Revision Lecture

Language & Logic

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Overview

- Exam
  - info, syllabus, resources
- Translations from English
  - to propositional logic, predicate calculus
  - 2016 past paper examples
- Proofs
  - tips, strategies, common errors
- Questions, interruptions welcome…

Exam

- Final module mark:
  - 20% continuous assessment (see Canvas) + 80% exam
- Exam – same format as last year
  - 1.5 hours
  - 4 questions (answer all; approx. equal weighting)
- Question types
  - best guide is previous written exercises/assignments/exams
  - also need to study lecture material
- No appendix/supplementary material
  - in particular: need to learn/remember inference rules
  - fixed set of (14) inference rules – see e.g. Ex 8 or Assm 3

Module syllabus (all examinable)

- Formal & natural languages
  - grammars, parse trees
- Propositional logic
  - syntax, semantics, translations, validity
  - truth tables
  - natural deduction proofs
- Predicate calculus
  - variables/quantifiers, identity, translations
  - natural deduction proofs
  - proving vs. disproving; equivalences
- Proof by induction
  - structural induction

Revision resources

- Exercise sheets (1–9) and assessments (1–3)
  - all model solutions on Canvas
- Past papers
  - available on my.bham
  - content all applicable; some minor differences in notation for papers prior to last year
- Books
  - *Logic,* Paul Tomassi (1999)
    - see earlier Canvas announcement “More example questions”
  - *Logic,* Wilfred Hodges (1997)
  - *Logic in Computer Science*, Michael Huth and Mark Ryan (2004)

Translations from English

- Translating English to propositional logic/predicate calculus
  - first, identify and clearly state propositions/predicates used
  - and for predicate calculus, also give domain
  - prop. logic: true/false statements (atomic propositions)
  - pred. calculus: properties/relations (predicates), names (particular instances), generalisations (variables & quantifiers)

- Transation tips
  - look for key "logical" words ("and", "or", "not", "therefore", …)
  - look for common patterns
    - e.g., "all Xs are Ys", "some Xs are Ys", "no Xs are Ys", "at most one..."
  - look for counterexamples (to equivalence between English/logic)

- Examples
  - see Lec 7 & Ex 5; also Lec 10 (relation, identity) & Assm 3

Feedback from Ex 5

- Examples Ex 5 Q1

(a) All white animals are mice
$\forall x[W(x) \rightarrow M(x)]$

- Common "template"

(b) Basil is a white mouse
$M(b) \wedge W(b)$

- No need for any quantifiers
- Re-use the same predicates, M(x) and W(x), here

- Examples Ex 5 Q2

(k) Everyone who loves Chris loves someone who loves John
$\forall x[L(x,c) \rightarrow \exists y[(L(x,y) \wedge L(y,j))]]$

- Same template as Q1(a), but more complex inner formula
- x and y must be distinct variables

Past paper (2016)

- Q3a (sentences in predicate calculus)

"All philosophers enjoy logic. Some mathematicians enjoy logic. All scientists are mathematicians or philosophers. Bob and Harry are scientists. If Bob is mathematician then he enjoys logic. It's the case that if a philosopher enjoys logic then they'll also enjoy ethics. Harry enjoys ethics. Fred is a philosopher."

- Q3b (sentence in predicate calculus)

For an academic to understand their own field, they must first study it. Write an expression which captures the idea that if somebody is either a logician or philosopher then they must have first studied either logic or philosophy respectively.

Proofs – Some tips

- Format clearly and carefully
  - numbering, annotations, dependencies, sub-proof boxes
  - cross out and rewrite if necessary
- Also very helpful for sanity checks
  - e.g. only dependencies for premises (usually all) should remain at the end of a proof (and so expect an empty set for a theorem)
- Check your proofs carefully
  - writing proofs can be hard; checking them should not be
  - are the inference rules used correctly?
  - are any inference rule conditions respected?
- Practice!

Some common mistakes

1. Only use sub-proofs (and hypotheses) where needed
   – only for certain inference rules; always know which one you want

$$\frac{A \vdash B}{A \to B} \to\text{-introduction} \qquad \frac{A \vdash \bot}{\neg A} \neg\text{-introduction}$$

$$\frac{A \vee B \quad A \vdash C \quad B \vdash C}{C} \vee\text{-elimination}$$

$$\frac{\exists x[\phi(x)] \quad \phi(a) \vdash C}{C} \exists\text{-elimination*}$$

Some common mistakes

1. Only use sub-proofs (and hypotheses) where needed
   - only for certain inference rules; always know which one you want

2. Never close two sub-proofs simultaneously

3. Inference rules only apply to the main connective of a formula

4. Only use the basic set of inference rules (not derived inference rules or "known" equivalences)

   Unless… you provide a separate proof of validity
   - e.g., De Morgan, law of excluded middle, etc.
   - and then use Theorem Introduction
   - e.g., if we know: $\vdash ((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$ (Modus Tollens)
   - we can insert: $(((P \wedge Q) \rightarrow R) \wedge \neg R) \rightarrow \neg(P \wedge Q)$

Example

- Prove: $P \lor \neg Q,\ P \to R,\ S \to Q \vdash R \lor \neg S$

1.	$P \lor \neg Q$	Premise	{1}
2.	P	Hypothesis	{2}
3.	$P \to R$	Premise	{3}
4.	R	$\to$-elimination$_{3,2}$	{2,3}
5.	$R \lor \neg S$	$\lor$-introduction$_{4}$	{2,3}
6.	$\neg Q$	Hypothesis	{6}
7.	$S \to Q$	Premise	{7}
8.	$(S \to Q) \land \neg Q$	$\land$-introduction$_{7,6}$	{6,7}
9.	$(S \to Q) \land \neg Q \to \neg S$	Theorem (Modus Tollens)	
10.	$\neg S$	$\to$-elimination$_{9,8}$	{6,7}
11.	$R \lor \neg S$	$\lor$-introduction$_{10}$	{6,7}
12.	$R \lor \neg S$	$\lor$-elimination$_{1,2,5,6,11}$	{1,3,7}

Modus Tollens:

$\vdash ((A \to B) \land \neg B) \to \neg A$

Proof strategies

- Basic strategy
  - what do you know?
    - premises/hypotheses, whatever proved so far
  - what are you trying to prove?
    - conclusion? contradiction?
  - work both forwards and backwards
  - elimination rules for premises, introduction rules for conclusion
  - rules of thumb for which inference rules to apply (see later)
  - changes each time you enter a sub-proof
    - consider e.g. $\rightarrow$-intro

Proof strategies…

- “Golden rules”
  - rules of thumb for propositional logic (and predicate calculus)
  1. if there is a $\rightarrow$ in the conclusion, try $\rightarrow$-introduction
  2. if there is a premise of the form $A \vee B$, try $\vee$-elimination
  3. otherwise, try negation introduction (reduction ad absurdum)

- For predicate calculus, also look at the quantifiers
  - universal ($\forall$-quantified)
  - existential ($\exists$-quantified)
  - unquantified (about individuals)

Rules of thumb

- When the only quantifiers in the premises are universal…

Premises	**Conclusion**	**Then use…**
universal, unquantified	unquantified	$\forall$-elimination to infer properties of individuals, then follow golden rules
	existential	$\forall$-elimination to infer properties of individuals, golden rules to prove conclusion for an individual, then $\exists$-introduction
universal	universal	$\forall$-elimination to infer properties for arbitrary a, golden rules to prove conclusion for a, then $\forall$-introduction

- In summary:
  - $\forall$-elimination first, then prove conclusion
- A common pattern is:
  - $\forall$-elimination then $\forall$-introduction

Proof strategies

- When there are existential quantifiers in the premises…

Premises	First…	Conclusion	Then…
existential	Assume a "typical" disjunct of the existential for arbitrary a	existential or universal or …	Golden rules to prove conclusion for a (e.g. ∃–introduction or ∀–introduction), then ∃–elimination
existential, universal	Assume a "typical" disjunct of the existential for arbitrary a, ∀–elimination to infer further properties of a		

- Note:
  - proof of conclusion is inside ∃–elimination sub–proof
- A common pattern is:
  - ∃–introduction inside ∃–elimination

Examples

1. $\forall x\,[\,D(x) \vee B(x)\,],\ \forall x\,[\,D(x) \rightarrow B(x)\,]\ :\ \forall x\,[\,B(x)\,]$

2. $\forall y\,[\,G(y) \rightarrow H(y)\,]\ :\ \exists x\,[\,G(x)\,] \rightarrow \exists z\,[\,H(z)\,]$

3. $\exists x\,[\,\neg F(x)\,] \vee \exists y\,[\,G(y)\,]\ :\ \exists z\,[\,F(z) \rightarrow G(z)\,]$

Common errors

- Mis-application of inference rules

- Unsatisfied conditions
  - on ∀-introduction
  - and ∃-elimination

- ∀-introduction
  - infer $\forall x\ [\ \phi(x)\ ]$ from $\phi(a)$ for a "typical" a
  - condition: a must not occur in any dependency of $\phi(a)$
- ∃-elimination
  - infer C from $\exists x\ [\ \phi(x)\ ]$ by inferring C for a "typical" a
  - condition: a must not occur in C
    or in any dependency of C except $\phi(a)$

Example – $\forall$-introduction

- Prove: $\forall x\,[\,P(x) \rightarrow Q(x)\,],\ \forall x\,[\,P(x)\,]\ :\ \forall x\,[\,Q(x)\,]$

1.	$\forall x\,[\,P(x) \rightarrow Q(x)\,]$	Premise	{1}
2.	$\forall x\,[\,P(x)\,]$	Premise	{2}
3.	$P(a) \rightarrow Q(a)$	$\forall$–elimination$_1$	{1}
4.	$P(a)$	$\forall$–elimination$_2$	{2}
5.	$Q(a)$	$\rightarrow$–elimination$_{3,4}$	{1,2}
6.	$\forall x\,[\,Q(x)\,]$	$\forall$–introduction$_5$	{1,2}

No use of a

Formula $\phi(a)$

Dependencies

condition: a must not occur in any dependency of $\phi(a)$

Example – ∃-elimination

- Prove: $\exists x\, [\, F(x) \wedge G(x)\,] \,:\, \exists x\, [\, F(x)\,]$

No other dependencies (apart from $\phi(a)$)

Formula $\phi(a)$

No use of a

1.	$\exists x\, [\, F(x) \wedge G(x)\,]$	Premise	{1}
2.	$\boxed{F(a) \wedge G(a)}$	Hypothesis	{2}
3.	$F(a)$	$\wedge$-elimination$_2$	{2}
4.	$\boxed{\exists x\, [\, F(x)\,]}$	$\exists$-introduction$_3$	{2}
5.	$\exists x\, [\, F(x)\,]$	$\exists$-elimination$_{1,2,4}$	{1}

condition: a must not occur in C
or in any dependency of C except $\phi(a)$

Questions

- Questions welcome
  - Facebook, email, Canvas
- Office hours
  - more next week (TBA)