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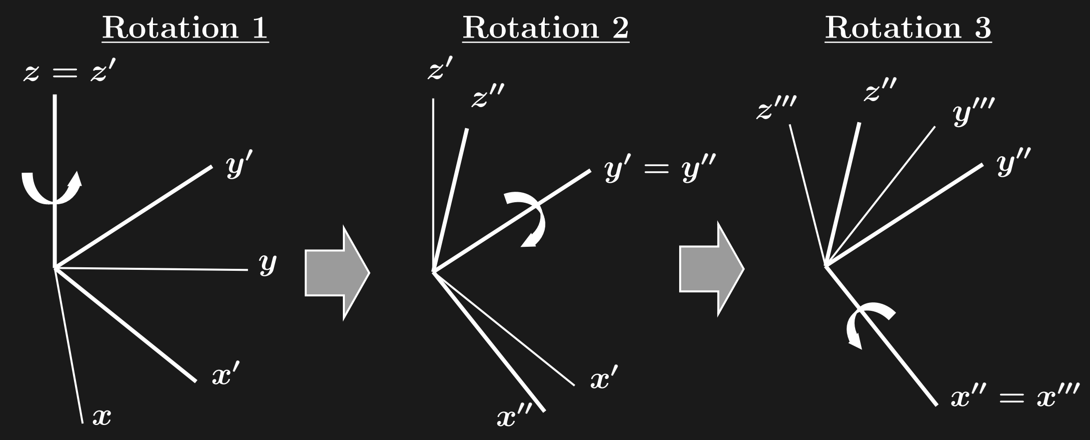
MODULE 2 — Assignment

Problem 1:

Type I Euler angles are also known as the aircraft Euler angles. These are comprised of (1) a rotation ψ about the fixed Z-axis, resulting in a primed axis system; (2) a rotation θ about the y'-axis resulting in a double-primed system; and (3) a rotation ϕ about the x"-axis, resulting in the final xyz body-fixed frame.

- **a Neatly sketch this sequence of rotations.**
- **b Derive the rotation matrix that maps the XYZ frame to the body-fixed xyz frame.**

The following sketch shows the rotations:



$\longrightarrow$ *Answer*

This rotation matrix is defined by the following three sequential rotations:

$$R_1 = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix}$$

The overall rotation matrix R that maps the XYZ frame to the body-fixed xyz frame is:

$$R = R_3(\phi)\ R_2(\theta)\ R_1(\psi)$$

Which becomes:

$$\boldsymbol{R(\phi, \theta, \psi)} = \begin{bmatrix} \cos(\psi)\cos(\theta) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & \cos(\theta)\sin(\phi) \\ \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) & \cos(\phi)\cos(\theta) \end{bmatrix}$$

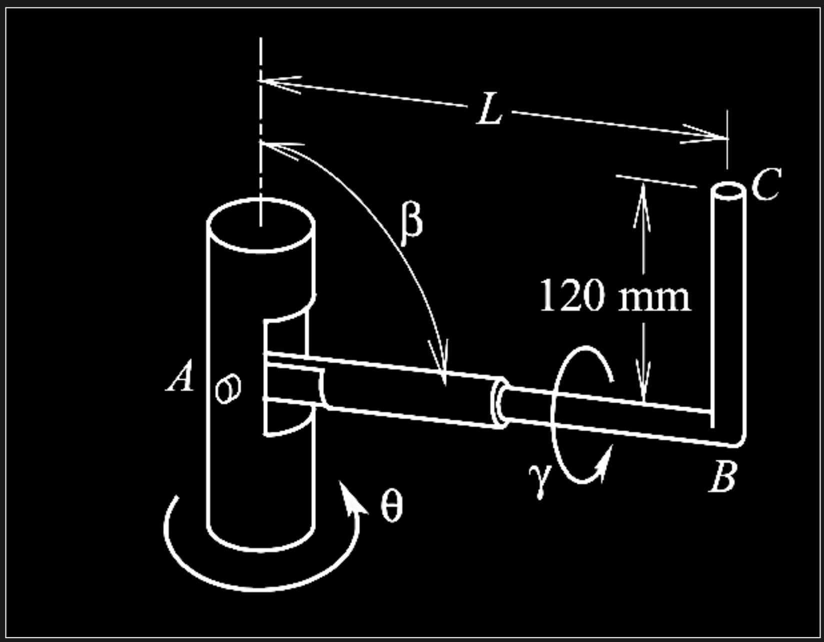
$\longrightarrow \mathcal{A}$nswer

This makes sense, as if I take the transpose of this matrix, I see the matrix in the space-fixed form that I am used to seeing the transform in the aerospace industry:

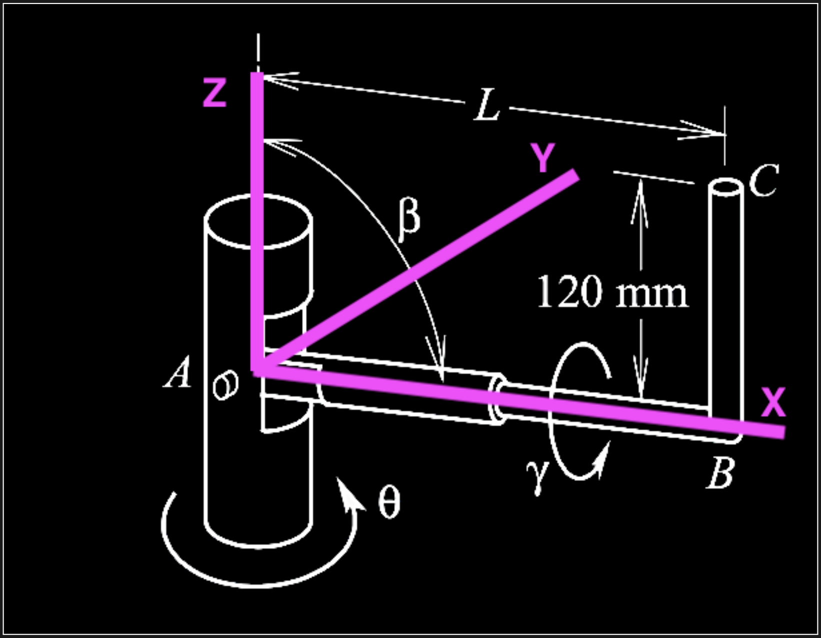
$$R(\phi, \theta, \psi)' = \begin{bmatrix} \cos(\psi)\ \cos(\theta) & \cos(\psi)\ \sin(\phi)\ \sin(\theta) - \cos(\phi)\ \sin(\psi) & \sin(\phi)\ \sin(\psi) + \cos(\phi)\ \cos(\psi)\ \sin(\theta) \\ \cos(\theta)\ \sin(\psi) & \cos(\phi)\ \cos(\psi) + \sin(\phi)\ \sin(\psi)\ \sin(\theta) & \cos(\phi)\ \sin(\psi)\ \sin(\theta) - \cos(\psi)\ \sin(\phi) \\ -\sin(\theta) & \cos(\theta)\ \sin(\phi) & \cos(\phi)\ \cos(\theta) \end{bmatrix}$$

Problem 2: EXERCISE 3.18

A hydraulic cylinder allows the length of arm AB to vary, and servomotors control the rotation angles θ about the vertical, β about pin A, and γ about axis AB, with $\gamma = 0$ corresponding to bar BC being situated in the vertical plane as shown. In the initial position $L = 250$ mm, $\theta = 0$, $\beta = 90°$,and $\gamma = 0$. In the final position, $\theta = \beta = 120°$, $\gamma = 90°$, and $L = 500$ mm. Determine the corresponding displacement of end C.



We will use the coordinate frame as defined in the following figure:



Where

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R_A \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The rotation about these axes is

$$R_A = \begin{bmatrix} \cos(\beta - 90) & 0 & -\sin(\beta - 90) \\ 0 & 1 & 0 \\ \sin(\beta - 90) & 0 & \cos(\beta - 90) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This gives

$$R_{A_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{A_f} = \begin{bmatrix} -\frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

The displacement of point A to C is defined as

$$\Delta \bar{r}_C = \bar{r}_{C/A_f} - \bar{r}_{C/A_i}$$

Where

$$\bar{r}_{C/A_i} = \begin{bmatrix} L_i \\ -120\sin(\gamma_i) \\ 120\cos(\gamma_i) \end{bmatrix} = \begin{bmatrix} 250 \\ 0 \\ 120 \end{bmatrix} \text{ mm}$$

$$\bar{r}_{C/A_f} = \begin{bmatrix} L_f \\ -120\sin(\gamma_f) \\ 120\cos(\gamma_f) \end{bmatrix} = \begin{bmatrix} 500 \\ 120 \\ 0 \end{bmatrix} \text{ mm}$$

Therefore

$$\Delta \bar{r}_C = \bar{r}_{C/A_f} - \bar{r}_{C/A_i} = \begin{bmatrix} 250 \\ 120 \\ -120 \end{bmatrix} \text{ mm}$$

We can compute the displacement of point C in terms of the original axes we defined as

$$\Delta \bar{r}'_C = {R_{A_f}}^T \Delta \bar{r}_C + \left({R_{A_f}}^T - {R_{A_i}}^T \right) \bar{r}_{C/A_i} = \begin{bmatrix} -570.43 \\ 315.00 \\ -370.00 \end{bmatrix} \text{ mm}$$

So the displacement is:

$$\boldsymbol{\Delta \bar{r}'_C} = \begin{bmatrix} \mathbf{-570.43} \\ \mathbf{315.00} \\ \mathbf{-370.00} \end{bmatrix} \textbf{ mm}, \quad \left| \boldsymbol{\Delta \bar{r}'_C} \right| = \mathbf{749.34 \text{ mm}}$$

$\longrightarrow$ *Answer*

Submitted by Austin Barrilleaux on September 10, 2024.