Algebra **Summer**

Algebra Problems

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Contents

1. Preliminaries

1.1. Naive Set Theory

Problem 1.2

First, note that for every $a \in S$, $a \in [a]_\sim$. Since every equivalence class is a subset of S, it follows that $S = \bigcup_{C \in \mathcal{P}_\sim} C$.

Take two partitions $[a]_\sim$ and $[b]_\sim$. If they are disjoint, we are done. Suppose they aren't. Then $c \in [a]_\sim$ sec $[b]_\sim$. By transitivity and symmetry, for any $x \in [b]_\sim$, $x \sim b \sim c \sim a$. Thus, $[b]_\sim \subseteq [a]. \sim$. By symmetry, $[b]_\sim = [a]_\sim$. Thus, distinct equivalence classes are disjoint.

This concludes the proof that equivalence classes form a partition of S.

Problem 1.3

Let $\mathcal{P}$ be a partition on S. Furthermore, for any $a \in S$, define $\mathcal{P}_a$ to be the unique set in the partition containing a. Then we can the equivalence relation $\underset{\mathcal{P}}{\sim}$ by $a \underset{\mathcal{P}}{\sim} b$ iff $b \in \mathcal{P}_a$.

This is reflexive because, trivially, $a \in \mathcal{P}_a$.

This is also symmetric. Note that as partitions are a collection of disjoint sets, $\mathcal{P}_a \cap \mathcal{P}_b \neq \emptyset$ implies that $\mathcal{P}_a = \mathcal{P}_b$. Thus, $a \in \mathcal{P}_a = \mathcal{P}_b$.

Finally, this is also transitive due to the transitivity and symmetry of set equality. Particularly, note that, as in the previous part, $\mathcal{P}_a = \mathcal{P}_b$ and $\mathcal{P}_b = \mathcal{P}_c$. Thus, $c \in \mathcal{P}_c = \mathcal{P}_a$.

Problem 1.6

We first show that $\sim$ is an equivalence relation. It is reflexive because for any $a \in \mathbb{R}$, $a - a = 0 \in \mathbb{Z}$. It is symmetric because $\mathbb{Z}$ is closed under multiplication i.e. $z \in \mathbb{Z} \Longrightarrow -z \in \mathbb{Z}$. It is transitive because $\mathbb{Z}$ is closed under addition, particularly for $a, b, c \in \mathbb{R}$, if $a \sim b, b \sim c$ then $c - a = (c - b) + (b - a) \in \mathbb{Z}$.

We claim that $\mathbb{R}/\sim \cong [0, 1)$. Note that any $x, y \in [0, 1)$ are such that $x \nsim y$ as $x - y \leq x < 1$. Thus, each element of $[0, 1)$ corresponds to a distinct equivalence class. Next, given any $z \in \mathbb{R}$, we claim that there exists $x \in [0, 1)$ such that $z \in [x]_\sim$. Particularly, take $x = z - \lfloor z \rfloor$. Then $z - x \in \mathbb{Z}$ and $0 \leq z - \lfloor z \rfloor < 1$.

Note that $\approx$ is an equivalence relation for pretty much the same reasons as $\sim$ above. Furthermore, $\mathbb{R} \times \mathbb{R}/\approx \cong [0, 1) \times [0, 1)$ by a similar reasoning to above.

1.2. Functions Between Sets

Problem 2.1

There are $n!$ many bijections between a set S and itself, with $|S| = n$. We prove this by induction. Let $P(n)$ be the statement that there are $n!$ bijections from S to S' with $|S| = |S'| = n$.

$P(1)$ is clearly true as the only bijection (in fact, the only function) from a singleton $S = \{s\}$ to another singleton $S' = \{s'\}$ is the function $f : S \longrightarrow S'$ defined by $f(s) = s'$.

Assuming $P(k)$ we wish to show that there $P(k + 1)$ holds too. Suppose $S = \{s_1, ..., s_k, s_{k+1}\}$. We can categorize bijections by where they map the first element s_1. There are n different categories as $|S'| = n$. We then claim that every category has $(n - 1)!$ elements. Consider some category that is defined by the fact that it maps $s_1 \mapsto s_{i'}$ for some $s_{i'} \in S'$. Hence, every function f in this category, restricts to a bijection $f|_{S - s_1} : S - s_1 \longrightarrow S' - s_{i'}$. By the induction hypothesis, there are precisely $(n - 1)!$ choices for these restrictions. So every category, has $(n - 1)!$ functions and in total, there are $n(n - 1)! = n!$ many bijections from S to S'. So, $P(k + 1)$ holds.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Problem 2.2: *f has a right inverse if and only if it is surjective.*

We deal with the forward direction first. Suppose $f : A \longrightarrow B$ has a right inverse $g : B \longrightarrow A$. Then for every $b \in B$, $g(b) \in A$ is such that $f(g(b)) = b$. Thus, every element of B is the image, under f, of at least one element of A i.e. f is surjective.

Now, consider the backward direction. If f is surjective then for every $b \in B$, the fiber $f^{-1}(b)$ is non-empty. So, define the function $g : B \longrightarrow A$ as follows: for every b, pick an element of the fiber $f^{-1}(b)$ and assign it to $g(b)$. Then, note that for every $b \in B$, $g(b) \in f^{-1}(b)$ implies that $(f \circ g)(b) = b$. Thus, $f \circ g = \mathrm{id}_B$ and g is the right inverse of f.

Problem 2.4

This follows from showing:

a. $\cong$ is reflexive.
   For any set A, $A \cong A$ as $\mathrm{id}_A : A \longrightarrow A$ is a bijection.
b. $\cong$ is symmetric.
   Take any sets A, B such that $A \cong B$. Then there exists a bijection $f : A \longrightarrow B$. Consequently, $f^{-1} : B \longrightarrow A$ is a bijection from B to A and $B \cong A$.
c. $\cong$ is transitive.
   Take any sets A, B, C such that $A \cong B$ and $B \cong C$. Then, there exists bijections $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Consequently, $h = g \circ f : A \longrightarrow C$ is also a bijection since it has inverse $f^{-1} \circ g^{-1}$

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ f = \mathrm{id}_A, \quad (g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ g^{-1} = \mathrm{id}_C.$$

   Thus, $A \cong C$.

Problem 2.5

Definition 1.2.1: A function $f : A \longrightarrow B$ is an *epimorphism* for all sets Z and all function $\alpha', \alpha'' : B \longrightarrow Z$

$$\alpha' \circ f = \alpha'' \circ f \Longrightarrow \alpha' = \alpha''.$$

Proposition 1.2.1: A function is surjective iff it is an epimorphism

Proof: We start with the forward direction. Suppose $f : A \longrightarrow B$ is surjective. Then we know that f has a right inverse g. So,

$$\begin{aligned} \alpha' \circ f &= \alpha'' \circ f \\ \Longrightarrow \alpha' \circ f \circ g &= \alpha'' \circ f \circ g \\ \Longrightarrow \alpha' &= \alpha''. \end{aligned}$$

Now, we consider, the backward direction. Suppose $f : A \longrightarrow B$ is an epimorphism. Pick $b \in B$ and define $\alpha'_b : B \longrightarrow \{0, 1\}, \alpha''_b : B \longrightarrow \{0, 1\}$ to be $\alpha'_b = \mathbb{1}_B, \alpha''_b = \mathbb{1}_{B-b}$. Then, $\alpha'(b) \neq \alpha''(b)$ so $\alpha' \circ f = \alpha'' \circ f$. For every $x \in f^{-1}(B - b)$, $(\alpha' \circ f)(x) = (\alpha'' \circ f)(x) = 1$. So, if $x \in A$, $(\alpha' \circ f)(x) \neq (\alpha'' \circ f)(x)$ necessarily implies that $x \in f^{-1}(b)$. Since b was chosen arbitrarily, this proves that f is surjective.

□

Problem 2.9

Suppose $A \cong A'$ and $B \cong B'$ with $A \cap B = \emptyset, A' \cap B' = \emptyset$. Let $g_1 : A \longrightarrow A'$ and $g_2 : B \longrightarrow B'$ be isomorphisms. Consider the map $f : A \cup B \longrightarrow A' \cup B'$ defined by

$$f(x) = \begin{cases} g_1(x) \text{ if } x \in A \\ g_2(x) \text{ otherwise} \end{cases}$$

Then, f is a bijection and hence, $A \cup B \cong A' \cup B'$.

Problem 2.10

We give a combinatorial argument. Note that a function from A to B must map an element of A to one of $|B|$ many elements of B. Since this choice has to be made for each of the $|A|$ many elements of A, there are a total of $|B|^{|A|}$ functions in B^A.

Problem 2.11

Let $\mathcal{P}$ denote the power set of A. Consider the map $\mathcal{F} : 2^A \longrightarrow \mathcal{P}$ defined by

$$\mathcal{F}(f) = \{x \in A \mid f(x) = 1\}$$

First, we show $\mathcal{F}$ is injective. Suppose $f, g \in 2^A$ with $\mathcal{F}(f) = \mathcal{F}(g)$. Then for every $x \neg \in \mathcal{F}(f)$, $f(x) = g(x) = 0$ and for every $x \in \mathcal{F}(f)$, $f(x) = g(x) = 1$. Thus, $f = g$.

Next, we show that $\mathcal{F}$ is surjective. Consider any $S \subseteq 2^A$. Then $1_S \in 2^A$ and is such that $\mathcal{F}(1_S) = S$.

1.3. Categories

Problem 3.1

For clarity denote by $\circ_C$ and $\circ_{C^{\text{op}}}$ the composition laws for C and C^{op} respectively. Consider $f \in \operatorname{Hom}_{C^{\text{op}}}(A, B)$ and $g \in \operatorname{Hom}_{C^{\text{op}}}(B, C)$. Then note that $f \in \operatorname{Hom}_C(B, A)$ and $g \in \operatorname{Hom}_C(C, B)$. So, there exists $f \circ_C g \in \operatorname{Hom}_C(C, A)$. Then, note that $f \circ_C g \in \operatorname{Hom}_{C^{\text{op}}}(A, C)$ and hence, we can define $\circ_{C^{\text{op}}}$ by

$$g \circ_{C^{\text{op}}} f = f \circ_C g \in \operatorname{Hom}_{C^{\text{op}}}(A, C).$$

We now prove that composition defined this way satisfies the required properties.

a. associativity
   Consider $f \in \operatorname{Hom}_{C^{\text{op}}}(A, B)$, $g \in \operatorname{Hom}_{C^{\text{op}}}(B, C)$ and $h \in \operatorname{Hom}_{C^{\text{op}}}(C, D)$. Then,

$$(h \circ_{C^{\text{op}}} g) \circ_{C^{\text{op}}} f = f \circ_C (g \circ_C h)$$

a. identity
b. disjointness

Problem 3.3

Problem 3.5

Problem 3.6

Problem 3.7

Problem 3.8

Problem 3.9

Problem 3.11

1.4. Morphisms

Problem 4.1

Problem 4.2

1.5. Universal Properties

Problem 5.2

Problem 5.3

Problem 5.5

Problem 5.6

Problem 5.12