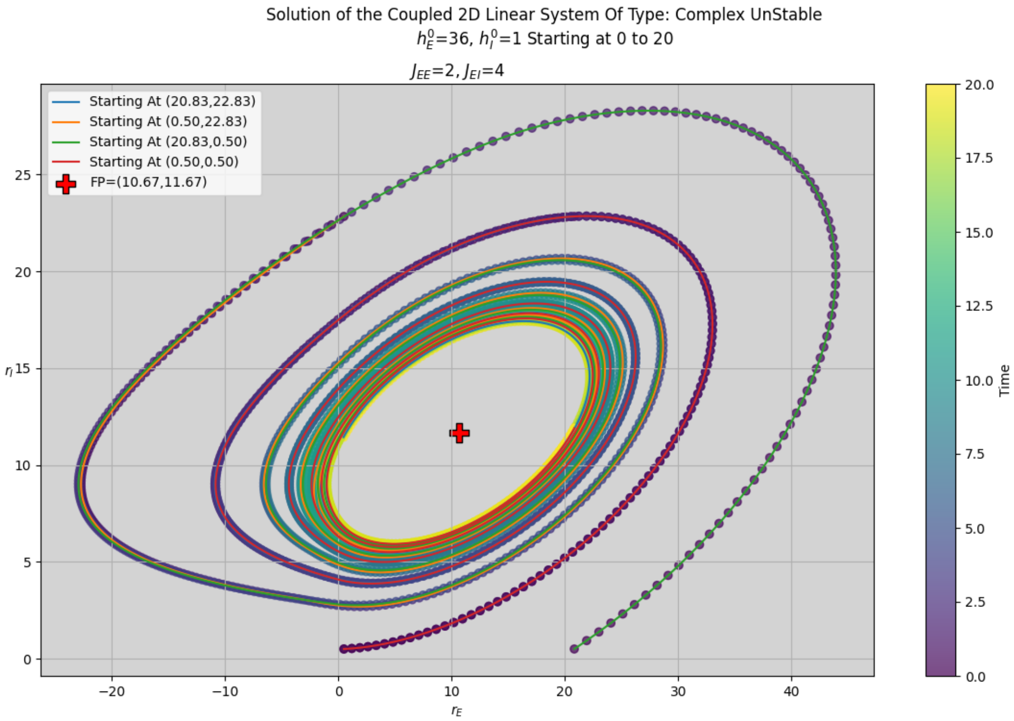
So as per request, Here is the simulation of the system:

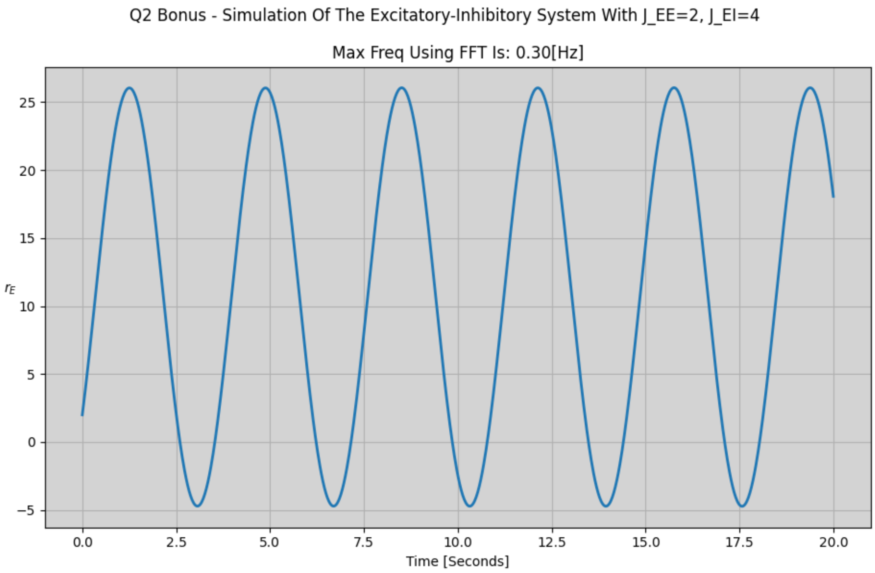


The oscillations of r_E plotted VS time. Using FFT analysis, we get the frequency is ~0.25. Moreover, we can see in the graph it takes about 4 seconds to complete a cycle.



For fun, I've also simulated the linear system and it looks pretty much the same:


Solution of the Coupled 2D Linear System Of Type: Complex UnStable
h_E^0=36, h_I^0=1 Starting at 0 to 20
J_{EE}=2, J_{EI}=4
Starting At (20.83,22.83)
Starting At (0.50,22.83)
Starting At (20.83,0.50)
Starting At (0.50,0.50)
FP=(10.67,11.67)
r_I
r_E
Time


Q2 Bonus - Simulation Of The Excitatory-Inhibitory System With J_EE=2, J_EI=4
Max Freq Using FFT Is: 0.30[Hz]
r_E
Time [Seconds]

2.4 Qualitative Observation

2.4.1 Stability Analysis

We saw here a complex system with a few types of attractors and repellers including limit-cycles, convergence to fixed points, divergence to infinity. Also, it's not shown here but I also came across oscillating divergence. For some values, any small change of parameters can cause dramatic changes, for others, not so much. We've learned about the different regimes of the dynamics, how we can approximate the dynamics as linear dynamics in the different regimes and how those linear systems behave. We saw this does not correspond 1-1 with the system and still, it's good enough to tell us a lot about how it works.

After such a long H.W. assignment, it's hard to say we've missed anything at all and it seems like an eternity had passed while I was working on this. Still, I would say we haven't plotted the "Phase portraits" of the dynamics. Also, we left out a lot of

2.4.2 Connectivity Structure And Non-Linearities

In the E-I we first saw oscillations. There's negative feedback with time delay (as Prof' Kleinfeld referred to it) which causes oscillations. Basically, when the wight of the synapses ($J_{\alpha\beta}$) is in a "good" ratio, and the excitatory (E) initial input is bigger than the inhibitory (I), the E neuron excites the I neuron, causing it to increase its firing rate. In turn, the I neuron inhibits the E neuron which fires less, causing the I neuron to fire less and thus E is less inhibited and fires more again. This is the nature of the cycle.

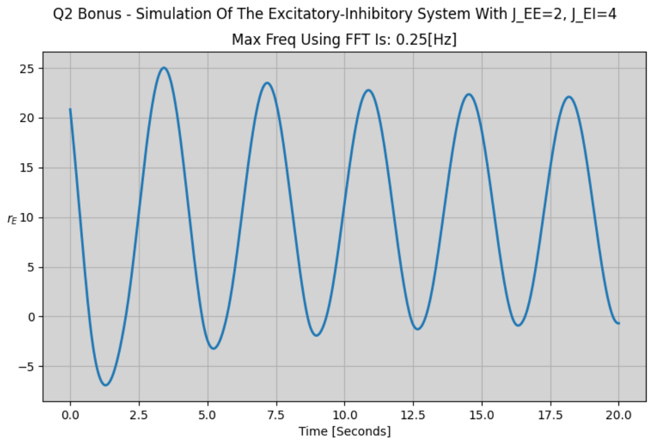
Using the ReLu function as a non-linearity, enabled us to have a few regimes of fixed points. Using $g(x) = x$ instead would have caused some odd behaviours such as - what's the meaning of the feedback loop for negative values. Also, using the sigmoid function would limit the effect of the inhibitory and excitatory inputs to between 0 and 1 (times the wight of the synapse), this may have not been sufficiant enough feedback to power oscillations.

3 Fourier Meets Oscillations

3.1 Numerical Part

3.1.1 r_EIn Time

The duration of a single oscillation is about 6 seconds.

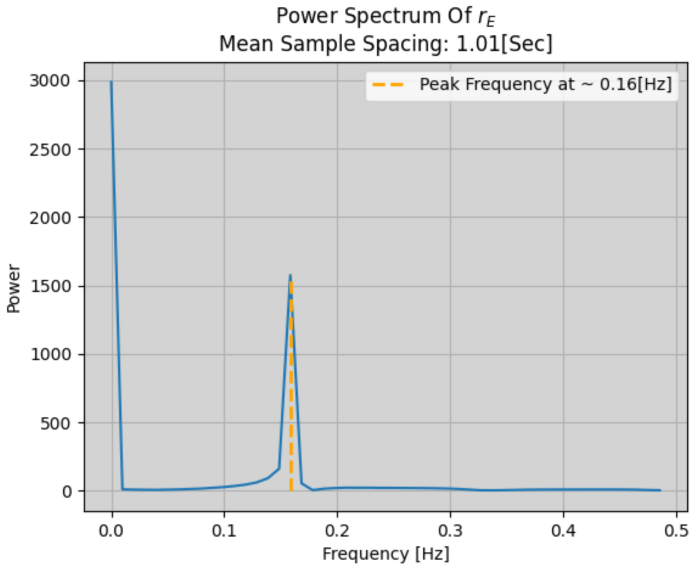


3.1.2 Q3.1.2-3: Power Spectrum Analysis Using The FFT Algorithm

A wave with $k = 3\left[\frac{1}{sec}\right]$ takes $\frac{1}{3}$ of a second to complete a single oscillation. When $k = 6\left[\frac{1}{sec}\right]$it takes $\frac{1}{6}$ of a second to complete a single oscillation. In this frequencies vector, the max is $k = 0.49\left[\frac{1}{sec}\right]$, meaning the wave takes $\frac{1}{0.49} \approx 2.04$ second to complete an oscillation.

As we can see in power spectrum's plot, the peak power is around $k = 0.16\left[\frac{1}{sec}\right]$, meaning about $\frac{1}{0.16} \approx 6.25$ second to complete an oscillation, As was approximated from the the signal's plot.

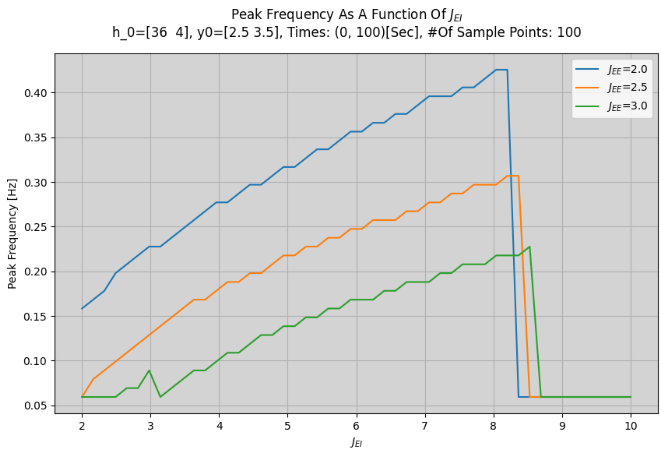
Here is the plot of the signal's power spectrum. Note I used sampling space is 1 second across 100 seconds and this is why the max possible power is so low. Increasing the sample rate does not change anything other than the possible frequencies. There's no need to zoom in in this graph. Moreover, as the power spectrum is symmetric around $f = 0[Hz]$, we only need to look at the positive part of the plot.



3.1.3 See Section 3.1.2

3.1.4 Frequency Of Oscillations As A Function Of J_{EI}

Here we can see the frequency of the oscillation as a function of J_{EI} for 3 values of J_{EE}. We can see the drop where we shift to the "real eigenvalues - unstable solution" regime from section 2.1.3.



3.2 Explaining The Numerical Results Via The Linear Model

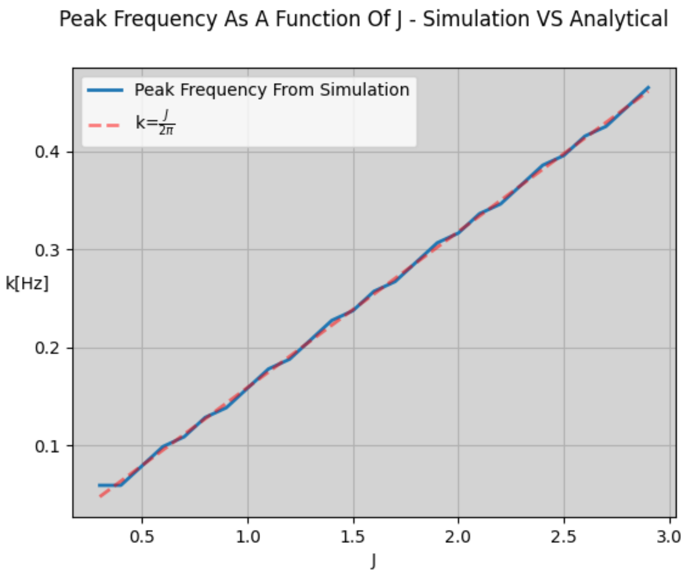
3.2.1 Eigenvalues And Oscillation In A 2D Linear System

$$M = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

Now - $tr(M) = 0$ and $det(M) = 0 - (-J * J) = J^2$ we get then that the eigenvalues are:

$$\mu_{1,2} = \frac{0 \pm \sqrt{0^2 - 4J^2}}{2} = \pm \frac{2J\sqrt{-1}}{2} = \pm Ji$$

So, the expected frequency of this system is: $k = \frac{J}{2\pi}$. Now looking at the simulation results VS the analytical expectation we can see they're pretty much the same:

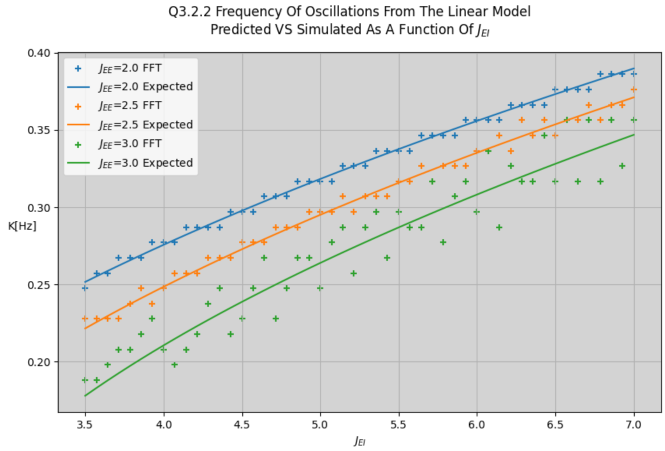


3.2.2 Frequency Of Oscillations From The Linear Model - Predicted VS Simulated

Using the linear model :

$$M = \begin{bmatrix} -1 + J_{EE} & -J_{EI} \\ 1 & -1 \end{bmatrix}$$

We run the simulation for 3 values of J_{EE} $(2, 2.5 \text{ and } 3)$ with initial condition $(h_E(t = 0), h_I(t = 0)) = (2.5, 3.5)$, with constant external input $(h_E^0, h_I^0) = (36, 4)$ from T=0[Sec] to T=100[Sec] with sample rate of 1 sample per second. We calculate the eigenvalues of matrix M. As this is an oscillating 2D system, we get eigenvalues of the form $a \pm bi$. We calculate $k = \frac{b}{2\pi}$ as was instructed and also extract the frequency of the simulation using FFT analysis and plot them. We can see in the plot that the expected values (lines) pretty match what happens in the simulations ("+" signs):



3.3 Qualitative Observation

3.3.1 Frequency Of Oscillations Increased With J_{EI} And decreased With J_{EE}.

As seen in the graph of 3.1.4. , J_{EE} increase means the E neuron have a bigger effect on itself, the relative part of the negative feedback decreases. Also, J_{EI} increase means the exact opposite, The relative effect of the negative feedback is greater. It's worth getting back to the explanation in section 2.4.2. Looking at the negative feedback creating the feedback loop, we can see how increasing the amount an E neuron acting on itself in effect, diminishes the significant of the the negative feedback and thus, the oscillations decrease. From the same perspective we can see how an increase in the negative feedback increases the oscillations.

3.3.2 Linear Model Describes The Non-Linear Dynamics

All in all we were able to capture the essence of the dynamics using the linear model. Albeit very limited to regions and it does not really looks like the dynamics, just gives us the gist of how it behaves.

3.3.3 For Larger Values Of J_{EE} The Linear Model Diverges From Expectations

As we can see in the graphs of 3.2.1 and 3.2.2, the FFT analysis of the linear model's simulation is around what we'd expect though for increasing values of J_{EE}, the error increases as well.