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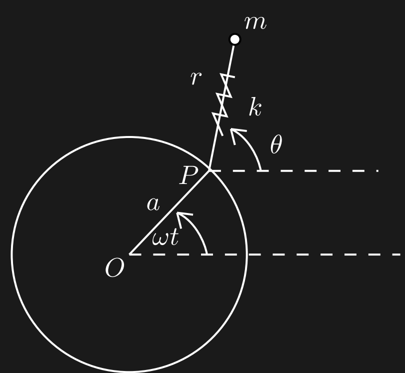
MODULE 9 — Assignment

Problem 1

Derive the equations of motion for a rotating spring-pendulum shown below. The spring- pendulum is attached at point P.



For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2$$

Therefore:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

By inspection of the sketch:

$$x = a\cos(\omega t) + r\cos(\theta)$$
$$y = a\sin(\omega t) + r\sin(\theta)$$

Taking the time derivative of both:

$$\dot{x} = \dot{r}\cos(\theta) - r\,\dot{\theta}\sin(\theta) - a\,\omega\,\sin(\omega\,t)$$
$$\dot{y} = \dot{r}\sin(\theta) + r\,\dot{\theta}\cos(\theta) + a\,\omega\,\cos(\omega\,t)$$

This gives:

$$T = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + a^2\,\omega^2 + 2\,a\,\omega\,\sin(\theta - \omega\,t)\,\dot{r} + 2\,a\,\omega\,r\,\cos(\theta - \omega\,t)\,\dot{\theta}\right)$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2}k\,(r - r_0)^2$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + a^2\,\omega^2 + 2\,a\,\omega\,\sin(\theta - \omega\,t)\,\dot{r} + 2\,a\,\omega\,r\,\cos(\theta - \omega\,t)\,\dot{\theta}\right) - \frac{1}{2}k\,(r - r_0)^2$$

We will solve for the equations of motion along the two generalized coordinates, θ and r. Starting with r, we will solve:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial r} = m\,\left(r\,\dot{\theta}^2 + a\,\omega\,\cos(\theta - \omega\,t)\,\dot{\theta}\right) - \,k\,(\,r - \,r_0)$$

$$\frac{\partial L}{\partial \dot{r}} = m\,\left(\dot{r} + a\,\omega\,\sin(\theta - \omega\,t)\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\,\left(\ddot{r} - a\,\omega\,\cos(\theta - \omega\,t)\,\left(\omega - \dot{\theta}\right)\right)$$

This results in the equation of motion:

$$\ddot{r} - r\,\dot{\theta}^2 - a\,\omega^2\,\cos(\theta - \omega\,t) + \frac{k}{m}\,(r - r_0) = 0$$

Solving along θ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = m\,\left(a\,\omega\,\cos\left(\omega\,t-\theta\right)\,\dot{r}+a\,\omega\,\sin\left(\omega\,t-\theta\right)\,r\,\dot{\theta}\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} =, m\,\left(r^2\,\dot{\theta}+a\,\omega\,\cos\left(\theta-\omega\,t\right)\,r\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\,\left(r^2\,\ddot{\theta}+2\,r\,\dot{\theta}\,\dot{r}+a\,\omega\,\cos\left(\theta-\omega\,t\right)\,\dot{r}+a\,\omega\,r\,\sin\left(\theta-\omega\,t\right)\,\left(\omega-\dot{\theta}\right)\right)$$

This results in the equation of motion:

$$r\,\ddot{\theta}+2\,\dot{\theta}\,\dot{r}+a\,\omega^2\,\sin\left(\theta-\omega\,t\right)=0$$

The equations of motion for the system are:

$$\boldsymbol{\ddot{r}-r\,\dot{\theta}^2-a\,\omega^2\,\cos\left(\theta-\omega\,t\right)+\frac{k}{m}\left(r-r_0\right)=0}$$
$$\boldsymbol{r\,\ddot{\theta}+2\,\dot{\theta}\,\dot{r}+a\,\omega^2\,\sin\left(\theta-\omega\,t\right)=0}$$

$\longrightarrow$ *Answer*

The following MATLAB script was used to solve this problem:

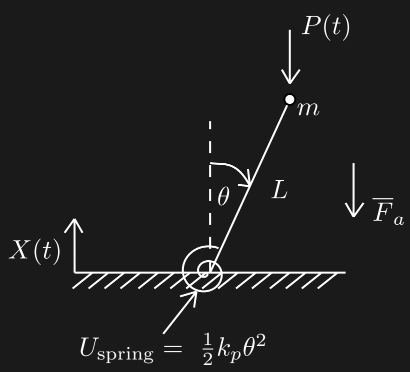
```
clc,clear
syms t m a k omega r_0
syms theta(t)
syms r(t)

% Position
x = a*cos(omega*t)+r*cos(theta);
y = a*sin(omega*t)+r*sin(theta);
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k*(r-r_0)^2;
% Lagrange Function
```

```
L = T - V;
% Coordinate derivatives
r_dot     = diff(r,t);
theta_dot = diff(theta,t);
% r coordinate derivatives
dL_dr     = diff(L,r);
dL_dr_dot = diff(L,r_dot);
dL_dr_dot_dt = diff(dL_dr_dot,t);
% theta coordinate derivatives
dL_dtheta     = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = diff(dL_dtheta_dot,t);
% Resulting equations of motion
EOM1 = simplify(dL_dr_dot_dt-dL_dr);
EOM2 = simplify(dL_dtheta_dot_dt-dL_dtheta);
```

Problem 2

Derive the equations of motion for the base-excited structure subject to an axial load $P(t)$.



For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2$$

Therefore:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

By inspection of the sketch:

$$x = L\sin(\theta)$$
$$y = L\cos(\theta)$$

Taking the time derivative of both:

$$\dot{x} = L\,\cos\left(\theta\right)\,\dot{\theta}$$
$$\dot{y} = -L\,\sin\left(\theta\right)\,\dot{\theta}$$

This gives:

$$T = \frac{1}{2} m\,L^2\,\dot{\theta}^2$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2} k_p \theta^2 + may$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2} m\,L^2\,\dot{\theta}^2 - \left(\frac{1}{2} k_p \theta^2 + may\right)$$

We will solve for the equation of motion along the generalized coordinate, θ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = L\,m\,a\,\sin\left(\theta\right) - k_p\,\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = L^2\,m\,\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = L^2\,m\,\ddot{\theta}$$

$$
\begin{aligned}
Q_\theta &= \sum_{k=1}^{N=3} F_k \frac{\partial x_k}{\partial \theta} \\
&= \begin{bmatrix} 0 \\ -P\left(t\right) \end{bmatrix} \cdot \begin{bmatrix} L\, \cos\left(\theta\right) \\ -L\, \sin\left(\theta\right) \end{bmatrix} \\
&= L \sin(\theta(t)) P(t)
\end{aligned}
$$

This results in the system equation of motion:

$$
m\, L^2\, \ddot{\theta} - m\, a\, \sin\left(\theta\right)\, L + k_p\, \theta = L\;\, P(t)\;\, \sin(\theta)
$$

Or:

$$
m\, L\, \ddot{\theta} - m\, a\, \sin\left(\theta\right) + k_p\, \theta = P(t)\;\, \sin(\theta)
$$

The following MATLAB script was used to solve this problem:

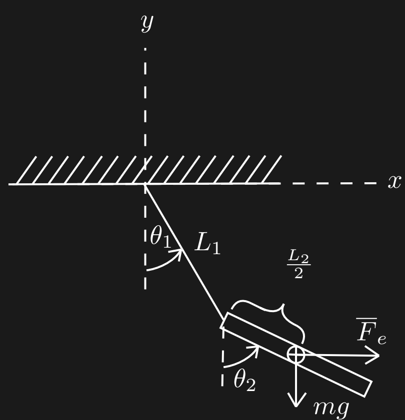
```
clc,clear
syms t m a k omega r_0
syms theta(t)
syms r(t)

% Position
x = a*cos(omega*t)+r*cos(theta);
y = a*sin(omega*t)+r*sin(theta);
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k*(r-r_0)^2;
% Lagrange Function
L = T - V;
% Coordinate derivatives
r_dot     = diff(r,t);
theta_dot = diff(theta,t);
% r coordinate derivatives
dL_dr     = diff(L,r);
dL_dr_dot = diff(L,r_dot);
dL_dr_dot_dt = diff(dL_dr_dot,t);
```

```
% theta coordinate derivatives
dL_dtheta     = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = diff(dL_dtheta_dot,t);
% Resulting equations of motion
EOM1 = simplify(dL_dr_dot_dt-dL_dr);
EOM2 = simplify(dL_dtheta_dot_dt-dL_dtheta);
```

Problem 3

The double pendulum. Find the equations governing a rigid bar attached to a string as shown in the figure.



a. This system starts with 9 degrees of freedom. What constraints exist that reduce this system to only two degrees of freedom?

b. Show that the Lagrangian is:

$$\boldsymbol{L = \frac{1}{2}m\left[L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{1}{3}L_2^2\dot{\theta}_2^2\right] - mg\left[L_1(1-\cos\theta_1) + \frac{L_2}{2}(1-\cos\theta_2)\right]}$$

For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}_2^2$$

Therefore:

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} \left(\frac{1}{12} \right) L_2^2 \dot{\theta}_2^2$$

By inspection of the sketch:

$$x = L_1 \sin(\theta_1) + \frac{L_2}{2} \sin(\theta_2)$$
$$y = \left(L_1 + \frac{L_2}{2} \right) - L_1 \cos(\theta_1) - \frac{L_2}{2} \cos(\theta_2)$$

Note that these equations define $(x, y)_0$ at the location of the c.g. of the bar when $\theta_1 = \theta_2 = 0$.

Taking the time derivative of both:

$$\dot{x} = L_1 \, \cos(\theta_1) \, \dot{\theta}_1 + \frac{1}{2} \, L_2 \, \cos(\theta_2) \, \dot{\theta}_2$$
$$\dot{y} = L_1 \, \sin(\theta_1) \, \dot{\theta}_1 + \frac{1}{2} \, L_2 \, \sin(\theta_2) \, \dot{\theta}_2$$

This gives:

$$T = \frac{1}{2} \, m \, \left[{L_1}^2 \, \dot{\theta}_1^2 + L_1 \, L_2 \, \dot{\theta}_2 \, \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{1}{3} \, {L_2}^2 \, \dot{\theta}_2^2 \right]$$

The potential energy of the system, V is defined by:

$$V = mgh$$

Therefore:

$$V = mg \left(L_1 + \frac{1}{2} \, L_2 - L_1 \, \cos(\theta_1) - \frac{1}{2} \, L_2 \, \cos(\theta_2) \right)$$
$$= mg \left[L_1 \, \left(1 - \cos(\theta_1) \right) + \frac{1}{2} L_2 \left(1 - \, \cos(\theta_2) \right) \right]$$

Given that the Lagrange is $L = T - V$:

$$\boldsymbol{L = \frac{1}{2}\, m\, \left[L_1{}^2\, \dot{\theta}_1^2 + L_1\, L_2\, \dot{\theta}_2\, \dot{\theta}_1 \cos\left(\theta_2 - \theta_1\right) + \frac{1}{3}\, L_2{}^2\, \dot{\theta}_2^2 \right] - mg \left[L_1\, \left(1 - \cos\left(\theta_1\right)\right) + \frac{1}{2} L_2 \left(1 - \cos\left(\theta_2\right)\right) \right]}$$

$\longrightarrow \mathcal{A}\mathsf{nswer}$

This matches the Lagrangian in the question prompt.

c. Use Lagrange's equations to derive the equations of motion (don't forget about the external force!)

Looking at the external forces:

Submitted by Austin Barrilleaux on October 27, 2024.