Exam for Programming Language Principals, Design and Implementation (Extended)

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After inserting your student ID and the module title in the preamble, write your answers below.

Question 1

(a)

$$M = \lambda f : \mathbb{B} \to \mathbb{B}.\lambda g : \mathbb{B} \to \mathbb{B}.\lambda x : \mathbb{B}.\lambda y : \mathbb{B}.\mathtt{if}\ x\ \mathtt{then}\ fy\ \mathtt{else}\ gy$$

(i)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : \mathbb{B}}\,\mathtt{VAR} \qquad \dfrac{\dfrac{}{\Gamma \vdash f : \mathbb{B} \to \mathbb{B}}\,\mathtt{VAR} \qquad \dfrac{}{\Gamma \vdash y : \mathbb{B}}\,\mathtt{VAR}}{\Gamma \vdash fy : \mathbb{B}}\,\mathtt{APP} \qquad \Pi_1}{\Gamma, y : \mathbb{B} \vdash \mathtt{if}\ x\ \mathtt{then}\ fy\ \mathtt{else}\ gy : \mathbb{B}}}{\Gamma, x : \mathbb{B} \vdash \lambda y : \mathbb{B}.\mathtt{if}\ x\ \mathtt{then}\ fy\ \mathtt{else}\ gy : \mathbb{B} \to \mathbb{B}}\,\mathtt{ABS}}{\Gamma, g : \mathbb{B} \to \mathbb{B} \vdash \lambda x : \mathbb{B}.\lambda y : \mathbb{B}.\mathtt{if}\ x\ \mathtt{then}\ fy\ \mathtt{else}\ gy : \mathbb{B} \to \mathbb{B} \to \mathbb{B}}\,\mathtt{ABS}}{f : \mathbb{B} \to \mathbb{B} \vdash M_1 : (\mathbb{B} \to \mathbb{B}) \to \mathbb{B} \to \mathbb{B} \to \mathbb{B}}\,\mathtt{ABS}}{\{\} \vdash M : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B}) \to \mathbb{B} \to \mathbb{B} \to \mathbb{B}}\,\mathtt{ABS}$$

Where:

- $M_1 = \lambda g : \mathbb{B} \to \mathbb{B}.\lambda x : \mathbb{B}.\lambda y : \mathbb{B}.\mathtt{if}\ x\ \mathtt{then}\ fy\ \mathtt{else}\ gy$
- $\Pi_1 =$

$$\dfrac{\dfrac{}{\Gamma \vdash g : \mathbb{B} \to \mathbb{B}}\,\mathtt{VAR} \qquad \dfrac{}{\Gamma \vdash y : \mathbb{B}}\,\mathtt{VAR}}{\Gamma \vdash gy : \mathbb{B}}\,\mathtt{APP}$$

(ii) To produce the exclusive function from M we can define the first order parameters F and G as follows:

$$F = \lambda y : \mathbb{B}.\mathtt{if}\ y\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true} : \mathbb{B} \to \mathbb{B}$$

$$G = \lambda y : \mathbb{B}.\mathtt{if}\ y\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{false} : \mathbb{B} \to \mathbb{B}$$

Alternatively, G can simply be defined as the boolean identity function $\lambda y : \mathbb{B}.y : \mathbb{B} \to \mathbb{B}$. This is the definition I will use in later parts of the question.

(iii) The fact that this expression is well-typed under the Simply-typed λ-Calculus means

Finish this question

(iv)

$$\dfrac{\dfrac{}{\lambda f:\mathbb{B}\to\mathbb{B}.M_1F \to_v \lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }\ x \texttt{ then } Fy \texttt{ else } gy}\beta}{\dfrac{\lambda f:\mathbb{B}\to\mathbb{B}.M_1FG \to_v (\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }\ x \texttt{ then } Fy \texttt{ else } gy)G}{(MFG)\texttt{false true} \to_v ((\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }\ x \texttt{ then } Fy \texttt{ else } gy)G)\texttt{false true}}\texttt{CTX}_{(\bullet)\texttt{false true}}}\texttt{CTX}_{\bullet G} \tag{1}$$

$$\dfrac{\dfrac{}{(\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }\ x \texttt{ then } Fy \texttt{ else } gy)G \to_v \lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }\ x \texttt{ then } Fy \texttt{ else } Gy}\beta}{((\lambda g:\mathbb{B}\to\mathbb{B}\lambda x:\mathbb{B}\texttt{if }\ x \texttt{ then } Fy \texttt{ else } gy)G)\texttt{false true} \to_v (\lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }\ x \texttt{ then } Fy \texttt{ else } Gy)\texttt{false true}}\texttt{CTX}_{(\bullet)\texttt{false true}} \tag{2}$$

$$\dfrac{\dfrac{}{(\lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }\ x \texttt{ then } Fy \texttt{ else } Gy)\texttt{false} \to_v \lambda y:\mathbb{B}\texttt{if false then } Fy \texttt{ else } Gy}\beta}{(\lambda x:\mathbb{B}.\lambda y:\mathbb{B}.\texttt{if }\ x \texttt{ then } Fy \texttt{ else } gy)\texttt{false true} \to_v (\lambda y:\mathbb{B}\texttt{if false then } Fy \texttt{ else } Gy)\texttt{true}}\texttt{CTX}_{\bullet\texttt{true}} \tag{3}$$

$$\dfrac{}{(\lambda y:\mathbb{B}\texttt{if false then } Fy \texttt{ else } Gy)\texttt{true} \to_v \texttt{if false then } F\texttt{true else } G\texttt{true}}\beta \tag{4}$$

$$\dfrac{}{\texttt{if false then } F\texttt{true else } G\texttt{true} \to_v G\texttt{true}}\texttt{IteF} \tag{5}$$

$$\dfrac{}{(\lambda y:\mathbb{B}.y)\texttt{true} \to_v \texttt{true}}\beta \tag{6}$$

Where:

- M_1 is as defined the same as above.

$\texttt{true} \in V$, $\therefore$ $(MFG)\texttt{false true}$ computes to a value.

(b)

$$\texttt{Stack} = \forall\alpha.(\mathbb{N}\to\alpha\to\alpha)\to\alpha\to\alpha$$

(i)

$$\lambda\alpha.\lambda f:\mathbb{N}\to\alpha\to\alpha.\lambda x:\alpha.f0(f0(f1x))$$

(ii)

$$\texttt{peek} = \lambda d:\mathbb{N}.\lambda s:\texttt{Stack}.s\{\mathbb{N}\to\mathbb{N}\}GId$$

Where:

- $G = \lambda n:\mathbb{N}.\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn$
- $I = \lambda x:\mathbb{N}.x$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1 \qquad \Pi_2}{\Gamma \vdash s\{\mathbb{N} \to \mathbb{N}\}G : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})}\;\texttt{APP} \qquad \dfrac{\dfrac{}{\Gamma, x : \mathbb{N} \vdash x : \mathbb{N}}\;\texttt{VAR}}{\Gamma \vdash I : \mathbb{N} \to \mathbb{N}}\;\texttt{ABS}}{\Gamma \vdash ((s\{\mathbb{N} \to \mathbb{N}\}G)I) : \mathbb{N} \to \mathbb{N}}\;\texttt{APP} \qquad \dfrac{}{\Gamma \vdash d : \mathbb{N}}\;\texttt{VAR}}{\dfrac{\Gamma, s : \texttt{Stack} \vdash ((s\{\mathbb{N} \to \mathbb{N}\}G)I)d : \mathbb{N} \to \texttt{Stack} \to \mathbb{N} : \mathbb{N}}{d : \mathbb{N} \vdash \lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId : \mathbb{N} \to \texttt{Stack} \to \mathbb{N} : \texttt{Stack} \to \mathbb{N}}\;\texttt{ABS}}\;\texttt{APP}}{\{\} \vdash \lambda d : \mathbb{N}.\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId : \mathbb{N} \to \texttt{Stack} \to \mathbb{N}}\;\texttt{ABS}
$$

Where:

- $\Pi_1 =$

$$
\dfrac{\dfrac{}{\Gamma \vdash s : \texttt{State}}\;\texttt{VAR}}{\Gamma \vdash s\{\mathbb{N} \to \mathbb{N}\} : (\mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})) \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})}\;\texttt{TAPP}
$$

- $\Pi_2 =$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : \mathbb{N} \to \mathbb{N}}\;\texttt{VAR} \qquad \dfrac{}{\Gamma \vdash n : \mathbb{N}}\;\texttt{VAR}}{\Gamma, x : \mathbb{N} \vdash gn : \mathbb{N}}\;\texttt{APP}}{\dfrac{\Gamma, g : \mathbb{N} \to \mathbb{N} \vdash \lambda x : \mathbb{N}.gn : \mathbb{N} \to \mathbb{N}}{\Gamma, n : \mathbb{N} \vdash \lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}}\;\texttt{ABS}}\;\texttt{ABS}}{\Gamma \vdash G : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}}\;\texttt{ABS}
$$

(iii) Prove:

$$
\texttt{peek}\ \ ds_2 \to_v^* m
$$

$$
\dfrac{\dfrac{}{(\lambda d : \mathbb{N}.\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId)d \to_v \lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId}\;\beta}{\texttt{peek}\ \ ds_2 \to_v (\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId)s_2}\;\texttt{CTX}_{\bullet s_2}
$$

$$
\dfrac{}{(\lambda s : \texttt{Stack}.s\{\mathbb{N} \to \mathbb{N}\}GId)s_2 \to_v s_2\{\mathbb{N} \to \mathbb{N}\}GId}\;\beta
$$

$$
\dfrac{\dfrac{}{\begin{array}{c}\lambda\alpha.\lambda f : \mathbb{N} \to \alpha \to \alpha.\lambda x : \alpha.fn(fmx)\{\mathbb{N} \to \mathbb{N}\} \to_v \\ \lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : (\mathbb{N} \to \mathbb{N}).fn(fmx)\end{array}}\;T_\beta}{\begin{array}{c}((s_2\{\mathbb{N} \to \mathbb{N}\}G)I)d \to_v \\ (((\lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : \mathbb{N} \to \mathbb{N}.fn(fmx))G)I)d\end{array}}\;\texttt{CTX}_{(\bullet G)I)d}
$$

$$
\dfrac{\dfrac{}{\begin{array}{c}(\lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : \mathbb{N} \to \mathbb{N}.fn(fmx))G \to_v \\ \lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx)\end{array}}\;\beta}{\begin{array}{c}(((\lambda f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}).\lambda x : \mathbb{N} \to \mathbb{N}.fn(fmx))G)I)d \to_v \\ ((\lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx))I)d\end{array}}\;\texttt{CTX}_{((\bullet)I)d}
$$

$$
\dfrac{\dfrac{}{(\lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx))I \to_v Gn(GmI)}\;\beta}{((\lambda x : \mathbb{N} \to \mathbb{N}.Gn(Gmx))I)d \to_v (Gn(GmI)d)}\;\texttt{CTX}_{(\bullet)d}
$$

$$
\dfrac{\dfrac{}{(\lambda n : \mathbb{N}.\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)n \to_v \lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn}\;\beta}{(Gn(GmI)d) \to_v ((\lambda g : \mathbb{N} \to \mathbb{N}.\lambda x : \mathbb{N}.gn)(GmI))d}\;\texttt{CTX}_{(\bullet(GmI))d}
$$

$$\dfrac{\dfrac{}{(\lambda n:\mathbb{N}\lambda g:\mathbb{N}\to\mathbb{N}\lambda x:\mathbb{N}.gn)m \to_v \lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gm}\,\beta}{((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(GmI))d \to_v ((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gm)I))d}\,\mathtt{CTX}_{((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(\bullet I))d}$$

$$\dfrac{\dfrac{}{(\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gm)I \to_v \lambda x:\mathbb{N}.Im}\,\beta}{((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gm)I))d \to_v ((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(\lambda x:\mathbb{N}.Im))d}\,\mathtt{CTX}_{((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(\bullet))d}$$

$$\dfrac{\dfrac{}{(\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(\lambda x:\mathbb{N}.Im) \to_v \lambda x:\mathbb{N}.(\lambda x:\mathbb{N}Im)n}\,\beta}{((\lambda g:\mathbb{N}\to\mathbb{N}.\lambda x:\mathbb{N}.gn)(\lambda x:\mathbb{N}.Im))d \to_v (\lambda x:\mathbb{N}.(\lambda x:\mathbb{N}.Im)n)d}\,\mathtt{CTX}_{(\bullet)d}$$

$$\dfrac{}{(\lambda x:\mathbb{N}.(\lambda x:\mathbb{N}.Im)n)d \to_v (\lambda x:\mathbb{N}.Im)n}\,\beta$$

$$\dfrac{}{(\lambda x:\mathbb{N}.Im)n \to_v Im}\,\beta$$

$$\dfrac{}{Im \equiv (\lambda x:\mathbb{N}.x)n \to_v m}\,\beta$$

(iv) an abstract *stack* datatype can be defined as follows:

$$\begin{aligned}\mathtt{pack}\ &\langle \mathtt{Stack}, \langle s_0, \langle \mathtt{push}, \langle \mathtt{peek}, \mathtt{pop}\rangle\rangle\rangle\rangle\\ &\ \mathtt{as}\\ &\exists \mathtt{stack}.\mathtt{stack} \times (\mathbb{N} \to \mathtt{stack} \to \mathtt{stack}\times\\ &(\mathtt{stack} \to \mathbb{N} \to \mathbb{N} \times \mathtt{stack} \to \mathtt{stack}))\end{aligned}$$

Check type of "pop"

Where:

- $s_0 = \lambda\alpha.\lambda f:\mathbb{N}\to\alpha\to\alpha.\lambda x:\alpha.x$
- $\mathtt{Stack} = \forall\alpha.(\mathbb{N}\to\alpha\to\alpha)\to\alpha\to\alpha$
- `push`, `peek` and `pop` are defined the same as in the exam booklet, over (concrete) the `Stack` type.

Question 2

$$Y = \lambda y.(\lambda f.t(\lambda z.ffz))(\lambda f.t(\lambda z.ffz))$$

Question 3

Add more questions if necessary. foo

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