Austin Barrilleaux
Whiting School of Engineering
Johns Hopkins University

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MODULE 3 — Practice Assignment

Problem 1

Solve the following practice problems in the 9th edition textbook.

- **Chapter 3:**

    - **10-1 (c) (use MATLAB if / as needed)**

The following differential equation represents a linear time-invariant system. Write the following dynamic equation (state equations and output equations) in vector-matrix form:

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) + \int_0^t y(\tau)d\tau = r(\tau)$$

We can establish the following relationships:

$$x_1(t) = \int_0^t y(\tau)d\tau$$
$$x_2(t) = \dot{x}_1 = y(t)$$
$$x_3(t) = \dot{x}_2 = \frac{dy(t)}{dt}$$
$$x_4(t) = \dot{x}_3 = \frac{d^2y(t)}{dt^2}$$
$$\dot{x}_4 = \frac{d^3y(t)}{dt^3}$$

We can rewrite the dynamic equation in question as:

$$\frac{d^3y(t)}{dt^3} = -5\frac{d^2y(t)}{dt^2} - 3\frac{dy(t)}{dt} - y(t) - \int_0^t y(\tau)d\tau + r(\tau)$$

The standard matrix state equations are:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Given the above relationships, we can write the dynamic equation as components of the standard matrix equations in the following matrix form:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Therefore, the state equation is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(\tau)$$

$\longrightarrow \mathcal{A}$nswer

And the output equation is:

$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix}$$

$\longrightarrow \mathcal{A}$nswer

Problem 1

Consider the block diagram of the following feedback control system:
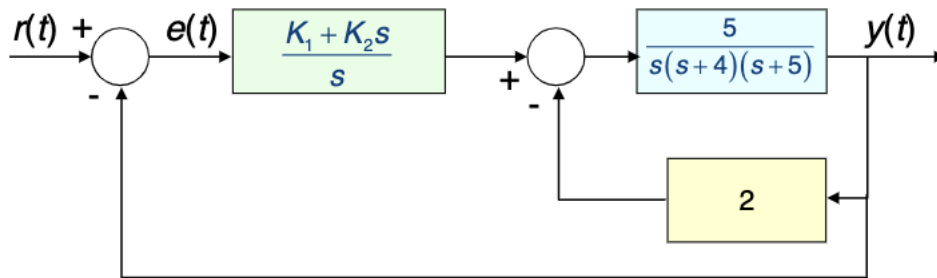

Figure 1: Problem 2

(a) Find the open-loop TF Y(s) / E(s) and the closed-loop TF Y(s) / R(s)

The open loop transfer function is simply the multiplication of the green and blue blocks:

$$\frac{Y(s)}{E(s)} = \left(\frac{K_1 + K_2 s}{s}\right)\left(\frac{5}{s(s+4)(s+5)}\right)$$

Or

$$\frac{Y(s)}{E(s)} = \frac{5(K_1 + K_2 s)}{s^2(s+4)(s+5)}$$

$$\longrightarrow \mathcal{A}\text{nswer}$$

The to solve for the closed loop transfer function, first we reduce the blue and yellow blocks. They reduce to:

$$\frac{\frac{5}{s(s+4)(s+5)}}{1 + 2\frac{5}{s(s+4)(s+5)}} = \frac{5}{s(s+4)(s+5) + 10}$$

This combines with the green block to get:

$$\frac{5(K_1 + K_2 s)}{s^2(s+4)(s+5) + 10s}$$

Which is the last block in a unity feedback loop, giving us the final transfer function of

$$\frac{Y(s)}{R(s)} = \frac{\frac{5(K_1+K_2 s)}{s^2(s+4)(s+5)+10s}}{1 + \frac{5(K_1+K_2 s)}{s^2(s+4)(s+5)+10s}} = \frac{5(K_1 + K_2 s)}{s^2(s + 4)(s + 5) + 10s + 5(K_1 + K_2 s)}$$

The closed loop transfer function is:

$$\boldsymbol{\frac{Y(s)}{R(s)} = \frac{5(K_1 + K_2 s)}{s^2(s + 4)(s + 5) + 10s + 5(K_1 + K_2 s)}}$$

$$\longrightarrow \mathcal{A}\text{nswer}$$

(b) Write the dynamic equations in the form:

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t)$$
$$y(t) = C\bar{x}(t) + D\bar{u}(t)$$

To do this for the open loop case, we must first expand the numerator and the denominator to get:

$$\frac{Y(s)}{E(s)} = \frac{5K_1 + 5K_2 s}{s^4 + 9s^3 + 20s^2}$$

Then separate the equation into the form:

$$\frac{Y(s)}{E(s)} = \frac{Y(s)}{V(s)}\frac{V(s)}{E(s)} = \left(\frac{5K_1 + 5K_2 s}{1}\right)\left(\frac{1}{s^4 + 9s^3 + 20s^2}\right)$$

We can put this into the state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$\longrightarrow$ $\mathcal{A}$nswer

$$y(t) = \begin{bmatrix} 5K_1 & 5K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\longrightarrow$ $\mathcal{A}$nswer

To do this for the closed loop case, we must first expand the numerator and the denominator to get:

$$\frac{Y(s)}{R(s)} = \frac{5K_1 + 5K_2 s}{s^4 + 9s^3 + 20s^2 + (5K_2 + 10)s + 5K_1}$$

Then separate the equation into the form:

$$\frac{Y(s)}{E(s)} = \frac{Y(s)}{V(s)} \frac{V(s)}{E(s)} = \left(\frac{5K_1 + 5K_2 s}{1} \right) \left(\frac{1}{s^4 + 9s^3 + 20s^2 + (5K_2 + 10)s + 5K_1} \right)$$

We can put this into the state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5K_1 & -(5K_2 + 10) & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$\longrightarrow$ $\mathcal{A}$nswer

$$y(t) = \begin{bmatrix} 5K_1 & 5K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\longrightarrow$ $\mathcal{A}$nswer

Submitted by Austin Barrilleaux on September 16, 2023.