Austin Barrilleaux
Whiting School of Engineering
Johns Hopkins University

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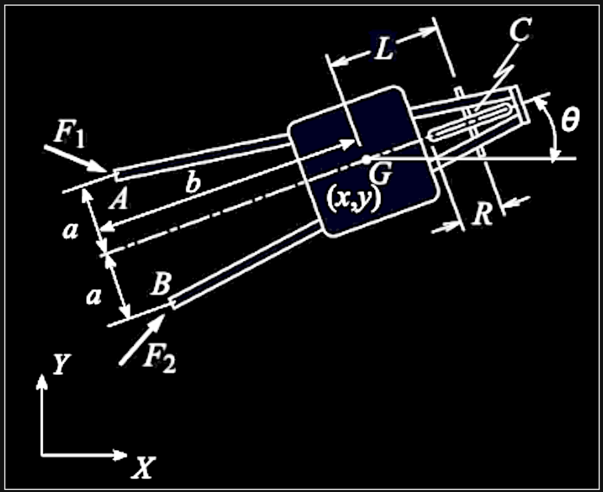
MODULE 14 — Final Exam

Problem 1

The wheelbarrow is pushed in the horizontal plane by forces $\boldsymbol{F_1} = [\boldsymbol{F_{1_x}}, \boldsymbol{F_{1_y}}]^T$ and $\boldsymbol{F_2} = [\boldsymbol{F_{2_x}}, \boldsymbol{F_{2_y}}]^T$ acting at the ends of the handles. See Fig. 1 for the graphical explanation of the system. The chassis has mass m, with its center of mass situated at point $\boldsymbol{G}$ on the centerline. The centroidal moment of inertia of the chassis about a vertical axis (the $\boldsymbol{Z}$-axis) is $\boldsymbol{I}$. The wheel, which may be approximated as a thin disk (hence inertia can be ignored), rolls without slipping. The position of G is denoted as $(\boldsymbol{x}, \boldsymbol{y})$ and the steering angle is denoted as $\boldsymbol{\theta}$.



Convenient generalized coordinates for the wheelbarrow are the absolute position coordinates of point G, $q_1 = x$, $q_2 = y$, and the angle $q_3 = \theta$ locating the x-axis relative to a fixed XYZ reference frame.

The position of the chassis is:

$$r_G = \left\langle \begin{array}{c} x \\ y \\ 0 \end{array} \right\rangle$$

Its derivative yields:

$$v_G = \left\langle \begin{array}{c} \dot{x} \\ \dot{y} \\ 0 \end{array} \right\rangle$$

The position of the wheel is:

$$r_C = \left\langle \begin{array}{c} x + L\,\cos(\theta) \\ y + L\,\sin(\theta) \\ 0 \end{array} \right\rangle$$

Its derivative yields:

$$v_C = \left\langle \begin{array}{c} \dot{x} - L\,\sin(\theta)\,\dot{\theta} \\ \dot{y} + L\,\cos(\theta)\,\dot{\theta} \\ 0 \end{array} \right\rangle$$

(a) Derive the velocity constraint form.

The velocity constraint exists due to the no-slip condition where the wheel contacts the ground, directly below the cm of the wheel. Due to this constraint, the velocity in the j direction must be zero. This is described as:

$$\begin{aligned} v_C \cdot j &= 0 \\ &= \left\langle \begin{array}{c} \dot{x} - L\,\sin(\theta)\,\dot{\theta} \\ \dot{y} + L\,\cos(\theta)\,\dot{\theta} \\ 0 \end{array} \right\rangle \cdot \left\langle \begin{array}{r} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{array} \right\rangle \\ &= -\sin(\theta)\,\dot{x} + \cos(\theta)\,\dot{y} + L\,\dot{\theta} = 0 \end{aligned}$$

The velocity constraint is:

$$\boldsymbol{-\sin(\theta)\ \dot{x} + \cos(\theta)\ \dot{y} + L\,\dot{\theta} = 0}$$

$\longrightarrow$ $\mathcal{A}$nswer

Putting this in the form of the constraint equation:

$$\sum_{j=1}^{N} a_{ij}\dot{q}_j + b_i = a_{11}\dot{q}_1 + a_{12}\dot{q}_2 + a_{13}\dot{q}_3 + b_1 = 0$$

We see that:

$$\boldsymbol{a_{11} = -\sin(\theta)}$$
$$\boldsymbol{a_{12} = \cos(\theta)}$$
$$\boldsymbol{a_{13} = L}$$

$\longrightarrow$ $\mathcal{A}$nswer

(b) Derive the Lagrange equations of motion. Express equations with given parameters in the problem.

Given that the inertia of the wheel can be ignored, and as a result there is no rotational or translational energy component for the wheel, the kinetic energy of the system is described as:

$$T = \frac{1}{2}m\,(v_G \cdot v_G) + \frac{1}{2}I\dot{\theta}^2$$

Where potential energy, $V = 0$, so $L = T$.

This evaluates to::

$$L = \frac{1}{2}\mathrm{I}\,\dot{\theta}^2 + \frac{1}{2}m\,\dot{x}^2 + \frac{1}{2}m\,\dot{y}^2$$

Solving for the generalized forces, the virtual work for the system is:

$$\begin{aligned}
\delta W &= \sum_{j=1}^{N} Q_j \delta q_j \\
&= F_1 \cdot \delta r + F_2 \cdot \delta r + M \cdot \delta\theta \\
&= (F_1 + F_2) \cdot \left\langle \begin{array}{c} \delta x \\ \delta y \end{array} \right\rangle + M \cdot \delta\theta \\
&= (F_1 + F_2) \cdot I \;\delta x + (F_1 + F_2) \cdot J \;\delta y + \left(r_{A/G} \times F_1 + r_{B/G} \times F_2\right) \delta\theta
\end{aligned}$$

Therefore, the generalized forces are:

$$\begin{aligned}
Q_x &= (F_1 + F_2) \cdot I \\
Q_y &= (F_1 + F_2) \cdot J \\
Q_\theta &= \left(r_{A/G} \times F_1 + r_{B/G} \times F_2\right)
\end{aligned}$$

Because the forces were not provided as vectors, I cannot describe the terms more precisely.

Solving for the equations of motion along the generalized coordinate, x:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q_x + a_{11}\lambda_1$$

The component parts of this equation are:

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\,\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\,\ddot{x}$$

This results in the equation of motion:

$$m\,\ddot{x} = (F_1 + F_2) \cdot I - \sin(\theta)\,\lambda_1$$

Solving for the equations of motion along the generalized coordinate, y:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = Q_y + a_{12}\lambda_1$$

The component parts of this equation are:

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = m\,\dot{y}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\,\ddot{y}$$

This results in the equation of motion:

$$m\,\ddot{y} = (F_1 + F_2) \cdot J + \cos(\theta)\,\lambda_1$$

Solving for the equations of motion along the generalized coordinate, θ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta + a_{13}\lambda_1$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \mathrm{I}\,\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \mathrm{I}\,\ddot{\theta}$$

This results in the equation of motion:

$$\mathrm{I}\,\ddot{\theta} = r_{A/G} \times F_1 + r_{B/G} \times F_2 + L\lambda_1$$

The Lagrange equations of motion are:

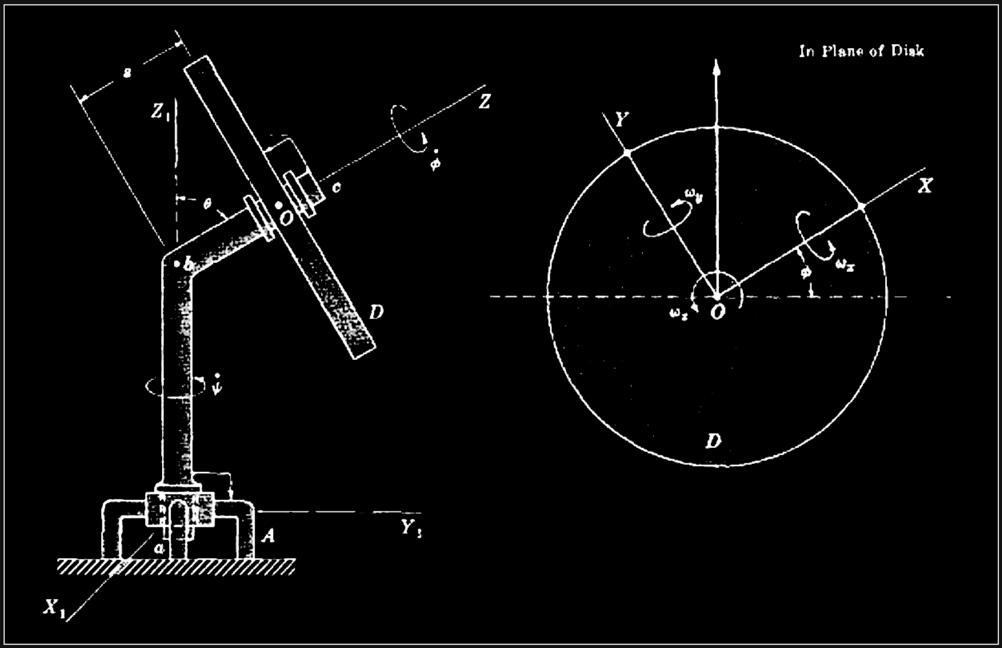
$$\boldsymbol{m\,\ddot{x} = (F_1 + F_2) \cdot I - \sin(\theta)\,\lambda_1}$$
$$\boldsymbol{m\,\ddot{y} = (F_1 + F_2) \cdot J + \cos(\theta)\,\lambda_1}$$
$$\boldsymbol{\mathrm{I}\,\ddot{\theta} = r_{A/C} \times F_1 + r_{B/C} \times F_2 + L\lambda_1}$$

$\longrightarrow$ $\mathcal{A}$nswer

The following MATLAB script was used to solve this problem:

```
clc,clear
syms t R L x(t) y(t) theta(t) omega_j

i = zRot(theta)*[1;0;0];
j = zRot(theta)*[0;1;0];
k = zRot(theta)*[0;0;1];

r_G = [x;y;0] %[output:7715b359]

v_G = diff(r_G,t) %[output:09bf8722]

r_C = r_G + L*i %[output:3aa47ebf]
v_C = diff(r_C,t) %[output:8f99bf64]

% Velocity constraint
simplify(expand(transpose(v_C)*j)) == 0 %[output:5be28138]

% Lagranges Equations
syms I m m_wheel
T = (1/2)*m*transpose(v_G)*v_G +... %[output:group:2c67c242] %[output:363080ab]
    (1/2)*I*diff(theta,t)^2 %+ ... %[output:group:2c67c242] %[output:363080ab]
    % (1/2)*m_wheel*transpose(v_C)*v_C %+ ...
    % (1/2)*((1/2)*m_wheel*R^2*omega_j^2) +...
    % (1/2)*((1/4)*m_wheel*R^2*diff(theta,t)^2)

T = subs(T,R^2*omega_j^2,transpose(v_C)*v_C) %[output:5b407340]
V = 0;
L = simplify(T-V) %[output:1e8027f4]
% x coordinate derivatives
dL_dx     = diff(L,x) %[output:48f9b153]
dL_dx_dot = simplify(diff(L,diff(x,t))) %[output:6b312451]
dL_dx_dot_dt = simplify(diff(dL_dx_dot,t)) %[output:928919b3]
EOM_1 = simplify(dL_dx_dot_dt-dL_dx) %[output:847d8a11]
% y coordinate derivatives
dL_dy     = diff(L,y) %[output:70570010]
dL_dy_dot = simplify(diff(L,diff(y,t))) %[output:8d481783]
dL_dy_dot_dt = simplify(diff(dL_dy_dot,t)) %[output:5cc8668c]
EOM_2 = simplify(dL_dy_dot_dt-dL_dy) %[output:0c151e55]

% theta coordinate derivatives
dL_dtheta     = diff(L,theta) %[output:320c304b]
dL_dtheta_dot = simplify(diff(L,diff(theta,t))) %[output:082c4aae]
dL_dtheta_dot_dt = simplify(diff(dL_dtheta_dot,t)) %[output:05df8b81]
EOM_3 = simplify(dL_dtheta_dot_dt-dL_dtheta) %[output:3e207815]

function R = zRot(ang)
R = [ cos(ang) -sin(ang) 0;
sin(ang) cos(ang) 0;
0 0 1];
end
```

Problem 2



Consider the system in Fig. 2. Frame $\{X_1Y_1Z_1\}$ is the spatially fixed frame, and $\{XYZ\}$ is the body-fixed frame attached to the disk (the origin O is at the center of mass of the disk). A is at rest. The origin O of the body-fixed frame is at the center of mass of the disk. The bar ab (hence the elbow) is rotating about the Z_1-axis and its rotation angle is denoted as ψ. The length of the bar ab is L. Angle θ is constant, as you can see in the figure. Finally the disk is rotating about the Z-axis and its rotation angle is denoted as ϕ. Mass of the disk is M, and the moment of inertia matrix (body quantity) is

$$I_b = \text{diag}[I_x, I_x, I_z]$$

We only consider the motion of the disk. In other words, we ignore the mass and inertia of the elbow. For this problem, use the approach in Module 13.

(a) Compute the kinetic energy of the system (Hint: use the $\boldsymbol{ZXZ}$ Euler angles to describe the orientation of the disk (i.e., $\{\boldsymbol{XYZ}\}$) relative to $\{\boldsymbol{X_1Y_1Z_1}\}$).

Using the ZXZ Euler angles to represent the orientation of the disk:

$$
\begin{aligned}
R_{ZXZ} &= R_Z(\psi)R_X(\theta)R_Z(\phi) \\
&= \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\phi)\cos(\psi) - \cos(\theta)\sin(\phi)\sin(\psi) & -\cos(\psi)\sin(\phi) - \cos(\phi)\cos(\theta)\sin(\psi) & \sin(\psi)\sin(\theta) \\ \cos(\phi)\sin(\psi) + \cos(\psi)\cos(\theta)\sin(\phi) & \cos(\phi)\cos(\psi)\cos(\theta) - \sin(\phi)\sin(\psi) & -\cos(\psi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \cos(\phi)\sin(\theta) & \cos(\theta) \end{bmatrix}
\end{aligned}
$$

We can compute the body angular velocity as:

$$
\omega_b = \mathbf{vect}\left(R^T\dot{R}\right) = \left\langle \begin{matrix} \sin(\phi)\sin(\theta)\,\dot{\psi} \\ \cos(\phi)\sin(\theta)\,\dot{\psi} \\ \dot{\phi} + \cos(\theta)\,\dot{\psi} \end{matrix} \right\rangle
$$

The location of the center of the disk is:

$$
r = \left\langle \begin{matrix} L + s\,\sin(\psi)\sin(\theta) \\ -s\,\cos(\psi)\sin(\theta) \\ s\,\cos(\theta) \end{matrix} \right\rangle
$$

Taking its derivative, the velocity of the center of mass of the disk is:

$$
v = \left\langle \begin{matrix} s\,\cos(\psi)\sin(\theta)\,\dot{\psi} \\ s\,\sin(\psi)\sin(\theta)\,\dot{\psi} \\ 0 \end{matrix} \right\rangle
$$

The translational component kinetic energy for the system is:

$$
T_{\text{trans}} = \frac{1}{2}M\,(v \cdot v) = \frac{1}{2}M\,s^2\sin(\theta)^2\,\dot{\psi}^2
$$

The rotational kinetic energy of the system is computed as:

$$T_{\text{rot}} = \frac{1}{2}\omega_b^T I_b \omega_b = \frac{1}{2} I_z \left(\dot{\phi} + \cos\left(\theta\right)\,\dot{\psi}\right)^2 + \frac{1}{2} I_x \sin\left(\theta\right)^2 \dot{\psi}^2$$

Summing these two terms, we get that the total kinetic energy for the system is:

$$\boldsymbol{T = \frac{1}{2} I_z\,\left(\dot{\phi} + \cos\left(\theta\right)\,\dot{\psi}\right) + \frac{1}{2} I_x\,\sin\left(\theta\right)^2\,\dot{\psi}^2 + \frac{1}{2} M\,s^2\,\sin\left(\theta\right)^2\,\dot{\psi}^2}$$

$\longrightarrow$ *Answer*

(b) Taking the body-fixed frame as before but with origin at b, compute the kinetic energy.

The location of the center of the disk starting at b is:

$$r = \left\langle \begin{array}{c} s\,\sin\left(\psi\right)\sin\left(\theta\right) \\ -s\,\cos\left(\psi\right)\sin\left(\theta\right) \\ s\,\cos\left(\theta\right) \end{array} \right\rangle$$

Taking its derivative, we get the same velocity equation as above because the L term falls out of the derivative since it is a constant:

$$v = \left\langle \begin{array}{c} s\,\cos\left(\psi\right)\sin\left(\theta\right)\,\dot{\psi} \\ s\,\sin\left(\psi\right)\sin\left(\theta\right)\,\dot{\psi} \\ 0 \end{array} \right\rangle$$

This results in the same kinetic energy equation as in part (a):

$$\boldsymbol{T = \frac{1}{2} I_z\,\left(\dot{\phi} + \cos\left(\theta\right)\,\dot{\psi}\right) + \frac{1}{2} I_x\,\sin\left(\theta\right)^2\,\dot{\psi}^2 + \frac{1}{2} M\,s^2\,\sin\left(\theta\right)^2\,\dot{\psi}^2}$$

$\longrightarrow$ *Answer*

The following MATLAB script was used to solve this problem:

```
clc,clear
syms t psi(t) theta phi(t) s M L

R = simplify(zRot(psi)*xRot(theta)*zRot(phi)) %[output:35ab0f9c]

r = R*[0;0;s]+ [L;0;0] % Comment second term for part (b) %[output:347c8fcc]
syms x(t) y(t) z(t)
```

```
v = diff(r,t) %[output:87e8cbe2]
T_trans = simplify((1/2)*M*(transpose(v)*v)) %[output:3721bd79]

R_dot = simplify(diff(R),1000) %[output:888bba5c]
Omega_b = simplify(transpose(R)*R_dot,1000);
Omega_b = Omega_b(t) %[output:9272f05d]
omega_b = vect(Omega_b) %[output:8e6b6e34]
syms I_x I_z
I_b = diag([I_x,I_x,I_z]) %[output:8c771519]

T_rot = simplify((1/2)*... %[output:group:16760b0a] %[output:42697f62]
    transpose(omega_b)*(I_b)*omega_b,1000) %[output:group:16760b0a] %[output:42697f62]
T = simplify(T_trans + T_rot,1000) %[output:88b7afa8]
function R = xRot(ang)
R = [ 1 0 0;
0 cos(ang) -sin(ang);
0 sin(ang) cos(ang)];
end

function R = yRot(ang)
R = [ cos(ang) 0 sin(ang);
0 1 0;
-sin(ang) 0 cos(ang)];
end

function R = zRot(ang)
R = [ cos(ang) -sin(ang) 0;
sin(ang) cos(ang) 0;
0 0 1];
end

function omega = vect(Omega)
omega = [Omega(3,2);Omega(1,3);Omega(2,1)];
end
```

Submitted by Austin Barrilleaux on December 10, 2024.