Coinduction

Sunday, 11 April 2021 6:55 PM

(Co) inductive types have constructors

Inductive $\mathbb{N}$:=
| 0 : True $\to \mathbb{N}$
| S : $\mathbb{N} \to \mathbb{N}$

T :=
| C_1 : $A_1 \to T$
⋮
| C_n : $A_n \to T$

Have associated destructor

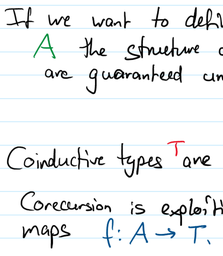
$des(0) = \langle *, \square \rangle$
$des(Sm) = \langle \square, m \rangle$

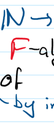
$des_T(t) = (k, a : A_k)$ s.t. $t = C_k(a)$
(which constructor was used to build t)

Build an (infinite) tree where the branches are the possible outputs of the destructor

Elements of coinductive type are paths through this tree.

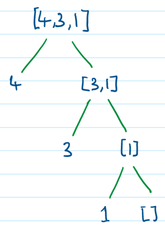
E.g. conatural numbers



Note  is a path

So $co\mathbb{N} = \mathbb{N} \cup \{\omega\}$
where $succ(\omega) = \omega$

Alternatively, we can visualise elements of an inductive type as well-founded trees.



E.g. Coinductive Stream[A] :=
| cons: $A \to$ Stream $\to$ Stream

elements are infinite lists of elements of A.

Then, elements of the corresponding coinductive type are (not necessarily well-founded trees).

Categorically, definitions/proofs by induction exploit the fact that a structure S is an initial algebra for some functor $F : Set \to Set$.

If we want to define $f : \mathbb{N} \to A$, then give A the structure of an F-algebra. Then we are guaranteed uniqueness of $f : \mathbb{N} \to A$. (by initiality)

Coinductive types T are final coalgebras for a functor.

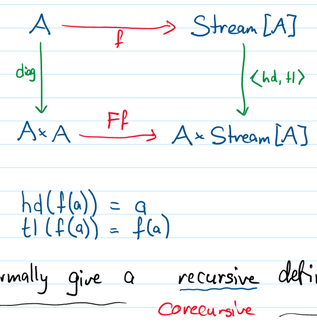
Corecursion is exploiting this finality to define maps $f : A \to T$.

E.g. defining $f : A \to Stream[A]$
$a \mapsto (a, a, a, \dots)$

The functor is $F : X \mapsto A \times X$ $\quad FA = A^2$

Want map $\alpha : A \to A^2$: pick $a \mapsto (a, a)$

Then, there is unique $f : A \to X$ s.t.



i.e. $hd(f(a)) = a$
$tl(f(a)) = f(a)$

To informally give a recursive definition, we say what f does on each constructor. corecursive each destructor does on $f(a)$

what happens when constructors are inputs.
what the destructors of outputs are
↳ if destructor gives term of coinductive type T, then corecursive defn must have $des(f(a)) = f(\text{——})$

Can be done similarly: $f : A^2 \to Str[A]$, $(a,b) \mapsto (a, b, a, b, a, \dots)$

$hd(f(a,b)) = a$
$tl(f(a,b)) = f(b,a)$

$f : A^{<\omega} \to Str(A)$
$\sigma \mapsto \sigma\sigma\sigma\sigma\cdots$

$f : Str[A]^2 \to Str[A]$
$\langle (a_i), (b_i) \rangle \mapsto (a_0, b_0, a_1, b_1, \dots)$

$hd(f(\alpha,\beta)) = hd(\alpha)$
$tl(f(\alpha,\beta)) = f(\beta, tl(\alpha))$

$Str[A]^2 \to A \times Str[A]^2$
$\alpha, \beta \mapsto hd(\alpha), \beta, tl(\alpha)$

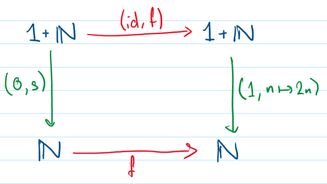
$f : Str[A] \to Str[A]$
$(a_0, a_1, a_2, \dots) \mapsto (a_0, a_2, a_4, \dots)$

$hd(f(\alpha)) = hd(\alpha)$
$tl(f(\alpha)) = f(tl(tl(\alpha)))$

E.g. $F : X \mapsto 1 + X$

F-algebras $A, \alpha_0 \in A, \alpha_S : A \to A$

$\mathbb{N}, 1, n \mapsto 2n$

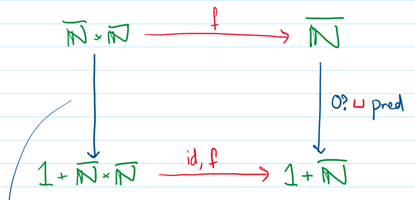


$f(0) = 1$
$f(s(n)) = 2 f(n)$
$n \mapsto 2^n$

$F : X \mapsto 1 + X$

E.g. conaturals $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$
$succ(\infty) = pred(\infty) = \infty$ $\quad \infty = s(s(\overbrace{\cdots}^{\omega} s(0)))$

Subtraction $\overline{\mathbb{N}} \times \overline{\mathbb{N}} \to \overline{\mathbb{N}}$



$\alpha(a,b) = \begin{cases} * & \text{if } a \le b \\ a,sb & \text{else} \end{cases}$

$f(a,b) = 0 \iff a \le b$
$pred(f(a,b)) = f(a, sb)$

Can we corecursively define relations on a coinductive type A?

Coinduction as a proof technique.

Let A be a coinductive type.

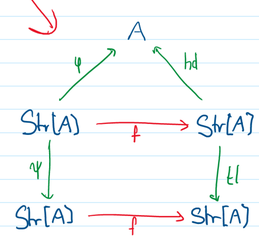
If $a, b : A$ are not equal, they can be distinguished by applying destructors of A finitely many times.

If $aRb \to da \; R \; db$ for all destructors d of A then aRb for all $a, b : A$? No.

A bisimulation on A^ω is a relation $R \subseteq A^\omega \times A^\omega$ such that
$\sigma R \tau \Rightarrow hd(\sigma) = hd(\tau)$ & $tl(\sigma) \; R \; tl(\tau)$

Coinduction: all bisimulations $R \subseteq$ "$=$"
if $\sigma R \tau$ for some bisimulation R,
then $\sigma = \tau$.

This works to prove any proposition which is an equality between elements of A,



$hd(f(\alpha)) = \varphi(\alpha)$ $\quad \varphi = hd$
$tl(f(\alpha)) = f(\psi(\alpha))$ $\quad \psi = tl(tl)$