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MODULE 2 — Practice Assignment

Problem 1

Solve the following practice problems in the 9th edition textbook.

- **Chapter 3:**
  - **3-12(a)**

The block diagram of an electric train control is shown in Fig. 1. The system parameters and variables are:

- $e_r(t)$ = **voltage representing the desired train speed,** V
- $v(t)$ = **speed of train, ft/sec**
- M = **Mass of train = 30,000 lb/sec**^{2}
- K = **amplifier gain**
- k_t = **gain of speed indicator = 0.15 V/ft/sec**

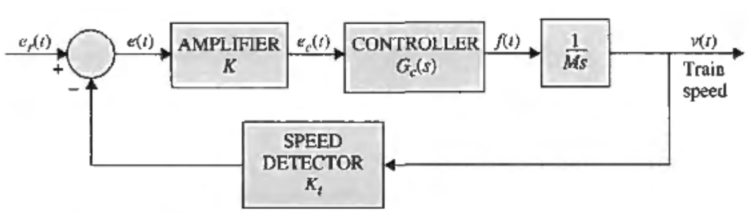


Figure 1: Problem 3-12

To determine the transfer function of the controller, we apply a step function of 1 volt to the input of the controller, that is, $e_c(t) = u_s(t)$. The output of the controller is measured and described by the following equation:

$$f(t) = 100(1 - 0.3e^{-6t} - 0.7e^{-10t})u_s(t)$$

(a) Find the transfer function $G_c(s)$ of the controller.

From Fig. 1, we can see that:

$$\frac{F(s)}{E_c(s)} = G_c(s)$$

Since:

$$f(t) = 100(1 - 0.3e^{-6t} - 0.7e^{-10t})u_s(t)$$

Then using the Laplace Transform Table from the book:

$$F(t) = \mathcal{L}^{-1}\left[100 - 30e^{-6t} - 70e^{-10t}\right] = \frac{100}{s} - \frac{30}{s+6} - \frac{70}{s+10}$$

To get a common denominator for $F(t)$:

$$F(t) = \frac{100(s+6)(s+10) - 30s(s+10) - 70s(s+6)}{s(s+6)(s+10)}$$

$$F(t) = \frac{100(s^2+16s+60) - 30(s^2+10s) - 70(s^2+6s)}{s(s+6)(s+10)}$$

$$F(t) = \frac{100(16s+60) - 30(10s) - 70(6s)}{s(s+6)(s+10)}$$

$$F(t) = \frac{880s + 6000}{s(s+6)(s+10)}$$

$$F(t) = \frac{880\left(s + \frac{600}{88}\right)}{s(s+6)(s+10)}$$

Since:

$$e_c(t) = u_s(t)$$

Then using the Laplace Transform Table from the book:

$$E_c(s) = \mathcal{L}^{-1}\left[u_s(t)\right] = \frac{1}{s}$$

Using these, $G_c(s)$ is:

$$G_c(s) = \frac{F(s)}{E_c(s)} = \frac{\frac{880\left(s+\frac{600}{88}\right)}{s(s+6)(s+10)}}{\frac{1}{s}}$$

Or more simply:

$$\boldsymbol{G_c(s) = \frac{880\left(s+\frac{600}{88}\right)}{(s+6)(s+10)}}$$

$\longrightarrow$ *Answer*

(b) Derive the forward-path transfer function $\frac{V(s)}{E(s)}$ of the system. The feedback path is opened in this case.

From Fig. 1, we can see that:

$$\frac{V(s)}{E(s)} = KG_c(s)\frac{1}{Ms}$$

Given the values for $G_c(s)$ and M from part (a) and the problem prompt respectively, and the fact that there is no value for K:

$$\frac{V(s)}{E(s)} = K\frac{880\left(s+\frac{600}{88}\right)}{(s+6)(s+10)}\frac{1}{30,000s}$$

$\frac{V(s)}{E(s)}$ can be written as:

$$\boldsymbol{\frac{V(s)}{E(s)} = \frac{880K\left(s+\frac{600}{88}\right)}{30,000s(s+6)(s+10)}}$$

$\longrightarrow$ *Answer*

(c) Derive the closed-loop transfer function $\frac{V(s)}{E_r(s)}$ of the system.

Using block diagram reduction techniques, we can get that the transfer function is:

$$\frac{V(s)}{E_r(s)} = \frac{KG_c(s)\frac{1}{Ms}}{1 + KG_c(s)\frac{1}{Ms}K_t}$$

Which can be more simply written as:

$$\frac{V(s)}{E_r(s)} = \frac{KG_c(s)}{Ms + KG_c(s)K_t}$$

Given the values for $G_c(s)$ from part (a) and M and K_t from the problem prompt, and the fact that there is no value for K:

$$\frac{V(s)}{E_r(s)} = \frac{K\frac{880\left(s+\frac{600}{88}\right)}{(s+6)(s+10)}}{30,000s + K\frac{880\left(s+\frac{600}{88}\right)}{(s+6)(s+10)}0.15}$$

Which can be simplified to:

$$\frac{V(s)}{E_r(s)} = \frac{K880\left(s+\frac{600}{88}\right)}{30,000s(s+6)(s+10) + K880\left(s+\frac{600}{88}\right)0.15}$$

Further:

$$\frac{V(s)}{E_r(s)} = \frac{K\frac{88}{3000}\left(s+\frac{600}{88}\right)}{s(s+6)(s+10) + K\frac{88}{3000}\left(s+\frac{600}{88}\right)0.15}$$

Expanding the denominator, the closed-loop transfer function $\frac{V(s)}{E_r(s)}$ of the system is:

$$\boldsymbol{\frac{V(s)}{E_r(s)} = \frac{K\frac{88}{3000}\left(s+\frac{600}{88}\right)}{s^3 + 16s^2 + s\left(K\frac{44}{10000} + 60\right) + K\frac{3}{100}}}$$

$\longrightarrow$ $\mathcal{A}$nswer

(c) Assuming that K is set at a value so that the train will not run away (unstable), find the steady-state speed of the train in feet per second when the input is $e_r(t) = u_s(t)V$.

The speed of train, $v(t)$ will approach a value at steady state when $t \to \infty$. Therefore we must solve the limit:

$$\lim_{t \to +\infty} v(t) = v_{ss}$$

Given $e_r(t) = u_s(t)V$:

$$E_r(s) = \frac{1}{s}$$

If we substitute this into the transfer function from part (c), we get the following:

$$sV(s) = \frac{K\frac{88}{3000}\left(s + \frac{600}{88}\right)}{s^3 + 16s^2 + s\left(K\frac{44}{10000} + 60\right) + K\frac{3}{100}}$$

From the final value theorem we know that:

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s)$$

Therefore:

$$\lim_{t \to +\infty} v(t) = \lim_{s \to 0} sV(s) = v_{ss}$$

So:

$$v_{ss} = \lim_{s \to 0} \frac{K\frac{88}{3000}\left(s + \frac{600}{88}\right)}{s^3 + 16s^2 + s\left(K\frac{44}{10000} + 60\right) + K\frac{3}{100}}$$

Analyzing the limit:

$$v_{ss} = \frac{K\frac{88}{3000}\left(\frac{600}{88}\right)}{K\frac{3}{100}} = \frac{\frac{600}{3000}}{\frac{3}{100}} = \frac{20}{3}$$

The steady-state speed of the train is:

$$\boldsymbol{v_{ss}(t) = \frac{20}{3}}\text{ft/s}$$

$\longrightarrow$ *Answer*

Problem 2

Using Mason's Gain Formula, determine the transfer function from input $R(s)$ to output $Y(s)$ given the following signal flow graph:

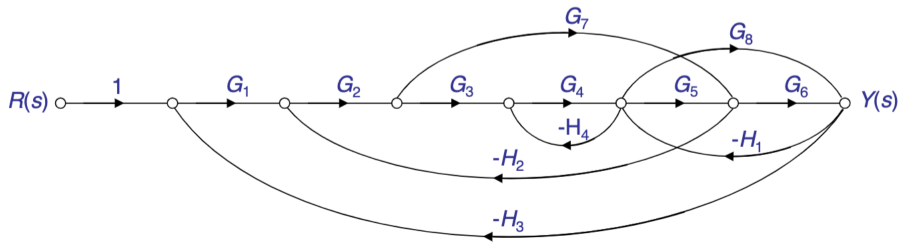


Figure 2: Problem 2: Signal Flow Graph

There are three forward paths:

$$M_1 = G_1G_2G_3G_4G_5G_6$$

$$M_2 = G_1G_2G_7G_6$$

$$M_3 = G_1G_2G_3G_4G_8$$

There are four loops:

$$L_{11} = G_5G_6(-H_1)$$

$$L_{21} = G_2G_3G_4G_5(-H_2)$$

$$L_{21} = G_1G_2G_3G_4G_5G_6(-H_3)$$

$$L_{41} = G_4(-H_4)$$

All loops touch M_1 and M_3:

$$\Delta_1 = \Delta_3 = 1$$

Loop M_2 does not touch L_{41}:

$$\Delta_2 = 1 - L_{41}$$

Pairs of non-touching loops are:

- L_{11} does not touch L_{21}
- L_{21} does not touch L_{31}
- L_{21} does not touch L_{41}
- L_{31} does not touch L_{41}

Triplets of non-touching loops are:

- L_{21}, L_{31} and L_{41} share no common nodes

Δ is calculated as:

$\Delta = 1 - (L_{11} + L_{21} + L_{31} + L_{41}) + (L_{11}L_{21} + L_{21}L_{31} + L_{21}L_{41} + L_{31}L_{41}) - (L_{21}L_{31}L_{41})$

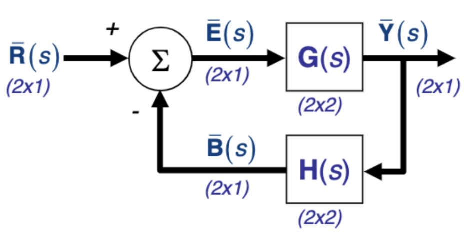
Therefore, the Transfer Function is:

$$\boldsymbol{M = \frac{Y(s)}{R(s)} = \frac{M_1 + M_2(1 - L_{41}) + M_3}{1 - (L_{11} + L_{21} + L_{31} + L_{41}) + (L_{11}L_{21} + L_{21}L_{31} + L_{21}L_{41} + L_{31}L_{41}) - (L_{21}L_{31}L_{41})}}$$

$\longrightarrow$ *Answer*

Problem 3

Consider the multivariable feedback system below and compute the multivariable closed-loop transfer function



$$\mathbf{G}(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{bmatrix}, \quad \mathbf{H}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Figure 3: Problem 3: Multivariable Feedback System

The transfer function of the closed loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)Hs}$$

$G(s)Hs$ is calculated as:

$$G(s)Hs = G(s) = \begin{pmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{pmatrix}$$

Therefore:

$$1 + G(s)Hs = 1 + G(s) = \begin{pmatrix} 1 + \frac{1}{s+1} & 1 - \frac{1}{s} \\ 3 & 1 + \frac{1}{s+2} \end{pmatrix}$$

Getting common denominators:

$$1 + G(s)Hs = \begin{pmatrix} \frac{s+2}{s+1} & \frac{s-1}{s} \\ 3 & \frac{s+3}{s+2} \end{pmatrix}$$

The inverse of which is:

$$[1+G(s)Hs]^{-1} = \frac{1}{(\frac{s+2}{s+1})(\frac{s+3}{s+2}) - 3\frac{s-1}{s}} \begin{pmatrix} \frac{s+3}{s+2} & -\frac{s-1}{s} \\ -3 & \frac{s+2}{s+1} \end{pmatrix} = \frac{s(s+1)}{s(s+3) - 3(s+1)(s-1)} \begin{pmatrix} \frac{s+3}{s+2} & -\frac{s-1}{s} \\ -3 & \frac{s+2}{s+1} \end{pmatrix}$$

Which becomes:

$$[1 + G(s)Hs]^{-1} = \frac{s(s+1)}{-2s^2 + 3s + 3} \begin{pmatrix} \frac{s+3}{s+2} & -\frac{s-1}{s} \\ -3 & \frac{s+2}{s+1} \end{pmatrix}$$

Therefore, the transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{s(s+1)}{-2s^2 + 3s + 3} \begin{pmatrix} \frac{1}{s+1} & -\frac{1}{s} \\ 2 & \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} \frac{s+3}{s+2} & -\frac{s-1}{s} \\ -3 & \frac{s+2}{s+1} \end{pmatrix}$$

Or:

$$\frac{Y(s)}{R(s)} = \frac{s(s+1)}{-2s^2 + 3s + 3} \begin{pmatrix} \frac{1}{s+1}\frac{s+3}{s+2} + 3\frac{1}{s} & -\frac{1}{s+1}\frac{s-1}{s} - \frac{1}{s}\frac{s+2}{s+1} \\ 2\frac{s+3}{s+2} - 3\frac{1}{s+2} & -2\frac{s-1}{s} + \frac{1}{s+2}\frac{s+2}{s+1} \end{pmatrix}$$

Simplifying:

$$\frac{Y(s)}{R(s)} = \frac{s(s+1)}{-2s^2+3s+3}\begin{pmatrix} \frac{s(s+3)+3(s+1)(s+2)}{s(s+1)(s+2)} & -\frac{2s+1}{s(s+1)} \\ \frac{2s+3}{s+2} & \frac{s(s+2)-(2s-2)(s+1)(s+2)}{s(s+1)(s+2)} \end{pmatrix}$$

Simplifying one last time, we get that the closed loop transfer function is:

$$\boldsymbol{\frac{Y(s)}{R(s)} = \frac{1}{-2s^2+3s+3}\begin{pmatrix} \frac{2(2s^2+6s+3)}{(s+2)} & -2s+1 \\ \frac{(2s^2+3s)(s+1)}{s+2} & -2s^2+s+2 \end{pmatrix}}$$

$\longrightarrow \mathcal{A}$nswer

Submitted by Austin Barrilleaux on September 10, 2023.