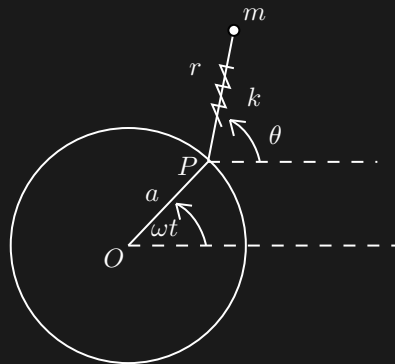


October 28, 2024

MODULE 9 — Assignment

Problem 1

Derive the equations of motion for a rotating spring-pendulum shown below. The spring-pendulum is attached at point P .



For this problem, the kinetic energy of the system, T , is defined as:

$$T = \frac{1}{2}mv^2$$

Therefore:

$$T = \frac{1}{2}m (\dot{x}^2 + \dot{y}^2)$$

By inspection of the sketch:

$$\begin{aligned}x &= a \cos(\omega t) + r \cos(\theta) \\y &= a \sin(\omega t) + r \sin(\theta)\end{aligned}$$

Taking the time derivative of both:

$$\begin{aligned}\dot{x} &= \dot{r} \cos(\theta) - r \dot{\theta} \sin(\theta) - a \omega \sin(\omega t) \\ \dot{y} &= \dot{r} \sin(\theta) + r \dot{\theta} \cos(\theta) + a \omega \cos(\omega t)\end{aligned}$$

This gives:

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + a^2 \omega^2 + 2 a \omega \sin(\theta - \omega t) \dot{r} + 2 a \omega r \cos(\theta - \omega t) \dot{\theta} \right)$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2} k (r - r_0)^2$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + a^2 \omega^2 + 2 a \omega \sin(\theta - \omega t) \dot{r} + 2 a \omega r \cos(\theta - \omega t) \dot{\theta} \right) - \frac{1}{2} k (r - r_0)^2$$

We will solve for the equations of motion along the two generalized coordinates, θ and r . Starting with r , we will solve:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial r} = m \left(r \dot{\theta}^2 + a \omega \cos(\theta - \omega t) \dot{\theta} \right) - k (r - r_0)$$

$$\frac{\partial L}{\partial \dot{r}} = m (\dot{r} + a \omega \sin(\theta - \omega t))$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m \left(\ddot{r} - a \omega \cos(\theta - \omega t) (\omega - \dot{\theta}) \right)$$

This results in the equation of motion:

$$\ddot{r} - r \dot{\theta}^2 - a \omega^2 \cos(\theta - \omega t) + \frac{k}{m} (r - r_0) = 0$$

Solving along θ

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = m \left(a \omega \cos(\omega t - \theta) \dot{r} + a \omega \sin(\omega t - \theta) r \dot{\theta} \right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m \left(r^2 \dot{\theta} + a \omega \cos(\theta - \omega t) r \right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m \left(r^2 \ddot{\theta} + 2 r \dot{\theta} \dot{r} + a \omega \cos(\theta - \omega t) \dot{r} + a \omega r \sin(\theta - \omega t) (\omega - \dot{\theta}) \right)$$

This results in the equation of motion:

$$r \ddot{\theta} + 2 \dot{\theta} \dot{r} + a \omega^2 \sin(\theta - \omega t) = 0$$

The equations of motion for the system are:

$$\begin{aligned} \ddot{r} - r \dot{\theta}^2 - a \omega^2 \cos(\theta - \omega t) + \frac{k}{m} (r - r_0) &= 0 \\ r \ddot{\theta} + 2 \dot{\theta} \dot{r} + a \omega^2 \sin(\theta - \omega t) &= 0 \end{aligned}$$

→ Answer

The following MATLAB script was used to solve this problem:

```
clc,clear
syms t m a k omega r_0
syms theta(t)
syms r(t)

% Position
x = a*cos(omega*t)+r*cos(theta);
y = a*sin(omega*t)+r*sin(theta);
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k*(r-r_0)^2;
% Lagrange Function
```

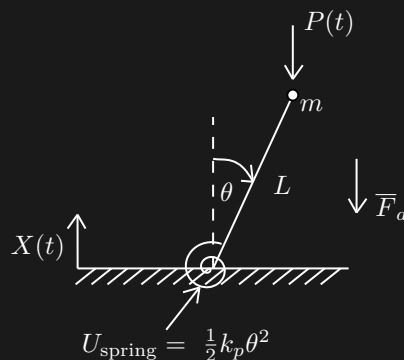
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L = T - V;
% Coordinate derivatives
r_dot = diff(r,t);
theta_dot = diff(theta,t);
% r coordinate derivatives
dL_dr = diff(L,r);
dL_dr_dot = diff(L,r_dot);
dL_dr_dot_dt = diff(dL_dr_dot,t);
% theta coordinate derivatives
dL_dtheta = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = diff(dL_dtheta_dot,t);
% Resulting equations of motion
EOM1 = simplify(dL_dr_dot_dt-dL_dr);
EOM2 = simplify(dL_dtheta_dot_dt-dL_dtheta);

```

Problem 2

Derive the equations of motion for the base-excited structure subject to an axial load $P(t)$.



For this problem, the kinetic energy of the system, T , is defined as:

$$T = \frac{1}{2} m v^2$$

Therefore:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

By inspection of the sketch:

$$\begin{aligned}x &= L \cos(\theta) + X(t) \\y &= L \sin(\theta)\end{aligned}$$

Taking the time derivative of both:

$$\begin{aligned}\dot{x} &= X'(t) - L \sin(\theta) \dot{\theta} \\ \dot{y} &= L \cos(\theta) \dot{\theta}\end{aligned}$$

This gives:

$$T = \frac{1}{2}m \left(X'(t)^2 + L^2 \dot{\theta}^2 - 2L \sin(\theta) X'(t) \dot{\theta} \right)$$

The potential energy of the system, V is defined by:

$$V = \frac{1}{2}k_p \theta^2 + m(-a)x$$

Given that the Lagrange is $L = T - V$:

$$L = \frac{1}{2}m \left(X'(t)^2 + L^2 \dot{\theta}^2 - 2L \sin(\theta) X'(t) \dot{\theta} \right) - \left(\frac{1}{2}k_p \theta^2 - ma (L \cos(\theta) + X(t)) \right)$$

We will solve for the equation of motion along the generalized coordinate, θ :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta} = -L m a \sin(\theta) - L m \cos(\theta) \dot{\theta} X'(t) - k_p \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \left(2 L^2 \dot{\theta} - 2 L \sin(\theta) X'(t) \right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \left(-2 L^2 \ddot{\theta} + 2 \sin(\theta) L X''(t) + 2 \cos(\theta) \dot{\theta} L X'(t) \right)$$

$$\begin{aligned}
Q_\theta &= \sum_{k=1}^{N=3} F_k \frac{\partial x_k}{\partial \theta} \\
&= \begin{bmatrix} 0 \\ -P(t) \end{bmatrix} \cdot \begin{bmatrix} L \cos(\theta) \\ -L \sin(\theta) \end{bmatrix} \\
&= L \sin(\theta(t)) P(t)
\end{aligned}$$

This results in the system equation of motion:

$$m L^2 \ddot{\theta} - m L \sin(\theta) X''(t) + k_p \theta + m a L \sin(\theta) = P(t) L \sin(\theta)$$

Or:

$$m L^2 \ddot{\theta} + k_p \theta + (a - X''(t)) m L \sin(\theta) = P(t) L \sin(\theta)$$

→ Answer

The following MATLAB script was used to solve this problem:

```

clc,clear
syms t m a k omega r_0
syms theta(t)
syms r(t)

% Position
x = a*cos(omega*t)+r*cos(theta);
y = a*sin(omega*t)+r*sin(theta);
% velocity
x_dot = diff(x,t);
y_dot = diff(y,t);
% Kinetic Energy
T = 0.5*m*(simplify(expand(x_dot^2 + y_dot^2)));
% Kinetic energy from spring
V = 0.5*k*(r-r_0)^2;
% Lagrange Function
L = T - V;
% Coordinate derivatives
r_dot = diff(r,t);
theta_dot = diff(theta,t);
% r coordinate derivatives

```

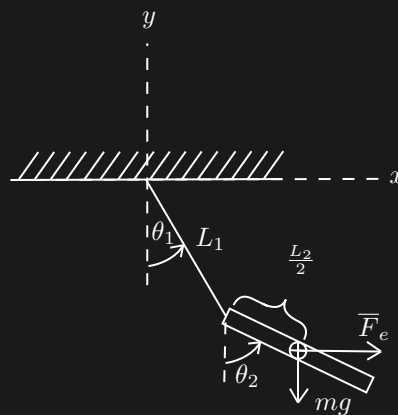
```

dL_dr      = diff(L,r);
dL_dr_dot  = diff(L,r_dot);
dL_dr_dot_dt = diff(dL_dr_dot,t);
% theta coordinate derivatives
dL_dtheta   = diff(L,theta);
dL_dtheta_dot = diff(L,theta_dot);
dL_dtheta_dot_dt = diff(dL_dtheta_dot,t);
% Resulting equations of motion
EOM1 = simplify(dL_dr_dot_dt-dL_dr);
EOM2 = simplify(dL_dtheta_dot_dt-dL_dtheta);

```

Problem 3

The double pendulum. Find the equations governing a rigid bar attached to a string as shown in the figure.



a. This system starts with 9 degrees of freedom. What constraints exist that reduce this system to only two degrees of freedom?

The system is composed of two objects; a mass particle and a ridged body. The mass, m_1 , particle has three translational dimensions, x_1 , y_1 and z_1 . The ridged body, m_2 , particle has three translational dimensions, x_2 , y_2 and z_2 , and three rotational dimensions, R_{x_2} , R_{y_2} and R_{z_2} .

Because this is a 2-D problem, the following four constraints are inherent:

$$z_1 = z_2 = R_{x_2} = R_{y_2} = C = 0$$

Via inspection of the diagram we can see the following 3 constraints:

$$L_1^2 = x_1^2 + y_1^2$$

$$\left(\frac{L_1}{2}\right)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$R_{z_2} = \theta_2 = \tan^{-1} \left(\frac{x_2 - x_1}{y_2 - y_1} \right)$$

With **9 degrees of freedom for the unconstrained system** and **7 constraints**, the system has **2 degrees of freedom**.

→ Answer

b. Show that the Lagrangian is:

$$L = \frac{1}{2}m \left[L_1^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \frac{1}{3} L_2^2 \dot{\theta}_2^2 \right] - mg \left[L_1 (1 - \cos \theta_1) + \frac{L_2}{2} (1 - \cos \theta_2) \right]$$

For this problem, the kinetic energy of the system, T , is defined as:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}_2^2$$

Therefore:

$$T = \frac{1}{2}m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \left(\frac{1}{12} \right) L_2^2 \dot{\theta}_2^2$$

By inspection of the sketch:

$$\begin{aligned} x &= L_1 \sin(\theta_1) + \frac{L_2}{2} \sin(\theta_2) \\ y &= \left(L_1 + \frac{L_2}{2} \right) - L_1 \cos(\theta_1) - \frac{L_2}{2} \cos(\theta_2) \end{aligned}$$

Note that these equations define $(x, y)_0$ at the location of the c.g. of the bar when $\theta_1 = \theta_2 = 0$.

Taking the time derivative of both:

$$\begin{aligned}\dot{x} &= L_1 \cos(\theta_1) \dot{\theta}_1 + \frac{1}{2} L_2 \cos(\theta_2) \dot{\theta}_2 \\ \dot{y} &= L_1 \sin(\theta_1) \dot{\theta}_1 + \frac{1}{2} L_2 \sin(\theta_2) \dot{\theta}_2\end{aligned}$$

This gives:

$$T = \frac{1}{2} m \left[L_1^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{1}{3} L_2^2 \dot{\theta}_2^2 \right]$$

The potential energy of the system, V is defined by:

$$V = mgh$$

Therefore:

$$\begin{aligned}V &= mg \left(L_1 + \frac{1}{2} L_2 - L_1 \cos(\theta_1) - \frac{1}{2} L_2 \cos(\theta_2) \right) \\ &= mg \left[L_1 (1 - \cos(\theta_1)) + \frac{1}{2} L_2 (1 - \cos(\theta_2)) \right]\end{aligned}$$

Given that the Lagrange is $L = T - V$:

$$\begin{aligned}L &= \frac{1}{2} m \left[L_1^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{1}{3} L_2^2 \dot{\theta}_2^2 \right] \\ &\quad - mg \left[L_1 (1 - \cos(\theta_1)) + \frac{1}{2} L_2 (1 - \cos(\theta_2)) \right]\end{aligned}$$

→ Answer

This matches the Lagrangian in the question prompt.

c. Use Lagrange's equations to derive the equations of motion (don't forget about the external force!)

Looking at the external forces:

Submitted by Austin Barrilleaux on October 28, 2024.