

Formative Assignment 0

The Untyped λ -Calculus

Sam Barrett, 1803086

October 6, 2020

1. (a) Prove that $(\mathbf{add} \ \underline{2} \ \underline{3})$ β -reduces to $\underline{5}$, i.e. that $(\mathbf{add} \ \underline{2} \ \underline{3}) \mapsto_{\beta}^* \underline{5}$. Describe each \mapsto_{β} step.

$$\begin{aligned}
 & (\mathbf{add} \ \underline{2} \ \underline{3}) \\
 &= ((\lambda a. \lambda b. \lambda f. \lambda x. a f (b f x)) \ \underline{2} \ \underline{3}) \\
 &\mapsto_{\beta} ((\lambda b. \lambda f. \lambda x. \underline{2} f (b f x)) \ \underline{3}) \\
 &\mapsto_{\beta} (\lambda f. \lambda x. \underline{2} f (\underline{3} f x)) \\
 &= (\lambda f. \lambda x. (\lambda f. \lambda x. f (f x)) f ((\lambda f. \lambda x. f (f (f x))) f x)) \\
 &\mapsto_{\beta} (\lambda f. \lambda x. (\lambda x. f (f x)) ((\lambda x. f (f (f x))) x)) \\
 &\mapsto_{\beta} \lambda f. \lambda x. (\lambda x. f (f x)) (f (f (f x))) \\
 &\mapsto_{\beta} \lambda f. \lambda x. (f (f (f (f (f x))))) \\
 &= \underline{5}
 \end{aligned}$$

- (b) Define a multiplication operation `mul` and prove $(\text{mul } \underline{2} \ \underline{3})$ β -reduces to $\underline{6}$, i.e., that $(\text{mul } \underline{2} \ \underline{3}) \mapsto_{\beta}^* \underline{6}$. Describe each \mapsto_{β} step.
Definition of `mul`:

$$\text{mul} = \lambda a. \lambda b. a(\text{add } b) \underline{0}$$

$$\begin{aligned}
& \text{mul } \underline{2} \ \underline{3} \\
&= ((\lambda a. \lambda b. a(\text{add } b) \underline{0}) \underline{2} \ \underline{3}) \\
&\mapsto_{\beta} \underline{2}(\text{add } \underline{3}) \underline{0} \\
&= \lambda f. \lambda x. f(fx)(\text{add } \underline{3}) \underline{0} \\
&\mapsto_{\beta} \lambda x. (\text{add } \underline{3})((\text{add } \underline{3})x) \underline{0} \\
&\mapsto_{\beta} (\text{add } \underline{3})((\text{add } \underline{3}) \underline{0}) \\
&= (\lambda a. \lambda b. \lambda f. \lambda x. a f(bfx)) \underline{3} (\text{add } \underline{3} \ \underline{0}) \\
&\mapsto_{\beta} (\lambda b. \lambda f. \lambda x. \underline{3} f(bfx)) (\text{add } \underline{3} \ \underline{0}) \\
&\mapsto_{\beta} (\lambda f. \lambda x. \underline{3} f(\text{add } \underline{3} \ \underline{0} \ fx)) \\
&= (\lambda f. \lambda x. (\lambda f. \lambda x. f(f(fx)))) f(\text{add } \underline{3} \ \underline{0} \ fx) \\
&\mapsto_{\beta} \lambda f. \lambda x. (\lambda x. f(f(fx))) (\text{add } \underline{3} \ \underline{0} \ fx) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(\text{add } \underline{3} \ \underline{0} \ fx)))) \\
&= \lambda f. \lambda x. (f(f(f(\lambda a. \lambda b. \lambda f. \lambda x. a f(bfx) \underline{3} \ \underline{0} \ fx)))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(\lambda f. \lambda x. \underline{3} f(bfx) \underline{0} \ fx)))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(\lambda f. \lambda x. \underline{3} f(\underline{0} \ fx) fx)))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(\underline{3} f(\underline{0} \ fx)))) \\
&= \lambda f. \lambda x. (f(f(f((\lambda f. \lambda x. f(f(fx)) f)(\underline{0} \ fx)))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(\lambda x. f(f(f(fx)))(\underline{0} \ fx)))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(f(f(f(\underline{0} \ fx)))))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(f(f(f(\lambda f. \lambda x. xfx)))))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(f(f(f(\lambda x. xx)))))) \\
&\mapsto_{\beta} \lambda f. \lambda x. (f(f(f(f(f(fx))))) \\
&= \underline{6}
\end{aligned}$$

2. *

$$M ::= x \mid \lambda x.M \mid MM \mid \mathbf{zero} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \mid \mathbf{isZero}(M, N, P)$$
$$\mathbf{pred}(M) \mapsto_{\beta} \mathbf{zero} \text{ if } M \mapsto_{\beta} \mathbf{zero}$$
$$\mathbf{pred}(M) \mapsto_{\beta} N \text{ if } M \mapsto_{\beta} \mathbf{succ}(N)$$
$$\mathbf{isZero}(M, N, P) \mapsto_{\beta} N \text{ if } M \mapsto_{\beta} \mathbf{zero}$$
$$\mathbf{isZero}(M, N, P) \mapsto_{\beta} P \text{ if } \exists Q. M \mapsto_{\beta} \mathbf{succ}(Q)$$

3.

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$\text{aux} = \lambda a.\lambda x.\lambda y.(\text{isZero } y \text{ T F})x(a \text{ succ}(x) \text{ pred}(y))$$

$$\text{add}' = Y \text{ aux}$$

$$\begin{aligned}
& \text{add}' \ 2 \ 3 \\
= & (Y \ \text{aux}) \ 2 \ 3 \\
= & ((\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))\text{aux}) \ 2 \ 3 \\
\mapsto_{\beta} & (\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)) \ 2 \ 3 \\
\mapsto_{\beta} & (\text{aux}(\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 2 \ 3 \\
= & ((\lambda a.\lambda x.\lambda y.(\text{isZero } y \ x(a \text{succ}(x) \text{pred}(y))))(\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 2 \ 3 \\
\mapsto_{\beta} & (\lambda x.\lambda y.(\text{isZero } y \ x((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(x) \text{pred}(y))) \ 2 \ 3 \\
\mapsto_{\beta} & (\text{isZero } 3 \ 2 ((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(2) \text{pred}(3)) \\
= & ((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 3 \ 2 \\
\mapsto_{\beta} & (\text{aux}(\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 3 \ 2 \\
= & ((\lambda a.\lambda x.\lambda y.(\text{isZero } y \ x(a \text{succ}(x) \text{pred}(y))))(\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 3 \ 2 \\
\mapsto_{\beta} & ((\lambda x.\lambda y.(\text{isZero } y \ x((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(x) \text{pred}(y)))) \ 3 \ 2 \\
\mapsto_{\beta} & (\text{isZero } 2 \ 3 (((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(3) \text{pred}(2))) \\
= & ((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 4 \ 1 \\
\mapsto_{\beta} & (\text{aux}(\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 4 \ 1 \\
= & ((\lambda a.\lambda x.\lambda y.(\text{isZero } y \ x(a \text{succ}(x) \text{pred}(y))))((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))) \ 4 \ 1 \\
\mapsto_{\beta} & (\lambda x.\lambda y.(\text{isZero } y \ x(((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(x) \text{pred}(y)))) \ 4 \ 1 \\
\mapsto_{\beta} & (\text{isZero } 1 \ 4 (((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(4) \text{pred}(1))) \\
= & ((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 5 \ 0 \\
\mapsto_{\beta} & (\text{aux}(\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx))) \ 5 \ 0 \\
= & ((\lambda a.\lambda x.\lambda y.(\text{isZero } y \ x(a \text{succ}(x) \text{pred}(y))))((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))) \ 5 \ 0 \\
\mapsto_{\beta} & (\lambda x.\lambda y.(\text{isZero } y \ x(((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(x) \text{pred}(y)))) \ 5 \ 0 \\
\mapsto_{\beta} & (\text{isZero } 0 \ 5 (((\lambda x.\text{aux}(xx))(\lambda x.\text{aux}(xx)))\text{succ}(5) \text{pred}(0))) \\
= & 5
\end{aligned}$$