Mathematics **Summer**

Random Graphs

a first course

Authors: Shaleen Baral

Contents

1. Introduction to Asymptotics

Definition 1.1 Asymptotic Equivalence. We say that $f(n)$ is *asymptotically equivalent* to $g(n)$ and write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$.

Definition 1.2 . We write $f(n) \in O(g(n))$ when there is a $C > 0$ such that for all sufficiently large n,

$$|f(n)| \leq C|g(n)|.$$

Definition 1.3 . We write $f(n) = \Omega(g(n))$ when there is a $c > 0$ such that for all sufficiently large n,

$$|f(n)| \geq c|g(n)|.$$

Lemma 1.1: Equivalently, $f(n) = O(g(n))$ iff $\limsup_{n \longrightarrow \infty} |f(n)|/|g(n)| < \infty$.

Proof: For convenience let $Q(n) = |f(n)|/|g(n)|$. First, the foward direction. We note that we have $0 \leq Q(n) \leq C$. As $Q(n)$ is bounded $\limsup Q(n)$ clearly exists and is finite (the sequence $\{\sup_{n \geq k} Q(n)\}_{k \in N}$ is decreasing and as it is bounded by below, must converge).

Conversely, assume $\limsup Q(n) < \infty$. Let $C = \max(\limsup Q(n) + 1, 1)$. As $C > \limsup Q(n)$, it is an eventual upper bound for $Q(n)$. That is to say, there exists $N \in \mathbb{N}_{>0}$ such that for all $n \geq N$

$$|f(n)| \leq C|g(n)|.$$

□

Lemma 1.2: Equivalently, $f(n) = \Omega(g(n))$ iff $\liminf_{n \longrightarrow \infty} |f(n)|/|g(n)| < \infty$.

Proof: Same idea as above.

□

Definition 1.1 . We write $f(n) = \Theta(g(n))$ when there are constants $c, C > 0$ such that

$$c|g(n)| \leq f(n) \leq C|g(n)|.$$

Equivalently, $f(n) = \Theta(g(n))$ iff $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$.

Lemma 1.3: If $f_1, f_2 \in O(g)$ then $f_1 + f_2 = O(g)$.

Proof: There exists $C_1, C_2, N_1, N_2 > 0$ such that for $n > N_1$ and $n > N_2$

$$|f_1| \leq C_1 |g|$$
$$|f_2| \leq C_2 |g|.$$

Then, for $N = \max(N_1, N_2)$, we can say that if $n > N$ then

$$|f_1 + f_2| \leq (C_1 + C_2)\ |g|.$$

□

Lemma 1.4: If $f_1, f_2 \in \Omega(g)$ then $f_1 + f_2 \in \Omega(g)$ too.

Proof: Same idea as above.

□

Lemma 1.5: If $f_1, f_2 \in \Theta(g)$ then $f_1 + f_2 \in \Theta(g)$ too.

Proof: Follows from prior two lemmas and definition of Θ.

□

Definition 1.2 . We write $f(n) = o(g(n))$ (or $f(n) \ll g(n)$) if $f(n)/g(n) \to 0$ as $n \to \infty$.

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