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November 10, 2023

MODULE 11 — Practice Assignment

Problem 1

Do the following homework problems from the text book:

- **8-39 (a)**

8-39. The forward-path transfer functions of unity-feedback control systems are given in the following equations. Plot the Bode diagram of $\mathbf{G(j\omega)/K}$, and do the following:
(1) Find the value of K so that the gain margin of the system is 20 dB.
(2) Find the value of K so that the phase margin of the system is 45°.

(a) $$\mathbf{\frac{K}{s(1+0.1s)(1+0.5s)}}$$

We can plot the Bode diagram using the `margin()` function in MATLAB, when $K = 1$. Evaluating `margin(1/(s*(1 + 0.1*s)*(1 + 0.5*s)))` generates the following plot:

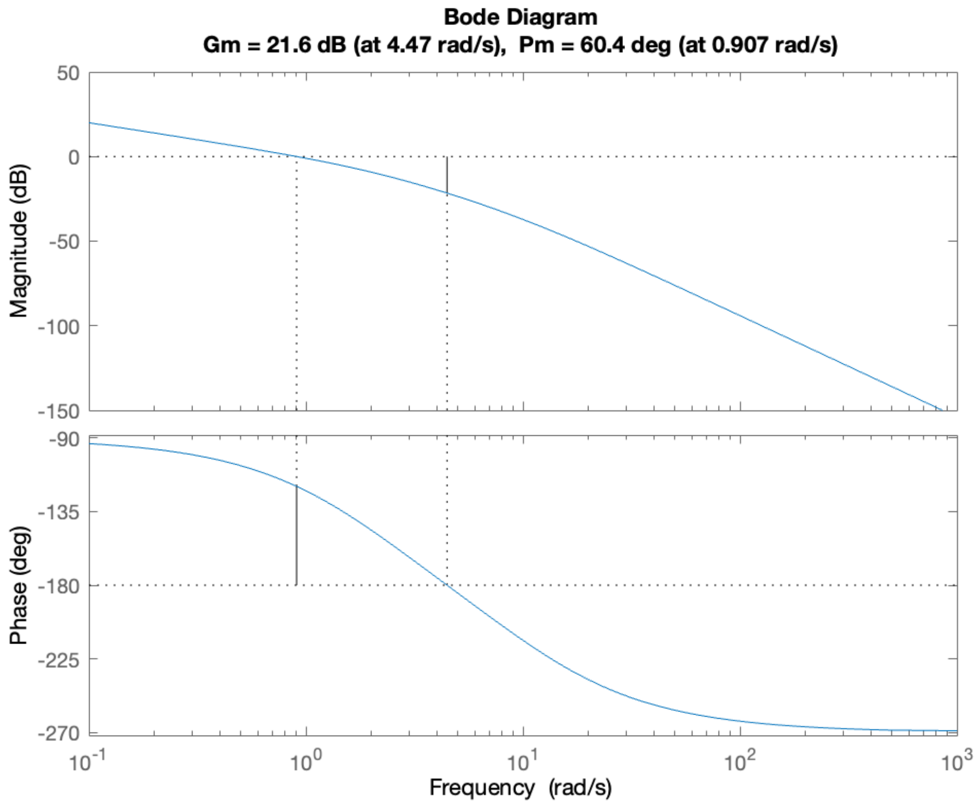

Figure 1: Bode Plot: K = 1

The transfer function in question can be written as:

$$G(j\omega) = \frac{K}{-0.05j\omega^3 - 0.6\omega^2 + j\omega} = \frac{K}{-0.6\omega^2 - j(0.05\omega^3 - \omega)}$$

The magnitude of $G(j\omega)$, is calculated as:

$$|G(j\omega)| = \frac{|K|}{|-0.05j\omega^3 - 0.6\omega^2 + j\omega|} = \frac{K}{\sqrt{0.0025\omega^6 + 0.26\omega^4 + \omega^2}}$$

Setting the magnitude of the gain equal to 1, we can solve for the gain crossover frequency. For this we will solve initially with $K = 1$:

$$1 = \frac{1}{\sqrt{0.0025\omega_g^6 + 0.26\omega_g^4 + \omega_g^2}}$$

This becomes:

$$0.0025\omega_g^6 + 0.26\omega_g^4 + \omega_g^2 - 1^2 = 0$$

Solving for the roots of this equation, using the MATLAB `roots()` function:

$$\texttt{roots}([0.0025, 0, 0.26, 0, 1, 0, -1]) = \begin{bmatrix} 0.0000 & +j9.9979 \\ 0.0000 & -j9.9979 \\ 0.0000 & +j2.2055 \\ 0.0000 & -j2.2055 \\ -0.9070 & \\ \mathbf{0.9070} & \end{bmatrix}$$

The gain crossover frequency is $\omega_g = 0.9070$ rad/s.

Taking the initial transfer function, we can rewrite it as:

$$\begin{aligned} G(j\omega) &= \frac{K}{-0.6\omega^2 - j(0.05\omega^3 - \omega)} \left(\frac{-0.6\omega^2 + j(0.05\omega^3 - \omega)}{-0.6\omega^2 + j(0.05\omega^3 - \omega)} \right) \\ &= \frac{K(-0.6\omega^2 + j(0.05\omega^3 - \omega))}{0.36\omega^4 + (0.05\omega^3 - \omega)^2} \end{aligned}$$

The phase angle for this equation is:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{0.05\omega^3 - \omega}{-0.6\omega^2} \right) = \tan^{-1}\left(\frac{0.05\omega^2 - 1}{-0.6\omega} \right)$$

We can determine the phase crossover frequency as:

$$\mathrm{Im}\{G(j\omega)\} = 0.05\omega_p^2 - 1 = 0$$

Which solves for ω_p as:

$$\omega_p = \sqrt{20} = 4.4721$$

The phase crossover frequency is $\omega_p = 4.4721$ rad/s.

The gain margin can be calculated as:

$$|G(j\omega_p)| = 10/120$$

$$-20\log_{10}(10/120) = -21.5836 \text{ dB}$$

If we want to make the gain margin 20 dB, we have to increase the gain magnitude at phase crossover frequency from -21.5836 to -20, which is an increase of 1.5836 dB. We can calculate this gain factor as:

$$K = 10^{(1.5836/20)} = 1.2$$

The value of $\boldsymbol{K}$ so that the gain margin of the system is **20** dB, is $\boldsymbol{K} = \mathbf{1.2}$.

$\longrightarrow$ $\mathcal{A}$nswer

We can confirm this using the `margin()` function in MATLAB. Evaluating `margin(1.2/(s*(1 + 0.1*s)*(1 + 0.5*s)))` generates the following plot:

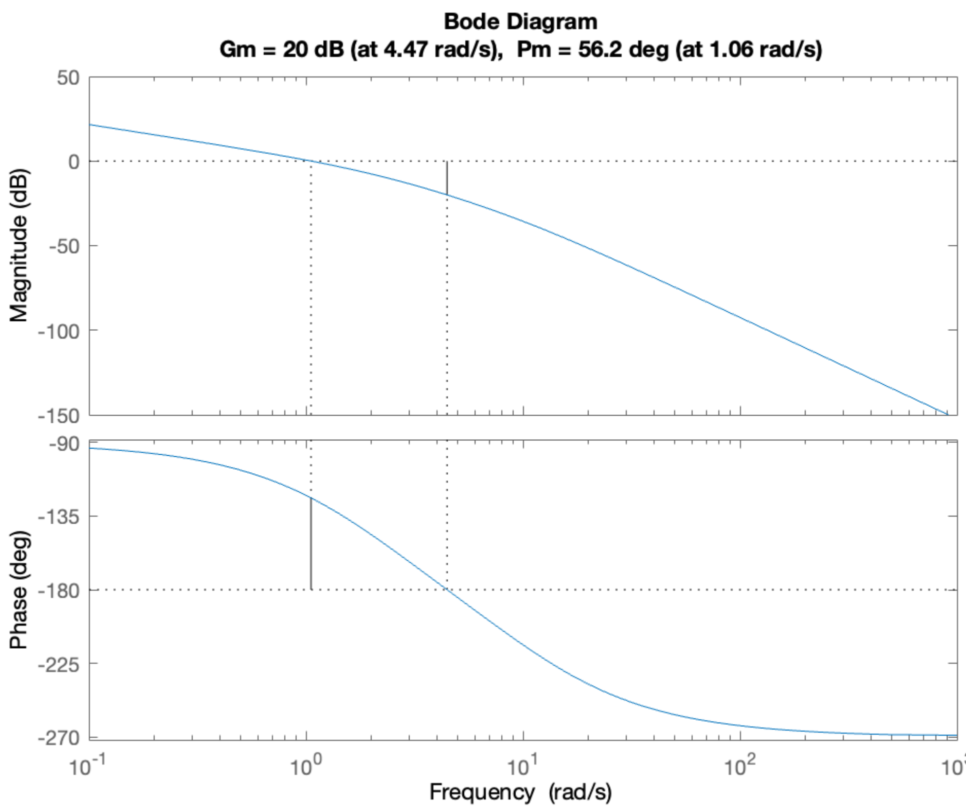

Figure 2: Bode Plot: K = 1.2

We can see that the gain margin is indeed 20 dB.

We can solve for the phase margin as:

$$\angle G(j\omega_g) = \tan^{-1}\left(\frac{0.05\omega_g^2 - 1}{-0.6\omega_g}\right)$$

$$\angle G(j0.907) = \tan^{-1}\left(\frac{0.05(0.907)^2 - 1}{-0.6(0.907)}\right) = 60.4231^\circ$$

If we want to make the phase margin equal to 45°, we need to set the phase angle equation equal to 45°, to solve for the frequency at which the phase angle is equal to 45°. We can write this as:

$$\angle G(j\omega_g) = 45^\circ = \tan^{-1}\left(\frac{0.05\omega_g^2 - 1}{-0.6\omega_g}\right)$$

This can be rewritten as:

$$\tan(45^\circ) = 1 = \left(\frac{0.05\omega_{45^\circ}^2 - 1}{-0.6\omega_{45^\circ}}\right)$$

Further:

$$0.05\omega_{45^\circ}^2 + 0.6\omega_{45^\circ} - 1$$

Solving for the roots of this equation, using the MATLAB `roots()` function:

$$\texttt{roots}([0.05, 0.6, -1]) = \begin{bmatrix} -13.4833 \\ \mathbf{1.4833} \end{bmatrix}$$

The frequency at which the phase angle is equal to 45°, is 1.4833 rad/s.
If we want this to be the gain crossover frequency, we need to set the magnitude equal to 1, where $\omega = 1.4833$:

$$1 = \frac{K}{\sqrt{0.0025(1.4833)^6 + 0.26(1.4833)^4 + (1.4833)^2}}$$

Solving this, $\boldsymbol{K = 1.8669}$ gives us a phase angle of $\mathbf{45°}$.

$\longrightarrow$ $\mathcal{A}$nswer

We can confirm this using the `margin()` function in MATLAB. Evaluating `margin(1.8669/(s*(1 + 0.1*s)*(1 + 0.5*s)))` generates the following plot:

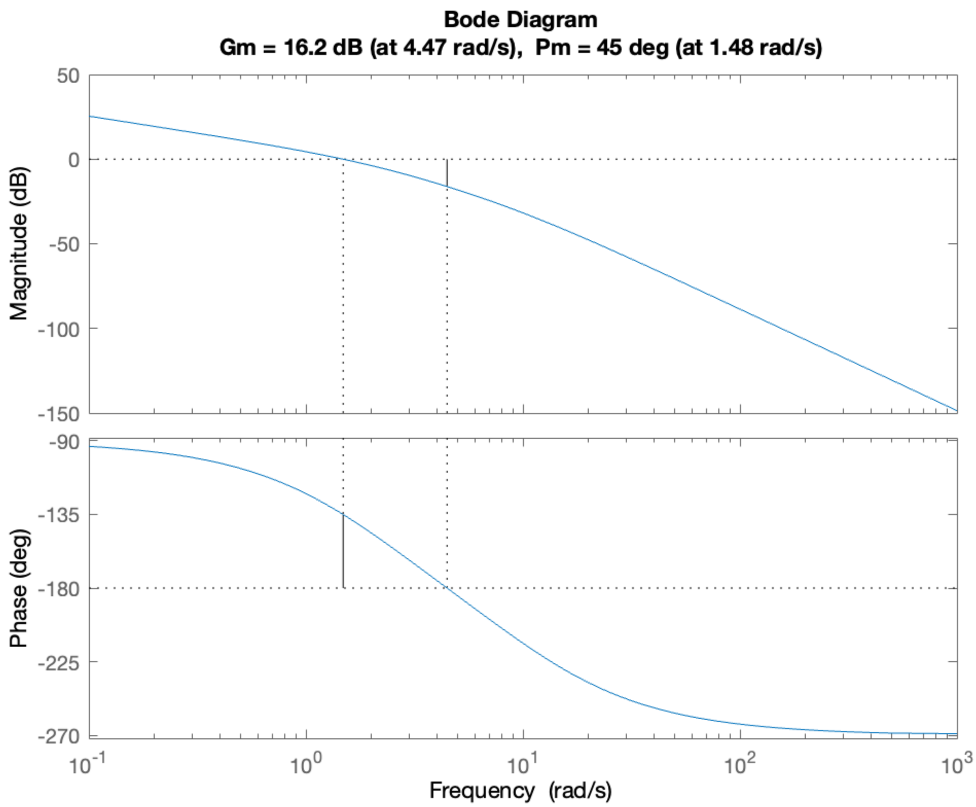

Figure 3: Bode Plot: K = 1.8669

We can see that the phase margin is indeed 45°.

Problem 2

Consider the following feedback system:

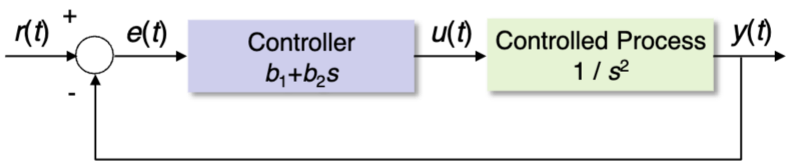

Figure 4: Feedback System

(a) Compute the sensitivity function, $\mathbf{S_G^T}$(s), and the complementary sensitivity function, T(s).

For this system:

$$D(s)G(s) = \frac{b_1 + b_2 s}{s^2}$$

The sensitivity function is:

$$\boldsymbol{S_G^T(s) = [1 + D(s)G(s)]^{-1} = \frac{s^2}{b_1 + b_2 s + s^2}}$$

$\longrightarrow$ $\mathcal{A}$nswer

The complimentary sensitivity function is the closed-loop transfer function. Therefor:

$$\boldsymbol{T(s) = \frac{D(s)G(s)}{1 + D(s)G(s)} = \frac{b_1 + b_2 s}{b_1 + b_2 s + s^2}}$$

$\longrightarrow$ $\mathcal{A}$nswer

(b) Assume a critically damped system ($\zeta = 1$) and a (normalized) natural frequency $\omega_n = 1$. Using MATLAB, plot the magnitude response of the sensitivity and complimentary sensitivity functions S(s) and T(s) on the same plot. At what frequency does S(s) = T(s)? What can be said about the roles of S(s) and T(s) at this frequency?

Assuming a critically damped system ($\zeta = 1$) and a (normalized) natural frequency $\omega_n = 1$, for this system:

$$b_1 = \omega_n^2 = 1^2 \quad \rightarrow \quad b_1 = 1$$

$$2\zeta\omega_n s = b_2 s \quad \rightarrow \quad b_2 = 2\zeta\omega_n = 2(1)(1) = 2$$

The sensitivity function is:

$$S(s) = \frac{s^2}{1 + 2s + s^2}$$

The complimentary sensitivity function is:

$$T(s) = \frac{1 + 2s}{1 + 2s + s^2}$$

Plotting both of these in MATLAB:

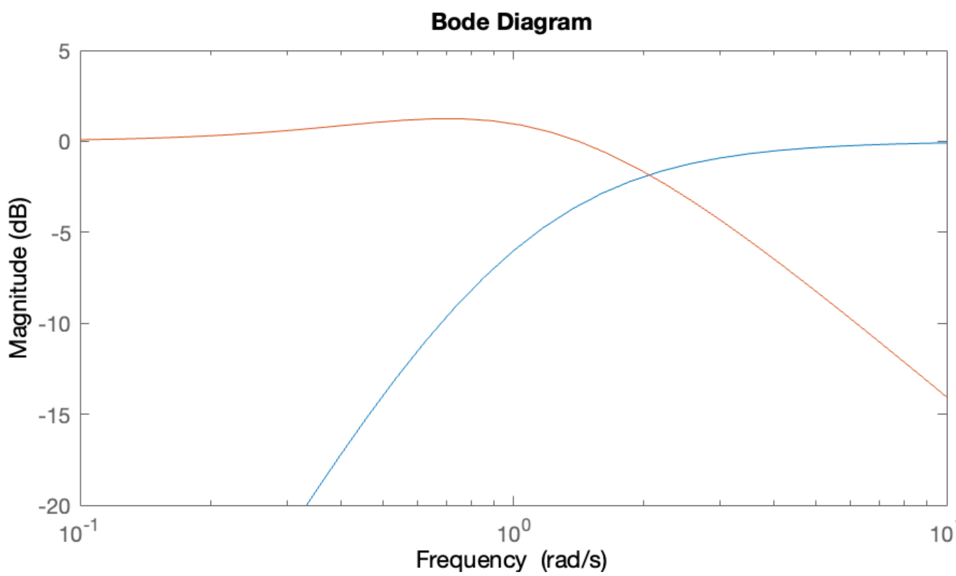

Figure 5: Sensitivity & Complementary Sensitivity

The frequency when $\boldsymbol{S(s) = T(s)}$ is, $\boldsymbol{\omega = 2.05887}$ rad/s.

$\longrightarrow$ *Answer*

When designing a system there are two desires, good tracking and disturbance rejection which is occurs when $\boldsymbol{S(s)}$ is small and $\boldsymbol{T(s)}$ is large, and good noise rejection when $\boldsymbol{S(s)}$ is large and $\boldsymbol{T(s)}$ is small. The point at which $\boldsymbol{S(s) = T(s)}$ is the point at which both of those desires share equal priority. If a larger or smaller frequency is chosen, then one or the other of those is getting prioritized at the expense of the other.

$\longrightarrow$ *Answer*

Problem 2

Consider the following (non-min phase) feedback system, where the CLTF, T(s), is stable for: $\mathbf{-2 < K < 1}$

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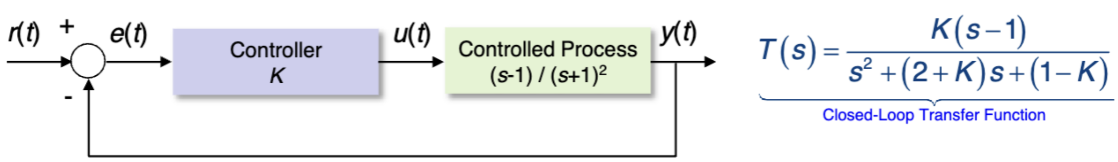

Figure 6: Non-Min Phase Feedback System

(a) Compute the expression for SSE

The closed loop transfer function is:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K(s-1)}{s^2 + (s+K)s + (1-K)}$$

The steady state response is:

$$y_{ss}(t) = \lim_{s\to 0} sY(s)\bigg|_{R(s)=1/s} = s\left(\frac{1}{s}\right)\left[\frac{K(s-1)}{s^2 + (s+K)s + (1-K)}\right]_{s=0}$$

This simplifies to:

$$y_{ss}(t) = \left[\frac{K(s-1)}{s^2+(s+K)s+(1-K)}\right]_{s=0}$$

Evaluating when $s=0$:

$$y_{ss}(t) = \frac{-K}{1-K}$$

The steady state error is:

$$\boldsymbol{e_{ss}(t) = r(t) - y_{ss}(t) = 1 - \frac{-K}{1-K}}$$

$\longrightarrow$ $\mathcal{A}$nswer

(b) At what (stable) gain is the SSE=0: $[\mathbf{e_{ss}(t) = 0}]$?

If we set $e_{ss}(t) = 0$, we can solve for the gain:

$$\boldsymbol{e_{ss}(t) = 0 = 1 - \frac{-K}{1-K} \quad \rightarrow \quad 1 = \frac{K}{1-K} \quad \rightarrow \quad K = \frac{1}{2}}$$

$\longrightarrow$ $\mathcal{A}$nswer

(c) Using the gain found above, use MATLAB to plot the system step response to a negative step $\mathbf{R(s) = -1/s}$. What's unusual about this response?

We can plot a negative step response using the following MATLAB command:

```
K = 1/2
T = K * (s-1)/(s^2 + (2 + K)*s + (1-K))
step(T,stepDataOptions('StepAmplitude',-1))
```

It produces this step response:

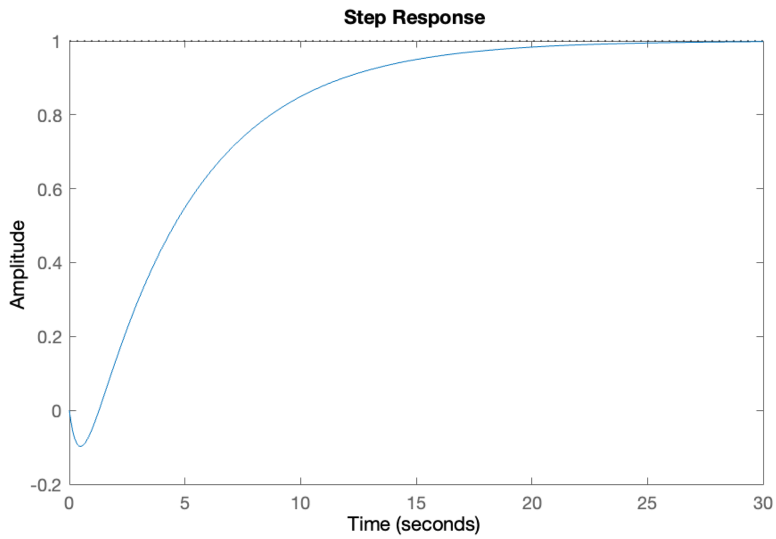

Figure 7: Negative Step Response

The fact that the response initially moves in the opposite direction of the reference signal is unusual.

(d) Create a table of SSE magnitude, % undershoot (observed from step response plot) and settling time (via step response plot) for K=[0.25,0.45, 0.5, 0.55, 0.75]

K	SSE Mag	% Undershoot	Settling Time (s)
0.25	0.6666	15.2627	11.0657
0.45	0.1818	10.7957	17.0232
0.50	0.0000	9.7404	19.2101
0.55	0.2222	8.6896	21.8672
0.75	2.0000	4.6662	42.8983

From the results of the table, what would you say about the sensitivity of this system to K?

From the results of the table, I would say that the system is very sensitive to K. Any variation of K away from $K = 1/2$ causes the system to incur steady state error. As K increases, settling time increases greatly. As K decreases, undershoot goes up. For all three parameters, K has an impact.

$\longrightarrow$ $\mathcal{A}$nswer

Submitted by Austin Barrilleaux on November 10, 2023.