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MODULE 14 — Practice Assignment

Problem 1

Solve the following 9th Edition textbook problems:

- **10-18 (a,c,f)**
- **10-59 (a)**

(10-18) Check the contro1ihi1ity of the following systems:

(a) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

The controllability matrix (determined using MATLAB) is:

$$\begin{aligned} \mathbf{Q}_c &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 \\ 5 & -10 \end{bmatrix} \end{aligned}$$

The rank of $\mathbf{Q}_c$ determined using the `rank()` function in MATLAB is:

$$\texttt{rank}(\mathbf{Q}_c) = 2$$

Since the $\boldsymbol{n} \times \boldsymbol{n}$ controllability matrix has a rank of $\boldsymbol{n}$, **the system is controllable**.

$\longrightarrow$ $\mathcal{A}$nswer

(c) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The controllability matrix (determined using MATLAB) is:

$$\begin{aligned} \mathbf{Q}_c &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 & -4 & -2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -6 & 0 & 12 & 0 \end{bmatrix} \end{aligned}$$

We know that, for controllability, the controllability matrix $\mathbf{Q_c}$ **must be square**. Since it is not, we know **the system is not controllable**.

$\longrightarrow$ *Answer*

Regardless, we can still check the rank of $\mathbf{Q}_c$ using the `rank()` function in MATLAB:

$$\texttt{rank}(\mathbf{Q}_c) = 2$$

This confirms the above conclusion.

(f) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} u$$

The controllability matrix (determined using MATLAB) is:

$$\begin{aligned} \mathbf{Q}_c &= \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} & \mathbf{A}^4\mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} 4 & -6 & 9 & -14 & 24 \\ 2 & -3 & 4 & -4 & 0 \\ 1 & -2 & 4 & -8 & 16 \\ 3 & -15 & 75 & -375 & 1875 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

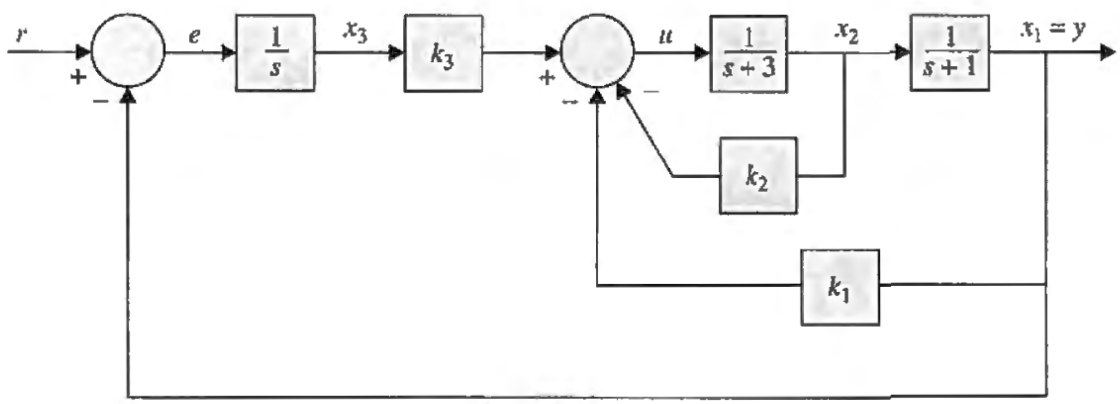
The rank of $\mathbf{Q}_c$ determined using the `rank()` function in MATLAB is:

$$\texttt{rank}(\mathbf{Q}_c) = 4$$

Since the $\boldsymbol{n} \times \boldsymbol{n}$ controllability matrix does not have a rank of $\boldsymbol{n}$ $(\boldsymbol{n} = \mathbf{5}, \boldsymbol{n} \neq \mathbf{4})$, **the system is not controllable**.

$\longrightarrow$ $\mathcal{A}$nswer

(10-59) The block diagram of a control system with state feedback is shown in the following figure. The feedback gains k_1, k_2, and k_3 are real constants.



(a) Find the values of the feedback gains so that:

- **The steady-state error e_{ss} [$e(t)$ is the error signal] due to a step input is zero.**

- **The characteristic equation roots are at $-1+j$, $-1-j$, and -10.**

If we reduce the block diagram to a unity feedback system, we get that the forward path transfer function, $G(s)$ is:

$$G(s) = \frac{k_3}{s^3 + (k_2+4)s^2 + (k_1+k_2+3)s}$$

The closed loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{k_3}{s^3 + (k_2+4)s^2 + (k_1+k_2+3)s + k_3}$$

For e_{ss} to be zero, when the input is a step function, K_p must be infinite:

$$K_p = \lim_{s\to 0} G(s) = \frac{k_3}{s^3 + (k_2+4)s^2 + (k_1+k_2+3)s} = \infty$$

Since all values in the denominator multiply by s, $\mathbf{e_{ss}}$ **is zero for all values of** $\mathbf{k_1}$**,** $\mathbf{k_2}$**, and** $\mathbf{k_3}$ since:

$$\boldsymbol{e_{ss} = \frac{R}{1+K_p} = \frac{R}{1+\infty} = 0}$$

$\longrightarrow$ *Answer*

The characteristic equation is:

$$\text{CE:} \quad s^3 + (k_2+4)s^2 + (k_1+k_2+3)s + k_3 = 0$$

And we want the characteristic equation roots to be $-1+j$, $-1-j$, and -10. We make the following equivalence:

$$s^3 + (k_2+4)s^2 + (k_1+k_2+3)s + k_3 = s^3 + 12s^2 + 22s + 20$$

If we equate like terms of s:

$$\begin{aligned} &(k_2+4) = 12 && \rightarrow \ k_2 = 8 \\ &(k_1+k_2+3) = 22 && \rightarrow \ k_1 = 11 \\ &k_3 = 20 && \rightarrow \ k_3 = 20 \end{aligned}$$

Find the values of the feedback gains are:

$$\boldsymbol{k_1 = 11}$$
$$\boldsymbol{k_2 = 8}$$
$$\boldsymbol{k_3 = 20}$$

$\longrightarrow$ *Answer*

Submitted by Austin Barrilleaux on December 5, 2023.