Exercise Class 4 – Solutions
More Natural Deduction for Propositional Logic

Model solutions for the required proofs are given below, although these are not unique (and not necessarily the best or the shortest proofs possible!). Also included are some explanations of the ideas behind the proofs. If you're stuck on a question, trying reading part or all of the explanation first and try to derive the proof yourself. If you're still stuck, the explanation should guide you through what is done.

1. The argument is:

$$P \to Q, Q \to R \ : \ P \to (Q \wedge R)$$

Since the conclusion is an implication, we use the $\to$-introduction rule, with the hypothesis being the antecedent (P) of the implication. We can then prove $Q \wedge R$ using $\to$-elimination and $\wedge$-introduction.

A proof of validity is:

1.	P	Hypothesis	{1}
2.	$P \to Q$	Premise	{2}
3.	Q	$\to$-elimination$_{2,1}$	{1,2}
4.	$Q \to R$	Premise	{4}
5.	R	$\to$-elimination$_{4,3}$	{1,2,4}
6.	$Q \wedge R$	$\wedge$-introduction$_{3,5}$	{1,2,4}
7.	$P \to (Q \wedge R)$	$\to$-introduction$_{1,6}$	{2,4}

2. The argument is:

$$: \ ((P \to Q) \wedge \neg Q) \to \neg P$$

This is a theorem, so there are no premises. But we proceed as usual. As above, the conclusion is an implication, so we start with the $\to$-introduction rule and, since we then need to prove a negation ($\neg P$), we use $\neg$-introduction. Notice that we end up with an empty set of premise dependencies (because there are no premises).

A proof of validity is:

1.	$(P \to Q) \wedge \neg Q$	Hypothesis	{1}
2.	P	Hypothesis	{2}
3.	$P \to Q$	$\wedge$-elimination$_1$	{1}
4.	Q	$\to$-elimination$_{3,2}$	{1,2}
5.	$\neg Q$	$\wedge$-elimination$_1$	{1}
6.	$\perp$	$\wedge$-introduction$_{4,5}$	{1,2}
7.	$\neg P$	$\neg$-introduction$_{2,6}$	{1}
8.	$((P \to Q) \wedge \neg Q) \to \neg P$	$\to$-introduction$_{1,7}$	{}

3. The argument is:

$$\neg R, P \to Q, R \to \neg Q, P \lor R \ : \ Q$$

The final premise is a disjunction, $P \lor R$, so we start with $\lor$-elimination. Intuitively, we see that, if the left-hand side is true, we're able to deduce the conclusion easily using the second premise. The right-hand side is actually impossible (since premise 1 also tells us that R is false). This suggests to use proof by contradiction for that case. Notice also that this means the third premise is irrelevant (and therefore is not found in the final list of dependencies).

A proof of validity is:

1.	$P \lor R$	Premise	{1}
2.	P	Hypothesis	{2}
3.	$P \to Q$	Premise	{3}
4.	Q	$\to$-elimination$_{3,2}$	{2,3}
5.	R	Hypothesis	{5}
6.	$\neg Q$	Hypothesis	{6}
7.	$\neg R$	Premise	{7}
8.	$\bot$	$\land$-introduction$_{5,7}$	{5,7}
9.	$\neg\neg Q$	$\neg$-introduction$_{6,8}$	{5,7}
10.	Q	$\neg\neg$-elimination$_{9}$	{5,7}
11.	Q	$\lor$-elimination$_{1,2,4,5,10}$	{1,3,7}

4. The argument is:

$$: \ ((P \lor Q) \land (P \lor R)) \to (P \lor (Q \land R))$$

We start with a hypothesis taken from the antecedent of the implication we are trying to prove and, since this is a conjunction, break it into two propositions. Since we now have disjunctions in the premises, and no obvious strategy for the conclusion, we use $\lor$-elimination on the premises, splitting our proof into the different possible cases. Since there are two disjunctions, there are four potential cases, which we can deal with using nested instances of $\lor$-elimination. In fact, when P is true, we don't care whether Q or R is true, so we don't actually need to consider all 4 cases. Our desired conclusion is $P \lor (Q \land R)$ and a good strategy here is to try and prove that full proposition in all cases, adding an extra disjunct where needed using $\lor$-introduction

A proof of validity is:

1.	$(P \lor Q) \land (P \lor R)$	Hypothesis	{1}
2.	$P \lor Q$	$\land$-elimination$_1$	{1}
3.	$P \lor R$	$\land$-elimination$_1$	{1}
4.	P	Hypothesis	{4}
5.	$P \lor (Q \land R)$	$\lor$-introduction$_4$	{4}
6.	Q	Hypothesis	{6}
7.	P	Hypothesis	{7}
8.	$P \lor (Q \land R)$	$\lor$-introduction$_7$	{7}
9.	R	Hypothesis	{9}
10.	$Q \land R$	$\land$-introduction$_{6,9}$	{6,9}
11.	$P \lor (Q \land R)$	$\lor$-introduction$_{10}$	{6,9}
12.	$P \lor (Q \land R)$	$\lor$-elimination$_{3,7,8,9,11}$	{1,6}
13.	$P \lor (Q \land R)$	$\lor$-elimination$_{2,4,5,6,12}$	{1}
14.	$((P \lor Q) \land (P \lor R)) \to (P \lor (Q \land R))$	$\to$-introduction$_{1,13}$	{}

5. The argument is:

$$: P \vee \neg P$$

The Law of Excluded Middle ($P \vee \neg P$) is a very fundamental notion, but it's not an axiom – we can prove it using our standard inference rules. We use proof by contradiction, i.e., $\neg$-introduction, so we start by assuming that the theorem is false. and then aim to derive a contradiction. More precisely, we'll show that $\neg(P \vee \neg P)$ means that both $\neg P$ and P are true. We achieve that using two further instances of $\neg$-introduction.

A proof of validity is:

1.	$\neg(P \vee \neg P)$	Hypothesis	{1}
2.	$\neg P$	Hypothesis	{2}
3.	$P \vee \neg P$	$\vee$-introduction$_2$	{2}
4.	$\perp$	$\wedge$-introduction$_{3,1}$	{1,2}
5.	$\neg\neg P$	$\neg$-introduction$_{2,4}$	{1}
6.	P	$\neg\neg$-elimination$_5$	{1}
7.	P	Hypothesis	{7}
8.	$P \vee \neg P$	$\vee$-introduction$_7$	{7}
9.	$\perp$	$\wedge$-introduction$_{8,1}$	{1,7}
10.	$\neg P$	$\neg$-introduction$_{7,9}$	{1}
11.	$\perp$	$\wedge$-introduction$_{6,10}$	{1}
12.	$\neg\neg(P \vee \neg P)$	$\neg$-introduction$_{1,11}$	{}
13.	$P \vee \neg P$	$\neg\neg$-elimination$_{12}$	{}

You can also shorten this proof a little, by removing the second inner sub-proof, and getting a contradiction in an alternative manner. Can you see how?

6. The argument is:

$$P \to Q, \neg P \to Q \ : \ Q$$

In the previous question, we proved that $\vdash P \vee \neg P$. This means we can insert that fact into this proof using theorem introduction. After doing that, we get the result quite easily using $\vee$-elimination and $\to$-elimination.

A proof of validity is:

1.	$P \to Q$	Premise	{1}
2.	$\neg P \to Q$	Premise	{2}
3.	$P \vee \neg P$	Theorem introduction (LEM)	{}
4.	P	Hypothesis	{4}
5.	Q	$\to$-elimination$_{1,4}$	{1,4}
6.	$\neg P$	Hypothesis	{6}
7.	Q	$\to$-elimination$_{2,6}$	{2,6}
8.	Q	$\vee$-elimination$_{3,4,5,6,7}$	{1,2}

It's also not too hard to prove this directly, using several instances of $\neg$-introduction.