Assignment 1
Types

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1.

$$M ::= \lambda x.M|MM|\texttt{true}|\texttt{false}|\texttt{if}\ M\ \texttt{then}\ M\ \texttt{else}\ M| \\ \langle M,M\rangle|\texttt{spread}(M,M)|\texttt{left}(M)|\texttt{right}(M)$$

$$T ::= \mathbb{B}|T \to T|T \times T|T+T$$

(a) Extend call-by-value small-step operational semantics of the Curry-style Simply Typed λ-Calculus
Values, $V ::= \lambda x : T.M|\texttt{true}|\texttt{false}|\langle M,M\rangle|\texttt{left}(M)|\texttt{right}(M)$ Evaluation Contexts

$$C ::= \bullet|CM|VC|\texttt{if}\ C\ \texttt{then}\ M\ \texttt{else}\ M|\texttt{spread}(C,M)|\texttt{case}(C,M,M)$$

Rules:

$$\frac{}{(\lambda x : T.M)V \to_v M[x\backslash V]}\ \beta$$

$$\frac{}{\texttt{if true then}\ M\ \texttt{else}\ N \to_v M}\ \text{ITET}$$

$$\frac{}{\texttt{if false then}\ M\ \texttt{else}\ N \to_v N}\ \text{ITEF}$$

$$\frac{M \to_v N}{C[M] \to_v C[N]}\ \texttt{CTX}_C$$

$$\frac{M \to_v \langle P,Q\rangle}{\texttt{spread}(M,N) \to_v NPQ}\ \text{SP}$$

$$\frac{M \to_v \texttt{left}(N)}{\texttt{case}(M,P,Q) \to_v NP}\ \text{CL}$$

$$\frac{M \to_v \texttt{right}(N)}{\texttt{case}(M, P, Q) \to_v NQ}\text{ CR}$$

(b) Prove: **Note: unless stated otherwise Π_n stands for the RHS of the bottom of the previous proof tree**

$$\texttt{spread}((\lambda x.\langle x, x\rangle)\texttt{true}, \lambda y.\lambda z.\texttt{if}\ y\ \texttt{then}\ \texttt{left}(z)\ \texttt{else}\ \texttt{right}(z)) \\ \to_v^* \texttt{left(true)}$$

$$\frac{\dfrac{\langle\texttt{true,true}\rangle}{\lambda x.\langle x, x\rangle)\texttt{true}}\ \beta}{\Pi_1}\ \texttt{spread}(\bullet, M) \qquad (1)$$

Where Π_1 is defined as:

$$\texttt{spread}((\lambda x.\langle x, x\rangle)\texttt{true}, \lambda y.\lambda z.\texttt{if}\ y\ \texttt{then}\ \texttt{left}(z)\ \texttt{else}\ \texttt{right}(z)) \\ \to_v \texttt{spread}(\langle\texttt{true,true}\rangle, \texttt{if}\ y\ \texttt{then}\ \texttt{left}(z)\ \texttt{else}\ \texttt{right}(z))$$

$$\frac{}{\Pi_1 \to_v (\lambda y.\lambda z.\texttt{if}\ y\ \texttt{then}\ \texttt{left}(z)\ \texttt{else}\ \texttt{right}(z))\texttt{true true}}\text{ SP} \qquad (2)$$

$$\frac{}{\Pi_2 \to_v \lambda z.\texttt{if true then left}(z)\ \texttt{else right}(z)}\ \beta \qquad (3)$$

$$\frac{}{\Pi_3 \to_v \texttt{if true then left(true) else right(true)}}\ \beta \qquad (4)$$

$$\frac{}{\Pi_4 \to_v \texttt{left(true)}}\text{ ITET} \qquad (5)$$

(c) Typing Rules:

$$\frac{}{\Gamma, x : T \vdash x : T}\text{ VAR}$$

$$\frac{\Gamma, x : T \vdash M : U}{\Gamma \vdash \lambda x : T.M : T \to U}\text{ ABS}$$

$$\frac{\Gamma \vdash M : T \to U \qquad \Gamma \vdash N : T}{\Gamma \vdash MN : U}\text{ APP}$$

$$\frac{}{\Gamma \vdash \texttt{true} : \mathbb{B}}\text{ T}$$

$$\frac{}{\Gamma \vdash \texttt{false} : \mathbb{B}}\text{ F}$$

$$\frac{\Gamma \vdash M : \mathbb{B} \qquad \Gamma \vdash N : T \qquad \Gamma \vdash P : T}{\Gamma \vdash \texttt{if}\ M\ \texttt{then}\ N\ \texttt{else}\ P : T}\text{ ITE}$$

$$\frac{\Gamma \vdash M : T \qquad \Gamma \vdash M : U}{\Gamma \vdash \langle M, N\rangle : T \times U}\text{ PAIR}$$

$$\frac{\Gamma \vdash M : T \times U \qquad \Gamma \vdash N : T \to U \to Z}{\Gamma \vdash \texttt{spread}(M, N) : Z}\ \text{SP}$$

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash \texttt{left}(M) : T + U}\ \text{L}$$

$$\frac{\Gamma \vdash M : U}{\Gamma \vdash \texttt{right}(M) : T + U}\ \text{R}$$

$$\frac{\Gamma \vdash M : T \qquad \Gamma \vdash P : T \to Z}{\Gamma \vdash \texttt{case}(\texttt{left}(M), P, Q) : Z}\ \text{CASEL}$$

$$\frac{\Gamma \vdash M : U \qquad \Gamma \vdash Q : U \to Z}{\Gamma \vdash \texttt{case}(\texttt{right}(M), P, Q) : Z}\ \text{CASER}$$

(d) Prove:

$$\texttt{spread}((\lambda x.\langle x, x\rangle)\texttt{true}, \lambda y.\lambda z.\texttt{if}\ \ y\ \ \texttt{then}\ \texttt{left}(z)\ \texttt{else}\ \texttt{right}(z))$$

has type $\mathbb{B} + \mathbb{B}$

$$\frac{\overline{\texttt{true} : \mathbb{B}}\ \text{T}}{\texttt{left}(\texttt{true}) : \mathbb{B} + U}\ \text{L}$$

Where U is of **any** type, $\therefore$ we can say $U \mapsto \mathbb{B}$ giving us $\texttt{left}(\texttt{true}) : \mathbb{B} + \mathbb{B}$