1. Propiedades

Propiedad 1 (Linealidad de la esperanza)**.** *Para cualesquiera $a, b \in \mathbb{R}$ y variables aleatorias X_i, X_j:*

$$\mathbb{E}[aX_i + bX_j] = a\mathbb{E}[X_i] + b\mathbb{E}[X_j] \tag{1}$$

Demostración. Por definición de esperanza y linealidad de la integral de Lebesgue:

$$\begin{aligned}
\mathbb{E}[aX_i + bX_j] &= \int_\Omega (aX_i(\omega) + bX_j(\omega))\, d\mathbb{P}(\omega) \\
&= a \int_\Omega X_i(\omega)\, d\mathbb{P}(\omega) + b \int_\Omega X_j(\omega)\, d\mathbb{P}(\omega) \\
&= a\mathbb{E}[X_i] + b\mathbb{E}[X_j]
\end{aligned}$$

□

Definición 1 (Varianza)**.** *Para cualquier variable aleatoria X con $\mathbb{E}[X^2] < \infty$:*

$$\mathbb{V}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \tag{2}$$

Teorema 1 (Varianza de combinación lineal)**.** *Dados X_i, X_j variables aleatorias y $a, b \in \mathbb{R}$:*

1. Si $X_i \perp X_j$:

$$\mathbb{V}(aX_i + bX_j) = a^2\mathbb{V}(X_i) + b^2\mathbb{V}(X_j) \tag{3}$$

2. En general:

$$\mathbb{V}(aX_i + bX_j) = a^2\mathbb{V}(X_i) + b^2\mathbb{V}(X_j) + 2ab\operatorname{cov}(X_i, X_j) \tag{4}$$

Demostración. 1. Caso independiente:

$$\begin{aligned}
\mathbb{V}(aX_i + bX_j) &= \mathbb{E}\left[(aX_i + bX_j - a\mathbb{E}[X_i] - b\mathbb{E}[X_j])^2\right] \\
&= \mathbb{E}\left[(a(X_i - \mathbb{E}[X_i]) + b(X_j - \mathbb{E}[X_j]))^2\right] \\
&= a^2\mathbb{E}\left[(X_i - \mathbb{E}[X_i])^2\right] + b^2\mathbb{E}\left[(X_j - \mathbb{E}[X_j])^2\right] \\
&\quad + 2ab\,\underbrace{\mathbb{E}\left[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])\right]}_{=0 \text{ por independencia}} \\
&= a^2\mathbb{V}(X_i) + b^2\mathbb{V}(X_j)
\end{aligned}$$

2. Caso general:

$$\begin{aligned}
\mathbb{V}(aX_i + bX_j) &= \mathbb{E}\left[(a(X_i - \mathbb{E}[X_i]) + b(X_j - \mathbb{E}[X_j]))^2\right] \\
&= a^2\mathbb{E}\left[(X_i - \mathbb{E}[X_i])^2\right] + b^2\mathbb{E}\left[(X_j - \mathbb{E}[X_j])^2\right] \\
&\quad + 2ab\,\mathbb{E}\left[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])\right] \\
&= a^2\mathbb{V}(X_i) + b^2\mathbb{V}(X_j) + 2ab\operatorname{cov}(X_i, X_j)
\end{aligned}$$

□

Propiedad 2 (Propiedad de covarianza)**.** *Para $a, b \in \mathbb{R}$ y variables aleatorias X_i, X_j:*

$$\operatorname{cov}(aX_i, bX_j) = ab\ \operatorname{cov}(X_i, X_j) \tag{5}$$

Demostración. Desarrollando la definición:

$$\begin{aligned}
\operatorname{cov}(aX_i, bX_j) &= \mathbb{E}\left[(aX_i - \mathbb{E}[aX_i])(bX_j - \mathbb{E}[bX_j])\right] \\
&= \mathbb{E}\left[a(X_i - \mathbb{E}[X_i]) \cdot b(X_j - \mathbb{E}[X_j])\right] \\
&= ab\,\mathbb{E}\left[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])\right] \\
&= ab\ \operatorname{cov}(X_i, X_j)
\end{aligned}$$

□

2. Propiedades de vectores aleatorios

Propiedad 3. *Sea* $\boldsymbol{X} = (X_1, X_2, \dots, X_p)^\top \in \mathbb{R}^p$ *un vector aleatorio con* $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{\mu}$ *y matriz de covarianza* $\boldsymbol{\Sigma}$*. Para cualquier* $\boldsymbol{c} = (c_1, c_2, \dots, c_p)^\top \in \mathbb{R}^p$ *y* $\boldsymbol{A} \in \mathbb{R}^{m\times p}$*, se verifica:*

1. Linealidad de la esperanza

$$\mathbb{E}\left[\boldsymbol{c}^\top \boldsymbol{X}\right] = \sum_{i=1}^{p} c_i \mathbb{E}[X_i] = \boldsymbol{c}^\top \boldsymbol{\mu}$$

Por definición de producto escalar:

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{c}^\top \boldsymbol{X}\right] &= \mathbb{E}\left[\sum_{i=1}^{p} c_i X_i\right] \\
&= \sum_{i=1}^{p} c_i \mathbb{E}[X_i] \quad \text{(linealidad de la esperanza)} \\
&= \boldsymbol{c}^\top \boldsymbol{\mu}
\end{aligned}$$

2. Matriz de covarianza

$$\boldsymbol{\Sigma} = \mathbb{E}\left[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top\right] \in \mathbb{R}^{p\times p}$$

Elemento (i, j) de la matriz:

$$\begin{aligned}
\Sigma_{ij} &= \mathbb{E}\left[(X_i - \mu_i)(X_j - \mu_j)\right] \\
&= \operatorname{Cov}(X_i, X_j)
\end{aligned}$$

Verificando que:

$$\boldsymbol{\Sigma} = \mathbb{E}\left[\begin{pmatrix} (X_1 - \mu_1)^2 & \cdots & (X_1 - \mu_1)(X_p - \mu_p) \\ \vdots & \ddots & \vdots \\ (X_p - \mu_p)(X_1 - \mu_1) & \cdots & (X_p - \mu_p)^2 \end{pmatrix}\right]$$

3. Varianza de combinación lineal

$$\mathbb{V}(\boldsymbol{c}^\top \boldsymbol{X}) = \boldsymbol{c}^\top \boldsymbol{\Sigma} \boldsymbol{c}$$

Desarrollando la definición de varianza:

$$\begin{aligned}
\mathbb{V}(\boldsymbol{c}^\top \boldsymbol{X}) &= \mathbb{E}\left[(\boldsymbol{c}^\top \boldsymbol{X} - \boldsymbol{c}^\top \boldsymbol{\mu})^2\right] \\
&= \mathbb{E}\left[(\boldsymbol{c}^\top (\boldsymbol{X} - \boldsymbol{\mu}))^2\right] \\
&= \mathbb{E}\left[\boldsymbol{c}^\top (\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top \boldsymbol{c}\right] \\
&= \boldsymbol{c}^\top \mathbb{E}\left[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top\right] \boldsymbol{c} \quad \text{(linealidad)} \\
&= \boldsymbol{c}^\top \boldsymbol{\Sigma} \boldsymbol{c}
\end{aligned}$$

4. Esperanza de transformación lineal

$$\mathbb{E}[\boldsymbol{A}\boldsymbol{X}] = \boldsymbol{A}\boldsymbol{\mu}$$

Elemento i-ésimo del vector $\mathbb{E}[\boldsymbol{A}\boldsymbol{X}]$:

$$\begin{aligned}
(\mathbb{E}[\boldsymbol{A}\boldsymbol{X}])_i &= \mathbb{E}\left[\sum_{j=1}^{p} A_{ij} X_j\right] \\
&= \sum_{j=1}^{p} A_{ij} \mathbb{E}[X_j] \quad \text{(linealidad)} \\
&= \sum_{j=1}^{p} A_{ij} \mu_j \\
&= (\boldsymbol{A}\boldsymbol{\mu})_i
\end{aligned}$$

5. Covarianza de transformación lineal

$$\mathbb{V}(\boldsymbol{A}\boldsymbol{X}) = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^\top$$

Usando la definición de matriz de covarianza:

$$\begin{aligned}
\mathbb{V}(\boldsymbol{A}\boldsymbol{X}) &= \mathbb{E}\left[(\boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\boldsymbol{\mu})(\boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\boldsymbol{\mu})^\top\right] \\
&= \mathbb{E}\left[\boldsymbol{A}(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top \boldsymbol{A}^\top\right] \\
&= \boldsymbol{A}\mathbb{E}\left[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top\right] \boldsymbol{A}^\top \quad \text{(linealidad)} \\
&= \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^\top
\end{aligned}$$