Exercise Class 6 – Solutions
Natural Deduction for Predicate Calculus

Model solutions for the required proofs are given below, although these are not unique (and not necessarily the best or the shortest proofs possible!).

1. The argument is:

   Alfie is a cat who is not black. Therefore not all cats are black.

   First, we translate into predicate calculus. We use the following predicates and constant symbols:

   - $C(x) = x$ is a cat
   - $B(x) = x$ is black
   - $a =$ Alfie

   and the argument is then:

   $$C(a) \wedge \neg B(a) : \neg\forall x[C(x) \rightarrow B(x)]$$

   Since the conclusion is a negation, we prove it using ¬-introduction, obtaining a contradiction by applying ∀-elimination to the hypothesis.

   A proof of validity is:

1.	$\forall x[C(x) \rightarrow B(x)]$	Hypothesis	{1}
2.	$C(a) \rightarrow B(a)$	∀-elimination$_1$	{1}
3.	$C(a) \wedge \neg B(a)$	Premise	{3}
4.	$C(a)$	∧-elimination$_3$	{3}
5.	$\neg B(a)$	∧-elimination$_3$	{3}
6.	$B(a)$	→-elimination$_{2,4}$	{1,3}
7.	$\perp$	∧-introduction$_{6,5}$	{1,3}
8.	$\neg\forall x[C(x) \rightarrow B(x)]$	¬-introduction$_{1,7}$	{3}

2. The argument is:

   $$\forall x[F(x)] \ : \ \exists x[F(x) \vee G(x)]$$

   We aim to use ∃-introduction to prove the conclusion, which means we need to show that $F(a) \vee G(a)$ is true for some a. We can do that by applying ∀-elimination to the premise.

   A proof of validity is:

1.	$\forall x[F(x)]$	Premise	{1}
2.	$F(a)$	∀-elimination$_1$	{1}
3.	$F(a) \vee G(a)$	∨-introduction$_2$	{1}
4.	$\exists x[F(x) \vee G(x)]$	∃-introduction$_3$	{1}

3. The argument is:

All cars are red. All red cars are fast. Therefore all cars are fast.

First, we translate into predicate calculus. We use the following predicates:

- $C(x) = x$ is a car
- $R(x) = x$ is red
- $F(x) = x$ is fast

and the argument is then:

$$\forall x[C(x) \to R(x)], \forall x[(C(x) \land R(x)) \to F(x)] \;:\; \forall x[C(x) \to F(x)]$$

We can prove this using first $\forall$-elimination and then $\forall$-introduction.

A proof of validity is:

1.	$\forall x[C(x) \to R(x)]$	Premise	$\{1\}$
2.	$\forall x[(C(x) \land R(x)) \to F(x)]$	Premise	$\{2\}$
3.	$C(a) \to R(a)$	$\forall$-elimination$_1$	$\{1\}$
4.	$(C(a) \land R(a)) \to F(a)$	$\forall$-elimination$_2$	$\{2\}$
5.	$C(a)$	Hypothesis	$\{5\}$
6.	$R(a)$	$\to$-elimination$_{3,5}$	$\{1,5\}$
7.	$C(a) \land R(a)$	$\land$-introduction$_{5,6}$	$\{1,5\}$
8.	$F(a)$	$\to$-elimination$_{4,7}$	$\{1,2,5\}$
5.	$C(a) \to F(a)$	$\to$-introduction$_{5,8}$	$\{1,2\}$
6.	$\forall x[C(x) \to F(x)]$	$\forall$-introduction$_5$	$\{1,2\}$

4. The argument is:

$$\exists x[P(x) \to Q(x)], \forall x[P(x)] \;:\; \exists x[Q(x)]$$

Since there is an existential quantification on the left-hand side, we will need to use $\exists$-elimination, meaning that we will have a subproof using a hypothesis $P(a) \to Q(a)$. Within the subproof, we aim to prove the overall conclusion, which requires first $\forall$-elimination then $\exists$-introduction.

A proof of validity is:

1.	$\exists x[P(x) \to Q(x)]$	Premise	$\{1\}$
2.	$P(a) \to Q(a)$	Hypothesis	$\{2\}$
3.	$\forall x[P(x)]$	Premise	$\{3\}$
4.	$P(a)$	$\forall$-elimination$_3$	$\{3\}$
5.	$Q(a)$	$\to$-elimination$_{2,4}$	$\{2,3\}$
6.	$\exists x[Q(x)]$	$\exists$-introduction$_5$	$\{2,3\}$
7.	$\exists x[Q(x)]$	$\exists$-elimination$_{1,6}$	$\{1,3\}$

5. The argument is:

$$\forall x[F(x) \to (H(x) \wedge J(x))],\ \forall x[\neg(H(x) \wedge J(x))]\ :\ \forall x[\neg F(x)]$$

This is another example where we use $\forall$-elimination and then $\forall$-introduction, but with slightly more to do in between.

A proof of validity is:

1.	$\forall x[F(x) \to (H(x) \wedge J(x))]$	Premise	{1}
2.	$\forall x[\neg(H(x) \wedge J(x))]$	Premise	{2}
3.	$F(a) \to (H(a) \wedge J(a))$	$\forall$-elimination$_1$	{1}
4.	$\neg(H(a) \wedge J(a))$	$\forall$-elimination$_2$	{2}
5.	$F(a)$	Hypothesis	{5}
6.	$H(a) \wedge J(a)$	$\to$-elimination$_{3,5}$	{1,5}
7.	$\perp$	$\wedge$-introduction$_{6,4}$	{1,2,5}
8.	$\neg F(a)$	$\neg$-introduction$_{5,7}$	{1,2}
9.	$\forall x[\neg F(x)]$	$\forall$-introduction$_8$	{1,2}

6. The argument is:

$$\neg\exists x[\neg F(x)]\ :\ \forall x[F(x)]$$

To use $\forall$-introduction to prove the conclusion, we need to show that $F(a)$ is true for some arbitrary a. We do that by contradiction.

A proof of validity is:

1.	$\neg F(a)$	Hypothesis	{1}
2.	$\exists x[\neg F(x)]$	$\exists$-introduction$_1$	{1}
3.	$\neg\exists x[\neg F(x)]$	Premise	{3}
4.	$\perp$	$\wedge$-introduction$_{2,3}$	{1,3}
5.	$\neg\neg F(a)$	$\neg$-introduction$_{1,4}$	{3}
6.	$F(a)$	$\neg\neg$-elimination$_5$	{3}
7.	$\forall x[F(x)]$	$\forall$-introduction$_6$	{3}

7. The argument is:

Everybody who loves everybody loves themself.

First, we translate into predicate calculus, using the following predicate:

- $L(x, y) = x$ loves y

and the argument is then:

$$: \forall x[\forall y[L(x, y)] \to L(x, x)]$$

To use $\forall$-introduction for the outer universal quantification, we must prove $\forall y[L(a, y)] \to L(a, a)$ for an arbitrary a. Since that is an implication, we start by assuming the antecedent, as usual.

A proof of validity is:

1.	$\forall y[L(a, y)]$	Hypothesis	{1}
2.	$L(a, a)$	$\forall$-elimination$_1$	{1}
3.	$\forall y[L(a, y)] \to L(a, a)$	$\to$-introduction$_{1,2}$	{}
4.	$\forall x[\forall y[L(x, y)] \to L(x, x)]$	$\forall$-introduction$_3$	{}