Formative Assignment 0
The Untyped λ-Calculus

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1. (a) Prove that ($\texttt{add}\ \underline{2}\ \underline{3}$) β-reduces to $\underline{5}$, i.e. that ($\texttt{add}\ \underline{2}\ \underline{3}$) $\mapsto_\beta^* \underline{5}$. Describe each $\mapsto_\beta$ step.

$$
\begin{aligned}
&(\texttt{add}\ \underline{2}\ \underline{3}) \\
&= && ((\lambda a.\lambda b.\lambda f.\lambda x.af(bfx))\ \underline{2}\ \underline{3}) \\
&\mapsto_\beta && ((\lambda b.\lambda f.\lambda x.\underline{2}f(bfx))\underline{3}) \\
&\mapsto_\beta && (\lambda f.\lambda x.\underline{2}f(\underline{3}fx)) \\
&= && (\lambda f.\lambda x.(\lambda f.\lambda x.f(fx))f((\lambda f.\lambda x.f(f(fx)))fx)) \\
&\mapsto_\beta && (\lambda f.\lambda x.(\lambda x.f(fx))((\lambda x.f(f(fx)))x) \\
&\mapsto_\beta && \lambda f.\lambda x(\lambda x.f(fx))(f(f(fx))) \\
&\mapsto_\beta && \lambda f.\lambda x.(f(f(f(f(fx))))) \\
&= && \underline{5}
\end{aligned}
$$

(b) Define a multiplication operation `mul` and prove ($\texttt{mul}\ \underline{2}\ \underline{3}$) β-reduces to $\underline{6}$, i.e., that ($\texttt{mul}\ \underline{2}\ \underline{3}$) $\mapsto_\beta^* \underline{6}$. Describe each $\mapsto_\beta$ step.

Definition of `mul`:

$$\texttt{mul} = \lambda a.\lambda b.a(\texttt{add}\ b)\underline{0}$$

$$\begin{aligned}
&\texttt{mul}\ \underline{2}\ \underline{3} \\
= &\quad ((\lambda a.\lambda b.a(\texttt{add}\ b)\underline{0})\underline{2}\ \underline{3}) \\
\mapsto_\beta &\quad \underline{2}(\texttt{add}\ \underline{3})\underline{0} \\
= &\quad \lambda f.\lambda x.f(fx)(\texttt{add}\ \underline{3})\underline{0} \\
\mapsto_\beta &\quad \lambda x.(\texttt{add}\ \underline{3})((\texttt{add}\ \underline{3})x)\underline{0} \\
\mapsto_\beta &\quad (\texttt{add}\ \underline{3})((\texttt{add}\ \underline{3})\underline{0}) \\
= &\quad (\lambda a.\lambda b.\lambda f.\lambda x.af(bfx))\underline{3}(\texttt{add}\ \underline{3}\ \underline{0}) \\
\mapsto_\beta &\quad (\lambda b.\lambda f.\lambda x.\underline{3}f(bfx))(\texttt{add}\ \underline{3}\ \underline{0}) \\
\mapsto_\beta &\quad (\lambda f.\lambda x.\underline{3}f(\texttt{add}\ \underline{3}\ \underline{0}\ fx)) \\
= &\quad (\lambda f.\lambda x.(\lambda f.\lambda x.f(f(fx))))f(\texttt{add}\ \underline{3}\ \underline{0}\ fx) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(\lambda x.f(f(fx)))(\texttt{add}\ \underline{3}\ \underline{0}\ fx) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(\texttt{add}\ \underline{3}\ \underline{0}\ fx)))) \\
= &\quad \lambda f.\lambda x.(f(f(f(\lambda a.\lambda b.\lambda f.\lambda x.af(bfx)\underline{3}\ \underline{0}\ fx)))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(\lambda f.\lambda x.\underline{3}\ f(bfx)\underline{0}\ fx)))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(\lambda f.\lambda x.\underline{3}\ f(\underline{0}\ fx)fx)))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(\underline{3}\ f(\underline{0}\ fx))))) \\
= &\quad \lambda f.\lambda x.(f(f(f((\lambda f.\lambda x.f(f(fx))f)(\underline{0}\ fx))))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(\lambda x.f(f(f(fx)))(\underline{0}\ fx))))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(f(f(f(\underline{0}\ fx))))))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(f(f(f(\lambda f.\lambda x.xfx))))))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(f(f(f(\lambda x.xx))))))) \\
\mapsto_\beta &\quad \lambda f.\lambda x.(f(f(f(f(f(fx)))))) \\
= &\quad \underline{6}
\end{aligned}$$

2. *

$$M ::= x|\lambda x.M|MM|\texttt{zero}|\texttt{succ}(M)|\texttt{pred}(M)|\texttt{isZero}(M,N,P)$$

$\texttt{pred}(M) \mapsto_\beta \texttt{zero}$ if $M \mapsto_\beta \texttt{zero}$

$\texttt{pred}(M) \mapsto_\beta N$ if $M \mapsto_\beta \texttt{succ}(N)$

$\texttt{isZero}(M,N,P) \mapsto_\beta N$ if $M \mapsto_\beta \texttt{zero}$

$\texttt{isZero}(M,N,P) \mapsto_\beta P$ if $\exists Q.M \mapsto_\beta \texttt{succ}(Q)$

3.

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$\texttt{aux} = \lambda a.\lambda x.\lambda y.(\texttt{isZero}\; y\; \texttt{T}\; \texttt{F})x(a\; \texttt{succ}(x)\; \texttt{pred}(y))$$

$$\texttt{add'} = \;\texttt{Y}\,\texttt{aux}$$

$$
\begin{aligned}
& \texttt{add'}\,\bar{2}\,\bar{3} \\
= \quad & (\texttt{Y aux})\bar{2}\,\bar{3} \\
= \quad & ((\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))\texttt{aux})\bar{2}\,\bar{3} \\
\mapsto_\beta \quad & (\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx))\bar{2}\,\bar{3} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{2}\,\bar{3} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\,y\,\texttt{T F})x(a\texttt{succ}(x)\texttt{pred}(y)))(\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{2}\,\bar{3} \\
\mapsto_\beta \quad & (\lambda x.\lambda y.(\texttt{isZero}\,y\,\texttt{T F})x((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\texttt{succ}(x)\texttt{pred}(y))\bar{2}\,\bar{3} \\
\mapsto_\beta \quad & (\texttt{isZero}\,\bar{3}\,\texttt{T F})\bar{2}\,((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\texttt{succ}(\bar{2})\texttt{pred}(\bar{3}) \\
= \quad & ((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{3}\,\bar{2} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{3}\,\bar{2} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\,y\,\texttt{T F})x(a\texttt{succ}(x)\,\texttt{pred}(y))))\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{3}\,\bar{2} \\
\mapsto_\beta \quad & ((\lambda x.\lambda y.(\texttt{isZero}\,y\,\texttt{T F})x((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\texttt{succ}(x)\,\texttt{pred}(y)))\bar{3}\,\bar{2} \\
\mapsto_\beta \quad & (\texttt{isZero}\,\bar{2}\,\texttt{T F})\bar{3}(((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\texttt{succ}(\bar{3})\texttt{pred}(\bar{2})) \\
= \quad & ((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{4}\,\bar{1} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{4}\,\bar{1} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\,y\texttt{T F})x(a\texttt{succ}(x)\texttt{pred}(y)))((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx))))\bar{4}\,\bar{1} \\
\mapsto_\beta \quad & (\lambda x.\lambda y.(\texttt{isZero}\,y\,\texttt{T F})x(((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\texttt{succ}(x)\texttt{pred}(y)))\bar{4}\,\bar{1} \\
\mapsto_\beta \quad & (\texttt{isZero}\,\bar{1}\,\texttt{T F})\bar{4}(((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\texttt{succ}(\bar{4})\texttt{pred}(\bar{1})) \\
= \quad & ((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{5}\,\bar{0} \\
\mapsto_\beta \quad & (\texttt{aux}(\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\bar{5}\,\bar{0} \\
= \quad & ((\lambda a.\lambda x.\lambda y.(\texttt{isZero}\,y\,\texttt{T F})x(a\texttt{succ}(x)\texttt{pred}(y)))((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx))))\bar{5}\,\bar{0} \\
\mapsto_\beta \quad & (\lambda x.\lambda y.(\texttt{isZero}\,y\,\texttt{T F})x(((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx))))\texttt{succ}(x)\texttt{pred}(y))\bar{5}\,\bar{0} \\
\mapsto_\beta \quad & (\texttt{isZero}\,\bar{0}\,\texttt{T F})\bar{5}(((\lambda x.\texttt{aux}(xx))(\lambda x.\texttt{aux}(xx)))\texttt{succ}(\bar{5})\texttt{pred}(\bar{0})) \\
= \quad & \bar{5}
\end{aligned}
$$