

# Algebra

Aluffi

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# 1. Preliminaries

## 1.1. Naive Set Theory

**Definition 1.1.1:** The *ordered pair*  $(s, t)$  can be defined as the set  $\{s, \{s, t\}\}$ . This retains both the elements of the tuple but also conveys an ordering.

**Definition 1.1.2:** The *disjoint union* of two sets  $S, T$  is the set  $S \sqcup T$  obtained by first producing ‘copies’  $S'$  and  $T'$  and then taking the union.

**Definition 1.1.3:** The *product* of two sets  $S, T$  is the set  $S \times T$  defined as

$$S \times T = \{(s, t) \text{ such that } s \in S, t \in T\}.$$

**Definition 1.1.4:** A *relation* on a set  $S$  is a subset  $R$  of the product  $S \times S$ . If  $(a, b) \in R$ , we write  $aRb$ .

**Definition 1.1.5:** An *equivalence relation* on a set  $S$  is any relation  $\sim$  satisfying the following properties

- a. *reflexivity*:  $\forall a \in S. a \sim a$
- b. *symmetry*:  $\forall a \in S. \forall b \in S. a \sim b \iff b \sim a$
- c. *transitivity*:  $\forall a \in S. \forall b \in S. \forall c \in S. a \sim b, b \sim c \implies a \sim c.$

**Definition 1.1.6:** A *partition* of  $S$  is a family of *disjoint* nonempty subsets of  $S$  whose union is  $S$ .

**Definition 1.1.7:** Let  $\sim$  be an equivalence relation on  $S$ . Then for every  $a \in S$ , the *equivalence class* of  $a$  is the subset  $S$  defined by

$$[a]_{\sim} = \{b \in S \mid b \sim a\}.$$

Further, the equivalence classes form a partition  $\mathcal{P}_{\sim}$  of  $S$ .

**Lemma 1.1.1:** Every partition of  $S$  corresponds to an equivalence relation.

**Definition 1.1.8:** The *quotient* of the set  $S$  with respect to the equivalence relation  $\sim$  is the set

$$S \sim = \mathcal{P}_{\sim}$$

of equivalence classes of elements of  $S$  with respect to  $\sim$ .

## 1.2. Functions

**Definition 1.2.1:** The graph of  $f$  is the set

$$\Gamma_f = \{(a, b) \in A \times B \mid b = f(a)\} \subseteq A \times B.$$

Officially, a function is its graph together with information of the source  $A$  and the target  $B$  of  $f$ .

**Definition 1.2.2:** A function is a relation  $\Gamma \subseteq A \times B$  such that  $\forall a \in A, \exists! b \in B$  with  $(a, b) \in \Gamma$ . To denote  $f$  is a function from  $A$  to  $B$  we write  $f : A \rightarrow B$ .

**Definition 1.2.3:** The collection of all functions from a set  $A$  to a set  $B$  is denoted  $B^A$ .

*Example:* Every set  $A$  comes equipped with the *identity function*,  $\text{id}_A : A \rightarrow A$ , whose graph is the diagonal in  $A \times A$ . It is defined by  $\forall a \in A. \text{id}_A(a) = a$ .

**Definition 1.2.4:** If  $S \subseteq A$ , for  $f : A \rightarrow B$ , we define  $f(S) \subseteq B$  as

$$f(S) = \{b \in B \mid \exists a \in S. b = f(a)\}$$

**Definition 1.2.5:** The *restriction* of  $f : A \rightarrow B$  to  $S \subseteq A$ , denoted  $f|_S$  is the function  $S \rightarrow B$  defined by

$$\forall s \in S. f|_S(s) = f(s).$$

*Remark:* The restriction can be equivalently described as  $f \circ i$  where  $i : S \rightarrow A$  is the inclusion. Further,  $f(S) = \text{im}(f|_S)$ .

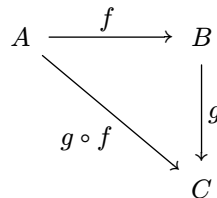
*Example (Multisets):* A *multiset* is like a set but allows for multiple instances of each element. A multiset may be defined by giving a function  $m : A \rightarrow \mathbb{N}^*$ , where  $\mathbb{N}^*$  is the set of positive integers. The corresponding multiset consists of the elements  $a \in A$ , each taken  $m(a)$  times.

*Example (Indexed Sets):* One may think of an *indexed set*  $\{a_i\}_{i \in I}$  as set whose elements are denoted by  $a_i$  for  $i$  ranging over some ‘set of indices’  $I$ . Instead, it is more proper to think of an indexed set as a function  $a : I \rightarrow A$ , with the understanding that  $a_i$  is a shorthand for  $a(i)$ . One benefit is that this allows us to consider  $a_0, a_1$  as distinct elements of  $\{a_i\}_{i \in \mathbb{N}}$  even if  $a_0 = a_1$  as elements of  $A$ .

**Definition 1.2.6:** If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are functions then so is the operation  $g \circ f$  defined by

$$\forall a \in A. (g \circ f)(a) = g(f(a)).$$

Pictorially, the following diagram *commutes*

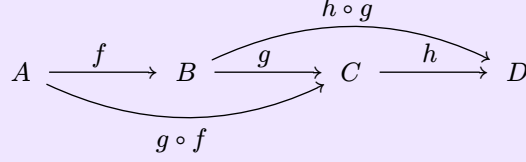


*Remark:* A diagram *commutes* when the result of following a path of arrows from any point of the diagram to any other point only depends on the starting and ending points and not on the particular path chosen.

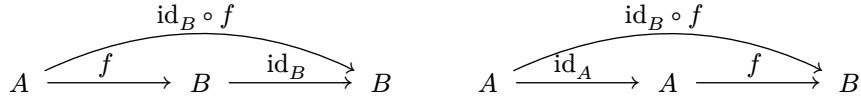
**Lemma 1.2.1:** Composition of functions is associative. That is to say, if  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$  then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Graphically, the following diagram commutes



*Example:* If  $f : A \rightarrow B$  then  $\text{id}_B \circ f = f$  and  $f \circ \text{id}_A = f$ . Graphically, the following diagrams commute

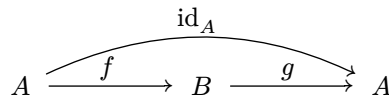


**Definition 1.2.7:** A function  $f : A \rightarrow B$  is *injective* if  $\forall a', a'' \in A, a' \neq a'' \Rightarrow f(a') \neq f(a'')$ .

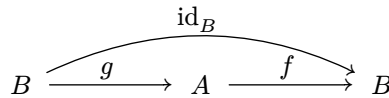
**Definition 1.2.8:** A function  $f : A \rightarrow B$  is *surjective* if  $\forall b \in B, \exists a \in A$  such that  $b = f(a)$ . That is to say  $f$  covers  $B$  and equivalently,  $\text{im}(f) = B$ .

**Definition 1.2.9:** If  $f : A \rightarrow B$  is both injective and surjective then it is a *bijection*. Then we often write  $f : A \xrightarrow{\sim} B$ . We also say that  $A$  and  $B$  are *isomorphic* and denote this by  $A \cong B$ .

**Definition 1.2.10:** A function  $g : B \rightarrow A$  is a *left inverse* of  $f : A \rightarrow B$  if  $g \circ f = \text{id}_A$ . Graphically, the following diagram commutes



**Definition 1.2.11:** A function  $f : A \rightarrow B$  is a *right inverse* of  $g : B \rightarrow A$  if  $f \circ g = \text{id}_B$ . Graphically, the following diagram commutes



**Definition 1.2.12:** We call  $g : B \rightarrow A$  an *inverse* of  $f : A \rightarrow B$  if  $g$  is both a left and right inverse of  $f$ . Then  $g$  may also be denoted  $f^{-1}$ .

**Proposition 1.2.1:** Assume  $A \neq \emptyset$  and let  $f : A \rightarrow B$  be a function. Then

- a.  $f$  has a left inverse iff it is injective.
- b.  $f$  has a right inverse iff it is surjective.

**Corollary 1.2.1.1:** A function  $f : A \rightarrow B$  is a bijection if and only if it has a (two-sided) inverse.

*Remark:* An injective but not surjective function has no right inverse. If the source has more than two elements, there will be more than one left inverse.

*Remark:* A surjective function but not injective function will have multiple inverses. These are called *sections*.

**Definition 1.2.13:** Let  $f : A \rightarrow B$  be any function and  $S \subseteq B$  be a subset. Then  $f^{-1}(S)$  is defined by

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}.$$

If  $S = \{q\}$  is a singleton then  $f^{-1}(T) = f^{-1}(q)$  is denoted the *fiber* of  $f$  over  $q$ .

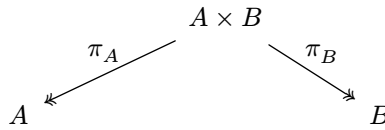
*Remark:* In this language:  $f$  is a bijection iff it has nonempty fiber over all elements of  $B$  and every fiber is a singleton.

**Definition 1.2.14:** A function  $f : A \rightarrow B$  is a *monomorphism* (or *monic*) if for all sets  $Z$  and all functions  $\alpha', \alpha'' : Z \rightarrow A$

$$f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''.$$

**Proposition 1.2.2:** A function is injective iff it is a monomorphism.

*Example (Projection):* Let  $A, B$  be sets. Then there are *natural projections*  $\pi_A, \pi_B$

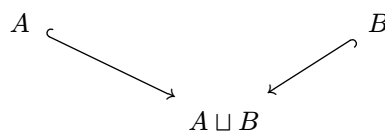


defined by

$$\forall (a, b) \in A \times B. \quad \pi_A((a, b)) = a, \quad \pi_B((a, b)) = b.$$

These maps are clearly surjective.

*Example (Direct Sum Injection):* There are natural injections from  $A, B$  to their disjoint union  $A \sqcup B$



obtained by sending  $a \in A$  (resp.  $b \in B$ ) to the corresponding element in the isomorphic copy  $A'$  of  $A$  (resp.  $B'$  of  $B$ ) in  $A \sqcup B$ .

*Example (Equivalence Relation Projection):* Let  $\sim$  be an equivalence relation on  $A$ . Then there is a surjective canonical projection

$$A \longrightarrow A / \sim$$

obtained by sending every  $a \in A$  to its equivalence class  $[a]_{\sim} \in A / \sim$ .

**Lemma 1.2.2:** Every function  $f : A \rightarrow B$  defines an equivalence relation  $\sim$  on  $A$  as follows: for every  $a', a'' \in A$ ,

$$a' \sim a'' \iff f(a') = f(a'').$$

**Proposition 1.2.3** (Canonical Decomposition): Let  $f : A \rightarrow B$  be any function and define  $\sim$  as above. Then  $f$  decomposes as follows:

$$A \xrightarrow{\quad} A / \sim \xrightarrow[\tilde{f}]{\quad} \text{im}(f) \hookrightarrow B$$

$f$

The first function is the canonical projection  $A \rightarrow A / \sim$ . The third function is the inclusion  $\text{im } f \subseteq B$ . The bijection  $\tilde{f}$  in the middle is defined by

$$\tilde{f}([a]_{\sim}) = f(a)$$

for all  $a \in A$ .

### 1.3. Categories

**Definition 1.3.1:** A category  $\mathcal{C}$  consists of

- a class  $\text{Obj}(\mathcal{C})$  of *objects* the category.
- for every two objects  $A, B$  of  $\mathcal{C}$ , a set  $\text{Hom}_{\mathcal{C}}(A, B)$  of *morphisms* with the following properties
  - for every object  $A$  of  $\mathcal{C}$ , there exists (at least) one morphism  $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$ , the ‘identity’ on  $A$ .
  - two morphisms  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$  determine a morphism  $fg \in \text{Hom}_{\mathcal{C}}(A, C)$ . That is for every triple of objects  $A, B, C$  of  $\mathcal{C}$  there is a function (of sets)

$$\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

and the image of the pair  $(f, g)$  is denoted  $fg$ .

- this composition law is associative: if  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ ,  $g \in \text{Hom}_{\mathcal{C}}(B, C)$  and  $h \in \text{Hom}_{\mathcal{C}}(C, D)$  then  $(hg)f = h(gf)$ .
- the identity morphisms are identities with respect to composition: that is for all  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  we have

$$f \cdot 1_A = f, \quad 1_B \cdot f = f.$$

- the sets  $\text{Hom}_{\mathcal{C}}(A, B)$  and  $\text{Hom}_{\mathcal{C}}(C, D)$  are disjoint unless  $A = C$  and  $B = D$ .

**Definition 1.3.2:** A morphism of an object  $A$  of a category  $C$  to itself is called an *endomorphism*. Furthermore,  $\text{Hom}_C(A, A)$  is also denoted  $\text{End}_{C(A)}$ .

**Definition 1.3.3:** A *diagram commutes* if all ways to traverse it lead to the same results of composing morphisms along the way.

*Example (Set):* By  $\text{Set}$  we denote the category of sets, where

- $\text{Obj}(\text{Set}) =$  the class of all sets;
- for  $A, B$  in  $\text{Obj}(\text{Set})$ ,  $\text{Hom}_{\text{Set}}(A, B) = B^A$ .

Composition of morphisms is defined to be the same as the composition of set-functions.

*Example (Relations):* Suppose  $\sim$  is a reflexive and transitive relation on some set  $S$ . Then, we can encode this data into a category.

- *objects:* the elements of  $S$ ;
- *morphisms:* if  $a, b$  are objects, then let  $\text{Hom}(a, b)$  be the set consisting of the element  $(a, b) \in S \times S$  if  $a \sim b$  and let  $\text{Hom}(a, b) = \emptyset$  otherwise.

For composition, let  $a, b, c$  be objects and  $f \in \text{Hom}(a, b)$  and  $g \in \text{Hom}(b, c)$ . Then,  $gf \in \text{Hom}(a, c)$  is defined to be

$$gf = (a, c).$$

This is a *small category*.

*Example (Partial Ordering of Sets):* Let  $S$  be a set. Define another (*small*) category  $\hat{S}$  by

- $\text{Obj}(\hat{S}) = \mathcal{P}(S)$ ;
- for  $A, B$  objects of  $\hat{S}$ , let  $\text{Hom}_{\hat{S}}(A, B)$  be the pair  $(A, B)$  if  $A \subseteq B$  and let  $\text{Hom}_{\hat{S}}(A, B) = \emptyset$  otherwise.

For composition, let  $A, B, C$  be objects and  $f \in \text{Hom}_{\hat{S}}(A, B)$  and  $g \in \text{Hom}_{\hat{S}}(B, C)$ . Then,  $gf \in \text{Hom}_{\hat{S}}(A, C)$  is defined to be

$$gf = (A, C).$$

*Example (Slice Category):* Let  $C$  be a category and let  $A$  be an object of  $C$ . We will define a category  $C_A$  whose objects are certain *morphisms* in  $C$  and whose morphisms are certain *diagrams* of  $C$ .

- $\text{Obj}(C_A) =$  all morphisms from any object of  $C$  to  $A$ ; that is, an object of  $C$  is an element  $f \in \text{Hom}_C(Z, A)$  for some object  $Z$  of  $C$ . Pictorially, an object of  $C_A$  is an arrow  $Z \xrightarrow{f} A$  in  $C$ ;

$$\begin{array}{c} Z \\ \downarrow f \\ A \end{array}$$

- for objects  $f_1, f_2$  of  $C_A$ , that is two arrows

$$\begin{array}{ccc} Z_1 & & Z_2 \\ \downarrow f_1 & & \downarrow f_2 \\ A & & A \end{array}$$

in  $C$ . Morphisms  $f_1 \rightarrow f_2$  are defined to be *commutative diagrams*

$$\begin{array}{ccc} Z_1 & \xrightarrow{\sigma} & Z_2 \\ & \searrow f_1 & \swarrow f_2 \\ & A & \end{array}$$

in the *ambient* category  $C$ . Alternatively, morphisms  $f_1 \rightarrow f_2$  corresponds to those morphisms  $\sigma : Z_1 \rightarrow Z_2$  in  $C$  such that  $f_1 = f_2 \sigma$ .

Categories constructed in these manners are known as *slice categories*, which are particular cases of *comma categories*.

*Example (Concrete Slice Category)*: Suppose  $C$  is the category with  $S = \mathbb{Z}$  and using the relation  $\leq$ . Choose an object  $A = 3$  of  $C$ . Then the objects of  $C_A$  are morphisms in  $C$  with target 3, that is, pairs  $(n, 3) \in \mathbb{Z} \times \mathbb{Z}$  with  $n \leq 3$ . There is a morphism

$$(m, 3) \longrightarrow (n, 3)$$

if and only if  $m \leq n$ . In this example,  $C_A$  may be harmlessly identified with the *subcategory* of integers  $\leq 3$  with the *same* morphisms as in  $C$ .

*Example (Co-Slice Category)*: We can consider a construction similar to slice categories but one where we take objects to be morphisms in a category  $C$  *from* a fixed object  $A$  to all objects in  $C$ . Morphisms are again defined to be suitable commutative diagrams. This construction is known as *the coslice category*.

*Example (Concrete Co-Slice Category)*: Let  $C = \mathbf{Set}$  and  $A =$  fixed singleton  $= \{\star\}$ . Call the constructed co-slice category  $\mathbf{Set}^*$ .

An object in  $\mathbf{Set}^*$  is then a morphism  $f : \{\star\} \rightarrow S$  in  $\mathbf{Set}$  where  $S$  is any set. The information of an object in  $\mathbf{Set}^*$  consists of a nonempty set  $S$  and an element  $s \in S$  – that is, the element  $f(\star)$ . This element determines and is determined by,  $f$ . So, we can denote objects of  $\mathbf{Set}^*$  as pairs  $(S, s)$  where  $S$  is any set and  $s \in S$  is any element of  $S$ .

A morphism between two such objects,  $(S, s) \rightarrow (T, t)$  corresponds to a set function  $\sigma : S \rightarrow T$  *such that*  $\sigma(s) = t$ .

Objects of  $\mathbf{Set}^*$  are called *pointed sets*.

*Example ( $C_{A,B}$ )*: Start from a category  $C$  and two objects  $A, B$  of  $C$ . We then define a new category  $C_{A,B}$  by a similar procedure with which we defined  $C_A$ .

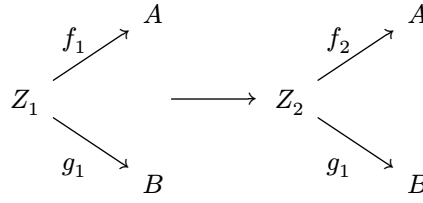
- $\text{Obj}(C_{A,B}) =$  diagrams

$$\begin{array}{ccc} & & A \\ & \nearrow f & \\ Z & & \\ & \searrow g & \\ & & B \end{array}$$

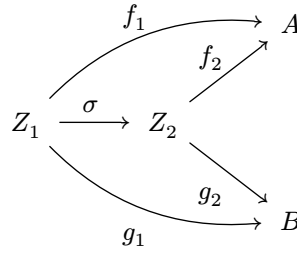
in  $C$ ;



- morphisms



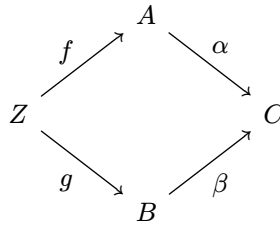
are commutative diagrams



Alternatively, morphisms in  $C_{A,B}$  corresponds to those morphisms  $\sigma : Z_1 \rightarrow Z_2$  in  $C$  such that  $f_1 = f_2 \sigma$  and  $g_1 = g_2 \sigma$ .

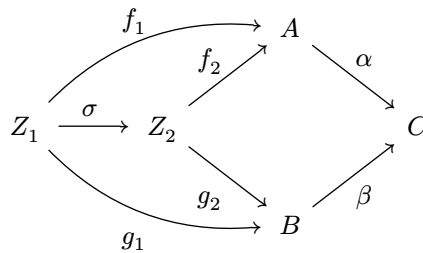
*Example* (Fibered  $C_{A,B}$ ): Start with a given category  $C$  and choose two fixed morphisms  $\alpha : A \rightarrow C, \beta : B \rightarrow C$  in  $C$  with the same target  $C$ . We can then consider a category  $C_{\alpha,\beta}$  as follows

- $\text{Obj}(C_{\alpha,\beta}) = \text{commutative diagrams}$



in  $C$ ;

- morphisms correspond to commutative diagrams



## 1.4. Morphisms

Throughout this section let  $C$  be a category.

**Definition 1.4.1** (Isomorphism): A morphism  $f \in \text{Hom}_C(A, B)$  is an *isomorphism* if it has a (two-sided) inverse: that is  $\exists g \in \text{Hom}_C(B, A)$  such that

$$gf = 1_A \quad fg = 1_B.$$

**Proposition 1.4.1:** The inverse of an isomorphism is unique.

*Remark:* Due to uniqueness, we may unambiguously refer to the inverse of an isomorphism  $f$  as  $f^{-1}$ .

**Proposition 1.4.2:**

- a. Each identity  $1_A$  is an isomorphism and is its own inverse,
- b. If  $f$  is an isomorphism, the  $f^{-1}$  is an isomorphism and further  $(f^{-1})^{-1} = f$ ,
- c. If  $f \in \text{Hom}_C(A, B)$  and  $g \in \text{Hom}_C(B, C)$  are isomorphisms, then the composition  $gf$  is an isomorphism and  $(gf)^{-1} = f^{-1}g^{-1}$ .

**Definition 1.4.2:** Two objects  $A, B$  of a category  $C$  are *isomorphic* if there is an isomorphism  $f : A \rightarrow B$ .

**Corollary 1.4.2.1:** Isomorphism is an equivalence relation on the objects of a category.

*Example:* Isomorphisms in **Set** are precisely bijective functions.

*Example:* In the category  $C$  obtained from the relation  $\leq$  on  $\mathbb{Z}$  there is a morphism  $a \rightarrow b$  and  $b \rightarrow a$  only if  $a \leq b$  and  $b \leq a$ —that is, if  $a = b$ . So an isomorphism must act from an object to itself and in  $C$  there is only one such object  $1_a$ .

*Example:* There are categories in which every morphism is an isomorphism. These are known as *groupoids*.

**Definition 1.4.3:** An *automorphism* of an object  $A$  of a category  $C$  is an isomorphism from  $A$  to itself. The set of automorphisms is denoted  $\text{Aut}_C(A)$  and is a subset of  $\text{End}_C(A)$ .

*Remark:* Equipped with composition,  $\text{Aut}_C(A)$  is a group!

**Definition 1.4.4:** Let  $C$  be a category. A morphism  $f \in \text{Hom}_C(A, B)$  is a *monomorphism* if for all objects  $Z$  of  $C$  and all morphisms  $\alpha', \alpha'' \in \text{Hom}_C(Z, A)$ ,

$$f \circ \alpha' = f \circ \alpha'' \implies \alpha' = \alpha''.$$

**Definition 1.4.5:** Let  $C$  be a category. A morphism  $f \in \text{Hom}_C(A, B)$  is an *epimorphism* if the following holds if for all objects  $Z$  of  $C$  and all morphisms  $\beta', \beta'' \in \text{Hom}_C(B, Z)$ ,

$$\beta' \circ f = \beta'' \circ f \implies \beta' = \beta''.$$

*Example:* In  $C$ , injective functions are *monomorphisms* whereas surjective functions are *epimorphisms*.

*Example:* In the category  $C$  obtained from the relation  $\leq$  on  $\mathbb{Z}$ , every morphism is both a monomorphism and an epimorphism. However, there is at most one isomorphism between any two pair of objects in  $C$ .

*Remark:* The previous example shows how the property of *Set*, wherein a function is an isomorphism iff it is a monomorphism and epimorphism, doesn't generalize to all categories. It can be shown that this is true in *abelian categories* (but *Set* isn't an example of one!)

*Remark:* The property of *Set* that a function is an epimorphism iff it has a right inverse doesn't generalize to all categories either.