Formative Assignment 0
The Untyped λ-Calculus

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1. (a) Prove that $(\texttt{add}\ \underline{2}\ \underline{3})$ β-reduces to $\underline{5}$, i.e. that $(\texttt{add}\ \underline{2}\ \underline{3}) \mapsto_\beta^* \underline{5}$. Describe each $\mapsto_\beta$ step.

$$
\begin{aligned}
&(\texttt{add}\ \underline{2}\ \underline{3}) \\
&= && ((\lambda a.\lambda b.\lambda f.\lambda x.af(bfx))\ \underline{2}\ \underline{3}) \\
&\mapsto_\beta && ((\lambda b.\lambda f.\lambda x.\underline{2}f(bfx))\underline{3}) \\
&\mapsto_\beta && (\lambda f.\lambda x.\underline{2}f(\underline{3}fx)) \\
&= && (\lambda f.\lambda x.(\lambda f.\lambda x.f(fx))f((\lambda f.\lambda x.f(f(fx)))fx)) \\
&\mapsto_\beta && (\lambda f.\lambda x.(\lambda x.f(fx))((\lambda x.f(f(fx)))x) \\
&\mapsto_\beta && \lambda f.\lambda x(\lambda x.f(fx))(f(f(fx))) \\
&\mapsto_\beta && \lambda f.\lambda x.(f(f(f(f(fx))))) \\
&= && \underline{5}
\end{aligned}
$$

(b) Define a multiplication operation `mul` and prove ($\texttt{mul}\ \underline{2}\ \underline{3}$) β-reduces to $\underline{6}$, i.e., that ($\texttt{mul}\ \underline{2}\ \underline{3}$) $\mapsto^*_\beta \underline{6}$. Describe each $\mapsto_\beta$ step.

Definition of `mul`:

$$\lambda a.\lambda b.a(\texttt{add}\ b)\underline{0}$$

$$\begin{aligned}
&\texttt{mul}\ \underline{2}\ \underline{3} \\
=\ & ((\lambda a.\lambda b.a(\texttt{add}\ b)\underline{0})\underline{2}\ \underline{3}) \\
\mapsto_\beta\ & \underline{2}(\texttt{add}\ \underline{3})\underline{0} \\
=\ & \lambda f.\lambda x.f(fx)(\texttt{add}\ \underline{3})\underline{0} \\
\mapsto_\beta\ & \lambda x.(\texttt{add}\ \underline{3})((\texttt{add}\ \underline{3})x)\underline{0} \\
\mapsto_\beta\ & (\texttt{add}\ \underline{3})((\texttt{add}\ \underline{3})\underline{0}) \\
=\ & (\lambda a.\lambda b.\lambda f.\lambda x.af(bfx))\underline{3}(\texttt{add}\ \underline{3}\ \underline{0}) \\
\mapsto_\beta\ & (\lambda b.\lambda f.\lambda x.\underline{3}f(bfx))(\texttt{add}\ \underline{3}\ \underline{0}) \\
\mapsto_\beta\ & (\lambda f.\lambda x.\underline{3}f(\texttt{add}\ \underline{3}\ \underline{0}\ fx)) \\
=\ & (\lambda f.\lambda x.(\lambda f.\lambda x.f(f(fx))))f(\texttt{add}\ \underline{3}\ \underline{0}\ fx) \\
\mapsto_\beta\ & \lambda f.\lambda x.(\lambda x.f(f(fx)))(\texttt{add}\ \underline{3}\ \underline{0}\ fx) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(\texttt{add}\ \underline{3}\ \underline{0}\ fx)))) \\
=\ & \lambda f.\lambda x.(f(f(f(\lambda a.\lambda b.\lambda f.\lambda x.af(bfx)\underline{3}\ \underline{0}\ fx)))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(\lambda f.\lambda x.\underline{3}\ f(bfx)\underline{0}\ fx)))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(\lambda f.\lambda x.\underline{3}\ f(\underline{0}\ fx)fx)))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(\underline{3}\ f(\underline{0}\ fx))))) \\
=\ & \lambda f.\lambda x.(f(f(f((\lambda f.\lambda x.f(f(fx))f)(\underline{0}\ fx))))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(\lambda x.f(f(f(fx)))(\underline{0}\ fx))))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(f(f(f(\underline{0}\ fx))))))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(f(f(f(\lambda f.\lambda x.xfx))))))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(f(f(f(\lambda x.xx))))))) \\
\mapsto_\beta\ & \lambda f.\lambda x.(f(f(f(f(f(fx)))))) \\
=\ & \underline{6}
\end{aligned}$$

2.