5. Natural Deduction



Language & Logic

Dave Parker

University of Birmingham 2017/18

Recap

- Notation for arguments (in sequent form)
 - P1, P2 : C
 - P1, P2 \vdash C (if proven to be valid)
 - where P1, P2, C are formula in propositional logic
- Proofs (of argument validity)
 - using natural deduction
 - with a given set of inference rules (syntactic transforms)
 - so far: list of premise introductions & inference rule applications
 - not necessary unique (nor easy to construct)
 - but easy to check, rigorously once, constructed
 - fixed proof notation

Proof for Question 5 from Exercise sheet 2

$\neg P \to \neg Q, P \to Z, \neg \neg Q : Z$			
$1. \neg P \rightarrow \neg Q$ $2. \neg \neg Q$ $3. \neg \neg P$ $4. P$ $5. P \rightarrow Z$	Premise Premise Modus Tollens _{1,2} Double Negative Elimination ₃ Premise	$\{1\}$ $\{2\}$ $\{1,2\}$ $\{1,2\}$ $\{5\}$	
6. Z	Modus Ponens _{5,4}	$\{1,2,5\}$	

- There are variants, but we will stick largely to the above
 - perhaps with abbreviations for rule annotations

Proof for Question 5 from Exercise sheet 2

$\neg P \to \neg Q, P \to Z, \neg \neg Q : Z$			
1.	$\neg P \to \neg Q$	Premise	{1}
2.	$\neg \neg Q$	Premise	$\{2\}$
3.	$\neg \neg P$	$Modus Tollens_{1,2}$	$\{1,\!2\}$
4.	P	Double Negative Elimination ₃	$\{1,\!2\}$
5.	$P \to Z$	Premise	{5}
6.	Z	Modus Ponens _{5,4}	$\{1,2,5\}$

Lines of propositions used in the inference rule

Lines of all premises on which this depends

Proof for Question 5 from Exercise sheet 2

$\neg P \to \neg Q, P \to Z, \neg \neg Q : Z$	
1. $\neg P \rightarrow \neg Q$ Premise 2. $\neg Q$ Premise 3. $\neg P$ Modus Tollens _{1,2} 4. P Double Negative Elimination ₃ 5. $P \rightarrow Z$ Premise 6. Z Modus Ponens _{5,4}	{1} {2} {1,2} {1,2} {5} {5} {1,2,5}

Inference rules can be applied to any propositions (including compounds)

Proof for Question 5 from Exercise sheet 2

$\neg P \to \neg Q, P \to Z, \neg \neg Q : Z$			
1. $\neg P \rightarrow \neg Q$	Premise	{1}	
$2. \neg \neg Q$	Premise	$\{2\}$	
$3. \neg \neg P$	$Modus Tollens_{1,2}$	$\{1,\!2\}$	
4. P	Double Negative Elimination ₃	$\{1,\!2\}$	
5. $P \rightarrow Z$	Premise	$\{5\}$	
6. Z	Modus Ponens _{5,4}	$\{1,2,5\}$	

Don't skip steps: every line should be the application of <u>one</u> inference rule

Proof strategies

- How do we go about constructing a proof?
- Simple examples so far
 - look at the available propositions and rules to manipulate them
 - work backwards from the conclusion
 - or forwards from the premises
 - look at the main connective of each proposition

Example

Proof for Question 2 from Exercise sheet 2

Today

- This lecture (and the next):
 - more systematic list of inference rules
 - more complex inference rules & proof techniques
 - and strategies to apply them

Inference rules

- We will always assume a fixed set of inference rules
- Previously, used an arbitrary set:
 - Modus Ponens & Modus Tollens
 - double negative elimination
 - − ∧-introduction & ∧-elimination
- Time to be systematic...
 - 4 connectives $(\neg, \land, \lor, \rightarrow)$
 - each one has rules for introduction and elimination
 - which introduce and eliminate the connective, respectively
 - e.g. to form a conclusion or break up a premise

Introduction & Elimination

Conjunction

- − ∧-introduction
- − ∧-elimination

Disjunction

- − ∨-introduction
- − ∨-elimination

Implication

- →-introduction
- →-elimination

Negation

- − ¬-introduction
- ¬¬−elimination

Introduction & Elimination

Conjunction

- − ∧-introduction already seen
- − ∧-elimination already seen

Disjunction

- − ∨-introduction
- − ∨-elimination

Implication

- →-introduction
- →-elimination already seen (Modus Ponens)

Negation

- − ¬-introduction
- $\neg \neg$ -elimination already seen (double negative elimination)



Introduction & Elimination

Conjunction

- − ∧-introduction already seen
- − ∧-elimination already seen

Disjunction

- − ∨-introduction new
- − ∨-elimination new (case analysis)

Implication

- →-introduction new (conditional proof)
- →-elimination already seen (Modus Ponens)

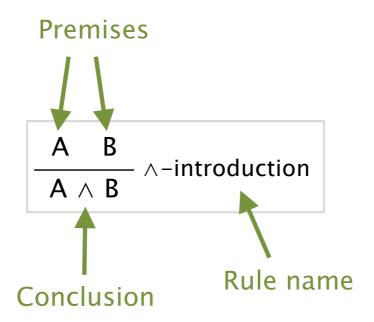
Negation

- ¬-introduction new (reductio ad absurdum)
- $\neg \neg$ -elimination already seen (double negative elimination)

But <u>no</u> Modus Tollens

Inference rules

• From now now, we now switch to a more common style for presenting inference rules:



Inference rules seen so far

Conjunction (∧)

$$\begin{array}{c|c} A & B \\ \hline A \wedge B & \\ \hline \end{array} \wedge \text{-introduction}$$

$$\frac{\begin{array}{c} A \wedge B \\ \hline A \end{array}}{\wedge -elimination}$$

$$\frac{A \wedge B}{B} \wedge \text{-elimination}$$

- →-elimination
 - (sometimes called "Modus Ponens")

$$\frac{A \rightarrow B \quad A}{B} \rightarrow -elimination$$

• ¬¬-elimination(sometimes abbreviated to DNE)

$$\frac{\neg\neg A}{A}$$
 $\neg\neg$ -elimination

∨-Introduction

∨ -introduction

$$\frac{A}{A \vee B} \vee -introduction$$

$$\frac{A}{B \vee A} \vee -introduction$$

for any formula B

- Simple, but slightly counterintuitive
 - we'll illustrate some uses of it later
- Proof notation
 - same as inference rules seen so far
 - (single) proof line index is added to rule annotation
 - and premise dependencies are copied from that line

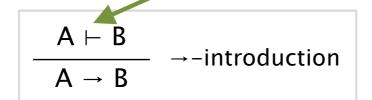
Example (V-Introduction)

• P, P \rightarrow Q : (P \wedge Q) \vee R

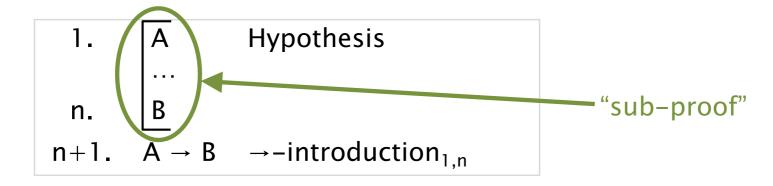
→-introduction

- Allows us to prove an implication A → B
 - also known as conditional proof
 - A is the antecedent
 - B is the consequent

Recall: ⊢ means "if we assume the LHS, we can prove the RHS"



- New proof technique:
 - make a hypothesis A (also called an assumption)
 - temporarily assume that A is true, and prove B (a sub-proof)
 - infer A → B and discharge the hypothesis



Example (conditional proof)

•
$$P \rightarrow Q, Q \rightarrow R : P \rightarrow (Q \land R)$$

- 1. $P \rightarrow Q$ Premise 2. $Q \rightarrow R$ Premise 3. P Hypothesis 4. Q \rightarrow -elimination_{1,3} 5. R \rightarrow -elimination_{2,4} 6. $Q \wedge R$ \wedge -introduction_{4,5} 7. $P \rightarrow (Q \wedge R)$ \rightarrow -introduction_{3,6}
- Dependencies for hypotheses done the same as for premises

 {1,3}

 {1,2,3}

 {1,2,3}

 Dependency on hypothesis removed when

Notes:

- lines around hypothesis + sub-proof
- sub-proof can use propositions proved outside, but <u>not</u> vice-versa
- cannot complete the proof while inside the sub-proof!

"discharged"

Example (nested conditional)

• $(P \land Q) \rightarrow R : P \rightarrow (Q \rightarrow R)$

∨-Elimination

Allows us to use a premise whose main connective is

$$\frac{A \lor B \quad A \vdash C \quad B \vdash C}{C} \quad \lor \text{-elimination}$$

- Proof technique
 - two separate sub-proofs,
 one assuming B
 - show that the same proposition C can be deduced from both
 - "case analysis"

1.	$A \vee B$	
2.	A	Hypothesis
n.	С	
n. n+1.	В	Hypothesis
m.	С	
m+1.	C	\vee -elimination _{1,2,n,n+1,m}

Example (\tau-elimination)

• $P \lor Q : Q \lor P$

1.	$P \vee Q$	Premise	{1}
2.	ГР	Hypothesis	{2}
3.	$Q \vee P$	∨-introduction	{2}
4.	Q	Hypothesis	{4 }
5.	$Q \vee P$	∨-introduction	{4}
6.	$\overline{Q} \vee P$	∨-elimination _{1,2,3,4,5}	{1}

Note:

- there are 5 line number indices for the rule annotation
- the dependencies for line 6 are derived by <u>combining</u> the dependencies for lines 1, 3 and 5, then removing the dependencies for lines 2 and 4

\neg -introduction

- Allows us to prove a negation ¬A
 - to prove $\neg A$, first assume that A is true...
 - and then show it leads to a contradiction
 - proof by contradiction, or reductio ad absurdum (RAA)
- A contradiction is any proposition of the form $B \wedge \neg B$
 - we use the symbol \perp ("bottom") to denote any such contradiction

$$A \vdash \bot$$
 \neg -introduction

- Alternatively...
 - to prove C, assume $\neg C$, then prove a contradiction
 - (in fact, the rule above would yield $\neg\neg C$, from which we infer C)

Example (\neg -introduction)

•
$$P \rightarrow Q$$
, $P \rightarrow \neg Q$: $\neg P$

1.

$$P$$
 Hypothesis
 {1}

 2.
 $P \rightarrow Q$
 Premise
 {2}

 3.
 Q
 \rightarrow -elimination_{2,1}
 {1,2}

 4.
 $P \rightarrow \neg Q$
 Premise
 {4}

 5.
 $\neg Q$
 \rightarrow -elimination_{4,1}
 {1,4}

 6.
 \bot
 \land -introduction_{3,5}
 {1,2,4}

 7.
 $\neg P$
 \neg -introduction_{1,6}
 {2,4}

This is will very often be the last inference rule used, to form a contradiction

Dependency on hypothesis removed at this point

Example (\neg -introduction)

• $\neg (P \land \neg Q) : P \rightarrow Q$

Summary

- Inference rules
 - introduction and elimination rules for each connective
- New rules
 - − ∨-introduction
 - →-introduction, i.e. conditional proof
 - − ∨-elimination, i.e. case analysis
 - − ¬-introduction, i.e. reduction ad absurdum
- New proof techniques/concepts
 - hypotheses (assumptions), sub-proofs & discharging
 - proof by contradiction
- Exercise class on Tue/Thur
 - practice proofs using these new rules