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MODULE 11 — Assignment

Problem 1: Solve Ginsberg 9.28

The absolute velocity of a particle may be represented by the components v_x, v_y, and v_z relative to the axes of a moving reference system xyz. Suppose that the angular velocity $\bar{\omega}$ of xyz and the velocity $\bar{v}_O$ of the origin of xyz are known as functions of time. Derive the Gibbs-Appell equations of motion relating the quasi-velocities $\dot{\gamma}_1 = v_x$, $\dot{\gamma}_2 = v_y$, and $\dot{\gamma}_3 = v_z$ to the resultant force acting on the particle.

Where:

$$\bar{\omega} = \left\langle \begin{array}{c} \omega_x \\ \omega_y \\ \omega_z \end{array} \right\rangle$$

Given that:

$$\bar{v} = \left\langle \begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right\rangle = \left\langle \begin{array}{c} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{array} \right\rangle$$

Solving for acceleration:

$$\begin{aligned} \bar{a} &= \frac{\partial \bar{v}}{\partial t} + \bar{\omega} \times \bar{v} \\ &= \frac{\partial \dot{\gamma}}{\partial t} + \bar{\omega} \times \dot{\gamma} \\ &= \left\langle \begin{array}{c} \ddot{\gamma}_1 - \dot{\gamma}_2\,\omega_z + \dot{\gamma}_3\,\omega_y \\ \ddot{\gamma}_2 + \dot{\gamma}_1\,\omega_z - \dot{\gamma}_3\,\omega_x \\ \ddot{\gamma}_3 - \dot{\gamma}_1\,\omega_y + \dot{\gamma}_2\,\omega_x \end{array} \right\rangle \end{aligned}$$

Given that the Gibbs-Appell function for a system of particles is:

$$S = \sum_p \frac{1}{2} m \bar{a}_p \cdot \bar{a}_p$$

For this single particle case:

$$\begin{aligned} S &= \frac{1}{2} m \left(\bar{a} \cdot \bar{a} \right) \\ &= \frac{1}{2} m \left[\left(\ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x \right)^2 + \left(\ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x \right)^2 + \left(\ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y \right)^2 \right] \end{aligned}$$

Where the equations of motion are calculated as:

$$\frac{\partial S}{\partial \ddot{\gamma}_j} = \Gamma_j = \Gamma_1$$

The virtual work associated with the forces applied to the particle is:

$$\delta W = \sum \bar{F} \cdot \delta \bar{r} = \sum_{j1}^{K} \Gamma_j \; \delta \gamma_j = \sum \bar{F} \cdot \left\langle \begin{array}{c} \delta \gamma_1 \\ \delta \gamma_2 \\ \delta \gamma_3 \end{array} \right\rangle$$

The equation of motion is solved for as:

$$\frac{\partial S}{\partial \ddot{\gamma}} = m \left\langle \begin{array}{c} \ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y \\ \ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x \\ \ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x \end{array} \right\rangle = \Gamma_1 = \left\langle \begin{array}{c} \sum F_x \\ \sum F_y \\ \sum F_z \end{array} \right\rangle$$

Or:

$$\begin{aligned} \boldsymbol{m \left(\ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y \right) = \sum F_x} \\ \boldsymbol{m \left(\ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x \right) = \sum F_y} \\ \boldsymbol{m \left(\ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x \right) = \sum F_z} \end{aligned}$$

$\longrightarrow \mathcal{A}\text{nswer}$

Where:

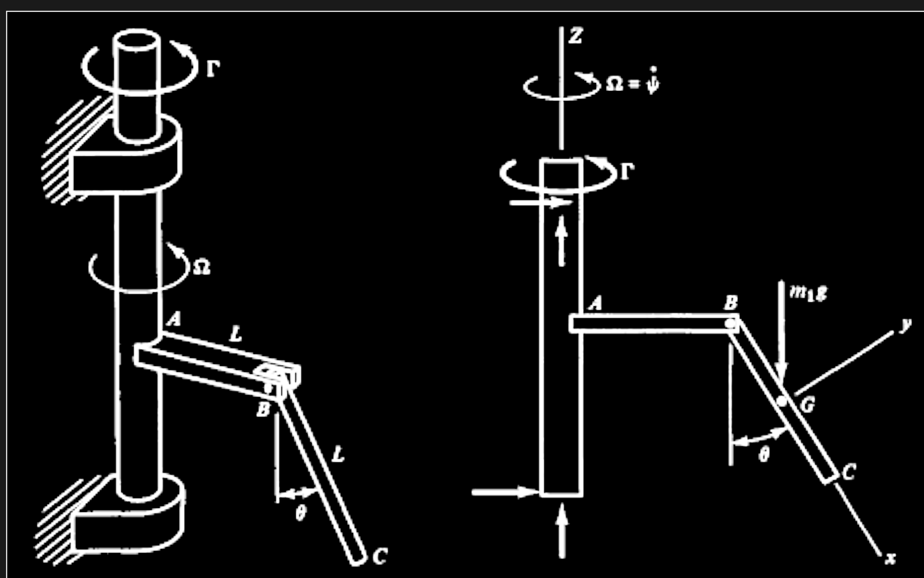
$$\dot{\gamma} = \left\langle \begin{array}{c} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{array} \right\rangle = v = \left\langle \begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right\rangle = v_0 + \bar{\omega} \times \bar{r} = \left\langle \begin{array}{c} v_{0_x} - \omega_z \, y + \omega_y \, z \\ v_{0_y} + \omega_z \, x - \omega_x \, z \\ v_{0_z} - \omega_y \, x + \omega_x \, y \end{array} \right\rangle$$

Where:

Problem 2:

Use the Gibbs-Appell approach to find the equations of motion for this problem.

A torque Γ applied to the vertical shaft of the T-bar causes the rotation rate Ω about the vertical axis to increase in proportion to the angle $\boldsymbol{\theta}$ by which bar BC swings outward, that is, $\Omega = c\boldsymbol{\theta}$. The mass of bar BC is $\boldsymbol{m_1}$ and the moment of inertia of the T-bar about its axis of rotation is $\boldsymbol{I_2}$. Determine the equations of motion for the system, and for the torque Γ.



Example 7.1
Free-body diagram.

$$\begin{pmatrix} \frac{L\,\cos(\psi)\,\cos(\theta)\,\frac{\partial}{\partial t}\dot{\gamma}_1}{2} - \frac{L\,\cos(\psi)\,\sin(\theta)\,\dot{\gamma}_1^2}{2} - \frac{L\,c^2\,\cos(\psi)\,\theta^2\,\left(\sin(\theta)+2\right)}{2} - \frac{L\,c\,\sin(\psi)\,\dot{\gamma}_1\,\left(\sin(\theta)+2\right)}{2} - L\,c\,\cos\left(\theta\right)\,\sin\left(\psi\right)\,\dot{\gamma}_1\,\theta \\ \frac{L\,\cos(\theta)\,\sin(\psi)\,\frac{\partial}{\partial t}\dot{\gamma}_1}{2} - \frac{L\,\sin(\psi)\,\sin(\theta)\,\dot{\gamma}_1^2}{2} - \frac{L\,c^2\,\sin(\psi)\,\theta^2\,\left(\sin(\theta)+2\right)}{2} + \frac{L\,c\,\cos(\psi)\,\dot{\gamma}_1\,\left(\sin(\theta)+2\right)}{2} + L\,c\,\cos\left(\psi\right)\,\cos\left(\theta\right)\,\dot{\gamma}_1\,\theta \\ \frac{L\,\sin(\theta)\,\frac{\partial}{\partial t}\dot{\gamma}_1}{2} + \frac{L\,\cos(\theta)\,\dot{\gamma}_1^2}{2} \end{pmatrix}$$

Submitted by Austin Barrilleaux on November 11, 2024.