1910SU **Summer**

Group Discussion: Integration

Solutions

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Recall the statement of the *First Fundamental Theorem of Calculus*,

Proposition 0.1: Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all $x \in [a, b]$, by

$$F(x) = \int_a^x f(t)dt.$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b) and

$$F'(x) = f(x)$$

for all $x \in (a, b)$. That is, F is an antiderivative of f on (a, b).

1. First Fundamental Theorem of Calculus

1.1. $a = 0, f(x) = x^2$

$$\begin{aligned} A(x) &= \int_0^x t^2 dt \\ &= \left.\frac{t^3}{3}\right|_a^x \\ &= \frac{x^3}{3} \end{aligned}$$

$$\frac{d}{dx}A(x) = x^2$$

1.2. $a = 2, f(x) = \frac{1}{x}$

$$\begin{aligned} A(x) &= \int_2^x \frac{1}{x} dt \\ &= \ln x|_2^x \\ &= \ln x - \ln 2 \end{aligned}$$

$$\frac{d}{dx}A(x) = \frac{1}{x}$$

1.3. $a = \pi, f(x) = \cos x$

$$
\begin{aligned}
A(x) &= \int_{\pi}^{x} \cos x dt \\
&= \sin x|_{\pi}^{x} \\
&= \sin x - \sin \pi \\
&= \sin x
\end{aligned}
$$

2. Chain Rule

2.1. $A(x) = \int_0^x t^3 dt$

By the First Fundamental Theorem of Calculus,

$$\frac{d}{dx} A(x) = x^3.$$

2.2. $A(x^2) = \int_0^{x^2} t^3 dt$

By the Chain Rule,

$$\frac{d}{dx} A(x^2) = A'(x^2) \frac{d}{dx} x^2 = x^6 \cdot 2x = 2x^7.$$

2.3. $B(x) = \int_x^0 t^3 dt$

By the First Fundamental Theorem of Calculus,

$$
\begin{aligned}
\frac{d}{dx} B(x) &= \frac{d}{dx} \int_x^0 t^3 dt \\
&= -\frac{d}{dx} \int_0^x t^3 dt \\
&= -x^3.
\end{aligned}
$$

2.4. $B(x^2) = \int_{x^2}^0 t^3 dt$

By the Chain Rule,

$$\frac{d}{dx} B(x^2) = B'(x^2) \frac{d}{dx} x^2 = -x^6 \cdot 2x = -2x^7.$$

2.5. $\frac{d}{dx} \int_0^{x^2} f(t) dt$

Define

$$A(x) = \int_0^x f(t) dt.$$

Then, by the Chain Rule,

$$\frac{d}{dx} A(x^2) = A'(x^2) \frac{d}{dx} x^2 = 2f(x^2)x.$$

2.6. $\frac{d}{dx} \int_{p(x)}^{q(x)} f(t) dt$

This is essentially the same idea as the previous barts, albeit generalized slightly. Consider first the decomposition,

$$\begin{aligned}\int_{p(x)}^{q(x)} f(t)dt &= \int_0^{q(x)} f(t)dt + \int_{p(x)}^{0} f(t)dt \\ &= \int_0^{q(x)} f(t)dt - \int_0^{p(x)} f(t)dt\end{aligned}$$

Then,

$$\begin{aligned}\frac{d}{dx}\int_{p(x)}^{q(x)} f(t)dt &= \frac{d}{dx}\int_0^{q(x)} f(t)dt - \frac{d}{dx}\int_0^{p(x)} f(t)dt \\ &= f(q(x))q'(x) - f(p(x))p'(x).\end{aligned}$$