Alternative Bases for Circuits

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1 Alternative Bases

So far we have looked a single (standard) basis which uses the gates $\{\neg, \vee, \wedge\}$. However, we can design circuits using arbitrary gate types.

Definition 1 *A* ***basis*** *is a finite set $\Omega = \{f_i\}_{i<k}$ of Boolean functions (also referred to as connectives). We say that a basis Ω is ((functionally)* ***complete*** */* ***adequate****) if every Boolean function is computed by an expression or formula formed from variables and elements in Ω.*

Definition 2 Ω-*Circuits.*

Let Ω be a basis. An Ω-circuit is defined just like a Boolean circuit except with non-source nodes labelled by arbitrary elements $f : \{0,1\}^n \to \{0,1\}$ from Ω with precisely n incoming edges.

The size and depth of an Ω circuit is the same as for a Boolean circuit.

- The **size** of an Ω-Circuit: length of longest path from a source node to the *sink*
- The **depth** of an Ω-circuit: the number of nodes.

2 The change-of-basis theorem

Theorem 1 *Let Ω_1 and Ω_2 be two* ***complete*** *bases. For any Ω_1-circuit C_1 there is an $\Omega)_2$-circuit C_2 computing a* ***logically equivalent*** Boolean *function with:*

- $|C_2| = O(|C_1|)$
- $depth(C_2) = O(depth(C_1))$

Proof 1 *(sketch)*

For each element $f_i \in \Omega_1$, let F_i be an Ω_2 circuit of size s_i and depth d_i computing f_i.

Let $s = \max_i s_i$ and $d = \max_i d_i$

Construct C_2 by replacing each non-source node labelled f_i by the Ω_2-circuit F_i. We can now say that $|C_2| \leq s \cdot |C_1|$ and depth$(C_2) \leq d \cdot$depth(C_1)

3 Ω-Rebalancing

Say we have a complete basis $\Delta = \{0, 1, \delta\}$ containing the Boolean constants 0 and 1 along with the function $\delta : \{0,1\}^3 \to \{0,1\}$ given by:

$$\delta(x, y, z) := (\neg x \wedge y) \vee (x \wedge z)$$

Theorem 2 *Spira Balancing*

Let Ω be a basis. For every Ω-formula A of size n, there is a logically equivalent Δ-formula A' s.t. $|A'| = n^{O(1)}$ and depth$(A') = O(\log n) = O(\log |A'|)$

We will show this for the case of **binary** complete bases containing $\{0, 1\}$:

Lemma 1 *Let T be a binary tree. There must exist some subtree S of T s.t.:*

$$\frac{1}{3}|T| < |S| \leq \frac{2}{3}|T|$$

Proof 2 *(sketch of Lemma 1)*

*Starting from the root of a binary tree, repeatedly take the largest immediate subtree until one falls into the range specified in Lemma 1. At each step the subtree **at most halves** in size, and so the first time the subtree has size $\leq \frac{2}{3}|T|$ it must also have size $> \frac{1}{3}|T|$*

Proof 3 *(Spira Balancing)*

We will continue by induction on n.

Let A be a Ω-formula and let B be a distinguished subformula of A s.t. $\frac{1}{3}|A| < |B| \leq \frac{2}{3}|A|$ (exists by Lemma 1)

We can say that $A = C(B)$, Where this notation means that A is the combination of the subtree B and the remaining tree C. so $\therefore \frac{1}{3}|A| < |C(x)| \leq \frac{2}{3}|A|$ Meaning some amount in this range of A exists in C as $|B| + |C| = |A|$.

We can now say that A is logically equivalent to the Δ-formula:

We do this by taking a case analysis on the value of B which can be 1 or 0 to form:

$$\delta(B', C(0)', C(1)')$$

Which means:

$$\delta = \begin{cases} C(0)' & B = 0 \\ C(1)' & B = 1 \end{cases}$$

Where $B', C(0)', C(1)'$ are obtained by the inductive hypothesis on n.