Mathematics **Summer**

Graph Theory

from Diestel.

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1. Basics

1.1. Graphs

Definition 1.1.1 . A *graph* a pair $G = (V, E)$ such that $E \subseteq [V]^2$. For clarity, we assume that $V \cap E = \emptyset$. The elements of V are the *vertices* of the graph G and the elements of E are its *edges*.

Definition 1.1.2 . The *order* of a graph, written $|G|$, is the number of vertices of G. The number of edges of G is denotes by $||G||$. Graphs are *finite*, *infinite*, *countable* and so on according to their order.

Example: The *empty graph* is $(\emptyset, \emptyset)$, also denotes as $\emptyset$ simply.

Example: A graph of order 0 or 1 is also known as a *trivial* graph.

Definition 1.1.3 . A vertex v is *incident* with an edge e if $v \in e$; then e is an edge *at* v. The two vertices incident with an edge are its *endvertices* or *ends*, and an edge *joins* its ends.

Definition 1.1.4 . An edge $\{x, y\}$ is usually written as xy (or yx). If $x \in X$ and $y \in Y$, then xy is an X-Y *edge*. The set of all X-Y edges in a set E is denoted by $E(X, Y)$.

Remark: Instead of $E(\{x\}, Y)$ and $E(X, \{y\})$, we write $E(x, Y)$ and $E(X, y)$. The set of all the edges in E at a vertex v is denoted by $E(v)$.

Definition 1.1.5 . Two vertices x, y of G are *adjacent* or *neighbors* if xy is an edge of G. Two edges $e \neq f$ are adjacent if they have an end in common.

Example: If all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on n vertices is denoted K^n.

Definition 1.1.6 . A set of vertices or edges is *independent* if no two of its elements are adjacent. Independent sets of vertices are also called *stable*.

Definition 1.1.7 . Let $G = (V, E)$ and $G' = (V', E')$. A map $\varphi : V \to V'$ is a *homomorphism* from G to G' if it preserves the adjacency of vertices, that is, if $\{\varphi(x), \varphi(y)\} \in E'$ whenever $\{x, y\} \in E$.

Lemma 1.1.1: For every vertex x' in the image of $\varphi : G \to G'$, its inverse image $\varphi^{-1}(x')$ is an independent set of vertices in G.

Definition 1.1.1 . If φ is bijective and its inverse φ^{-1} is also a homomorphism (i.e. $xy \in E \iff \varphi(x)\varphi(y) \in E'$ for all $x, y \in V$), we call φ an *isomorphism*. We also say G and G' are isomorphic as denoted by $G \simeq G'$ (or even simpler, $G = G'$, when we only care about the *isomorphism type* of a given graph)

Definition 1.1.9 . An isomorphism from G to itself is an *automorphism* of G.

Definition 1.1.10 . A class of graphs that is closed under isomorphism is called a *graph property*.

Example: Containing a triangle is a graph property.

Definition 1.1.11 . A map taking graphs as arguments is called a *graph invariant* if it assigns equal values to isomorphic graphs.

Example: The number of vertices and the number of edges are graph invariants. The greatest number of pairwise adjacent vertices is also another one.

Definition 1.1.12 . We define $G \cup G' = (V \cup V', E \cup E')$ and $G \cap G' = (V \cap V', E \cap E')$. If $G \cap G' = \emptyset$ then G and G' are *disjoint*.

Definition 1.1.13 . If $V' \subseteq V$ and $E' \subseteq E$, then G' is *subgraph* of G (and G a *supergraph* of G'), written $G' \subseteq G$. If $G' \subseteq G$ but $G' \neq G$ then G' is a *proper subgraph* of G.

Remark:

Definition 1.1.14 . If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an induced subgraph of G; we say that V' *induces* or *spans* G' in G.

Remark: If $U \subseteq V$ is any set of vertices, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U.

2. Matchings, Covering, Packing

3. Connectivity

4. Flows

5. Hamilton Cycles