9. Proof Strategies for Predicate Calculus

Language & Logic

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This week

- Assignment 2 marks/feedback/solutions out now

- No Tue/Thu exercise classes this week
  - I will release a practice exercise sheet (Ex 7)
  - questions welcome: Facebook/office hours/…
  - follow-up next week

- Office hours for this week
  - Tue 1-2 (the usual Tuesday office hour)
  - also Tue 11-12 (i.e. the usual exercise class slot)
  - no office hour on Thursday
  - I have moved office: I'm now in room 133

Overview

- Assessment 2 – Feedback
- Equivalences
- Theorem & sequent introduction
- Proof strategies for predicate calculus
- Proving vs. disproving

Assessment 2 - Feedback

- Some common mistakes:

1. Only use sub-proofs (and hypotheses) where needed
   - only for certain inference rules; always know which one you want

Sub-proofs

- Inference rules needing sub-proofs/hypotheses

$$\frac{A \vdash B}{A \to B} \to\text{-introduction} \qquad \frac{A \vdash \bot}{\neg A} \neg\text{-introduction}$$

$$\frac{A \vee B \quad A \vdash C \quad B \vdash C}{C} \vee\text{-elimination}$$

$$\frac{\exists x[\phi(x)] \quad \phi(a) \vdash C}{C} \exists\text{-elimination*}$$

Assessment 2 – Feedback

- Some common mistakes:

1. Only use sub-proofs (and hypotheses) where needed
   - only for certain inference rules; always know which one you want
2. Never close two sub-proofs simultaneously
3. Proof formatting/annotation
   - always put boxes around sub-proofs
   - always check your dependencies
4. Inference rules only apply to the main connective of a formula
5. Don't use derived inference rules (or equivalences), unless you provide a separate proof of validity
   - e.g. modus tollens, law of excluded middle, etc.

Equivalences

- Formulas C and D are equivalent, written $C \equiv D$ iff:
  - their truth value is the same for every possible interpretation
  - alternatively, it means that both $C \vdash D$ and $D \vdash C$

- Some common equivalences:

1. Double negation
   - $\neg\neg A \equiv A$
2. Implication
   - $A \rightarrow B \;\equiv\; \neg A \vee B$
3. De Morgan's laws
   - $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and $\neg(A \vee B) \equiv \neg A \wedge \neg B$
4. Quantifier duality:
   - $\neg \forall x\, [\, \dots \,] \equiv \exists x\, [\, \neg(\dots)\,]$ and $\neg \exists x\, [\, \dots \,] \equiv \forall x\, [\, \neg(\dots)\,]$

Equivalences

- Equivalences allow us to rewrite formulas or subformulas
  - e.g., since $\neg(A \wedge B) \equiv \neg A \vee \neg B$
  - we know that: $P \rightarrow \neg(Q \wedge (R \wedge S)) \equiv P \rightarrow (\neg Q \vee \neg(R \wedge S))$
- But…
  - we make no direct use of these in our natural deduction proofs
- Still useful to know
  - e.g. to guide/structure the proof
  - or to confirm validity if this is unknown
- Example: Assessment 2 Q4

$$\neg(P \wedge (Q \vee R)) \ : \ \neg P \vee (\neg Q \wedge \neg R)$$

Theorem & sequents

- Theorem:
  - a formula that we can prove to be always true
- An example of a theorem (modus tollens):
  - $\vdash ((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$
- We could also write this in sequent notation as:
  - $A \rightarrow B, \neg B \vdash \neg A$

- In natural deduction proofs, you are allowed to:
  - use theorem/sequent introduction
  - <u>if</u> you have provided a separate (natural deduction) proof of it

- Theorem introduction inserts a known theorem
- Sequent introduction inserts a conclusion that can be derived

Theorem/sequent introduction

- A theorem:
  - $\vdash (A \wedge B) \to \neg(\neg A \vee \neg B)$ [De Morgan]
- Prove, using theorem introduction:
  - $P, Q, R : \neg(\neg(P \wedge Q) \vee \neg R)$

- A valid argument (sequent):
  - $A \wedge B \vdash \neg(\neg A \vee \neg B)$ [De Morgan]
- Prove again, using sequent introduction:
  - $P, Q, R : \neg(\neg(P \wedge Q) \vee \neg R)$

- This uses (uniform) substitutions
  - e.g. replacing A with $(P \wedge Q)$ and B with R

Example: Theorem introduction

- Prove: $P \vee \neg Q,\ P \to R,\ S \to Q\ \vdash\ R \vee \neg S$

1.	$P \vee \neg Q$	Premise	$\{1\}$
2.	P	Hypothesis	$\{2\}$
3.	$P \to R$	Premise	$\{3\}$
4.	R	$\to$-elimination$_{3,2}$	$\{2,3\}$
5.	$R \vee \neg S$	$\vee$-introduction$_{4}$	$\{2,3\}$
6.	$\neg Q$	Hypothesis	$\{6\}$
7.	$S \to Q$	Premise	$\{7\}$
8.	$((S \to Q) \wedge \neg Q) \to \neg S$	Theorem (modus tollens)	
9.	$(S \to Q) \wedge \neg Q$	$\wedge$-introduction$_{7,6}$	$\{6,7\}$
10.	$\neg S$	$\to$-elimination$_{9,8}$	$\{6,7\}$
11.	$R \vee \neg S$	$\vee$-introduction$_{10}$	$\{6,7\}$
12.	$R \vee \neg S$	$\vee$-elimination$_{1,2,5,6,11}$	$\{1,3,7\}$

Using:
$\vdash ((A \to B) \wedge \neg B) \to \neg A$

Example: Sequent introduction

- Prove: $P \vee \neg Q, P \rightarrow R, S \rightarrow Q \vdash R \vee \neg S$

1.	$P \vee \neg Q$	Premise	{1}
2.	P	Hypothesis	{2}
3.	$P \rightarrow R$	Premise	{3}
4.	R	$\rightarrow$-elimination$_{3,2}$	{2,3}
5.	$R \vee \neg S$	$\vee$-introduction$_{4}$	{2,3}
6.	$\neg Q$	Hypothesis	{6}
7.	$S \rightarrow Q$	Premise	{7}
8.	$\neg S$	Theorem (modus tollens)	{6,7}
9.	$R \vee \neg S$	$\vee$-introduction$_{10}$	{6,7}
10.	$R \vee \neg S$	$\vee$-elimination$_{1,2,5,6,11}$	{1,3,7}

Using:
$(A \rightarrow B, \neg B \vdash \neg A$

Predicate calculus – Recap

- Natural deduction for predicate calculus
  - extends natural deduction for propositional logic
- $\forall$–elimination
  - infer a particular instance $\phi(a)$ from $\forall x\ [\ \phi(x)\ ]$
- $\exists$–introduction
  - infer $\exists x\ [\ \phi(x)\ ]$ from a particular instance $\phi(a)$
- $\forall$–introduction
  - infer $\forall x\ [\ \phi(x)\ ]$ from $\phi(a)$ for a "typical" a
  - condition: a must not occur in any dependency of $\phi(a)$
- $\exists$–elimination
  - infer C from $\exists x\ [\ \phi(x)\ ]$ by inferring C for a "typical" a
  - condition: a must not occur in C
    or in any dependency of C except $\phi(a)$

Example – ∀-introduction

- Prove: $\forall x\,[\,P(x) \rightarrow Q(x)\,],\ \forall x\,[\,P(x)\,]\ :\ \forall x\,[\,Q(x)\,]$

1.	$\forall x\,[\,P(x) \rightarrow Q(x)\,]$	Premise	{1}
2.	$\forall x\,[\,P(x)\,]$	Premise	{2}
3.	$P(a) \rightarrow Q(a)$	$\forall$-elimination$_1$	{1}
4.	$P(a)$	$\forall$-elimination$_2$	{2}
5.	$Q(a)$	$\rightarrow$-elimination$_{3,4}$	{1,2}
6.	$\forall x\,[\,Q(x)\,]$	$\forall$-introduction$_5$	{1,2}

No use of a

Formula $\phi(a)$

Dependencies

condition: a must not occur in any dependency of $\phi(a)$

Example – ∃-elimination

- Prove: $\exists x\,[\,F(x) \wedge G(x)\,]\ :\ \exists x\,[\,F(x)\,]$

Formula $\phi(a)$

No use of a

No other dependencies (apart from $\phi(a)$)

1.	$\exists x\,[\,F(x) \wedge G(x)\,]$	Premise	{1}
2.	$\boxed{F(a) \wedge G(a)}$	Hypothesis	{2}
3.	$F(a)$	$\wedge$-elimination$_2$	{2}
4.	$\boxed{\exists x\,[\,F(x)\,]}$	$\exists$-introduction$_3$	{2}
5.	$\exists x\,[\,F(x)\,]$	$\exists$-elimination$_{1,2,4}$	{1}

condition: a must not occur in C
or in any dependency of C except $\phi(a)$

Proof strategies

- Recall the rules of thumb ("golden rules")
  when writing natural deduction proofs for propositional logic
  1. if there is a → in the conclusion, try →-introduction
  2. if there is a premise of the form A∨B, try ∨-elimination
  3. otherwise, try negation ¬-introduction
- Generally:
  - elimination rules for premises, introduction rules for conclusion

- We extend these rules to predicate calculus proofs
- Formulas in predicate calculus arguments are:
  - universal (∀-quantified), existential (∃-quantified)
    or unquantified (about individuals)

Rules of thumb

- When the only quantifiers in the premises are universal…

Premises	Conclusion	Then use…
universal, unquantified	unquantified	∀-elimination to infer properties of individuals, then follow golden rules
	existential	∀-elimination to infer properties of individuals, golden rules to prove conclusion for an individual, then ∃-introduction
universal	universal	∀-elimination to infer properties for arbitrary a, golden rules to prove conclusion for a, then ∀-introduction

- In summary:
  - ∀-elimination first, then prove conclusion
- A common pattern is:
  - ∀-elimination then ∀-introduction

Proof strategies

- When there are existential quantifiers in the premises…

<table>
<tr><th>Premises</th><th>First…</th><th>Conclusion</th><th>Then…</th></tr>
<tr><td>existential</td><td>Assume a “typical” disjunct of the existential for arbitrary a</td><td rowspan="2">existential
or
universal
or
…</td><td rowspan="2">Golden rules to prove conclusion for a (e.g. ∃–introduction or ∀–introduction), then ∃–elimination</td></tr>
<tr><td>existential, universal</td><td>Assume a “typical” disjunct of the existential for arbitrary a, ∀–elimination to infer further properties of a</td></tr>
</table>

- Note:
  - proof of conclusion is inside ∃–elimination sub-proof
- A common pattern is:
  - ∃–introduction inside ∃–elimination

Examples

1. $\forall x\,[\,D(x) \vee B(x)\,],\ \forall x\,[\,D(x) \rightarrow B(x)\,]\ :\ \forall x\,[\,B(x)\,]$

2. $\forall y\,[\,G(y) \rightarrow H(y)\,]\ :\ \exists x\,[\,G(x)\,] \rightarrow \exists z\,[\,H(z)\,]$

3. $\exists x\,[\,\neg F(x)\,] \vee \exists y\,[\,G(y)\,]\ :\ \exists z\,[\,F(z) \rightarrow G(z)\,]$

Proving versus disproving

- So far: asked to prove arguments/theorems known to be valid
  - what if we do not know if it is valid?
- Example (propositional logic):
  - prove or disprove the theorem:
  - : $(P \rightarrow Q) \vee (\neg Q \rightarrow \neg P)$
- How do we answer this?
  - first step: decide if it is valid or not
  - second step: use appropriate technique to prove/disprove
- How to disprove a theorem? (i.e. how to prove invalid/false)
  - we need a counterexample
  - (the same is true for disproving arguments, or equivalences)

Counterexamples (propositional logic)

- For propositional logic
  - a counterexample is a truth valuation for atomic propositions which makes the formula false
  - and we can search for it using a truth table

- Example: Prove or disprove the theorem:
  - : $(P \rightarrow Q) \vee (\neg Q \rightarrow \neg P)$
  - counterexample: P=true, Q=false

Counterexamples (predicate calculus)

- For predicate calculus
  - a counterexample is an interpretation for (some domain) which makes the formula false
- Example: Prove or disprove the theorem:
  - : $(\exists x\ [\ F(x)\ ] \lor \exists x\ [\ G(x)\ ]) \rightarrow (\exists x\ [\ F(x) \land G(x)\ ])$
  - counterexample: single constant a where $F(a)=\text{true}$, $G(a)=\text{false}$
- What about arguments?
  - to disprove, show counterexample that makes the premises true and the conclusion false
- What about equivalences?
  - to prove $A \equiv B$, need to (separately) prove: $A \vdash B$ and $B \vdash A$
  - to disprove, disprove either $A \vdash B$ or $B \vdash A$