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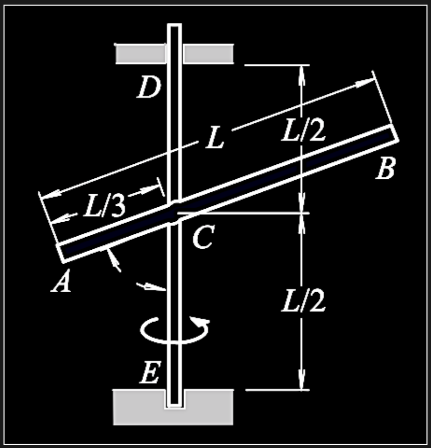
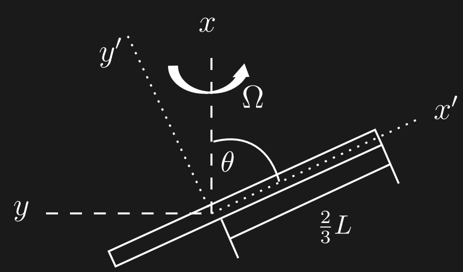
MODULE 6 — Assignment

EXERCISE 5.26

Thin bar ACB is welded to a shaft that rotates at the constant angular speed Ω, so the angle θ between the bar and the shaft is constant.
(a) Derive expressions for the angular momentum $\bar{H}_C$ and the kinetic energy of the bar. Draw a sketch of $\bar{H}_C$.
(b) Based on an analysis of the manner in which $\bar{H}_C$ in Part (a) rotates, derive an expression for $\frac{\partial}{\partial t}\bar{H}_C$.
(c) Use Eq. (5.3.4) to evaluate $\frac{\partial}{\partial t}\bar{H}_C$, and compare it with the result of Part (b).



The following sketch shows the two frames of the system:



The rotation matrix that converts the $\{xyz\}$ frame to the $\{x'y'z'\}$ frame is:

$$R = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Constructing the angular velocity vector:

$$\bar{\omega} = \Omega \; i$$

In the body frame is:

$$\bar{\omega} = R \; \Omega \; i = \begin{bmatrix} \Omega \; \cos(\theta) \; i' \\ \Omega \; \sin(\theta) \; j' \\ 0 \; k' \end{bmatrix}$$

From textbook Appendix, the centroidal inertia mass properties of the shaft are:

$$I_{xx} = 0$$
$$I_{yy} = \frac{1}{12} m \, L^2$$
$$I_{zz} = \frac{1}{12} m \, L^2$$

This expressed as the inertia tensor is:

$$I_{x'y'z'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m \, L^2 & 0 \\ 0 & 0 & \frac{1}{12} m \, L^2 \end{bmatrix}$$

The distance from the body frame is:

$$d_{x'y'z'} = \begin{bmatrix} \frac{1}{6} L & 0 & 0 \end{bmatrix}$$

Using the parallel axis theorem to get the parallel axis transformation of inertia matrix relative to the center of the frame of reference in question:

$$I_{pat} = m \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{36}\,L^2 & 0 \\ 0 & 0 & \frac{1}{36}\,L^2 \end{bmatrix}$$

This makes the inertia tensor at point C:

$$I_C = I_{x'y'z'} + I_{pat} = m \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{9}\,L^2 & 0 \\ 0 & 0 & \frac{1}{9}\,L^2 \end{bmatrix}$$

From this we can compute the angular momentum as:

$$\boldsymbol{\bar{H}_C = I_C\ \bar{\omega} = \frac{1}{9} m\,L^2\,\Omega, \sin\left(\theta\right)\ j'}$$

$\longrightarrow$ *Answer*

Since the bar rotates about the x axis, the rotation component in the terminal frame that is orthogonal to the j'-axis is $\Omega\cos(\theta)\ i'$, therefore:

$$\boldsymbol{\dot{\bar{H}}_C = \frac{1}{9} m\,L^2\,\Omega^2 \cos\left(\theta\right) \sin\left(\theta\right)\ k'}$$

$\longrightarrow$ *Answer*

If we use Eq. (5.3.4) to evaluate this, we get the same answer:

$$\dot{\bar{H}}_C = \cancelto{0}{\frac{\partial}{\partial t}\bar{H}_C} + \bar{\omega} \times \bar{H}_C$$

$$\boldsymbol{\dot{\bar{H}}_C} = \begin{bmatrix} \boldsymbol{0\ i'} \\ \boldsymbol{0\ j'} \\ \boldsymbol{\frac{1}{9} m\,L^2\,\Omega^2 \cos\left(\theta\right)\ \sin\left(\theta\right)\ k'} \end{bmatrix}$$
$$= \boldsymbol{\frac{1}{9} m\,L^2\,\Omega^2 \cos\left(\theta\right)\ \sin\left(\theta\right)\ k'}$$

$\longrightarrow$ *Answer*

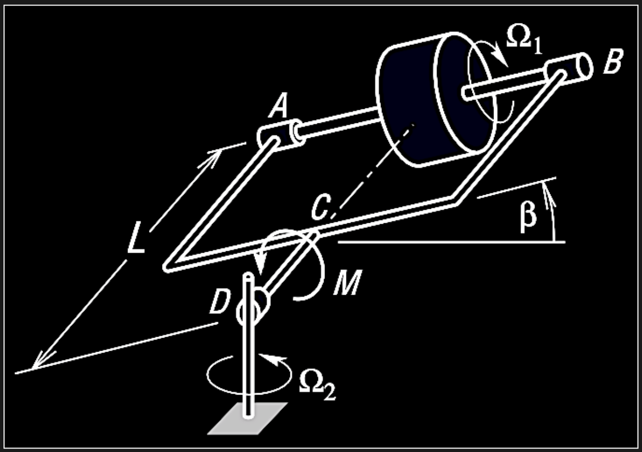
Matlab Code Used

The following MATLAB code was used to solve this problem:

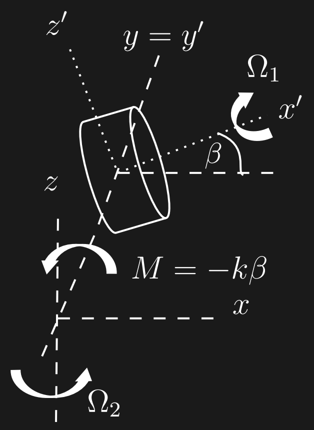
```
%% EXERCISE 5.26
% Thin bar ACB is welded to a shaft that rotates at the constant
% angular speed, so the angle theta between the bar and the
% shaft is constant.
% (a) Derive expressions for the angular momentum H-C and the
%     kinetic energy of the bar. Draw a sketch of H-C.
% (b) Based on an analysis of the manner in which H-C in Part (a)
%     rotates, derive an expression for dH-C/dt.
% (c) Use Eq. (5.3.4) to evaluate dH -C/dt, and compare it
%     with the result of Part (b).

syms m L t
syms theta real
syms Omega real

R = Dynamics.Inertial2Body.zRot(-theta);

omega_bar = R*Omega*[1;0;0];

I = diag([0,(1/12)*m*L^2,(1/12)*m*L^2]);

I_pa = Dynamics.Inertia.parallelAxisTransform(m,[(2*L/3-L/2),0,0]);

H_C = (I+I_pa)*omega_bar;

H_C_dot = diff(H_C,t) + cross(omega_bar,H_C);
```

EXERCISE 6.8

The torque M acting on the gimbal of the gyroscopic turn indicator is exerted by a torsional spring, so $M = -k\beta$. The precession rate Ω_2 is a specified function of time, and the spin rate Ω_1 is held constant by a servomotor. Let I_1 denote the moment of inertia of the flywheel about axis AB, and let I_2 be the centroidal moment of inertia perpendicular to axis AB. Derive the differential equation of motion for β.



We will define the following frames for the system:



Construct the angular velocity vector $\bar{\omega}$ of xyz by vectorially adding the simple rotation rates according to:

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3$$

This gives us:

$$\bar{\omega} = \Omega_2 \; k + \dot{\beta}(-j') + \Omega_1(-i')$$

We can equate k, given the following rotation as:

$$R(-\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$k = R(-\beta) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin(\beta) \;\; i' \\ 0 \; j \\ \cos(\beta) \;\; k' \end{bmatrix}$$

This allows us to express the angular velocity vector as:

$$\bar{\omega} = \Omega_2 \; \left(\sin(\beta) \; i' + \cos(\beta) \; k'\right) - \dot{\beta} \; j' - \Omega_1 \; i'$$

Or:

$$\bar{\omega} = \begin{bmatrix} \left\{\sin(\beta) \; \Omega_2 - \Omega_1\right\} \;\; i' \\ \left\{-\dot{\beta}\right\} \;\; j' \\ \left\{\cos(\beta) \; \Omega_2\right\} \;\; k' \end{bmatrix}$$

Next solve for angular acceleration:

$$\bar{\alpha} = \sum_n \left(\dot{\omega}_n \bar{e}_n + \bar{\Omega}_n \times \omega_n \bar{e}_n\right)$$

Where:

$$\begin{aligned} \omega_{n=1} &= -\Omega_1 \; i', & \dot{\omega}_{n=1} &= 0, & \Omega_{n=1} &= \Omega_2 R \; k - \dot{\beta} \; j' \\ \omega_{n=2} &= -\dot{\beta} \; j', & \dot{\omega}_{n=2} &= -\ddot{\beta} \; j', & \Omega_{n=2} &= \Omega_2 R \; k \\ \omega_{n=3} &= \Omega_2 R \; k, & \dot{\omega}_{n=3} &= \dot{\Omega}_2 R \; k, & \Omega_{n=3} &= 0 \end{aligned}$$

Substituting these into $\bar{\alpha}$:

$$\bar{\alpha} = \begin{bmatrix} \left\{\dot{\Omega}_2 \sin(\beta) +, \Omega_2\, \dot{\beta} \cos(\beta)\right\} i' \\ \left\{-\ddot{\beta} - \Omega_1\, \Omega_2\, \cos(\beta)\right\} j' \\ \left\{\dot{\Omega}_2 \cos(\beta) - \Omega_2\, \dot{\beta} \sin(\beta) - \Omega_1\, \dot{\beta}\right\} k' \end{bmatrix}$$

Since the torque M is acting in the j' direction, we can refer to equation 6.1.6 in the textbook:

$$\sum \bar{M}_{cg}\ j' = k\beta = I_{yy}\alpha_y + (I_{zz} - I_{xx})\, \omega_x \omega_z$$

Where the rotation of the frame, which is not rigidly fixed to the flywheel, is:

$$\omega = \Omega_2\ k + \dot{\beta}(-j') = \begin{bmatrix} \Omega_2 \sin(\beta)\ \ i' \\ -\dot{\beta}\ j' \\ \Omega_2 \cos(\beta)\, k' \end{bmatrix}$$

And:

$$I_{xx} = I_1$$
$$I_{yy} = I_{zz} = I_2$$

This evaluates to:

$$\boldsymbol{k\beta = \cos(\beta)\ \sin(\beta)\ {\Omega_2}^2\ (I_1 - I_2) - I_2\ddot{\beta} - I_2\Omega_1\ \cos(\beta)\ \Omega_2}$$

$\longrightarrow$ *Answer*

Matlab Code Used

The following MATLAB code was used to solve this problem:

```
%% EXERCISE 6.8
% The torque M acting on the gimbal of the gyroscopic turn indicator is
% exerted by a torsional spring, so M = -k beta. The precession rate
% Omega 2 is a specified function of time, and the spin rate Omega 1 is
% held constant by a servomotor. Let I1 denote the moment of inertia of
% the flywheel about axis AB, and let I2 be the centroidal moment of
% inertia perpendicular to axis AB. Derive the differential equation
```

```
% of motion for beta.

clc,clear
syms k t beta(t)

syms Omega_1 Omega_2(t)

syms I_1 I_2

R = yRot(-beta);

omega_bar = Omega_2*R*[0;0;1] + diff(beta,t)*[0;-1;0]...
                + Omega_1*[-1;0;0];

Omega_N = Omega_2(t)*R(t)*[0;0;1];

alpha_bar = cross(omega_bar,Omega_1*[-1;0;0]) +...
            diff(Omega_2,t)*R*[0;0;1] +...
            simplify(cross(Omega_N,Omega_2*R*[0;0;1])) +...
            diff(diff(beta,t),t)*[0;-1;0] +...
            cross( Omega_N ,diff(beta,t)*[0;-1;0]);

system = ([0;1;0]*k*beta(t) == I_2*alpha_bar(t)-...
    (I_2-I_1)*Omega_N(1)*Omega_N(3));

system(2)

function R = yRot(ang)
R = [ cos(ang) 0 -sin(ang) ; %% Neg body 2 in
0 1 0 ;
sin(ang) 0 cos(ang) ];
end
```

Submitted by Austin Barrilleaux on October 9, 2024.