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MODULE 13 — Assignment

Problem 1

Compute the angular velocity for the rotation parameterized with the $\boldsymbol{ZXZ}$ Euler angles. Compute both the body and the spatial angular velocity. Note that the rotation matrix is

$$\boldsymbol{R_{ZXZ}(\psi,\theta,\phi) = R_3(\psi)R_1(\theta)R_3(\phi)}$$

$$\begin{aligned} R_{ZXZ} &= R_Z(\psi)R_X(\theta)R_Z(\phi) \\ &= \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\phi)\cos(\psi) - \cos(\theta)\sin(\phi)\sin(\psi) & -\cos(\psi)\sin(\phi) - \cos(\phi)\cos(\theta)\sin(\psi) & \sin(\psi)\sin(\theta) \\ \cos(\phi)\sin(\psi) + \cos(\psi)\cos(\theta)\sin(\phi) & \cos(\phi)\cos(\psi)\cos(\theta) - \sin(\phi)\sin(\psi) & -\cos(\psi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \cos(\phi)\sin(\theta) & \cos(\theta) \end{bmatrix} \end{aligned}$$

We can compute the spatial angular velocity by:

$$\boldsymbol{\omega_s} = \textbf{vect}\left(\dot{\boldsymbol{R}}\boldsymbol{R^T}\right) = \left\langle \begin{matrix} \cos(\psi)\ \dot{\theta} + \sin(\psi)\ \sin(\theta)\ \dot{\phi} \\ \sin(\psi)\ \dot{\theta} - \cos(\psi)\ \sin(\theta)\ \dot{\phi} \\ \cos(\theta)\ \dot{\phi} + \dot{\psi} \end{matrix} \right\rangle$$

$\longrightarrow$ *Answer*

We can compute the body angular velocity by:

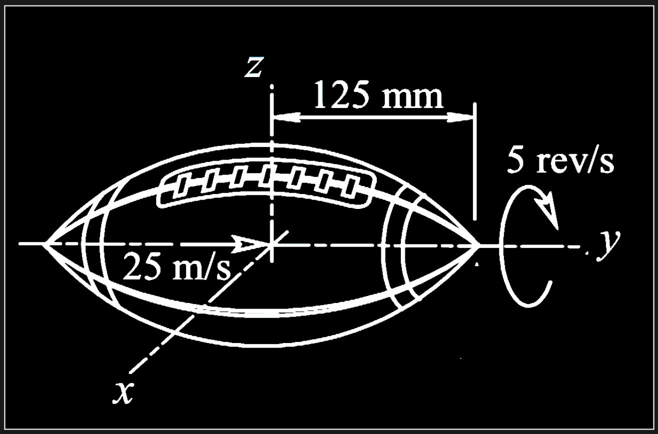
$$\boldsymbol{\omega_b} = \textbf{vect}\left(\boldsymbol{R^T}\dot{\boldsymbol{R}}\right) = \left\langle \begin{matrix} \cos(\phi)\ \dot{\theta} + \sin(\phi)\ \sin(\theta)\ \dot{\psi} \\ \cos(\phi)\ \sin(\theta)\ \dot{\psi} - \sin(\phi)\ \dot{\theta} \\ \cos(\theta)\ \dot{\psi} + \dot{\phi} \end{matrix} \right\rangle$$

$\longrightarrow$ *Answer*

The following MATLAB script was used to solve this problem:

```
syms t psi(t) theta(t) phi(t)

R = simplify(zRot(psi)*xRot(theta)*zRot(phi)) %[output:5d5dec6e]
R_dot = simplify(diff(R),1000) %[output:7e3fbfdb]
Omega_b = simplify(transpose(R)*R_dot,1000);
Omega_b = Omega_b(t) %[output:08922e58]
omega_b = vect(Omega_b) %[output:02d3fe7a]

Omega_s = simplify(R_dot*transpose(R),1000);
Omega_s = Omega_s(t) %[output:05052463]
omega_s = vect(Omega_s) %[output:4731dd26]

function R = xRot(ang)
R = [ 1 0 0;
0 cos(ang) -sin(ang);
0 sin(ang) cos(ang)];
end

function R = yRot(ang)
R = [ cos(ang) 0 sin(ang);
0 1 0;
-sin(ang) 0 cos(ang)];
end

function R = zRot(ang)
R = [ cos(ang) -sin(ang) 0;
sin(ang) cos(ang) 0;
0 0 1];
end

function omega = vect(Omega)
omega = [Omega(3,2);Omega(1,3);Omega(2,1)];
```

Problem 2



A football of mass m is flying at a velocity of 25 m/s along the y-axis of the body fixed frame. The radii of gyration about the axes of the body-fixed frame are 40 mm and 70 mm along the x-/z-and the y-axis, respectively.

The inertia properties of the football, given the radii of gyration given in the prompt are are:

$$
\begin{aligned}
I_b &= \begin{bmatrix} m(0.04)^2 & 0 & 0 \\ 0 & m(0.07)^2 & 0 \\ 0 & 0 & m(0.04)^2 \end{bmatrix} \\
&= m \begin{bmatrix} 0.0016 & 0 & 0 \\ 0 & 0.0049 & 0 \\ 0 & 0 & 0.0016 \end{bmatrix}
\end{aligned}
$$

(a) Compute the kinetic energy of the ball. Use the ZYZ Euler angles to represent the orientation of the ball.

The translational kinetic energy is computed as:

$$T_{\text{trans}} = \frac{1}{2} m\,(v \cdot v)$$

Where $v = [0, 25, 0]^T$ m/s, the translational kinetic energy is:

$$T_{\text{trans}} = 312.5\,m \quad \left[\text{kg}\left(\frac{\text{m}}{\text{s}}\right)^2\right] = 312.5\,m \quad [\text{J}]$$

Using the ZYZ Euler angles to represent the orientation of the ball:

$$\begin{aligned}
R_{ZYZ} &= R_Z(\psi)R_Y(\theta)R_Z(\phi) \\
&= \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\phi)\,\cos(\psi)\,\cos(\theta) - \sin(\phi)\,\sin(\psi) & -\cos(\phi)\,\sin(\psi) - \cos(\psi)\,\cos(\theta)\,\sin(\phi) & \cos(\psi)\,\sin(\theta) \\ \cos(\psi)\,\sin(\phi) + \cos(\phi)\,\cos(\theta)\,\sin(\psi) & \cos(\phi)\,\cos(\psi) - \cos(\theta)\,\sin(\phi)\,\sin(\psi) & \sin(\psi)\,\sin(\theta) \\ -\cos(\phi)\,\sin(\theta) & \sin(\phi)\,\sin(\theta) & \cos(\theta) \end{bmatrix}
\end{aligned}$$

Where computing the body angular velocity:

$$\omega_b = \mathbf{vect}\left(R^T\dot{R}\right) = \left\langle \begin{array}{c} \sin(\phi)\,\dot{\theta} - \cos(\phi)\,\sin(\theta)\,\dot{\psi} \\ \cos(\phi)\,\dot{\theta} + \sin(\phi)\,\sin(\theta)\,\dot{\psi} \\ \cos(\theta)\,\dot{\psi} + \dot{\phi} \end{array} \right\rangle$$

The rotational kinetic energy is computed as:

$$\begin{aligned}
T_{\text{rot}} &= \frac{1}{2}\omega_b^T I_b\,\omega_b \\
&= \frac{m}{20000}\Big(16\,\dot{\phi}^2 + 49\,\dot{\psi}^2 + 16\,\dot{\theta}^2 - 33\,\cos(\phi)^2\,\dot{\psi}^2 + 33\,\cos(\phi)^2\,\dot{\theta}^2 - 33\,\cos(\theta)^2\,\dot{\psi}^2 \\
&\qquad +33\,\cos(\phi)^2\cos(\theta)^2\,\dot{\psi}^2 + 32\,\cos(\theta)\,\dot{\psi}\,\dot{\phi} + 66\,\cos(\phi)\,\sin(\phi)\,\sin(\theta)\,\dot{\theta}\,\dot{\psi}\Big)
\end{aligned}$$

The total kinetic energy is:

$$\begin{aligned}
\boldsymbol{T} &= \boldsymbol{T}_{\text{trans}} + \boldsymbol{T}_{\text{rot}} \\
&= \mathbf{312.5}\,\boldsymbol{m} + \\
&\quad \frac{\boldsymbol{m}}{\mathbf{20000}}\Big(\mathbf{16}\,\dot{\boldsymbol{\phi}}^{\mathbf{2}} + \mathbf{49}\,\dot{\boldsymbol{\psi}}^{\mathbf{2}} + \mathbf{16}\,\dot{\boldsymbol{\theta}}^{\mathbf{2}} - \mathbf{33}\,\mathbf{cos}\,(\boldsymbol{\phi})^{\mathbf{2}}\,\dot{\boldsymbol{\psi}}^{\mathbf{2}} + \mathbf{33}\,\mathbf{cos}\,(\boldsymbol{\phi})^{\mathbf{2}}\,\dot{\boldsymbol{\theta}}^{\mathbf{2}} \\
&\qquad - \mathbf{33}\,\mathbf{cos}\,(\boldsymbol{\theta})^{\mathbf{2}}\,\dot{\boldsymbol{\psi}}^{\mathbf{2}} + \mathbf{33}\,\mathbf{cos}\,(\boldsymbol{\phi})^{\mathbf{2}}\,\mathbf{cos}\,(\boldsymbol{\theta})^{\mathbf{2}}\,\dot{\boldsymbol{\psi}}^{\mathbf{2}} + \mathbf{32}\,\mathbf{cos}\,(\boldsymbol{\theta})\,\dot{\boldsymbol{\psi}}\,\dot{\boldsymbol{\phi}} \\
&\qquad + \mathbf{66}\,\mathbf{cos}\,(\boldsymbol{\phi})\,\mathbf{sin}\,(\boldsymbol{\phi})\,\mathbf{sin}\,(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}}\,\dot{\boldsymbol{\psi}}\Big)
\end{aligned}$$

$\longrightarrow \mathcal{A}$nswer

(b) If the ball is spinning about the y-axis of the body-fixed frame at 5 rev/s, as shown in the figure, what is the kinetic energy?

If the ball is spinning about the y-axis of the body-fixed frame at 5 rev/s, as shown in the figure, we can state that the body angular velocity is:

$$\omega_b = \left\langle \begin{matrix} 0 \\ 5 \\ 0 \end{matrix} \right\rangle \quad \left[\frac{\text{rev}}{\text{s}}\right] = \left\langle \begin{matrix} 0 \\ 10\pi \\ 0 \end{matrix} \right\rangle \quad \left[\frac{\text{rad}}{\text{s}}\right]$$

For this body rate, the rotational kinetic energy is:

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\omega_b^T I_b \, \omega_b \\ &= 0.245 \, m \, \pi^2 \quad \left[\text{kg}\left(\frac{\text{m}}{\text{s}}\right)^2\right] \\ &= 0.245 \, m \, \pi^2 \quad [J] \end{aligned}$$

The pairing this with the translational kinetic energy from part (a), the total kinetic energy is:

$$\boldsymbol{T = T_{\text{trans}} + T_{\text{rot}} = \left(312.5 \, m + 0.245 \, m \, \pi^2\right) \quad [J]}$$

$\longrightarrow$ *Answer*

The following MATLAB script was used to solve this problem:

```
syms m

sympref('FloatingPointOutput',false);

I_xx = m*(0.040)^2 %[output:00868c7e]
I_yy = m*(0.070)^2 %[output:8144f992]
I_zz = I_xx %[output:204e095f]
I = diag([I_xx,I_yy,I_zz]) %[output:673bbe43]
v_dot = [0;25;0] %[output:2a53fbbb]

T_trans = 0.5*m*dot(v_dot,v_dot) %[output:1db220e2]

syms t psi(t) theta(t) phi(t)
R = simplify(zRot(psi)*yRot(theta)*zRot(phi)) %[output:89624325]
```

```
R_dot = simplify(diff(R),1000) %[output:5ac9e925]

Omega_b = simplify(transpose(R)*R_dot,1000);
Omega_b = Omega_b(t) %[output:2a4d44e0]
omega_b = vect(Omega_b) %[output:0e51b43a]

T_rot = simplify(expand(... %[output:group:44ff5573] %[output:62e47f9d]
    0.5*transpose(omega_b)*I*omega_b)) %[output:group:44ff5573] %[output:62e47f9d]
omega_b = sym(2*pi*[0;5;0]) %[output:81347777]
sympref('FloatingPointOutput',false);
T_rot = simplify(expand(... %[output:group:7a8650c2] %[output:3d38bcc6]
    0.5*transpose(omega_b)*I*omega_b)) %[output:group:7a8650c2] %[output:3d38bcc6]

function R = xRot(ang)
R = [ 1 0 0;
0 cos(ang) -sin(ang);
0 sin(ang) cos(ang)];
end

function R = yRot(ang)
R = [ cos(ang) 0 sin(ang);
0 1 0;
-sin(ang) 0 cos(ang)];
end

function R = zRot(ang)
R = [ cos(ang) -sin(ang) 0;
sin(ang) cos(ang) 0;
0 0 1];
end

function omega = vect(Omega)
omega = [Omega(3,2);Omega(1,3);Omega(2,1)];
end
```

Submitted by Austin Barrilleaux on November 22, 2024.