4. Propositional Logic

Language & Logic

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Assignments

- Continuous assessment (20% of module)
  - 3 assignments in weeks 3, 6 and 10
    - worth 6%, %6 and 8%, respectively
- Assignment 1 out now, due Fri 13 Oct (5pm)
  - topic: grammars and truth tables
  - submission via Canvas
    - handwritten and scanned (via phone, tablet, scanner, etc.) or typeset (e.g. with Word, LibreOffice, Latex), PDF preferred
    - any problems – please ask
  - see solutions for Exercise Sheet 1 and practice quizzes

Overview

- Last lecture: a first formal look at logic
  - atomic propositions and connectives
  - constructing truth tables, for...
  - 1. semantics of connectives and propositions
  - 2. checking validity of arguments

- Today
  - propositional logic as a formal language
  - semantics of propositions and arguments
  - proof: introduction to natural deduction

Propositional logic

- Logical connectives:
  - negation: $\neg$ (not)
  - conjunction: $\wedge$ (and)
  - disjunction: $\vee$ (or)
  - implication: $\rightarrow$ (if then / implies)

Alternative notation (e.g., Tomassi book)

- negation: $\sim\phi$
- conjunction: $\phi \mathbin{\&} \psi$
- disjunction: $\phi \mathrm{v} \psi$

- Formulas (or "sentences") in propositional logic
  - if P is an atomic proposition, then P is a formula
  - if ϕ and ψ are formula, then these are all formulas:

    $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \rightarrow \psi$
  - these are atomic and compound formulas, respectively

- To be more precise, these are well-formed formulas

Propositional logic – syntax

- The syntax of propositional logic is specified by the grammar
  - F → Ap | [¬] F | F [∧] F | F [∨] F | F [→] F
  - Ap → [P] | [Q] | [R] | …

- Question
  - P ∧ Q ∨ R
  - is this a well-formed formula? what does it mean?

- Question
  - If Arachnids have eight legs then Crabs spin webs and Scorpions live underwater
  - how do we write this in propositional logic?

Avoiding ambiguity

- Use parentheses to avoid ambiguity
  - $(P \wedge Q) \vee R$ vs. $P \wedge (Q \vee R)$
  - $A \rightarrow (C \wedge S)$ vs. $(A \rightarrow C) \wedge S$

- General rule:
  - enclose all compound sub-formulas in parentheses

- Precedence
  - negation applies to the smallest formula following
  - e.g. $\neg P \vee Q$ means $(\neg P) \vee Q$ not $\neg(P \vee Q)$
  - but: we won't make assumptions about precedence of $\wedge$ and $\vee$

Some terminology

- The scope of a connective
  - the connective itself, plus what it connects
- The main connective of a formula
  - the connective whose scope is the whole formula
- Example
  - $(\neg(P \wedge Q)) \rightarrow (\neg P \vee \neg Q)$
- In terms of parse trees
  - scope = sub-tree
  - main connective = root node

Semantics of propositions

- We have used truth tables to illustrate the semantics (meaning) of propositions
  - i.e., the truth value (true or false) of the proposition for each possible truth assignment (to atomic propositions)

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- Properties of propositions
  - valid: true for every possible truth assignment
  - satisfiable: true for some possible truth assignment
  - unsatisfiable: false for every possible truth assignment
  - contingent: true for some assignment and false for another

- Note
  - don't confuse validity of propositions and arguments

Relationships between propositions

- Logical equivalence
  - $\phi \equiv \psi$
  - ϕ and ψ have the same truth value for every possible truth assignment
  - e.g. $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

- Logical entailment
  - $\phi \models \psi$
  - for every truth assignment where ϕ is true, ψ is also true
  - e.g. $P \wedge Q \models P$

P	Q	$P \wedge Q$	P
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

- Extends to arguments...

Arguments in propositional logic

- Example
  - Premise 1: $H \rightarrow O$
  - Premise 2: $\neg O$
  - Conclusion: $\neg H$

- Notation: written as a sequent
  - $H \rightarrow O, \neg O : \neg H$
  - i.e., comma-separated list of premises, colon, conclusion
  - (where premises/conclusion are formulas in propositional logic)

- If the argument is valid (and been proven to be), we write:
  - $H \rightarrow O, \neg O \vdash \neg H$
  - i.e., the : is replaced by the turnstile symbol $\vdash$ ("entails")

Proofs in propositional logic

- For formal proofs, we need two things:

1. A formal language
   - for representing propositions, arguments
   - for now, we are using propositional logic
2. A proof theory
   - to prove (“infer”, “deduce”) whether an argument is valid
   - first option (last week): truth tables
   - another option (next few weeks): natural deduction

- Why natural deduction?
  - scales better when there are many atomic propositions
  - can also be adapted to predicate logic

Natural deduction

- Natural deduction
  - "natural" style of constructing a proof (like a human would)
  - syntactic (rather than semantic) proof method
  - proofs are constructed by applying inference rules
  - showing that the conclusion can be inferred from the premises
- Some example inference rules (written as sequents):
  - Modus ponens: $P \rightarrow Q, P \vdash Q$
  - Modus tollens: $P \rightarrow Q, \neg Q \vdash \neg P$
  - $\wedge$-Introduction: $P, Q \vdash P \wedge Q$
  - $\wedge$-Elimination: $P \wedge Q \vdash P$
- Note:
  - the set of inference rules available for use may vary

Inference rules

- Inference rules need to be truth preserving
  - as shown before, we can check this with a truth table

- E.g. for modus tollens:
  - $P \rightarrow Q, \neg Q \vdash \neg P$

		P1	P2	C
P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Writing proofs

- A proof comprises a number of lines, each of which has
  - line number (consecutive)
  - formula (in propositional logic, for now)
  - rule annotation (which inference rule (or premise) used?)
  - dependency numbers (depends on which other lines?)
- Basic idea
  - work from the premises towards the conclusion
- Example
  - proof for $P, \neg Q \vdash P \wedge \neg Q$

1.	P	Premise	$\{1\}$
2.	$\neg Q$	Premise	$\{2\}$
3.	$P \wedge \neg Q$	$\wedge$-Introduction$_{1,2}$	$\{1,2\}$

Summary

- Propositional logic
  - atomic propositions, combined with logical connectives
  - formal language for expressing propositions
  - ambiguity, precedence, scope, main connective
  - semantics, equivalence, entailment

- Arguments
  - written as sequents
  - can be proved using natural deduction
  - sequence of inference rule applications

- Exercise class: Tue 11am or Thur 10am
  - topic: natural deduction