8. Proofs in Predicate Calculus

Language & Logic

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Today

- Recap: predicate calculus
  - syntax, translation, examples
- Next main topic: Proofs in predicate calculus
  - syntax vs. semantics
  - validity for arguments in predicate calculus
  - syntactic vs. semantic validity
  - soundness & completeness
  - natural deduction for predicate calculus: first steps
- See also Chap 6 (and some of Chap 5) of Tomassi book

Recap: Predicate calculus

- Key ingredients
  - variables x,y,z
  - constant symbols a,b,c
  - quantifiers: universal: $\forall$ ("for all"); existential: $\exists$ ("there exists")
  - predicates (unary, binary, n-ary, nullary)
  - e.g. P(x), P(a), L(m,x), …
- Extension of propositional logic
  - retains all usual connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$
- Translation from natural language
  - first establish what the domain is
  - then identify constants and predicates
  - templates for translating several common patterns

Ex 5 examples

- Examples Ex 5 Q1

(a) All white animals are mice
$\forall x[W(x) \to M(x)]$

- Common "template"

(b) Basil is a white mouse
$M(b) \wedge W(b)$

- No need for any quantifiers
- Re-use the same predicates, M(x) and W(x), here
- Can't write (M(W(b))!

- Examples Ex 5 Q2

(k) Everyone who loves Chris loves someone who loves John
$\forall x[L(x,c) \to \exists y[(L(x,y) \wedge L(y,j))]]$

- Same template as Q1(a), but more complex inner formula
- x and y must be distinct variables

Semantics: Interpretations

- Syntax (allowable sentences) vs. semantics (meaning)
- An interpretation is an assignment of meaning to the symbols of a formal language
  - usually provides a way to determine the truth value of a sentence
- Recall… In propositional logic:
  - sentence = proposition, e.g. $P \vee Q$
  - interpretation = truth assignment (i.e., an assignment of true/false to each atomic proposition)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Predicate calculus semantics

- How do we define the semantics of the predicate calculus?
  - i.e, what is the truth value of a sentence (formula)?
  - and what is the validity of an argument?

- We need 2 things:

1. a domain (also called "domain of discourse" or "universe")
   - a <u>non-empty</u> set of objects/entities
   - e.g. people, cars, program executions, the natural numbers, …
   - consider e.g. "all computer scientists are logical"
     - translated as $\forall x\ [\ L(x)\ ]$ or $\forall x\ [\ C(x) \rightarrow L(x)\ ]$?
2. an interpretation
   - defines the meaning of predicates in terms of the domain
   - is Alan logical? is he a computer scientist? etc.

Semantics of a formula

- Given a domain and an interpretation:
  - we can assign a truth value to each sentence/formula
- Example sentence (assuming the domain = people)
  - $\forall x\ [\ C(x) \rightarrow L(x)\ ]$
- An example interpretation:
  - constants:
    - a = Alan; b = Bella
  - predicates:
    - C(a) and C(b) are true
    - L(a) is false but L(b) is true
  - for this interpretation:
    - is $\forall x\ [\ C(x) \rightarrow L(x)\ ]$ true?
    - is $\exists x\ [\ C(x) \wedge L(x)\ ]$ true?

Validity of an argument

- As for propositional logic, we mostly care about arguments and their validity, rather than particular interpretations
  - e.g. $\forall x\ [\ C(x) \rightarrow L(x)\ ],\ C(a) : L(a)$ – is this valid?

- An argument in predicate logic is valid if and only if
  - for every possible domain and every possible interpretation, whenever the premises are all true, the conclusion is true

- An argument in predicate logic is invalid if and only if
  - for some domain, there is a possible interpretation under which all the premises are true and the conclusion is false

Example arguments

- (Taken from the quiz in week 1)

All men are mortal.

Some men are brave.

Therefore, some men are mortal and brave.

All men are brave.

No man is a philosopher.

Therefore, no philosopher is brave.

Semantic and syntactic validity

- For propositional logic, we saw two methods to show validity of an argument
  - truth table construction & analysis (semantic validity)
  - natural deduction proofs (syntactic validity)

- Semantic validity
  - an argument is valid iff, whenever all the premises are true, then the conclusion is also true
- Syntactic validity
  - an argument is valid iff the conclusion can be derived from the premises by means of stipulated rules of inference

- Notation
  - semantic validity: $A \rightarrow B, \neg B \models \neg A$
  - syntactic validity: $A \rightarrow B, \neg B \vdash \neg A$

Soundness & completeness

- Soundness
  - means any syntactically valid argument is semantically valid
  - i.e. $A_1, \ldots, A_n \vdash B$ implies $A_1, \ldots, A_n \vDash B$
  - means that our proofs are correct

- Completeness
  - $A_1, \ldots, A_n \vDash B$ implies $A_1, \ldots, A_n \vdash B$
  - means we can construct a proof of any valid argument

- Natural deduction
  - for propositional logic (and predicate calculus)
  - is sound and complete
  - which means we can prove <u>any</u> valid argument

Natural deduction

- How do we prove validity in the predicate calculus?
  - no way to enumerate all possible domains/interpretations
  - so use natural deduction (which, again, is sound and complete)

- Natural deduction for predicate calculus
  - extension of the case for propositional logic
  - i.e., we can use all the existing inference rules
  - (and all exist theorems hold)

- We add 4 new inference rules
  - $\forall$-elimination (universal-elimination)
  - $\exists$-introduction (existential-introduction)
  - $\forall$-introduction (universal-introduction)
  - $\exists$-elimination (existential-elimination)

∀-elimination

- Basic idea:
  - given a universal quantification, we can infer a particular instance

- Example

  All computer scientists are logical.

  Alan is a computer scientist.

  Therefore, Alan is logical.

- Argument $\forall x\ [\ C(x) \rightarrow L(x)\ ],\ C(a) : L(a)$

- Proof:

1.	$\forall x\ [\ C(x) \rightarrow L(x)\ ]$	Premise	{1}
2.	$C(a) \rightarrow L(a)$	$\forall$-elimination$_1$	{1}
3.	$C(a)$	Premise	{2}
4.	$L(a)$	$\rightarrow$-elimination$_{2,3}$	{1,2}

∀-elimination

- The inference rule:

$$\frac{\forall x\ [\ \phi(x)\ ]}{\phi(a)}\ \forall\text{-elimination}$$

- Notation (here and in the later rules):
  - $\phi(x)$ is a formula in which variable x appears
  - (x is called a free variable)
  - $\phi(a)$ is $\phi(x)$ with each instance of x replaced by constant a

- Intuition for ∀-elimination:
  - universal quantification is like a conjunction
  - $\forall x\ [\ P(x)\ ]\ \equiv\ P(a) \wedge P(b) \wedge P(c) \wedge \ldots$
  - so ∀-elimination is like ∧-elimination

∃-introduction

- Basic idea
  - from a particular instance, we can infer an existential quantification

- The inference rule:

$$\frac{\phi(a)}{\exists x\,[\ \phi(x)\ ]}\ \exists\text{-introduction}$$

  - where $\phi(x)$ and $\phi(a)$ are as defined for the previous rule

- Intuition
  - existential quantification is like a disjunction
  - $\exists x\,[\ P(x)\ ] \equiv P(a) \vee P(b) \vee P(c) \vee \ldots$
  - so ∃-introduction is similar to ∨-introduction

Example – ∃-introduction

- Is this argument valid?
  - $\forall x\,[\,P(x)\,] : \exists x\,[\,P(x)\,]$

- Yes (remember that domains are always non-empty)
- Here is the proof (using our 2 new inference rules):

1.	$\forall x\,[\,P(x)\,]$	Premise	{1}
2.	$P(a)$	$\forall$-elimination$_1$	{1}
3.	$\exists x\,[\,P(x)\,]$	$\exists$-introduction$_2$	{1}

- Question: Is this argument valid?
  - $\forall x\,[\,P(x) \rightarrow Q(x)\,] : \exists x\,[\,Q(x)\,]$

$\forall$-introduction

- The inference rule:

$$\frac{\phi(a)}{\forall x\ [\ \phi(x)\ ]}\ \forall\text{-introduction}$$

- But there are conditions…

- Under what circumstances can we infer this?
  - a needs to be a "typical" example of x
  - i.e., we must know nothing else about a

- To be more precise…
  - $\forall$-introduction can only be applied to a formula $\phi(a)$
    if a does not appear in any of the dependencies for $\phi(a)$,

Example – $\forall$-introduction

- Prove: $\forall x\,[\,P(x) \rightarrow Q(x)\,],\ \forall x\,[\,P(x)\,]\ :\ \forall x\,[\,Q(x)\,]$

1.	$\forall x\,[\,P(x) \rightarrow Q(x)\,]$	Premise	{1}
2.	$\forall x\,[\,P(x)\,]$	Premise	{2}
3.	$P(a) \rightarrow Q(a)$	$\forall$-elimination$_1$	{1}
4.	$P(a)$	$\forall$-elimination$_2$	{2}
5.	$Q(a)$	$\rightarrow$-elimination$_{3,4}$	{1,2}
6.	$\forall x\,[\,Q(x)\,]$	$\forall$-introduction$_5$	{1,2}

No use of a

Formula $\phi(a)$

Dependencies

- Note: this is a common pattern in predicate calculus proofs
  - $\forall$-elimination (for new symbol a) then $\forall$-introduction

$\forall$-introduction: Why the condition?

- Why do we need the condition on the a?

- An example: Is this argument valid?
  - $P(a), \neg P(b) \;:\; \forall x\,[\,P(x)\,]$

Dependencies

- No. Here is an (incorrect) proof

1.	$\boxed{P(a)}$	Premise	{1}
2.	$\forall x\,[\,P(x)\,]$	$\forall$-introduction$_1$	{1}

Contains a

- The condition for $\forall$-introduction is not satisfied
  - so the inference rule cannot be applied here

∃–elimination

- Intuition for ∃–elimination:
  - recall: existential quantification is like a disjunction
  - $\exists x\ [\ P(x)\ ] \equiv P(a) \vee P(b) \vee P(c) \vee \ldots$
  - so ∃–elimination is like ∨–elimination

$$\frac{\exists x\ [\ \phi(x)\ ] \quad \phi(a) \vdash C}{C}\ \exists\text{–elimination}$$

- Like for ∀–introduction
  - a needs to be a "typical" example of x
- So:
  - the constant a in $\phi(a)$ must <u>not</u> appear in C, nor in any of the dependencies of C (except the hypothesis $\phi(a)$ itself)

Example – ∃-elimination

- Prove: $\exists x\, [\, F(x) \wedge G(x)\,] \;:\; \exists x\, [\, F(x)\,]$

Formula $\phi(a)$

No use of a

No other dependencies (apart from $\phi(a)$)

1.	$\exists x\, [\, F(x) \wedge G(x)\,]$	Premise	{1}
2.	$\boxed{F(a) \wedge G(a)}$	Hypothesis	{2}
3.	$F(a)$	$\wedge$-elimination$_2$	{2}
4.	$\boxed{\exists x\, [\, F(x)\,]}$	$\exists$-introduction$_3$	{2}
5.	$\exists x\, [\, F(x)\,]$	$\exists$-elimination$_{1,2,4}$	{1}

- Note: this is also a common pattern in predicate calculus proofs
  - ∃-introduction inside ∃-elimination

Examples – ∃-elimination

- Prove: $\exists x\ [\ F(x) \wedge G(x)\ ] : \exists x\ [\ G(x) \wedge F(x)\ ]$

- Prove: $\forall x\ [\ G(x) \rightarrow G(b)\ ], \exists x\ [\ G(x)\ ]\ :\ G(b)$

Summary

- Natural deduction for predicate calculus
  - extends natural deduction for propositional logic

- $\forall$-elimination
  - infer a particular instance $\phi(a)$ from $\forall x\ [\ \phi(x)\ ]$
- $\exists$-introduction
  - infer $\exists x\ [\ \phi(x)\ ]$ from a particular instance $\phi(a)$
- $\forall$-introduction
  - infer $\forall x\ [\ \phi(x)\ ]$ from $\phi(a)$ for a "typical" a
- $\exists$-elimination
  - infer C from $\exists x\ [\ \phi(x)\ ]$ by inferring C for a "typical" a