1910SU **Summer**

Group Discussion: L'Hôpital's Rule

Solutions

Authors: Shaleen Baral

1. Limits with Exponents.

1.1. Compute the limit $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

First we compute the following.

$$\lim_{x\to\infty}\ln\left(\left(1+\frac{2}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)$$

This is an indeterminate form of the type $\infty\cdot 0$. We write the expression above to resemble the form $\frac{0}{0}$ so that we can use L'Hôpital's rule.

$$= \lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}\right)$$

Taking the derivatives of the numerator and denominator, we obtain

$$= \lim_{x\to\infty} -x^2\cdot\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}$$
$$= \lim_{x\to\infty}\frac{2}{1+\frac{2}{x}} = 2.$$

Thus,

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^{\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)} = e^2.$$

1.2. Compute the limit $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$

First we compute the following.

$$\lim_{x\to\infty}\ln\left(\left(1+\frac{1}{x^2}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x^2}\right)$$

This is again an indeterminate form of the time $\infty\cdot 0$. We write the expression above to resemble the form $\frac{0}{0}$ so that we can use L'Hôpital's rule.

$$= \lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x^2}\right)}{\frac{1}{x}}\right)$$

Taking the derivatives of the numerator and denominator, we obtain

$$= \lim_{x\to\infty} -x^2\cdot\frac{1}{1+\frac{1}{x^2}}\cdot\frac{-2}{x^3}$$
$$= \lim_{x\to\infty}\frac{2}{x^3+x} = 0.$$

Thus,

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x^2}\right)} = e^0 = 1.$$

1.3. Compute the limit $\lim_{x\to 0^+} x^{\sin x}$

First we compute the following.

$$\lim_{x\downarrow 0^+} \sin x \ln x$$

This is an indeterminate form of the type $0 \cdot (-\infty)$. We rewrite this to resemble $\frac{\infty}{\infty}$ so that we can use L'Hôpital's rule.

$$= \lim_{x\downarrow 0^+} \frac{\ln x}{\csc x}$$

Taking the derivative of the numerator and denominator gives us,

$$= \lim_{x\downarrow 0^+} \frac{1/x}{-\cot x \csc x}$$
$$= \lim_{x\downarrow 0^+} -\frac{\sin^2 x}{x\cos x}$$

This is an indeterminate form of the type $\frac{0}{0}$. We apply L'Hopital's rule again.

$$= \lim_{x\downarrow 0^+} -\frac{2\sin x \cdot \cos x}{\cos x + x\sin x} = \frac{0}{1} = 0.$$

Thus, we have

$$\lim_{x\to 0^+} x^{\sin x} = e^{\lim_{x\to 0^+} \sin x \ln x} = e^0 = 1.$$

2. (Non) Indeterminate Form 0^∞.

As $\lim_{x\to a} g(x) = \infty$ and $\lim_{x\to a} \ln g(x) = \lim_{x\to 0^+} \ln x = -\infty$. So,

$$\lim_{x\to a} g(x)\ln f(x) = -\infty.$$

Remark: Note that, technically, we need to assume that $f(x)$ approaches 0 from above as otherwise $\ln f$ will be undefined.

Thus, $\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = \lim_{y\to -\infty} e^y = 0.$