Analysis **Summer**

Measure, Integration and Real Analysis

Axler

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Contents

1. Riemann Integration

1.1. Review: Riemann Integral

Definition 1.1.1 (partition): Suppose $a, b \in \mathbb{R}$ with $a < b$. A *partition* of $[a, b]$ is a finite list of the form $x_0, x_1, ..., x_n$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

We use a partition $x_1, x_1, ..., x_n$ of $[a, b]$ to think of $[a, b]$ as a union of closed subintervals,

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n].$$

Definition 1.1.2 (notation for infimum and supremum of a function): If f is a real-valued function and A is a subset of the domain of f, then

$$\inf_A f = \inf\{f(x) : x \in A\} \text{ and } \sup_A f = \sup\{f(x) : x \in A\}.$$

Definition 1.1.3 (lower and upper Riemann sums): Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P is a partition $x_0, ..., x_n$ of $[a, b]$. The *lower Riemann sum* $L(f, P, [a, b])$ and the *upper Riemann sum* $U(f, P, [a, b])$ are defined by

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

and

$$U(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f.$$

Lemma 1.1.1 (inequalities with Riemann sums): Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a, b]$ such that the list are defining P is a sublist of the list defining P'. Then

$$L(f, P, [a, b]) \leq L(f, P', [a, b]) \leq U(f, P', [a, b]) \leq U(f, P, [a, b]).$$

Lemma 1.1.2 (lower Riemann sums $\leq$ upper Riemann sums): Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a, b]$. Then

$$L(f, P, [a, b]) \leq U(f, P', [a, b]).$$

Definition 1.1.4 (lower and upper Riemann integrals): Supose $f : [a, b] \to \mathbb{R}$ is a bounded function. The *lower Riemann integral* $L(f, [a, b])$ and hte *upper Riemann integral* $U(f, [a, b])$ of f are defined by

$$L(f, [a, b]) = \sup_P L(f, P, [a, b])$$

and

$$U(f, [a, b]) = \inf_P U(f, P, [a, b])$$

where the supremum and infimum above are taken over all partitions P of $[a, b]$.

Lemma 1.1.3 (lower Riemann integral $\leq$ upper Riemann integral): Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Then

$$L(f, [a, b]) \leq U(f, [a, b]).$$

Definition 1.1.5 (Riemann integrable; Riemann integral):

a. A bounded function on a closed bounded interval is called *Riemann integrable* if its lower Riemann integral equals its upper Riemann integral.
b. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then the *Riemann integral* $\int_a^b f$ is defined by

$$\int_a^b f = L(f, [a, b]) = U(f, [a, b]).$$

Proposition 1.1.1 (continuous functions are Riemann integrable): Every continuous real-valued function on each closed bounded interval is Riemann integrable.

Lemma 1.1.4 (bounds on Riemann integral): Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Then

$$(b-a) \inf_{[a,b]} f \leq \int_a^b f \leq (b-a) \sup_{[a,b]} f.$$

1.2. Riemann Integral Is Not Good Enough

There are three issues we discuss

a. Riemann integration does not handle functions with many discontinuities;
b. Riemann integration does not handle unbounded functions;
c. Riemann integration does not work well with limits.

Example (a function that is not Riemann integrable): Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

If $[a, b] \subseteq [0, 1]$ with $a < b$, then

$$\inf_{[a,b]} f = 0 \quad \text{and} \quad \sup_{[a,b]} f = 1$$

because $[a, b]$ contains an irrational number and contains a rational number. Thus, $L(f, P, [0, 1]) = 0$ and $U(f, P, [0, 1]) = 1$ for any partition P of $[0, 1]$. Since $L(f, [0, 1]) \neq U(f, [0, 1])$, we conclude that f is not Riemann integrable.

Example (Riemann integration does not work with unbounded functions): Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

If $x_0, x_1, ..., x_n$ is a partition of $[0, 1]$, then $\sup_{[x_0, x_1]} f = \infty$. Then, $U(f, P, [0, 1]) = \infty$ for every partition P of $[0, 1]$.

However, we should consider the area under the graph of f to be 2 and not ∞ as

$$\lim_{a \downarrow 0} \int_a^1 f = \lim_{a \downarrow 0} (2 - 2\sqrt{a}) = 2.$$

Calculus courses fix with this issue by just defining $\int_0^1 \frac{1}{\sqrt{x}}$ dx to be $\lim_{a \downarrow 0} \int_a^1 \frac{1}{\sqrt{x}}$ dx.

Example (area seems to make sense, but Riemann integral is not defined): Let $r_1, r_2, ...$ be a sequence that includes each rational number in $(0, 1)$ exactly once and includes no other numbers. For $k \in \mathbb{Z}^+$, define $f_k : [0, 1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} \frac{1}{\sqrt{x - r_k}} & \text{if } x > r_k, \\ 0 & \text{if } x \leq r_k. \end{cases}$$

Then define $f : [0, 1] \to [0, \infty]$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{f_{k(x)}}{2^k}.$$

Since every nonempty open subinterval of $[0, 1]$ contains a rational number, f is unbounded on every such subinterval. Thus, the Riemann integral of f is undefined on every subinterval of $[0, 1]$ with more than one element. However, the area under the graph of each f_k is less than 2. Then by the definition of f, the area under the graph of f should be less than 2.

Example (Riemann integration does not work well with pointwise limits): Let $r_1, r_2, ...$ be a sequence that includes each rational number in $[0, 1]$ exactly once and that includes no other numbers. For $k \in \mathbb{Z}^+$, define $f_k : [0, 1] \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \{r_1, ..., r_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Each f_k is Riemann integrable and $\int_0^1 f_k = 0$.

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

Then

$$\lim_{k \to \infty} f_k(x) = f(x) \text{ for each } x \in [0, 1].$$

However, f is not Riemann integrable even though f is the pointwise limit of a sequence of integrable functions bounded by 1.

There is a condition under which Riemann integrals behave well with limits– though, this positive result has the undesirable hypothesis of the limit function f being Riemann integrable.

Proposition 1.2.1 (interchanging Riemann integral and limit): Suppose $a, b, M \in \mathbb{R}$ with $a < b$. Suppose $f_1, f_2, \dots$ is a sequence of Riemann integrable functions on $[a, b]$ such that

$$|f_k(x)| \leq M$$

for all $k \in \mathbb{Z}^+$ and all $x \in [a, b]$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x \in [a, b]$. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f = \lim_{k\to\infty} \int_a^b f_k.$$

2. Measures

2.1. Outer Measure on $\mathbb{R}$

Definition 2.1.1 (length of open interval): The *length* $\ell(I)$ of an open interval I is define by

$$\ell(I) = \begin{cases} b - a & \text{if } I = (a, b) \text{ for some } a, b \in \mathbb{R} \text{ with } a < b \\ 0 & \text{if } I = \emptyset \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbb{R} \\ \infty & \text{if } I = (-\infty, \infty) \end{cases}$$

Definition 2.1.2 (outer measure): The *outer measure* $|A|$ of a set $A \subseteq \mathbb{R}$ is defined by

$$|A| = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) \mid I_1, I_2, \dots \text{ are open intervals such that } A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

Example (finite sets have outer meaure 0): Let $A = \{a_1, \dots, a_n\}$ be a finite subset of $\mathbb{R}$. Suppose $\varepsilon > 0$. Define the sequence of $I_1, I_2, \dots$ of open intervals by

$$I_k = \begin{cases} (a_k - \varepsilon, a_k + \varepsilon) & \text{if } k \leq n \\ \emptyset & \text{if } k > n. \end{cases}$$

Then $I_1, I_2, \dots$ is a sequence of open interval whose union contains A. Then, $\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon n$. Hence $|A| \leq 2\varepsilon n$. Since ε is an arbitrary positive number, this implies that $|A| = 0$.

2.1.1. Good Properties of Outer Measure

Proposition 2.1.1.1: Every countable subset of $\mathbb{R}$ has outer measure 0.

Proposition 2.1.1.2: Suppose A and B are subsets of $\mathbb{R}$ with $A \subseteq B$. Then $|A| \leq |B|$.

Definition 2.1.1.1 (translation): If $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then the *translation* $t + A$ is defined by

$$t + A = \{t + a \mid a \in A\}.$$

Proposition 2.1.1.3 (translation invariant): Suppose $t \in \mathbb{R}$ and $A \subseteq \mathbb{R}$. Then $|t + A| = |A|$.

Proposition 2.1.1.4 (countable subadditivity): Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $\mathbb{R}$. Then

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \sum_{k=1}^{\infty} |A_k|.$$

2.1.2. Outer Measure of Closed Bounded Interval

Definition 2.1.2.1 (open cover; finite subcover): Suppose $A \subseteq \mathbb{R}$

a. A collection $\mathcal{C}$ of open subsets of $\mathbb{R}$ is called an *open cover* of A if A is contained in the union of all the sets in $\mathcal{C}$.
b. An open cover $\mathcal{C}$ of A is said to have a *finite subcover* if A is contained in the union of some finite list of sets in $\mathcal{C}$.

Proposition 2.1.2.1 (Heine-Borel Theorem): Every open cover of a closed bounded subset of $\mathbb{R}$ has a finite subcover.

Proposition 2.1.2.2 (outer measure of a closed interval)): Suppose $a, b \in \mathbb{R}$, with $a < b$. Then $|[a, b]| = b - a$.

Proposition 2.1.2.3 (nontrivial intervals are uncountable): Every interval in $\mathbb{R}$ that contains at least two distint elements is uncountable.

2.1.3. Outer Measure is Not Additive

Proposition 2.1.3.1 (non-additivity of outer measure): There exist disjoint subsets A and B of $\mathbb{R}$ such that $|A \cup B| \neq |A| + |B|$.

2.2. Measurable Spaces and Functions

Proposition 2.2.1 (nonexistence of extension of length to all subsets of $\mathbb{R}$): There does not exist a function μ with all the following properties.

a. μ is a function from the set of subsets of $\mathbb{R}$ to $[0, \infty]$.
b. $\mu(I) = \ell(I)$ for every open interval I of $\mathbb{R}$.
c. $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \ldots$ of subsets of $\mathbb{R}$.
d. $\mu(t + A) = \mu(A)$ for every $A \subseteq \mathbb{R}$ and every $t \in \mathbb{R}$.

2.2.1. σ-Algebras

Definition 2.2.1.1 (σ-algebra): Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. THen $\mathcal{S}$ is called a *σ-algebra* on X if the following three conditions are satisfied:
- $\emptyset \in \mathcal{S}$;
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$;
- if $E_1, E_2, \ldots$ is a seuqence of elements of $\mathcal{S}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$.

Example: The following are some σ-algebras on a set X.
- $\{\emptyset, X\}$
- $\mathcal{P}(X)$
- The set of all subsets E of X such that E is countable or $X \setminus E$ is countable.

Proposition 2.2.1.1 (σ-algebras are closed under countable intersection): Suppose $\mathcal{S}$ is a σ-algebra on a set X. Then

a. $X \in \mathcal{S}$;
b. if $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$ and $D \cap E \in \mathcal{S}$ and $D \setminus E \in \mathcal{S}$;
c. if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$.

Definition 2.2.1.2 (measurable space; measurable set):
- A *measurable space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X.
- An element of $\mathcal{S}$ is called an *$\mathcal{S}$-measurable set*, or just a *measurable set* if $\mathcal{S}$ is clear from the context.

2.2.2. Borel Subsets of $\mathbb{R}$

Proposition 2.2.2.1 (smallest σ-algebra containing a collection of subsets): Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X.

Example: For a set X with $\mathcal{A} = \{\{x\} \mid x \in X\}$, the smallest σ-algebra containing $\mathcal{A}$ is the finite-cofinite σ-algebra.

Definition 2.2.2.1 (Borel set): The smallest σ-algebra on $\mathbb{R}$ containing all open subsets of $\mathbb{R}$ is called the collection of *Borel subsets* of $\mathbb{R}$. An element of this σ-algebra is called a *Borel set*.

Example:
- Every closed subset of $\mathbb{R}$ is a Borel set because every closed subset of $\mathbb{R}$ is the complement of an open subset of $\mathbb{R}$.
- Every countable subset of $\mathbb{R}$ is a Borel subset because if $B = \{x_1, x_2, \ldots\}$, then $B = \bigcup_{k=1}^{\infty} \{x_k\}$, which is a Borel set because each $\{x_k\}$ is a closed set.
- Every half-open interval $[a, b)$ (where $a, b \in \mathbb{R}$) is a Borel set because $[a, b) = \bigcap_{k=1}^{\infty} \left(a - \frac{1}{k}, b\right)$.
- If $f : \mathbb{R} \to \mathbb{R}$ is a function, then the set of points at which f is continuous is the intersection of a sequence of open sets and thus is a Borel set.

Remark: There is no finite procedure involving countable unions, countable intersection and complements for constructing the collection of Borel subsets.

2.2.3. Inverse Images

Definition 2.2.3.1 (inverse image; $f^{-1}(A)$): If $f : X \to Y$ is a function nd $A \subseteq Y$, then the set $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}.$$

Proposition 2.2.3.1 (algebra of inverse images): Suppose $f : X \to Y$ is a function. Then

a. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ for every $A \subseteq Y$;
b. $f^{-1}\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of Y;
c. $f^{-1}\left(\bigcap_{A \in \mathcal{A}} A\right) = \bigcap_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of Y.

Proposition 2.2.3.2 (inverse image of a composition): Suppose $f : X \to Y$ and $g : Y \to W$ are functions. Then

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)).$$

. 2.2.4. Measurable Functions

Definition 2.2.4.1 (measurable function): Suppose $(X, \mathcal{S})$ is a measurable sapce. A function $f : X \to \mathbb{R}$ is called $\mathcal{S}$-measurable if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subseteq \mathbb{R}$

Definition 2.2.4.2 (characteristic function;χ_E): Suppose E is a subset of a set X. The *characteristic function of E* is the function $\chi_E : X \to \mathbb{R}$ defined by

$$\chi_{E(x)} = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Note that,

$$\chi_E^{-1}(B) = \begin{cases} E & \text{if } 0 \notin B \text{ and } 1 \in B, \\ X \setminus E & \text{if } 0 \in B \text{ and } 1 \notin B, \\ X & \text{if } 0 \in B \text{ and } 1 \in B, \\ \emptyset & \text{if } 0 \notin B \text{ and } 1 \notin B. \end{cases}$$

Then,

Lemma 2.2.4.1: χ_E is an $\mathcal{S}$-measurable function if and only if E in $\mathcal{S}$.

Proposition 2.2.4.1 (condition for measurable function): Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to \mathbb{R}$ is a function such that

$$f^{-1}((a, \infty)) \in \mathcal{S}$$

for all $a \in \mathbb{R}$. Then f is an $\mathcal{S}$-measurable function.

In general, we can say the following things.

Lemma 2.2.4.2 (image of a σ-algebra): Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to Y$ a function. Then, the following defines a σ-algebra on Y

$$\mathcal{F} = \{A \subseteq Y \mid f^{-1}(A) \in \mathcal{S}\}$$

So, the family from Proposition 2.2.4.1 can be replaced by any family of sets such that the smallest σ-algebra containing it also contains the Borel subsets of $\mathbb{R}$.

Definition 2.2.4.3 (Borel measurable function): Suppose $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is called *Borel measurable* if $f^{-1}(B)$ is a Borel set for every Borel set $B \subseteq \mathbb{R}$.

Proposition 2.2.4.2 (every continuous function is Borel measurable): Every continuous real-valued function defined on a Borel subset of $\mathbb{R}$ is a Borel measurable function.

Definition 2.2.4.4 (increasing functions; strictly increasing): Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is afunction

- f is called *increasing* if $f(x) \leq f(y)$ for all $x, y \in X$ with $x < y$.
- f is called *strictly increasing* if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

Proposition 2.2.4.3 (every increasing function is Borel measurable): Every increasing function defined on a Borel subset of $\mathbb{R}$ is a Borel measurable function.

Proposition 2.2.4.4 (composition of measurable functions): Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to \mathbb{R}$ is an $\mathcal{S}$-measurable function. Suppose g is a real-valued measurable function defined on a subset of $\mathbb{R}$ that includes the range of f. Then $g \circ f : X \to \mathbb{R}$ is an $\mathcal{S}$-measurable function.

Proposition 2.2.4.5 (algebraic operations with measurable functions): Suppose $(X, \mathcal{C})$ is a measurable space and $f, g : X \to \mathbb{R}$ are $\mathcal{S}$-measurable. Then

a. $f + g, f - g$ and fg are $\mathcal{S}$-measurable functions;
b. if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

Proposition 2.2.4.6 (limit of $\mathcal{S}$-measurable functions): Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_@, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbb{R}$. Suppose $\lim_{k\to\infty} f_{k(x)}$ exists for each $x \in X$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \lim_{k\to\infty} f_{k(x)}.$$

Then f is an $\mathcal{S}$-measurable function.

Definition 2.2.4.5 (Borel subsets of $[-\infty, \infty]$): A subset of $[-\infty, \infty]$ is called a *Borel set* if its intersection with $\mathbb{R}$ is a Borel set.

Definition 2.2.4.6 (measurable function): Suppose $(X, \mathcal{S})$ is a measurable space. A function $f : X \to [-\infty, \infty]$ is called $\mathcal{S}$*-measurable* if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subseteq [-\infty, \infty]$.

Proposition 2.2.4.7 (condition for measurable function): Suppose $(X, \mathcal{S})$ is a measurable sapce and $f : X \to [-\infty, \infty]$ is a function such that

$$f^{-1}((a, \infty]) \in \mathcal{S})$$

for all $a \in \mathbb{R}$. Then f is an $\mathcal{S}$-measurable function.

Proposition 2.2.4.8 (infimum and supremum of a sequence of $\mathcal{S}$-measurable functions): Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a seuqence of $\mathcal{S}$-measurable functions from X to $[-\infty, \infty]$. Define $g, h : X \to [-\infty, \infty]$ by

$$g(x) = \inf\left\{f_{k(x)} \mid k \in \mathbb{Z}^+\right\} \quad \text{and} \quad h(x) = \sup\left\{f_{k(x)} \mid k \in \mathbb{Z}^+\right\}.$$

Then g and h are $\mathcal{S}$-measurable functions.

2.3. Measures and Their Properties

Definition 2.3.1 (measure): Suppose X is a set and $\mathcal{S}$ is a σ-algebra on X. A *measure* on $(X, \mathcal{S})$ is a function $\mu : \mathcal{S} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

for every disjoint sequence $E_1, E_2, \ldots$ of sets in $\mathcal{S}$.

Example:
- If X is a set, then *counting measure* is the measure μ defined on the σ-algebra of all subsets of X by setting $\mu(E) = n$ if E is a finite set containing exactly n elements and $\mu(E) = \infty$ if E is not a finite set.
- Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $c \in X$. Define the *Dirac* measure δ_c on $(X, \mathcal{S})$ by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

- Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $\omega : X \to [0, \infty]$ is a function. Define a measure μ on $(X, \mathcal{S})$ by

$$\mu(E) = \sum_{x \in E} w(x)$$

  for $E \in \mathcal{S}$. THe sum is defined as the supremum of all finite subsums $\sum_{x \in D} w(x)$ as D ranges over all finite subsets of E.
- Suppose X is a set and $\mathcal{S}$ is the σ-algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Define a measure on μ on $(X, \mathcal{S})$ by

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable}, \\ 3 & \text{if } E \text{ is uncountable}. \end{cases}$$

- Suppose $\mathcal{S}$ is the σ-algebra on $\mathbb{R}$ consisting of all subsets of $\mathbb{R}$. Then the function that takes a set $E \subseteq \mathbb{R}$ to $|E|$ (the outer measure of E) is not a measure because it is not finitely additive.
- Suppose $\mathcal{B}$ is the σ-algebra on $\mathbb{R}$ consisting of all Borel subsets of $\mathbb{R}$. The outer measure is a measure on $(\mathbb{R}, \mathcal{B})$ (proven below).

Definition 2.3.2 (measure space): A *measure space* is an ordered triple $(X, \mathcal{S}, \mu)$, where X is a set, $\mathcal{S}$ is a σ-algebra on X, and μ is a measure on $(X, \mathcal{S})$.

2.3.1. Properties of Measures

Proposition 2.3.1.1 (measure preserves order; measure of a set difference): Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $D, E \in \mathcal{S}$ are such that $D \subseteq E$. Then

a. $\mu(D) \leq \mu(E)$;
b. $\mu(E \setminus D) = \mu(E) - \mu(D)$ provided that $\mu(D) < \infty$.

Remark: The hypothesis $\mu(D) < \infty$ is required for part (b) to avoid undefined expressions of the form $\infty - \infty$.

Proposition 2.3.1.2 (countable subadditivity): Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1, E_2, \ldots \in \mathcal{S}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Proposition 2.3.1.3 (measure of an increasing union): Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \subseteq E_2 \subseteq \cdots$ is an increasing sequence of sets in $\mathcal{S}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k \to \infty} \mu(E_k)$$

Proposition 2.3.1.4 (measure of a decreasing intersection): Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \supseteq E_2 \supseteq \cdots$ is a decreasing sequence of sets in $\mathcal{S}$, with $\mu(E_1) < \infty$. Then

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k \to \infty} \mu(E_k).$$

Remark: The hypothesis $\mu(E_1) < \infty$ is necessary.

Proposition 2.3.1.5 (measure of a union): Suppose $(X, \mathcal{S}, \mu)$ is a measure space $D, E \in \mathcal{S}$, with $\mu(D \cap E) < \infty$. Then

$$\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E).$$

2.4. Lebesgue Measure

2.4.1. Additivity of Outer Measure on Borel Sets

2.4.2. Lebesgue Measurable Sets

2.4.3. Cantor Set and Cantor Function

2.5. Convergence of Measurable Functions