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September 17, 2023

MODULE 3 — Practice Assignment

Problem 1

Solve the following practice problems in the 9th edition textbook.

- **Chapter 3:**
  - **10-1 (c) (use MATLAB if / as needed)**

The following differential equation represents a linear time-invariant system. Write the following dynamic equation (state equations and output equations) in vector-matrix form:

$$\boldsymbol{\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) + \int_0^t y(\tau)d\tau = r(\tau)}$$

We can establish the following relationships:

$$\begin{aligned} x_1(t) =& \int_0^t y(\tau)d\tau \\ x_2(t) =& \dot{x}_1 = y(t) \\ x_3(t) =& \dot{x}_2 = \frac{dy(t)}{dt} \\ x_4(t) =& \dot{x}_3 = \frac{d^2y(t)}{dt^2} \\ & \dot{x}_4 = \frac{d^3y(t)}{dt^3} \end{aligned}$$

We can rewrite the dynamic equation in question as:

$$\frac{d^3y(t)}{dt^3} = -5\frac{d^2y(t)}{dt^2} - 3\frac{dy(t)}{dt} - y(t) - \int_0^t y(\tau)d\tau + r(\tau)$$

The standard matrix state equations are:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Given the above relationships, we can write the dynamic equation as components of the standard matrix equations in the following matrix form:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Therefore, the state equation is:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ \dot{\boldsymbol{x}}_3 \\ \dot{\boldsymbol{x}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{-3} & \mathbf{-5} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{r(\tau)}$$

$\longrightarrow$ $\mathcal{A}$nswer

And the output equation is:

$$\boldsymbol{y(t)} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{bmatrix}$$

$\longrightarrow$ $\mathcal{A}$nswer

Problem 1

Consider the block diagram of the following feedback control system:

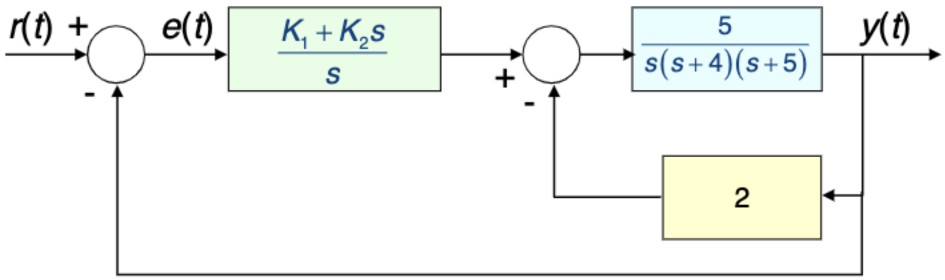


Figure 1: Problem 2

(a) Find the open-loop TF Y(s) / E(s) and the closed-loop TF Y(s) / R(s)

The open loop transfer function is simply the multiplication of the green and blue blocks:

$$\frac{Y(s)}{E(s)} = \left(\frac{K_1 + K_2 s}{s}\right)\left(\frac{5}{s(s+4)(s+5)}\right)$$

Or

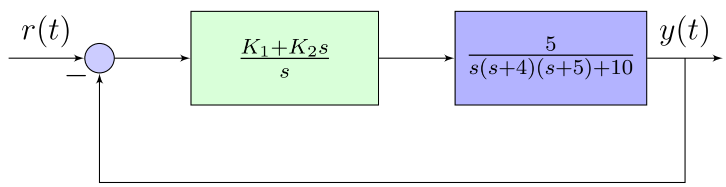
$$\boldsymbol{\frac{Y(s)}{E(s)} = \frac{5(K_1 + K_2 s)}{s^2(s+4)(s+5)}}$$

$\longrightarrow$ $\mathcal{A}$nswer

The to solve for the closed loop transfer function, first we reduce the blue and yellow blocks. They reduce to:

$$\frac{\frac{5}{s(s+4)(s+5)}}{1 + 2\frac{5}{s(s+4)(s+5)}} = \frac{5}{s(s+4)(s+5) + 10}$$

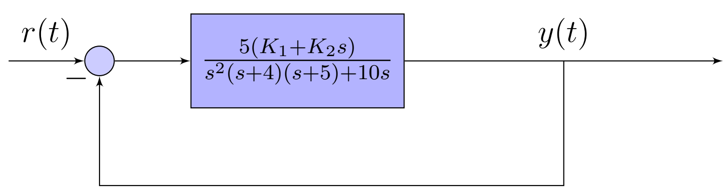
This is reflected in the following block diagram:



This combines with the green block to get:

$$\frac{5(K_1 + K_2 s)}{s^2(s+4)(s+5)+10s}$$

This is reflected in the following block diagram:



Which is the last block in a unity feedback loop, giving us the final transfer function of:

$$\frac{Y(s)}{R(s)} = \frac{\frac{5(K_1+K_2 s)}{s^2(s+4)(s+5)+10s}}{1+\frac{5(K_1+K_2 s)}{s^2(s+4)(s+5)+10s}} = \frac{5(K_1 + K_2 s)}{s^2(s+4)(s+5)+10s+5(K_1+K_2 s)}$$

The closed loop transfer function is:

$$\boldsymbol{\frac{Y(s)}{R(s)} = \frac{5(K_1 + K_2 s)}{s^2(s+4)(s+5)+10s+5(K_1+K_2 s)}}$$

$\longrightarrow$ $\mathcal{A}$nswer

(b) Write the dynamic equations in the form:

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t)$$
$$y(t) = C\bar{x}(t) + D\bar{u}(t)$$

To do this for the open loop case, we must first expand the numerator and the denominator to get:

$$\frac{Y(s)}{E(s)} = \frac{5K_1 + 5K_2 s}{s^4 + 9s^3 + 20s^2}$$

Then separate the equation into the form:

$$\frac{Y(s)}{E(s)} = \frac{Y(s)}{V(s)}\frac{V(s)}{E(s)} = \left(\frac{5K_1 + 5K_2 s}{1}\right)\left(\frac{1}{s^4 + 9s^3 + 20s^2}\right)$$

We can put this into the state space form:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ \dot{\boldsymbol{x}}_3 \\ \dot{\boldsymbol{x}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{-20} & \mathbf{-9} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \boldsymbol{r(t)}$$

$\longrightarrow \mathcal{A}$nswer

$$\boldsymbol{y(t)} = \begin{bmatrix} \boldsymbol{5K_1} & \boldsymbol{5K_2} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{bmatrix}$$

$\longrightarrow \mathcal{A}$nswer

To do this for the closed loop case, we must first expand the numerator and the denominator to get:

$$\frac{Y(s)}{R(s)} = \frac{5K_1 + 5K_2 s}{s^4 + 9s^3 + 20s^2 + (5K_2 + 10)s + 5K_1}$$

Then separate the equation into the form:

$$\frac{Y(s)}{E(s)} = \frac{Y(s)}{V(s)}\frac{V(s)}{E(s)} = \left(\frac{5K_1 + 5K_2 s}{1}\right)\left(\frac{1}{s^4 + 9s^3 + 20s^2 + (5K_2 + 10)s + 5K_1}\right)$$

We can put this into the state space form:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ \dot{\boldsymbol{x}}_3 \\ \dot{\boldsymbol{x}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\mathbf{5}\boldsymbol{K}_1 & -(\mathbf{5}\boldsymbol{K}_2 + \mathbf{10}) & -\mathbf{20} & -\mathbf{9} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{r}(\boldsymbol{t})$$

$\longrightarrow$ $\mathcal{A}$nswer

$$\boldsymbol{y}(\boldsymbol{t}) = \begin{bmatrix} \mathbf{5}\boldsymbol{K}_1 & \mathbf{5}\boldsymbol{K}_2 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \end{bmatrix}$$

$\longrightarrow$ $\mathcal{A}$nswer

(c) Apply the final value theorem to find the steady-state value of the output when the input is a unit step function (assume the CL system is stable)

From the final value theorem we know that:

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s)$$

Therefore:

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s)$$

For the open loop case, given $r(t) = u_s(t)Y$:

$$R(s) = \frac{1}{s}$$

If we substitute this into the closed loop transfer function, we get:

$$sY(s) = \frac{5K_1 + 5K_2 s}{s^4 + 9s^3 + 20s^2 + (5K_2 + 10)s + 5K_1}$$

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{5K_1 + 5K_2 s}{s^4 + 9s^3 + 20s^2 + (5K_2 + 10)s + 5K_1} = \frac{5K_1}{5K_1} = 1$$

Submitted by Austin Barrilleaux on September 17, 2023.