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MODULE 11 — Practice Assignment

Problem 1

Do the following homework problems from the text book:

- **8-39 (a)**

8-39. The forward-path transfer functions of unity-feedback control systems are given in the following equations. Plot the Bode diagram of G(jω)/K, and do the following: (1) Find the value of K so that the gain margin of the system is 20 dB. (2) Find the value of K so that the phase margin of the system is 45°.

(a) $$\frac{\mathbf{K}}{\mathbf{s(1+0.1s)(1+0.5s)}}$$

The transfer function in question can be written as:

$$G(j\omega) = \frac{K}{-0.05j\omega^3 - 0.6\omega^2 + j\omega} = \frac{K}{-0.6\omega^2 - j(0.05\omega^3 - \omega)}$$

The magnitude of $G(j\omega)$, is calculated as:

$$|G(j\omega)| = \frac{|K|}{|-0.05j\omega^3 - 0.6\omega^2 + j\omega|} = \frac{K}{\sqrt{0.0025\omega^6 + 0.26\omega^4 + \omega^2}}$$

Setting the magnitude of the gain equal to 1, we can solve for the gain crossover frequency. For this we will solve initially with $K = 1$:

$$1 = \frac{1}{\sqrt{0.0025\omega_g^6 + 0.26\omega_g^4 + \omega_g^2}}$$

This becomes:

$$0.0025\omega_g^6 + 0.26\omega_g^4 + \omega_g^2 - 1^2 = 0$$

Solving for the roots of this equation, using the MATLAB `roots()` function:

$$\texttt{roots}([0.0025, 0, 0.26, 0, 1, 0, -1]) = \begin{bmatrix} 0.0000 & +j9.9979 \\ 0.0000 & -j9.9979 \\ 0.0000 & +j2.2055 \\ 0.0000 & -j2.2055 \\ -0.9070 & \\ \mathbf{0.9070} & \end{bmatrix}$$

The gain crossover frequency is $\omega_g = 0.9070$ rad/s.

Taking the initial transfer function, we can rewrite it as:

$$\begin{aligned} G(j\omega) &= \frac{K}{-0.6\omega^2 - j(0.05\omega^3 - \omega)} \left(\frac{-0.6\omega^2 + j(0.05\omega^3 - \omega)}{-0.6\omega^2 + j(0.05\omega^3 - \omega)} \right) \\ &= \frac{K(-0.6\omega^2 + j(0.05\omega^3 - \omega))}{0.36\omega^4 + (0.05\omega^3 - \omega)^2} \end{aligned}$$

The phase angle for this equation is:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{0.05\omega^3 - \omega}{-0.6\omega^2} \right) = \tan^{-1}\left(\frac{0.05\omega^2 - 1}{-0.6\omega} \right)$$

We can determine the phase crossover frequency as:

$$\text{Im}\{G(j\omega)\} = 0.05\omega_p^2 - 1 = 0$$

Which solves for ω_p as:

$$\omega_p = \sqrt{20} = 4.4721$$

The phase crossover frequency is $\omega_p = 4.4721$ rad/s.

The gain margin can be calculated as:

$$|G(j\omega_p)| = 10/120$$

$$-20\log_{10}(10/120) = -21.5836 \text{ dB}$$

If we want to make the gain margin 20 dB, we have to increase the gain magnitude at phase cross over frequency from -21.5836 to -20, which is an increase of 1.5836 dB. We can calculate this gain factor as:

$$K = 10^{(1.5836/20)} = 1.2$$

The value of $\boldsymbol{K}$ so that the gain margin of the system is **20** dB, is $\boldsymbol{K = 1.2}$.

$\longrightarrow$ $\mathcal{A}$nswer

We can solve for the phase margin as:

$$\angle G(j\omega_g) = \tan^{-1}\left(\frac{0.05\omega_g^2 - 1}{-0.6\omega_g}\right)$$

$$\angle G(j0.907) = \tan^{-1}\left(\frac{0.05(0.907)^2 - 1}{-0.6(0.907)}\right) = 60.4231^\circ$$

Submitted by Austin Barrilleaux on November 8, 2023.