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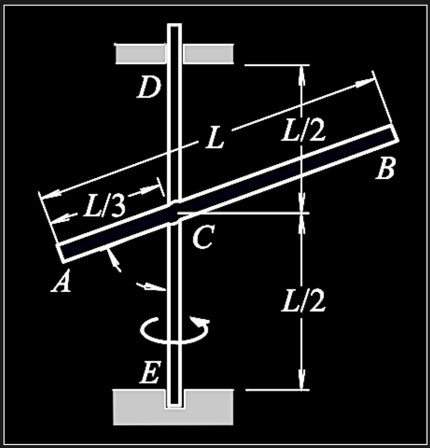
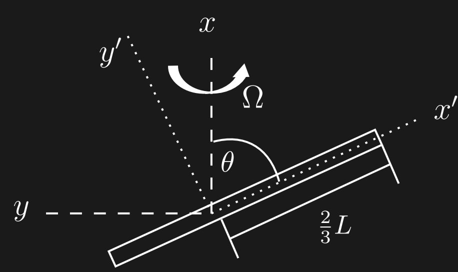
MODULE 6 — Assignment

EXERCISE 5.26

Thin bar ACB is welded to a shaft that rotates at the constant angular speed Ω, so the angle θ between the bar and the shaft is constant.
(a) Derive expressions for the angular momentum $\bar{H}_C$ and the kinetic energy of the bar. Draw a sketch of $\bar{H}_C$.
(b) Based on an analysis of the manner in which $\bar{H}_C$ in Part (a) rotates, derive an expression for $\frac{\partial}{\partial t}\bar{H}_C$.
(c) Use Eq. (5.3.4) to evaluate $\frac{\partial}{\partial t}\bar{H}_C$, and compare it with the result of Part (b).



The following sketch shows the two frames of the system:



The rotation matrix that converts the $\{xyz\}$ frame to the $\{x'y'z'\}$ frame is:

$$R = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Constructing the angular velocity vector:

$$\bar{\omega} = \Omega\ i$$

In the body frame is:

$$\bar{\omega} = R\ \Omega\ i = \begin{bmatrix} \Omega\ \cos(\theta)\ \ i' \\ \Omega\ \sin(\theta)\ \ j' \\ 0\ k' \end{bmatrix}$$

From textbook Appendix, the centroidal inertia mass properties of the shaft are:

$$I_{xx} = 0$$
$$I_{yy} = \frac{1}{12} m\, L^2$$
$$I_{zz} = \frac{1}{12} m\, L^2$$

This expressed as the inertia tensor is:

$$I_{x'y'z'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m\, L^2 & 0 \\ 0 & 0 & \frac{1}{12} m\, L^2 \end{bmatrix}$$

The distance from the body frame is:

$$d_{x'y'z'} = \begin{bmatrix} \frac{1}{6} L & 0 & 0 \end{bmatrix}$$

Using the parallel axis theorem to get the parallel axis transformation of inertia matrix relative to the center of the frame of reference in question:

$$I_{pat} = m \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{36}\,L^2 & 0 \\ 0 & 0 & \frac{1}{36}\,L^2 \end{bmatrix}$$

This makes the inertia tensor at point C:

$$I_C = I_{x'y'z'} + I_{pat} m \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{9}\,L^2 & 0 \\ 0 & 0 & \frac{1}{9}\,L^2 \end{pmatrix}$$

From this we can compute the angular momentum as:

$$\boldsymbol{\bar{H}_C = I_C\ \bar{\omega} = \frac{1}{9} m\, L^2\, \Omega, \sin\left(\theta\right)\ \ j'}$$

$\longrightarrow$ $\mathcal{A}$nswer

Since the bar rotates about the x axis, the rotation component in the terminal frame that is orthogonal to the j'-axis is $\Omega\cos(\theta)\ i'$, therefore:

$$\boldsymbol{\dot{\bar{H}}_C = \frac{1}{9} m\, L^2\, \Omega^2 \cos\left(\theta\right) \sin\left(\theta\right)\ \ k'}$$

$\longrightarrow$ $\mathcal{A}$nswer

If we use Eq. (5.3.4) to evaluate this, we get the same answer:

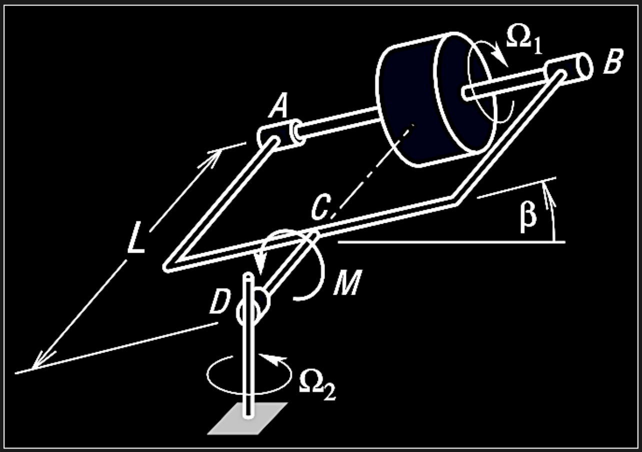
$$\dot{\bar{H}}_C = \cancelto{0}{\frac{\partial}{\partial t}\bar{H}_C} + \bar{\omega} \times \bar{H}_C$$

$$\begin{aligned}\boldsymbol{\dot{\bar{H}}_C} &= \begin{bmatrix} \boldsymbol{0\ i'} \\ \boldsymbol{0\ j'} \\ \boldsymbol{\frac{1}{9} m\, L^2\, \Omega^2 \cos\left(\theta\right)\ \sin\left(\theta\right)\ \ k'} \end{bmatrix} \\ &= \boldsymbol{\frac{1}{9} m\, L^2\, \Omega^2 \cos\left(\theta\right)\ \sin\left(\theta\right)\ \ k'}\end{aligned}$$

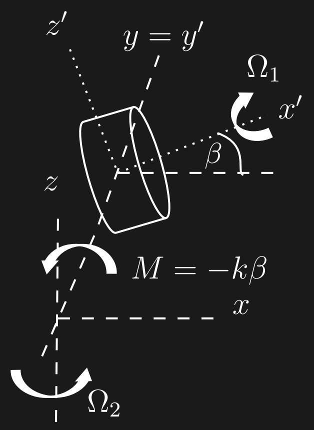
$\longrightarrow$ $\mathcal{A}$nswer

EXERCISE 6.8

The torque M acting on the gimbal of the gyroscopic turn indicator is exerted by a torsional spring, so $M = -k\beta$. The precession rate Ω_2 is a specified function of time, and the spin rate Ω_1 is held constant by a servomotor. Let I_1 denote the moment of inertia of the flywheel about axis AB, and let I_2 be the centroidal moment of inertia perpendicular to axis AB. Derive the differential equation of motion for β.



We will define the following frames for the system:



Construct the angular velocity vector $\bar{\omega}$ of xyz by vectorially adding the simple rotation rates according to:

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3$$

This gives us:

$$\bar{\omega} = \Omega_2 \ k + \dot{\beta}(-j') + \Omega_1(-i')$$

We can equate k, given the following rotation as:

$$R(-\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$k = R(-\beta) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin(\beta) \ \ i' \\ 0 \ j \\ \cos(\beta) \ \ k' \end{bmatrix}$$

This allows us to express the angular velocity vector as:

$$\bar{\omega} = \Omega_2 \ \left(\sin(\beta) \ i' + \cos(\beta) \ k'\right) - \dot{\beta} \ j' - \Omega_1 \ i'$$

Or:

$$\bar{\omega} = \begin{bmatrix} \left\{\sin(\beta) \ \Omega_2 - \Omega_1\right\} \ i' \\ \left\{-\dot{\beta}\right\} \ j' \\ \left\{\cos(\beta) \ \Omega_2\right\} \ k' \end{bmatrix}$$

Next solve for angular acceleration:

$$\bar{\alpha} = \sum_n \left(\dot{\omega}_n \bar{e}_n + \bar{\Omega}_n \times \omega_n \bar{e}_n\right)$$

Where:

$$\begin{aligned} \omega_{n=1} &= -\Omega_1 \ i', & \dot{\omega}_{n=1} &= 0, & \Omega_{n=1} &= \Omega_2 R \ k - \dot{\beta} \ j' \\ \omega_{n=2} &= -\dot{\beta} \ j', & \dot{\omega}_{n=2} &= -\ddot{\beta} \ j', & \Omega_{n=2} &= \Omega_2 R \ k \\ \omega_{n=3} &= \Omega_2 R \ k, & \dot{\omega}_{n=3} &= \dot{\Omega}_2 R \ k, & \Omega_{n=3} &= 0 \end{aligned}$$

Substituting these into $\bar{\alpha}$:

$$\bar{\alpha} = \begin{bmatrix} \left\{\dot{\Omega}_2 \sin(\beta) +, \Omega_2\, \dot{\beta} \cos(\beta)\right\} i' \\ \left\{-\ddot{\beta} - \Omega_1\, \Omega_2\, \cos(\beta)\right\} j' \\ \left\{\dot{\Omega}_2 \cos(\beta) - \Omega_2\, \dot{\beta} \sin(\beta) - \Omega_1\, \dot{\beta}\right\} k' \end{bmatrix}$$

Since the torque M is acting in the j' direction, we can refer to equation 6.1.6 in the textbook:

$$\sum \bar{M}_{cg}\ j' = k\beta = I_{yy}\alpha_y + (I_{zz} - I_{xx})\,\omega_x\omega_z$$

Where the rotation of the frame, which is not rigidly fixed to the flywheel, is:

$$\omega = \Omega_2\ k + \dot{\beta}(-j') = \begin{bmatrix} \Omega_2 \sin(\beta)\ \ i' \\ -\dot{\beta}\ j' \\ \Omega_2 \cos(\beta)\, k' \end{bmatrix}$$

And:

$$I_{xx} = I_1$$
$$I_{yy} = I_{zz} = I_2$$

This evaluates to:

$$\boldsymbol{k\beta = \cos(\beta)\ \sin(\beta)\ \Omega_2{}^2\ (I_1 - I_2) - I_2\ddot{\beta} - I_2\Omega_1\ \cos(\beta)\ \Omega_2}$$

$\longrightarrow$ $\mathcal{A}$nswer

Attached

The MATLAB scripts used to solve both problems are included in the submission.

Submitted by Austin Barrilleaux on October 8, 2024.