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MODULE 12 — Practice Assignment

Problem 1

Solve the following 9th Edition textbook problems:

- **9-49 (a)**
- **9-50 (c,d)**

9-49: Consider that the controller in the liquid-level control system shown in Fig. 9P-10 is a single-stage phase-lag controller:

$$\mathbf{G_c(s) = \frac{1 + aTs}{1 + Ts}, \quad a < 1}$$

$$\mathbf{G_p(s) = \frac{10N}{s(s+1)(s+10)}}$$

(a) For $\mathbf{N = 20}$, select the values of a and T so that the two complex roots of the characteristic equation correspond to a relative damping ratio of approximately 0.707. Plot the unit-step response of the output $\mathbf{y(t)}$. Find the attributes of the unit-step response. Plot the Bode plot of $\mathbf{G_c(s)G_p(s)}$ and determine the phase margin of the designed system.

This makes the process:

$$G_p(s) = \frac{200}{s(s+1)(s+10)}$$

The compensated system is:

$$G_c(s)G_p(s) = \frac{200(1 + aTs)}{s(s+1)(s+10)(1+Ts)}$$

Rewriting the uncompensated process as:

$$G_p(s) = \frac{K}{s(s+1)(s+10)}$$

K_{SSE} to satisfy the SSE requirement is $K_{\text{SSE}} = 200$.

Looking at the root locus of the uncompensated system where $K = 1$, we can that $K_{\%\text{OS}}$ for a damping ratio of 0.707 is $K_{\%\text{OS}} = 4.54$:

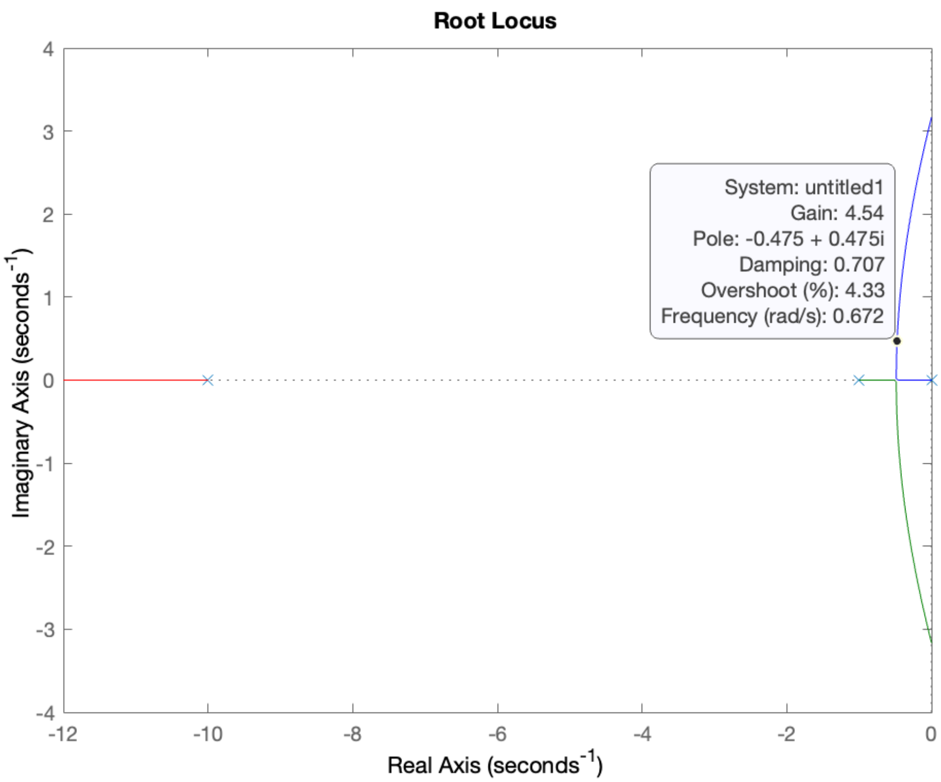

Figure 1: P9-49 (a): Root Locus

We can calculate a for the compensated system as:

$$a = \frac{K_{\%\text{OS}}}{K_{\text{SSE}}} = \frac{4.54}{200} = 0.0227$$

To determine a value for T as a general guideline, the frequency $\frac{1}{aT}$ should be approximately one decade below ω_g', the crossover frequency of the forward path transfer function when $K = K_{\%\mathrm{OS}}$. Looking at the bode plot for $\mathbf{G_p(s)} = \frac{\mathbf{4.54}}{\mathbf{s(s+1)(s+10)}}$ in MATLAB:

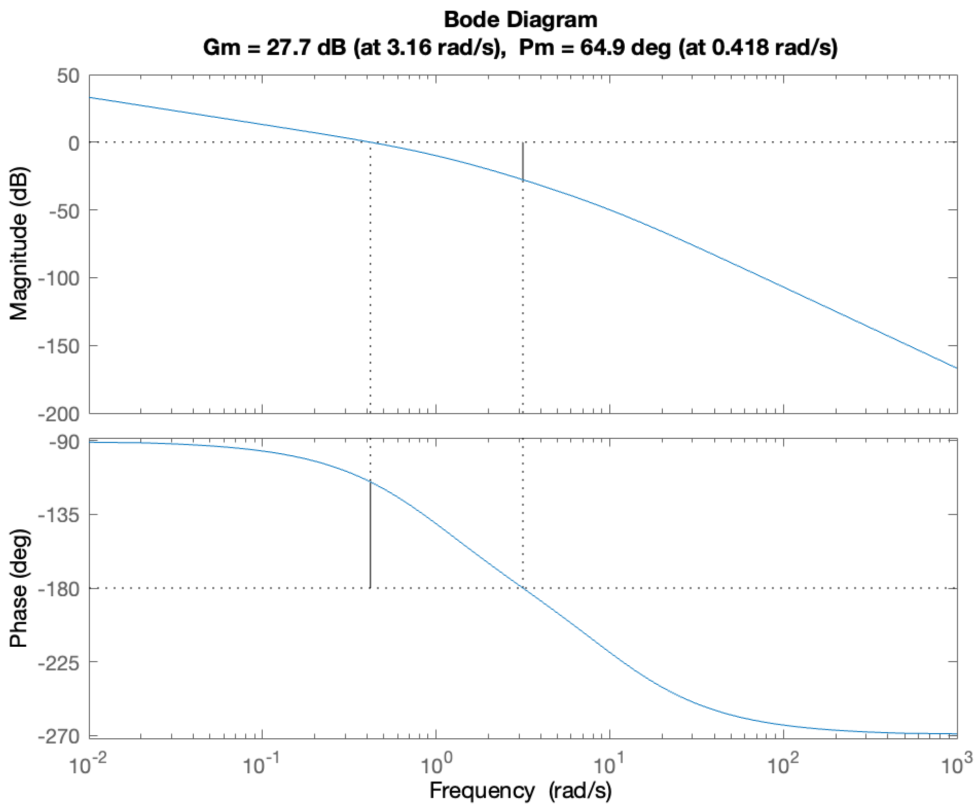

Figure 2: P9-49 (a): Bode Plot, G_p, $K = 4.54$

Therefore, T should be calculated as:

$$T = \left(\frac{\omega_g' a}{10}\right)^{-1} = \left(\frac{(0.418)(0.0227)}{10}\right)^{-1} = 1053.90 \approx 1050$$

$$\boldsymbol{a = 0.0227} \text{ and } \boldsymbol{T = 1050}$$

$\longrightarrow$ $\mathcal{A}$nswer

Looking at the closed-loop step-response of the system in MATLAB:

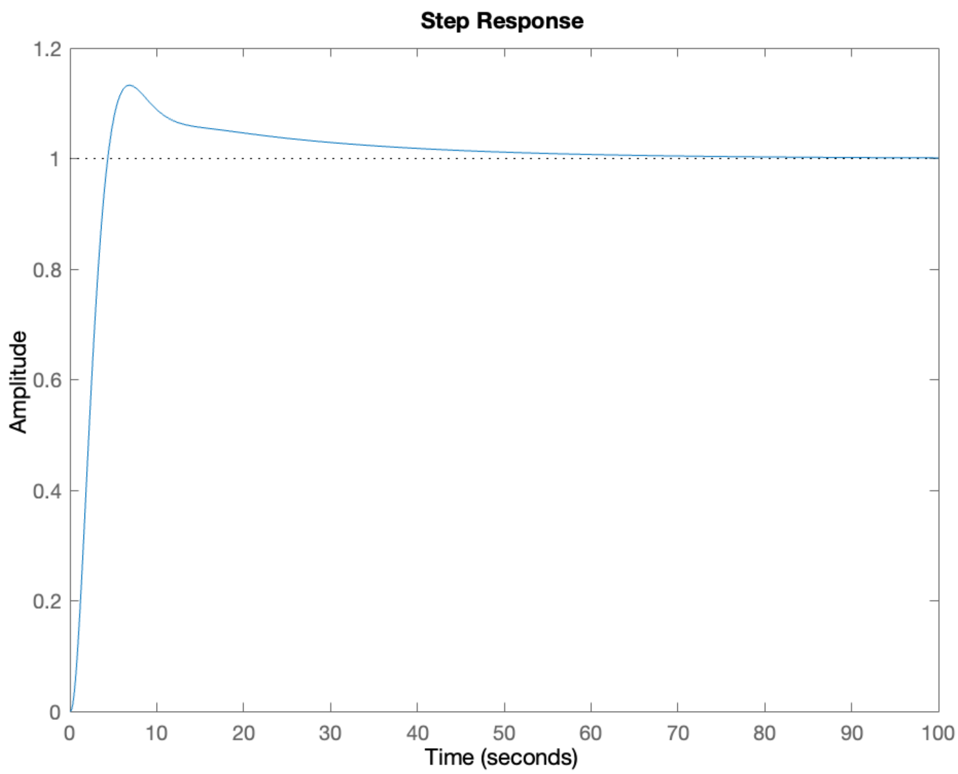

Figure 3: P9-49 (a): Step Response

The attributes of the unit-step response, using the `stepinfo()` function in MATLAB, are:

$$\texttt{stepinfo()} = \left[\begin{array}{rl} \textbf{RiseTime:} & \textbf{2.8926} \\ \textbf{TransientTime:} & \textbf{38.05733} \\ \textbf{SettlingTime:} & \textbf{38.05733} \\ \textbf{SettlingMin:} & \textbf{0.9061} \\ \textbf{SettlingMax:} & \textbf{1.1325} \\ \textbf{Overshoot:} & \textbf{13.2498} \\ \textbf{Undershoot:} & \textbf{0} \\ \textbf{Peak:} & \textbf{1.1325} \\ \textbf{PeakTime:} & \textbf{6.8196} \end{array}\right]$$

$\longrightarrow \mathcal{A}$nswer

Plotting the Bode plot for the forward path transfer function, $G_c(s)G_p(s)$, where:

$$G_c(s)G_p(s) = \frac{4.54(s+0.041)}{s(s+1)(s+10)(s+0.00095)}$$

We get the plot:

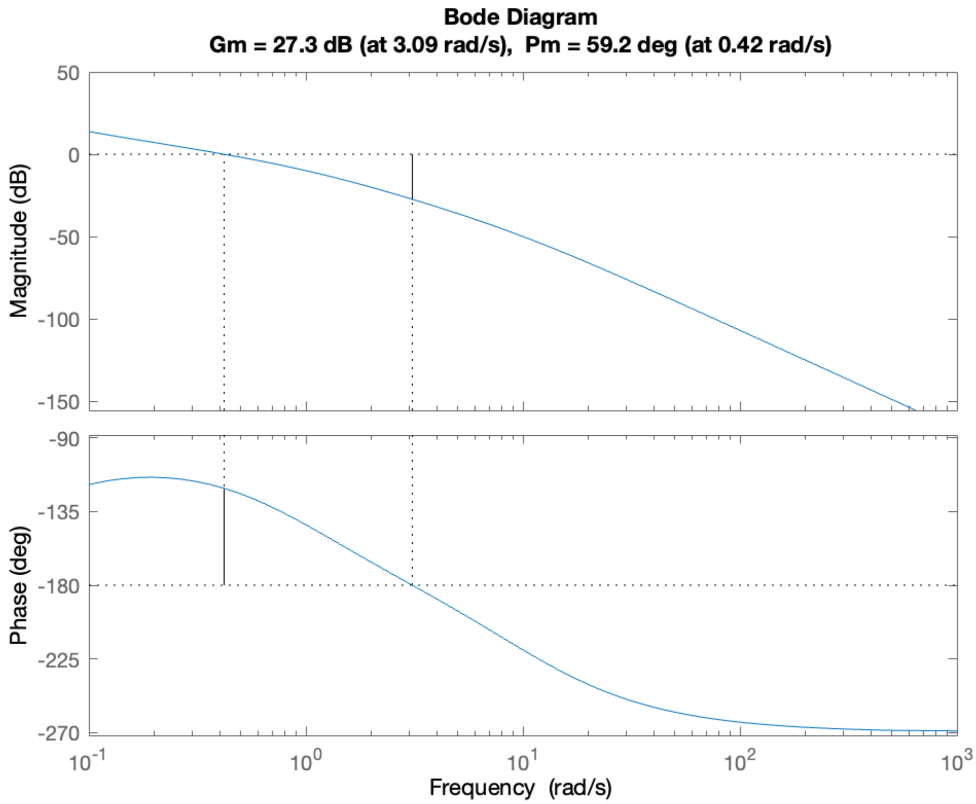

Figure 4: P9-49 (a): Bode Plot

The Gain and Phase Margin of the system, respectively, are $\boldsymbol{GM} = \mathbf{27.3\ dB}$ and $\boldsymbol{PM} = \mathbf{59.2^\circ}$.

$\longrightarrow$ $\mathcal{A}$nswer

9-50: The controlled process of a unity-feedback control system is:

$$\mathbf{G_p(s) = \frac{K}{s(s+5)^2}}$$

The series controller has the transfer function:

$$\mathbf{G_c(s) = \frac{1 + aTs}{1 + Ts}}$$

(c) Design a phase-lag controller (a < 1) so that the following performance specifications are satisfied:

- **Ramp-error constant $\mathbf{K_v = 10}$**
- **Maximum Overshoot < 1 %**
- **Rise time, $\mathbf{t_r < 2}$ sec**
- **Settling time, ts < 2.5 sec**

Find the PM, GM, M_r and BW of the designed system.

For the system, since $K_v = 10$, we can solve for K via the following:

$$K_v = \lim_{s \to 0} sG(s) = \frac{K(1 + aTs)s}{s(s+5)^2(1+Ts)} = \frac{K}{5^2} = 10$$

Solving for K, $K = 250$, which is K_{SSE} to satisfy the SSE requirement.
To calculate the damping ratio, ζ that would yield a 1% overshoot, we can use the formula:

$$\zeta = \frac{-\log\left(\frac{\%\text{OS}}{100}\right)}{\sqrt{\pi^2 + \log\left(\frac{\%\text{OS}}{100}\right)^2}} = \frac{-\log\left(\frac{1}{100}\right)}{\sqrt{\pi^2 + \log\left(\frac{1}{100}\right)^2}} = 0.826$$

Looking at the root locus of the uncompensated system where $K = 1$, we can that $K_{\%\text{OS}}$ for a damping ratio of 0.826 is $K_{\%\text{OS}} = 24.5$:

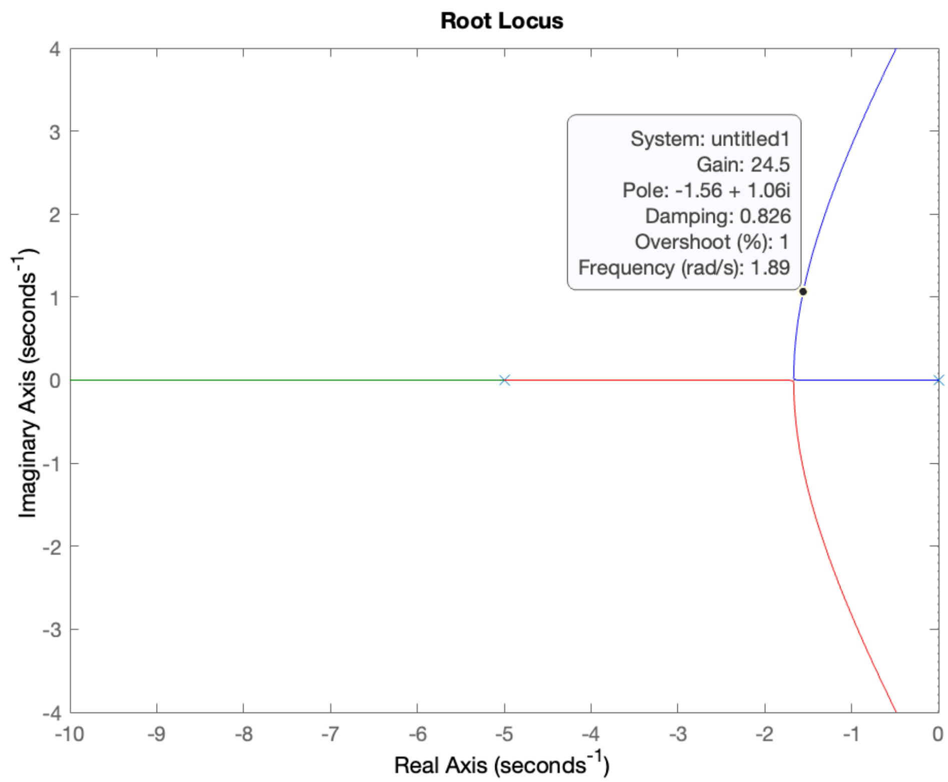

Figure 5: P9-50 (c): Root Locus

We can calculate a for the compensated system as:

$$a = \frac{K_{\%\text{OS}}}{K_{\text{SSE}}} = \frac{24.5}{250} = 0.098$$

To get a K that will satisfy both %OS and t_r, we will apply a fudge factor, γ of 2% to this value:

$$a = \gamma \frac{K_{\%\text{OS}}}{K_{\text{SSE}}} = \left(\frac{1}{1+\text{SM}}\right) \frac{K_{\%\text{OS}}}{K_{\text{SSE}}} = \left(\frac{1}{1.02}\right) \frac{24.5}{250} = 0.0961$$

As in the above problem, to determine a value for T as a general guideline, the frequency $\frac{1}{aT}$ should be approximately one decade below ω_g', the crossover frequency of the forward path transfer function when $K = K_{\%OS}$. Looking at the bode plot for $\mathbf{G_p(s)} = \frac{\mathbf{250a}}{\mathbf{s(s+5)^2}}$ in MATLAB:

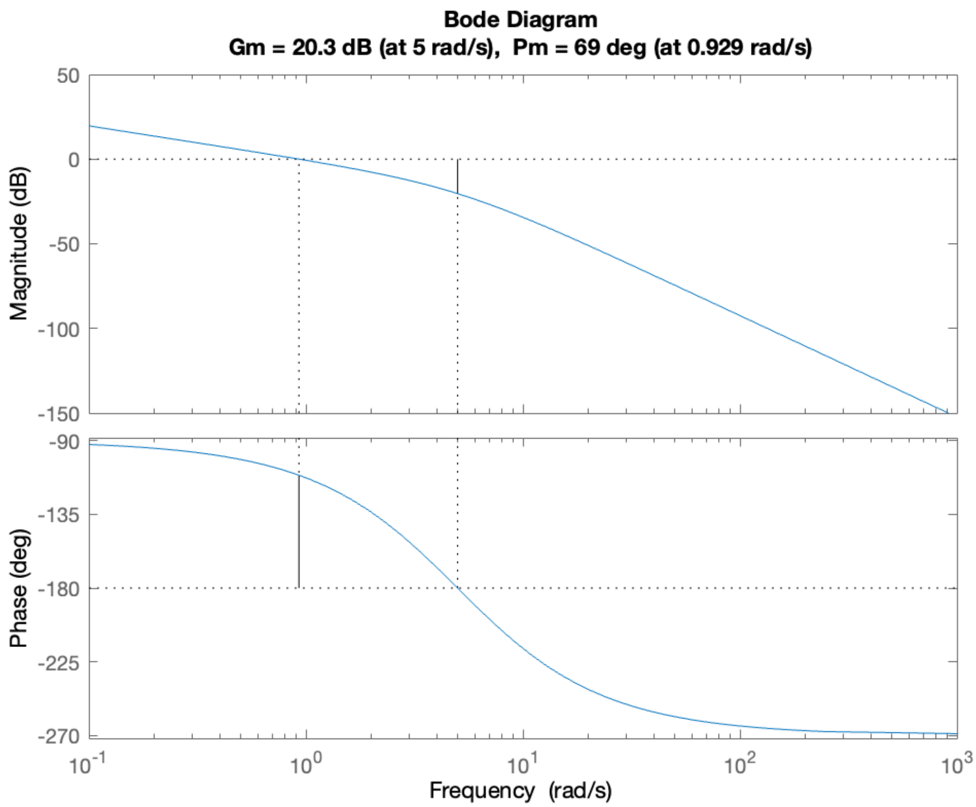

Figure 6: P9-50 (c): Bode Plot, G_p, $K = 250(0.0961)$

Therefore, T should be initially calculated as:

$$T = \left(\frac{\omega_g' a}{10}\right)^{-1} = \left(\frac{(0.929)(0.0961)}{10}\right)^{-1} = 112.03 \approx 100$$

The attributes of the unit-step response at this value of T, using the `stepinfo()` function in MATLAB, are:

$$
\texttt{stepinfo()} = \begin{bmatrix} \textbf{RiseTime:} & 1.2694 \\ \textbf{TransientTime:} & 16.3925 \\ \textbf{SettlingTime:} & 16.3925 \\ \textbf{SettlingMin:} & 0.9026 \\ \textbf{SettlingMax:} & 1.0962 \\ \textbf{Overshoot:} & 9.6168 \\ \textbf{Undershoot:} & 0 \\ \textbf{Peak:} & 1.0962 \\ \textbf{PeakTime:} & 3.2774 \end{bmatrix}
$$

This value of T does not provide an ideal response for the system, as overshoot and settling time are too high. From here, if we continuously increase T by 100 until we get an acceptable response, we see that when $T = 4400$, all of the requirements are pulled into range. To provide a little extra cushion, when $T = 4500$:

$$
\texttt{stepinfo()} = \begin{bmatrix} \textbf{RiseTime:} & 1.4394 \\ \textbf{TransientTime:} & 2.3314 \\ \textbf{SettlingTime:} & 2.3314 \\ \textbf{SettlingMin:} & 0.9051 \\ \textbf{SettlingMax:} & 1.0099 \\ \textbf{Overshoot:} & 0.9908 \\ \textbf{Undershoot:} & 0 \\ \textbf{Peak:} & 1.0099 \\ \textbf{PeakTime:} & 3.2681 \end{bmatrix}
$$

The values of a and T for a response that meets the requested requirements:

$$
\boldsymbol{a = 0.0961} \text{ and } \boldsymbol{T = 4500}
$$

$\longrightarrow$ $\mathcal{A}$nswer

For this design of the unity feedback system:

$$
G_c(s)G_p(s) = \frac{24.01(s + 0.0023)}{s(s+5)^2(s+0.0002)}
$$

The closed-loop step-response can be seen in the following MATLAB plot:

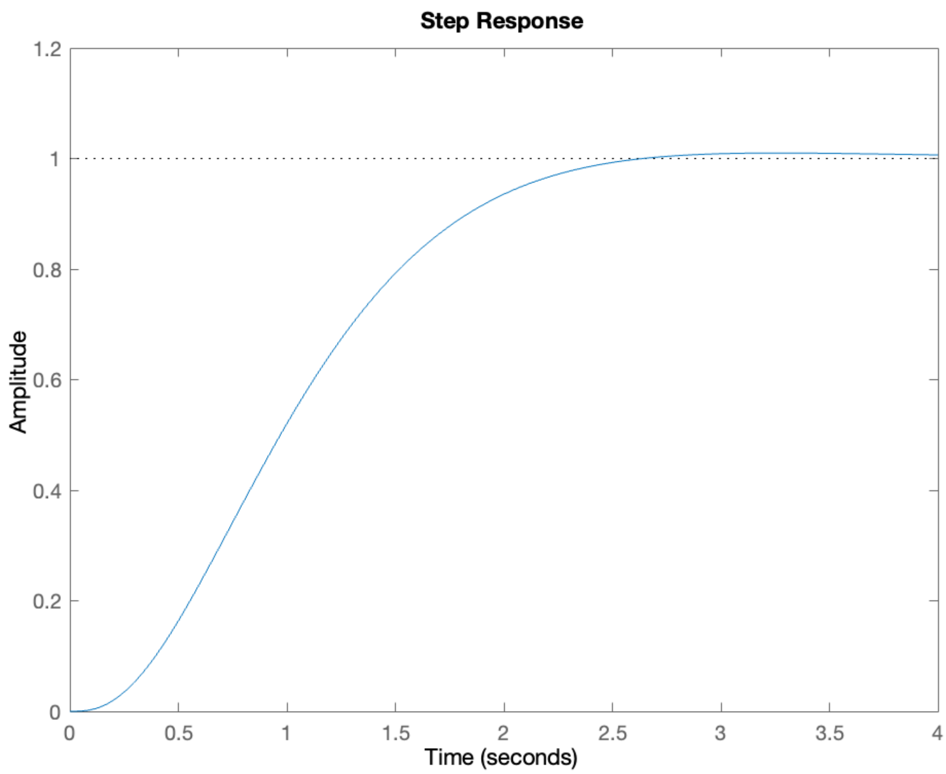

Figure 7: P9-50 (c): Step Response

Gain and Phase Margin for the system can be found using the `margin()` function in MATLAB:

$$\texttt{[GM, PM] = margin():} \begin{bmatrix} \boldsymbol{GM =} & \mathbf{20.3402\ dB} \\ \boldsymbol{PM =} & \mathbf{68.8257^\circ} \end{bmatrix}$$

$\longrightarrow \mathcal{A}$nswer

M_r for the system can be found using the `getPeakGain()` function in MATLAB:

MR = getPeakGain(): $[\boldsymbol{M_r = 1.0011}]$

$\longrightarrow$ $\mathcal{A}$nswer

BW for the system can be found using the `bandwidth()` function in MATLAB:

BW = bandwidth(): $\left[\boldsymbol{BW = 1.4883 \text{ rad/sec}}\right]$

$\longrightarrow$ $\mathcal{A}$nswer

(d) Design the phase-lag controller in the frequency domain so that the following performance specifications are satisfied:

- **Ramp-error constant $\mathbf{K_v = 10}$**
- **Phase margin $\geq \mathbf{70^\circ}$**

Check the unit-step response attributes of the designed system and compare with those obtained in part (c).

Note that, as in part (c), $K = 250$.

The Bode plot of the uncompensated system, $G(s) = \frac{250}{s(s+5)^2}$ is:

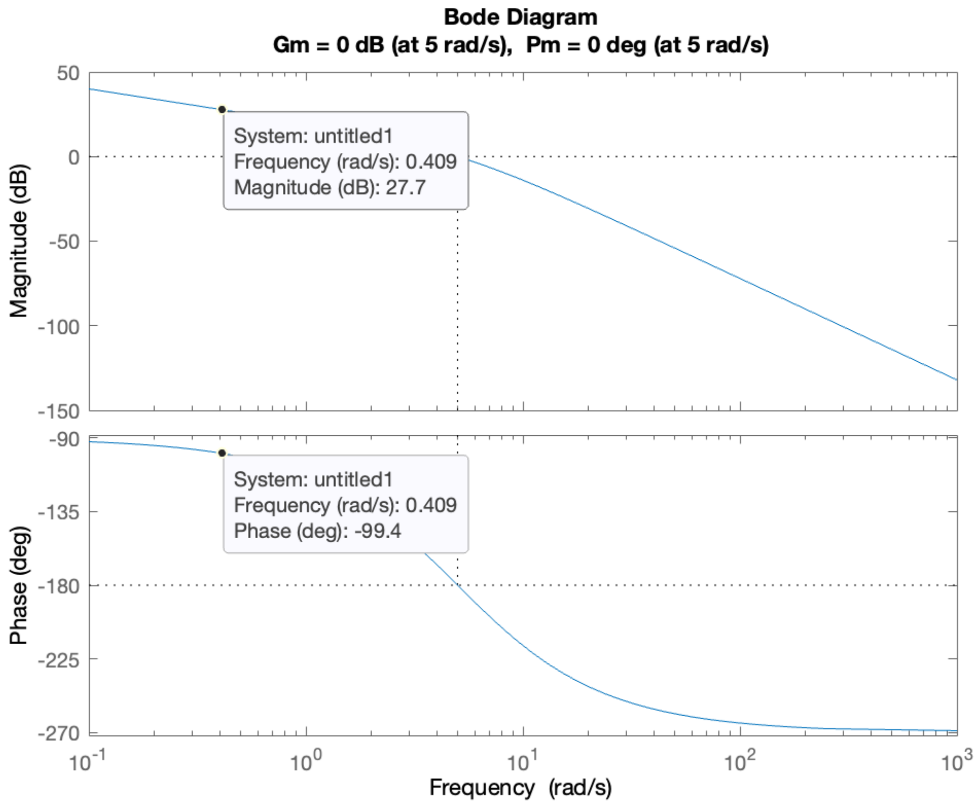

Figure 8: P9-50 (d): Bode Plot, G_p, $K = 250$

If we want to make the Phase Margin of the system $\geq 70°$, since phase margin is initially zero, we need to increase the phase margin by more than $70°$. To add margin, we will increase it by $80°$. To do so, looking at the Bode plot, we see that to get a phase margin of $80°$, we must increase the gain margin by 27.7 dB.

This allows us to solve for the controller value of a as:

$$a = 10^{\left(\frac{-GM}{20}\right)} = 10^{\left(\frac{-27.7}{20}\right)} = 0.0412$$

If we set the value for $\frac{1}{aT}$ to be approximately one decade below ω_g', the crossover frequency of the forward path transfer function, T is calculated as:

$$T = \left(\frac{\omega_g' a}{10}\right)^{-1} = \left(\frac{(0.409)(0.0412)}{10}\right)^{-1} = 593.4436 \approx 600$$

$$\boldsymbol{a = 0.0412} \text{ and } \boldsymbol{T = 600}$$

$\longrightarrow \mathcal{A}$nswer

For this design of the unity feedback system:

$$G_c(s)G_p(s) = \frac{9.89(s + 0.0404)}{s(s+5)^2(s+0.0017)}$$

The attributes of the unit-step response for these values of a and T, using the `stepinfo()` function in MATLAB, are:

$$\texttt{stepinfo()} = \begin{bmatrix} \textbf{RiseTime:} & 3.6073 \\ \textbf{TransientTime:} & 41.6464 \\ \textbf{SettlingTime:} & 41.6464 \\ \textbf{SettlingMin:} & 0.9001 \\ \textbf{SettlingMax:} & 1.0705 \\ \textbf{Overshoot:} & 7.0494 \\ \textbf{Undershoot:} & 0 \\ \textbf{Peak:} & 1.0705 \\ \textbf{PeakTime:} & 11.4026 \end{bmatrix}$$

In part (c), the rise and settling times for the response were much faster, and the overshoot for this response was significantly higher at about 7%. Given that the system in part (c) nearly met the 70° PM requirement, it's design has a much more desirable response.

A plot of the step-response of the closed-loop system can be seen in the following MATLAB plot:

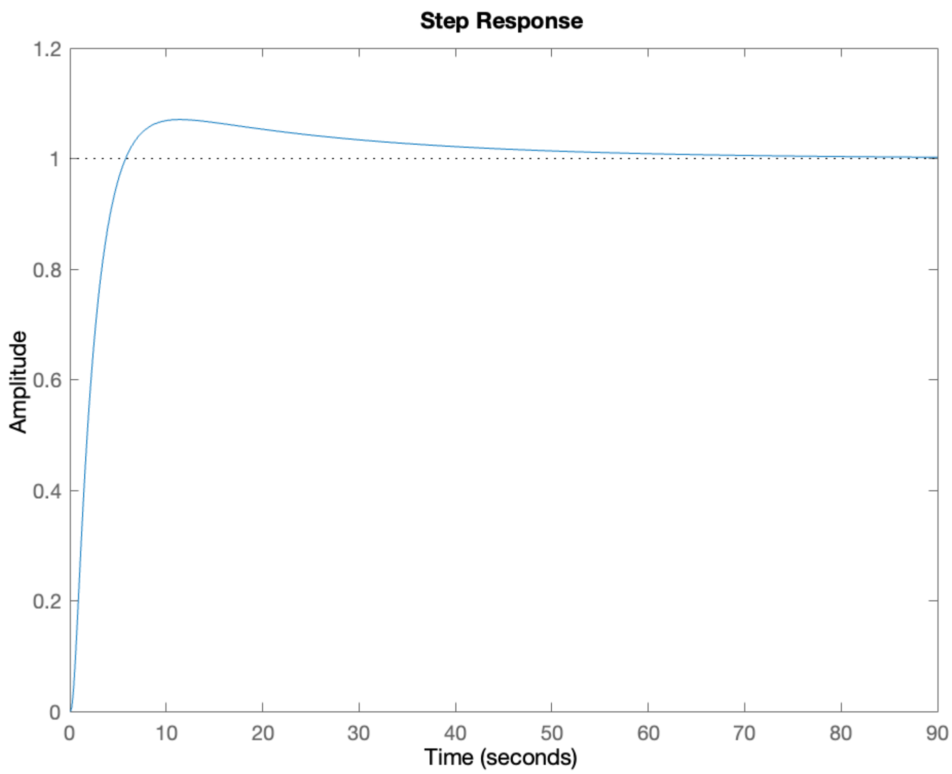

Figure 9: P9-50 (d): Step Response

Submitted by Austin Barrilleaux on December 1, 2023.