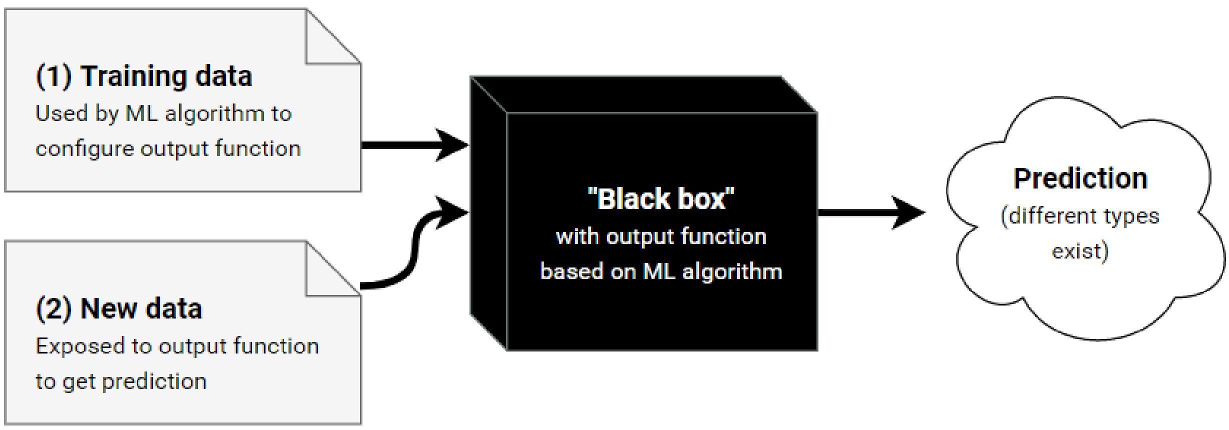
Lecture 2

Linear Regression Models



Cat Hearts example:

Experience E

- The dataset consists of n data points
  - $((x_1,y_1),...,(x_n,y_n))\in \R^d\times \R)$
  - $x_i \in \R^d$ is the *"input"* for the i^{th} data point as a feature vector with d elements, d being the # of dimensions in the feature space, in this case 1.
  - $y_i \in \R$ is the *"output"* for the i^{th} data point, in this case the weight of the corresponding cat heart.

Learning Task, T

- In this example, our task is: **Linear Regression**

- Find a *"model"*, i.e. a function:
    - $f : \R^d \rightarrow \R$
- s.t. our future observations produce output *"close to"* the true output.

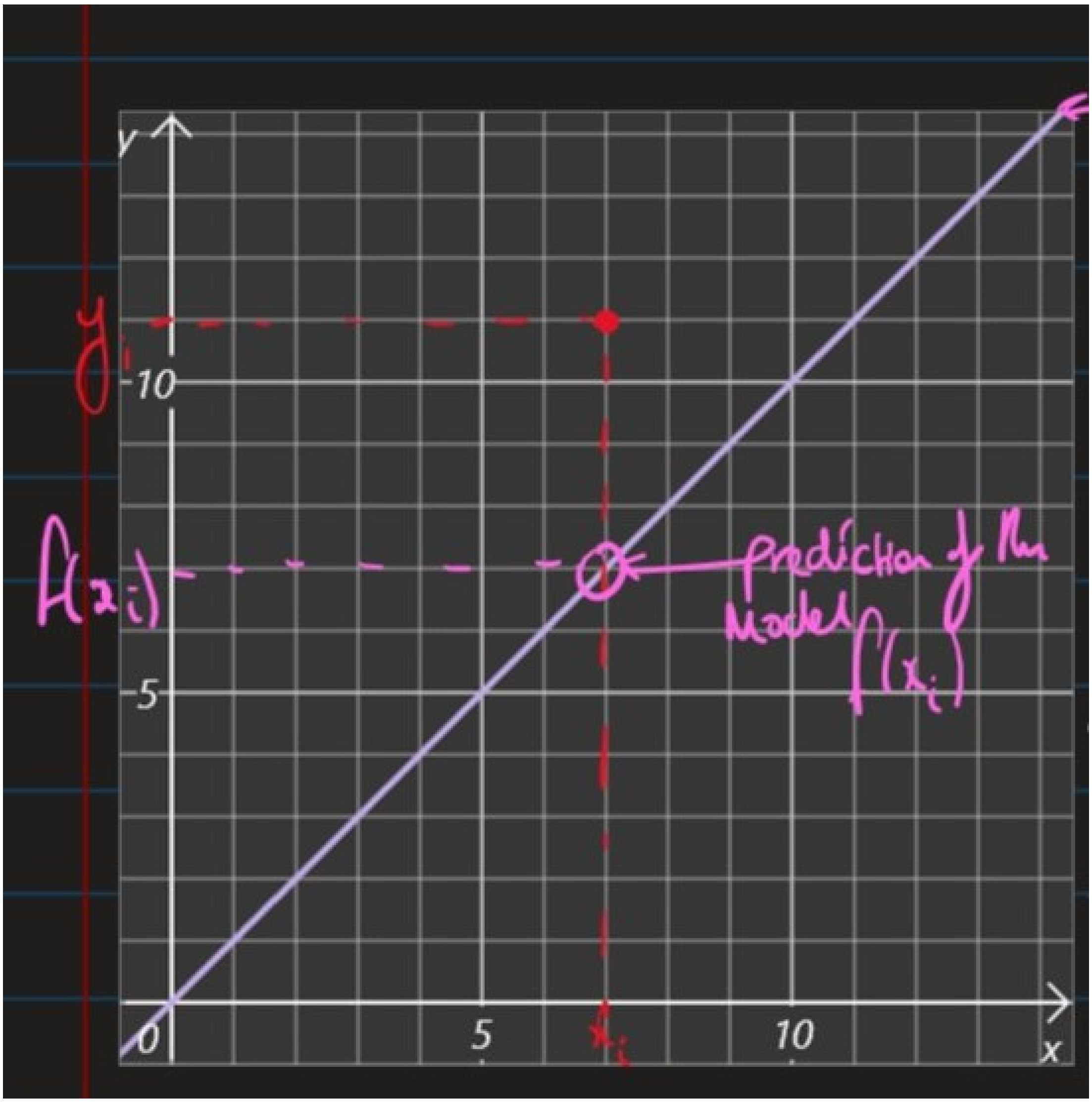
Linear Regression Model

- A linear regression model has the form:
    - $f(x) = (\sum_{i=1}^{d}w_i \cdot x_i)+ b$
    - where:
        - $x \in \R^d$ is the input vector (feature)
        - $w \in \R^d$ is the weight vector (parameters)
        - $b \in \R$ is a bias (parameter)
        - $f(x) \in \R$ is the predicted output

- In our cat example we have:
    - $d=1$ as "body weight" is our only feature
    - $b=0$ as from intuition we expect a cat of 0 weight to have a heart of 0 weight.
    - Our model has one parameter: w

Performance Measure, P

- Want a function, $J(w)$ which quantifies the error in the predictions for a given parameter w

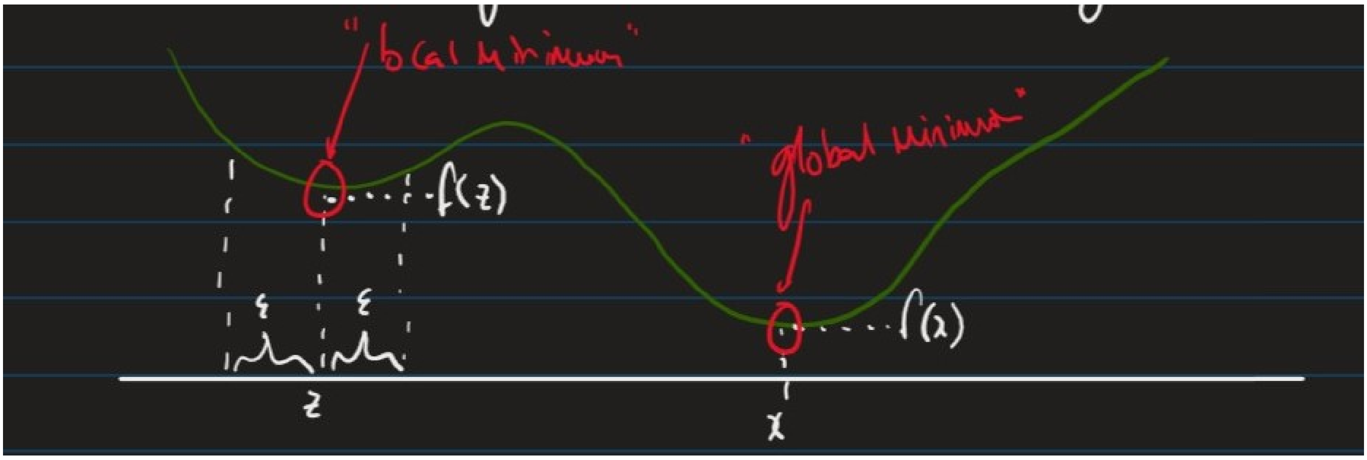


- The following empirical loss function, J takes into account the errors $\forall n$ data points.
  - $J(w) = (1/2N)\sum_{i=1}^N(y_i-wx_i)^2$
  - where the summation term is squared so that:
    - we ignore the sign
    - we penalise large errors more
- To find the optimum weight, solve:
  - $\frac{\delta J}{\delta w}$ = 0

Unconstrained Optimisation (Minimisation)

Given a continuous function:

- $f: \R^d \rightarrow \R$, as our *loss function*
- an element $x \in \R^d$ is called:
  - A **global** minimum of f iff:
    - $\forall y \in \R^d, f(x) \leq f(y) $
  - A **local** minimum of f iff:
    - $\exists \epsilon > 0, \forall y \in \R^d$ if $\forall i \in {1,...,d} , | x_i - y_i | < \epsilon \implies f(x) \leq f(y)$



Theorem: For any continous function, $f: \R \rightarrow \R$, if x is a local optimum, $f'(x) = 0$

Definition: The 1^{st} derivative of a function $f: \R \rightarrow \R$ is $f'(x) = \lim_{\Delta x \rightarrow 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$

Differentiation Rules

1. $(cf(x))' = cf'(x)$
2. $(x^k)'$ = kx^{k-1}, if $k \neq 0$
3. $(f(x)+g(x))' = f'(x) + g'(x)$
4. $(f(g(x))' = f'(g(x))g'(x))$ $\leftarrow$ **chain rule**

Approach 1: Ordinary least squares

- Optimise J by solving $J'(w) = 0$
  - $J(w) = \frac{1}{2N}\sum_{i=1}^N(y_i - wx_i)^2$
  - $J'(w) = \frac{1}{N}\sum_{i=1}^{N}(wx_i - y_i)x_i$
  - $J'(w) = 0$
    - $\frac{1}{N}\sum_{i=1}^{N}(wx_i - y_i)x_i = 0$
    - $w\sum_{i=1}^{N}(x_i)^2 = \sum_{i=1}^{N}x_iy_i$
    - $w = \frac{\sum_{i=1}^{N}x_iy_i}{\sum_{i=1}^{N}x_i^2}$
      - This only has one solution $\therefore$ a global minimum.

Approach 2: Gradient descent

- Often difficult / impossible to solve $J'(w) = 0$ for non-linear models with many parameters

Idea:

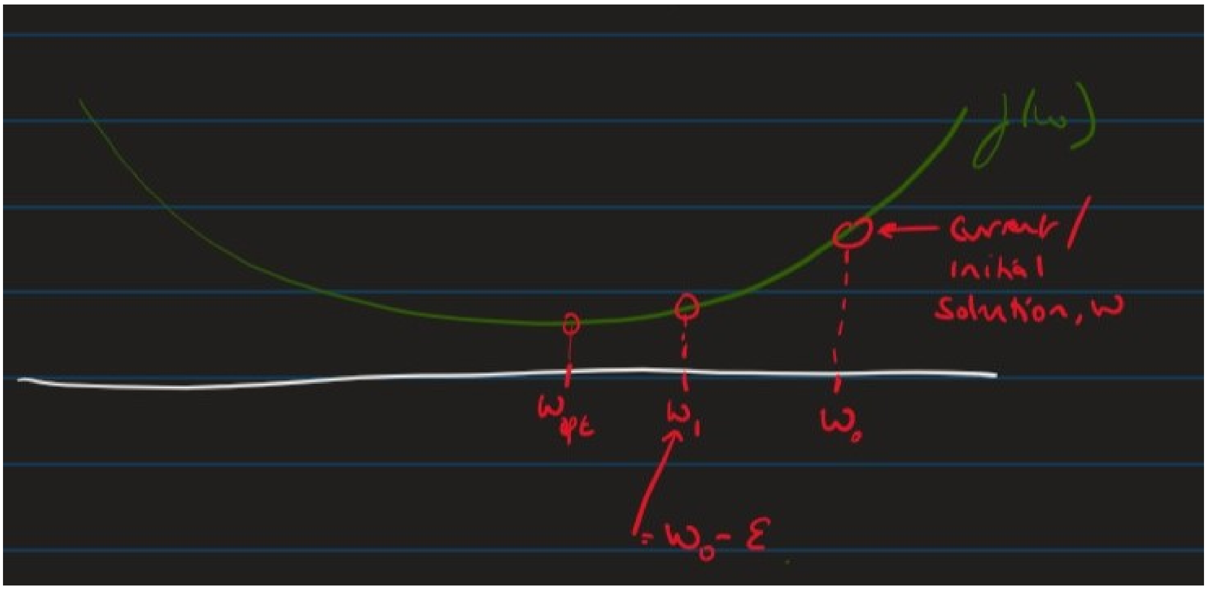
- Start with an initial guess

- While $J'(w) \neq 0$:

  - move *slightly* in the *right direction*

- To make this viable we need to define:

    - *"what is the right direction?"*
    - *"what is slightly?"*



Attempt 1 (failed)

$ w \leftarrow initial \ weight $ repeat: if $J'(w) < 0 $ $ w \leftarrow w + \epsilon $ elseif $J'(w) > 0$ $w \leftarrow w - \epsilon$

- where ϵ is the learning rate set manually. **(hyper-parameter)**

Issue with this attempt:

- w may oscillate in the interval $[w_{opt} - \epsilon, w_{opt}+ \epsilon]$
- w fails to converge

Attempt 2: Gradient Descent (1D)

$ w \leftarrow initial \ weight $ repeat: if $J'(w) < 0 $ $ w \leftarrow w - \epsilon \cdot J'(w)$