Algebra

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Contents

1. Preliminaries

1.1. Naive Set Theory

Definition 1.1.1: The *ordered pair* (s, t) can be defined as the set $\{s, \{s, t\}\}$. This retains both the elements of the tuple but also conveys an ordering.

Definition 1.1.2: The *disjoint union* of two sets S, T is the set $S \sqcup T$ obtained by first producing 'copies' S' and T' and then taking the union.

Definition 1.1.3: The *product* of two sets S, T is the set $S \times T$ defined as

$$S \times T = \{(s, t) \text{ such that } s \in S,, t \in T\}.$$

Definition 1.1.4: A *relation* on a set S is a subset R of the product $S \times S$. If $(a, b) \in R$, we write aRb.

Definition 1.1.5: An *equivalence relation* on a set S is any relation $\sim$ satisfying the following properties

a. *reflexivity*: $\forall a \in S.\ a \sim a$
b. *symmetry*: $\forall a \in S. \forall b \in S.\ a \sim b \Longleftrightarrow b \sim a$
c. *transitivity*: $\forall a \in S. \forall b \in S. \forall c \in S.\ a \sim b, b \sim c \Longrightarrow a \sim c$.

Definition 1.1.6: A *partition* of S is a family of *disjoint* nonempty subsets of S whose union is S.

Definition 1.1.7: Let $\sim$ be an equivalence relation on S. Then for every $a \in S$, the *equivalence class* of a is the subset S defined by

$$[a]_{\sim} = \{b \in S \mid b \sim a\}.$$

Further, the equivalence classes form a partition $\mathcal{P}_{\sim}$ of S.

Lemma 1.1.1: Every partition of S corresponds to an equivalence relation.

Definition 1.1.8: The *quotient* of the set S with respect to the equivalence relation $\sim$ is the set

$$S \sim= \mathcal{P}_{\sim}$$

of equivalence classes of elements of S with respect to $\sim$.

1.2. Functions

Definition 1.2.1: The graph of f is the set

$$\Gamma_f = \{(a, b) \in A \times B \mid b = f(a)\} \subseteq A \times B.$$

Officially, a function *is* its graph together with information of the source A and the target B of f.

Definition 1.2.2: A function is a relation $\Gamma \subseteq A \times B$ such that $\forall a \in A, \exists! b \in B$ with $(a, b) \in \Gamma$. To denote f is a function from A to B we write $f : A \longrightarrow B$.

Definition 1.2.3: The collection of all functions from a set A to a set B is denoted B^A.

Example: Every set A comes equipped with the *identity function*, $\mathrm{id}_A : A \longrightarrow A$, whose graph is the diagonal in $A \times A$. It is defined by $\forall a \in A.\ \ \mathrm{id}_A(a) = a$.

Definition 1.2.4: If $S \subseteq A$, for $f : A \longrightarrow B$, we define $f(S) \subseteq B$ as

$$f(S) = \{b \in B \mid \exists a \in S. b = f(a)\}$$

Definition 1.2.5: The *restriction* of $f : A \longrightarrow B$ to $S \subseteq A$, denoted $f\ |_S$ is the function $S \longrightarrow B$ defined by

$$\forall s \in S. \quad f\ |_S\ (s) = f(s).$$

Remark: The restriction can be equivalently described as $f \circ i$ where $i : S \longrightarrow A$ is the inclusion. Further, $f(S) = \mathrm{im}(f\ |_S)$.

Example (Multisets): A *multiset* is like a set but allows for multiple instances of each element. A multiset may be defined by giving a function $m : A \longrightarrow \mathbb{N}^\star$, where $N^\star$ is the set of positive integers. The corresponding multiset consists of the elements $a \in A$, each taken $m(a)$ times.

Example (Indexed Sets): One may think of an *indexed set* $\{a_i\}_{i \in I}$ as set whose elements are denoted by a_i for i ranging over some 'set of indices' I. Instead, it is more proper to think of an indexed set as a function $a : I \longrightarrow A$, with the understanding that a_i is a shorthand for $a(i)$. One benefit is that this allows us to consider a_0, a_1 as distinct elements of $\{a_i\}_{i \in \mathbb{N}}$ even if $a_0 = a_1$ as elements of A.

Definition 1.2.6: If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions then so is the operation $g \circ f$ defined by

$$\forall a \in A. \quad (g \circ f)(a) = g(f(a)).$$

Pictorially, the following diagram *commutes*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Remark: A diagram *commutes* when the result of following a path of arrows from any point of the diagram to any other point only depends on the starting and ending points and not on the particular path chosen.

Lemma 1.2.1: Composition of functions is associative. That is to say, if $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$ then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Graphically, the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h \circ g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

Example: If $f : A \longrightarrow B$ then $\mathrm{id}_B \circ f = f$ and $f \circ \mathrm{id}_A = f$. Graphically, the following diagrams commute

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "\mathrm{id}_B \circ f"] & B \arrow[r, "\mathrm{id}_B"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[rr, bend left, "\mathrm{id}_B \circ f"] & A \arrow[r, "f"] & B \end{tikzcd}$$

Definition 1.2.7: A function $f : A \longrightarrow B$ is *injective* if $\forall a', a'' \in A$, $a' \neq a'' \implies f(a') \neq f(a'')$.

Definition 1.2.8: A function $f : A \longrightarrow B$ is *surjective* if $\forall b \in B$, $\exists a \in A$ such that $b = f(a)$. That is to say f *covers* B and equivalently, $\mathrm{im}(f) = B$.

Definition 1.2.9: If $f : A \longrightarrow B$ is both injective and surjective then it is a *bijection*. Then we often write $f : A \xrightarrow{\sim} B$. We also say that A and B are *isomorphic* and denote this by $A \cong B$.

Definition 1.2.10: A function $g : B \longrightarrow A$ is a *left inverse* of $f : A \longrightarrow B$ if $g \circ f = \mathrm{id}_A$. Graphically, the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "\mathrm{id}_A"] & B \arrow[r, "g"] & A \end{tikzcd}$$

Definition 1.2.11: A function $f : A \longrightarrow B$ is a *right inverse* of $g : B \longrightarrow A$ if $f \circ g = \mathrm{id}_A$. Graphically, the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[rr, bend left, "\mathrm{id}_B"] & A \arrow[r, "f"] & B \end{tikzcd}$$

Definition 1.2.12: We call $g : B \longrightarrow A$ an inverse of $f : A \longrightarrow B$ if g is both a left and right inverse of f. Then g may also be denoted f^{-1}.

Proposition 1.2.1: Assume $A \neq \emptyset$ and let $f : A \longrightarrow B$ be a function. Then

a. f has a left inverse iff it is injective.
b. f has a right inverse iff it is surjective.

Corollary 1.2.1.1: A function $f : A \longrightarrow B$ is a bijection if and only if it has a (two-sided) inverse.

Remark: An injective but not surjective function has no right inverse. If the source has more than two elements, there will be more than one left inverse.

Remark: A surjective function but not injective function will have multiple inverses. These are called *sections*.

Definition 1.2.13: Let $f : A \longrightarrow B$ be any function and $S \subseteq B$ be a subset. Then $f^{-1}(S)$ is defined by

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}.$$

If $S = \{q\}$ is a singleton then $f^{-1}(T) = f^{-1}(q)$ is denoted the *fiber* of f over q.

Remark: In this language: f is a bijection iff it has nonempty fiber over all elements of B and every fiber is a singleton.

Definition 1.2.14: A function $f : A \longrightarrow B$ is a *monomorphism* (or *monic*) if for all sets Z and all functions $\alpha', \alpha'' : Z \longrightarrow A$

$$f \circ \alpha' = f \circ \alpha'' \Longrightarrow \alpha' = \alpha''.$$

Proposition 1.2.2: A function is injective iff it is a monomorphism.

Example (Projection): Let A, B be sets. Then there are *natural projections* π_A, π_B

$$\begin{tikzcd} & A \times B \arrow[dl, two heads, "\pi_A"'] \arrow[dr, two heads, "\pi_B"] & \\ A & & B \end{tikzcd}$$

defined by

$$\forall (a, b) \in A \times B. \quad \pi_A((a, b)) = a, \qquad \pi_B((a, b)) = b.$$

These maps are clearly surjective.

Example (Direct Sum Injection): There are natural injections from A, B to their disjoint union $A \sqcup B$

$$\begin{tikzcd} A \arrow[dr, hook] & & B \arrow[dl, hook'] \\ & A \sqcup B & \end{tikzcd}$$

obtained by sending $a \in A$ (resp. $b \in B$) to the corresponding element in the isomorphic copy A' of A (resp. B' of B) in $A \sqcup B$.

Example (Equivalence Relation Projection): Let $\sim$ be an equivalence relation on A. Then there is a surjective *canonical projection*

$$A \twoheadrightarrow A/\sim$$

obtained by sending every $a \in A$ to its equivalence class $[a]_\sim \in A/\sim$.

Lemma 1.2.2: Every function $f : A \longrightarrow B$ defines an equivalence relation $\sim$ on A as follows: for every $a', a'' \in A$,

$$a' \sim a'' \iff f(a') = f(a'').$$

Proposition 1.2.3 (Canonical Decomposition): Let $f : A \longrightarrow B$ be any function and define $\sim$ as above. Then f decomposes as follows:

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[rrr, bend left, "f"] & (A/\sim) \arrow[r, "\sim", "\tilde{f}"'] & \mathrm{im}(f) \arrow[r, hook] & B \end{tikzcd}$$

The first function is the canonical projection $A \longrightarrow A/\sim$. The third function is the inclusion $\mathrm{im}\, f \subseteq B$. The bijection $\tilde{f}$ in the middle is defined by

$$\tilde{f}\big([a]_\sim\big) = f(a)$$

for all $a \in A$.

1.3. Categories

Definition 1.3.1: A *category* C consists of

- a class $\mathrm{Obj}(C)$ of *objects* the category.
- for every two objects A, B of C, a set $\mathrm{Hom}_C(A, B)$ of *morphisms* with the following properties
  - for every object A of C, there exists (at least) one morphism $1_A \in \mathrm{Hom}_C(A, A)$, the 'identity' on A.
  - two morphisms $f \in \mathrm{Hom}_C(A, B)$ and $g \in \mathrm{Hom}_C(B, C)$ determine a morphism $fg \in \mathrm{Hom}_C(A, C)$. That is for every triple of objects A, B, C of C there is a function (of sets)

    $$\mathrm{Hom}_C(A, B) \times \mathrm{Hom}_C(B, C) \longrightarrow \mathrm{Hom}_C(A, C)$$

    and the image of the pair (f, g) is denoted fg.
  - this composition law is associative: if $f \in \mathrm{Hom}_C(A, B)$, $g \in \mathrm{Hom}_C(B, C)$ and $h \in \mathrm{Hom}(C)(C, D)$ then $(hg)f = h(gf)$.
  - the identity morphisms are identities with respect to composition: that is for all $f \in \mathrm{Hom}_C(A, B)$ we have

    $$f \cdot 1_A = f, \quad 1_B \cdot f = f.$$
  - the sets $\mathrm{Hom}_C(A, B)$ and $\mathrm{Hom}_C(C, D)$ are disjoint unless $A = C$ and $B = D$.

Definition 1.3.2: A morphism of an object A of a category C to itself is called an *endomorphism*. Furthermore, $\mathrm{Hom}_C(A, A)$ is also denoted $\mathrm{End}_{C(A)}$.

Definition 1.3.3: A *diagram commutes* if all ways to traverse it lead to the same results of composing morphisms along the way.

Example (Set*)*: By Set we denote the category of sets, where
- $\mathrm{Obj}(\mathsf{Set}) =$ the class of all sets;
- for A, B in $\mathrm{Obj}(\mathsf{Set})$, $\mathrm{Hom}_{\mathsf{Set}}(A, B) = B^A$.

Composition of morphisms is defined to be the same as the composition of set-functions.

Example (Relations): Suppose $\sim$ is a reflexive and transitive relation on some set S. Then, we can encode this data into a category.
- *objects*: the elements of S;
- *morphisms*: if a, b are objects, then let $\mathrm{Hom}(a, b)$ be the set consisting of the element $(a, b) \in S \times S$ if $a \sim b$ and let $\mathrm{Hom}(a, b) = \emptyset$ otherwise.

For composition, let a, b, c be objects and $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$. Then, $gf \in \mathrm{Hom}(a, c)$ is defined to be

$$gf = (a, c).$$

This is a *small category*.

Example (Partial Ordering of Sets): Let S be a set. Define another (*small*) category $\hat{S}$ by
- $\mathrm{Obj}\left(\hat{S}\right) = \mathcal{P}(S)$;
- for A, B objects of $\hat{S}$, let $\mathrm{Hom}_{\hat{S}}(A, B)$ be the pair (A, B) if $A \subseteq B$ and let $\mathrm{Hom}_{\hat{S}}(A, B) = \emptyset$ otherwise.

For composition, let A, B, C be objects and $f \in \mathrm{Hom}_{\hat{S}}(A, B)$ and $g \in \mathrm{Hom}_{\hat{S}}(B, C)$. Then, $gf \in \mathrm{Hom}_{\hat{S}}(A, C)$ is definted to be

$$gf = (A, C).$$

Example (Slice Category): Let C be a category and let A be an object of C. We will define a category C_A whose objects are certain *morphisms* in C and whose morphisms are certain *diagrams* of C.

- $\mathrm{Obj}(C_A) =$ all morphisms from any object of C to A; that is, an object of C is an element $f \in \mathrm{Hom}_C(Z, A)$ for some object Z of C. Pictorally, an object of C_A is an arrow $Z \xrightarrow{f} A$ *in* C;

$$\begin{tikzcd} Z \arrow[d, "f"'] \\ A \end{tikzcd}$$

- for objects f_1, f_2 of C_A, that is two arrows

$$\begin{tikzcd} Z_1 \arrow[d, "f_1"'] & Z_2 \arrow[d, "f_2"] \\ A & A \end{tikzcd}$$

in C. Morphisms $f_1 \to f_2$ are defined to be *commutative diagrams*

$$\begin{tikzcd} Z_1 \arrow[rr, "\sigma"] \arrow[dr, "f_1"'] & & Z_2 \arrow[dl, "f_2"] \\ & A & \end{tikzcd}$$

in the *ambient* category C. Alternatively, morphisms $f_1 \to f_2$ corresponds to those morphisms $\sigma : Z_1 \to Z_2$ in C such that $f_1 = f_2\sigma$.

Categories constructed in these manners are known as *slice categories*, which are particular cases of *comma categories.*

Example (Concrete Slice Category): Suppose C is the category with $S = \mathbb{Z}$ and using the relation $\leq$. Choose an object $A = 3$ of C. Then the objects of C_A are morphisms in C with target 3, that is, pairs $(n, 3) \in \mathbb{Z} \times \mathbb{Z}$ with $n \leq 3$. There is a morphism

$$(m, 3) \longrightarrow (n, 3)$$

if and only if $m \leq n$. In this example, C_A may be harmlessly identified with the *subcategory* of integers ≤ 3 with the *same* morphisms as in C.

Example (Co-Slice Category): We can consider a construction similar to slice categories but one where we take objects to be morphisms in a category C *from* a fixed object A to all objects in C. Morphisms are again defined to be suitable commutative diagrams. This construction is known as *the coslice category.*

Example (Concrete Co-Slice Category): Let $C = \mathsf{Set}$ and $A =$ fixed singleton $= \{\star\}$. Call the constructed co-slice category $\mathsf{Set}^\star$.

An object in $\mathsf{Set}^\star$ is then a morphism $f : \{\star\} \to S$ in Set where S is any set. The information of an object in $\mathsf{Set}^\star$ consists of a nonempty set S and an element $s \in S$ – that is, the element $f(\star)$. This element determines and is determined by, f. So, we can denote objects of $\mathsf{Set}^\star$ as pairs (S, s) where S is any set and $s \in S$ is any element of S.

A morphism between two such objects, $(S, s) \to (T, t)$ corresponds to a set function $\sigma : S \to T$ *such that* $\sigma(s) = t$.

Objects of $\mathsf{Set}^\star$ are called *pointed sets.*

Example ($C_{A,B}$): Start from a category C and two objects A, B of C. We then define a new category $C_{A,B}$ by a similar procedure with which we defined C_A.

- $\mathrm{Obj}(C_{A,B}) =$ diagrams

$$\begin{tikzcd} & A \\ Z \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$$

in C;

- morphisms

$$
\begin{tikzcd}
& A & & & A \\
Z_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & \arrow[r] & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_1"'] & \\
& B & & & B
\end{tikzcd}
$$

  are *commutative diagrams*

$$
\begin{tikzcd}
& & A \\
Z_1 \arrow[r, "\sigma"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "g_1"'] & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & \\
& & B
\end{tikzcd}
$$

Alternatively, morphisms in $C_{A,B}$ corresponds to those morphisms $\sigma : Z_1 \to Z_2$ in C such that $f_1 = f_2\sigma$ and $g_1 = g_2\sigma$.

*Example (*Fibered $C_{A,B}$*)*: Start with a given category C and choose two fixed morphisms $\alpha : A \to C, \beta : B \to C$ in C with the same target C. We can then consider a category $C_{\alpha,\beta}$ as follows

- $\mathrm{Obj}(C_{\alpha,\beta})$ = commutative diagrams

$$
\begin{tikzcd}
& A \arrow[dr, "\alpha"] & \\
Z \arrow[ur, "f"] \arrow[dr, "g"'] & & C \\
& B \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

  in C;
- morphisms correspond to commutative diagrams

$$
\begin{tikzcd}
& & A \arrow[dr, "\alpha"] & \\
Z_1 \arrow[r, "\sigma"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "g_1"'] & Z_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & & C \\
& & B \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

1.4. Morphisms

Throughout this section let C be a category.

Definition 1.4.1 (Isomorphism): A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is an *isomorphism* if it has a (two-sided) inverse: that is $\exists g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ such that

$$gf = 1_A \qquad fg = 1_B.$$

Proposition 1.4.1: The inverse of an isomorphism is unique.

Remark: Due to uniqueness, we may unambiguously refer to the inverse of an isomorphism f as f^{-1}.

Proposition 1.4.2:

a. Each identity 1_A is an isomorphism and is its own inverse,
b. If f is an isomorphism. the f^{-1} is an isomorphism and further $\left(f^{-1}\right)^{-1} = f$,
c. If $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ are isomorphisms, then the composition gf is an isomorphism and $(gf)^{-1} = f^{-1}g^{-1}$.

Definition 1.4.2: Two objects A, B of a category C are *isomorphic* if there is an isomorphism $f : A \to B$.

Corollary 1.4.2.1: Isomorphism is an equivalence relation on the objets of a category.

Example: Isomorphisms in Set are precisely bijective functions.

Example: In the category C obtained from the relation $\leq$ on $\mathbb{Z}$ there is a morphism $a \to b$ and $b \to a$ only if $a \leq b$ and $b \leq a$– that is, if $a = b$. So an isomorphism must act from an object to itself and in C there is only one such object 1_a.

Example: There are categories in which every morphism is an isomorphism. These are known as *groupoids*.

Definition 1.4.3: An *automorphism* of an object A of a category C is an isomorphism from A to itself. The set of automorphisms is denoted $\mathrm{Aut}_{\mathsf{C}}(A)$ and is a subset of $\mathrm{End}_{\mathsf{C}}(A)$.

Remark: Equipped with composition, $\mathrm{Aut}_{\mathsf{C}}(A)$ is a group!

Definition 1.4.4: Let C be a category. A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is a *monomorphism* if for all objects Z of C and all morphisms $\alpha', \alpha'' \in \mathrm{Hom}_{\mathsf{C}}(Z, A)$,

$$f \circ \alpha' = f \circ \alpha'' \Longrightarrow \alpha' = \alpha''.$$

Definition 1.4.5: Let C be a category. A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is an *epimorphism* if the following holds if for all objects Z of C and all morphisms $\beta', \beta'' \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$,

$$\beta' \circ f = \beta'' \circ f \Longrightarrow \beta' = \beta''.$$

Example: In C, injective functions are *monomorphisms* whereas surjective functions are *epimorphisms*.

Example: In the category C obtained from the relation $\leq$ on $\mathbb{Z}$, every morphism is both a monomorphism and an epimorphism. However, there is at most one isomorphism between any two pair of objects in C.

Remark: The previous example shows how the property of Set, wherin a function is an isomorphism iff it is a monomorphism and epimorphism, doesn't generalize to all categories. It can be shown that this is true in *abelian categories* (but Set isn't an example of one!)

Remark: The property of Set that a function is an epimorphism iff it has a right inverse doesn't generalize to all categories either.