

# Assignment 1

## Types

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1.

$$M ::= \lambda x.M \mid MM \mid \text{true} \mid \text{false} \mid \text{if } M \text{ then } M \text{ else } M \mid \langle M, M \rangle \mid \text{spread}(M, M) \mid \text{left}(M) \mid \text{right}(M)$$

$$T ::= \mathbb{B} \mid T \rightarrow T \mid T \times T \mid T + T$$

- (a) Extend call-by-value small-step operational semantics of the Curry-style Simply Typed  $\lambda$ -Calculus

Values,  $V ::= \lambda x : T.M \mid \text{true} \mid \text{false} \mid \langle M, M \rangle \mid \text{left}(M) \mid \text{right}(M)$  Evaluation Contexts

$$C ::= \bullet \mid CM \mid VC \mid \text{if } C \text{ then } M \text{ else } M \mid \text{spread}(C, M) \mid \text{case}(C, M, M)$$

Rules:

$$\begin{array}{c} \frac{}{(\lambda x : T.M)V \rightarrow_v M[x \setminus V]} \beta \\[10pt] \frac{}{\text{if true then } M \text{ else } N \rightarrow_v M} \text{ITET} \\[10pt] \frac{}{\text{if false then } M \text{ else } N \rightarrow_v N} \text{ITEF} \\[10pt] \frac{M \rightarrow_v N}{C[M] \rightarrow_v C[N]} \text{CTX}_C \\[10pt] \frac{M \rightarrow_v \langle P, Q \rangle}{\text{spread}(M, N) \rightarrow_v NPQ} \text{SP} \\[10pt] \frac{M \rightarrow_v \text{left}(N)}{\text{case}(M, P, Q) \rightarrow_v NP} \text{CL} \end{array}$$

$$\frac{M \rightarrow_v \text{right}(N)}{\text{case}(M, P, Q) \rightarrow_v NQ} \text{CR}$$

(b) Prove: **Note: unless stated otherwise  $\Pi_n$  stands for the RHS of the bottom of the previous proof tree**

$$\text{spread}((\lambda x. \langle x, x \rangle) \text{true}, \lambda y. \lambda z. \text{if } y \text{ then left}(z) \text{ else right}(z)) \rightarrow_v^* \text{left}(\text{true})$$

$$\frac{\frac{\langle \text{true}, \text{true} \rangle}{(\lambda x. \langle x, x \rangle) \text{true}} \beta}{\Pi_1} \text{spread}(\bullet, M) \quad (1)$$

Where  $\Pi_1$  is defined as:

$$\frac{\text{spread}((\lambda x. \langle x, x \rangle) \text{true}, \lambda y. \lambda z. \text{if } y \text{ then left}(z) \text{ else right}(z)) \rightarrow_v \text{spread}(\langle \text{true}, \text{true} \rangle, \text{if } y \text{ then left}(z) \text{ else right}(z))}{\Pi_1 \rightarrow_v (\lambda y. \lambda z. \text{if } y \text{ then left}(z) \text{ else right}(z)) \text{true true}} \text{SP} \quad (2)$$

$$\frac{\Pi_1 \rightarrow_v \lambda z. \text{if true then left}(z) \text{ else right}(z)}{\Pi_2 \rightarrow_v \lambda z. \text{if true then left}(z) \text{ else right}(z)} \beta \quad (3)$$

$$\frac{\Pi_2 \rightarrow_v \text{if true then left}(\text{true}) \text{ else right}(\text{true})}{\Pi_3 \rightarrow_v \text{if true then left}(\text{true}) \text{ else right}(\text{true})} \beta \quad (4)$$

$$\frac{\Pi_3 \rightarrow_v \text{left}(\text{true})}{\Pi_4 \rightarrow_v \text{left}(\text{true})} \text{ITET} \quad (5)$$

(c) Typing Rules:

$$\frac{}{\Gamma, x : T \vdash x : T} \text{VAR}$$

$$\frac{\Gamma, x : T \vdash M : U}{\Gamma \vdash \lambda x : T. M : T \rightarrow U} \text{ABS}$$

$$\frac{\Gamma \vdash M : T \rightarrow U \quad \Gamma \vdash N : T}{\Gamma \vdash MN : U} \text{APP}$$

$$\frac{}{\Gamma \vdash \text{true} : \mathbb{B}} \text{T}$$

$$\frac{}{\Gamma \vdash \text{false} : \mathbb{B}} \text{F}$$

$$\frac{\Gamma \vdash M : \mathbb{B} \quad \Gamma \vdash N : T \quad \Gamma \vdash P : T}{\Gamma \vdash \text{if } M \text{ then } N \text{ else } P : T} \text{ITE}$$

$$\frac{\Gamma \vdash M : T \quad \Gamma \vdash M : U}{\Gamma \vdash \langle M, N \rangle : T \times U} \text{PAIR}$$

$$\frac{\Gamma \vdash M : T \times U \quad \Gamma \vdash N : T \rightarrow U \rightarrow Z}{\Gamma \vdash \text{spread}(M, N) : Z} \text{ SP}$$

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash \text{left}(M) : T + U} \text{ L}$$

$$\frac{\Gamma \vdash M : U}{\Gamma \vdash \text{right}(M) : T + U} \text{ R}$$

$$\frac{\Gamma \vdash M : T \quad \Gamma \vdash P : T \rightarrow Z}{\Gamma \vdash \text{case}(\text{left}(M), P, Q) : Z} \text{ CASEL}$$

$$\frac{\Gamma \vdash M : U \quad \Gamma \vdash Q : U \rightarrow Z}{\Gamma \vdash \text{case}(\text{right}(M), P, Q) : Z} \text{ CASER}$$

(d) Prove:

$\text{spread}((\lambda x. \langle x, x \rangle) \text{true}, \lambda y. \lambda z. \text{if } y \text{ then left}(z) \text{ else right}(z))$

has type  $\mathbb{B} + \mathbb{B}$

$$\frac{\frac{}{\text{true} : \mathbb{B}} \text{ T}}{\text{left}(\text{true}) : \mathbb{B} + U} \text{ L}$$

Where  $U$  is of **any** type,  $\therefore$  we can say  $U \mapsto \mathbb{B}$  giving us  $\text{left}(\text{true}) : \mathbb{B} + \mathbb{B}$

2. Is:

$(\lambda x. \lambda y. ((xy)(x\text{true}))$

typeable within the Simply Typed  $\lambda$ -Calculus?

Wand's Algorithm:

**Note:** where ' $C = C'$ ',  $C$  has not changed since the previous line

$$C = \emptyset, G = \langle \cdot; M_0; \alpha_0 \rangle \quad (1)$$

$$C = \{\alpha_0 = \alpha_1 \rightarrow \alpha_2\}, G = \langle x : \alpha_1; \lambda y. ((xy)(x\text{true})); \alpha_2 \rangle \quad (2)$$

$$C = \{\alpha_0 = \alpha_1 \rightarrow \alpha_2, \alpha_2 = \alpha_3 \rightarrow \alpha_4\}, G = \langle x : \alpha_1, y : \alpha_3; (xy)(x\text{true}); \alpha_4 \rangle \quad (3)$$

$$C = C, G = \{\langle x : \alpha_1, y : \alpha_3; xy; \alpha_5 \rightarrow \alpha_4 \rangle, \langle x : \alpha_1, y : \alpha_3; x\text{true}; \alpha_5 \rangle\} \quad (4)$$

$$\begin{aligned}
C &= C, \\
G &= \{\langle x : \alpha_1, y : \alpha_3; x; \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 \rangle, \langle x : \alpha_1, y : \alpha_3; y; \alpha_6 \rangle, \langle x : \alpha_1, y : \alpha_3; x\mathbf{true}; \alpha_5 \rangle\} \quad (5)
\end{aligned}$$

$$\begin{aligned}
C &= C, G = \{\langle x : \alpha_1, y : \alpha_3; x; \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 \rangle, \langle x : \alpha_1, y : \alpha_3; y; \alpha_6 \rangle, \\
&\quad \langle x : \alpha_1, y : \alpha_3; x; \alpha_7 \rightarrow \alpha_5 \rangle, \langle x : \alpha_1, y : \alpha_3; \mathbf{true}; \alpha_7 \rangle\} \quad (6)
\end{aligned}$$

$$\begin{aligned}
C &= \{\alpha_0 = \alpha_1 \rightarrow \alpha_2, \alpha_2 = \alpha_3 \rightarrow \alpha_4, \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 = \alpha_1\}, \\
G &= \{\langle x : \alpha_1, y : \alpha_3; y; \alpha_6 \rangle, \langle x : \alpha_1, y : \alpha_3; x; \alpha_7 \rightarrow \alpha_5 \rangle, \langle x : \alpha_1, y : \alpha_3; \mathbf{true}; \alpha_7 \rangle\} \quad (7)
\end{aligned}$$

$$\begin{aligned}
C &= \{\alpha_0 = \alpha_1 \rightarrow \alpha_2, \alpha_2 = \alpha_3 \rightarrow \alpha_4, \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 = \alpha_1, \alpha_6 = \alpha_3\}, \\
G &= \{\langle x : \alpha_1, y : \alpha_3; x; \alpha_7 \rightarrow \alpha_5 \rangle, \langle x : \alpha_1, y : \alpha_3; \mathbf{true}; \alpha_7 \rangle\} \quad (8)
\end{aligned}$$

$$\begin{aligned}
C &= \{\alpha_0 = \alpha_1 \rightarrow \alpha_2, \alpha_2 = \alpha_3 \rightarrow \alpha_4, \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 = \alpha_1, \alpha_6 = \alpha_3, \alpha_7 \rightarrow \alpha_5 = \alpha_1\}, \\
G &= \{\langle x : \alpha_1, y : \alpha_3; \mathbf{true}; \alpha_7 \rangle\} \quad (9)
\end{aligned}$$

$$\begin{aligned}
C &= \{\alpha_0 = \alpha_1 \rightarrow \alpha_2, \alpha_2 = \alpha_3 \rightarrow \alpha_4, \\
&\quad \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 = \alpha_1, \alpha_6 = \alpha_3, \alpha_7 \rightarrow \alpha_5 = \alpha_1, \alpha_7 = \mathbb{B}\}, \\
G &= \emptyset \quad (10)
\end{aligned}$$

Robinson's Algorithm:

$$\begin{aligned}
&\mathbf{unify}(\{\alpha_0 = \alpha_1 \rightarrow \alpha_2, \alpha_2 = \alpha_3 \rightarrow \alpha_4, \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 = \alpha_1, \\
&\quad \alpha_6 = \alpha_3, \alpha_7 \rightarrow \alpha_4 = \alpha_1, \alpha_7 = \mathbb{B}\}) \quad (11)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{unify}(\{\alpha_2 = \alpha_3 \rightarrow \alpha_4, \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 = \alpha_1, \alpha_6 = \alpha_3, \alpha_7 \rightarrow \alpha_4 = \alpha_1, \alpha_7 = \mathbb{B}\}) \circ \\
&\quad (\alpha_0 \mapsto \alpha_1 \rightarrow \alpha_2) \quad (12)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{unify}(\{\alpha_6 \rightarrow \alpha_6 \rightarrow \alpha_4 = \alpha_1, \alpha_6 = \alpha_3, \alpha_7 \rightarrow \alpha_4 = \alpha_1, \alpha_7 = \mathbb{B}\}) \circ \\
&\quad (\alpha_2 \mapsto \alpha_3 \rightarrow \alpha_4) \circ (\alpha_0 \mapsto \alpha_1 \rightarrow \alpha_2) \quad (13)
\end{aligned}$$

$$\text{unify}(\{\alpha_6 = \alpha_3, \alpha_7 \rightarrow \alpha_4 = \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4, \alpha_7 = \mathbb{B}\}) \circ (\alpha_1 \mapsto \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4) \circ (\alpha_2 \mapsto \alpha_3 \rightarrow \alpha_4) \circ (\alpha_0 \mapsto \alpha_1 \rightarrow \alpha_2) \quad (14)$$

$$\text{unify}(\{\alpha_7 \rightarrow \alpha_4 = \alpha_3 \rightarrow \alpha_5 \rightarrow \alpha_4, \alpha_7 = \mathbb{B}\}) \circ (\alpha_6 \mapsto \alpha_3) \circ (\alpha_1 \mapsto \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4) \circ (\alpha_2 \mapsto \alpha_3 \rightarrow \alpha_4) \circ (\alpha_0 \mapsto \alpha_1 \rightarrow \alpha_2) \quad (15)$$

$$\text{unify}(\{\alpha_7 = \mathbb{B}, \alpha_7 = \alpha_3 \rightarrow \alpha_5, \alpha_4 = \alpha_4\}) \circ (\alpha_6 \mapsto \alpha_3) \circ (\alpha_1 \mapsto \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4) \circ (\alpha_2 \mapsto \alpha_3 \rightarrow \alpha_4) \circ (\alpha_0 \mapsto \alpha_1 \rightarrow \alpha_2) \quad (16)$$

$$\text{unify}(\{\mathbb{B} = \alpha_3 \rightarrow \alpha_5, \alpha_4 = \alpha_4\}) \circ (\alpha_7 \mapsto \mathbb{B}) \circ (\alpha_6 \mapsto \alpha_3) \circ (\alpha_1 \mapsto \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4) \circ (\alpha_2 \mapsto \alpha_3 \rightarrow \alpha_4) \circ (\alpha_0 \mapsto \alpha_1 \rightarrow \alpha_2) \quad (17)$$

ERROR: Cannot have constraint of the form  $\mathbb{B} = T_1 \rightarrow T_2$  ! Therefore term is not typable

3. (a) Define a successor function  $\text{Succ} : \mathbb{N} \rightarrow \mathbb{N}$  that takes a number and computes its successor.

Where  $\mathbb{N}$  is the type  $\text{Nat} = \lambda\alpha.(\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$

$$\text{Succ} = \lambda a : \mathbb{N}. \lambda\alpha'. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f(a\{\alpha'\}fx)$$

And prove that it is well-typed:

$$\frac{\frac{\frac{\frac{\frac{\Pi_1}{\Gamma \vdash f : \alpha' \rightarrow \alpha'} \text{VAR}}{\Gamma, x : \alpha' \vdash f(a\{\alpha'\}fx) : \alpha'} \text{APP}}{\Gamma, f : \alpha' \rightarrow \alpha' \vdash \lambda x : \alpha'. f(a\{\alpha'\}fx) : \alpha' \rightarrow \alpha'} \text{ABS}}{\Gamma \vdash \lambda f : \alpha' \rightarrow \alpha'. \lambda x : \alpha'. f(a\{\alpha'\}fx) : (\alpha' \rightarrow \alpha') \rightarrow \alpha' \rightarrow \alpha'} \text{ABS}}{\frac{a : \mathbb{N} \vdash \lambda\alpha'. \lambda f : \alpha' \rightarrow \alpha'. \lambda x : \alpha'. f(a\{\alpha'\}fx) : \forall\alpha'. (\alpha' \rightarrow \alpha') \rightarrow \alpha' \rightarrow \alpha'}{\cdot \vdash \lambda a : \mathbb{N}. \lambda\alpha'. \lambda f : \alpha' \rightarrow \alpha'. \lambda x : \alpha'. f(a\{\alpha'\}fx) : \mathbb{N} \rightarrow \mathbb{N}} \text{TABS} \text{ABS}$$

Where  $\Pi_1$  is:

$$\frac{\frac{\frac{\Gamma \vdash f : \alpha' \rightarrow \alpha'}{\Gamma \vdash a\{\alpha'\}f : \alpha' \rightarrow \alpha'} \text{VAR}}{\Gamma \vdash a\{\alpha'\}f : \alpha' \rightarrow \alpha'} \text{APP} \quad \frac{\frac{\Gamma \vdash a : \forall\alpha. (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha}{\Gamma \vdash a\{\alpha'\} : (\alpha' \rightarrow \alpha') \rightarrow \alpha' \rightarrow \alpha'} \text{TAPP}}{\Gamma \vdash (a\{\alpha'\}f)x : \alpha'} \text{APP} \quad \frac{\Gamma \vdash x : \alpha'}{\Gamma \vdash (a\{\alpha'\}f)x : \alpha'} \text{VAR}$$

- (b) Define an addition function  $\text{Add} : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$  that makes use of the successor function  $\text{Succ}$

$$\text{Add} = \lambda a : \mathbb{N}. \lambda b : \mathbb{N}. \lambda\alpha'. \lambda f : \alpha' \rightarrow \alpha'. \lambda x : \alpha'. a\{\alpha'\}f(b\{\alpha'\}fx)$$