Exam for Programming Language Principals, Design and Implementation (Extended)

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After inserting your student ID and the module title in the preamble, write your answers below.

Question 1

(a)

$$M = \lambda f : \mathbb{B} \to \mathbb{B}.\lambda g : \mathbb{B} \to \mathbb{B}.\lambda x : \mathbb{B}.\lambda y : \mathbb{B}.\texttt{if } x \texttt{ then } fy \texttt{ else } gy$$

(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : \mathbb{B}}\;\texttt{VAR} \qquad \dfrac{\dfrac{}{\Gamma \vdash f : \mathbb{B} \to \mathbb{B}}\;\texttt{VAR} \qquad \dfrac{}{\Gamma \vdash y : \mathbb{B}}\;\texttt{VAR}}{\Gamma \vdash fy : \mathbb{B}}\;\texttt{APP} \qquad \Pi_1}{\Gamma, y : \mathbb{B} \vdash \texttt{if } x \texttt{ then } fy \texttt{ else } gy : \mathbb{B}}}{\Gamma, x : \mathbb{B} \vdash \lambda y : \mathbb{B}.\texttt{if } x \texttt{ then } fy \texttt{ else } gy : \mathbb{B} \to \mathbb{B}}\;\texttt{ABS}}{\Gamma, g : \mathbb{B} \to \mathbb{B} \vdash \lambda x : \mathbb{B}.\lambda y : \mathbb{B}.\texttt{if } x \texttt{ then } fy \texttt{ else } gy : \mathbb{B} \to \mathbb{B} \to \mathbb{B}}\;\texttt{ABS}}{f : \mathbb{B} \to \mathbb{B} \vdash M_1 : (\mathbb{B} \to \mathbb{B}) \to \mathbb{B} \to \mathbb{B} \to \mathbb{B}}\;\texttt{ABS}}{\{\} \vdash M : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B}) \to \mathbb{B} \to \mathbb{B} \to \mathbb{B}}\;\texttt{ABS}$$

Where:

- $M_1 = \lambda g : \mathbb{B} \to \mathbb{B}.\lambda x : \mathbb{B}.\lambda y : \mathbb{B}.\texttt{if } x \texttt{ then } fy \texttt{ else } gy$
- $\Pi_1 =$

$$\dfrac{\dfrac{}{\Gamma \vdash g : \mathbb{B} \to \mathbb{B}}\;\texttt{VAR} \qquad \dfrac{}{\Gamma \vdash y : \mathbb{B}}\;\texttt{VAR}}{\Gamma \vdash gy : \mathbb{B}}\;\texttt{APP}$$

(ii) To produce the exclusive function from M we can define the first order parameters F and G as follows:

$$F = \lambda y : \mathbb{B}.\texttt{if } y \texttt{ then false else true} : \mathbb{B} \to \mathbb{B}$$

$$G = \lambda y : \mathbb{B}.\texttt{if } y \texttt{ then true else false} : \mathbb{B} \to \mathbb{B}$$

Alternatively, G can simply be defined as the boolean identity function $\lambda y : \mathbb{B}.y : \mathbb{B} \to \mathbb{B}$. This is the definition I will use in later parts of the question.

(iii) The fact that this expression is well-typed under the Simply-typed λ-Calculus means

Finish this question

(iv)

$$\dfrac{\dfrac{}{\lambda f:\mathbb{B}\to\mathbb{B}.M_1F \to_v \lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }x\texttt{ then }Fy\texttt{ else }gy}\,\beta}{\lambda f:\mathbb{B}\to\mathbb{B}.M_1FG \to_v (\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }x\texttt{ then }Fy\texttt{ else }gy)G}\,\texttt{CTX}_{\bullet G}$$

$$\dfrac{}{(MFG)\texttt{false true} \to_v ((\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }x\texttt{ then }Fy\texttt{ else }gy)G)\texttt{false true}}\,\texttt{CTX}_{(\bullet)\texttt{false true}} \tag{1}$$

$$\dfrac{\dfrac{}{(\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }x\texttt{ then }Fy\texttt{ else }gy)G \to_v \lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }x\texttt{ then }Fy\texttt{ else }Gy}\,\beta}{((\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }x\texttt{ then }Fy\texttt{ else }gy)G)\texttt{false true} \to_v (\lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }x\texttt{ then }Fy\texttt{ else }Gy)\texttt{false true}}\,\texttt{CTX}_{(\bullet)\texttt{false true}} \tag{2}$$

$$\dfrac{\dfrac{}{(\lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }x\texttt{ then }Fy\texttt{ else }Gy)\texttt{false} \to_v \lambda y:\mathbb{B}\texttt{if false then }Fy\texttt{ else }Gy}\,\beta}{(\lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }x\texttt{ then }Fy\texttt{ else }gy)\texttt{false true} \to_v (\lambda y:\mathbb{B}\texttt{if false then }Fy\texttt{ else }Gy)\texttt{true}}\,\texttt{CTX}_{\bullet\texttt{true}} \tag{3}$$

$$\dfrac{}{(\lambda y:\mathbb{B}\texttt{if false then }Fy\texttt{ else }Gy)\texttt{true} \to_v \texttt{if false then }F\texttt{true}\texttt{ else }G\texttt{true}}\,\beta \tag{4}$$

$$\dfrac{}{\texttt{if false then }F\texttt{true}\texttt{ else }G\texttt{true} \to_v G\texttt{true}}\,\texttt{IteF} \tag{5}$$

$$\dfrac{}{(\lambda y:\mathbb{B}.y)\texttt{true} \to_v \texttt{true}}\,\beta \tag{6}$$

Where:

- M_1 is as defined the same as above.

$\texttt{true} \in V$, $\therefore$ $(MFG)\texttt{false true}$ computes to a value.

(b)

$$\texttt{Stack} = \forall\alpha.(\mathbb{N}\to\alpha\to\alpha)\to\alpha\to\alpha$$

(i)

$$\lambda\alpha.\lambda f:\mathbb{N}\to\alpha\to\alpha.\lambda x:\alpha.f0(f0(f1x))$$

(ii)

$$\texttt{peek} = \lambda d:\mathbb{N}.\lambda s:\texttt{Stack}.s\{\mathbb{N}\to\mathbb{N}\}GId$$

Where:

- $G = \lambda n:\mathbb{N}.\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn$
- $I = \lambda x:\mathbb{N}.x$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{\Gamma \vdash s\{\mathbb{N} \to \mathbb{N}\}G : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})}\,\texttt{APP} \qquad \dfrac{\dfrac{}{\Gamma, x : \mathbb{N} \vdash x : \mathbb{N}}\,\texttt{VAR}}{\Gamma \vdash I : \mathbb{N} \to \mathbb{N}}\,\texttt{ABS}}{\Gamma \vdash ((s\{\mathbb{N} \to \mathbb{N}\}G)I) : \mathbb{N} \to \mathbb{N}}\,\texttt{APP} \qquad \dfrac{}{\Gamma \vdash d : \mathbb{N}}\,\texttt{VAR}}{\Gamma, s : \texttt{Stack} \vdash ((s\{\mathbb{N} \to \mathbb{N}\}G)I)d : \mathbb{N} \to \texttt{Stack} \to \mathbb{N} : \mathbb{N}}\,\texttt{APP}}{d : \mathbb{N} \vdash \lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId : \mathbb{N} \to \texttt{Stack} \to \mathbb{N} : \texttt{Stack} \to \mathbb{N}}\,\texttt{ABS}}{\{\} \vdash \lambda d : \mathbb{N}.\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId : \mathbb{N} \to \texttt{Stack} \to \mathbb{N}}\,\texttt{ABS}
$$

Where:

- $\Pi_1 =$

$$
\dfrac{\dfrac{}{\Gamma \vdash s : \texttt{State}}\,\texttt{VAR}}{\Gamma \vdash s\{\mathbb{N} \to \mathbb{N}\} : (\mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})) \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})}\,\texttt{TAPP}
$$

- $\Pi_2 =$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : \mathbb{N} \to \mathbb{N}}\,\texttt{VAR} \qquad \dfrac{}{\Gamma \vdash n : \mathbb{N}}\,\texttt{VAR}}{\Gamma, x : \mathbb{N} \vdash gn : \mathbb{N}}\,\texttt{APP}}{\Gamma, g : \mathbb{N} \to \mathbb{N} \vdash \lambda x : \mathbb{N}.gn : \mathbb{N} \to \mathbb{N}}\,\texttt{ABS}}{\Gamma, n : \mathbb{N} \vdash \lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}}\,\texttt{ABS}}{\Gamma \vdash G : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}}\,\texttt{ABS}
$$

(iii) Prove:

$$
\texttt{peek}\ \ ds_2 \to_v^* m
$$

$$
\dfrac{\dfrac{}{(\lambda d : \mathbb{N}.\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId)d \to_v \lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId}\,\beta}{\texttt{peek}\ \ ds_2 \to_v (\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId)s_2}\,\texttt{CTX}_{\bullet s_2}
$$

$$
\dfrac{}{(\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId)s_2 \to_v s_2\{\mathbb{N} \to \mathbb{N}\}GId}\,\beta
$$

$$
\dfrac{\dfrac{}{\begin{gathered}\lambda\alpha.\lambda f : \mathbb{N} \to \alpha \to \alpha.\lambda x : \alpha.fn(fmx)\{\mathbb{N} \to \mathbb{N}\} \to_v \\ \lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : (\mathbb{N} \to \mathbb{N}).fn(fmx)\end{gathered}}\,T_\beta}{\begin{gathered}((s_2\{\mathbb{N} \to \mathbb{N}\}G)I)d \to_v \\ (((\lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : \mathbb{N} \to \mathbb{N}.fn(fmx))G)I)d\end{gathered}}\,\texttt{CTX}_{((\bullet G)I)d}
$$

$$
\dfrac{\dfrac{}{\begin{gathered}(\lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : \mathbb{N} \to \mathbb{N}.fn(fmx))G \to_v \\ \lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx)\end{gathered}}\,\beta}{\begin{gathered}(((\lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : \mathbb{N} \to \mathbb{N}.fn(fmx))G)I)d \to_v \\ ((\lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx))I)d\end{gathered}}\,\texttt{CTX}_{((\bullet)I)d}
$$

$$
\dfrac{\dfrac{}{(\lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx))I \to_v Gn(GmI)}\,\beta}{((\lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx))I)d \to_v (Gn(GmI)d)}\,\texttt{CTX}_{(\bullet)d}
$$

$$
\dfrac{\dfrac{}{(\lambda n : \mathbb{N}.\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)n \to_v \lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn}\,\beta}{(Gn(GmI)d) \to_v ((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)(GmI))d}\,\texttt{CTX}_{(\bullet(GmI))d}
$$

$$\dfrac{\dfrac{}{(\lambda n : \mathbb{N}\lambda g : \mathbb{N} \to \mathbb{N}\lambda x : \mathbb{N}.gn)m \to_v \lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gm}\,\beta}{\begin{array}{c}((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)(Gml))d \to_v \\ ((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gm)l))d\end{array}}\,\mathtt{CTX}_{((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(\bullet l))d}$$

$$\dfrac{\dfrac{}{(\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gm)l \to_v \lambda x : \mathbb{N}.lm}\,\beta}{\begin{array}{c}((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gm)l))d \to_v \\ ((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)(\lambda x : \mathbb{N}.lm))d\end{array}}\,\mathtt{CTX}_{((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(\bullet))d}$$

$$\dfrac{\dfrac{}{(\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)(\lambda x : \mathbb{N}.lm) \to_v \lambda x : \mathbb{N}.(\lambda x : \mathbb{N}lm)n}\,\beta}{\begin{array}{c}((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)(\lambda x : \mathbb{N}.lm))d \to_v \\ (\lambda x : \mathbb{N}.(\lambda x : \mathbb{N}.lm)n)d\end{array}}\,\mathtt{CTX}_{(\bullet)d}$$

$$\dfrac{}{(\lambda x : \mathbb{N}.(\lambda x : \mathbb{N}.lm)n)d \to_v (\lambda x : \mathbb{N}.lm)n}\,\beta$$

$$\dfrac{}{(\lambda x : \mathbb{N}lm)n \to_v lm}\,\beta$$

$$\dfrac{}{lm \equiv (\lambda x : \mathbb{N}x)n \to_v m}\,\beta$$

Question 2

Question 3

Add more questions if necessary.

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