Exercise Class 3 – Solutions
Natural Deduction for Propositional Logic

1. The argument is:

$$P,\ P \to Q\ :\ P \wedge (Q \vee R)$$

Since there is a disjunction in the conclusion, we use the new $\vee$-introduction inference rule, combined with $\to$-elimination (modus ponens) and $\wedge$-introduction, which you saw previously.

A proof of validity is:

1.	P	Premise	{1}
2.	$P \to Q$	Premise	{2}
3.	Q	$\to$-elimination$_{2,1}$	{1,2}
4.	$Q \vee R$	$\vee$-introduction$_{3}$	{1,2}
5.	$P \wedge (Q \vee R)$	$\wedge$-introduction$_{1,4}$	{1,2}

2. The argument is:

$$P \to (Q \wedge R)\ :\ P \to Q$$

Since the conclusion is an implication, we use the $\to$-introduction rule, with the hypothesis being the antecedent (P) of the implication.

A proof of validity is:

1.	P	Hypothesis	{1}
2.	$P \to (Q \wedge R)$	Premise	{2}
3.	$Q \wedge R$	$\to$-elimination$_{2,1}$	{1,2}
4.	Q	$\wedge$-elimination$_{3}$	{1,2}
5.	$P \to Q$	$\to$-introduction$_{1,4}$	{2}

3. The argument is:

$$P \to (Q \to R)\ :\ Q \to (P \to R)$$

Here, there are two nested implications in the conclusion, so we assume first Q, and then P, creating each implication with a separate sub-proof.

A proof of validity is:

1.	Q	Hypothesis	{1}
2.	P	Hypothesis	{2}
3.	$P \to (Q \to R)$	Premise	{3}
4.	$Q \to R$	$\to$-elimination$_{3,2}$	{2,3}
5.	R	$\to$-elimination$_{4,1}$	{1,2,3}
6.	$P \to R$	$\to$-introduction$_{2,5}$	{1,3}
7.	$Q \to (P \to R)$	$\to$-introduction$_{1,6}$	{3}

4. The argument is:

$$P \lor (Q \land R),\ S\ :\ (S \land P) \lor Q$$

Since the first premise is a disjunction, this suggests we need to use $\lor$-elimination. Because the conclusion is also a disjunction, we will need to use $\lor$-introduction too.

A proof of validity is:

1.	S	Premise	{1}
2.	$P \lor (Q \land R)$	Premise	{2}
3.	P	Hypothesis	{3}
4.	$S \land P$	$\land$-introduction$_{1,3}$	{1,3}
5.	$(S \land P) \lor Q$	$\lor$-introduction$_{4}$	{1,3}
6.	$Q \land R$	Hypothesis	{6}
7.	Q	$\land$-elimination$_{6}$	{6}
8.	$(S \land P) \lor Q$	$\lor$-introduction$_{7}$	{6}
9.	$(S \land P) \lor Q$	$\lor$-elimination$_{2,3,5,6,8}$	{1,2}

5. The argument is:

$$Q \to \neg P\ :\ \neg(P \land Q)$$

The conclusion is negated, which suggest use of the $\neg$-introduction rule. To do that, we assume the un-negated form $(P \land Q)$ as a hypothesis, and deduce a contradiction.

A proof of validity is:

1.	$P \land Q$	Hypothesis	{1}
2.	P	$\land$-elimination$_{1}$	{1}
3.	Q	$\land$-elimination$_{1}$	{1}
4.	$Q \to \neg P$	Premise	{4}
5.	$\neg P$	$\to$-elimination$_{4,3}$	{1,4}
6.	$\perp$	$\land$-introduction$_{2,5}$	{1,4}
7.	$\neg(P \land Q)$	$\neg$-introduction$_{1,6}$	{4}

6. The argument is:

$$\neg P \to Q,\ \neg Q\ :\ P$$

The conclusion is *not* negated here but, as hinted, we can use reduction ad absurdum (i.e., $\neg$-introduction) by assuming the negation of the conclusion and deducing a contradiction.

A proof of validity is:

1.	$\neg P$	Hypothesis	{1}
2.	$\neg P \to Q$	Premise	{2}
3.	Q	$\to$-elimination$_{2,1}$	{1,2}
4.	$\neg Q$	Premise	{4}
5.	$\perp$	$\land$-introduction$_{3,4}$	{1,2,4}
6.	$\neg\neg P$	$\neg$-introduction$_{1,5}$	{2,4}
7.	P	$\neg\neg$-elimination$_{6}$	{2,4}

7. The argument is:

$$P \vee Q,\ P \to R,\ \neg S \to \neg Q\ :\ R \vee S$$

Our starting point is the first premise $P \vee Q$. With some thought, we can see that: (i) if P is true, then R is true; and (ii) if Q is true, then S is true. This means that, if one of P or Q is true, then we know one of R or S is true. So, intuitively, the argument makes sense.

To prove this, we apply the $\vee$-elimination rule, which allows us to use the $P \vee Q$ premise, by considering it as two separate cases. For the first of the two cases (where P is true), it is easy to show that R is true and, using also $\vee$-introduction, that $R \vee S$ is true.

For the second of the two cases (where Q is true), we could try something similar, first showing that S must be true, using modus tollens, and then again applying $\vee$-introduction. However, we do not have modus tollens in our set of inference rules, so we have to infer S in a different way. We can do this in a sub-proof using $\neg$-introduction, assuming $\neg S$ and then deriving a contradiction.

A proof of validity is:

1.	$P \vee Q$	Premise	{1}
2.	P	Hypothesis	{2}
3.	$P \to R$	Premise	{3}
4.	R	$\to$-elimination$_{3,2}$	{2,3}
5.	$R \vee S$	$\vee$-introduction$_{4}$	{2,3}
6.	Q	Hypothesis	{6}
7.	$\neg S$	Hypothesis	{7}
8.	$\neg S \to \neg Q$	Premise	{8}
9.	$\neg Q$	$\to$-elimination$_{8,7}$	{7,8}
10.	$\perp$	$\wedge$-introduction$_{6,9}$	{6,7,8}
11.	$\neg\neg S$	$\neg$-introduction$_{7,10}$	{6,8}
12.	S	$\neg\neg$-elimination$_{11}$	{6,8}
13.	$R \vee S$	$\vee$-introduction$_{12}$	{6,8}
14.	$R \vee S$	$\vee$-elimination$_{1,2,5,6,13}$	{1,3,8}