Austin Barrilleaux
Whiting School of Engineering
Johns Hopkins University

October 15, 2023

MODULE 7 — Practice Assignment

Problem 1

Solve the following practice problems in the 9th edition textbook.

- **Chapter 5:**
  - **5-3 (b,d,f)**
  - **5-4 (c,d)**

5-3. Determine the step, ramp, and parabolic error constants of the following unity-feedback control systems. The forward-path transfer functions are given.

(b) $\mathbf{G(s) = \dfrac{100}{s(s^2 + 10s + 100)}}$

The step error constant K_p, is calculated as:

$$K_p = \lim_{s\to 0} \left[G(s)\right] = \lim_{s\to 0} \left[\frac{100}{s(s^2 + 10s + 100)}\right] = \frac{100}{0} = \infty$$

The ramp error constant K_v, is calculated as:

$$K_v = \lim_{s\to 0} \left[sG(s)\right] = \lim_{s\to 0} \left[\cancel{s}\frac{100}{\cancel{s}(s^2 + 10s + 100)}\right] = \frac{100}{100} = 1$$

The parabolic error constant K_a, is calculated as:

$$K_a = \lim_{s\to 0} \left[s^2G(s)\right] = \lim_{s\to 0} \left[\cancel{s} \cdot s\frac{100}{\cancel{s}(s^2 + 10s + 100)}\right] = 0\left(\frac{100}{100}\right) = 0$$

Therefore:

$$\boldsymbol{K_p = \infty, \quad K_v = 1, \quad K_a = 0}$$

$\longrightarrow \mathcal{A}$nswer

(d) $\mathbf{G(s) = \dfrac{100}{s^2(s^2+10s+100)}}$

The step error constant K_p, is calculated as:

$$K_p = \lim_{s\to 0}\big[G(s)\big] = \lim_{s\to 0}\left[\frac{100}{s^2(s^2+10s+100)}\right] = \frac{100}{0} = \infty$$

The ramp error constant K_v, is calculated as:

$$K_v = \lim_{s\to 0}\big[sG(s)\big] = \lim_{s\to 0}\left[\cancel{s}\frac{100}{\cancel{s}\cdot s(s^2+10s+100)}\right] = \frac{100}{0} = \infty$$

The parabolic error constant K_a, is calculated as:

$$K_a = \lim_{s\to 0}\big[s^2G(s)\big] = \lim_{s\to 0}\left[\cancel{s^2}\frac{100}{\cancel{s^2}(s^2+10s+100)}\right] = \frac{100}{100} = 1$$

Therefore:

$$\boldsymbol{K_p = \infty, \quad K_v = \infty, \quad K_a = 1}$$

$\longrightarrow \mathcal{A}$nswer

(d) $\mathbf{G(s) = \dfrac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}}$

The step error constant K_p, is calculated as:

$$K_p = \lim_{s\to 0}\big[G(s)\big] = \lim_{s\to 0}\left[\frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}\right] = \frac{K(1)(1)}{0} = \infty$$

The ramp error constant K_v, is calculated as:

$$K_v = \lim_{s\to 0}\big[sG(s)\big] = \lim_{s\to 0}\left[\cancel{s}\frac{K(1+2s)(1+4s)}{\cancel{s}\cdot s(s^2+s+1)}\right] = \frac{K(1)(1)}{0} = \infty$$

The parabolic error constant K_a, is calculated as:

$$K_a = \lim_{s\to 0}\left[s^2 G(s)\right] = \lim_{s\to 0}\left[\cancel{s^2}\frac{K(1+2s)(1+4s)}{\cancel{s^2}(s^2+s+1)}\right] = \frac{K(1)(1)}{1} = K$$

Therefore:

$$\boldsymbol{K_p = \infty,\quad K_v = \infty,\quad K_a = K}$$

$\longrightarrow$ *Answer*

5-4. For the unity-feedback control systems described in Problem 5-2, determine the steady-state error for a unit-step input, a unit-ramp input, and a parabolic input, $(\mathbf{t^2/2})\mathbf{u_s(t)}$. Check the stability of the system before applying the final-value theorem.

(c) $\mathbf{G(s) = \dfrac{10(s+1)}{s(s+5)(s+6)}}$

The unity feedback transfer function is:

$$M(s) = \frac{G(s)}{1+G(s)} = \frac{10\,s+10}{s^3+11\,s^2+40\,s+10}$$

Using the `roots()` function in MATLAB, we can see that the poles are:

$$\texttt{roots([1,11,40,10])} = \begin{bmatrix} -5.3653 & +2.8848i \\ -5.3653 & -2.8848i \\ -0.2695 & +0.0000i \end{bmatrix}$$

All poles are in the left-hand plane (LHP). Therefore, the system is **stable**.

$\longrightarrow$ *Answer*

The step error constant K_p, is calculated as:

$$K_p = \lim_{s\to 0}\left[G(s)\right] = \lim_{s\to 0}\left[\frac{10(s+1)}{s(s+5)(s+6)}\right] = \frac{10}{0} = \infty$$

The steady state error for a unit-step input, $e_{\mathrm{ss_p}}$, is calculated as:

$$\boldsymbol{e_{\mathrm{ss_p}} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

The ramp error constant K_v, is calculated as:

$$K_v = \lim_{s\to 0}\left[sG(s)\right] = \lim_{s\to 0}\left[\frac{10\cancel{s}(s+1)}{\cancel{s}(s+5)(s+6)}\right] = \frac{10}{30} = \frac{1}{3}$$

The steady state error for a unit-ramp input, $e_{\mathrm{ss_v}}$, is calculated as:

$$\boldsymbol{e_{\mathrm{ss_v}} = \frac{1}{K_v} = \frac{1}{1/3} = 3}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

The parabolic error constant K_a, is calculated as:

$$K_a = \lim_{s\to 0}\left[s^2G(s)\right] = \lim_{s\to 0}\left[\frac{10\cancel{s}s(s+1)}{\cancel{s}(s+5)(s+6)}\right] = \frac{0}{30} = 0$$

The steady state error for a parabolic input, $e_{\mathrm{ss_a}}$, is calculated as:

$$\boldsymbol{e_{\mathrm{ss_a}} = \frac{1}{K_a} = \frac{1}{0} = \infty}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

Overall the unit-step, unit-ramp, and parabolic input steady state error for the system are:

$$\boldsymbol{e_{\mathrm{ss_p}} = 0, \quad e_{\mathrm{ss_v}} = 3, \quad e_{\mathrm{ss_a}} = \infty}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

(c) $\mathbf{G(s) = \dfrac{100(s-1)}{s^2(s+5)(s+6)^2}}$

The unity feedback transfer function is:

$$M(s) = \frac{100\,s - 100}{s^5 + 17\,s^4 + 96\,s^3 + 180\,s^2 + 100\,s - 100}$$

Using the `roots()` function in MATLAB, we can see that the poles are:

$$\texttt{roots([1,17,96,180,100,-100])} = \begin{bmatrix} -7.1437 & +1.9796i \\ -7.1437 & -1.9796i \\ -1.5948 & +1.1275i \\ -1.5948 & -1.1275i \\ 0.4771 & +0.0000i \end{bmatrix}$$

There is a single pole in the right-hand plane (RHP). Therefore, the system is **unstable**, so **an error analysis would be meaningless**. There is no need to perform the analysis. **The steady state error is infinite.**

$\longrightarrow$ *Answer*

Problem 2

Consider the block diagram of the following missile attitude control system:

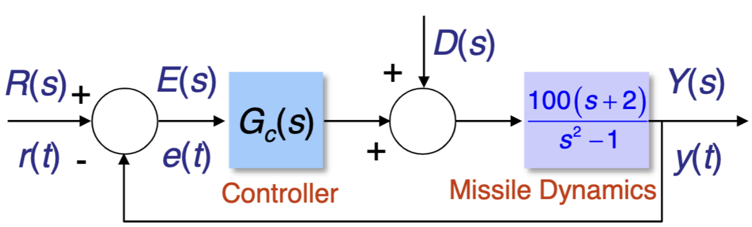


Figure 1: Missile Attitude Control System

(a) Let $G_c(s) = 1$, and find the SSE when $r(t)$ is a unit-step function.

For this system:

$$G(s) = \frac{100(s+2)}{(s^2-1)}$$

The transfer function for the system is:

$$\frac{Y(s)}{R(s)} = \frac{100\,s + 200}{s^2 + 100\,s + 199}$$

Using the `roots()` function in MATLAB, we can see that the poles are:

$$\texttt{roots([1,17,96,180,100,-100])} = \begin{bmatrix} -97.9687 & +0.0000i \\ -2.0313 & +0.0000i \end{bmatrix}$$

All poles are in the left-hand plane (LHP). Therefore, the system is **stable**.
The step error constant K_p, is calculated as:

$$K_p = \lim_{s\to 0} \left[G(s)\right] = \lim_{s\to 0} \left[\frac{100(s+2)}{(s^2-1)}\right] = \frac{100(2)}{-1} = -200$$

The steady state error for a unit-step input, $e_{\mathrm{ss_p}}$, is calculated as:

$$\boldsymbol{e_{\mathrm{ss_p}} = \frac{1}{1+K_p} = \frac{1}{1+(-200)} = \frac{1}{-199} = -0.005}$$

$\longrightarrow$ $\mathcal{A}$nswer

We can validate this by plotting the step response in MATLAB using the `step()` function, which produces the following figure:

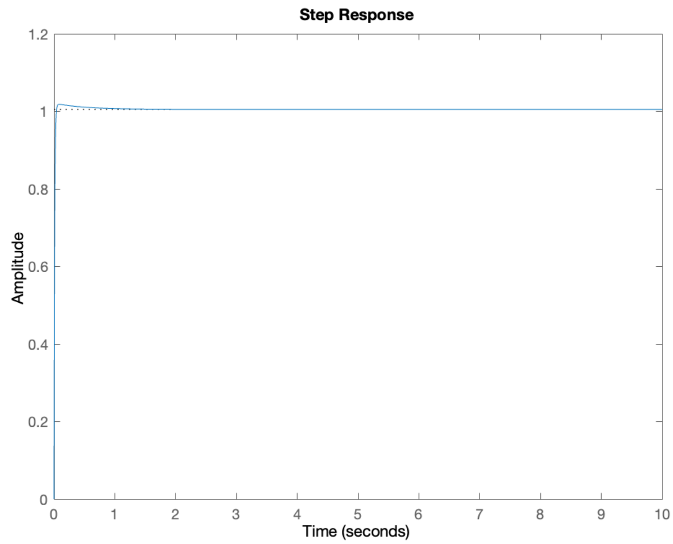


Figure 2: $G_c(s) = 1$, Step Response

The last value of $y(t)$ at the end of the plot is $y(t = 10) = 1.0050$, and the reference signal is 1. Since:

$$e(t) = \text{refernece signal} - y(t)$$

We can see that our above analysis resulted in the correct steady state error:

$$e(t) = 1 - 1.0050 = -0.005$$

(b) Let $G_c(s) = \frac{(s+\alpha)}{s}$, and find the SSE when $r(t)$ is a unit-step function

For this system:

$$G(s) = \frac{100(s+2)(s+\alpha)}{s(s^2-1)}$$

$$\frac{Y(s)}{R(s)} = \frac{100s^2 + (200+100\alpha)s + 200\alpha}{s^3 + 100\,s^2 + (199+100\alpha)\,s + 200\,\alpha}$$

The stability of the system depends on the value of α.

Evaluating the characteristic equation using Routh Hurwitz, we get the following table:

$\mathbf{s^3}$	1	$100\alpha + 199$
$\mathbf{s^2}$	100	200α
$\mathbf{s^1}$	$98\alpha + 199$	0
$\mathbf{s^0}$	200α	0

In order for there to be no sign changes in the left most column, for a stable system, $\alpha \geq 0$.

When $\alpha \geq 0$, the step error constant K_p, is calculated as:

$$K_p = \lim_{s \to 0} \left[G(s)\right] = \lim_{s \to 0} \left[\frac{100(s+2)(s+\alpha)}{s(s^2-1)}\right] = \begin{cases} -200 & \text{if } \alpha = 0 \\ \frac{200\alpha}{0} = \infty & \text{if } \alpha > 0 \end{cases}$$

The steady state error for a unit-step input, $e_{\mathrm{ss_p}}$, is calculated as:

$$\boldsymbol{e_{\mathrm{ss_p}}[\alpha = 0] = \frac{1}{1+K_p} = \frac{1}{1+(-200)} = \frac{1}{-199} = -0.005}$$

$\longrightarrow$ *Answer*

$$\boldsymbol{e_{\mathrm{ss_p}}[\alpha > 0] = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0}$$

$\longrightarrow$ *Answer*

For $\boldsymbol{\alpha < 0}$, since the system is unstable, **the steady state error is infinite.**

$$\boldsymbol{e_{\mathrm{ss_p}}[\alpha < 0] = \infty}$$

$\longrightarrow$ *Answer*

Problem 3

Consider the following non-unity feedback system:

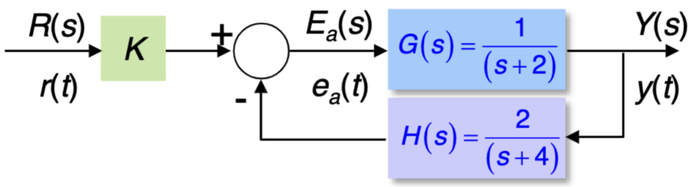


Figure 3: Non-Unity System

(a) Derive an equivalent unity-feedback forward path transfer function

The transfer function for this system is of the form:

$$\frac{Y(s)}{R(s)} = K\left(\frac{G(s)}{1 + G(s)\,H(s)}\right)$$

To put it in the unity feedback form $\frac{Q(s)}{1+Q(s)}$, we must solve:

$$K\frac{G(s)}{1 + G(s)\,H(s)} = \frac{Q(s)}{1 + Q(s)}$$

Which solves to:

$$Q(s) = \frac{KG(s)}{G(s)H(s) - K(s)G(s) + 1}$$

$$Q(s) = \frac{K(s+4)}{s^2 + (6-K)\,s + (10 - 4\,K)}$$

The unity feedback transfer function $\frac{Y(s)}{R(s)}$ is:

$$\boldsymbol{\frac{Y(s)}{R(s)} = \frac{Q(s)}{1 + Q(s)} = \frac{\frac{K(s+4)}{s^2+(6-K)\,s+(10-4\,K)}}{1 + \left(\frac{K(s+4)}{s^2+(6-K)\,s+(10-4\,K)}\right)}}$$

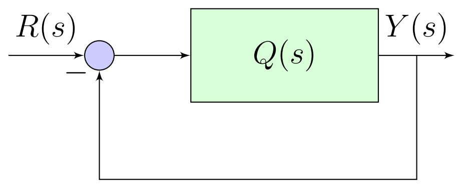
$\longrightarrow$ *Answer*

This can of course be simplified to:

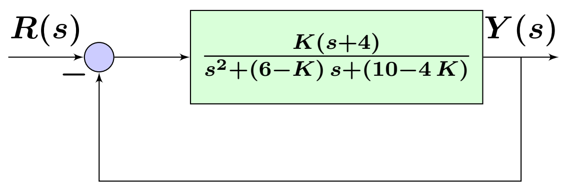
$$\frac{Y(s)}{R(s)} = \frac{K\;(s+4)}{s^2 + 6\,s + 10}$$

(b) Draw the block diagram of the equivalent unity-feedback system.

This can represented by the following block diagrams:



Or equivalently:



$\longrightarrow$ $\mathcal{A}$nswer

Submitted by Austin Barrilleaux on October 15, 2023.