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MODULE 8 — Practice Assignment

Problem 1

Solve the following practice problems in the 9th edition textbook.

- **Chapter 5:**
  - **7-1** (a)
  - **7-6** (a)

7-1. Find the angles of the asymptotes and the intersect of the asymptotes of the root loci of the following equations when K varies from $-\infty$ to ∞.

(a) $\mathbf{s^4 + 4s^3 + 4s^2 + (K+8)s + K = 0}$

Putting the above equation into the form:

$$1 + \frac{KQ(s)}{P(s)} = 0$$

We get:

$$Q(s) = s + 1$$

And:

$$P(s) = s^4 + 4s^3 + 4s^2 + 8s$$

The poles are, (using the `roots()` function in MATLAB):

$$\texttt{roots([1,4,4,8,0])} = \begin{bmatrix} 0.0000 & +0.0000j \\ -3.5098 & +0.0000j \\ -0.2451 & +1.4897j \\ -0.2451 & -1.4897j \end{bmatrix}$$

The zero is $s = -1$.

Factoring $P(s)$ results in 4 poles, and the equation has 1 zero:

For large values of s, the root locus for $K > 0$ are asymptotic to asymptotes with angles given by:

$$\theta_i = \frac{(2i+1)}{|n-m|} \times 180^\circ, \quad n \neq m, \quad i = 0, 1, 2 ... \; |n-m|-1$$

For this case:

$$|n-m|-1 = |4-1|-1 = 2 \rightarrow i = 0, 1, 2$$

Therefore, when $K > 0$:

$$\boldsymbol{\theta_0 = \frac{(2(0)+1)}{3} \times 180^\circ = 60^\circ, \quad [K > 0]}$$

$\longrightarrow$ Answer

$$\boldsymbol{\theta_1 = \frac{(2(1)+1)}{3} \times 180^\circ = 180^\circ, \quad [K > 0]}$$

$\longrightarrow$ Answer

$$\boldsymbol{\theta_2 = \frac{(2(2)+1)}{3} \times 180^\circ = 300^\circ, \quad [K > 0]}$$

$\longrightarrow$ Answer

For large values of s, the root locus for $K < 0$ are asymptotic to asymptotes with angles given by:

$$\theta_i = \frac{(2i)}{|n-m|} \times 180^\circ, \quad n \neq m, \quad i = 0, 1, 2 ... \; |n-m|-1$$

Where again:

$$|n-m|-1 = |4-1|-1 = 2 \rightarrow i = 0, 1, 2$$

Therefore, when $K < 0$:

$$\boldsymbol{\theta_0 = \frac{(2(0))}{3} \times 180^\circ = 0^\circ, \quad [K < 0]}$$

$\longrightarrow \mathcal{A}$nswer

$$\boldsymbol{\theta_1 = \frac{(2(1))}{3} \times 180^\circ = 120^\circ, \quad [K < 0]}$$

$\longrightarrow \mathcal{A}$nswer

$$\boldsymbol{\theta_2 = \frac{(2(2))}{3} \times 180^\circ = 240^\circ, \quad [K < 0]}$$

$\longrightarrow \mathcal{A}$nswer

The point of intersection of the asymptotes is given by:

$$\sigma_1 = \frac{\sum \text{real parts of poles of } G(s)H(s) - \sum \text{real parts of zeros of } G(s)H(s)}{n - m}$$

Where:

$$G(s)H(s) = \frac{KQ(s)}{P(s)}$$

Evaluating the sum of the real parts of the poles in MATLAB:

`sum(real(roots([1,4,4,8,0])))` $= -4$

Therefore:

$$\boldsymbol{\sigma_1 = \frac{(-4) - (-1)}{4 - 1} = -1}$$

$\longrightarrow \mathcal{A}$nswer

7-6. For the loop transfer functions that follow, find the angle of departure or arrival of the root loci at the designated pole or zero.

(a) $\mathbf{G(s)H(s) = \dfrac{Ks}{(s+1)(s^2+1)}}$

Angle of arrival $(\mathbf{K < 0})$ and angle of departure $(\mathbf{K > 0})$ at $\mathbf{s = j}$.

This equation can be rewritten as:

$$G(s)H(s) = \frac{Ks}{(s+1)(s+j)(s-j)}$$

Plotting the poles and the zeros:

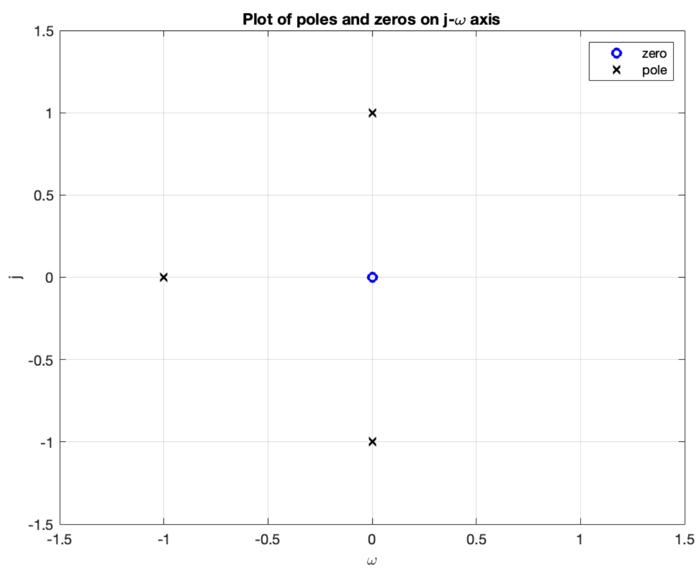

Figure 1: Poles and Zeros

The angle of arrival is calculated as:

$$K < 0: \quad \sum_{k=1}^{m} \theta_{z_k} - \sum_{j=1}^{m} \theta_{p_j} = 2i \times 180°$$

The angle of arrival $(K < 0)$ at the pole $s = j$ is:

$$90^\circ - (45^\circ + 90^\circ + \theta_a) = 2i \times 180^\circ$$

Solving for θ_a:

$$\theta_a = -45^\circ - 2i \times 180^\circ = -45^\circ - (0) = -45^\circ$$

The angle of departure is calculated as:

$$K > 0: \quad \sum_{k=1}^{m} \theta_{z_k} - \sum_{j=1}^{m} \theta_{p_j} = (2i+1) \times 180^\circ$$

The angle of departure ($K > 0$) at the pole $s = j$ is:

$$90^\circ - (45^\circ + 90^\circ + \theta_d) = (2i+1) \times 180^\circ$$

Solving for θ_d:

$$\theta_d = -45^\circ - (2i+1) \times 180^\circ = -45^\circ - (-180^\circ) = 135^\circ$$

The angle of arrival and angle of departure at the pole $s = j$ is:

$$\boldsymbol{\theta_a = -45^\circ, \quad \theta_d = 135^\circ}$$

$\longrightarrow$ $\mathcal{A}$nswer

Problem 1

The block diagram of a control system with tachometer feedback is shown in the following figure:

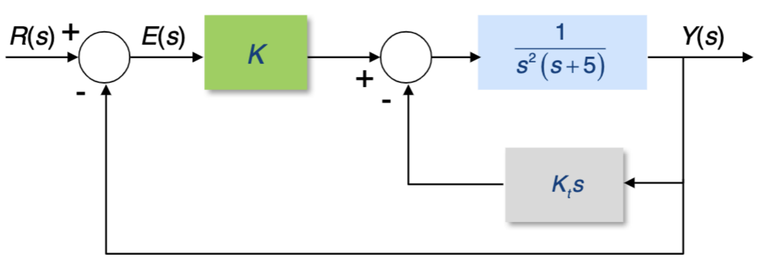

Figure 2: Block Diagram Of A Control System

(a) Construct the root loci of the CE for $\mathbf{K \geq 0}$ when $\mathbf{K_t = 0}$.

If $K_t = 0$, the transfer function of the control system is:

$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s^2(s+5)}}{1 + \frac{K}{s^2(s+5)}} = \frac{K}{s^2(s+5) + K}$$

Putting the characteristic equation into the form $1 + \frac{KQ(s)}{P(s)} = 0$:

$$Q(s) = 1$$

$$P(s) = s^2(s+5)$$

There are no zeros, there are 3 poles, at $s = 0, 0, -5$

For large values of s, the root locus for $K > 0$ are asymptotic to asymptotes with angles given by:

$$\theta_i = \frac{(2i+1)}{|n-m|} \times 180°, \quad n \neq m, \quad i = 0, 1, 2 ... \ |n-m| - 1$$

For this case:

$$|n-m| - 1 = |3-0| - 1 = 2 \rightarrow i = 0, 1, 2$$

Therefore, when $K > 0$:

$$\boldsymbol{\theta_0 = \frac{(2(0)+1)}{3} \times 180° = 60°, \quad [K > 0]}$$

$\longrightarrow$ *Answer*

$$\boldsymbol{\theta_1 = \frac{(2(1)+1)}{3} \times 180° = 180°, \quad [K > 0]}$$

$\longrightarrow$ *Answer*

$$\boldsymbol{\theta_2 = \frac{(2(2)+1)}{3} \times 180° = 300°, \quad [K > 0]}$$

$\longrightarrow$ *Answer*

The point of intersection of the asymptotes is given by:

$$\sigma_1 = \frac{\sum \text{real parts of poles of } G(s)H(s) - \sum \text{real parts of zeros of } G(s)H(s)}{n-m}$$

$$\sigma_1 = \frac{(-5)-(0)}{3} = -\frac{5}{3}$$

There are no complex poles, so there are no angles of arrival or departure.

Using the above information, we can create the following sketch:

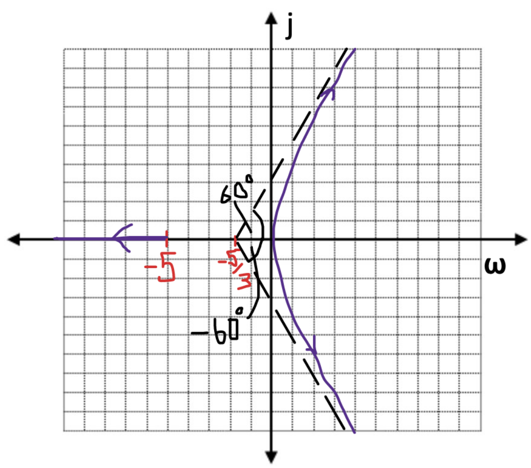

Figure 3: Root Locus when $K_t = 0$ (Sketch)

Using the `rlocus()` function in MATLAB, we can confirm that our sketch is correct with the following plot:

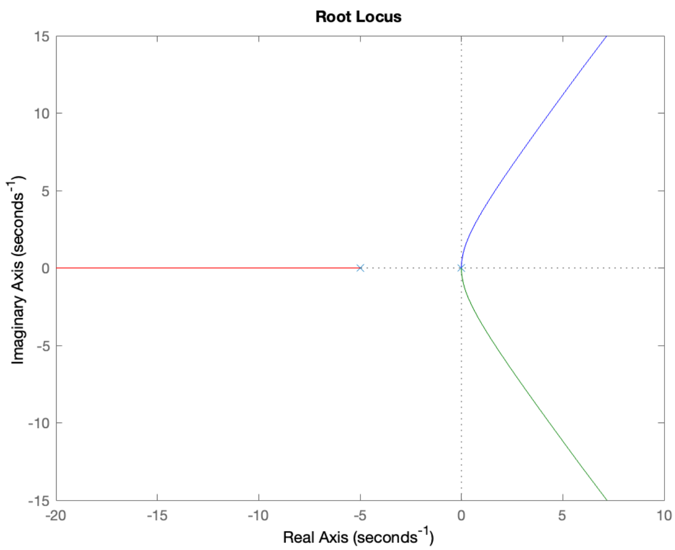

Figure 4: Root Locus when $K_t = 0$ (MATLAB)

(b) Set K = 10 and construct the root loci of the CE for $\mathbf{K_t \geq 0}$.

If $K_t = 10$, the transfer function of the control system is:

$$\frac{Y(s)}{R(s)} = \frac{10}{s^3 + 5\,s^2 + K_t\,s + 10}$$

Putting the characteristic equation into the form $1 + \frac{KQ(s)}{P(s)} = 0$:

$$Q(s) = s$$

$$P(s) = s^3 + 5s^2 + 10$$

There is a zero at $s = 0$.

The poles are, (using the `roots()` function in MATLAB):

$$\texttt{roots([1,5,0,10])} = \begin{bmatrix} -5.3494 & +0.0000j \\ 0.1747 & +1.3560j \\ 0.1747 & -1.3560j \end{bmatrix}$$

For large values of s, the root locus for $K > 0$ are asymptotic to asymptotes with angles given by:

$$\theta_i = \frac{(2i+1)}{|n-m|} \times 180^\circ, \quad n \neq m, \quad i = 0, 1, 2... \; |n-m| - 1$$

For this case:

$$|n-m| - 1 = |3-1| - 1 = 1 \rightarrow i = 0, 1$$

Therefore, when $K > 0$:

$$\boldsymbol{\theta_0 = \frac{(2(0)+1)}{2} \times 190^\circ = 90^\circ, \quad [K > 0]}$$

$\longrightarrow$ $\mathcal{A}$nswer

$$\boldsymbol{\theta_1 = \frac{(2(1)+1)}{2} \times 180^\circ = 270^\circ, \quad [K > 0]}$$

$\longrightarrow$ $\mathcal{A}$nswer

The point of intersection of the asymptotes is given by:

$$\sigma_1 = \frac{\sum \text{real parts of poles of } G(s)H(s) - \sum \text{real parts of zeros of } G(s)H(s)}{n-m}$$

$$\sigma_1 = \frac{(-5)-(0)}{2} = -\frac{5}{2} = -2.5$$

Since there are complex poles, we need to calculate the angles of departure. The angle of departure is calculated as:

$$K > 0: \quad \sum_{k=1}^{m} \theta_{z_k} - \sum_{j=1}^{m} \theta_{p_j} = (2i+1) \times 180^\circ$$

The angle of departure ($K > 0$) at the pole $s = (0.1747 + 1.3560j)$ is:

$$82.6587^\circ - (90^\circ + 13.7917^\circ + \theta_d) = (2i+1) \times 180^\circ$$

Solving for θ_d:

$$\theta_d = 21.1330° - (2i+1) \times 180° = 21.1330° - 180° = 158.8670°$$

Because of symmetry, the angle of departure ($K > 0$) at the pole $s = (0.1747 - 1.3560j)$ is:

$$\theta_d = -158.8670°$$

Using the above information, we can create the following sketch:

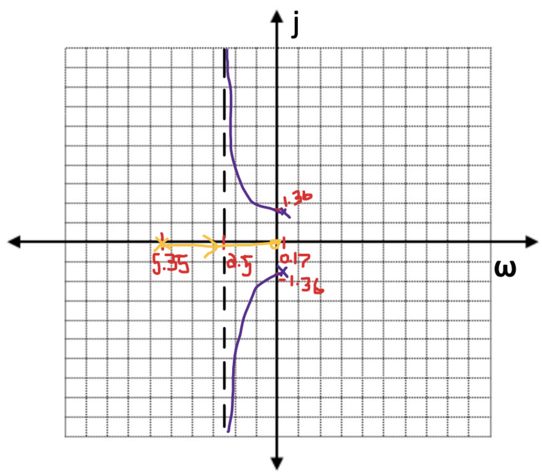

Figure 5: Root Locus when $K = 10$ (Sketch)

Using the `rlocus()` function in MATLAB, we can confirm that our sketch is correct with the following plot:

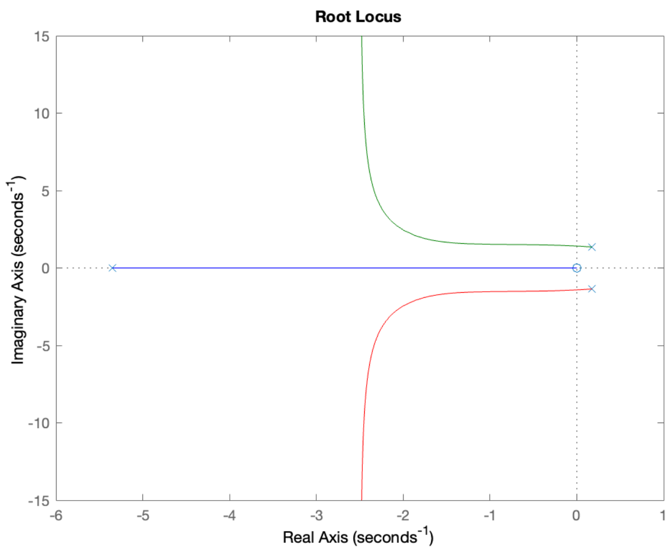

Figure 6: Root Locus when $K = 10$ (MATLAB)

Submitted by Austin Barrilleaux on October 21, 2023.