1910SU **Summer**

Group Discussion: Differentiation II

Solutions

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1. Quotient Rule

1.1. $\frac{d}{dx}\tan x$

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \sin x\left(\frac{d}{dx}\frac{1}{\cos x}\right) + \frac{1}{\cos x}\left(\frac{d}{dx}\sin x\right)\\ &= \sin x\left(-\frac{1}{\cos^2 x}\cdot(-\sin x)\right) + \frac{\cos x}{\cos x}\\ &= \frac{\sin^2 x}{\cos^2 x} + 1\\ &= \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

1.2. $\frac{d}{dx}\sec x$

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= -\frac{1}{\cos^2 x}\cdot(-\sin(x))\\ &= \frac{\sin x}{\cos^2 x}\\ &= \tan x\sec x.\end{aligned}$$

1.3. $\frac{d}{dx}\csc x$

$$\begin{aligned}\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= -\frac{\cos x}{\sin^2 x}\\ &= -\cot x\csc x.\end{aligned}$$

2. Chain Rule

2.1. $\frac{d}{dx}\arcsin x$

By definition,

$$\sin(\arcsin x) = x.$$

Differentiating both sides,

$$\frac{d}{dx}\sin(\arcsin x) = \frac{d}{dx}x.$$

To be a bit more explicit about the use of the Chain Rule, suppose $y = \arcsin x$. Then,

$$\implies \left(\frac{d}{dy}\sin y\right)\left(\frac{d}{dx}\arcsin x\right) = 1$$

$$\implies \frac{d}{dx}\arcsin x = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

How do we show that $\cos(\arcsin x) = \sqrt{1-x^2}$? There are two methods (though, maybe a better descriptor would be that they are two different perspectives for the same underlying proof):

2.1.1. Algebraically showing $\cos(\arcsin x) = \sqrt{1-x^2}$

Recall that

$$\sin^2\theta + \cos^2\theta = 1.$$

So,

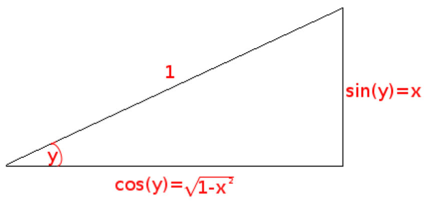
$$\cos\theta = \pm\sqrt{1-(\sin\theta)^2}.$$

In our case, $\theta = \arcsin x$. The range of $\arcsin x$ is $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In this range, cos is always positive, so it suffices to take just the positive square root to obtain

$$\cos(\arcsin x) = \sqrt{1-\sin^2(\arcsin x)} = \sqrt{1-x^2}.$$

2.1.2. Geometrically showing $\cos(\arcsin x) = \sqrt{1-x^2}$

Let $y = \arcsin x$. Then note that y represents the angle in a right-angled triangle with unit hypotenuse and perpendicular with side length $\sin(y) = x$. Pictorally,



Note that $\cos y$ then corresponds to the adjacent side whose length can be obtained by using Pythagoras's Theorem–

$$\begin{aligned}\cos y &= \sqrt{1-\sin^2 y}\\ &= \sqrt{1-x^2}\\ \implies \cos(\arcsin x) &= \sqrt{1-x^2}.\end{aligned}$$

2.2. $\frac{d}{dx}\arctan x$

By definition,

$$\tan(\arctan x) = x.$$

Differentiating both sides,

$$\frac{d}{dx}\tan(\arctan x) = \frac{d}{dx}x.$$

To be a bit more explicit about the use of the Chain Rule, suppose $y = \arccos x$. Then,

$$\implies \left(\frac{d}{dy}\tan y\right)\left(\frac{d}{dx}\arctan x\right) = 1$$
$$\implies \frac{d}{dx}\arctan x = \frac{1}{\sec^2 y} = \frac{1}{\sec^2 \arctan x} = \frac{1}{1+x^2}.$$

How do we show that $\sec^2(\arctan x) = 1 + x^2$? Again, there are two methods:

2.2.1. Algebraically showing $\sec^2(\arctan x) = 1 + x^2$.

Note that

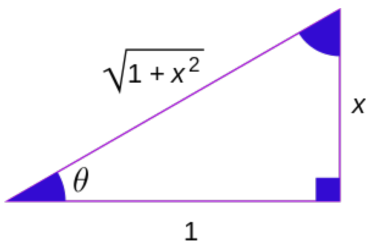
$$\sec^2\theta = \tan^2\theta + 1.$$

In our case, $\theta = \arctan x$. So,

$$\sec^2 \arctan x = x^2 + 1.$$

2.2.2. Geometrically showing $\sec^2(\arctan x) = 1 + x^2$.

Consider a right-angled triangle with angle θ such that the perpendicular side has length x and adjacent side has length 1. Then $\theta = \arctan x$. Pictorally,



So, the hypotenuse has length $\sqrt{x^2+1}$. Then,

$$\sec^2\theta = \frac{1}{\cos^2\theta} = \left(\sqrt{1+x^2}\right)^2 = 1 + x^2$$
$$\sec^2 \arctan x = 1 + x^2.$$