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MODULE 10 — Practice Assignment

Problem 1

Solve problem 8-19 in the 9th Edition textbook.

- **Given the following CE:** $\mathbf{s(s^3 + 2s^2 + s + 1) + K(s^2 + s + 1) = 0}$
- **(a) Hand-sketch the polar plot as discussed in Module 10 and apply the Nyquist criterion to determine the values of K for system stability.**
- **(b) Check the answers by means of the Routh-Hurwitz criterion.**

From the above CE:

$$1 + G(s)H(s) = s(s^3 + 2s^2 + s + 1) + K(s^2 + s + 1) = 0$$

Putting this into the form:

$$1 + G(s)H(s) = 1 + L(s)$$

We find that $L(s)$:

$$L(s) = \frac{K(s^2 + s + 1)}{s(s^3 + 2s^2 + s + 1)}$$

Or:

$$L(j\omega) = \frac{K\left(-\omega^2 + \omega j + 1\right)}{\omega^4 - 2\omega^3 j - \omega^2 + \omega j}$$

Simplifying to:

$$
\begin{aligned}
L(j\omega) &= \frac{K\,\left((1-\omega^2)+\omega j\right)}{(\omega^4-\omega^2)+j(\omega-2\omega^3)} \\
&= \frac{K\,\left((1-\omega^2)+\omega j\right)}{(\omega^4-\omega^2)+j(\omega-2\omega^3)} \\
&= \frac{K\,\left((1-\omega^2)+\omega j\right)}{(\omega^4-\omega^2)+j(\omega-2\omega^3)}\left(\frac{(\omega^4-\omega^2)-j(\omega-2\omega^3)}{(\omega^4-\omega^2)-j(\omega-2\omega^3)}\right) \\
&= \frac{K\left(-\omega^6+j(-\omega^5+2\omega^3-\omega)\right)}{(\omega^4-\omega^2)+(\omega-2\omega^3)}
\end{aligned}
$$

The Magnitude of $L(j\omega)$ is:

$$
|L(j\omega)| = \left|\frac{K\left(-\omega^6+j(-\omega^5+2\omega^3-\omega)\right)}{(\omega^4-\omega^2)+(\omega-2\omega^3)}\right|
$$

The Angle of $L(j\omega)$ is:

$$
\angle L(j\omega) = \tan^{-1}\left(\frac{-\omega^5+2\omega^3-\omega}{-\omega^6}\right) = 180^\circ - \tan^{-1}\left(\frac{-\omega^4+2\omega^2-1}{\omega^5}\right)
$$

From this, we can determine the following, evaluating the limit as $\omega = 0$:

$$
\lim_{\omega\to 0} L(j\omega) = \infty
$$

$$
\lim_{\omega\to 0} \angle L(j\omega) = 180 - \tan^{-1}\left(\frac{-1}{0}\right) = 270^\circ = -90^\circ
$$

From this, we can determine the following, evaluating the limit as $\omega = \infty$:

$$
\lim_{\omega\to\infty} L(j\omega) = 0
$$

$$
\lim_{\omega\to\infty} \angle L(j\omega) = 180 - \tan^{-1}\left(\frac{-1}{\infty}\right) = 180^\circ
$$

Finding intersection of the locus with the real axis via: $\mathrm{Im}[L(j\omega)] = 0$:

$$
-\omega^5+2\omega^3-\omega = 0 \quad\rightarrow\quad \omega^4-2\omega^2+1=0
$$

This can be factored to:

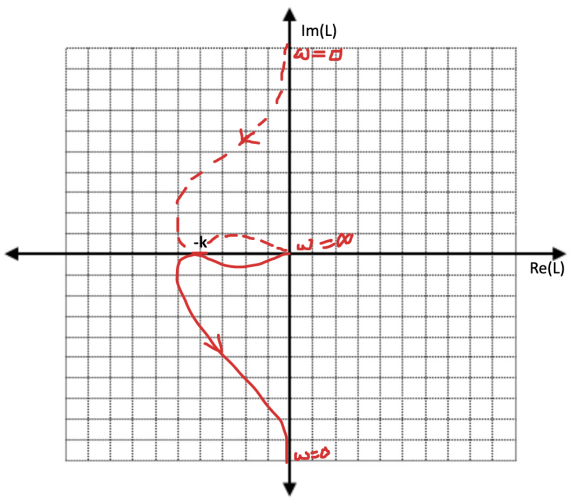
$$(\omega^2-1)^2=0 \quad \rightarrow \quad \omega=\pm 1 \text{ rad/s}$$

Substituting, $L(j\omega)$:

$$|L(j1)|=\frac{K(s^2+s+1)}{(s^4+2s^3+s^2+s)}=\frac{K(-1+j+1)}{(1-2j-1+j)}=K\frac{j}{-j}=-K$$

$$\angle L(j1)=\angle\frac{K(-1+j+1)}{(1-2j-1+j)}=180-\tan^{-1}\left(\frac{0}{-1}\right)=180^\circ$$

Plotting this, we get a plot that looks something like:



Applying the Nyquist Stability Criterion:

$$N=Z-P$$

Where:
N = number of encirclements of the $(-1, j0)$ point made by the $L(s)$ plot.
Z = number of zeros of $1+L(s)$ that are inside the Nyquist path, that is, the right-half s-plane.
P = number of poles of $1+L(s)$ that are inside the Nyquist path, that is, the right-half s-plane. (Notice that the poles of $1+L(s)$ are the same as that of $L(s)$.)

There are zero poles of $L(s)$ in the RHP (MATLAB: `roots([1 2 1 1 0])` = $0, -0.1226 \pm 0.7449j, -1.7549$), so $P=0$. For the system to be stable ($Z=0$), there must be zero encirclements of the $(-1, j0)$ point ($N=0$.)

As stated above, the Phase Angle of $L(j\omega)$ is:

$$\angle L(j\omega) = \tan^{-1}\left(\frac{-\omega^5 + 2\omega^3 - \omega}{-\omega^6}\right)$$

In the event that K is negative, the negative sign would impact this angle:

$$\angle L(j\omega) = \tan^{-1}\left(\frac{\omega^5 - 2\omega^3 + \omega}{-\omega^6}\right)$$

So when evaluating the angle as $\omega \to 0$, the angle would be positive $90°$ instead of $-90°$. In this case, the $(-1, j0)$ would be enclosed, so the system would be unstable since Z would not be zero.

Therefor, the **system is stable when $\boldsymbol{K > 0}$**, and the **system is unstable when $\boldsymbol{K < 0}$**. If $\boldsymbol{K = 1}$, the contour would lie on the point $\boldsymbol{(-1, j0)}$, so the **system would be marginally stable**.

$\longrightarrow$ *Answer*

Evaluating with Routh-Hurwitz, we get the following table:

$$\begin{array}{c|ccc} s^4 & 1 & K+1 & K \\ s^3 & 2 & K+1 & 0 \\ s^2 & \frac{K+1}{2} & K & 0 \\ s^1 & \frac{K^2-2K+1}{K+1} & 0 & 0 \\ s^0 & K & 0 & 0 \end{array}$$

This reinforces the above conclusion. The **system is stable when $\boldsymbol{K > 0}$**, and **unstable when $\boldsymbol{K < 0}$**.

$\longrightarrow$ *Answer*

The leftmost column of s^1 equates to:

$$\frac{K^2 - 2K + 1}{K + 1} = \frac{(K-1)^2}{K+1}$$

Set equal to zero:

$$\frac{(K-1)^2}{K+1} = 0$$

We see that when $\boldsymbol{K} = \mathbf{1}$, we get a row of all zeros implying complex poles on the imaginary axis. In this case, **the system is marginally stable.**

$\longrightarrow \mathcal{A}$nswer

Problem 2

Consider the open loop transfer function, below, which is placed in a unity-feedback configuration:

$$\mathbf{G(s)} = \frac{\mathbf{5(s-2)}}{\mathbf{s(s+1)(s-1)}}$$

- **(a) Draw the entire polar plot of this system.**
- **(b) Considering the portion of the polar plot from $\omega = +\infty$ to $\omega = 0+$, apply the Nyquist criterion to determine the stability of the system. If the system is unstable, determine the number of closed loop poles in the RHP.**

The open loop transfer function in question can be written as:

$$G(j\omega) = \frac{5(j\omega - 2)}{(j\omega)^3 - j\omega}$$

Simplifying to:

$$\begin{aligned}
G(j\omega) &= \frac{5(j\omega - 2)}{(j\omega)^3 - j\omega} \\
&= \frac{5(j\omega - 2)}{-j\omega^3 - j\omega} \\
&= \frac{5(j\omega - 2)}{-j(\omega^3 + \omega)} \\
&= \frac{5(j\omega - 2)}{-j(\omega^3 + \omega)} \left(\frac{j(\omega^3 + \omega)}{j(\omega^3 + \omega)} \right) \\
&= \frac{-5\omega^2(\omega^2 + 1) - 10j\omega(\omega^2 + 1)}{(\omega^3 + \omega)^2}
\end{aligned}$$

The Magnitude of $G(j\omega)$ is:

$$|G(j\omega)| = \left| \frac{-5\omega(\omega^2+1) - 10j(\omega^2+1)}{\omega(\omega^2+1)^2} \right|$$

The Angle of $G(j\omega)$ is:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{10\omega(\omega^2+1)}{5\omega^2(\omega^2+1)}\right) = \tan^{-1}\left(\frac{2}{\omega}\right)$$

From this, we can determine the following evaluating the limit as $\omega = 0$:

$$\lim_{\omega \to 0} L(j\omega) = \infty$$

$$\lim_{\omega \to 0} \angle L(j\omega) = \tan^{-1}\left(\frac{2}{0}\right) = 90^\circ$$

From this, we can determine the following evaluating the limit as $\omega = \infty$:

$$\lim_{\omega \to \infty} L(j\omega) = 0$$

$$\lim_{\omega \to 0} \angle L(j\omega) = \tan^{-1}\left(\frac{2}{\infty}\right) = 0^\circ$$

We can check for a real axis intersection:

$$10(\omega^2+1) = 0$$

Which simplifies to:

$$\omega^2 = -1 \quad \rightarrow \quad \omega = \pm j$$

No real axis crossing exists:

We can check for an imaginary axis intersection:

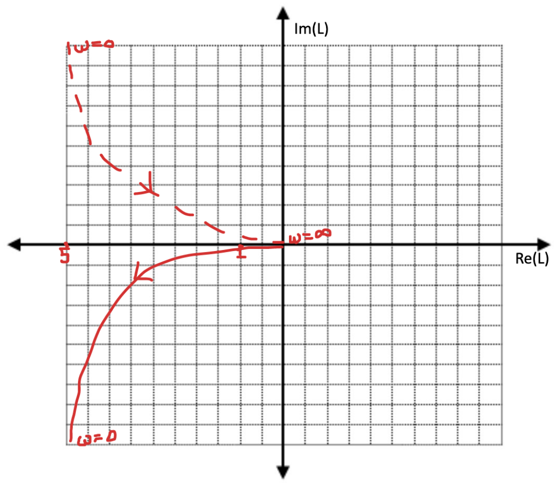
$$5\omega(\omega^2+1) = 0$$

Which simplifies to:

$$\omega^2 = -1 \quad \rightarrow \quad \omega = \pm j$$

No imaginary axis crossing exists:

Using this information, the polar plot will look like the following:



Applying the Nyquist Stability Criterion:

$$N = Z - P$$

Where:
N = number of encirclements of the $(-1, j0)$ point made by the $L(s)$ plot.
Z = number of zeros of $1 + L(s)$ that are inside the Nyquist path, that is, the right-half s-plane.
P = number of poles of $1 + L(s)$ that are inside the Nyquist path, that is, the right-half s-plane. (Notice that the poles of $1 + L(s)$ are the same as that of $L(s)$.)

There is one open loop pole in the RHP, so $P = 1$. (The open loop transfer function is not stable.) Looking at the Polar Plot, $(-1, j0)$ is not enclosed by the contour, so $N = 0$. Solving for Z:

$$Z = N + P = 0 + 1 = 1$$

Since $\boldsymbol{Z \neq 0}$, the **closed loop system is not stable!**. There is **1 closed loop pole in the RHP.**

$\longrightarrow$ $\mathcal{A}$nswer

Submitted by Austin Barrilleaux on November 5, 2023.