Formative Assignment 0
The Untyped λ-Calculus

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1. (a) Prove that ($\texttt{add}\ \underline{2}\ \underline{3}$) β-reduces to $\underline{5}$, i.e. that ($\texttt{add}\ \underline{2}\ \underline{3}$) $\mapsto_\beta^* \underline{5}$. Describe each $\mapsto_\beta$ step.

$$
\begin{array}{lr}
(\texttt{add}\ \underline{2}\ \underline{3}) & \\
= & ((\lambda a.\lambda b.\lambda f.\lambda x.af(bfx))\ \underline{2}\ \underline{3}) \\
\mapsto_\beta & ((\lambda b.\lambda f.\lambda x.\underline{2}f(bfx))\underline{3}) \\
\mapsto_\beta & (\lambda f.\lambda x.\underline{2}f(\underline{3}fx)) \\
= & (\lambda f.\lambda x.(\lambda f.\lambda x.f(fx))f((\lambda f.\lambda x.f(f(fx)))fx)) \\
\mapsto_\beta & (\lambda f.\lambda x.(\lambda x.f(fx))((\lambda x.f(f(fx)))x) \\
\mapsto_\beta & \lambda f.\lambda x(\lambda x.f(fx))(f(f(fx))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(f(fx))))) \\
= & \underline{5}
\end{array}
$$

(b) Define a multiplication operation `mul` and prove ($\mathtt{mul}\ \underline{2}\ \underline{3}$) β-reduces to $\underline{6}$, i.e., that ($\mathtt{mul}\ \underline{2}\ \underline{3}$) $\mapsto_\beta^* \underline{6}$. Describe each $\mapsto_\beta$ step.
Definition of `mul`:

$$\mathtt{mul} = \lambda a.\lambda b.a(\mathtt{add}\ b)\underline{0}$$

$$\begin{array}{rr}
\mathtt{mul}\ \underline{2}\ \underline{3} & \\
= & ((\lambda a.\lambda b.a(\mathtt{add}\ b)\underline{0})\underline{2}\ \underline{3}) \\
\mapsto_\beta & \underline{2}(\mathtt{add}\ \underline{3})\underline{0} \\
= & \lambda f.\lambda x.f(fx)(\mathtt{add}\ \underline{3})\underline{0} \\
\mapsto_\beta & \lambda x.(\mathtt{add}\ \underline{3})((\mathtt{add}\ \underline{3})x)\underline{0} \\
\mapsto_\beta & (\mathtt{add}\ \underline{3})((\mathtt{add}\ \underline{3})\underline{0}) \\
= & (\lambda a.\lambda b.\lambda f.\lambda x.af(bfx))\underline{3}(\mathtt{add}\ \underline{3}\ \underline{0}) \\
\mapsto_\beta & (\lambda b.\lambda f.\lambda x.\underline{3}f(bfx))(\mathtt{add}\ \underline{3}\ \underline{0}) \\
\mapsto_\beta & (\lambda f.\lambda x.\underline{3}f(\mathtt{add}\ \underline{3}\ \underline{0}\ fx)) \\
= & (\lambda f.\lambda x.(\lambda f.\lambda x.f(f(fx))))f(\mathtt{add}\ \underline{3}\ \underline{0}\ fx) \\
\mapsto_\beta & \lambda f.\lambda x.(\lambda x.f(f(fx)))(\mathtt{add}\ \underline{3}\ \underline{0}\ fx) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(\mathtt{add}\ \underline{3}\ \underline{0}\ fx)))) \\
= & \lambda f.\lambda x.(f(f(f(\lambda a.\lambda b.\lambda f.\lambda x.af(bfx)\underline{3}\ \underline{0}\ fx)))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(\lambda f.\lambda x.\underline{3}\ f(bfx)\underline{0}\ fx)))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(\lambda f.\lambda x.\underline{3}\ f(\underline{0}\ fx)fx)))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(\underline{3}\ f(\underline{0}\ fx))))) \\
= & \lambda f.\lambda x.(f(f(f((\lambda f.\lambda x.f(f(fx))f)(\underline{0}\ fx))))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(\lambda x.f(f(f(fx)))(\underline{0}\ fx))))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(f(f(f(\underline{0}\ fx))))))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(f(f(f(\lambda f.\lambda x.xfx))))))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(f(f(f(\lambda x.xx))))))) \\
\mapsto_\beta & \lambda f.\lambda x.(f(f(f(f(f(fx)))))) \\
= & \underline{6}
\end{array}$$

2. *

$$M ::= x|\lambda x.M|MM|\texttt{zero}|\texttt{succ}(M)|\texttt{pred}(M)|\texttt{isZero}(M, N, P)$$

$\texttt{pred}(M) \mapsto_\beta \texttt{zero}$ if $M \mapsto_\beta \texttt{zero}$

$\texttt{pred}(M) \mapsto_\beta N$ if $M \mapsto_\beta \texttt{succ}(N)$

$\texttt{isZero}(M, N, P) \mapsto_\beta N$ if $M \mapsto_\beta \texttt{zero}$

$\texttt{isZero}(M, N, P) \mapsto_\beta P$ if $\exists Q.M \mapsto_\beta \texttt{succ}(Q)$

3.

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$\texttt{aux} = \lambda a.\lambda x.\lambda y.(\texttt{isZero}\; y\; x(a\; \texttt{succ}(x)\; \texttt{pred}(y)))$$

$$\texttt{add'} = \; \texttt{Y}\, \texttt{aux}$$

$$
\begin{aligned}
& \texttt{add'}\; \bar{2}\; \bar{3} \\
= \quad & (\texttt{Y aux })\bar{2}\; \bar{3} \\
= \quad & ((\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))\texttt{aux })\bar{2}\; \bar{3} \\
\mapsto_\beta \quad & (\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ )\bar{2}\; \bar{3} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\bar{2}\; \bar{3} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\; y\; x(a\texttt{succ}(x)\texttt{pred}(y)))(\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ )))\bar{2}\; \bar{3} \\
\mapsto_\beta \quad & (\lambda x.\lambda y.(\texttt{isZero}\; y\; x((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\texttt{succ}(x)\texttt{pred}(y)))\bar{2}\; \bar{3} \\
\mapsto_\beta \quad & (\texttt{isZero}\; \bar{3}\; \bar{2}\; ((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\texttt{succ}(\bar{2})\texttt{pred}(\bar{3})) \\
= \quad & ((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\bar{3}\; \bar{2} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\bar{3}\; \bar{2} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\; y\; x(a\texttt{succ}(x)\; \texttt{pred}(y))))\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ )))\bar{3}\; \bar{2} \\
\mapsto_\beta \quad & ((\lambda x.\lambda y.(\texttt{isZero}\; y\; x((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\texttt{succ}(x)\; \texttt{pred}(y))))\bar{3}\; \bar{2} \\
\mapsto_\beta \quad & (\texttt{isZero}\; \bar{2}\; \bar{3}(((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\texttt{succ}(\bar{3})\texttt{pred}(\bar{2}))) \\
= \quad & ((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\bar{4}\; \bar{1} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\bar{4}\; \bar{1} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\; y\; x(a\texttt{succ}(x)\texttt{pred}(y)))((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))))\bar{4}\; \bar{1} \\
\mapsto_\beta \quad & (\lambda x.\lambda y.(\texttt{isZero}\; y\; x(((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\texttt{succ}(x)\texttt{pred}(y))))\bar{4}\; \bar{1} \\
\mapsto_\beta \quad & (\texttt{isZero}\; \bar{1}\; \bar{4}(((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\texttt{succ}(\bar{4})\texttt{pred}(\bar{1}))) \\
= \quad & ((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\bar{5}\; \bar{0} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\bar{5}\; \bar{0} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\; y\; x(a\texttt{succ}(x)\texttt{pred}(y)))((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ )))\bar{5}\; \bar{0} \\
\mapsto_\beta \quad & (\lambda x.\lambda y.(\texttt{isZero}\; y\; x(((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ )))\texttt{succ}(x)\texttt{pred}(y))))\bar{5}\; \bar{0} \\
\mapsto_\beta \quad & (\texttt{isZero}\; \bar{0}\; \bar{5}(((\lambda x.\texttt{aux}(xx)\ )(\lambda x.\texttt{aux}(xx)\ ))\texttt{succ}(\bar{5})\texttt{pred}(\bar{0}))) \\
= \quad & \bar{5}
\end{aligned}
$$