Austin Barrilleaux
Whiting School of Engineering
Johns Hopkins University

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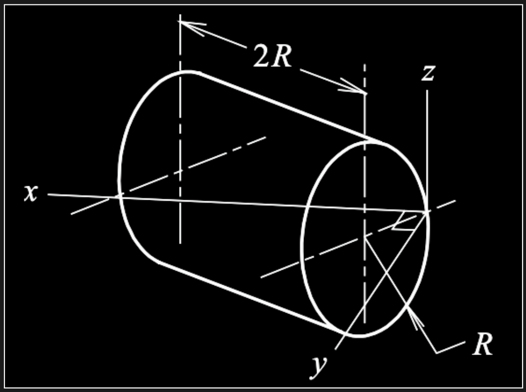
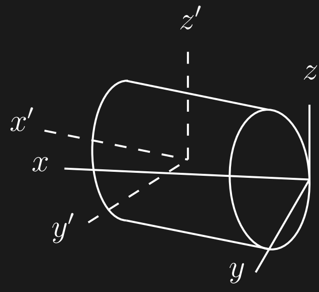
MODULE 5 — Assignment

Problem 1: EXERCISE 5.13

The x axis forms a diagonal intersecting the centroid of the homogeneous cylinder. Determine the inertia properties of the cylinder with respect to xyz.



For this problem, we will define the body frame of the cylinder as:



From textbook Appendix, the centroidal inertia mass properties of a homogeneous cylinder where in the body frame as I defined it are:

$$
\begin{aligned}
I_{xx} &= \frac{1}{2}mR^2 \\
I_{yy} &= \frac{1}{12}m\left(3R^2+h^2\right) &= \frac{7}{12}mR^2 \\
I_{zz} &= \frac{1}{12}m\left(3R^2+h^2\right) &= \frac{7}{12}mR^2
\end{aligned}
$$

This expressed as the inertia tensor is:

$$
I_{x'y'z'} = \begin{bmatrix} \frac{1}{2}mR^2 & 0 & 0 \\ 0 & \frac{7}{12}m\,R^2 & 0 \\ 0 & 0 & \frac{7}{12}m\,R^2 \end{bmatrix}
$$

The distance from the body frame is:

$$
d_{x'y'z'} = \begin{bmatrix} R \\ R \\ 0 \end{bmatrix}
$$

Using the parallel axis theorem to get the parallel axis transformation of inertia matrix relative to the center of the frame of reference in question:

$$
I_{pat} = m\begin{pmatrix} \left(y^2+z^2\right) & -xy & -xz \\ -xy & \left(x^2+z^2\right) & -yz \\ -xz & -yz & \left(x^2+y^2\right) \end{pmatrix} = m\begin{pmatrix} R^2 & -R^2 & 0 \\ -R^2 & R^2 & 0 \\ 0 & 0 & 2\,R^2 \end{pmatrix}
$$

The body to inertial rotation matrix is simply a z-axis rotation of $\frac{\pi}{4}$:

$$
R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

All together, we can compute the inertia properties of the cylinder with respect to xyz:

$$
\begin{aligned}
\boldsymbol{I_{xyz}} &= \boldsymbol{R^T}\left(\boldsymbol{I_{x'y'z'}} + \boldsymbol{I_{pat}}\right)\boldsymbol{R} \\
&= \boldsymbol{m\,R^2}\begin{bmatrix} \mathbf{0.5417} & \mathbf{0.0417} & \mathbf{0} \\ \mathbf{0.0417} & \mathbf{2.5417} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2.5833} \end{bmatrix}
\end{aligned}
$$

$\longrightarrow$ *Answer*

Explain the physical reasons for the off diagonal terms.

The off-diagonal terms of an inertia tensor, also called the products of inertia, describe the degree to which mass is distributed symmetrically relative to three coordinate planes. These terms effectively describe how rotation around one axis is coupled with the rotation around another.

$\longrightarrow$ *Answer*

Problem 2:

A rigid body has an inertia matrix given by:

$$
\boldsymbol{I} = \begin{bmatrix} \mathbf{400} & \mathbf{0} & \mathbf{-125} \\ \mathbf{0} & \mathbf{350} & \mathbf{0} \\ \mathbf{-125} & \mathbf{0} & \mathbf{100} \end{bmatrix}
$$

Find then principal moments of inertia and the transformation matrix that diagonalizes $\boldsymbol{I}$.

Solving for the eigenvectors of the inertia matrix:

$$
\begin{vmatrix} 400-\lambda & 0 & -125 \\ 0 & 350-\lambda & 0 \\ -125 & 0 & 100-\lambda \end{vmatrix} = -\lambda^3 + 850\,\lambda^2 - 199375\,\lambda + 8531250 = 0
$$

Solving for the roots of this equation, we get that $\lambda = 54.7438,\ 350,\ 445.2562$. This means that the principal moments of inertia are:

$$
\boldsymbol{I} = \begin{bmatrix} \mathbf{54.7438} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{350} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{445.2562} \end{bmatrix}
$$

$\longrightarrow$ *Answer*

We can deduce from the initial inertia matrix that we can get to the principal moment of inertia matrix via a y-axis rotation:

$$
R = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}
$$

The relationship between the provided inertia matrix and the principal moment of inertia matrix, I_D, is:

$$
I = R\ I_D R^T
$$

This evaluates to:

$$
\begin{bmatrix} 400 & 0 & -125 \\ 0 & 350 & 0 \\ -125 & 0 & 100 \end{bmatrix} = \begin{bmatrix} 54.7438\cos(\theta)^2 + 445.2562\sin(\theta)^2 & 0 & 390.5125\ \cos(\theta)\ \sin(\theta) \\ 0 & 350 & 0 \\ 390.5125\ \cos(\theta)\ \sin(\theta) & 0 & 445.2562\cos(\theta)^2 + 54.7438\sin(\theta)^2 \end{bmatrix}
$$

If we evaluate:

$$
400 = 54.7438\cos(\theta)^2 + 445.2562\sin(\theta)^2
$$

We get that $\theta = -1.2234$. This gives us a rotation matrix of:

$$
\boldsymbol{R} = \begin{bmatrix} \mathbf{0.3404} & \mathbf{0} & \mathbf{-0.9403} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0.9403} & \mathbf{0} & \mathbf{0.3404} \end{bmatrix}
$$

$\longrightarrow$ *Answer*

If we evaluate the following, we can prove that the rotation matrix can transform the principal moment of inertia matrix to the original inertia matrix:

$$R\ I_D R^T = \begin{bmatrix} 400 & 0 & -125 \\ 0 & 350 & 0 \\ -125 & 0 & 100 \end{bmatrix}$$

This proves that the transformation matrix, R, is valid and does diagonalizes I.

This diagonal matrix has the following properties:

$$R^T\ R = R\ R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|R| = 1$$

The columns of the rotation matrix, which constitute the eigenvectors, are unit vectors:

$$R = [v_1, v_2, v_3] \quad \rightarrow \quad ||v_1|| = ||v_2|| = ||v_3|| = 1$$

Submitted by Austin Barrilleaux on October 1, 2024.