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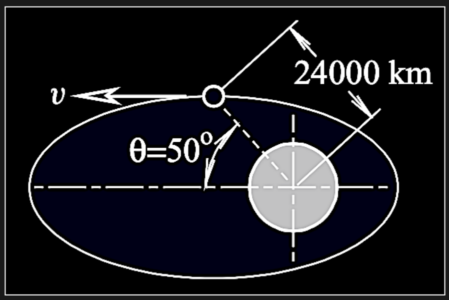
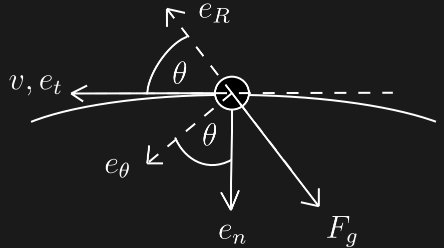
MODULE 8 — Midterm Exam

Problem 1

A satellite is in an orbit about the Earth. The magnitude of the acceleration of this body is $g(R_e/R)^2$, where R is the distance from the body to the center of the Earth, $R_e = 6370$ km is the radius of the Earth, and $g = 9.807$ m/s^2. At the position shown, the speed of the body is $v = 27\ 000$km/h.
(a) Determine the rate of change of the speed and the radius of curvature of the orbit at this position.
(b) Determine $\dot{R}$, $\ddot{R}$, $\dot{\theta}$, and $\ddot{\theta}$ at this position.



Sketching the following block diagram where, by inspection, the unit tangent vector e_t is parallel with the velocity vector, and the normal direction unit vector e_n extends toward the center of curvature (which is toward the center of the elliptical orbit.)



From this, we can define the unit tangent vector as:

$$\bar{e}_t = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \bar{e}_R \\ \sin(\theta) & \bar{e}_\theta \end{bmatrix}$$

Because $\dot{v}$ is the tangential component of acceleration, we find that:

$$\dot{v} = \bar{a} \cdot \bar{e}_t = g\left(\frac{R_e}{R}\right)^2 \begin{bmatrix} -1 & \bar{e}_R \\ 0 & \bar{e}_\theta \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) & \bar{e}_R \\ \sin(\theta) & \bar{e}_\theta \end{bmatrix} = -9.807\left(\frac{6370 \times 10^3}{24000 \times 10^3}\right)^2 \cos(\theta)$$

This evaluates to:

$$\boldsymbol{\dot{v} = -0.4441\ \mathrm{m/s^2}}$$

$\longrightarrow$ *Answer*

Looking at the equation for acceleration:

$$\bar{a} = \dot{v}\bar{e}_t + \frac{v^2}{\rho}\bar{e}_n$$

To solve for the radius of curvature:

$$\frac{v^2}{\rho}\bar{e}_n = \bar{a} - \dot{v}\bar{e}_t \ \rightarrow \ \rho = \frac{v^2}{|\bar{a} - \dot{v}\bar{e}_t|}$$

The radius of curvature is:

$$\rho = \frac{v^2}{|\bar{a} - \dot{v}\bar{e}_t|} = \frac{\left(27\ 000 \times 10^3/3600\right)^2}{\left|-9.807\left(\frac{6370\times10^3}{24000\times10^3}\right)^2 \begin{bmatrix} -1 & \bar{e}_R \\ 0 & \bar{e}_\theta \end{bmatrix} - (-0.4441)\begin{bmatrix} \cos(\theta) & \bar{e}_R \\ \sin(\theta) & \bar{e}_\theta \end{bmatrix}\right|}$$

This evaluates to:

$$\boldsymbol{\rho = 1.0629 \times 10^8\ \mathrm{m}}$$

$\longrightarrow$ *Answer*

Using the following velocity equation:

$$\bar{v} = \dot{R}\ \bar{e}_R + R\dot{\theta}\ \bar{e}_\theta$$

Looking at the sketch:

$$\bar{v} = \left(\frac{27\ 000 \times 10^3}{3600}\right)\begin{bmatrix} \cos(\theta)\ \ \bar{e}_R \\ \sin(\theta)\ \ \bar{e}_\theta \end{bmatrix}$$

From these two equations we get that:

$$\boldsymbol{\dot{R} = \left(\frac{27\ 000 \times 10^3}{3600}\right)\cos(\theta) = 4820.9\ \mathrm{m/s}}$$

$$\boldsymbol{\dot{\theta} = \left(\frac{27\ 000 \times 10^3}{3600}\right)\sin(\theta)\,/R = 0.0002393\ \mathrm{rad/s}}$$

$\longrightarrow$ $\mathcal{A}$nswer

Using the following acceleration equation:

$$\bar{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\bar{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\bar{e}_\theta$$

Where:

$$\bar{a} = -9.807\left(\frac{6370 \times 10^3}{24000 \times 10^3}\right)^2\begin{bmatrix} -1\ \bar{e}_R \\ 0\ \bar{e}_\theta \end{bmatrix}$$

We get that:

$$\boldsymbol{\ddot{R} = \bar{a} \cdot \bar{e}_R + R\dot{\theta}^2 = 0.6845\ \mathrm{m/s^2}}$$

$$\boldsymbol{\ddot{\theta} = \left(\cancelto{0}{\bar{a} \cdot \bar{e}_\theta} - 2\dot{R}\dot{\theta}\right)/R = -9.617 \times 10^8\ \mathrm{rad/s^2}}$$

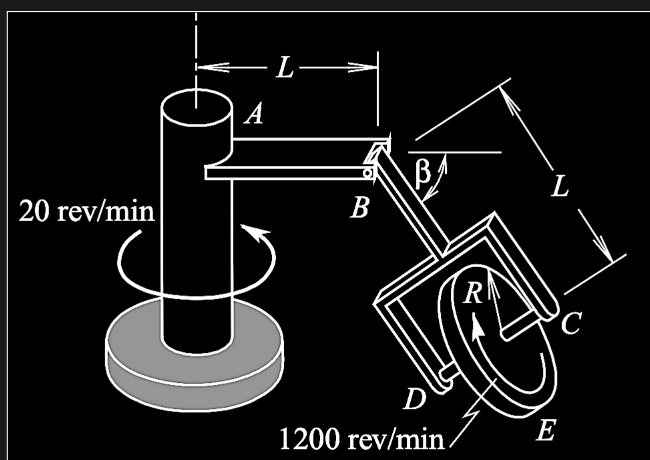
$\longrightarrow$ $\mathcal{A}$nswer

The following MATLAB script was used to solve this problem:

```
%% EXERCISE 2.43
clc,clear
syms t
syms theta(t)
assume(t,{'real','positive'})

e_t = zRot(theta)*[1;0;0];
e_n = zRot(theta)*[0;0;0];

R = 24000E3;
Re = 6370E3;
g = 9.807;
a_bar = -g*(Re/R)^2*[1;0;0];

v_dot = subs(transpose(a_bar)*e_t,theta(t),50*pi/180); % [ans]
v_bar = double((27000E3/3600)*subs(e_t,theta(t),50*pi/180));
rho = ((27000E3/3600)^2)/... [ans] (2.1.10)
subs(norm(a_bar - v_dot*e_t),theta(t),50*pi/180); % m

R_dot = v_bar(1); % m/s^2 (2.3.11)
theta_dot = v_bar(2)/R; % rad/s^2 (2.3.11)

R_ddot = a_bar(1)+ R*theta_dot^2; % m/s^2 (2.3.13)
theta_ddot = (a_bar(2) - 2*R_dot*theta_dot)/R; % rad/s^2 (2.3.13)

function R = zRot(ang)
    R = [ cos(ang) -sin(ang) 0;
          sin(ang) cos(ang) 0;
                 0        0 1];
end
```

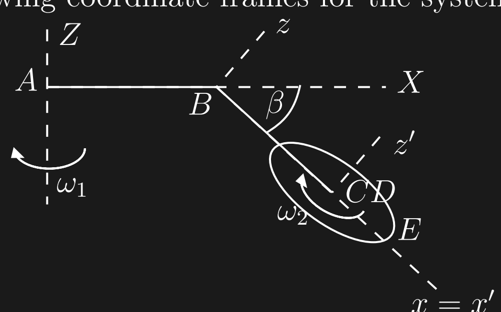
Problem 2

The disk spins about its axis C D at 1200 rev/min as the system rotates about the vertical axis at 20 rev/min. Both rates are constant. The angle of elevation of the arm supporting the disk is such that $\dot{\beta} = 10$ rad/s and $\ddot{\beta} = -500$ rad/s^2 when $\beta = 36.87°$. Determine the velocity and acceleration of point E, which is the lowest point on the perimeter of the disk. Note: the solution for velocity in the text is for a clockwise rotation about the vertical axis



We will solve this problem consistent with the note in the prompt.

We will define the following coordinate frames for the system:



You will note that the {xyz} frame and {x'y'z'} frame are identical for this problem, so I will use {xyz} to express the coordinate frame at both points from this point. Constructing the angular velocity vector $\bar{\omega}$ of $\{xyz\}$ by vectorially adding the simple rotation rates according to:

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3$$

This gives us:

$$\bar{\omega} = -\omega_1 K + \dot{\beta} j - \omega_2 k$$

Using the following coordinate transformation:

$$R_{\text{rot}} = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$K = R_{\text{rot}} \begin{bmatrix} 0 & I \\ 0 & J \\ 1 & K \end{bmatrix} = \begin{bmatrix} -\sin(\beta) & i \\ 0 & j \\ \cos(\beta) & k \end{bmatrix} = -\sin(\beta)\, i + \cos(\beta)\, k$$

This gives us the angular velocity vector in the form:

$$\bar{\omega} = \begin{bmatrix} \omega_1 \sin(\beta) & i \\ \dot{\beta} & j \\ -\omega_2 - \omega_1 \cos(\beta) & k \end{bmatrix}$$

For the angular rotations, the angular velocity is:

$$\Omega_1 = -\omega_1 K = \begin{bmatrix} \omega_1 \sin(\beta) & i \\ 0 & j \\ -\omega_1 \cos(\beta) & k \end{bmatrix}$$

$$\Omega_2 = -\omega_1 K + \dot{\beta} j = \begin{bmatrix} \omega_1 \sin(\beta) & i \\ \dot{\beta} & j \\ -\omega_1 \cos(\beta) & k \end{bmatrix}$$

$$\Omega_3 = \bar{\omega}$$

Using the following relative positions:

$$\bar{r}_{B/A} = L \begin{pmatrix} \cos(\beta) \ i \\ 0 \ j \\ \sin(\beta) \ k \end{pmatrix}$$
$$\bar{r}_{CD/B} = L \ i$$
$$\bar{r}_{E/CD} = R \ i$$

Referencing the velocity equation:

$$\bar{v}_P = \bar{v}_O + \bar{\omega} \times \bar{r}_{P/O}$$

We solve for the velocities along the system from A to E:

$$\bar{v}_B = \Omega_1 \times \bar{r}_{B/A} = \begin{bmatrix} 0 \\ -2.0944\, L \\ 0 \end{bmatrix}$$
$$\bar{v}_{CD} = \bar{v}_B + \Omega_2 \times \bar{r}_{CD/B} = \begin{bmatrix} 0 \\ -3.7699\, L \\ -10\, L \end{bmatrix}$$
$$\bar{v}_E = \bar{v}_{CD} + \Omega_3 \times \bar{r}_{E/CD} = \begin{bmatrix} 0 \\ -3.7699\, L - 127.34\, R \\ -10\, L - 10\, R \end{bmatrix}$$

Therefore:

$$\boldsymbol{\bar{v}_E = -\left(3.7699\, L + 127.34\, R\right) j - 10\left(L + R\right) k}$$

$\longrightarrow$ $\mathcal{A}$nswer

We can solve for the angular rotation at the rotation points using the equation for angular acceleration:

$$\bar{\alpha} = \sum_n \left(\dot{\omega}_n \bar{e}_n + \bar{\Omega}_n \times \omega_n \bar{e}_n\right)$$

The relative angular accelerations across the system for points A to E are:

$$\bar{\alpha}_1 = \Omega_1 \times \Omega_1 = 0$$
$$\bar{\alpha}_2 = \bar{\alpha}_1 + \ddot{\beta} j + \Omega_2 \times \dot{\beta} j$$
$$\bar{\alpha}_3 = \bar{\alpha}_2 + \Omega_3 \times \omega_2(-k) = \bar{\alpha}$$

Using the acceleration equation:

$$\bar{a}_P = \bar{a}_O + \bar{\alpha} \times \bar{r}_{P/O} + \bar{\omega} \times \left(\bar{\omega} \times \bar{r}_{P/O}\right)$$

Solving for the acceleration across the system for points A to E:

$$\bar{a}_B = 0 + \bar{\alpha}_1 \times \bar{r}_{B/A} + \Omega_1 \times \left(\Omega_1 \times \bar{r}_{B/A}\right) = \begin{bmatrix} -3.5092\,L \\ 0 \\ -2.6319\,L \end{bmatrix}$$

$$\bar{a}_{CD} = \bar{a}_B + \bar{\alpha}_2 \times \bar{r}_{CD/B} + \Omega_2 \times \left(\Omega_2 \times \bar{r}_{CD/B}\right) = \begin{bmatrix} -106.32\,L \\ 25.133\,L \\ 495.26\,L \end{bmatrix}$$

$$\bar{a}_E = \bar{a}_{CD} + \bar{\alpha}_3 \times \bar{r}_{E/CD} + \Omega_3 \times \left(\Omega_3 \times \bar{r}_{E/CD}\right) = \begin{bmatrix} -106.32\,L - 16315\,R \\ 25.133\,L + 25.133\,R \\ 495.26\,L + 182.07\,R \end{bmatrix}$$

Therefore:

$$\boldsymbol{a_E = -(106.32\,L + 16315\,R)\,i + 25.133(L + R)\,j + (495.26\,L + 182.07\,R)\,k}$$

$\longrightarrow$ $\mathcal{A}$nswer

These solutions match the text.

The following MATLAB script was used to solve this problem:

```
clear
syms L R

x = [1;0;0];
y = [0;1;0];
```

```
z = [0;0;1];

omega_1 = 20*2*pi/60; % rad/s
omega_2 = 1200*2*pi/60; % rad/s

beta = 36.87*pi/180; % rad
beta_dot = 10; % rad/s
beta_ddot = -500; % rad/s^2

R_rot = yRot(beta) ;
Z = R_rot*z;

Omega_1 = -omega_1*Z;
Omega_2 = -omega_1*Z+beta_dot*y;
Omega_3 = -omega_1*Z+beta_dot*y-omega_2*z; % omega_bar

r_B_A = L*[cos(beta);0;sin(beta)];
r_CD_B = L*x;
r_E_CD = R*x;

v_B = 0 + cross(Omega_1,r_B_A);
v_CD = v_B + cross(Omega_2,r_CD_B);
v_E = v_CD + cross(Omega_3,r_E_CD);

alpha_1 = cross(Omega_1,Omega_1);
alpha_2 = alpha_1 + beta_ddot*y + cross(Omega_2,beta_dot*y);
alpha_3 = alpha_2 + cross(Omega_3,-omega_2*z); % alpha_bar

a_B = 0 + cross(alpha_1,r_B_A) + ...
    cross(Omega_1,cross(Omega_1,r_B_A));
a_CD = a_B + cross(alpha_2,r_CD_B) + ...
    cross(Omega_2,cross(Omega_2,r_CD_B));
a_E = a_CD + cross(alpha_3,r_E_CD) + ...
    cross(Omega_3,cross(Omega_3,r_E_CD));

function R = yRot(ang)
R = [ cos(ang) 0 sin(ang);
0 1 0;
-sin(ang) 0 cos(ang)]';
end
```

Submitted by Austin Barrilleaux on October 19, 2024.