10. Predicate Calculus: Wrapping up

Language & Logic

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Remainder of the module

- Today (week 9): wrapping up predicate calculus
  - proving vs. disproving
  - identity (and quantities, definite descriptions)

- Remaining weeks
  - Weeks 10 & 11: proofs about programs; structural induction

- Exercise classes this week on Tue/Thu
  - Ex 8 (on material from this lecture)
  - and Ex 7 (from last week: predicate calculus proof strategies)

- Continuous assessment
  - assignment 3 (week 10)

Recap: Rules of thumb

- Previous rules of thumb (“golden rules”) for propositional logic still applicable for predicate calculus proofs
- Now need to consider quantifiers in premises/conclusion
- Case 1: only quantifiers in the premises are universal
  - ∀-elimination first, then prove conclusion
  - e.g. ∀-elimination then ∀-introduction
- Case 2: some quantifiers in the premises are existential
  - proof of conclusion is inside ∃-elimination subproof
  - e.g. ∃-introduction inside ∃-elimination
- See Exercise Sheet 7 for examples

Proving versus disproving

- So far: asked to prove arguments/theorems known to be valid
  - what if we do not know if it is valid?
- Example (propositional logic):
  - prove or disprove the theorem:
  - : $(P \rightarrow Q) \lor (\neg Q \rightarrow \neg P)$
- How do we answer this?
  - first step: decide if it is valid or not
  - second step: use appropriate technique to prove/disprove
- How to disprove a theorem? (i.e. how to prove invalid/false)
  - we need a counterexample
  - (the same is true for disproving arguments, or equivalences)

Counterexamples (propositional logic)

- For propositional logic
  - a counterexample is a truth valuation for atomic propositions which makes the formula false
  - and we can search for it using a truth table

- Example: Prove or disprove the theorem:
  - : $(P \rightarrow Q) \vee (\neg Q \rightarrow \neg P)$
  - counterexample: P=true, Q=false

Counterexamples (predicate calculus)

- For predicate calculus
  - a counterexample is an interpretation for (some domain) which makes the formula false
- Example: Prove or disprove the theorem:
  - : $(\exists x\ [\ F(x)\ ] \vee \exists x\ [\ G(x)\ ]) \rightarrow (\exists x\ [\ F(x) \wedge G(x)\ ])$
  - counterexample: single constant a where $F(a)=\text{true}$, $G(a)=\text{false}$
- What about arguments?
  - to disprove, show counterexample that makes the premises true and the conclusion false
- What about equivalences?
  - to prove $A \equiv B$, need to (separately) prove: $A \vdash B$ and $B \vdash A$
  - to disprove, disprove either $A \vdash B$ or $B \vdash A$

Relations

- We can represent relations (between objects) using predicates
  - $L(x,y)$ = x loves y
  - $T(x,y)$ = x is taller than y
  - $B(x,y)$ = x and y have the same birthday

- Possible properties of relations
  - reflexivity: $L(a,a)$
  - symmetry: $L(a,b)$ if and only if $L(b,a)$
  - transitivity: if $L(a,b)$ and $L(b,c)$ then $L(a,c)$
  - (for all a, b, c)

- An equivalence relation
  - is a relation that is reflexive, symmetric and transitive

Identity

- We add the identity equivalence relation to predicate calculus
  - $x=y$ means x is the same object as y (i.e. x is an alias for y)
  - standard binary relation, but written using infix notation: $x=y$
  - example: $\forall x\ [\ x=a \rightarrow P(x)\ ]$

- Interpretation
  - clearly, $a=a$, $b=b$, etc. but for some interpretations $a=b$ too

- Equally (or perhaps even more) useful
  - also non-equivalence: $\neg(x=y)$

- Identity significantly increases the expressive power of predicate calculus
  - usage: exceptions, uniqueness, quantities, definite descriptions

Examples – Identity

- Example
  - John is the tallest person
  - where $T(x,y)$ = x is taller than y; and j = John
- Translation:
  - $\forall x\ [\ \neg(x=j) \rightarrow T(j,x)\ ]$
  - not: $\forall x\ [\ T(j,x)\ ]$ (why?)

- Example
  - everybody except John and Chris loves Mary
  - where $L(x,y)$ = x loves y; j = John; c = Chris; m = Mary
- Translation:
  - $\forall x\ [\ \neg(x=j \vee x=c) \rightarrow L(x,m)\ ]$

Quantities

- So far: some, all or none objects have property Q
  - $\exists x\,[\,Q(x)\,]$, $\forall x\,[\,Q(x)\,]$, $\neg\exists x\,[\,Q(x)\,]$
  - in other words: “at least 1 Q”, “all Q”, “0 Qs”
- Or, restricting attention to objects with property P:
  - $\exists x\,[\,P(x) \wedge Q(x)\,]$, $\forall x\,[\,P(x) \rightarrow G(x)\,]$, $\neg\exists x\,[\,P(x) \wedge Q(x)\,]$

- At least two?
  - $\exists x\,[\exists y\,[\,(Q(x) \wedge Q(y)) \wedge \neg(x=y)\,]]$
- At most one Q?
  - $\forall x\,[\,\forall y\,[\,(Q(x) \wedge Q(y)) \rightarrow (x=y)\,]]$
- Exactly one Q?
  - $\exists x\,[\,Q(x)] \wedge \forall x\,[\,\forall y\,[\,(Q(x) \wedge Q(y)) \rightarrow (x=y)\,]]$
  - or: $\exists x\,[\,Q(x) \wedge \forall y\,[\,Q(y)) \rightarrow (x=y)\,]]$

Definite descriptions

- Definite descriptions
  - refer to specific individuals without constant symbols

- Examples

  - the person who loves Mary also loves Chris

  - the tallest person and the shortest person share a birthday

Inference rules for identity

- Identity-introduction

$$\frac{}{a=a}\;=\text{-introduction}$$

  – (i.e. reflexivity)

- Identity-elimination

$$\frac{a=b \quad \phi(a)}{\phi(b)}\;=\text{-elimination}$$

  – “indiscernibility”

- Simple, intuitive rules, but proofs can still be tricky

Example proofs for identity

- 1. Prove:
  - $F(a), \neg G(b), \forall x\ [\ F(x) \rightarrow G(x)\ ]\ :\ \neg(a=b)$

- 2. Prove:
  - transitivity of identity

Summary

- Predicate calculus proof strategies
  - rules of thumb, grouped into two main cases
  - depends on whether there are existential quantifiers in premises

- Proving / disproving
  - disproving a theorem/argument needs a counterexample

- Identity
  - special binary (equivalence) relation: $x=y$
  - adds significant expressive power to predicate calculus
  - aliases, non-equivalences, quantities, definite descriptions, …
  - two new inference rules for introduction/elimination