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MODULE 5 — Practice Assignment

Problem 1

Solve the following practice problems in the 9th edition textbook.

- **Chapter 2:**
  - **2-33** (a-f)
  - **2-39**

2-33. Without using the Routh-Hurwitz criterion, determine if the following systems are asymp- totically stable. marginally stable, or unstable. In each case, the closed-loop system transfer function is given.

(a) $\mathbf{M(s) = \dfrac{10(s+2)}{s^3 + 3s^2 + 5s}}$

Using the `roots()` function MATLAB, we get that the roots are:

$$\texttt{roots([1,3,4,0])} = \begin{pmatrix} 0 \\ -1.5000 + 1.6583\mathrm{i} \\ -1.5000 - 1.6583\mathrm{i} \end{pmatrix}$$

There is a real pole at $s = 0$.

The system is **marginally stable**.

$\longrightarrow \mathcal{A}$nswer

(b) $\mathbf{M(s) = \dfrac{(s-1)}{(s+5)(s^2+2)}}$

Using the `roots()` function MATLAB, we get that the roots are:

$$\texttt{roots(conv([1,5],[1,0,2]))} = \begin{pmatrix} -5 \\ 0.0 + 1.4142\text{i} \\ 0.0 - 1.4142\text{i} \end{pmatrix}$$

There are complex conjugate poles on the imaginary axis (real parts of s equal to zero).

The system is **marginally stable**.

$\longrightarrow$ $\mathcal{A}$nswer

(c) $\mathbf{M(s) = \dfrac{K}{s^3+5s+5}}$

Using the `roots()` function MATLAB, we get that the roots are:

$$\texttt{roots([1,0,5,5])} = \begin{pmatrix} 0.4344 + 2.3593\text{i} \\ 0.4344 - 2.3593\text{i} \\ -0.8688 \end{pmatrix}$$

There are complex conjugate poles in the right-hand plane (RHP).

The system is **unstable**.

$\longrightarrow$ $\mathcal{A}$nswer

(d) $\mathbf{M(s) = \dfrac{100(s-1)}{(s+5)(s^2+2s+2)}}$

Using the `roots()` function MATLAB, we get that the roots are:

$$\texttt{roots(conv([1,5],[1,2,2]))} = \begin{pmatrix} -5 \\ -1 + 1\text{i} \\ -1 - 1\text{i} \end{pmatrix}$$

All poles exist in the left-hand plane (LHP).

The system is **stable**.

$\longrightarrow$ $\mathcal{A}$nswer

(e) $\mathbf{M(s) = \dfrac{100}{s^3 - 2s^2 + 3s + 10}}$

Using the `roots()` function MATLAB, we get that the roots are:

$$\texttt{roots([1,-2,3,10])} = \begin{pmatrix} 1.6694 + 2.1640i \\ 1.6694 - 2.1640i \\ -1.3387 \end{pmatrix}$$

There are complex conjugate poles in the right-hand plane (RHP).

The system is **unstable**.

$\longrightarrow$ $\mathcal{A}$nswer

(f) $\mathbf{M(s) = \dfrac{10(s + 12.5)}{s^4 + s^3 + 50s^2 + s + 10^6}}$

Using the `roots()` function MATLAB, we get that the roots are:

$$\texttt{roots([1,3,50,1,10\^{}6])} = \begin{pmatrix} -22.8487 + 22.6376i \\ -22.8487 - 22.6376i \\ 21.3487 + 22.6023i \\ 21.3487 - 22.6023i \end{pmatrix}$$

There are complex conjugate poles in the right-hand plane (RHP).

The system is **unstable**.

$\longrightarrow$ $\mathcal{A}$nswer

2-39. The loop transfer function of a single-loop feedback control system is given as

$$\mathbf{G(s)H(s) = \frac{K(s + 5)}{s(s + 2)(1 + Ts)}}$$

The parameters K and T may be represented in a plane with K as the horizontal axis and T as the vertical axis. Determine the regions in the T-versus-K parameter plane where the closed-loop system is asymptotically stable and where it is unstable. Indicate the boundary on which the system is marginally stable.

The closed loop transfer function of the above loop transfer function is:

$$\frac{\mathbf{G(s)H(s)}}{\mathbf{1+G(s)H(s)}} = \frac{\frac{\mathbf{K(s+5)}}{\mathbf{s(s+2)(1+Ts)}}}{\mathbf{1+\frac{K(s+5)}{s(s+2)(1+Ts)}}} = \frac{K(s+5)}{s(s+2)(1+Ts)+K(s+5)}$$

This makes the characteristic equation:

$$s(s+2)(1+Ts)+K(s+5)=0$$

Which can be written as:

$$Ts^3+(1+2T)s^2+(2+K)s+5K=0$$

$\mathbf{s^3}$	T	(2 + K)
$\mathbf{s^2}$	(1 + 2T)	5K
$\mathbf{s^1}$	$\frac{(2+K)(1+2T)-5TK}{(1+2T)}$	0
$\mathbf{s^0}$	5K	0

The third left-most row is calculated as:

$$-\frac{\begin{vmatrix} T & (2+K) \\ (1+2T) & 5K \end{vmatrix}}{(1+2T)} = \frac{(2+K)(1+2T)-5TK}{(1+2T)}$$

The fourth left-most row is simply equal to the coefficient of s^0.

Given this Routh array, the following must be true so that no signs change occours in the left-most column.

$$T>0$$

$$(T+\frac{1}{2})>0$$

$$(2+K)(1+2T)-5TK>0$$

$$K>0$$

Taking $(2+K)(1+2T)-5TK>0$, we can write this as:

$$2 + K + 4T + 2TK - 5TK > 0$$

Which simplifies to:

$$2 + K + 4T + -3TK > 0$$

Solving for K:

$$2 + 4T + K(1 - 3T) > 0$$

$$2 + 4T + > K(3T - 1)$$

$$\frac{(2 + 4T)}{(3T - 1)} > K$$

The condition of stability exists when $T > 0$, $K > 0$ and $K < \frac{(2+4T)}{(3T-1)}$. The following figure shows the stability region:

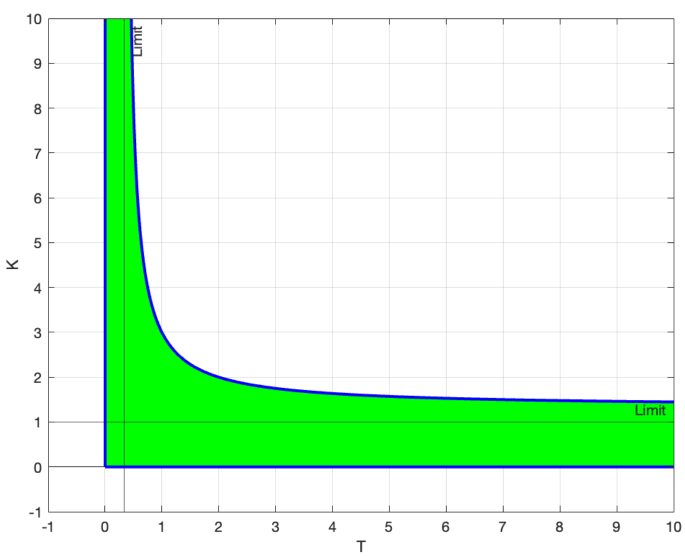

Figure 1: Stability Region

Populating these three boundaries with test values, I can evaluate the roots of the characteristic equation:

If I use values of $T = 0$ and $K = 1$:

$$\texttt{roots(subs([T,(1 + 2*T),(2 + K),5*K],[T,K],[0,1]))} = \begin{pmatrix} -2.5000 - 2.9580\mathrm{i} \\ -2.5000 + 2.9580\mathrm{i} \end{pmatrix}$$

Indicating that the system is **stable** along the boundary $\boldsymbol{T = 0}$, since both poles are in the left-hand plane (LHP).

$\longrightarrow \mathcal{A}$nswer

If I use values of $T = 1$ and $K = 0$:

$$\texttt{roots(subs([T,(1 + 2*T),(2 + K),5*K],[T,K],[1,0]))} = \begin{pmatrix} -3 \\ 0.0 - 2.2361\mathrm{i} \\ 0.0 + 2.2361\mathrm{i} \end{pmatrix}$$

Indicating that the system is **marginally stable** along the boundary $\boldsymbol{K = 0}$, as there are complex conjugate poles with real components equal to zero.

$\longrightarrow \mathcal{A}$nswer

If we evaluate $K = \frac{(2+4T)}{(3T-1)}$ with the value $T = 2$, we get $K = 2$. Evaluating the characteristic equation with these values:

$$\texttt{roots(subs([T,(1 + 2*T),(2 + K),5*K],[T,K],[2,2]))} = \begin{pmatrix} -2.5000 \\ 0.0 - 1.4142\mathrm{i} \\ 0.0 + 1.4142\mathrm{i} \end{pmatrix}$$

Indicating that the system is **marginally stable** along the boundary $\boldsymbol{K = 0}$, as there are complex conjugate poles with real components equal to zero.

$\longrightarrow \mathcal{A}$nswer

If I use values of $T = 0$ and $K = 0$:

$$\texttt{roots(subs([T,(1 + 2*T),(2 + K),5*K],[T,K],[0,0]))} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Indicating that the system is **marginally stable**at the point $\boldsymbol{K = 0}$ and $\boldsymbol{T = 0}$, as there is a pole with a value of zero.

$\longrightarrow \mathcal{A}$nswer

Problem 2

Consider the following transfer function and do the following

$$\mathbf{G(s) = \frac{Y(s)}{R(s)} = \frac{3s + 2}{2s^3 + 4s^2 + 5s + 1}}$$

(a) Employ the Routh-Hurwitz criterion to determine if this system stable or unstable?

I wrote the following MATLAB function to solve for a Routh-Array given the coefficients of a characteristic equation:

```
function table = routhHurwitz(CE)

    arguments
        CE (1,:) double % Characteristic Equation
    end

    % Get number of rows and columns
    num_row = size(CE,2);
    num_col = ceil(num_row/2); % Rounds up

    % Initialize table as all zeros:
    table = zeros(num_row,num_col);

    % Assign the first two rows:
    row = 1;
    col = 1;
    for k = 1:num_row
        % Assign coeffcient to table
        table(row,col) = CE(k);
        % Assign rows and  for next iteration
        if row == 1
            row = 2;
        elseif row == 2
            row = 1;
            col = col+1;
        end
    end

    % Compute rest ob table
    for iterRow = 3:num_row-1
```

```
        for iterCol = 1:num_col-1
            % Intialize matrix for determinant
            matrix = zeros(2,2);
            % Populate matrix for determinant
            matrix(1,1) = table(iterRow-2,1);
            matrix(2,1) = table(iterRow-1,1);
            matrix(1,2) = table(iterRow-2,iterCol+1);
            matrix(2,2) = table(iterRow-1,iterCol+1);
            % Compute table value
            table(iterRow,iterCol) = ...
                -det(matrix)/table(iterRow-1,1);
        end
    end
    % Last table value is equal to coefficient of
    % s^0
    table(end,1) = CE(1,end);

end
```

If I run it using the command `routhHurwitz([2,4,5,1])`, I get the following result:

$\mathbf{s^3}$	2	5
$\mathbf{s^2}$	4	1
$\mathbf{s^1}$	4.5	0
$\mathbf{s^0}$	1	0

Since there are no sign changes in the left-most column, the system is **stable**.

$\longrightarrow$ *Answer*

If I evaluate the command `roots([2,4,5,1])` in MATLAB, I get the following:

$$\texttt{roots([2,4,5,1])} = \begin{pmatrix} -0.8796 + 1.1414i \\ -0.8796 - 1.1414i \\ -0.2408 \end{pmatrix}$$

All poles exist in the left-hand plane (LHP).

The system is **stable**.

⟶ *Answer*

* Note that the function I wrote is not rubust, as if any of the determinants evaluate zero, the resulting table will be incorrect.

(b) Define the numerator and denominator in MATLAB, and use the STEP command to plot the system unit step response

I evaluated the following in MATLAB:

```
% Problem 2 (b)

num = [3,2];
den = [2,4,5,1];

sys = tf(num,den);
step(sys)
```

This produced the following plot:

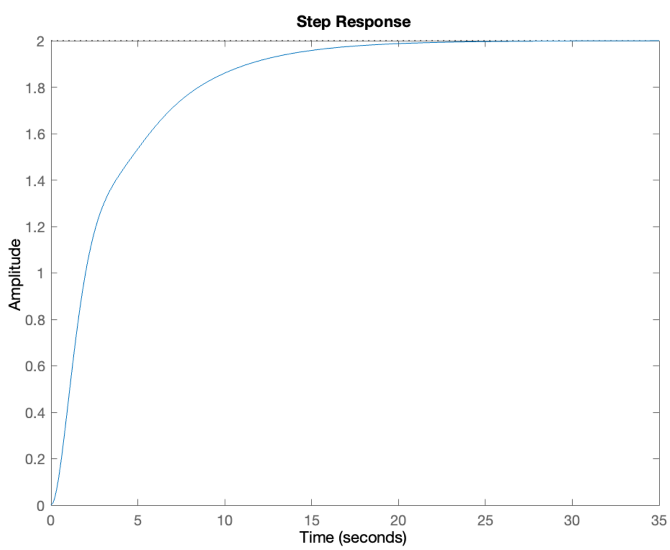

Figure 2: Step Response

(c) Use the final value theorem to determine the steady state value of the system - does this agree with the step response?

From the final value theorem we know that:

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s)$$

Therefore:

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s)$$

For the open loop case, given $r(t) = u_s(t)Y$:

$$R(s) = \frac{1}{s}$$

If we substitute this into the closed loop transfer function, we get:

$$sY(s) = \frac{3s + 2}{2s^3 + 4s^2 + 5s + 1}$$

Therefore:

$$\boldsymbol{\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{3s + 2}{2s^3 + 4s^2 + 5s + 1} = \frac{2}{1} = 2}$$

$\longrightarrow$ $\mathcal{A}$nswer

The steady state value of the system is **2**, and this **agrees** with the plot that I produced in part (c).

$\longrightarrow$ $\mathcal{A}$nswer

Submitted by Austin Barrilleaux on September 30, 2023.