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**MODULE 1 — Practice Assignment**

## Problem 1

**Solve the following practice problems in the 9th edition textbook.**

- **Chapter 2:**
  - **2-16(a)**
  - **2-22(a)**

**2-16(a): Find the Laplace transform of the following function:**

$$g(t) = 5te^{-5t}u_s(t)$$

The unit-step function that is defined as:

$$u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The Laplace transform is:

$$G(s) = \mathcal{L}\{5te^{-5t}u_s(t)\} = \int_{t=0}^{\infty} \left(5te^{-5t}\right) e^{-st} dt$$

This can be written as:

$$G(s) = 5 \int_{t=0}^{\infty} te^{-(s+5)t} dt$$

Using integration by parts:

$$G(s) = \left. \frac{5te^{-(s+5)t}}{-(s+5)} \right|_{0}^{\infty} - \frac{5}{-(s+5)} \int_{t=0}^{\infty} e^{-(s+5)t} dt$$

Which further evaluates to:

$$G(s) = \left.\frac{5te^{-(s+5)t}}{-(s+5)}\right|_0^\infty - \frac{5}{-(s+5)}\left.\frac{e^{-(s+5)t}}{-(s+5)}\right|_0^\infty$$

Or:

$$G(s) = \left.\frac{5te^{-(s+5)t}}{-(s+5)}\right|_0^\infty - \left.\frac{5e^{-(s+5)t}}{(s+5)^2}\right|_0^\infty$$

Evaluating the limits:

$$G(s) = \left[\frac{\cancelto{0}{5\infty e^{-(s+5)\infty}}}{-(s+5)} - \frac{\cancelto{0}{50e^{-(s+5)0}}}{-(s+5)}\right]^0 - \left[\frac{\cancelto{0}{5e^{-(s+5)\infty}}}{(s+5)^2} - \frac{\cancelto{}{5e^{-(s+5)0}}}{(s+5)^2}\right]^1$$

This results in the solution:

$$\boldsymbol{G(s) = \frac{5}{(s+5)^2}}$$

$\longrightarrow$ *Answer*

Using the following Laplace Transform from the table in the textbook, we can get the same answer:

$$te^{-\alpha t} \Leftrightarrow \frac{1}{(s+\alpha)^2}$$

**2-22(a): Solve the following differential equations by means of the Laplace transform. Assume zero initial conditions.**

$$\frac{d^2 f(t)}{dt^2} + 5\frac{df(t)}{dt} + 4f(t) = e^{-2t}u_s(t)$$

Take the Laplace transform of both sides:

$$s^2F(s) - sf(0) - f'(0) + 5sF(s) - f(0) + 4F(s) = \frac{1}{s+2}$$

Evaluate initial conditions:

$$s^2F(s) - \cancelto{0}{sf(0)} - \cancelto{0}{f'(0)} + 5sF(s) - \cancelto{0}{f(0)} + 4F(s) = \frac{1}{s+2}$$

Which gives us:

$$s^2F(s) + 5sF(s) + 4F(s) = \frac{1}{s+2}$$

Solving for $F(s)$, the solution is:

$$\boldsymbol{F(s) = \frac{s^2 + 5s + 4}{s+2}}$$

$\longrightarrow$ $\mathcal{A}$nswer

## Problem 2

**Consider the nonlinear system, below, and linearize the system about the nominal point x=2**

$$\dot{x} = 2x^2 - x$$

Linearizing via Taylor Series expansion:

$$\dot{x} = f(x) \approx f(x_n) + \left.\frac{df(x)}{dx}\right|_{x=x_n} (x - x_n)$$

where:

$$f(x) = 2x^2 - x$$

and:

$$\frac{df(x)}{dx} = 4x - 1$$

Using this, we can solve that:

$$\dot{x} \approx (2(2)^2 - 2) + (4(2) - 1)(x - 2)$$

This simplifies to following linear equation:

$$\dot{\boldsymbol{x}} \approx \boldsymbol{7x - 8}$$

$\longrightarrow$ $\mathcal{A}$nswer

## Problem 3

**Compute the Laplace Transform of $f(t) = \sin(\omega t)$ using the one-sided transform integral.**

The Laplace transform is:

$$F(s) = \mathcal{L}\{\sin(\omega t)\} = \int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt$$

Using integration by parts this becomes:

$$F(s) = -\frac{e^{-st}}{s}\sin(\omega t)\Big|_0^{\infty} - \int_{t=0}^{\infty} \cos(\omega t)\omega \frac{-e^{-st}}{s} dt$$

Using integration by parts again this becomes:

$$F(s) = -\frac{e^{-st}}{s}\sin(\omega t)\Big|_0^{\infty} - \cos(\omega t)\omega\frac{e^{-st}}{s^2} - \int_{t=0}^{\infty} \sin(\omega t)\omega^2 \frac{e^{-st}}{s^2} dt$$

Rewriting this equation as:

$$\int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt = -\frac{e^{-st}}{s}\sin(\omega t)\Big|_0^{\infty} - \cos(\omega t)\omega\frac{e^{-st}}{s^2}\Big|_0^{\infty} - \int_{t=0}^{\infty} \sin(\omega t)\omega^2 \frac{e^{-st}}{s^2} dt$$

Evaluating the first two terms in the equation:

$$\begin{aligned}
\int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt = &- \left[\cancelto{0}{\frac{e^{-s(\infty)}}{s}\sin(\omega\infty)} - \cancelto{0}{\frac{e^{-s((0))}}{s}\sin(\omega(0))}\right] \\
&- \left[\cancelto{0}{\cos(\omega\infty)\omega\frac{e^{-s\infty}}{s^2}} - \cancelto{\frac{\omega}{s^2}}{\cos(\omega(0))\omega\frac{e^{-s(0)}}{s^2}}\right] \\
&- \int_{t=0}^{\infty} \sin(\omega t)\omega^2 \frac{e^{-st}}{s^2} dt
\end{aligned}$$

The equation therefore reduces to:

$$\int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2} - \int_{t=0}^{\infty} \sin(\omega t) \omega^2 \frac{e^{-st}}{s^2} dt$$

If we add $\int_{t=0}^{\infty} \sin(\omega t) \omega^2 \frac{e^{-st}}{s^2} dt$ to both sides:

$$\int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt + \int_{t=0}^{\infty} \sin(\omega t) \omega^2 \frac{e^{-st}}{s^2} dt = \frac{\omega}{s^2}$$

And gather terms:

$$\left(1 + \frac{\omega^2}{s^2}\right) \int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{s^2}$$

We can solve for $\int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt$:

$$\int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt = \frac{\frac{\omega}{s^2}}{\left(1 + \frac{\omega^2}{s^2}\right)}$$

If we multiply the top and bottom of the left-hand side by $s^2$ we get the solution:

$$\int_{t=0}^{\infty} \sin(\omega t) e^{-st} dt = \frac{\omega}{(s^2 + \omega^2)}$$

Therefore:

$$\boldsymbol{F(s) = \frac{\omega}{(s^2 + \omega^2)}}$$

$\longrightarrow$ *Answer*

This matches the transform from the table in the textbook.

## Problem 4

**Compute the inverse Laplace transform of the following transfer function using the PFE method and the transform tables in the text:**

$$G(s) = \frac{1}{s(s+1)^3(s+2)}$$

First perform the partial fraction expansion of the transfer function:

$$\frac{1}{s(s+1)^3(s+2)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3} + \frac{E}{(s+2)}$$

Multiply both side by $s(s+1)^3(s+2)$:

$$\begin{aligned} 1 =& A(s+1)^3(s+2) + Bs(s+1)^2(s+2) + \\ & Cs(s+1)(s+2) + Ds(s+2) + Es(s+1)^3 \end{aligned}$$

Expand:

$$\begin{aligned} 1 =& As^4 + 5As^3 + 9As^2 + 7As + 2A + Bs^4 + 4Bs^3 + 5Bs^2 + 2Bs + \\ & Cs^3 + 3Cs^2 + 2Cs + Ds^2 + 2Ds + E^4 + 3Es^3 + 3Es^2 + Es \end{aligned}$$

Collect like powers of $s$:

$$\begin{aligned} 1 =& s^4(A + B + E) + \\ & s^3(5A + 4B + C + 3E) + \\ & s^2(9A + 5B + 3C + D + 3E) + \\ & s(7A + 2B + 2C + 2D + E) + \\ & 2A \end{aligned}$$

Equate the coefficients of like powers on both sides to create a system of equations:

$$\begin{aligned} 1 =& 2A \\ 0 =& (7A + 2B + 2C + 2D + E) \\ 0 =& (9A + 5B + 3C + D + 3E) \\ 0 =& (5A + 4B + C + 3E) \\ 0 =& (A + B + E) \end{aligned}$$

This expressed as a matrix is:

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & 0 & 0 & 1 \\ 7 & 2 & 2 & 2 & 1 & 0 \\ 9 & 5 & 3 & 1 & 3 & 0 \\ 5 & 4 & 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

The following steps put the matrix into reduced row echelon form.

Steps: R1/2

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 7 & 2 & 2 & 2 & 1 & 0 \\ 9 & 5 & 3 & 1 & 3 & 0 \\ 5 & 4 & 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Steps: R2 - 7 $\cdot$ R1, R3 - 9 $\cdot$ R1, R4 - 5 $\cdot$ R1, R5 - R1

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 2 & 2 & 2 & 1 & -\frac{7}{2} \\ 0 & 5 & 3 & 1 & 3 & -\frac{9}{2} \\ 0 & 4 & 1 & 0 & 3 & -\frac{5}{2} \\ 0 & 1 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Steps: R2 - 2 $\cdot$ R5, R3 - 5 $\cdot$ R5, R4 - 4 $\cdot$ R5

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 2 & 2 & -1 & -\frac{5}{2} \\ 0 & 0 & 3 & 1 & -2 & -\frac{4}{2} \\ 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Steps: R2 - 2 $\cdot$ R4, R3 - 3 $\cdot$ R4

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 2 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Steps: R2 - 2 $\cdot$ R3

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Steps: R5 - R2

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right]$$

Steps: R4 + R2

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right]$$

Steps: Order so that left-hand side is identity:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Given this, transfer function is:

$$G(s) = \frac{1}{2s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^3} + \frac{1}{2(s+2)}$$

Using the following Laplace transforms:

$$u_s(t) \Leftrightarrow \frac{1}{s}$$

$$e^{-\alpha t} \Leftrightarrow \frac{1}{s+\alpha}$$
$$t^n e^{-\alpha t} \Leftrightarrow \frac{n!}{(s+\alpha)^{n+1}}$$

The inverse Laplace Transform of the transfer function is:

$$\boldsymbol{\mathcal{L}^{-1}\{G(s)\} = \frac{1}{2}u_s(t) - e^{-t} - \frac{1}{2}t^2 e^{-t} + \frac{1}{2}e^{-2t}}$$

$\longrightarrow \mathcal{A}$nswer

*Submitted by Austin Barrilleaux on September 3, 2023.*