

Algebra

Aluffi

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1. Preliminaries

1.1. Naive Set Theory

Definition 1.1.1: The *ordered pair* (s, t) can be defined as the set $\{s, \{s, t\}\}$. This retains both the elements of the tuple but also conveys an ordering.

Definition 1.1.2: The *disjoint union* of two sets S, T is the set $S \sqcup T$ obtained by first producing ‘copies’ S' and T' and then taking the union.

Definition 1.1.3: The *product* of two sets S, T is the set $S \times T$ defined as

$$S \times T = \{(s, t) \text{ such that } s \in S, t \in T\}.$$

Definition 1.1.4: A *relation* on a set S is a subset R of the product $S \times S$. If $(a, b) \in R$, we write aRb .

Definition 1.1.5: An *equivalence relation* on a set S is any relation \sim satisfying the following properties

- a. *reflexivity*: $\forall a \in S. a \sim a$
- b. *symmetry*: $\forall a \in S. \forall b \in S. a \sim b \iff b \sim a$
- c. *transitivity*: $\forall a \in S. \forall b \in S. \forall c \in S. a \sim b, b \sim c \implies a \sim c$.

Definition 1.1.6: A *partition* of S is a family of *disjoint* nonempty subsets of S whose union is S .

Definition 1.1.7: Let \sim be an equivalence relation on S . Then for every $a \in S$, the *equivalence class* of a is the subset S defined by

$$[a]_{\sim} = \{b \in S \mid b \sim a\}.$$

Further, the equivalence classes form a partition \mathcal{P}_{\sim} of S .

Lemma 1.1.1: Every partition of S corresponds to an equivalence relation.

Definition 1.1.8: The *quotient* of the set S with respect to the equivalence relation \sim is the set

$$S \sim = \mathcal{P}_{\sim}$$

of equivalence classes of elements of S with respect to \sim .

1.2. Functions Between Sets

Definition 1.2.1: The graph of f is the set

$$\Gamma_f = \{(a, b) \in A \times B \mid b = f(a)\} \subseteq A \times B.$$

Officially, a function is its graph together with information of the source A and the target B of f .

Definition 1.2.2: A function is a relation $\Gamma \subseteq A \times B$ such that $\forall a \in A, \exists! b \in B$ with $(a, b) \in \Gamma$. To denote f is a function from A to B we write $f : A \rightarrow B$.

Definition 1.2.3: The collection of all functions from a set A to a set B is denoted B^A .

Example: Every set A comes equipped with the *identity function*, $\text{id}_A : A \rightarrow A$, whose graph is the diagonal in $A \times A$. It is defined by $\forall a \in A. \text{id}_A(a) = a$.

Definition 1.2.4: If $S \subseteq A$, for $f : A \rightarrow B$, we define $f(S) \subseteq B$ as

$$f(S) = \{b \in B \mid \exists a \in S. b = f(a)\}$$

Definition 1.2.5: The *restriction* of $f : A \rightarrow B$ to $S \subseteq A$, denoted $f|_S$ is the function $S \rightarrow B$ defined by

$$\forall s \in S. f|_S(s) = f(s).$$

Remark: The restriction can be equivalently described as $f \circ i$ where $i : S \rightarrow A$ is the inclusion. Further, $f(S) = \text{im}(f|_S)$.

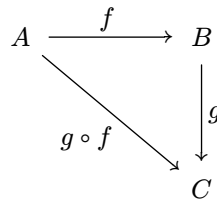
Example (Multisets): A *multiset* is like a set but allows for multiple instances of each element. A multiset may be defined by giving a function $m : A \rightarrow \mathbb{N}^*$, where \mathbb{N}^* is the set of positive integers. The corresponding multiset consists of the elements $a \in A$, each taken $m(a)$ times.

Example (Indexed Sets): One may think of an *indexed set* $\{a_i\}_{i \in I}$ as set whose elements are denoted by a_i for i ranging over some ‘set of indices’ I . Instead, it is more proper to think of an indexed set as a function $a : I \rightarrow A$, with the understanding that a_i is a shorthand for $a(i)$. One benefit is that this allows us to consider a_0, a_1 as distinct elements of $\{a_i\}_{i \in \mathbb{N}}$ even if $a_0 = a_1$ as elements of A .

Definition 1.2.6: If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions then so is the operation $g \circ f$ defined by

$$\forall a \in A. (g \circ f)(a) = g(f(a)).$$

Pictorially, the following diagram *commutes*

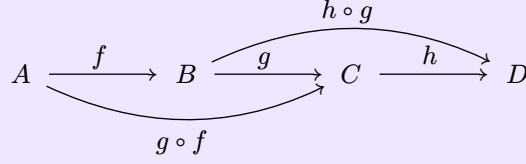


Remark: A diagram *commutes* when the result of following a path of arrows from any point of the diagram to any other point only depends on the starting and ending points and not on the particular path chosen.

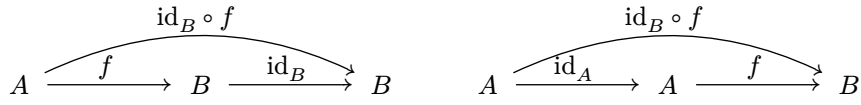
Lemma 1.2.1: Composition of functions is associative. That is to say, if $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$ then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Graphically, the following diagram commutes



Example: If $f : A \rightarrow B$ then $\text{id}_B \circ f = f$ and $f \circ \text{id}_A = f$. Graphically, the following diagrams commute

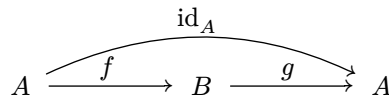


Definition 1.2.7: A function $f : A \rightarrow B$ is *injective* if $\forall a', a'' \in A, a' \neq a'' \implies f(a') \neq f(a'')$.

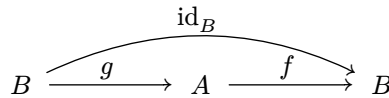
Definition 1.2.8: A function $f : A \rightarrow B$ is *surjective* if $\forall b \in B, \exists a \in A$ such that $b = f(a)$. That is to say f covers B and equivalently, $\text{im}(f) = B$.

Definition 1.2.9: If $f : A \rightarrow B$ is both injective and surjective then it is a *bijection*. Then we often write $f : A \xrightarrow{\sim} B$. We also say that A and B are *isomorphic* and denote this by $A \cong B$.

Definition 1.2.10: A function $g : B \rightarrow A$ is a *left inverse* of $f : A \rightarrow B$ if $g \circ f = \text{id}_A$. Graphically, the following diagram commutes



Definition 1.2.11: A function $f : A \rightarrow B$ is a *right inverse* of $g : B \rightarrow A$ if $f \circ g = \text{id}_B$. Graphically, the following diagram commutes



Definition 1.2.12: We call $g : B \rightarrow A$ an *inverse* of $f : A \rightarrow B$ if g is both a left and right inverse of f . Then g may also be denoted f^{-1} .

Proposition 1.2.1: Assume $A \neq \emptyset$ and let $f : A \rightarrow B$ be a function. Then

- a. f has a left inverse iff it is injective.
- b. f has a right inverse iff it is surjective.

Corollary 1.2.1.1: A function $f : A \rightarrow B$ is a bijection if and only if it has a (two-sided) inverse.

Remark: An injective but not surjective function has no right inverse. If the source has more than two elements, there will be more than one left inverse.

Remark: A surjective function but not injective function will have multiple inverses. These are called *sections*.

Definition 1.2.13: Let $f : A \rightarrow B$ be any function and $S \subseteq B$ be a subset. Then $f^{-1}(S)$ is defined by

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}.$$

If $S = \{q\}$ is a singleton then $f^{-1}(T) = f^{-1}(q)$ is denoted the *fiber* of f over q .

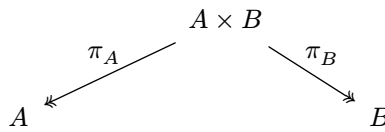
Remark: In this language: f is a bijection iff it has nonempty fiber over all elements of B and every fiber is a singleton.

Definition 1.2.14: A function $f : A \rightarrow B$ is a *monomorphism* (or *monic*) if for all sets Z and all functions $\alpha', \alpha'' : Z \rightarrow A$

$$f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''.$$

Proposition 1.2.2: A function is injective iff it is a monomorphism.

Example (Projection): Let A, B be sets. Then there are *natural projections* π_A, π_B

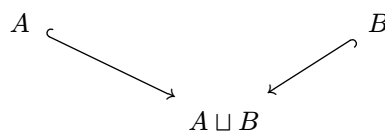


defined by

$$\forall (a, b) \in A \times B. \quad \pi_A((a, b)) = a, \quad \pi_B((a, b)) = b.$$

These maps are clearly surjective.

Example (Direct Sum Injection): There are natural injections from A, B to their disjoint union $A \sqcup B$



obtained by sending $a \in A$ (resp. $b \in B$) to the corresponding element in the isomorphic copy A' of A (resp. B' of B) in $A \sqcup B$.

Example (Equivalence Relation Projection): Let \sim be an equivalence relation on A . Then there is a surjective canonical projection

$$A \longrightarrow A/\sim$$

obtained by sending every $a \in A$ to its equivalence class $[a]_{\sim} \in A/\sim$.

Lemma 1.2.2: Every function $f : A \rightarrow B$ defines an equivalence relation \sim on A as follows: for every $a', a'' \in A$,

$$a' \sim a'' \iff f(a') = f(a'').$$

Proposition 1.2.3 (Canonical Decomposition): Let $f : A \rightarrow B$ be any function and define \sim as above. Then f decomposes as follows:

$$A \xrightarrow{\quad} A/\sim \xrightarrow[\tilde{f}]{\quad} \text{im}(f) \hookrightarrow B$$

f

The first function is the canonical projection $A \rightarrow A/\sim$. The third function is the inclusion $\text{im } f \subseteq B$. The bijection \tilde{f} in the middle is defined by

$$\tilde{f}([a]_{\sim}) = f(a)$$

for all $a \in A$.

1.3. Categories

Definition 1.3.1: A category \mathcal{C} consists of

- a class $\text{Obj}(\mathcal{C})$ of *objects* the category.
- for every two objects A, B of \mathcal{C} , a set $\text{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* with the following properties
 - for every object A of \mathcal{C} , there exists (at least) one morphism $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$, the ‘identity’ on A .
 - two morphisms $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, C)$ determine a morphism $fg \in \text{Hom}_{\mathcal{C}}(A, C)$. That is for every triple of objects A, B, C of \mathcal{C} there is a function (of sets)

$$\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

and the image of the pair (f, g) is denoted fg .

- this composition law is associative: if $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $g \in \text{Hom}_{\mathcal{C}}(B, C)$ and $h \in \text{Hom}_{\mathcal{C}}(C, D)$ then $(hg)f = h(gf)$.
- the identity morphisms are identities with respect to composition: that is for all $f \in \text{Hom}_{\mathcal{C}}(A, B)$ we have

$$f \cdot 1_A = f, \quad 1_B \cdot f = f.$$

- the sets $\text{Hom}_{\mathcal{C}}(A, B)$ and $\text{Hom}_{\mathcal{C}}(C, D)$ are disjoint unless $A = C$ and $B = D$.

Definition 1.3.2: A morphism of an object A of a category C to itself is called an *endomorphism*. Furthermore, $\text{Hom}_C(A, A)$ is also denoted $\text{End}_{C(A)}$.

Definition 1.3.3: A *diagram commutes* if all ways to traverse it lead to the same results of composing morphisms along the way.

Example (Set): By Set we denote the category of sets, where

- *objects:* $\text{Obj}(\text{Set}) =$ the class of all sets;
- *morphisms:* For A, B in $\text{Obj}(\text{Set})$, $\text{Hom}_{\text{Set}}(A, B) = B^A$;
- *composition:* Composition of morphisms is defined to be the same as the composition of functions;
- *identity:* For any object A of Set , the identity is defined to be the identity function on A .

Example (Relations): Suppose \sim is a reflexive and transitive relation on some set S . Then, we can encode this data into a category, C .

- *objects:* The elements of S ;
- *morphisms:* If a, b are objects, then let $\text{Hom}(a, b)$ be the set consisting of the element $(a, b) \in S \times S$ if $a \sim b$ and let $\text{Hom}(a, b) = \emptyset$ otherwise;
- *composition:* For composition, let a, b, c be objects and $f \in \text{Hom}(a, b)$ and $g \in \text{Hom}(b, c)$. Then, $gf \in \text{Hom}(a, c)$ is defined to be

$$gf = (a, c).$$

- *identity:* By reflexivity, we are guaranteed that $(a, a) \in \text{Hom}_C(A, A)$. Then, for any object A of C , we define the identity to be $1_A = (a, a)$.

Remark: This is a *small category*;

Example (Partial Ordering of Sets): Let S be a set. Define another (*small*) category \hat{S} by

- *objects:* $\text{Obj}(\hat{S}) = \mathcal{P}(S)$;
- *morphisms:* For A, B objects of \hat{S} , let $\text{Hom}_{\hat{S}}(A, B)$ be the pair (A, B) if $A \subseteq B$ and let $\text{Hom}_{\hat{S}}(A, B) = \emptyset$ otherwise.
- *composition:* Let A, B, C be objects and $f \in \text{Hom}_{\hat{S}}(A, B)$ and $g \in \text{Hom}_{\hat{S}}(B, C)$. Then, $gf \in \text{Hom}_{\hat{S}}(A, C)$ is defined to be

$$gf = (A, C).$$

- *identity:* For any set A , $A \subseteq A$ and so, $(A, A) \in \text{Hom}_{\hat{S}}(A, A)$. The, for every object A of C , we define the identity to be $1_A = (A, A)$.

Example (C_A): Let C be a category and let A be an object of C . We will define a category C_A whose objects are certain *morphisms* in C and whose morphisms are certain *diagrams* of C .

- *objects:* $\text{Obj}(C_A) =$ all morphisms from any object of C to A ; that is, an object of C is an element $f \in \text{Hom}_C(Z, A)$ for some object Z of C . Pictorially, an object of C_A is an arrow $Z \xrightarrow{f} A$ in C ;

$$\begin{array}{c} Z \\ \downarrow f \\ A \end{array}$$

- *morphisms:* For objects f_1, f_2 of C_A , that is two arrows

$$\begin{array}{ccc} Z_1 & & Z_2 \\ f_1 \downarrow & & \downarrow f_2 \\ A & & A \end{array}$$

in C . Morphisms $f_1 \rightarrow f_2$ are defined to be *commutative diagrams*

$$\begin{array}{ccc} Z_1 & \xrightarrow{\sigma} & Z_2 \\ f_1 \searrow & & \swarrow f_2 \\ & A & \end{array}$$

in the *ambient* category C . Alternatively, morphisms $f_1 \rightarrow f_2$ corresponds to those morphisms $\sigma : Z_1 \rightarrow Z_2$ in C such that $f_1 = f_2 \sigma$.

- *composition*:
- *identity*:

Categories constructed in these manners are known as *slice categories*, which are particular cases of *comma categories*.

Example (C_A II): Suppose C is the category with $S = \mathbb{Z}$ and using the relation \leq . Choose an object $A = 3$ of C . Then the objects of C_A are morphisms in C with target 3, that is, pairs $(n, 3) \in \mathbb{Z} \times \mathbb{Z}$ with $n \leq 3$. There is a morphism

$$(m, 3) \longrightarrow (n, 3)$$

if and only if $m \leq n$. In this example, C_A may be harmlessly identified with the *subcategory* of integers ≤ 3 with the *same* morphisms as in C .

Example (C^A): We can consider a construction similar to slice categories but one where we take objects to be morphisms in a category C from a fixed object A to all objects in C . Morphisms are again defined to be suitable commutative diagrams. This construction is known as *the coslice category*.

Example (C^A II): Let $C = \text{Set}$ and $A = \text{fixed singleton} = \{\star\}$. Call the constructed co-slice category Set^* .

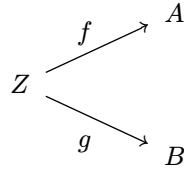
An object in Set^* is then a morphism $f : \{\star\} \rightarrow S$ in Set where S is any set. The information of an object in Set^* consists of a nonempty set S and an element $s \in S$ – that is, the element $f(\star)$. This element determines and is determined by, f . So, we can denote objects of Set^* as pairs (S, s) where S is any set and $s \in S$ is any element of S .

A morphism between two such objects, $(S, s) \rightarrow (T, t)$ corresponds to a set function $\sigma : S \rightarrow T$ such that $\sigma(s) = t$.

Objects of Set^* are called *pointed sets*.

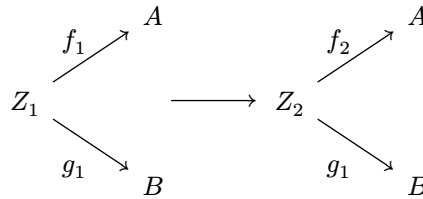
Example ($C_{A,B}$): Start from a category C and two objects A, B of C . We then define a new category $C_{A,B}$ by a similar procedure with which we defined C_A .

- *objects*: $\text{Obj}(C_{A,B}) = \text{diagrams}$

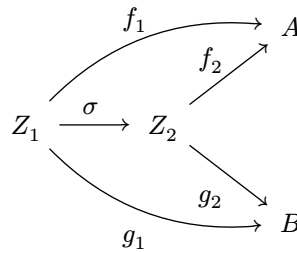


in C ;

- *morphisms*:



are *commutative diagrams*

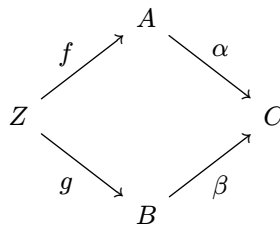


Alternatively, morphisms in $C_{A,B}$ corresponds to those morphisms $\sigma : Z_1 \rightarrow Z_2$ in C such that $f_1 = f_2 \sigma$ and $g_1 = g_2 \sigma$.

- *composition*:
- *identity*:

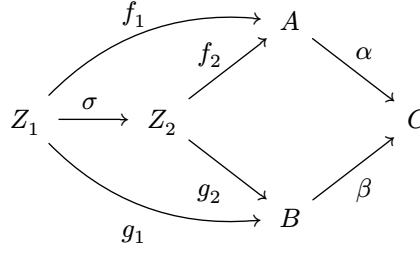
Example (Fibered $C_{A,B}$): Start with a given category C and choose two fixed morphisms $\alpha : A \rightarrow C, \beta : B \rightarrow C$ in C with the same target C . We can then consider a category $C_{\alpha,\beta}$ as follows

- *objects*: $\text{Obj}(C_{\alpha,\beta}) = \text{commutative diagrams}$



in C ;

- *morphisms*: morphisms correspond to commutative diagrams



- *composition*:
- *identity*:

1.4. Morphisms

Throughout this section let C be a category.

Definition 1.4.1 (Isomorphism): A morphism $f \in \text{Hom}_C(A, B)$ is an *isomorphism* if it has a (two-sided) inverse: that is $\exists g \in \text{Hom}_C(B, A)$ such that

$$gf = 1_A \quad fg = 1_B.$$

Proposition 1.4.1: The inverse of an isomorphism is unique.

Remark: Due to uniqueness, we may unambiguously refer to the inverse of an isomorphism f as f^{-1} .

Proposition 1.4.2:

- Each identity 1_A is an isomorphism and is its own inverse,
- If f is an isomorphism. the f^{-1} is an isomorphism and further $(f^{-1})^{-1} = f$,
- If $f \in \text{Hom}_C(A, B)$ and $g \in \text{Hom}_C(B, C)$ are isomorphisms, then the composition gf is an isomorphism and $(gf)^{-1} = f^{-1}g^{-1}$.

Definition 1.4.2: Two objects A, B of a category C are *isomorphic* if there is an isomorphism $f : A \rightarrow B$.

Corollary 1.4.2.1: Isomorphism is an equivalence relation on the objects of a category.

Example: Isomorphisms in **Set** are precisely bijective functions.

Example: In the category C obtained from the relation \leq on \mathbb{Z} there is a morphism $a \rightarrow b$ and $b \rightarrow a$ only if $a \leq b$ and $b \leq a$ —that is, if $a = b$. So an isomorphism must act from an object to itself and in C there is only one such object 1_a .

Example: There are categories in which every morphism is an isomorphism. These are known as *groupoids*.

Definition 1.4.3: An *automorphism* of an object A of a category C is an isomorphism from A to itself. The set of automorphisms is denoted $\text{Aut}_C(A)$ and is a subset of $\text{End}_C(A)$.

Remark: Equipped with composition, $\text{Aut}_C(A)$ is a group!

Definition 1.4.4: Let C be a category. A morphism $f \in \text{Hom}_C(A, B)$ is a *monomorphism* if for all objects Z of C and all morphisms $\alpha', \alpha'' \in \text{Hom}_C(Z, A)$,

$$f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''.$$

Definition 1.4.5: Let C be a category. A morphism $f \in \text{Hom}_C(A, B)$ is an *epimorphism* if the following holds if for all objects Z of C and all morphisms $\beta', \beta'' \in \text{Hom}_C(B, Z)$,

$$\beta' \circ f = \beta'' \circ f \implies \beta' = \beta''.$$

Example: In C , injective functions are *monomorphisms* whereas surjective functions are *epimorphisms*.

Example: In the category C obtained from the relation \leq on \mathbb{Z} , every morphism is both a monomorphism and an epimorphism. However, there is at most one isomorphism between any two pair of objects in C .

Remark: The previous example shows how the property of Set , wherein a function is an isomorphism iff it is a monomorphism and epimorphism, doesn't generalize to all categories. It can be shown that this is true in *abelian categories* (but Set isn't an example of one!)

Remark: The property of Set that a function is an epimorphism iff it has a right inverse doesn't generalize to all categories either.

1.5. Universal Properties

Definition 1.5.1: We say that an object I of category C is *initial* in C if for every object A of C there exists exactly one isomorphism $I \rightarrow A$ in C — that is, for every object A of C , $\text{Hom}_C(I, A)$ is a singleton.

Definition 1.5.2: We say that an object F of category C is *final* in C if for every object A of C there exists exactly one isomorphism $A \rightarrow F$ in C — that is, for every object A of C , $\text{Hom}_C(A, F)$ is a singleton.

Definition 1.5.3: An object F of category C is *terminal* in C if it is either initial or final.