Austin Barrilleaux
Whiting School of Engineering
Johns Hopkins University

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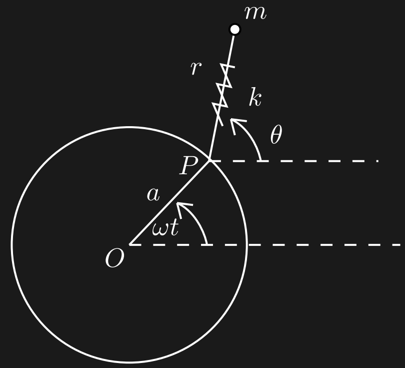
MODULE 9 — Assignment

Problem 1

Derive the equations of motion for a rotating spring-pendulum shown below. The spring- pendulum is attached at point P.



For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2$$

Therefore:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

By inspection of the sketch:

$$x = a\cos(\omega t) + r\cos(\theta)$$
$$y = a\sin(\omega t) + r\sin(\theta)$$

Taking the time derivative of both:

$$\dot{x} = \dot{r}\cos(\theta) - r\,\dot{\theta}\sin(\theta) - a\,\omega\,\sin(\omega\,t)$$
$$\dot{y} = \dot{r}\sin(\theta) + r\,\dot{\theta}\cos(\theta) + a\,\omega\,\cos(\omega\,t)$$

This gives:

$$T = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + a^2\,\omega^2 + 2\,a\,\omega\,\sin(\theta-\omega\,t)\,\dot{r} + 2\,a\,\omega\,r\,\cos(\theta-\omega\,t)\,\dot{\theta}\right)$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2}k\,(r-r_0)^2$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + a^2\,\omega^2 + 2\,a\,\omega\,\sin(\theta-\omega\,t)\,\dot{r} + 2\,a\,\omega\,r\,\cos(\theta-\omega\,t)\,\dot{\theta}\right) - \frac{1}{2}k\,(r-r_0)^2$$

We will solve for the equations of motion along the two generalized coordinates, θ and r. Starting with r, we will solve:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial r} = m\,\left(r\,\dot{\theta}^2 + a\,\omega\,\cos(\theta-\omega\,t)\,\dot{\theta}\right) - k\,(r - r_0)$$

$$\frac{\partial L}{\partial \dot{r}} = m\,\left(\dot{r} + a\,\omega\,\sin(\theta-\omega\,t)\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\,\left(\ddot{r} - a\,\omega\,\cos(\theta-\omega\,t)\,\left(\omega-\dot{\theta}\right)\right)$$

This results in the equation of motion:

$$\ddot{r} - r\,\dot{\theta}^2 - a\,\omega^2\,\cos(\theta-\omega\,t) + \frac{k}{m}\,(r-r_0) = 0$$

Solving along θ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = m\,\left(a\,\omega\,\cos\left(\omega\,t-\theta\right)\,\dot{r}+a\,\omega\,\sin\left(\omega\,t-\theta\right)\,r\,\dot{\theta}\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} =, m\,\left(r^2\,\dot{\theta}+a\,\omega\,\cos\left(\theta-\omega\,t\right)\,r\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\,\left(r^2\,\ddot{\theta}+2\,r\,\dot{\theta}\,\dot{r}+a\,\omega\,\cos\left(\theta-\omega\,t\right)\,\dot{r}+a\,\omega\,r\,\sin\left(\theta-\omega\,t\right)\,\left(\omega-\dot{\theta}\right)\right)$$

This results in the equation of motion:

$$r\,\ddot{\theta}+2\,\dot{\theta}\,\dot{r}+a\,\omega^2\,\sin\left(\theta-\omega\,t\right)=0$$

The equations of motion for the system are:

$$\boldsymbol{\ddot{r}-r\,\dot{\theta}^2-a\,\omega^2\,\cos\left(\theta-\omega\,t\right)+\frac{k}{m}\left(r-r_0\right)=0}$$
$$\boldsymbol{r\,\ddot{\theta}+2\,\dot{\theta}\,\dot{r}+a\,\omega^2\,\sin\left(\theta-\omega\,t\right)=0}$$

$\longrightarrow$ *Answer*

The following MATLAB script was used to solve this problem:

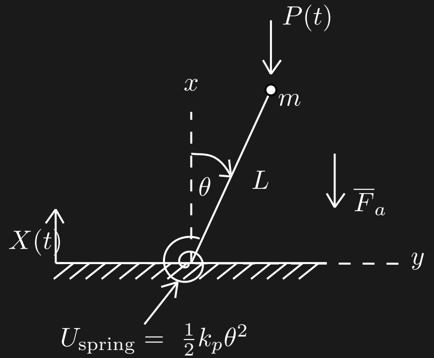
```
clc,clear
syms t m a k omega r_0
syms theta(t)
syms r(t)

% Position
x = a*cos(omega*t)+r*cos(theta);
y = a*sin(omega*t)+r*sin(theta);
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k*(r-r_0)^2;
% Lagrange Function
```

```
L = T - V;
% Coordinate derivatives
r_dot     = diff(r,t);
theta_dot = diff(theta,t);
% r coordinate derivatives
dL_dr     = diff(L,r);
dL_dr_dot = diff(L,r_dot);
dL_dr_dot_dt = diff(dL_dr_dot,t);
% theta coordinate derivatives
dL_dtheta     = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = diff(dL_dtheta_dot,t);
% Resulting equations of motion
EOM1 = simplify(dL_dr_dot_dt-dL_dr);
EOM2 = simplify(dL_dtheta_dot_dt-dL_dtheta);
```

Problem 2

Derive the equations of motion for the base-excited structure subject to an axial load $P(t)$.



For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2$$

Therefore:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

By inspection of the sketch:

$$x = L\cos(\theta) + X(t)$$
$$y = L\sin(\theta)$$

Taking the time derivative of both:

$$\dot{x} = X'(t) - L\,\sin(\theta)\,\dot{\theta}$$
$$\dot{y} = L\,\cos(\theta)\,\dot{\theta}$$

This gives:

$$T = \frac{1}{2}m\left(X'(t)^2 + L^2\dot{\theta}^2 - 2L\sin(\theta)X'(t)\dot{\theta}\right)$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2}k_p\theta^2 + m(-a)x$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2}m\left(X'(t)^2 + L^2\dot{\theta}^2 - 2L\sin(\theta)X'(t)\dot{\theta}\right) - \left(\frac{1}{2}k_p\theta^2 - ma\left(L\cos(\theta) + X(t)\right)\right)$$

We will solve for the equation of motion along the generalized coordinate, θ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = -L\,m\,a\,\sin(\theta) - L\,m\,\cos(\theta)\,\dot{\theta}\,X'(t) - k_p\,\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\,L^2\,\dot{\theta} - m\,L\,\sin(\theta)\,X'(t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\,L^2\,\ddot{\theta} - m\,\sin(\theta)\,L\,X''(t) - m\,\cos(\theta)\dot{\theta}\,L\,X'(t)$$

$$
\begin{aligned}
Q_\theta &= \sum_{k=1}^{N=3} F_k \frac{\partial x_k}{\partial \theta} \\
&= \begin{bmatrix} 0 \\ -P(t) \end{bmatrix} \cdot \begin{bmatrix} L\,\cos(\theta) \\ -L\,\sin(\theta) \end{bmatrix} \\
&= L\sin(\theta(t))P(t)
\end{aligned}
$$

This results in the system equation of motion:

$$m\,L^2\,\ddot{\theta} - m\,L\sin(\theta)\ X''(t) + k_p\,\theta + m\,a\,L\sin(\theta) = P(t)\ L\sin(\theta)$$

Or:

$$\boldsymbol{m\,L^2\,\ddot{\theta} + \left[a - X''(t)\right] m\,L\sin(\theta) + k_p\,\theta = P(t)\ L\sin(\theta)}$$

$\longrightarrow \mathcal{A}$nswer

The following MATLAB script was used to solve this problem:

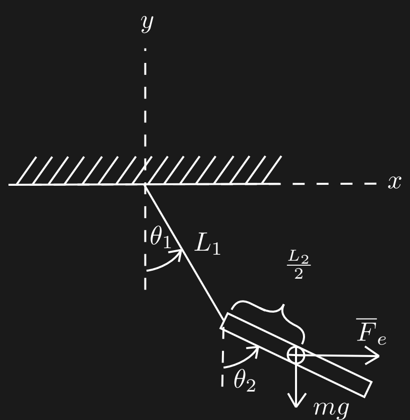
```
clc,clear
syms t m a k omega r_0
syms theta(t)
syms r(t)

% Position
x = a*cos(omega*t)+r*cos(theta);
y = a*sin(omega*t)+r*sin(theta);
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k*(r-r_0)^2;
% Lagrange Function
L = T - V;
% Coordinate derivatives
r_dot     = diff(r,t);
theta_dot = diff(theta,t);
% r coordinate derivatives
```

```
dL_dr      = diff(L,r);
dL_dr_dot = diff(L,r_dot);
dL_dr_dot_dt = diff(dL_dr_dot,t);
% theta coordinate derivatives
dL_dtheta      = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = diff(dL_dtheta_dot,t);
% Resulting equations of motion
EOM1 = simplify(dL_dr_dot_dt-dL_dr);
EOM2 = simplify(dL_dtheta_dot_dt-dL_dtheta);
```

Problem 3

The double pendulum. Find the equations governing a rigid bar attached to a string as shown in the figure.



a. This system starts with 9 degrees of freedom. What constraints exist that reduce this system to only two degrees of freedom?

The system is composed of two objects; a mass particle and a ridged body. The mass, m_1, particle has three translational dimensions, x_1, y_1 and z_1. The ridged body, m_2, particle has three translational dimensions, x_2, y_2 and z_2, and three rotational dimensions, R_{x_2}, R_{y_2} and R_{z_2}.

Because this is a 2-D problem, the following four constraints are inherent:

$$z_1 = z_2 = R_{x_2} = R_{y_2} = C = 0$$

Via inspection of the diagram we can see the following 3 constraints:

$$L_1{}^2 = x_1{}^2 + y_1{}^2$$

$$\left(\frac{L_1}{2}\right)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$R_{z_2} = \theta_2 = \tan^{-1}\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$

With **9 degrees of freedom for the unconstrained system** and **7 constraints**, the system has **2 degrees of freedom**.

$\longrightarrow$ $\mathcal{A}$nswer

b. Show that the Lagrangian is:

$$\boldsymbol{L = \frac{1}{2}m\left[L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1) + \frac{1}{3}L_2^2\dot{\theta}_2^2\right] - mg\left[L_1(1 - \cos\theta_1) + \frac{L_2}{2}(1 - \cos\theta_2)\right]}$$

For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}_2^2$$

Therefore:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\left(\frac{1}{12}\right)L_2^2\dot{\theta}_2^2$$

By inspection of the sketch:

$$x = L_1\sin(\theta_1) + \frac{L_2}{2}\sin(\theta_2)$$
$$y = \left(L_1 + \frac{L_2}{2}\right) - L_1\cos(\theta_1) - \frac{L_2}{2}\cos(\theta_2)$$

Note that these equations define $(x, y)_0$ at the location of the c.g. of the bar when $\theta_1 = \theta_2 = 0$.

Taking the time derivative of both:

$$\dot{x} = L_1 \cos(\theta_1)\ \dot{\theta}_1 + \frac{1}{2} L_2 \cos(\theta_2)\ \dot{\theta}_2$$
$$\dot{y} = L_1 \sin(\theta_1)\ \dot{\theta}_1 + \frac{1}{2} L_2 \sin(\theta_2)\ \dot{\theta}_2$$

This gives:

$$T = \frac{1}{2} m \left[{L_1}^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{1}{3} {L_2}^2 \dot{\theta}_2^2 \right]$$

The potential energy of the system, V is defined by:

$$V = mgh$$

Therefore:

$$\begin{aligned} V &= mg \left(L_1 + \frac{1}{2} L_2 - L_1 \cos(\theta_1) - \frac{1}{2} L_2 \cos(\theta_2) \right) \\ &= mg \left[L_1 \ \big(1 - \cos(\theta_1)\big) + \frac{1}{2} L_2 \left(1 - \cos(\theta_2)\right) \right] \end{aligned}$$

Given that the Lagrange is $L = T - V$:

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{\frac{1}{2} m \left[{L_1}^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{1}{3} {L_2}^2 \dot{\theta}_2^2 \right]} \\ &\quad \boldsymbol{- mg \left[L_1 \ \big(1 - \cos(\theta_1)\big) + \frac{1}{2} L_2 \left(1 - \cos(\theta_2)\right) \right]} \end{aligned}$$

$\longrightarrow$ *Answer*

This matches the Lagrangian in the question prompt.

c. Use Lagrange's equations to derive the equations of motion (don't forget about the external force!)

Looking at the external forces:

Submitted by Austin Barrilleaux on October 28, 2024.