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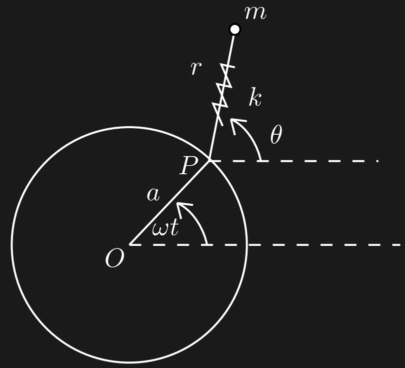
MODULE 9 — Assignment

Problem 1

Derive the equations of motion for a rotating spring-pendulum shown below. The spring- pendulum is attached at point *P*.



For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2$$

Therefore:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

By inspection of the sketch:

$$x = a\cos(\omega t) + r\cos(\theta)$$
$$y = a\sin(\omega t) + r\sin(\theta)$$

Taking the time derivative of both:

$$\dot{x} = \dot{r}\cos(\theta) - r\,\dot{\theta}\sin(\theta) - a\,\omega\,\sin(\omega\,t)$$
$$\dot{y} = \dot{r}\sin(\theta) + r\,\dot{\theta}\cos(\theta) + a\,\omega\,\cos(\omega\,t)$$

This gives:

$$T = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + a^2\,\omega^2 + 2\,a\,\omega\,\sin(\theta - \omega\,t)\,\dot{r} + 2\,a\,\omega\,r\,\cos(\theta - \omega\,t)\,\dot{\theta}\right)$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2}k\,(r - r_0)^2$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + a^2\,\omega^2 + 2\,a\,\omega\,\sin(\theta - \omega\,t)\,\dot{r} + 2\,a\,\omega\,r\,\cos(\theta - \omega\,t)\,\dot{\theta}\right) - \frac{1}{2}k\,(r - r_0)^2$$

We will solve for the equations of motion along the two generalized coordinates, θ and r. Starting with r, we will solve:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial r} = m\,\left(r\,\dot{\theta}^2 + a\,\omega\,\cos(\theta - \omega\,t)\,\dot{\theta}\right) - \,k\,(\,r - \,r_0)$$

$$\frac{\partial L}{\partial \dot{r}} = m\,\left(\dot{r} + a\,\omega\,\sin(\theta - \omega\,t)\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\,\left(\ddot{r} - a\,\omega\,\cos(\theta - \omega\,t)\,\left(\omega - \dot{\theta}\right)\right)$$

This results in the equation of motion:

$$\ddot{r} - r\,\dot{\theta}^2 - a\,\omega^2\,\cos(\theta - \omega\,t) + \frac{k}{m}\,(r - r_0) = 0$$

Solving along θ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = m\,\left(a\,\omega\,\cos\left(\omega\,t-\theta\right)\,\dot{r}+a\,\omega\,\sin\left(\omega\,t-\theta\right)\,r\,\dot{\theta}\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} =, m\,\left(r^2\,\dot{\theta}+a\,\omega\,\cos\left(\theta-\omega\,t\right)\,r\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\,\left(r^2\,\ddot{\theta}+2\,r\,\dot{\theta}\,\dot{r}+a\,\omega\,\cos\left(\theta-\omega\,t\right)\,\dot{r}+a\,\omega\,r\,\sin\left(\theta-\omega\,t\right)\,\left(\omega-\dot{\theta}\right)\right)$$

This results in the equation of motion:

$$r\,\ddot{\theta}+2\,\dot{\theta}\,\dot{r}+a\,\omega^2\,\sin\left(\theta-\omega\,t\right)=0$$

The equations of motion for the system are:

$$\boldsymbol{\ddot{r}-r\,\dot{\theta}^2-a\,\omega^2\,\cos\left(\theta-\omega\,t\right)+\frac{k}{m}\left(r-r_0\right)=0}$$
$$\boldsymbol{r\,\ddot{\theta}+2\,\dot{\theta}\,\dot{r}+a\,\omega^2\,\sin\left(\theta-\omega\,t\right)=0}$$

$\longrightarrow$ $\mathcal{A}$nswer

The following MATLAB script was used to solve this problem:

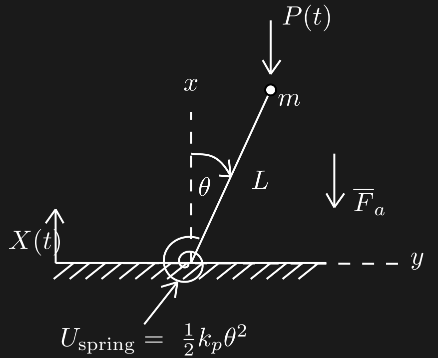
```
clc,clear
syms t m a k omega r_0
syms theta(t)
syms r(t)

% Position
x = a*cos(omega*t)+r*cos(theta);
y = a*sin(omega*t)+r*sin(theta);
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k*(r-r_0)^2;
% Lagrange Function
```

```
L = T - V;
% Coordinate derivatives
r_dot     = diff(r,t);
theta_dot = diff(theta,t);
% r coordinate derivatives
dL_dr     = diff(L,r);
dL_dr_dot = diff(L,r_dot);
dL_dr_dot_dt = diff(dL_dr_dot,t);
% theta coordinate derivatives
dL_dtheta     = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = diff(dL_dtheta_dot,t);
% Resulting equations of motion
EOM1 = simplify(dL_dr_dot_dt-dL_dr);
EOM2 = simplify(dL_dtheta_dot_dt-dL_dtheta);
```

Problem 2

Derive the equations of motion for the base-excited structure subject to an axial load $P(t)$.



For this problem, the kinetic energy of the system, T, is defined as:

$$T = \frac{1}{2}mv^2$$

Therefore:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

By inspection of the sketch:

$$x = L\cos(\theta) + X(t)$$
$$y = L\sin(\theta)$$

Taking the time derivative of both:

$$\dot{x} = X'(t) - L\,\sin(\theta)\;\dot{\theta}$$
$$\dot{y} = L\,\cos(\theta)\;\dot{\theta}$$

This gives:

$$T = \frac{1}{2}m\left(X'(t)^2 + L^2\dot{\theta}^2 - 2L\sin(\theta)X'(t)\dot{\theta}\right)$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2}k_p\theta^2 + m(-a)x$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2}m\left(X'(t)^2 + L^2\dot{\theta}^2 - 2L\sin(\theta)X'(t)\dot{\theta}\right) - \left(\frac{1}{2}k_p\theta^2 - ma\left(L\cos(\theta) + X(t)\right)\right)$$

We will solve for the equation of motion along the generalized coordinate, θ:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = Q_\theta$$

The component parts of this equation are:

$$\frac{\partial L}{\partial\theta} = -L\,m\,a\,\sin(\theta) - L\,m\,\cos(\theta)\;\dot{\theta}\,X'(t) - k_p\,\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = m\,L^2\,\dot{\theta} - m\,L\,\sin(\theta)\;X'(t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = m\,L^2\,\ddot{\theta} - m\,\sin(\theta)\,L\,X''(t) - m\,\cos(\theta)\dot{\theta}\,L\,X'(t)$$

Where $r = [x\ y]^T$, the external force is computed as:

$$
\begin{aligned}
Q_\theta &= \sum_{k=1}^{N=3} F_k \frac{\partial x_k}{\partial \theta} \\
&= \begin{bmatrix} 0 \\ -P\left(t\right) \end{bmatrix} \cdot \begin{bmatrix} L\,\cos\left(\theta\right) \\ -L\,\sin\left(\theta\right) \end{bmatrix} \\
&= L\sin(\theta(t))P(t)
\end{aligned}
$$

This results in the system equation of motion:

$$
m\,L^2\,\ddot{\theta} - m\,L\sin\left(\theta\right)\;X''\left(t\right) + k_p\,\theta + m\,a\,L\sin\left(\theta\right) = P(t)\;L\sin(\theta)
$$

Or:

$$
\boldsymbol{m\,L^2\,\ddot{\theta} + \left[a - X''\left(t\right)\right] m\,L\sin\left(\theta\right) + k_p\,\theta = P(t)\;L\sin(\theta)}
$$

$\longrightarrow \mathcal{A}$nswer

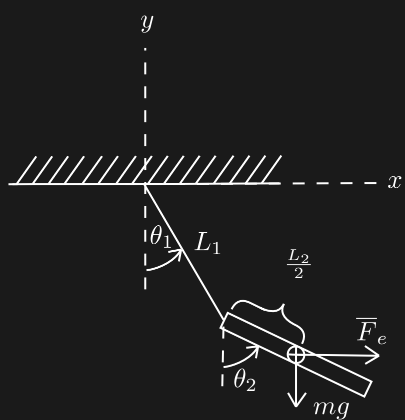
The following MATLAB script was used to solve this problem:

```
clc,clear
syms t m L k_p a X(t)
syms theta(t)
% Position
y = L*sin(theta);
x = L*cos(theta)+X;
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k_p*theta^2 + m*(-a)*x;
% Lagrange Function
L = T - V;
% Coordinate derivatives
theta_dot = diff(theta,t);
% theta coordinate derivatives
dL_dtheta     = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = simplify(diff(dL_dtheta_dot,t));
```

```
% Resulting equations of motion excluding external forces
EOM = simplify(dL_dtheta_dot_dt-dL_dtheta);
% External force:
syms P(t); assume(P(t),'real')
r = [x;y];
dr_dtheta = diff(r,theta);
F = (P*[-1;0])' * dr_dtheta; % dot product
```

Problem 3

The double pendulum. Find the equations governing a rigid bar attached to a string as shown in the figure.



a. This system starts with 9 degrees of freedom. What constraints exist that reduce this system to only two degrees of freedom?

The system is composed of two objects; a mass particle and a ridged body. The mass, m_1, particle has three translational dimensions, x_1, y_1 and z_1. The ridged body, m_2, particle has three translational dimensions, x_2, y_2 and z_2, and three rotational dimensions, R_{x_2}, R_{y_2} and R_{z_2}.

Because this is a 2-D problem, the following four constraints are inherent:

$$z_1 = z_2 = R_{x_2} = R_{y_2} = C = 0$$

Via inspection of the diagram we can see the following 3 constraints:

$$ {L_1}^2 = {x_1}^2 + {y_1}^2 $$

$$ \left(\frac{L_1}{2}\right)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 $$

$$ R_{z_2} = \theta_2 = \tan^{-1}\left(\frac{x_2 - x_1}{y_2 - y_1}\right) $$

With **9 degrees of freedom for the unconstrained system** and **7 constraints**, the system has **2 degrees of freedom**.

$\longrightarrow$ *Answer*

b. Show that the Lagrangian is:

$$ \boldsymbol{L = \frac{1}{2}m\left[L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_2 - \theta_1\right) + \frac{1}{3}L_2^2\dot{\theta}_2^2\right] - mg\left[L_1\left(1 - \cos\theta_1\right) + \frac{L_2}{2}\left(1 - \cos\theta_2\right)\right]} $$

For this problem, the kinetic energy of the system, T, is defined as:

$$ T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}_2^2 $$

Therefore:

$$ T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\left(\frac{1}{12}\right)L_2^2\dot{\theta}_2^2 $$

By inspection of the sketch:

$$ \begin{aligned} x &= L_1\sin(\theta_1) + \frac{L_2}{2}\sin(\theta_2) \\ y &= \left(L_1 + \frac{L_2}{2}\right) - L_1\cos(\theta_1) - \frac{L_2}{2}\cos(\theta_2) \end{aligned} $$

Note that these equations define $(x, y)_0$ at the location of the c.g. of the bar when $\theta_1 = \theta_2 = 0$.

Taking the time derivative of both:

$$
\begin{aligned}
\dot{x} &= L_1 \cos(\theta_1)\, \dot{\theta}_1 + \frac{1}{2} L_2 \cos(\theta_2)\, \dot{\theta}_2 \\
\dot{y} &= L_1 \sin(\theta_1)\, \dot{\theta}_1 + \frac{1}{2} L_2 \sin(\theta_2)\, \dot{\theta}_2
\end{aligned}
$$

This gives:

$$
T = \frac{1}{2} m \left[{L_1}^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{1}{3} {L_2}^2 \dot{\theta}_2^2 \right]
$$

The potential energy of the system, V is defined by:

$$
V = mgh
$$

Therefore:

$$
\begin{aligned}
V &= mg \left(L_1 + \frac{1}{2} L_2 - L_1 \cos(\theta_1) - \frac{1}{2} L_2 \cos(\theta_2) \right) \\
&= mg \left[L_1 \, \big(1 - \cos(\theta_1)\big) + \frac{1}{2} L_2 \, \big(1 - \cos(\theta_2)\big) \right]
\end{aligned}
$$

Given that the Lagrange is $L = T - V$:

$$
\begin{aligned}
\boldsymbol{L} = \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{m} & \left[\boldsymbol{L_1}^{\mathbf{2}} \dot{\boldsymbol{\theta}}_{\mathbf{1}}^{\mathbf{2}} + \boldsymbol{L_1} \boldsymbol{L_2} \dot{\boldsymbol{\theta}}_{\mathbf{2}} \dot{\boldsymbol{\theta}}_{\mathbf{1}} \cos(\boldsymbol{\theta_2} - \boldsymbol{\theta_1}) + \frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{L_2}^{\mathbf{2}} \dot{\boldsymbol{\theta}}_{\mathbf{2}}^{\mathbf{2}} \right] \\
& - \boldsymbol{mg} \left[\boldsymbol{L_1} \, \big(\mathbf{1} - \cos(\boldsymbol{\theta_1})\big) + \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{L_2} \, \big(\mathbf{1} - \cos(\boldsymbol{\theta_2})\big) \right]
\end{aligned}
$$

$\longrightarrow \mathcal{A}$nswer

This matches the Lagrangian in the question prompt.

c. Use Lagrange's equations to derive the equations of motion (don't forget about the external force!)

We will solve for the equations of motion along the two generalized coordinates, θ_1 and θ_2. Starting with θ_1, we will solve:

$$
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = Q_{\theta_1}
$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta_1} = \frac{1}{2} L_1\, L_2\, m\, \sin\left(\theta_2 - \theta_1\right)\, \dot{\theta}_2\, \dot{\theta}_1 - L_1\, g\, m\, \sin\left(\theta_1\right)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{2} m \left(2\, {L_1}^2\, \dot{\theta}_1 + L_2\, \cos\left(\theta_1 - \theta_2\right)\, L_1\, \dot{\theta}_2\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{2} m \left(2\, {L_1}^2\, \ddot{\theta}_1 + L_1\, L_2\, \cos\left(\theta_1 - \theta_2\right)\, \ddot{\theta}_2 - L_1\, L_2\, \sin\left(\theta_1 - \theta_2\right)\, \left(\dot{\theta}_1 - \dot{\theta}_2\right) \dot{\theta}_2\right)$$

Where $r = [x\ y]^T$, solving for the external force:

$$\begin{aligned} Q_{\theta_1} &= \sum_{k=1}^{N=3} F_{e_k} \frac{\partial x_k}{\partial \theta_1} \\ &= \begin{bmatrix} F_e \\ 0 \end{bmatrix} \cdot \begin{bmatrix} L_1\, \cos\left(\theta_1\right) \\ L_1\, \sin\left(\theta_1\right) \end{bmatrix} \\ &= F_e\, L_1\, \cos\left(\theta_1\right) \end{aligned}$$

This results in the equation of motion:

$$m\, {L_1}^2\, \ddot{\theta}_1 + \frac{1}{2} L_2\, m\, \cos\left(\theta_1 - \theta_2\right)\, L_1\, \ddot{\theta}_2 + \frac{1}{2} L_2\, m\, \sin\left(\theta_1 - \theta_2\right)\, L_1\, {\dot{\theta}_2}^2 + g\, m\, \sin\left(\theta_1\right)\, L_1 = F_e\, L_1\, \cos\left(\theta_1\right)$$

Solving along θ_2, we will solve:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = Q_{\theta_2}$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta_2} = \frac{1}{2} L_1\, L_2\, m\, \sin\left(\theta_1 - \theta_2\right)\, \dot{\theta}_2\, \dot{\theta}_1 - \frac{1}{2} L_2\, g\, m\, \sin\left(\theta_2\right)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2} m \left(\frac{2}{3} {L_2}^2\, \dot{\theta}_2 + L_1\, \cos\left(\theta_1 - \theta_2\right)\, L_2\, \dot{\theta}_1\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2} m \left(\frac{2}{3} {L_2}^2\, \ddot{\theta}_2 + L_1\, L_2\, \cos\left(\theta_1 - \theta_2\right)\, \ddot{\theta}_1 - L_1\, L_2\, \sin\left(\theta_1 - \theta_2\right)\, \left(\dot{\theta}_1 - \dot{\theta}_2\right) \dot{\theta}_1\right)$$

Solving for the external force:

$$\begin{aligned}
Q_{\theta_2} &= \sum_{k=1}^{N=3} F_{e_k} \frac{\partial x_k}{\partial \theta_2} \\
&= \begin{bmatrix} F_e \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} L_2 \cos(\theta_2) \\ \frac{1}{2} L_2 \sin(\theta_2) \end{bmatrix} \\
&= \frac{1}{2} F_e L_2 \cos(\theta_2)
\end{aligned}$$

This results in the equation of motion:

$$\begin{aligned}
\frac{1}{3} m {L_2}^2 \ddot{\theta}_2 + \frac{1}{2} L_1 m \cos(\theta_1 - \theta_2)\, L_2 \ddot{\theta}_1 - \frac{1}{2} L_1 m \sin(\theta_1 - \theta_2)\, L_2 {\dot{\theta}_1}^2 + \frac{1}{2} g m \sin(\theta_2)\, L_2 \\
= \frac{1}{2} F_e L_2 \cos(\theta_2)
\end{aligned}$$

All together, the equations of motion for the system are:

$$\boldsymbol{\begin{aligned}
m {L_1}^2 \ddot{\theta}_1 + \frac{1}{2} L_2 m \cos(\theta_1 - \theta_2)\, L_1 \ddot{\theta}_2 + \frac{1}{2} L_2 m \sin(\theta_1 - \theta_2)\, L_1 {\dot{\theta}_2}^2 + g m \sin(\theta_1)\, L_1 \\
= F_e L_1 \cos(\theta_1) \\
\frac{2}{3} m {L_2}^2 \ddot{\theta}_2 + L_1 m \cos(\theta_1 - \theta_2)\, L_2 \ddot{\theta}_1 - L_1 m \sin(\theta_1 - \theta_2)\, L_2 {\dot{\theta}_1}^2 + g m \sin(\theta_2)\, L_2 \\
= F_e L_2 \cos(\theta_2)
\end{aligned}}$$

$\longrightarrow \mathcal{A}$nswer

The following MATLAB script was used to solve this problem:

```
clc,clear
syms t m g
syms L_1 L_2 theta_1(t) theta_2(t)
% Position
x = L_1*sin(theta_1)+ L_2/2*sin(theta_2);
y = (L_1 + L_2/2) - L_1*cos(theta_1)- L_2/2*cos(theta_2);
% Potential Energy
V = m*g*(y);
```

```matlab
V = collect(V,[L_1,L_2]);
% Velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)) +...
    (1/12)*L_2^2*diff(theta_2,t)^2);
% Lagrange Function
L = T-V;
% Coordinate derivatives
theta_1_dot = diff(theta_1,t);
theta_2_dot = diff(theta_2,t);
% theta_1 coordinate derivatives
dL_dtheta_1     = diff(L,theta_1);
dL_dtheta_1_dot = diff(L,theta_1_dot);
dL_dtheta_1_dot_dt = simplify(diff(dL_dtheta_1_dot,t));
% theta_2 coordinate derivatives
dL_dtheta_2     = diff(L,theta_2);
dL_dtheta_2_dot = diff(L,theta_2_dot);
dL_dtheta_2_dot_dt = simplify(diff(dL_dtheta_2_dot,t));
% Resulting equations of motion excluding external forces
EOM_1 = simplify(dL_dtheta_1_dot_dt-dL_dtheta_1);
EOM_2 = simplify(dL_dtheta_2_dot_dt-dL_dtheta_2);
% Compute external forces:
syms F_e; assume(F_e,'real')
r = [x;y];
% Partial With respect to theta_1
dr_dtheta_1 = diff(r,theta_1);
Q_theta_1 = (F_e*[1;0])' * dr_dtheta_1; % dot product
% Partial With respect to theta_2
dr_dtheta_2 = diff(r,theta_2);
Q_theta_2 = (F_e*[1;0])' * dr_dtheta_2; % dot product
```

Submitted by Austin Barrilleaux on October 29, 2024.