Assignment 1 - Network Graph Sciences Analysis

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Question 1

(a) $\mathbf{k} = A.1$
(b) $m = |E| = \frac{1}{2}k^T k = \frac{1}{2}\sum_{i=0}^{n} k_i$
(c) $N = AA^T$,

Question 2

Let V be partitioned into two disjoint sets V_1, V_2. Because there are no internal edges in each set V_1, V_2 :

$$
\begin{aligned}
\{\text{\# of out edges of } V_1\} &= \{\text{\# of out edges of } V_2\} \\
\sum_{i \in V_1} k_i &= \sum_{i \in V_2} k_i \\
|V_1| \frac{\sum_{i \in V_1} k_i}{|V_1|} &= |V_2| \frac{\sum_{i \in V_2} k_i}{|V_2|} i \\
|V_1| c_1 &= |V_2| c_2 \\
n_1 c_1 &= n_1 c_1
\end{aligned}
$$

Question 3

Graph : G=(V,E)
(a) By definition, the element A_{ij} of the square matrix contains the number of paths of length 3 that starts from node i and end at node j. A triangle is designed when the node participates simultaneously as the start node and the end node, so we are interested in diagonal elements A_{ii} : nodes involved in triangles.
Therefore, the trace of A^3 is related to the number of triangles. As :
- 3 nodes are involved in a triangle : we are then triple counting the number of triangles
- the graph is undirected : we are then doubling the number of triangles
So to conclude, the total number of triangles in the graph is

$$\Delta(G) = \frac{1}{6}tr(A^3)$$

(b) As we know the trace of a matrix if the sum of eigen values. We have :

$$\Delta(G) = \frac{1}{6}\sum_{i \in V} \lambda_i^3$$

(c) As we have just shown that the diagonal element A_{ii} is Δ_i the number of triangles that node i participates in, we just need to find a way to derive A_{ii}. The adjacency matrix A being real and symmetric, we have $A = U\Sigma U^T$ with $UU^T = I$. Then $A^3 = U\Sigma^3 U^T$. So with the eigenvectors $(\vec{u_1}, \dots, \vec{u_n})$ of the eigenvalues $(\lambda_1, ..., \lambda_n)$, and u_{ij} the ith element of $\vec{u_j}$, we can derive Δ_i as follows:

$$\begin{aligned}
\Delta_i = \frac{1}{2}A_{ii}^3 &= \frac{1}{2}\sum_{k \in V} u_{ik}(\Sigma^3 U)_{ki} \\
&= \frac{1}{2}\sum_{k \in V} u_{ik}(\sum_{l \in V} \Sigma_{kl}^3 u_{li}) \\
&= \frac{1}{2}\sum_{k \in V} u_{ik}(\lambda_k^3 u_{ki}) \\
\Delta_i &= \frac{1}{2}\sum_{k \in V} \lambda_k^3 u_{ki}^2
\end{aligned}$$

Question 4

(a) $G_{n,p}$undirected graph on n nodes and each edge (u, v) appears i.i.d. with probability p **(a)** There are $\binom{n}{3}$ triples of vertices . Each triple has statistically a probability of p of being a triangle. Let Δ_{ijk} be the random variable for the triangle with vertices i, j, and k being present equals to 1 if vertices participate to the same triangle. Then the number of triangles is the expectancy of $\sum_{ijk} \Delta_{ijk}$:

$$
\begin{aligned}
E[\sum_{ijk} \Delta_{ijk}] =* \sum_{ijk} E[\Delta_{ijk}] = \sum_{ijk} p^3 &= \binom{n}{3} p^3 \\
&= \frac{n(n-1)(n-2)}{3!} p^3 \\
&\underset{\infty}{\sim} \frac{1}{6}(np)^3 = \frac{1}{6}c^3
\end{aligned}
$$

*the expected value of a sum of random variables is the sum of the expected values, because the events are identically distributed.

(b) Mutatis mutandis :

$$
\begin{aligned}
E[\sum_{ijk} \Delta'_{ijk}] =* \sum_{ijk} E[\Delta'_{ijk}] = \sum_{ijk} \binom{3}{2} p^2 &= \binom{n}{3}\binom{3}{2} p^2 \\
&= \frac{n(n-1)(n-2)}{3!} \frac{3!}{2!} p^2 \\
&\underset{\infty}{\sim} \frac{1}{2} nc^2
\end{aligned}
$$

(c) $C = \frac{\frac{1}{6}c^3 3}{\frac{1}{2}nc^2} = \frac{c}{n} \sim p$

Question 5

(a)

$$x_i = \sum_{k=1}^{\infty} \sum_{j \in V} \alpha^k A_{ij}^k$$

(b) As we know above :

$$A_{ij}^k = \sum_{l \in V} \lambda_l^k u_{li} u_{lj}$$

Question 6

Let's reason by equivalence to prove $\boxed{\dfrac{1}{C_A} + \dfrac{n_A}{n} = \dfrac{1}{C_B} + \dfrac{n_B}{n}}$:

$$
\begin{aligned}
& \frac{1}{C_A} + \frac{n_A}{n} = \frac{1}{C_B} + \frac{n_B}{n} \\
\Longleftrightarrow \quad & \frac{\sum_j d_{Aj}}{n} + \frac{n_A}{n} = \frac{\sum_j d_{Bj}}{n} + \frac{n_B}{n} \\
\Longleftrightarrow \quad & \sum_j d_{Aj} + n_A = \sum_j d_{Bj} + n_B \\
\Longleftrightarrow \quad & \sum_{j \in A \cup B} d_{Aj} + n_A = \sum_{j \in A \cup B} d_{Bj} + n_B \\
\Longleftrightarrow \quad & \sum_{j \in A} d_{Aj} + \sum_{j \in B} d_{Aj} + n_A = \sum_{j \in A} d_{Bj} + \sum_{j \in B} d_{Bj} + n_B \\
\Longleftrightarrow \quad & \sum_{j \in A} d_{Aj} + \sum_{j \in B} (d_{Bj} + 1) + n_A = \sum_{j \in A} (d_{Aj} + 1) + \sum_{j \in B} d_{Bj} + n_B \\
\Longleftrightarrow \quad & \sum_{j \in A} d_{Aj} + \sum_{j \in B} d_{Bj} + n_B + n_A = \sum_{j \in A} d_{Aj} + n_A + \sum_{j \in B} d_{Bj} + n_B \quad \text{which is true}
\end{aligned}
$$

Question 9

(a) The produced Kronecker graph is connected, if:

$$b + c > 1$$

Here $b + c = 0.26 + 0.53 = 0.79$. So the produced Kronecker graph is NOT connected.

The graph has a giant connected component of size $\Theta(n)$ if:

$$(a + b)(b + c) > 1$$

Here $(a + b)(b + c) = (0.99 + 0.26)(0.26 + 0.53) = 0.9875$. So the produced Kronecker graph is NOT connected.

(b) Similar structural properties :

- Triangle participation distribution
- Visualization of nodes and edges of graph.