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MODULE 4 — Practice Assignment

Problem 1

Solve the following practice problems in the 9th edition textbook.

- **Chapter 3:**
  - **10-13 (a) (use MATLAB if / as needed)**

Given the dynamic equation:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Find the transformation $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{P}\bar{\boldsymbol{x}}(\boldsymbol{t})$ that transforms the state equations into the controllability canonical form (CCF).

The characteristic equation of this system is given by:

$$|sI - A| = \begin{vmatrix} s & 2 & 0 \\ 1 & s-2 & 0 \\ -1 & 0 & s-1 \end{vmatrix} = (s(s-2)(s-1)+0+0) - (0+0+2(s-1)) = s^3 - 3s^2 + 2$$

The M matrix is computed as, where $a_1 = 0$ and $a_2 = -3$:

$$M = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

And S is defined, defined as $S = [B \quad AB \quad A^2B]$, is:

$$AB = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 2 & 6 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix}$$

$\therefore$

$$S = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 6 \\ 1 & 1 & -1 \end{bmatrix}$$

Since the transform is calculated as $P = SM$:

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-4} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

$\longrightarrow$ *Answer*

Consider the square matrix, A, below, and:

a) **Determine the characteristic equation (CE)**

b) **Find the eigenvalues of A (roots of the CE)**

c) **Is this matrix A diagonalizable? Why or why not?**

d) **Compute the eigenvectors associated with the computed eigenvalues.**

e) **If possible, compute the Diagonal Canonical Form (DCF) of A.**

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$$

The characteristic equation is computed as:

$$|sI - A| = 0 \;\rightarrow\; |sI - A| = \begin{vmatrix} s-1 & -1 \\ 0 & s+1 \end{vmatrix} = (s-1)(s+1) = 0$$

Therefore the characteristic equation is:

$$\boldsymbol{s^2 - 1 = 0}$$

$\longrightarrow$ *A*nswer

The the eigenvalues can be computed using the following equation:

$$|A - \lambda I| = 0 \;\rightarrow\; |A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) = 0$$

This simplifies to:

$$(1-\lambda)(1+\lambda) = 0$$

From here, we can see that the eigenvalues are:

$$\boldsymbol{\lambda = 1, -1}$$

$\longrightarrow$ *A*nswer

We could have also determined this by looking at the characteristic equation.

The matrix A has distinct eigenvalues (all have multiplicity = 1), therefore the matrix is diagonalizable.

$\longrightarrow$ *A*nswer

Eigenvectors are determined by evaluating the following equation:

$$(\lambda_n I - A)p_n = 0$$

The eigenvector associated with $\lambda = 1$ is computed as:

$$\begin{bmatrix} 1-1 & 0-(-1) \\ 0 & 1-(-1) \end{bmatrix} \begin{bmatrix} p_{\lambda=1,1} \\ p_{\lambda=1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which simplifies to

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} p_{\lambda=1,1} \\ p_{\lambda=1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only vector that satisfies this is:

$$\boldsymbol{p_{\lambda=1}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

The eigenvector associated with $\lambda = -1$ is computed as:

$$\begin{bmatrix} -1-1 & 0-(-1) \\ 0 & -1-(-1) \end{bmatrix} \begin{bmatrix} p_{\lambda=-1,1} \\ p_{\lambda=-1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which simplifies to

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{\lambda=-1,1} \\ p_{\lambda=-1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only vector that satisfies this is:

$$\boldsymbol{p_{\lambda=-1}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

Unitized, this vector is:

$$\boldsymbol{p_{\lambda=-1}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{1+2^2}}} \approx \begin{bmatrix} \mathbf{0.4472} \\ \mathbf{0.8944} \end{bmatrix}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

Given these eigenvectors, we can show that matrix of eigenvectors, H is:

$$H = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

The diagonal of A, is Λ where:

$$\Lambda = H^{-1}AH$$

The inverse of H is:

$$\Lambda = \frac{1}{\det(H)} \begin{bmatrix} H_{(2,2)} & -H_{(1,2)} \\ -H_{(2,1)} & H_{(1,1)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

Therefore:

$$\Lambda = H^{-1}AH = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Which solves to:

$$\Lambda = H^{-1}AH = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Then:

$$\Lambda = H^{-1}AH = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Finally, the Diagonal Canonical Form of A is:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$$

$\longrightarrow$ *Answer*

This could have also been determined, by putting the eigenvalues into the form:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\lambda_1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda_2} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$$

$\longrightarrow$ *Answer*

Using MATLAB, find the transformation $\bar{x}(t) = H\bar{z}(t)$ so that the state equations are transformed to DCF if A has distinct eigenvalues, where system matrices A, B, and C are below

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

First in MATLAB, we can calculate the eigenvalues:

```
A = [ 0 2 0 ;
      1 2 0 ;
     -1 0 1 ];

eig(A)
```

This computes the eigenvalues as:

$$\lambda \approx 1.0000,\ 2.7321,\ -0.7321$$

The eigenvalues are distinct, so the matrix is diagonalizable.

In MATLAB, we can compute H using the `eig()` function:

```
A = [ 0 2 0 ;
      1 2 0 ;
     -1 0 1 ];

[V,~] = eig(A);

D = inv(V)*A*V;
```

Where:

$$\boldsymbol{H} = \boldsymbol{V} = \begin{bmatrix} \mathbf{0} & \mathbf{0.5591} & \mathbf{0.8255} \\ \mathbf{0} & \mathbf{0.7637} & \mathbf{-0.3022} \\ \mathbf{1.0000} & \mathbf{-0.3228} & \mathbf{0.4766} \end{bmatrix}$$

$\longrightarrow \mathcal{A}$nswer

And the Diagonal of A can be computed as D, where:

$$\Lambda = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.7321 & 0 \\ 0 & 0 & -0.7321 \end{bmatrix}$$

Alternatively, D (and subsequently Λ) can be directly computed as using `eig()` with:

```
A = [ 0 2 0 ;
      1 2 0 ;
     -1 0 1 ];

[~,D] = eig(A);
```

resulting in:

$$\Lambda = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.7321 & 0 \\ 0 & 0 & -0.7321 \end{bmatrix}$$

Submitted by Austin Barrilleaux on September 22, 2023.