Assignment 3 – Solutions
Predicate Calculus

Model solutions are given below. For proofs, note that these are not unique (and not necessarily the best or the shortest ones possible!).

1. We use the following predicates:

   - $F(x) = x$ speaks French, $D(x) = x$ speaks German, $H(x) = x$ speaks Hebrew
   - $L(x, y) = x$ loves y

   and the constant symbols:

   - $b =$ Bella
   - $c =$ Claire
   - $d =$ Daisy

   (a) If everyone who speaks French also speaks German and at least one person does not speak German, then somebody does not speak French.

   $(\forall x[F(x) \to G(x)] \wedge \exists x[\neg G(x)]) \to \exists x[\neg F(x)]$

   (b) Everyone who loves Bella also loves either Claire or Daisy, and none of them speak French, but at least one person who loves Daisy speaks German.

   $\forall x[L(x, b) \to (L(x, c) \vee L(x, d))] \wedge (\neg F(b) \wedge \neg F(c) \wedge \neg F(d)) \wedge \exists x[L(x, d) \wedge G(x)]$

   (c) If everyone either speaks French or loves Daisy, and Bella speaks neither French nor German, then somebody must love Daisy.

   $(\forall x[F(x) \vee L(x, d)] \wedge (\neg F(b) \wedge \neg G(b))) \to \exists x[L(x, d)]$

   (d) Everybody speaks French, except for the single person who speaks German and Hebrew.

   $\exists x[(G(x) \wedge H(x)) \wedge \forall y[(G(y) \wedge H(y)) \to (x = y)] \wedge \forall y[\neg(x = y) \to F(x)]]$

2. Parts (a) and (c) from Question 1 are valid theorems. Proofs of validity are given below.

1.	$\forall x[F(x) \to G(x)] \wedge \exists x[\neg G(x)]$	Hypothesis	$\{1\}$
2.	$\forall x[F(x) \to G(x)]$	$\wedge$-elimination$_1$	$\{1\}$
3.	$\exists x[\neg G(x)]$	$\wedge$-elimination$_1$	$\{1\}$
4.	$\neg G(a)$	Hypothesis	$\{4\}$
5.	$F(a) \to G(a)$	$\forall$-elimination$_2$	$\{1\}$
6.	$F(a)$	Hypothesis	$\{6\}$
7.	$G(a)$	$\to$-elimination$_{5,6}$	$\{1,6\}$
8.	$\bot$	$\wedge$-introduction$_{7,4}$	$\{1,4,6\}$
9.	$\neg F(a)$	$\neg$-introduction$_{6,8}$	$\{1,4\}$
10.	$\exists x[\neg F(x)]$	$\exists$-introduction$_9$	$\{1,4\}$
11.	$\exists x[\neg F(x)]$	$\exists$-elimination$_{3,4,10}$	$\{1\}$
12.	$(\forall x[F(x) \to G(x)] \wedge \exists x[\neg G(x)]) \to \exists x[\neg F(x)]$	$\to$-introduction$_{1,11}$	$\{\}$

1.	$\forall x[F(x) \vee L(x,d)] \wedge (\neg F(b) \wedge \neg G(b))$	Hypothesis	{1}
2.	$\forall x[F(x) \vee L(x,d)]$	$\wedge$-elimination$_1$	{1}
3.	$\neg F(b) \wedge \neg G(b)$	$\wedge$-elimination$_1$	{1}
4.	$\neg F(b)$	$\wedge$-elimination$_3$	{1}
5.	$F(b) \vee L(b,d)$	$\forall$-elimination$_2$	{1}
6.	$F(b)$	Hypothesis	{6}
7.	$\neg L(b,d)$	Hypothesis	{7}
8.	$\perp$	$\wedge$-introduction$_{6,4}$	{1,6}
9.	$\neg\neg L(b,d)$	$\neg$-introduction$_{7,8}$	{1,6}
10.	$L(b,d)$	$\neg\neg$-elimination$_9$	{1,6}
11.	$L(b,d)$	Hypothesis	{11}
12.	$L(b,d)$	$\vee$-elimination$_{5,6,10,11,11}$	{1}
13.	$\exists x[L(x,d)]$	$\exists$-introduction$_{12}$	{1}
14.	$(\forall x[F(x) \vee L(x,d)] \wedge (\neg F(b) \wedge \neg G(b))) \rightarrow \exists x[L(x,d)]$	$\rightarrow$-introduction$_{1,13}$	{}

3. The proposed equality is:

$$\forall x[\exists y[F(y) \wedge \neg G(x,y)]] \;\equiv\; \forall x[\neg\forall y[\neg F(y) \wedge G(x,y)]]$$

and we consider each direction separately.

The $\rightarrow$ direction *is* valid. A proof of validity is given below.

1.	$\forall x[\exists y[F(y) \wedge \neg G(x,y)]]$	Premise	{1}
2.	$\exists y[F(y) \wedge \neg G(a,y)]$	$\forall$-elimination$_1$	{1}
3.	$F(b) \wedge \neg G(a,b)$	Hypothesis	{3}
4.	$\forall y[\neg F(y) \wedge G(a,y)]$	Hypothesis	{4}
5.	$\neg F(b) \wedge G(a,b)$	$\forall$-elimination$_4$	{4}
6.	$\neg F(b)$	$\wedge$-elimination$_5$	{4}
7.	$F(b)$	$\wedge$-elimination$_3$	{3}
8.	$\perp$	$\wedge$-introduction$_{7,6}$	{3,4}
9.	$\neg\forall y[\neg F(y) \wedge G(a,y)]$	$\neg$-introduction$_{4,8}$	{3}
10.	$\neg\forall y[\neg F(y) \wedge G(a,y)]$	$\exists$-elimination$_{3,9}$	{1}
11.	$\forall x[\neg\forall y[\neg F(y) \wedge G(x,y)]]$	$\forall$-introduction$_{10}$	{1}

The $\leftarrow$ direction is *invalid.*

A counterexample is the interpretation that a single constant a, where $F(a)$ = true and $G(a,a)$ = true. This makes the right-hand side true and the left-hand side false.

4. The proposed equality is:

$$\forall x[\exists y[L(y,x) \wedge \forall z[L(z,x) \rightarrow (y=z)]]] \;\wedge\; \exists x[L(x,x)] \;\equiv\; \exists x[\neg\exists y[L(y,x) \wedge \neg(x=y)]]$$

and we consider each direction separately.

The $\rightarrow$ direction *is* valid. A proof of validity is given below.

1.	$\forall x[\exists y[L(y,x) \wedge \forall z[L(z,x) \rightarrow (y=z)]]] \;\wedge\; \exists x[L(x,x)]$	Premise	{1}
2.	$\forall x[\exists y[L(y,x) \wedge \forall z[L(z,x) \rightarrow (y=z)]]]$	$\wedge$-elimination$_1$	{1}
3.	$\exists x[L(x,x)]$	$\wedge$-elimination$_1$	{1}
4.	$L(a,a)$	Hypothesis	{4}
5.	$\exists y[L(y,a) \wedge \neg(a=y)]$	Hypothesis	{5}
6.	$L(b,a) \wedge \neg(a=b)$	Hypothesis	{6}
7.	$L(b,a)$	$\wedge$-elimination$_6$	{6}
8.	$\neg(a=b)$	$\wedge$-elimination$_6$	{6}
9.	$\exists y[L(y,a) \wedge \forall z[L(z,a) \rightarrow (y=z)]]$	$\forall$-elimination$_2$	{1}
10.	$L(c,a) \wedge \forall z[L(z,a) \rightarrow (c=z)]$	Hypothesis	{10}
11.	$\forall z[L(z,a) \rightarrow (c=z)]$	$\wedge$-elimination$_{10}$	{10}
12.	$L(b,a) \rightarrow (c=b)$	$\forall$-elimination$_{11}$	{10}
13.	$c=b$	$\rightarrow$-elimination$_{12,7}$	{6,10}
14.	$L(a,a) \rightarrow (c=a)$	$\forall$-elimination$_{11}$	{10}
15.	$c=a$	$\rightarrow$-elimination$_{14,4}$	{4,10}
16.	$a=b$	$=$-elimination$_{15,13}$	{4,6,10}
17.	$\bot$	$\wedge$-introduction$_{16,8}$	{4,6,10}
18.	$\bot$	$\exists$-elimination$_{9,10,17}$	{1,4,6}
19.	$\bot$	$\exists$-elimination$_{5,6,18}$	{1,4,5}
20.	$\neg\exists y[L(y,a) \wedge \neg(a=y)]$	$\neg$-introduction$_{5,19}$	{1,4}
21.	$\neg\exists y[L(y,a) \wedge \neg(a=y)]$	$\exists$-elimination$_{3,4,20}$	{1}
22.	$\exists x[\neg\exists y[L(y,x) \wedge \neg(x=y)]]$	$\exists$-introduction$_{21}$	{1}

The $\leftarrow$ direction is *invalid*.

A counterexample is the interpretation that a single constant a, where $L(a,a) =$ false. This makes the right-hand side true and the left-hand side false.