Austin Barrilleaux
Whiting School of Engineering
Johns Hopkins University

November 18, 2023

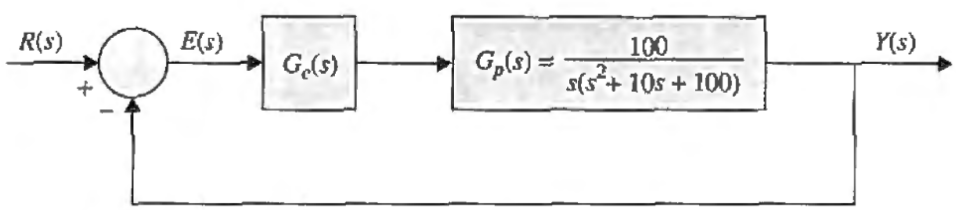
MODULE 11 — Practice Assignment

Problem 1

Solve the following 9th Edition textbook problems:

- **9-1**
- **9-43**

(9-1) The block diagram of a control system with a series controller below.



Find the transfer function of the controller $G_c(s)$ so that the following specifications are satisfied:

- **The ramp error constant K_v is 5.**
- **The closed-loop transfer function is of the form:**

$$\mathbf{M(s) = \frac{Y(s)}{R(s)} = \frac{K}{(s^2 + 20s + 200)(s + a)}}$$

where K and a are real constants. Find the values of K and a.

We know that the closed-loop transfer function is:

$$M(s) = \frac{G(s)}{1 + G(s)} = \frac{K}{(s^2 + 20s + 200)(s + a)}$$

Solving for $G(s)$:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{K}{s^3 + (20+a)s^2 + (200+20a)s + 200a - K}$$

For this system to have a constant ramp error, the system must be type 1, so in order to have a zero pole:

$$200a - K = 0 \quad \rightarrow \quad K = 200a$$

So $G(s)$ becomes:

$$G(s) = \frac{M(s)}{1 - M(s)} = \frac{200a}{s^3 + (20+a)s^2 + (200+20a)s}$$

The ramp error constant is computed as:

$$K_v = \lim_{s \to 0} sG(s) = \frac{s \cdot 200a}{s^3 + (20+a)s^2 + (200+20a)s} = \frac{200a}{s^2 + (20+a)s + (200+20a)}$$

Evaluating this:

$$K_v = \frac{200a}{\cancel{s^2}^{\,0} + \cancel{(20+a)s}^{\,0} + (200+20a)} = \frac{200a}{200+20a} = 5$$

Solving for $\boldsymbol{a}$, $\boldsymbol{a = 10}$ and $\boldsymbol{K = 2000}$.

$\longrightarrow$ *Answer*

From this we can solve for the transfer function of the controller, $G_c(s)$:

$$G(s) = G_c(s)G_p(s) \quad \rightarrow \quad G_c(s) = \frac{G(s)}{G_p(s)}$$

So:

$$G_c(s) = \frac{2000}{s(s^2 + 20s + 400)} \left(\frac{s(s^2 + 10s + 100)}{100} \right)$$

$G_c(s)$ reduces to:

$$\boldsymbol{G_c(s) = \frac{20(s^2 + 10s + 100)}{(s^2 + 20s + 400)}}$$

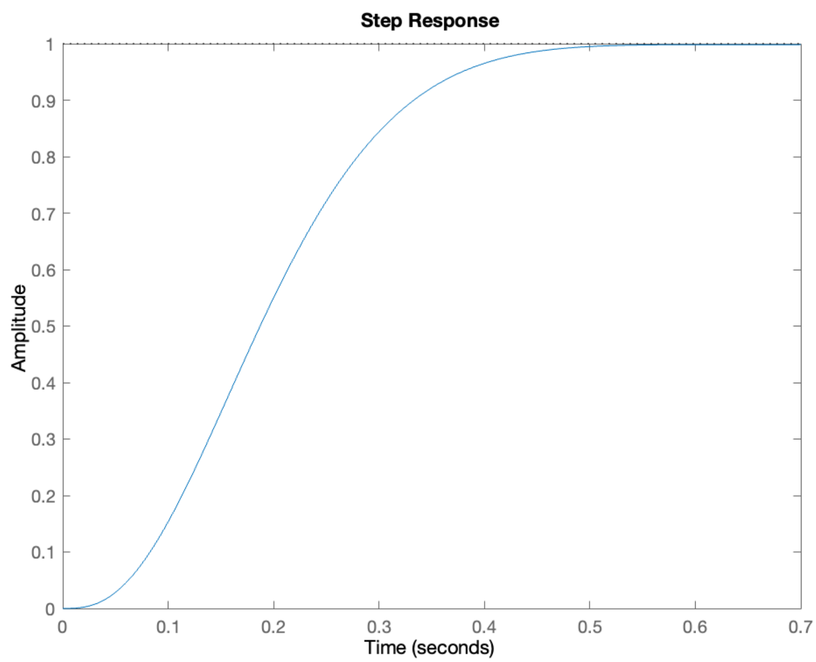
$\longrightarrow$ *Answer*

The design strategy is to place the closed-loop poles at $-10 + j10$ and $-10 - j10$, and then adjust the values of K and a to satisfy the steady-state requirement. The value of a is large so that it will not affect the transient response appreciably. Find the maximum overshoot of the designed system.

We can determine the maximum overshoot of the system by using the `stepinfo()` function in MATLAB. Doing so, gives us a **maximum overshoot** of **0**.

$\longrightarrow$ *Answer*

If we plot the response we can see that there is no overshoot:



(9-43) Consider that the process of a unity-feedback control system is:

$$\mathbf{G_p(s) = \frac{1000}{s(s+10)}}$$

Let the series controller be a single-stage phase-lead controller:

$$\mathbf{G_c(s) = \frac{1 + aTs}{1 + Ts}, \quad a > 1}$$

(a) Determine the values of a and T so that the zero of $\mathbf{G_c(s)}$ cancels the pole of Gp(s) at $\mathbf{s = -10}$. The damping ratio of the designed system should be unity. Find the attributes of the unit-step response of the designed system.

Combining $G_p(s)$ and $G_c(s)$:

$$G(s) = \frac{1000a(\frac{1}{Ta} + s)}{s(s + 10)(\frac{1}{T} + s)}$$

To cancel the $(s + 10)$ pole:

$$\frac{1}{Ta} = 10$$

The transfer function becomes:

$$G(s) = \frac{1000a}{s(\frac{1}{T} + s)}, \quad \frac{1}{Ta} = 10$$

The characteristic equation of the closed loop transfer function is:

$$s^2 + \frac{s}{T} + 1000a = 0$$

From this, we can define ζ and ω_n as:

$$\frac{1}{T} = 2\zeta\omega_n \quad \omega_n = \sqrt{1000a}$$

Therefore:

$$\frac{1}{T} = 10a = 2\zeta\omega_n = 2\sqrt{1000a}$$

$$\boldsymbol{100a^2 = 4000a \quad \rightarrow \quad a = 40}$$

$\longrightarrow$ $\mathcal{A}$nswer

$$\boldsymbol{T = \frac{1}{10a} = \frac{1}{400}}$$

$\longrightarrow$ $\mathcal{A}$nswer

The constant $\boldsymbol{a = 40}$ and $\boldsymbol{T = 0.0025}$.

$\longrightarrow$ $\mathcal{A}$nswer

The closed loop transfer function is:

$$M(s) = \frac{40000}{s^2 + 400s + 40000}$$

Solvinf for the attributes of the system:

$$\zeta = \frac{400}{2\omega_n} = \frac{400}{2\sqrt{40000}} = 1$$

Therefor, maximum overshoot is **0**.

$\longrightarrow$ $\mathcal{A}$nswer

The rise time can be approximated as:

$$\boldsymbol{t_r = \frac{0.8 + 2.5\zeta}{\omega_n} = 0.0165} \ \text{ sec}$$

$\longrightarrow$ $\mathcal{A}$nswer

The settling time can be approximated as:

$$\boldsymbol{t_s = \frac{4.5\zeta}{\omega_n} = 0.0225} \ \text{ sec}\boldsymbol{,} \quad \boldsymbol{\zeta > 0.69}$$

$\longrightarrow$ $\mathcal{A}$nswer

(b) Carry out the design in the frequency domain using the Bode plot. The design specifications are as follows:

- **Phase margin $> 75°$**
- **$M_r < 1.1$**

Find the attributes of the unit-step response of the designed system.

The uncompensated open loop transfer function is:

$$G(s) = \frac{1000}{s(s+10)}$$

The attributes of the system using `margin()` and `getPeakMargin()` functions in MATLAB, we see that $PM = 17.9642$, $GM = \infty$ and $M_r = 3.2026$

In order the meet the phase margin requirement, we need to shift it by:

$$75° - 17.9642° + 15° = 72.0358°$$

This includes a fudge factor of 5°.

a is calculated as:

$$a = \frac{1 + \sin(72.0358°)}{1 - \sin(72.0358°)} = 40.0251$$

The gain of the controller is calculated as:

$$-10\log_{10}(a) = -10\log_{10}(40.0251) = -16.0233 \text{ dB}$$

The gain magnitude is:

$$|G(j\omega)| = \left|\frac{1000}{j\omega(j\omega + 100)}\right| = \frac{1000}{\omega\sqrt{\omega^2 + 100}}$$

We can solve for the new gain crossover frequency as:

$$\frac{1000}{\omega\sqrt{\omega^2 + 10}} = 10^{\frac{-16.0233}{20}}$$

We can put this in the form:

$$\omega^4 + 100\omega^2 + \left(\frac{1000}{10^{\frac{-16.0233}{20}}}\right)^2 = 0$$

We can solve for ω_{max} via the following:

$$\omega_{\text{max}} = \sqrt{-10 + \sqrt{10^2 + \left(\frac{1000}{10^{\frac{-16.0233}{20}}}\right)^2}} = 79.4767 \text{ rad/s}$$

The relationship between ω_{max} and the compensator pole / zero location is:

$$\omega_{\text{max}} = \frac{1}{\tau\sqrt{a}} = \frac{1}{T\sqrt{a}} \quad , \qquad\qquad (\tau = T)$$

Therefore:

$$T = \frac{1}{\sqrt{a}\omega_{\text{max}}} = 0.0020 \text{ rad/s}$$

The controller transfer function is:

$$G_c(s) = \frac{1 + 0.0796s}{1 + 0.0020s}$$

We now get the following for $G(s)$:

$$G(s) = \frac{1000}{s(s+10)}\left(\frac{1 + 0.0796s}{1 + 0.0020s}\right) = \frac{79.6s + 1000}{0.001989s^3 + 1.02s^2 + 10s}$$

If we evaluate this using the `margin()` function in MATLAB, we get that $\boldsymbol{PM = 79.2502}$ and $\boldsymbol{GM = \infty}$.

$\longrightarrow$ *Answer*

If we evaluate it's corresponding closed loop transfer function using the `getPeakMargin()` function in MATLAB, we get that $\boldsymbol{M_r = 1.0096}$.

$\longrightarrow$ *Answer*

This controller design satisfies both of the constraints, that $\mathbf{PM > 75°}$ and $\mathbf{M_r < 1.1}$.

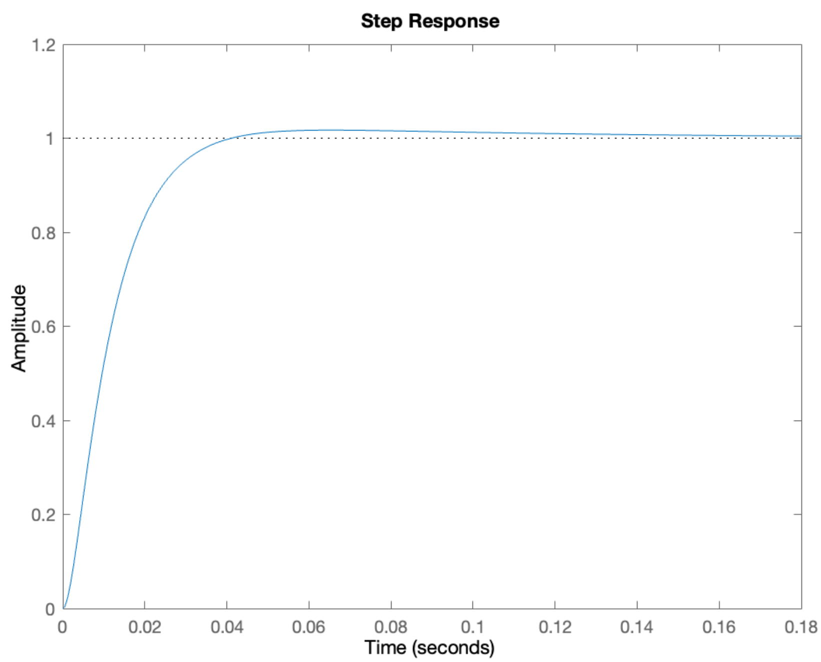
$\longrightarrow$ *Answer*

The attributes of the unit-step response of the designed system, found using the `stepinfo()` function in MATLAB are:

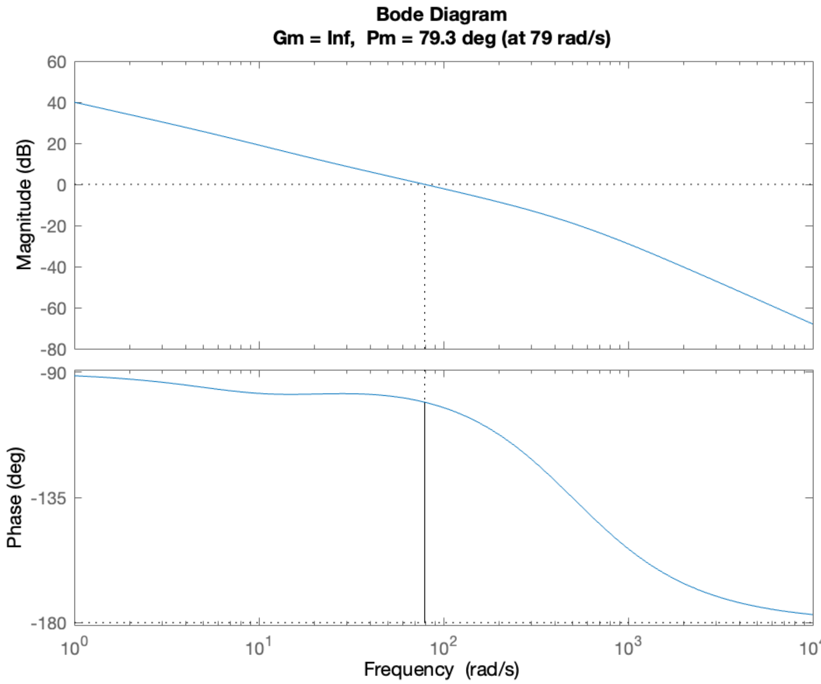
- $\boldsymbol{t_r = 0.0216}$ sec
- $\boldsymbol{t_s = 0.0348}$ sec
- **%OS = 1.7075** %

$\longrightarrow \mathcal{A}$nswer

The step response can be seen here:



The bode plot for the system is:



I attempted this part the first time with a fudge factor of 5°, which did not meet the design requirements.

Submitted by Austin Barrilleaux on November 18, 2023.