Summer

Cauchy-Schwarz Masterclass

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1. Starting with Cauchy

1.1. The 1-Trick and the Splitting Trick

By the Cauchy-Schwarz inequality, with $b_1 = \cdots = b_n = 1$, we get

$$a_1 + a_2 + \cdots + a_n \leq \sqrt{n}(a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}}.$$

For the second inequality, consider $x_k = a_k^{\frac{1}{3}}$ and $y_k = a_k^{\frac{2}{3}}$. Then, by Cauchy-Schwarz we have

$$\begin{aligned} \sum_{k=1}^{n} x_k y_k &\leq \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} y_k^2\right)^{\frac{1}{2}} \\ \Longrightarrow \sum_{k=1}^{n} a_k &\leq \left(\sum_{k=1}^{n} a_k^{\frac{2}{3}}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} a_k^{\frac{4}{3}}\right)^{\frac{1}{2}} \end{aligned}$$

1.2. Product of Averages and Averages of Products

By the Cauchy-Schwarz inequality and using the fact that $1 \leq a_k b_k$ implies $1 \leq \sqrt{a_k b_k}$, we have

$$\left(\sum_{k=1}^{n} p_k a_k\right)\left(\sum_{k=1}^{n} p_k b_k\right) \geq \left(\sum_{k=1}^{n} p_k a_k^{\frac{1}{2}} b_k^{\frac{1}{2}}\right)^2 \geq \left(\sum_{k=1}^{n} p_k\right)^2 = 1.$$

1.3. Why Not Three or More?

By applying the Cauchy-Schwarz inequality twice,

$$\begin{aligned} \sum_{k=1}^{n} a_k b_k c_k &\leq \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} b_k^2 c_k^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} b_k^4\right)^{\frac{1}{4}} \left(\sum_{k=1}^{n} c_k^4\right)^{\frac{1}{4}}. \end{aligned}$$

Thus,

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leq \left(\sum_{k=1}^{n} a_k^2\right)^2 \sum_{k=1}^{n} b_k^4 \sum_{k=1}^{n} c_k^4.$$

For the next inequality, we start with the observation that for any $k \in [n]$,

$$c_k \leq \left(\sum_{k=1}^{n} c_k^2\right)^{\frac{1}{2}}.$$

Using this and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{k=1}^{n} a_k b_k c_k &\leq \sum_{k=1}^{n} a_k b_k \left(\sum_{k=1}^{n} c_k^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} c_k^2\right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2.$$

1.4. Some Help From Symmetry

a. This pretty much follows from Problem 1.1. Consider,

$$\left(\frac{x+y}{x+y+z}\right)^{\frac{1}{2}} + \left(\frac{x+z}{x+y+z}\right)^{\frac{1}{2}} + \left(\frac{y+z}{x+y+z}\right)^{\frac{1}{2}} \leq 3^{\frac{1}{2}}\left(\frac{2(x+y+z)}{x+y+z}\right)^{\frac{1}{2}} = 6^{\frac{1}{2}}.$$

a. Again, we use the Cauchy-Schwarz inequality,

$$\begin{aligned}(x+y+z)^2 &= \left(\frac{x}{\sqrt{y+z}}\sqrt{y+z} + \frac{y}{\sqrt{x+z}}\sqrt{x+z} + \frac{z}{\sqrt{x+y}}(\sqrt{x+y})\right)^2 \\ &\leq 2(x+y+z)\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right) \\ \Longrightarrow x+y+z &\leq 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right).\end{aligned}$$

1.5. A Crystallographic Inequality with a Message

Consider, using the Cauchy-Schwarz inequality,

$$\begin{aligned}g^2(x) &= \left(\sum_{k=1}^{n} p_k^{\frac{1}{2}} p_k^{\frac{1}{2}} \cos(\beta_k x)\right) \\ &\leq \sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k \cos^2(\beta_k x) \\ &= \sum_{k=1} \frac{p_k}{2}(1+\cos(2\beta_k x)) \\ &\leq \frac{1}{2}\left(\sum_{k=1}^{n} p_k + \sum_{k=1}^{n} p_k \cos(2\beta_k x)\right) \\ &= \frac{1}{2}(1+g(2x)).\end{aligned}$$

1.6. A Sum of Inversion Preserving Summands

Momentarily, we shift focus on bounding $\sum_{k=1}^{n} \frac{1}{p_k}$ from below. This is also a consequence of Cauchy-Schwarz,

$$\sum_{k=1}^{n} \frac{1}{p_k} = \sum_{k=1}^{n} p_k \sum_{k=1}^{n} \frac{1}{p_k} \geq \left(\sum_{k=1}^{n} 1\right)^2 = n^2 \tag{1}$$

The equality occurs iff $\sqrt{p_k} = \frac{\alpha}{\sqrt{p_k}}$. That is, when all p_k are equal. This then implies that equality occurs iff $p_k = \frac{1}{n}$.

Then, by the Cauchy-Schwarz inequality and the above result,

$$\begin{aligned}\sum_{k=1}^{n}\left(p_k + \frac{1}{p_k}\right)^2 &\geq \frac{1}{n}\left(\sum_{k=1}^{n} p_k + \sum_{k=1}^{n} \frac{1}{p_k}\right)^2 = \frac{1}{n}\left(1 + \sum_{k=1}^{n} \frac{1}{p_k}\right)^2 \\ &\geq \frac{1}{n}(1+n^2)^2 \geq n^3 + 2n + \frac{1}{n}.\end{aligned} \tag{2}$$

It is easy to check that $p_k = \frac{1}{n}$ for all k is a sufficient condition for equality in (2). For it to be necessary, we just note that $p_k = \frac{1}{n}$ is a necessary condtion for equality in the second application of Cauchy-Schwarz in (2).

1.7. Flexibility of Form

Consider the inner product on $\mathbb{R}^2$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 5x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$$

It is an inner product since,

a. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ for non-zero $\boldsymbol{x} \in \mathbb{R}^2$

   Consider, if one of the coordinates of $\boldsymbol{x}$ is nonzero then,

$$5x_1^2 + 2x_1x_2 + 3x_2^2 = 4x_1^2 + 2x_2^2 + (x_1 + x_2)^2 > 0.$$

b. $\langle \alpha\boldsymbol{x} + \boldsymbol{v}, \boldsymbol{y} \rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{v}, \boldsymbol{y} \rangle$ for all $\alpha \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v} \in \mathbb{R}^2$

   Consider the following,

$$\begin{aligned}\langle \alpha\boldsymbol{x} + \boldsymbol{v}, \boldsymbol{y} \rangle &= 5(\alpha x_1 + v_1)y_1 + (\alpha x_1 + v_1)y_2 + (\alpha x_2 + v_2)y_1 + 3(\alpha x_2 + v_2)y_2 \\ &= \alpha(5x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2) + (5v_1y_1 + v_1y_2 + v_2y_1 + 3v_2y_2) \\ &= \alpha\langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{v}, \boldsymbol{y} \rangle.\end{aligned}$$

c. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$

   This follows from a routine computation,

$$\begin{aligned}\langle \boldsymbol{x}, \boldsymbol{y} \rangle &= 5x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2 \\ &= 5y_1x_1 + y_2x_1 + y_1x_2 + 3y_2x_2 \\ &= \langle \boldsymbol{y}, \boldsymbol{x} \rangle.\end{aligned}$$

Then by Cauchy-Schwarz on $\boldsymbol{x} = (x, y)$ and $\boldsymbol{\alpha} = (\alpha, \beta)$, we have

$$5x\alpha + x\beta + y\alpha + 3y\beta \le (5x^2 + 2xy + 3y^2)^{\frac{1}{2}}(5\alpha^2 + 2\alpha\beta + 3\beta^2)^{\frac{1}{2}}.$$

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