# 5. Natural Deduction

Language & Logic

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# Recap

- Notation for arguments (in sequent form)
  - P1, P2 : C
  - P1, P2 $\vdash$ C (if proven to be valid)
  - where P1, P2, C are formula in propositional logic

- Proofs (of argument validity)
  - using natural deduction
  - with a given set of inference rules (syntactic transforms)
  - so far: list of premise introductions & inference rule applications
  - not necessary unique (nor easy to construct)
  - but easy to check, rigorously once, constructed
  - fixed proof notation

# Exercise Sheet 2 Feedback

- Proof for Question 5 from Exercise sheet 2

$$\neg P \rightarrow \neg Q, P \rightarrow Z, \neg\neg Q \;:\; Z$$

| | | | |
|---|---|---|---|
| 1. $\neg P \rightarrow \neg Q$ | Premise | $\{1\}$ |
| 2. $\neg\neg Q$ | Premise | $\{2\}$ |
| 3. $\neg\neg P$ | Modus $\text{Tollens}_{1,2}$ | $\{1,2\}$ |
| 4. $P$ | Double Negative $\text{Elimination}_3$ | $\{1,2\}$ |
| 5. $P \rightarrow Z$ | Premise | $\{5\}$ |
| 6. $Z$ | Modus $\text{Ponens}_{5,4}$ | $\{1,2,5\}$ |

- There are variants, but we will stick largely to the above
  - perhaps with abbreviations for rule annotations

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| 3. | $\neg\neg P$ | Modus Tollens$_{1,2}$ | $\{1,2\}$ |
| 4. | $P$ | Double Negative Elimination$_{3}$ | $\{1,2\}$ |
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Lines of propositions used in the inference rule

Lines of all premises on which this depends

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| 4. | $P$ | Double Negative Elimination$_3$ | $\{1,2\}$ |
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Inference rules can be applied
to any propositions (including compounds)

# Exercise Sheet 2 Feedback

- Proof for Question 5 from Exercise sheet 2

$$\neg P \to \neg Q, P \to Z, \neg\neg Q \ : \ Z$$

| | | | |
|---|---|---|---|
| 1. | $\neg P \to \neg Q$ | Premise | $\{1\}$ |
| 2. | $\neg\neg Q$ | Premise | $\{2\}$ |
| 3. | $\neg\neg P$ | Modus Tollens$_{1,2}$ | $\{1,2\}$ |
| 4. | $P$ | Double Negative Elimination$_3$ | $\{1,2\}$ |
| 5. | $P \to Z$ | Premise | $\{5\}$ |
| 6. | $Z$ | Modus Ponens$_{5,4}$ | $\{1,2,5\}$ |

Don't skip steps: every line should be the application of one inference rule

# Proof strategies

- How do we go about constructing a proof?

- Simple examples so far
  - look at the available propositions and rules to manipulate them
  - work backwards from the conclusion
  - or forwards from the premises
  - look at the main connective of each proposition

# Example

- Proof for Question 2 from Exercise sheet 2

$$P, Q, R \;:\; P \wedge (Q \wedge R)$$

| | | | |
|---|---|---|---|
| 1. | $Q$ | Premise | $\{1\}$ |
| 2. | $R$ | Premise | $\{2\}$ |
| 3. | $Q \wedge R$ | $\wedge$-Introduction$_{1,2}$ | $\{1,2\}$ |
| 4. | $P$ | Premise | $\{4\}$ |
| 5. | $P \wedge (Q \wedge R)$ | $\wedge$-Introduction$_{4,3}$ | $\{1,2,4\}$ |

# Today

- This lecture (and the next):
  - more systematic list of inference rules
  - more complex inference rules & proof techniques
  - and strategies to apply them

# Inference rules

- We will always assume a fixed set of inference rules
- Previously, used an arbitrary set:
  - Modus Ponens & Modus Tollens
  - double negative elimination
  - $\wedge$-introduction & $\wedge$-elimination
- Time to be systematic…
  - 4 connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$)
  - each one has rules for introduction and elimination
  - which introduce and eliminate the connective, respectively
  - e.g. to form a conclusion or break up a premise

# Introduction & Elimination

- Conjunction
  - $\wedge$-introduction
  - $\wedge$-elimination
- Disjunction
  - $\vee$-introduction
  - $\vee$-elimination
- Implication
  - $\rightarrow$-introduction
  - $\rightarrow$-elimination
- Negation
  - $\neg$-introduction
  - $\neg\neg$-elimination

# Introduction & Elimination

- Conjunction
  - ∧-introduction – already seen
  - ∧-elimination – already seen
- Disjunction
  - ∨-introduction
  - ∨-elimination
- Implication
  - →-introduction
  - →-elimination – already seen (Modus Ponens)
- Negation
  - ¬-introduction
  - ¬¬-elimination – already seen (double negative elimination)

But no Modus Tollens

# Introduction & Elimination

- Conjunction
  - $\wedge$-introduction – already seen
  - $\wedge$-elimination – already seen
- Disjunction
  - $\vee$-introduction – new
  - $\vee$-elimination – new (case analysis)
- Implication
  - $\rightarrow$-introduction – new (conditional proof)
  - $\rightarrow$-elimination – already seen (Modus Ponens)
- Negation
  - $\neg$-introduction – new (reductio ad absurdum)
  - $\neg\neg$-elimination – already seen (double negative elimination)

But no Modus Tollens

# Inference rules

- From now now, we now switch to a more common style for presenting inference rules:

Premises

$$\frac{A \qquad B}{A \wedge B}\ \wedge\text{-introduction}$$

Conclusion

Rule name

# Inference rules seen so far

- Conjunction ($\wedge$)

$$\frac{A \quad B}{A \wedge B}\ \wedge\text{-introduction} \qquad \frac{A \wedge B}{A}\ \wedge\text{-elimination} \qquad \frac{A \wedge B}{B}\ \wedge\text{-elimination}$$

- $\rightarrow$-elimination
  - (sometimes called "Modus Ponens")

$$\frac{A \rightarrow B \quad A}{B}\ \rightarrow\text{-elimination}$$

- $\neg\neg$-elimination(sometimes abbreviated to DNE)

$$\frac{\neg\neg A}{A}\ \neg\neg\text{-elimination}$$

# $\vee$-Introduction

- $\vee$ -introduction

$$\frac{A}{A \vee B}\ \vee\text{-introduction} \qquad \frac{A}{B \vee A}\ \vee\text{-introduction}$$

  for any formula B

- Simple, but slightly counterintuitive
  - we'll illustrate some uses of it later

- Proof notation
  - same as inference rules seen so far
  - (single) proof line index is added to rule annotation
  - and premise dependencies are copied from that line

# Example ($\vee$-Introduction)

- $P, P \to Q \;:\; (P \wedge Q) \vee R$

# →-introduction

- Allows us to prove an implication $A \rightarrow B$
  - also known as conditional proof
  - A is the antecedent
  - B is the consequent

Recall: $\vdash$ means "if we assume the LHS, we can prove the RHS"

$$\frac{A \vdash B}{A \rightarrow B} \quad \rightarrow\text{-introduction}$$

- New proof technique:
  - make a hypothesis A (also called an assumption)
  - temporarily assume that A is true, and prove B (a sub-proof)
  - infer $A \rightarrow B$ and discharge the hypothesis

| | | |
|---|---|---|
| 1. | A | Hypothesis |
| | … | |
| n. | B | |
| n+1. | $A \rightarrow B$ | $\rightarrow$-introduction$_{1,n}$ |

"sub-proof"

# Example (conditional proof)

- $P \rightarrow Q, Q \rightarrow R \;:\; P \rightarrow (Q \wedge R)$

| | | | |
|---|---|---|---|
| 1. | $P \rightarrow Q$ | Premise | {1} |
| 2. | $Q \rightarrow R$ | Premise | {2} |
| 3. | $P$ | Hypothesis | {3} |
| 4. | $Q$ | $\rightarrow$-elimination$_{1,3}$ | {1,3} |
| 5. | $R$ | $\rightarrow$-elimination$_{2,4}$ | {1,2,3} |
| 6. | $Q \wedge R$ | $\wedge$-introduction$_{4,5}$ | {1,2,3} |
| 7. | $P \rightarrow (Q \wedge R)$ | $\rightarrow$-introduction$_{3,6}$ | {1,2} |

Dependencies for hypotheses done the same as for premises

Dependency on hypothesis removed when "discharged"

- Notes:
  - lines around hypothesis + sub-proof
  - sub-proof can use propositions proved outside, but not vice-versa
  - cannot complete the proof while inside the sub-proof!

# Example (nested conditional)

- $(P \wedge Q) \rightarrow R \;:\; P \rightarrow (Q \rightarrow R)$

# $\vee$–Elimination

- Allows us to use a premise whose main connective is $\vee$

$$\frac{A \vee B \quad A \vdash C \quad B \vdash C}{C} \; \vee\text{–elimination}$$

- Proof technique
  - two separate sub–proofs, one assuming A, one assuming B
  - show that the same proposition C can be deduced from both
  - "case analysis"

| | | |
|---|---|---|
| 1. | $A \vee B$ | |
| 2. | $A$ | Hypothesis |
| | … | |
| n. | $C$ | |
| n+1. | $B$ | Hypothesis |
| | … | |
| m. | $C$ | |
| m+1. | $C$ | $\vee$–elimination$_{1,2,n,n+1,m}$ |

# Example (∨–elimination)

- $P \vee Q \;:\; Q \vee P$

| | | | |
|---|---|---|---|
| 1. | $P \vee Q$ | Premise | {1} |
| 2. | $P$ | Hypothesis | {2} |
| 3. | $Q \vee P$ | ∨–introduction | {2} |
| 4. | $Q$ | Hypothesis | {4} |
| 5. | $Q \vee P$ | ∨–introduction | {4} |
| 6. | $Q \vee P$ | ∨–elimination$_{1,2,3,4,5}$ | {1} |

- Note:
  - there are 5 line number indices for the rule annotation
  - the dependencies for line 6 are derived by combining the dependencies for lines 1, 3 and 5, then removing the dependencies for lines 2 and 4

# $\neg$-introduction

- Allows us to prove a negation $\neg A$
  - to prove $\neg A$, first assume that $A$ is true…
  - and then show it leads to a contradiction
  - proof by contradiction, or reductio ad absurdum (RAA)
- A contradiction is any proposition of the form $B \wedge \neg B$
  - we use the symbol $\perp$ (“bottom”) to denote <u>any</u> such contradiction

$$\frac{A \vdash \perp}{\neg A} \quad \neg\text{-introduction}$$

- Alternatively…
  - to prove $C$, assume $\neg C$, then prove a contradiction
  - (in fact, the rule above would yield $\neg\neg C$, from which we infer $C$)

# Example (¬-introduction)

- $P \rightarrow Q, P \rightarrow \neg Q \ : \ \neg P$

| | | | |
|---|---|---|---|
| 1. | $P$ | Hypothesis | {1} |
| 2. | $P \rightarrow Q$ | Premise | {2} |
| 3. | $Q$ | $\rightarrow$-elimination$_{2,1}$ | {1,2} |
| 4. | $P \rightarrow \neg Q$ | Premise | {4} |
| 5. | $\neg Q$ | $\rightarrow$-elimination$_{4,1}$ | {1,4} |
| 6. | $\perp$ | $\wedge$-introduction$_{3,5}$ | {1,2,4} |
| 7. | $\neg P$ | $\neg$-introduction$_{1,6}$ | {2,4} |

This is will very often be the last inference rule used, to form a contradiction

Dependency on hypothesis <u>removed</u> at this point

# Example (¬-introduction)

- $\neg(P \wedge \neg Q)\ :\ P \rightarrow Q$

# Summary

- Inference rules
  - introduction and elimination rules for each connective
- New rules
  - $\vee$-introduction
  - $\rightarrow$-introduction, i.e. conditional proof
  - $\vee$-elimination, i.e. case analysis
  - $\neg$-introduction, i.e. reduction ad absurdum
- New proof techniques/concepts
  - hypotheses (assumptions), sub-proofs & discharging
  - proof by contradiction
- Exercise class on Tue/Thur
  - practice proofs using these new rules