Algebra **Summer**

Algebra Problems

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1. Preliminaries

1.1. Naive Set Theory

Problem 1.2

First, note that for every $a \in S$, $a \in [a]_\sim$. Since every equivalence class is a subset of S, it follows that $S = \bigcup_{C \in \mathcal{P}_\sim} C$.

Take two partitions $[a]_\sim$ and $[b]_\sim$. If they are disjoint, we are done. Suppose they aren't. Then $c \in [a]_\sim$ sec $[b]_\sim$. By transitivity and symmetry, for any $x \in [b]_\sim$, $x \sim b \sim c \sim a$. Thus, $[b]_\sim \subseteq [a]. \sim$. By symmetry, $[b]_\sim = [a]_\sim$. Thus, distinct equivalence classes are disjoint.

This concludes the proof that equivalence classes form a partition of S.

Problem 1.3

Let $\mathcal{P}$ be a partition on S. Furthermore, for any $a \in S$, define $\mathcal{P}_a$ to be the unique set in the partition containing a. Then we can the equivalence relation $\underset{\mathcal{P}}{\sim}$ by $a \underset{\mathcal{P}}{\sim} b$ iff $b \in \mathcal{P}_a$.

This is reflexive because, trivially, $a \in \mathcal{P}_a$.

This is also symmetric. Note that as partitions are a collection of disjoint sets, $\mathcal{P}_a \cap \mathcal{P}_b \neq \emptyset$ implies that $\mathcal{P}_a = \mathcal{P}_b$. Thus, $a \in \mathcal{P}_a = \mathcal{P}_b$.

Finally, this is also transitive due to the transitivity and symmetry of set equality. Particularly, note that, as in the previous part, $\mathcal{P}_a = \mathcal{P}_b$ and $\mathcal{P}_b = \mathcal{P}_c$. Thus, $c \in \mathcal{P}_c = \mathcal{P}_a$.

Problem 1.6

We first show that $\sim$ is an equivalence relation. It is reflexive because for any $a \in \mathbb{R}$, $a - a = 0 \in \mathbb{Z}$. It is symmetric because $\mathbb{Z}$ is closed under multiplication i.e. $z \in \mathbb{Z} \implies -z \in \mathbb{Z}$. It is transitive because $\mathbb{Z}$ is closed under addition, particularly for $a, b, c \in \mathbb{R}$, if $a \sim b, b \sim c$ then $c - a = (c - b) + (b - a) \in \mathbb{Z}$.

We claim that $\mathbb{R}/\sim \cong [0, 1)$. Note that any $x, y \in [0, 1)$ are such that $x \nsim y$ as $x - y \leq x < 1$. Thus, each element of $[0, 1)$ corresponds to a distinct equivalence class. Next, given any $z \in \mathbb{R}$, we claim that there exists $x \in [0, 1)$ such that $z \in [x]_\sim$. Particularly, take $x = z - \lfloor z \rfloor$. Then $z - x \in \mathbb{Z}$ and $0 \leq z - \lfloor z \rfloor < 1$.

Note that $\approx$ is an equivalence relation for pretty much the same reasons as $\sim$ above. Furthermore, $\mathbb{R} \times \mathbb{R}/\approx \cong [0, 1) \times [0, 1)$ by a similar reasoning to above.

1.2. Functions between sets

Problem 2.1

There are $n!$ many bijections between a set S and itself, with $|S| = n$. We prove this by induction. Let $P(n)$ be the statement that there are $n!$ bijections from S to S' with $|S| = |S'| = n$.

$P(1)$ is clearly true as the only bijection (in fact, the only function) from a singleton $S = \{s\}$ to another singleton $S' = \{s'\}$ is the function $f : S \longrightarrow S'$ defined by $f(s) = s'$.

Assuming $P(k)$ we wish to show that there $P(k+1)$ holds too. Suppose $S = \{s_1, ..., s_k, s_{k+1}\}$. We can categorize bijections by where they map the first element s_1. There are n different categories as $|S'| = n$. We then claim that every category has $(n-1)!$ elements. Consider some category that is defined by the fact that it maps $s_1 \mapsto s_{i'}$ for some $s_{i'} \in S'$. Hence, every function f in this category, restricts to a bijection $f|_{S - s_1} : S - s_1 \longrightarrow S' - s_{i'}$. By the induction hypothesis, there are precisely $(n-1)!$ choices for these restrictions. So every category, has $(n-1)!$ functions and in total, there are $n(n-1)! = n!$ many bijections from S to S'. So, $P(k+1)$ holds.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Problem 2.2: *f has a right inverse if and only if it is surjective.*

We deal with the forward direction first. Suppose $f : A \longrightarrow B$ has a right inverse $g : B \longrightarrow A$. Then for every $b \in B$, $g(b) \in A$ is such that $f(g(b)) = b$. Thus, every element of B is the image, under f, of at least one element of A i.e. f is surjective.

Now, consider the backward direction. If f is surjective then for every $b \in B$, the fiber $f^{-1}(b)$ is non-empty. So, define the function $g : B \longrightarrow A$ as follows: for every b, pick an element of the fiber $f^{-1}(b)$ and assign it to $g(b)$. Then, note that for every $b \in B$, $g(b) \in f^{-1}(b)$ implies that $(f \circ g)(b) = b$. Thus, $f \circ g = \mathrm{id}_B$ and g is the right inverse of f.

Problem 2.4

This follows from showing:

a. $\cong$ is reflexive.
   For any set A, $A \cong A$ as $\mathrm{id}_A : A \longrightarrow A$ is a bijection.
b. $\cong$ is symmetric.
   Take any sets A, B such that $A \cong B$. Then there exists a bijection $f : A \longrightarrow B$. Consequently, $f^{-1} : B \longrightarrow A$ is a bijection from B to A and $B \cong A$.
c. $\cong$ is transitive.
   Take any sets A, B, C such that $A \cong B$ and $B \cong C$. Then, there exists bijections $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Consequently, $h = g \circ f : A \longrightarrow C$ is also a bijection since it has inverse $f^{-1} \circ g^{-1}$

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ f = \mathrm{id}_A, \quad (g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ g^{-1} = \mathrm{id}_C.$$

   Thus, $A \cong C$.

Problem 2.5

Definition 1.2.1 . A function $f : A \longrightarrow B$ is an *epimorphism* for all sets Z and all function $\alpha', \alpha'' : B \longrightarrow Z$

$$\alpha' \circ f = \alpha'' \circ f \Longrightarrow \alpha' = \alpha''.$$

Proposition 1.2.1: A function is surjective iff it is an epimorphism

Proof: We start with the forward direction. Suppose $f : A \longrightarrow B$ is surjective. Then we know that f has a right inverse g. So,

$$\begin{aligned} &\alpha' \circ f = \alpha'' \circ f \\ \Longrightarrow\ &\alpha' \circ f \circ g = \alpha'' \circ f \circ g \\ \Longrightarrow\ &\alpha' = \alpha''. \end{aligned}$$

Now, we consider, the backward direction. Suppose $f : A \longrightarrow B$ is an epimorphism. Pick $b \in B$ and define $\alpha'_b : B \longrightarrow \{0, 1\}, \alpha''_b : B \longrightarrow \{0, 1\}$ to be $\alpha'_b = \mathbb{1}_B, \alpha''_b = \mathbb{1}_{B-b}$. Then, $\alpha'(b) \neq \alpha''(b)$ so $\alpha' \circ f = \alpha'' \circ f$. For every $x \in f^{-1}(B - b)$, $(\alpha' \circ f)(x) = (\alpha'' \circ f)(x) = 1$. So, if $x \in A$, $(\alpha' \circ f)(x) \neq (\alpha'' \circ f)(x)$ necessarily implies that $x \in f^{-1}(b)$. Since b was chosen arbitrarily, this proves that f is surjective.

□

Problem 2.9

Suppose $A \cong A'$ and $B \cong B'$ with $A \cap B = \emptyset, A' \cap B' = \emptyset$. Let $g_1 : A \longrightarrow A'$ and $g_2 : B \longrightarrow B'$ be isomorphisms. Consider the map $f : A \cup B \longrightarrow A' \cup B'$ defined by

$$f(x) = \begin{cases} g_1(x) \text{ if } x \in A \\ g_2(x) \text{ otherwise} \end{cases}$$

Then, f is a bijection and hence, $A \cup B \cong A' \cup B'$.

Problem 2.10

We give a combinatorial argument. Note that a function from A to B must map an element of A to one of $|B|$ many elements of B. Since this choice has to be made for each of the $|A|$ many elements of A, there are a total of $|B|^{|A|}$ functions in B^A.

Problem 2.11

Let $\mathcal{P}$ denote the power set of A. Consider the map $\mathcal{F} : 2^A \longrightarrow \mathcal{P}$ defined by

$$\mathcal{F}(f) = \{x \in A \mid f(x) = 1\}$$

First, we show $\mathcal{F}$ is injective. Suppose $f, g \in 2^A$ with $\mathcal{F}(f) = \mathcal{F}(g)$. Then for every $x \neg \in \mathcal{F}(f)$, $f(x) = g(x) = 0$ and for every $x \in \mathcal{F}(f)$, $f(x) = g(x) = 1$. Thus, $f = g$.

Next, we show that $\mathcal{F}$ is surjective. Consider any $S \subseteq 2^A$. Then $1_S \in 2^A$ and is such that $\mathcal{F}(1_S) = S$.