Exercise Sheet 7 – Solutions
More Natural Deduction for Predicate Calculus

Model solutions for the required proofs are given below, although these are not unique (and not necessarily the best or the shortest proofs possible!).

1. (a) The argument is:

$$\forall x[D(x) \lor B(x)],\ \forall x[D(x) \to B(x)]\ :\ \forall x[B(x)]$$

The premises and conclusion are all universally quantified, so we use $\forall$-elimination and then $\forall$-introduction. A proof of validity is:

1.	$\forall x[D(x) \lor B(x)]$	Premise	$\{1\}$
2.	$\forall x[D(x) \to B(x)$	Premise	$\{2\}$
3.	$D(a) \lor B(a)$	$\forall$-elimination$_1$	$\{1\}$
4.	$D(a)$	Hypothesis	$\{4\}$
5.	$D(a) \to B(a)$	$\forall$-elimination$_2$	$\{2\}$
6.	$B(a)$	$\to$-elimination$_{5,4}$	$\{2,4\}$
7.	$B(a)$	Hypothesis	$\{7\}$
8.	$B(a)$	$\lor$-elimination$_{3,4,6,7,7}$	$\{1,2\}$
9.	$\forall x[B(x)]$	$\forall$-introduction$_8$	$\{1,2\}$

(b) The argument is:

$$\forall y[G(y) \to H(y)]\ :\ \exists x[G(x)] \to \exists z[H(z)]$$

The conclusion to be proved is an implication, so we start a subproof to use $\to$-introduction. We are then in a position where we have one existentially and one universally quantified formula, and are trying to prove an existential. So, following the rules sketched in the lecture, we use $\exists$-introduction inside $\exists$-elimination, also making use of $\forall$-elimination within the subproof for $\exists$-elimination.

A proof of validity is:

1.	$\forall y[G(y) \to H(y)]$	Premise	$\{1\}$
2.	$\exists x[G(x)]$	Hypothesis	$\{2\}$
3.	$G(a)$	Hypothesis	$\{3\}$
4.	$G(a) \to H(a)$	$\forall$-elimination$_1$	$\{1\}$
5.	$H(a)$	$\to$-elimination$_{4,3}$	$\{1,3\}$
6.	$\exists z[H(z)]$	$\exists$-introduction$_5$	$\{1,3\}$
7.	$\exists z[H(x)]$	$\exists$-elimination$_{2,3,6}$	$\{1,2\}$
8.	$\exists x[G(x)] \to \exists z[H(z)]$	$\to$-introduction$_{2,7}$	$\{1\}$

(c) The argument is:

$$\exists x[\neg F(x)] \lor \exists y[G(y)] \ : \ \exists z[F(z) \to G(z)]$$

Before considering which quantifiers are present, we see that the premise is a disjunction, which means that we should use disjunction elimination. This gives us two subproofs, both of which have conclusion $\exists z[F(z) \to G(z)]$, and where we use $\exists$-introduction inside $\exists$-elimination.

A proof of validity is:

1.	$\exists x[\neg F(x)] \lor \exists y[G(y)]$	Premise	{1}
2.	$\mid\ \exists x[\neg F(x)]$	Hypothesis	{2}
3.	$\mid\ \mid\ \neg F(a)$	Hypothesis	{3}
4.	$\mid\ \mid\ \mid\ F(a)$	Hypothesis	{4}
5.	$\mid\ \mid\ \mid\ \mid\ \neg G(a)$	Hypothesis	{5}
6.	$\mid\ \mid\ \mid\ \mid\ \bot$	$\land$-introduction$_{4,3}$	{3,4}
7.	$\mid\ \mid\ \mid\ \neg\neg G(a)$	$\neg$-introduction$_{5,6}$	{3,4}
8.	$\mid\ \mid\ \mid\ G(a)$	$\neg\neg$-elimination$_{7}$	{3,4}
9.	$\mid\ \mid\ F(a) \to G(a)$	$\to$-introduction$_{4,8}$	{3}
10.	$\mid\ \mid\ \exists z[F(z) \to G(z)]$	$\exists$-introduction$_{9}$	{3}
11.	$\mid\ \exists z[F(z) \to G(z)]$	$\exists$-elimination$_{2,3,10}$	{2}
12.	$\mid\ \exists y[G(y)]$	Hypothesis	{12}
13.	$\mid\ \mid\ G(a)$	Hypothesis	{13}
14.	$\mid\ \mid\ \mid\ F(a)$	Hypothesis	{14}
15.	$\mid\ \mid\ \mid\ G(a)$	From line 13	{13}
16.	$\mid\ \mid\ F(a) \to G(a)$	$\to$-introduction$_{14,15}$	{13}
17.	$\mid\ \mid\ \exists z[F(z) \to G(z)]$	$\exists$-introduction$_{16}$	{13}
18.	$\mid\ \exists z[F(z) \to G(z)]$	$\exists$-elimination$_{12,13,17}$	{12}
19.	$\exists z[F(z) \to G(z)]$	$\lor$-elimination$_{1,2,11,12,18}$	{1}

2. The argument is:

$$\forall x[(P(x) \land Q(x)) \to R(x)], Q(a) \ : \ \exists x[P(x) \to R(x)]$$

The premises contain a universally quantified formula and an unquantified formula about symbol a. So we use $\forall$-elimination to prove further statements about a and then $\exists$-introduction to prove the conclusion. The inner subproof follows the structure of Question 1.

A proof of validity is:

1.	$\forall x[(P(x) \land Q(x)) \to R(x)]$	Premise	{1}
2.	$(P(a) \land Q(a)) \to R(a)$	$\forall$-elimination$_{1}$	{1}
3.	$Q(a)$	Premise	{3}
4.	$\mid\ P(a)$	Hypothesis	{4}
5.	$\mid\ P(a) \land Q(a)$	$\land$-introduction$_{4,3}$	{3,4}
6.	$\mid\ R(a)$	$\to$-elimination$_{2,5}$	{1,3,4}
7.	$P(a) \to R(a)$	$\to$-introduction$_{4,6}$	{1,3}
8.	$\exists x[P(x) \to R(x)]$	$\exists$-introduction$_{7}$	{1,3}

3. The argument is:

$$\exists x[F(x)] \vee \exists x[G(x)] \;:\; \exists x[F(x) \vee G(x)]$$

This has existential quantifiers in both the premises and conclusion, so we use $\exists$-introduction inside $\exists$-elimination. However the main connective in the premise is a disjunction, so we also need to use $\vee$-elimination.

A proof of validity is:

1.	$\exists x[F(x)] \vee \exists x[G(x)]$	Premise	{1}
2.	$\exists x[F(x)]$	Hypothesis	{2}
3.	$F(a)$	Hypothesis	{3}
4.	$F(a) \vee G(a)$	$\vee$-introduction$_{3}$	{3}
5.	$\exists x[F(x) \vee G(x)]$	$\exists$-introduction$_{4}$	{3}
6.	$\exists x[F(x) \vee G(x)]$	$\exists$-elimination$_{2,3,5}$	{2}
7.	$\exists x[G(x)]$	Hypothesis	{7}
8.	$G(a)$	Hypothesis	{8}
9.	$F(a) \vee G(a)$	$\vee$-introduction$_{8}$	{8}
10.	$\exists x[F(x) \vee G(x)]$	$\exists$-introduction$_{9}$	{8}
11.	$\exists x[F(x) \vee G(x)]$	$\exists$-elimination$_{7,8,10}$	{7}
12.	$\exists x[F(x) \vee G(x)]$	$\vee$-elimination$_{1,2,6,7,11}$	{1}

4. The argument is:

Some rich people are kind. All kind people are loved. Therefore, some rich people are loved.

First, we translate into predicate calculus, using the following predicates:

- $R(x) = x$ is rich, $K(x) = x$ is kind, $L(x) = x$ is loved

and the argument is then:

$$\exists x[R(x) \wedge K(x)], \forall x[K(x) \rightarrow L(x)] : \exists x[R(x) \wedge L(x)]$$

This has both existentially and universally quantified formulas in the premises, so we use a combination of $\exists$-elimination (first) and $\forall$-elimination (second), proving the conclusion inside the subproof using $\exists$-introduction.

A proof of validity is:

1.	$\forall x[K(x) \rightarrow L(x)]$	Premise	{1}
2.	$\exists x[R(x) \wedge K(x)]$	Premise	{2}
3.	$R(a) \wedge K(a)$	Hypothesis	{3}
4.	$R(a)$	$\wedge$-elimination$_{3}$	{3}
5.	$K(a)$	$\wedge$-elimination$_{3}$	{3}
6.	$K(a) \rightarrow L(a)$	$\forall$-elimination$_{1}$	{1}
7.	$L(a)$	$\rightarrow$-elimination$_{6,5}$	{1,3}
8.	$R(a) \wedge L(a)$	$\wedge$-introduction$_{4,7}$	{1,3}
9.	$\exists x[R(x) \wedge L(x)]$	$\exists$-introduction$_{8}$	{1,3}
10.	$\exists x[R(x) \wedge L(x)]$	$\exists$-elimination$_{2,3,9}$	{1,2}

5. The argument is:

$$\exists x[G(x)] \;:\; \neg\forall x[\neg G(x)]$$

This premise is existentially quantified so we use $\exists$-elimination. Inside, we use proof by contradiction ($\neg$-introduction) to prove the conclusion.

A proof of validity is:

1.	$\exists x[G(x)]$	Premise	$\{1\}$
2.	$G(a)$	Hypothesis	$\{2\}$
3.	$\forall x[\neg G(x)]$	Hypothesis	$\{3\}$
4.	$\neg G(a)$	$\forall$-elimination$_3$	$\{3\}$
5.	$\bot$	$\wedge$-introduction$_{2,4}$	$\{2,3\}$
6.	$\neg\forall x[\neg G(x)]$	$\neg$-introduction$_{3,5}$	$\{2\}$
7.	$\neg\forall x[\neg G(x)]$	$\exists$-elimination$_{1,2,6}$	$\{1\}$