Week 12 Continuous Assessment

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1.

2.

$$d(x, y, z) = 1 \iff x = y = 1 \text{ or } x = z = 0$$

(a)

$$\begin{aligned}
d(x, y, z) =& (x \wedge y) \vee (\neg x \wedge \neg z) \\
&\equiv \neg(\neg(x \wedge y) \wedge \neg(\neg x \wedge \neg z)) \\
&\equiv \neg((\neg x \vee \neg y) \wedge (x \vee z)) \\
&\equiv \neg((\neg x \wedge x) \vee (\neg x \wedge z) \vee (\neg y \wedge x) \vee (\neg y \wedge z)) \\
&\equiv \neg(\neg x \wedge z) \wedge \neg(\neg y \wedge x) \wedge \neg(\neg y \wedge z) \\
&\equiv (x \vee z) \wedge (y \vee x) \wedge (y \vee \neg z)
\end{aligned}$$

(b) i.

$$\begin{aligned}
& x \vee y \iff \\
& \quad d(0, \quad d(\\
& \qquad d(\neg x, \neg y, d(x, x, x)), \\
& \qquad d(\neg x, \neg y, d(x, x, x)), \\
& \qquad d(\neg x, \neg y, d(x, x, x)) \\
& \qquad) \\
& \quad , d(\neg x, \neg y, d(x, x, x)))
\end{aligned}$$

Where:

- $\neg x = d(0, d(x, x, x), x)$
- $\neg y = d(0, d(y, y, y), y)$

ii. $x \wedge y \iff d(x, y, d(x, x, x))$

3. By the change-of-basis theorem

   Let the circuits Ω_1 defined using the operators $\{\neg, \wedge, \vee\}$ and let the circuits defined over $\{d, 0\}$ be the set Ω_2

for each circuit $c_i \in \Omega_1$ let there be an equivalent circuit $c'_i \in \Omega_2$ of size s_i and depth l_i which computes c_i. Also let $s = \max_i s_i$ and $l = \max_i l_i$.

Given a circuit $C \in \Omega_1$, we can construct a logically equivalent circuit $C' \in \Omega_2$ by replacing all non-source nodes labelled with c_i by the circuit c'_i.

With $\Omega_1 = \{\neg, \vee, \wedge\}$ and $\Omega_2 = \{0, d\}$ we can show that every Ω_1 circuit C can be polynomially transformed to a Ω_2 circuit C'.

for the operator $\neg$, we have seen in (b) that this can be transformed into $d(0, d(x, x, x), x)$. This node can therefore be replaced with a circuit of depth 6.

The operator $\vee$ has been shown to be equivalent to the Ω_2 circuit of depth 9.

The $\wedge$ operator has been shown to be equivalent to a Ω_2 circuit of depth 6.

Therefore we can see that in this case $l = 9$ and so the depth of C' is at most $9 \cdot \texttt{depth}(C)$ which is in $O(\texttt{depth}(C))$