Exercise Class 6
Natural Deduction for Predicate Calculus

For this exercise class, we'll be writing natural deduction proofs for arguments expressed in predicate calculus. The inference rules available for use are shown overleaf: these include the 8 rules you have been using so far, and the 4 new ones for universal and existential quantification.

Construct a proof of validity for each of the arguments below. For arguments that are expressed in natural language, you first need to translate them into predicate calculus.

1. Warm up

   Alfie is a cat who is not black. Therefore not all cats are black.

   (assume the domain is animals)

2. A choice exists

   $$\forall x[F(x)] \;:\; \exists x[F(x) \vee G(x)]$$

3. Not necessarily true

   All cars are red. All red cars are fast. Therefore all cars are fast.

   (assume the domain is vehicles)

4. An example to them all

   $$\exists x[P(x) \to Q(x)], \forall x[P(x)] \;:\; \exists x[Q(x)]$$

5. Be careful

   $$\forall x[F(x) \to (H(x) \wedge J(x))],\; \forall x[\neg(H(x) \wedge J(x))] \;:\; \forall x[\neg F(x)]$$

6. Dualities

   $$\neg\exists x[\neg F(x)] \;:\; \forall x[F(x)]$$

7. Back to last week

   Everybody who loves everybody loves themself.

Inference Rules

Conjunction ($\wedge$)	**Disjunction ($\vee$)**
$\dfrac{A \quad B}{A \wedge B}$ $\wedge$-introduction $\dfrac{A \wedge B}{A}$ $\wedge$-elimination $\quad$ $\dfrac{A \wedge B}{B}$ $\wedge$-elimination	$\dfrac{A}{A \vee B}$ $\vee$-introduction $\quad$ $\dfrac{A}{B \vee A}$ $\vee$-introduction $\dfrac{A \vee B \quad A \vdash C \quad B \vdash C}{C}$ $\vee$-elimination
Implication ($\rightarrow$)	**Negation ($\neg$)**
$\dfrac{A \vdash B}{A \rightarrow B}$ $\rightarrow$-introduction $\dfrac{A \rightarrow B \quad A}{B}$ $\rightarrow$-elimination	$\dfrac{A \vdash \perp}{\neg A}$ $\neg$-introduction $\dfrac{\neg\neg A}{A}$ $\neg\neg$-elimination
Universal quantification ($\forall$)	**Existential quantification ($\exists$)**
$\dfrac{\phi(a)}{\forall x[\phi(x)]}$ $\forall$-introduction* $\dfrac{\forall x[\phi(x)]}{\phi(a)}$ $\forall$-elimination * For $\forall$-introduction, a must not occur in any dependency of $\phi(a)$.	$\dfrac{\phi(a)}{\exists x[\phi(x)]}$ $\exists$-introduction $\dfrac{\exists x[\phi(x)] \quad \phi(a) \vdash C}{C}$ $\exists$-elimination* *For $\exists$-elimination, a must not occur in C or any dependency of C except $\phi(a)$.