6. Natural Deduction (continued)

Language & Logic

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Module syllabus

- Syntax of formal & natural languages
  - grammars, parsing
- Propositional logic
  - truth tables, semantics, proofs via natural deduction
- Predicate calculus
  - proofs via natural deduction
- Program correctness
  - structural induction

Recap

- Natural deduction for propositional logic
  - introduction and elimination rules for each connective

- New rules
  - $\vee$-introduction
  - $\rightarrow$-introduction, i.e. conditional proof
  - $\vee$-elimination, i.e. case analysis
  - $\neg$-introduction, i.e. proof by contradiction

- New proof techniques/concepts
  - sub-proofs, hypotheses (assumptions) & discharging

Today

- Proof strategies
  - i.e. which inference rules to apply and when
- Primitive vs. derived inference rules
  - i.e. which rules do we assume and which can we prove?
- Theorems & theorem introduction
- Soundness and completeness

Basic proof strategy

- Choose the right inference rules
  - use introduction rules to add connectives to the conclusion
  - use elimination rules to work with premises
  - only apply them to the main connective

- Plan the proof
  - work backwards from the conclusion
  - and forwards from the premises
  - look carefully at the structure of the propositions
  - e.g. proving $Q \rightarrow (P \rightarrow R)$ vs. $Q \wedge (P \wedge R)$

- And remember each sub-proof changes the context…

Proof strategy: Rules of thumb

- Some rules of thumb (so-called “Golden Rules”) for proofs:

1. If the main connective in the conclusion is an implication, use implication introduction
   - i.e., assume its antecedent and try to prove the consequent
2. If the main connective in any of the premises is a disjunction, try to use disjunction elimination
   - i.e., assume each disjunct separately and try to prove the same conclusion with both
3. If all else fails, try negation introduction
   - i.e., assume the opposite of what you want to prove and deduce a contradiction

Example

- $\neg(P \wedge (\neg Q \vee R)) \ : \ P \rightarrow Q$

1.	P	Hypothesis	$\{1\}$
2.	$\neg Q$	Hypothesis	$\{2\}$
3.	$\neg Q \vee R$	$\vee$–introduction$_{2}$	$\{2\}$
4.	$P \wedge (\neg Q \vee R))$	$\wedge$–introduction$_{1,3}$	$\{1,2\}$
5.	$\neg(P \wedge (\neg Q \vee R))$	Premise	$\{5\}$
6.	$\perp$	$\wedge$–introduction$_{4,5}$	$\{1,2,5\}$
7.	$\neg\neg Q$	$\neg$–introduction$_{2,6}$	$\{1,5\}$
8.	Q	$\neg\neg$–elimination$_{7}$	$\{1,5\}$
9.	$P \rightarrow Q$	$\rightarrow$–introduction$_{1,8}$	$\{5\}$

Inference rules for propositional logic

$$\frac{A \quad B}{A \wedge B}\ \wedge\text{-introduction}$$

$$\frac{A \wedge B}{A}\ \wedge\text{-elimination}$$

$$\frac{A}{A \vee B}\ \vee\text{-introduction}$$

$$\frac{A \vee B \quad A \vdash C \quad B \vdash C}{C}\ \vee\text{-elimination}$$

$$\frac{A \vdash B}{A \rightarrow B}\ \rightarrow\text{-introduction}$$

$$\frac{A \rightarrow B \quad A}{B}\ \rightarrow\text{-elimination}$$

$$\frac{A \vdash \perp}{\neg A}\ \neg\text{-introduction}$$

$$\frac{\neg\neg A}{A}\ \neg\neg\text{-elimination}$$

(symmetric rules omitted for clarity)

Inference rules

- Which inference rules do we need to prove any valid argument? And do we need all of them?

- Primitive vs derived inference rules
  - primitive inference rules allow us to prove things that we would not be able to without them
  - derived rules do not allow us to prove anything new
  - and can be proved correct using primitive ones

- For example:

$$\frac{A \rightarrow B \quad \neg B}{\neg A}$$ "modus tollens"

Some derived inference rules

$$\frac{A \to B \quad \neg B}{\neg A}$$ "modus tollens"

$$\frac{A}{\neg\neg A}$$ $\neg\neg$-introduction

$$\frac{A \vee B \quad \neg A}{B}$$ "disjunctive syllogism"

$$\frac{\perp}{A}$$ "$\perp$ proves anything"

Proof (disjunctive syllogism)

- $A \vee B, \neg A \; : \; B$

1.	$A \vee B$	Premise	{1}
2.	$\neg A$	Premise	{2}
3.	A	Hypothesis	{3}
4.	$\neg B$	Hypothesis	{4}
5.	$\perp$	$\wedge$-introduction$_{2,3}$	{2,3}
6.	$\neg\neg B$	$\neg$-introduction$_{4,5}$	{2}
7.	B	$\neg\neg$-elimination$_{6}$	{2}
8.	B	Hypothesis	{8}
9.	B	$\vee$-elimination$_{1,3,7,8,8}$	{1,2}

Hypothesis not used at all

Theorems

- A theorem is a formula (a proposition) which can be proved
  - i.e., a valid argument with no premises

- Examples of theorems
  - $\vdash P \rightarrow P$
  - $\vdash P \vee \neg P$ ("Law of Excluded Middle")

- We can rewrite valid arguments as theorems
  - $\vdash ((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$ ("modus tollens")
  - $\vdash ((A \vee B) \wedge \neg A) \rightarrow B$ ("disjunctive syllogism")

- Closely related: a tautology is a formula that is always true
  - i.e., is true for any possible truth valuation (of atomic propositions)

Theorem introduction

- In general (exercises, exam questions, etc.)
  - use only the inference rules you are given/allowed
  - so what use are derived inference rules?

- Theorem introduction
  - insert a known theorem directly into a proof
  - may yield shorter, clearer or more elegant proofs
  - and provide a separate proof of the theorem (e.g. in exam)

- Using theorem introduction
  - apply a uniform substitution (like when applying inference rules)
  - e.g., if we know: $\vdash ((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$ (modus tollens)
  - we can insert, e.g.: $(((P \wedge Q) \rightarrow R) \wedge \neg R) \rightarrow \neg(P \wedge Q)$

Example

- Prove: $P \lor \neg Q,\ P \to R,\ S \to Q \ \vdash\ R \lor \neg S$

1.	$P \lor \neg Q$	Premise	{1}
2.	P	Hypothesis	{2}
3.	$P \to R$	Premise	{3}
4.	R	$\to$-elimination$_{3,2}$	{2,3}
5.	$R \lor \neg S$	$\lor$-introduction$_{4}$	{2,3}
6.	$\neg Q$	Hypothesis	{6}
7.	$S \to Q$	Premise	{7}
8.	$((S \to Q) \land \neg Q) \to \neg S$	Theorem (modus tollens)	
9.	$(S \to Q) \land \neg Q$	$\land$-introduction$_{7,6}$	{6,7}
10.	$\neg S$	$\to$-elimination$_{9,8}$	{6,7}
11.	$R \lor \neg S$	$\lor$-introduction$_{10}$	{6,7}
12.	$R \lor \neg S$	$\lor$-elimination$_{1,2,5,6,11}$	{1,3,7}

Using:
$\vdash ((A \to B) \land \neg B) \to \neg A$

Proofs

- Theorems
  - useful to know, even just to confirm validity or to guide the proof

- Equivalences
  - similarly, equivalences can be very helpful, e.g.:
  - $\neg(A \land B) \equiv \neg A \lor \neg B$
  - $\neg(A \lor B) \equiv \neg A \land \neg B$
  - $A \rightarrow B \equiv \neg A \lor B$

- Natural deduction for propositional logic (using the set of inference rules) is sound and complete
  - soundness means, if we can prove an argument, then it is valid
  - completeness means, if it is valid, then we can prove it

Semantic and syntactic validity

- We've seen two methods to show validity of an argument
  - truth table construction & analysis (semantic validity)
  - natural deduction proofs (syntactic validity)
- Semantic validity
  - an argument is valid iff it cannot be the case that all the premises are true and the conclusion false at the same time
- Syntactic validity
  - an argument is valid iff the conclusion can be derived from the premises by means of stipulated rules of inference
- Notation
  - semantic validity: $A \rightarrow B, \neg B \models \neg A$
  - syntactic validity: $A \rightarrow B, \neg B \vdash \neg A$

Soundness & completeness

- Soundness
  - any syntactically valid argument is semantically valid
  - i.e. $A_1, \ldots, A_n \vdash B$ implies $A_1, \ldots, A_n \vDash B$
  - (alternatively: all theorems are tautologies)

- Completeness
  - $A_1, \ldots, A_n \vDash B$ implies $A_1, \ldots, A_n \vdash B$
  - i.e., we can construct a proof of any valid argument in propositional logic using natural deduction
  - (alternatively: all tautologies are theorems)

Summary

- Proof strategies: rules of thumb
  - if there is a → in the conclusion, try →-introduction
  - if there is a premise of the form A∨B, try ∨-elimination
  - otherwise, try ¬-introduction

- Further concepts of natural deduction/propositional logic
  - primitive vs. derived inference rules
  - theorems (and tautologies)
  - soundness and completeness

- Practice / revision
  - exercise classes on Tue/Thur: practice on some harder proofs
  - see also examples in, e.g., Tomassi book