1910SU **Summer**

Group Discussion: Differentiation I

Solutions

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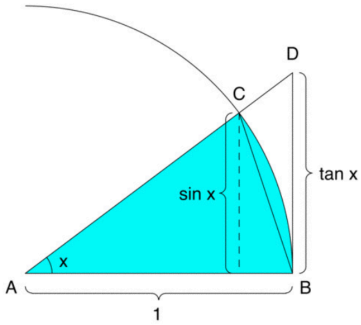
1. Trigonometric Derivatives

1.1. Prove the identity: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.



From the picture above, we can conclude

$$\triangle\, ABC \leq \widehat{ABC} \leq \triangle\, ABD.$$

Remark: Notationally, I am using $\widehat{ABC}$ to denote the area of the sector colored blue in the figure above.

Furthermore, the area of each of the shapes can be calculated as follows,

$$\triangle\, ABC = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 \cdot \sin x$$
$$\widehat{ABC} = \frac{1}{2} \cdot r^2 \cdot x = \frac{1}{2} \cdot x$$
$$\triangle\, ABD = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 \cdot \tan x$$

Substituting these values back into our initial inequality, we get the desired result.

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}.$$

From the first inequality, we obtain the desired upper bound,

$$\frac{\sin x}{2} \leq \frac{x}{2} \implies \frac{\sin x}{x} \leq \frac{2}{2} \implies \frac{\sin x}{x} \leq 1.$$

From the second inequality, we obtain the desired lower bound,

$$\frac{x}{2} \leq \frac{\tan x}{2} \implies x \leq \frac{\sin x}{\cos x} \implies \cos x \leq \frac{\sin x}{x}.$$

Thus,

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

By the Squeeze Theorem,

$$
\begin{aligned}
&\lim_{x\to 0}\cos x \le \lim_{x\to 0}\frac{\sin x}{x} \le \lim_{x\to 0} 1\\
\implies & 1 \le \lim_{x\to 0}\frac{\sin x}{x} \le 1\\
\implies & \lim_{x\to 0}\frac{\sin x}{x} = 1.
\end{aligned}
$$

□

1.2. Prove the identity $\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$.

We start by multiplying the numerator and denominator by $(\cos x + 1)$ as suggested by the hint.

$$\frac{\cos x - 1}{x}\cdot\frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

From $\sin^2 x + \cos^2 x = 1$, it follows that $\cos^2 x - 1 = -\sin^2 x$. So,

$$= -\frac{\sin^2 x}{x(\cos x + 1)}$$

The trick now is to write the expression above as a product of terms each of whose limit as $x \to 0$ is known.

$$= \frac{\sin x}{x}\cdot(-\sin(x))\cdot\frac{1}{\cos x + 1}$$

From the previous part, we know that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Furthermore, by continuity of sin, we note that $\lim_{x\to 0} -\sin x = -\sin 0 = 0$. Similarly, by continuity of $\frac{1}{\cos x+1}$ at $x = 0$, we have $\lim_{x\to 0}\frac{1}{\cos x+1} = \frac{1}{2}$. Thus, we may conclude the following.

$$
\begin{aligned}
\lim_{x\to 0}\frac{\cos x - 1}{x} &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0} -\sin(x)\right)\left(\lim_{x\to 0}\frac{1}{\cos x + 1}\right)\\
&= 1\cdot 0\cdot\frac{1}{2} = 0
\end{aligned}
$$

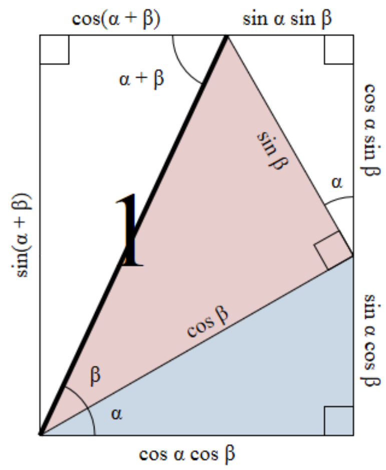
□

1.3. Prove that $\frac{d}{dx}\sin x = \cos x$ from first principles.

First, a quick note on the angle sum formula,

$$\sin(a + b) = \sin a\cos b + \cos a\sin b.$$

There are a lot of ways of proving it and its interesting to see how you go on to see nicer and nicer proofs of it as you learn more math (there are fairly neat proofs using linear algebra, the Taylor series, Euler's formula etc.) Below is one such very nice proof, perhaps even more extraordinary because of how elementary it is.



Now, lets move to the actual problem. This was also covered in the lecture notes but, for completion, the derivative of sin can be derived from first principles as follows.

$$\begin{aligned}
\lim_{h \longrightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} &= \lim_{h \longrightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
&= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right) \\
&= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1} \\
&= \cos x.
\end{aligned}$$

□

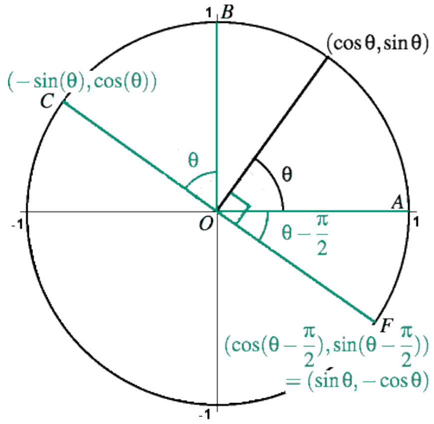
1.4. Prove $\frac{d}{dx} \cos x = -\sin x$.

$$\frac{d}{dx} \cos x \overset{(1)}{=} \frac{d}{dx} \sin\left(x + \frac{\pi}{2}\right) \overset{(2)}{=} \cos\left(x + \frac{\pi}{2}\right) \overset{(3)}{=} -\sin x.$$

Lets focus on equalities (1) and (3) first. They follow from the identities

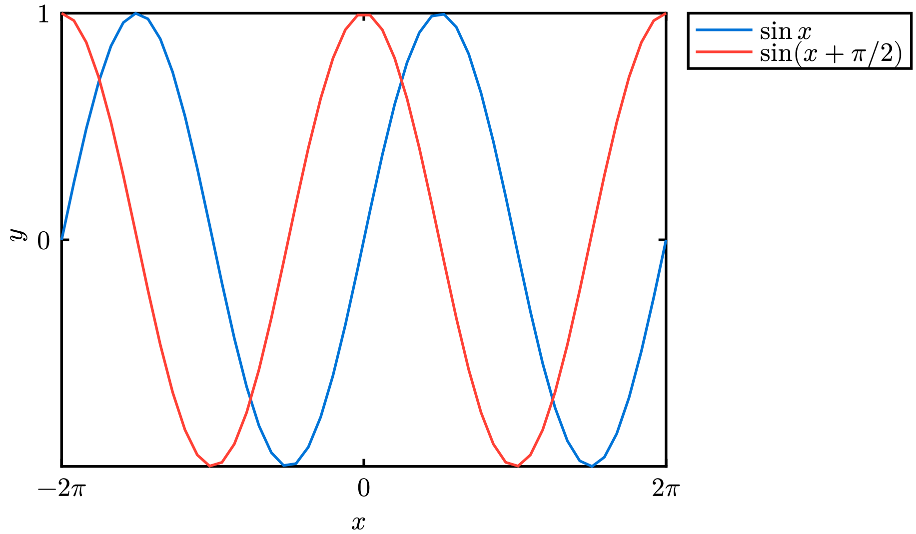
$$\cos x = \sin\left(x + \frac{\pi}{2}\right) \qquad \sin x = -\cos\left(x + \frac{\pi}{2}\right)$$

They may be verified using the angle sum formulae $\sin(a+b) = \sin a \cos b + \cos a \sin b$, $\cos(a+b) = \cos a \cos b - \sin a \sin b$. Alternatively, one may derive them directly by considering the unit circle and the following constructions.



Now, we move to justifying equality (2). Since $\sin\left(x + \frac{\pi}{2}\right)$ is a composition of the functions $x \mapsto \sin x$ and $x \mapsto x + \frac{\pi}{2}$, we have to justify this step a bit more than just citing our previous result that $\frac{d}{dx} \sin x = \cos x$. Of course, since a composition is involved, we can just use the chain rule here!

However, since we hadn't talked about the chain rule explicitly during this week, I want to highlight an alternative way of approaching (2). First note that, $\sin\left(x + \frac{\pi}{2}\right)$ is just $\sin x$ shifted to the left by $\frac{\pi}{2}$ units.



Intuitively, shifting doesn't change the shape of the graph itself. So, we may expect the derivative of $\sin\left(x + \frac{\pi}{2}\right)$ to perhaps just be the derivative of $\sin x$ shifted by the same amount i.e. $\cos\left(x + \frac{\pi}{2}\right)$. This intuition can be formalized from first principles as follows:

Assume f is a differentiable real-valued function. Let $g(x) = f(x + t)$ for some fixed $t \in \mathbb{R}$. Then,

$$\begin{aligned}\frac{d}{dx} g(x) &= \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x + t + h) - f(x + t)}{h}\end{aligned}$$

Let $y = x + t$.

$$\begin{aligned}&= \lim_{h \to 0} \frac{f(y + h) - f(y)}{h} \\ &= f'(y) = f'(x + t).\end{aligned}$$

$\square$

2. Derivative of e^x

2.1. Show that $\lim_{h \longrightarrow 0} \frac{e^h - 1}{h} = \lim_{u \longrightarrow 0} \frac{u}{\ln(u+1)}$ using the substitution $u = e^h - 1$.

Note that substituting $u = e^h - 1$ to the second expression gives us the first expression back.

$$\frac{u}{\ln(u+1)} = \frac{e^h - 1}{\ln(e^h)} = \frac{e^h - 1}{h}$$

Furthermore, note that as $h \to 0$, $u \to 0$ too. Thus,

$$\lim_{h \longrightarrow 0} \frac{e^h - 1}{h} = \lim_{u \to 0} \frac{u}{\ln(u+1)}.$$

$\square$

2.2. Equate the previous limit with $\lim_{u \longrightarrow 0} \frac{1}{\ln(u+1)^{\frac{1}{u}}}$.

Dividing the numerator and denominator by u, we get

$$\frac{u}{\ln(u+1)} = \frac{\frac{1}{u} \cdot u}{\frac{1}{u} \cdot \ln(u+1)} = \frac{1}{\ln(u+1)^{\frac{1}{u}}}$$

$\square$

2.3. Show that the previous limit is equal to 1.

Since ln is continuous, we have

$$\lim_{u \to 0} \ln(u+1)^{\frac{1}{u}} = \ln\left(\lim_{u \longrightarrow 0} (u+1)^{\frac{1}{u}}\right) = \ln e = 1.$$

Thus,

$$\lim_{u \to 0} \frac{1}{\ln(u+1)} = \frac{1}{\lim_{u \to 0} \ln(u+1)} = \frac{1}{1} = 1.$$

$\square$

2.4. Show that $\frac{d}{dx} e^x = e^x$.

Note that the previous parts have shown

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

We start from the definition of the derivative.

$$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \longrightarrow 0} \frac{e^h - 1}{h} = e^x.$$

$\square$

3. Pascal's Triangle

There are many ways to prove the required result, which is also known as *the Binomial Theorem*. I will give two proofs here, one is a *combinatorial proof* [1] and the other is *by induction*. I expect the second technique to be new to most of you– so, I wouldn't worry if not all of it makes sense in your first read through!

3.1. Combinatorial Proof

Recall that $\binom{n}{k}$ counts the number of ways of selecting k objects from n objects.

We want to know what the coefficient of $x^k h^{n-k}$ (for $k \in \{0, 1, ..., n\}$) will be in the expansion of

$$(x+h)^n = \underbrace{(x+h)(x+h)...(x+h)}_{n \text{ times}}.$$

We note that each $x^k h^{n-k}$ term arises by selecting an x term from k of the $(x+y)$ factors and a y term from all the remaining factors. This essentially amounts to selecting k items from n items. Thus, there are $\binom{n}{k}$ many $x^k h^{n-k}$ terms. Summing over all the possible values of k, we obtain the formula we desired

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^k h^{n-k}.$$

□

3.2. Proof by Induction

Let $P(n)$ denote the statement that the following equality holds

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^k h^{n-k}. \tag{1}$$

Remark: For example, $P(2)$ corresponds to the statement that

$$(x+h)^2 = x^2 + 2xh + h^2.$$

Base Case. Now, we start by showing that $P(1)$ is true. We simplify both the left and right hand side of (1) with $n = 1$ to obtain

$$\begin{aligned}
&(x+h)^1 = x + h \\
&\sum_{k=0}^{1} \binom{1}{k} x^k h^{1-k} = \binom{1}{0} x^0 h^1 + \binom{1}{1} x^1 h^0 = x + h.
\end{aligned}$$

Thus, $P(1)$ clearly holds.

Induction. Now, we show that if $P(n)$ holds for some $n \in \mathbb{N}$ then $P(n+1)$ is also true. So, assume $P(n)$ is true for some n. That is to say, the following equality is true,

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^k h^{n-k}.$$

Note then that

[1] Combinatorial proofs are often quite elegant. For an in-depth exploration, I recommend reading the book *Proofs that Really Count* by Benjamin & Quinn.

$$
\begin{aligned}
(x+h)^{n+1} &= (x+h)(x+h)^n \\
&= (x+h)\sum_{k=0}^{n}\binom{n}{k}x^k h^{n-k'} \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{k+1}h^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k h^{n-k+1}
\end{aligned}
$$

Let $k' = k+1$. Then,

$$
\begin{aligned}
&= \sum_{k'=1}^{n}\binom{n}{k-1}x^{k'}h^{n+1-k'} + \sum_{k=0}^{n}\binom{n}{k}x^k h^{n+1-k} \\
&= h^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]x^k h^{n+1-k}
\end{aligned}
$$

Then, we use the identity $\binom{n}{k-1}+\binom{n}{k}=\binom{n}{k+1}$ (this itself can be proven using a combinatorial proof or induction– I leave verifying this to you!)

$$
\begin{aligned}
&= h^{n+1} + \sum_{k=1}^{n}\binom{n}{k+1}x^k h^{n+1-k} \\
&= \sum_{k=0}^{n}\binom{n}{k+1}x^k h^{n+1-k}.
\end{aligned}
$$

Thus, $P(n+1)$ is also true.

Thus, by induction, we have shown that $P(n)$ is true for all $n \in \mathbb{N}$. (The idea is that the base case shows that $P(1)$ is true and the second step lets us show that $P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \ldots$ and so on)

Remark: By $\mathbb{N}$, I am denoting the set of *natural numbers*. These are all the positive integers.

$\square$