Austin Barrilleaux
Whiting School of Engineering
Johns Hopkins University

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MODULE 6 — Practice Assignment

Problem 1

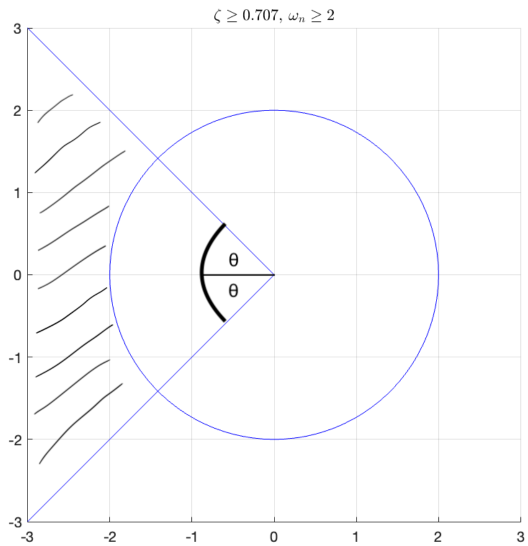
Solve the following practice problems in the 9th edition textbook.

- **Chapter 5:**
  - **5-1 (a,b)**
  - **5-18**
  - **5-19**

5-1. A pair of complex-conjugate poles in the s-plane is required to meet the various specifications that follow. For each specification, sketch the region in the s-plane in which the poles should be located.

(a) $\zeta \geq 0.707$, $\omega_n \geq 2$ **rad/sec, (positive damping)**

The following sketch shows the region in the s-plane in which the poles should be located (hand shaded area in black):



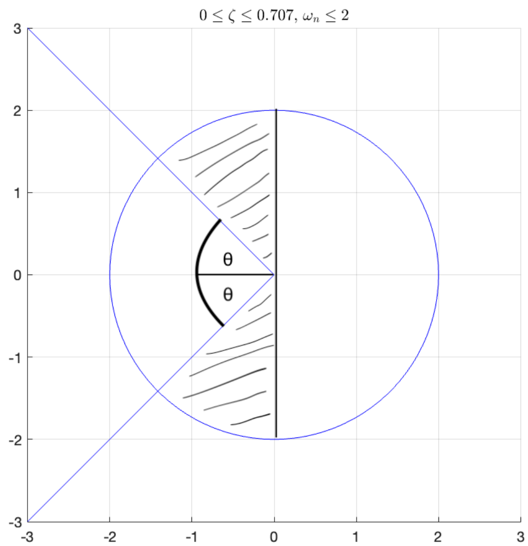
The plot was generated using MATLAB then annotated.
The value for θ was calculated as:

$$\theta = \cos^{-1}(\zeta) = \cos^{-1}(0.707) = 0.7855$$

This value for θ which is in radians is roughly equal to 45 degrees.

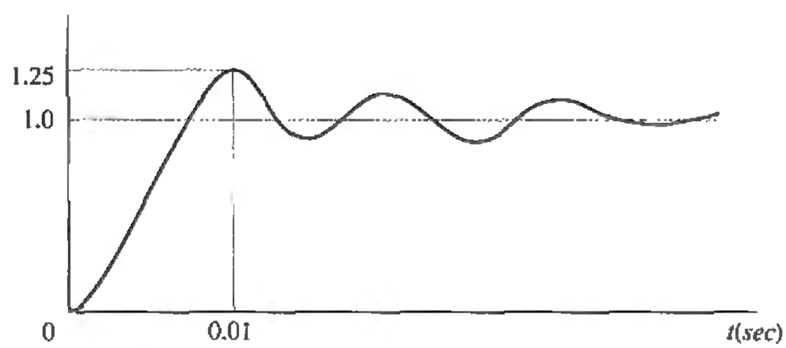
(a) $0 \leq \zeta \leq 0.707$**,** $\omega_n \leq 2$ **rad/sec, (positive damping)**

The following sketch shows the region in the s-plane in which the poles should be located (hand shaded area in black):



The value for θ, was calculated the same as in part (a), as the boundary of the region was the same.

5-18. The unit-step response of a linear control system is shown in Fig. SP- I8. Find the transfer function of a second-order prototype system to model the system.



We know that the equation for maximum overshoot is:

$$y(t_p) - 1 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Populating that $y(t_p) = 1.25$, the equation can be written as:

$$0.25 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Further, taking the natural log of both sides:

$$\ln(0.25) = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}}$$

Reorganizing this:

$$\ln(0.25)^2(1-\zeta^2) = \pi^2\zeta^2$$

Solving for $zeta$, we get:

$$\ln(0.25)^2 = (\ln(0.25)^2 + \pi^2)\zeta^2$$

Further:

$$\zeta^2 = \frac{\ln(0.25)^2}{(\ln(0.25)^2 + \pi^2)}$$

Or:

$$\zeta = \sqrt{\frac{\ln(0.25)^2}{(\ln(0.25)^2 + \pi^2)}} \approx 0.4037$$

Which evaluated in MATLAB is approximately $\zeta \approx 0.4037$.

Now that we have ζ, using the following equation for peak time, we can solve for ω_n:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Re-arranging:

$$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}}$$

Solving this in MATLAB using the solved value of ζ, the value for ω_n is:

$$\omega_n = \frac{\pi}{0.01\sqrt{1-\zeta^2}} \approx 343.3863 \;\; \text{rad/s}$$

The gives a transfer function of:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{117914.16}{s^2 + 277.25s + 117914.16}$$

The final result is:

$$\boldsymbol{\frac{Y(s)}{R(s)} = \frac{117914.16}{s^2 + 277.25s + 117914.16}}$$

$\longrightarrow$ *Answer*

The following MATLAB script provides this answer:

```
%% Problem 5-18
% maximum overshoot
y_tp = 1.25;
% Overshoot
os =  y_tp - 1;
% Compute Zeta
zeta = sqrt(log(os)^2/(pi^2 + log(os)^2));
% Peak time
tp = 0.01;
% Compute omega_n
omega_n = pi/(tp * sqrt(1-zeta^2));
% Create the transfer function
TF = tf(omega_n^2,[1,2*zeta*omega_n,omega_n^2]);
```

5-19. For the control system shown in Fig. 5P-13, find the values of K and K_t, so that the maximum overshoot of the output is approximately 4.3% and the rise time t_r, is approximately 0.2 sec. Simulate the system with any time-response simulation program to check the accuracy of your solutions.

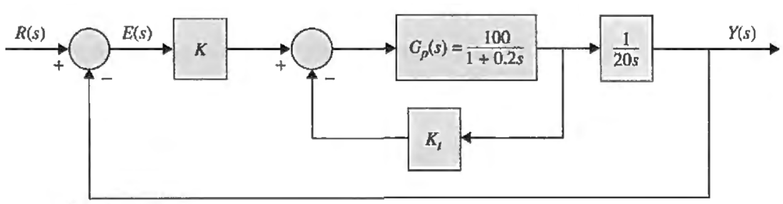


Figure 5P-13

This block diagram reduces to:

$$\frac{Y(s)}{R(s)} = \frac{100K}{4s^2 + 20s + 2000K_t s + 100K} = \frac{25K}{s^2 + (5 + 500K_t)s + 25K}$$

Putting this into the form:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

This means that:

$$\omega_n = 5\sqrt{K}$$

$$2\zeta\omega_n = (5 + 500K_t)$$

Populating the latter function with ω_n:

$$\zeta = \frac{(0.5 + 50K_t)}{sqrtK}$$

Since:

$$y(t_p) - 1 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Populating that $y(t_p) = 1.25$, the equation can be written as:

$$0.043 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Further, taking the natural log of both sides:

$$\ln(0.043) = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}}$$

Reorganizing this:

$$\ln(0.043)^2(1 - \zeta^2) = \pi^2\zeta^2$$

Solving for $zeta$, we get:

$$\ln(0.043)^2 = (\ln(0.043)^2 + \pi^2)\zeta^2$$

Further:

$$\zeta^2 = \frac{\ln(0.043)^2}{(\ln(0.043)^2 + \pi^2)}$$

Or:

$$\zeta = \sqrt{\frac{\ln(0.043)^2}{(\ln(0.043)^2 + \pi^2)}} \approx 0.7077$$

Given this value for ζ we can now solve for ω_n using:

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n}$$

Re-arranging:

$$\omega_n = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{t_r}$$

Solving this in MATLAB using the solved value of ζ, the value for ω_n is:

$$\omega_n = \frac{1 - 0.4167 \cdot 0.7077 + 2.917 \cdot 0.7077^2}{t_r} \approx 10.83 \quad \text{rad/s}$$

Since $\omega_n = 5\sqrt{K}$, this means that:

$$\boldsymbol{K = \frac{\omega_n^2}{25} = \frac{10.83^2}{25} \approx 4.69}$$

$\longrightarrow$ $\mathcal{A}$nswer

Using this value of K, we can now solve for K_t:

$$\zeta = \frac{(0.5 + 50K_t)}{sqrtK}$$

Re-arranging:

$$\boldsymbol{K_t = \frac{\zeta\sqrt{K} - 0.5}{50} \approx 0.0207}$$

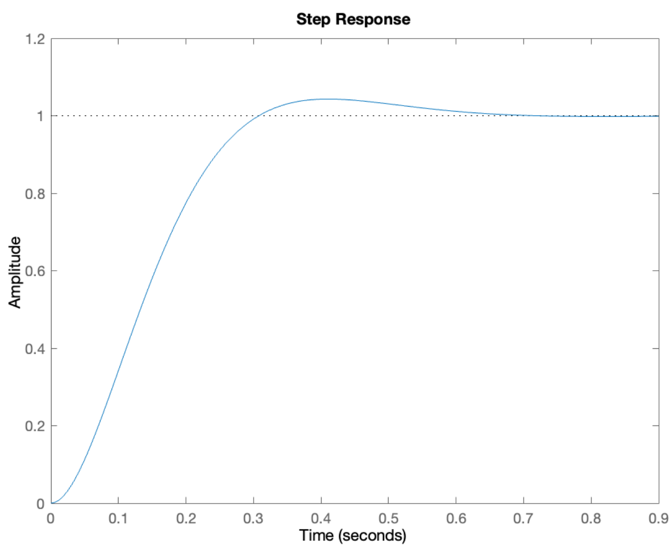
$\longrightarrow$ $\mathcal{A}$nswer

Using this to populate the transfer function, we get:

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\mathbf{25 \cdot 4.69}}{\boldsymbol{s^2 + (5 + 500 \cdot 0.0207)s + 25 \cdot 4.69}}$$

⟶ *Answer*

Plotting the step response using the `step()` function in MATLAB, we get the following response:



The max y value of the response is $y = 1.04299$, or an overshoot of 4.299%. The rise time is 0.1979 sec.

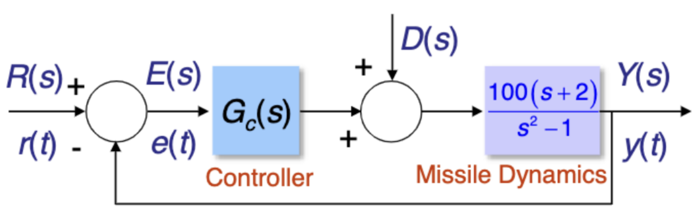
The following MATLAB script provides these answers:

```
%% Problem 5-18
% maximum overshoot
y_tp = 1.043;
% Overshoot
os =  y_tp - 1;
% Compute Zeta
zeta = sqrt(log(os)^2/(pi^2 + log(os)^2));
% Peak time
```

```
tr = 0.2;
% Compute omega_n
omega_n = (1 - 0.4167 *zeta + 2.917 *zeta^2)/tr;
% Solve for K
K = omega_n^2 / 25;
% Solve for K_t
K_t = (zeta * sqrt(K) - 0.5)/50;
% Create transfer function
TF = tf(25*K,[1,(5+500*K_t),25*K]);
% Generate step response plot
[y,tOut] = step(TF);
% Get step info
step_info = stepinfo(y,tOut);
```

Problem 1

Consider the following block diagram of a missile attitude control system:

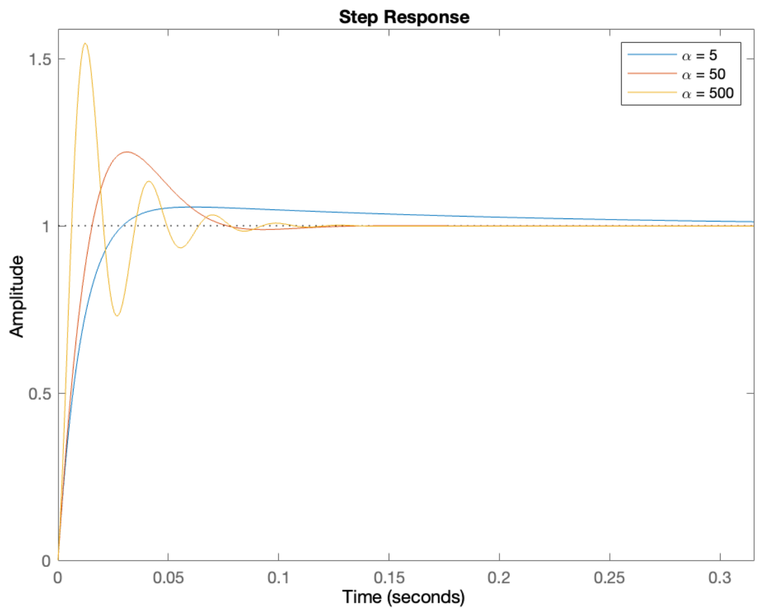


For, $G_c(s) = (s+\alpha)/s$, use MATLAB to plot the step responses of the system for $\alpha = 5, 50$ and 500, and comment on the step response of the three cases (assume zero I.C.'s)

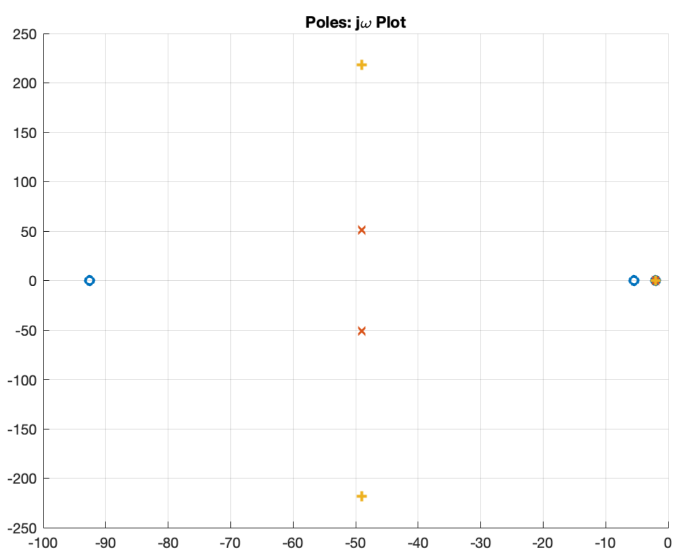
The transfer function for this system is:

$$\frac{Y(s)}{R(s)} = \frac{100s^2 + (200 + 100\alpha)s + 200\alpha}{s^3 + 100s^2 + (199 + 100\alpha)s + 200\alpha}$$

Plotting these in MATLAB, we get the following responses:



The poles of each response are included in the following figure:



When $\alpha = 5$, the system has only real poles (no complex poles). The system is slightly under-damped, with low overshoot. When $\alpha = 50$, the system has one real pole near zero, and a complex conjugate pair to the left of the real pole. The dominant pole for $\alpha = 50$ is slightly to the left of $\alpha = 5$. As this pole moved to the left, rise time and overshoot increase. This behavior relative to the dominant pole increases more for the case $\alpha = 500$, as it moves further left. For $\alpha = 500$ the complex conjugate pair has larger imaginary values (a larger ω_n), driving a much more transient response than $\alpha = 50$.

$\longrightarrow$ $\mathcal{A}$nswer

Submitted by Austin Barrilleaux on October 8, 2023.