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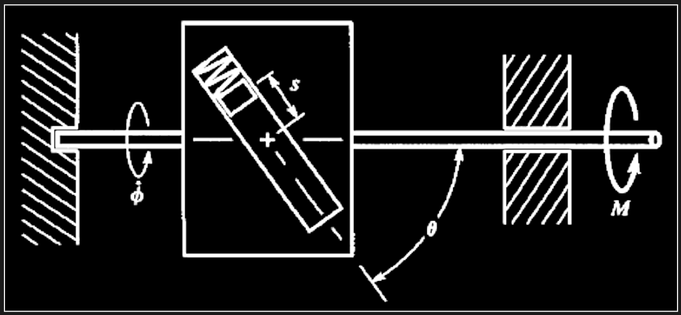
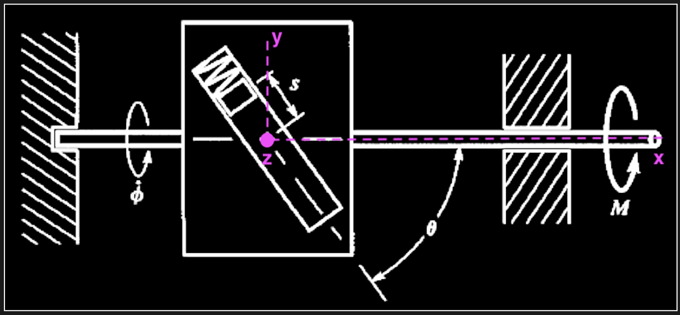
MODULE 10 — Assignment

Problem 1: Multibody System

The slider, whose mass is m_1 oscillates within the groove in the housing. The moment of inertia of the housing about the axis of rotation is I. The spring restraining the slider is unstretched when $s = 0$. Derive differential equations for the distance s and spin angle ϕ resulting from application of a torque $M(t)$ to the shaft.



The following coordinate frame will be used to solve this problem:



Based on the above coordinate frame, the position of the mass is defined by:

$$p_{m_1} = \begin{pmatrix} -\cos(\theta)\ s \\ \cos(\phi)\ s\ \sin(\theta) \\ \sin(\phi)\ s\ \sin(\theta) \end{pmatrix}$$

The velocity of m_1 is:

$$v_{m_1} = \begin{pmatrix} -\cos(\theta)\ \dot{s} \\ \cos(\phi)\ \sin(\theta)\ \dot{s} - \sin(\phi)\ s\ \sin(\theta)\ \dot{\phi} \\ \sin(\phi)\ \sin(\theta)\ \dot{s} + \cos(\phi)\ s\ \sin(\theta)\ \dot{\phi} \end{pmatrix}$$

The kinetic energy of the system is:

$$T = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m v_{m_1}^2$$

Or:

$$T = \frac{1}{2} m \left(s^2 \sin(\theta)^2\, \dot{\phi}^2 + \dot{s}^2 \right) + \frac{1}{2} I\, \dot{\phi}^2$$

The potential energy is:

$$V = \frac{1}{2} k s^2 + mg\, s\, \sin(\theta)\cos(\phi)$$

So the Lagrangian is:

$$L = \frac{1}{2} m \left(s^2 \sin(\theta)^2\, \dot{\phi}^2 + \dot{s}^2 \right) + \frac{1}{2} I\, \dot{\phi}^2 - \frac{1}{2} k s^2 - mg\, s\, \sin(\theta)\cos(\phi)$$

Solving for the equations of motion along the generalized coordinate, s:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0$$

The component parts of this equation are:

$$\frac{\partial L}{\partial s} = m\, s \sin(\theta)^2 \left(\dot{\phi}\right)^2 - m\, g\, \cos(\phi)\, \sin(\theta) - k\, s$$

$$\frac{\partial L}{\partial \dot{s}} = m\, \dot{s}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = m\, \ddot{s}$$

This results in the equation of motion:

$$m\,\ddot{s} - m\,s\sin\left(\theta\right)^2\,\dot{\phi}^2 + m\,g\,\cos\left(\phi\right)\,\sin\left(\theta\right) + k\,s = 0$$

Solving for the equations of motion along the generalized coordinate, ϕ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = M_i\frac{\partial \omega_i}{\partial \dot{\phi}} = M$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \phi} = m\,g\,\sin\left(\phi\right)\,s\,\sin\left(\theta\right)$$

$$\frac{\partial L}{\partial \dot{\phi}} = m\,s^2\sin\left(\theta\right)^2\,\dot{\phi} + I\,\dot{\phi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = I\,\ddot{\phi} + m\,s^2\sin\left(\theta\right)^2\,\ddot{\phi} + 2\,m\,s\sin\left(\theta\right)^2\,\dot{s}\,\dot{\phi}$$

This results in the equation of motion:

$$I\,\ddot{\phi} + m\,s^2\sin\left(\theta\right)^2\,\ddot{\phi} + 2\,m\,s\sin\left(\theta\right)^2\,\dot{s}\,\dot{\phi} - m\,g\,\sin\left(\phi\right)\,s\,\sin\left(\theta\right) = M$$

The equations of motion for the system are:

$$\boldsymbol{m\,\ddot{s} - m\,s\sin\left(\theta\right)^2\,\dot{\phi}^2 + m\,g\,\cos\left(\phi\right)\,\sin\left(\theta\right) + k\,s = 0}$$
$$\boldsymbol{I\,\ddot{\phi} + m\,s^2\sin\left(\theta\right)^2\,\ddot{\phi} + 2\,m\,s\sin\left(\theta\right)^2\,\dot{s}\,\dot{\phi} - m\,g\,\sin\left(\phi\right)\,s\,\sin\left(\theta\right) = M}$$

$\longrightarrow$ $\mathcal{A}$nswer

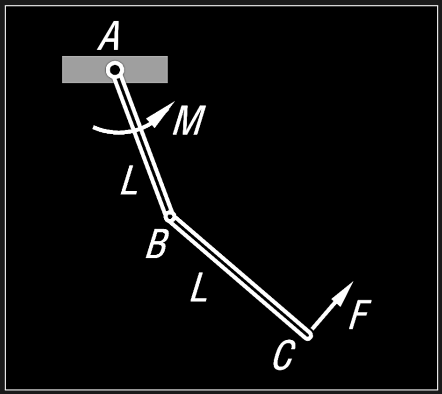
The following MATLAB function was used to solve this problem:

```
clc,clear
syms t m g theta phi(t)
syms s(t) I k
%[text] Position
p = s*[-cos(theta);
        sin(theta)*cos(phi);
        sin(theta)*sin(phi)];
%[text] Velocity
p_dot = diff(p,t) %[output:552a80dc]
```

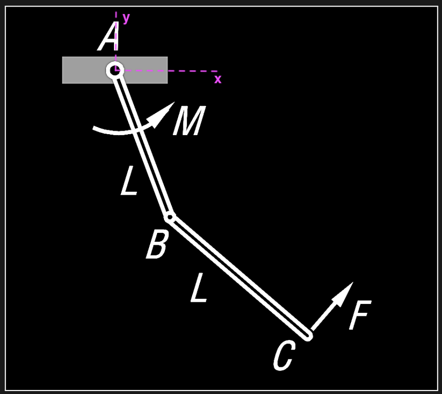
```
%[text] Kinetic Energy
T = (1/2)*I*diff(phi,t)^2 ... %[output:group:940b71c8] %[output:04ea3874]
    + (1/2)*m*simplify(... %[output:04ea3874]
       transpose(p_dot)*... %[output:04ea3874]
       p_dot,1000) %[output:group:940b71c8] %[output:04ea3874]
%[text] Potential Energy
V = (1/2)*k*s^2 + m*g*s*sin(theta)*cos(phi) %[output:3372388c]
%[text] Lagrange Function
L = T - V;
%[text] Derivative of s and phi:
s_dot = diff(s,t);
phi_dot = diff(phi,t);
%[text]  Coordinate Lagrange Function derivatives for s:
dL_ds        = diff(L,s) %[output:3c912068]
dL_ds_dot    = diff(L,s_dot) %[output:818bf61d]
dL_ds_dot_dt = diff(dL_ds_dot,t) %[output:2a416b8d]
%[text]  Coordinate Lagrange Function derivatives for phi:
dL_dphi        = diff(L,phi) %[output:2be53048]
dL_dphi_dot    = diff(L,phi_dot) %[output:4f128612]
dL_dphi_dot_dt = diff(dL_dphi_dot,t) %[output:8741427c]
%[text] Resulting equations of motion:
EOM_s   = simplify(dL_ds_dot_dt-dL_ds) %[output:2b62797e]
EOM_phi = simplify(dL_dphi_dot_dt-dL_dphi) %[output:7796cc0c]
```

Problem 2: Nonholonomic System

A known couple $M(t)$ is applied to the upper bar. Force F, which is applied perpendicularly to the lower bar, acts to make the velocity of end C always be parallel to the line from joint A to end B. The bars have equal mass m, and the system lies in the vertical plane. Use the method of Lagrange multipliers to derive the equations of motion.



The following coordinate frame will be used to solve this problem:



We will define the following positions:

$$p_{\mathrm{cg}_1} = \begin{bmatrix} \frac{1}{2}L\,\sin\left(1\theta_1\right) \\ \frac{1}{2}L\,\cos\left(\theta_1\right) \end{bmatrix}$$

$$p_b = \begin{bmatrix} L\,\sin\left(\theta_1\right) \\ L\,\cos\left(\theta_1\right) \end{bmatrix}$$

$$p_{\mathrm{cg}_2} = \begin{bmatrix} L\,\sin\left(\theta_1\right) + \frac{1}{2}L\,\sin\left(\theta_2\right) \\ L\,\cos\left(\theta_1\right) + \frac{1}{2}L\,\cos\left(\theta_2\right) \end{bmatrix}$$

$$p_c = \begin{bmatrix} L\,\sin\left(\theta_1\right) + L\,\sin\left(\theta_2\right) \\ L\,\cos\left(\theta_1\right) + L\,\cos\left(\theta_2\right) \end{bmatrix}$$

Taking their derivatives, we get the following velocities:

$$v_{\mathrm{cg}_1} = \begin{bmatrix} \frac{1}{2}L\,\cos\left(\theta_1\right)\,\dot{\theta}_1 \\ -\frac{1}{2}L\,\sin\left(\theta_1\right)\,\dot{\theta}_1 \end{bmatrix}$$

$$v_b = \begin{bmatrix} L\,\cos\left(\theta_1\right)\,\dot{\theta}_1 \\ -L\,\sin\left(\theta_1\right)\,\dot{\theta}_1 \end{bmatrix}$$

$$v_{\mathrm{cg}_2} = \begin{bmatrix} L\,\cos\left(\theta_1\right)\,\dot{\theta}_1 + \frac{1}{2}L\,\cos\left(\theta_2\right)\,\dot{\theta}_2 \\ -L\,\sin\left(\theta_1\right)\,\dot{\theta}_1 - \frac{1}{2}L\,\sin\left(\theta_2\right)\,\dot{\theta}_2 \end{bmatrix}$$

$$v_c = \begin{bmatrix} L\,\cos\left(\theta_1\right)\,\dot{\theta}_1 + L\,\cos\left(\theta_2\right)\,\dot{\theta}_2 \\ -L\,\sin\left(\theta_1\right)\,\dot{\theta}_1 - L\,\sin\left(\theta_2\right)\,\dot{\theta}_2 \end{bmatrix}$$

The constraint, which can be mathematically described as $v_c \times r_{B/A} = 0$, where $r_{B/A} = p_b$ evaluates to:

$$L^2\,\dot{\theta}_1 + L^2\,\cos\left(\theta_1 - \theta_2\right)\,\dot{\theta}_2 = 0$$

Or more simply:

$$\dot{\theta}_1 + \cos\left(\theta_1 - \theta_2\right)\,\dot{\theta}_2 = 0$$

Putting this in the form of the constraint equation:

$$\sum_{j=1}^{N} a_{ij}\dot{q}_j + b_i = a_{11}\dot{q}_1 + a_{12}\dot{q}_2 + b_1 = 0$$

We see that:

$$a_{11} = 1$$
$$a_{12} = \cos\left(\theta_1 - \theta_2\right)$$

The centriodal inertia properties of both bars are:

$$I = I_1 = I_2 = \frac{1}{12} m L^2$$

The kinetic energy of the system is:

$$T = \frac{1}{2} m v_{\mathrm{cg}_1}^2 + \frac{1}{2} I \dot{\theta}_1^{\,2} + \frac{1}{2} m v_{\mathrm{cg}_1}^2 + \frac{1}{2} I \dot{\theta}_2^{\,2}$$

This evaluates to:

$$T = \frac{2}{3}\, L^2\, m\, \dot{\theta}_1^{\,2} + \frac{1}{6} L^2\, m\, \dot{\theta}_2^{\,2} + \frac{1}{2} L^2\, m\, \cos\left(\theta_1 - \theta_2\right)\, \dot{\theta}_2\, \dot{\theta}_1$$

The potential energy of the system is:

$$V = -m\, g \left(L\, \cos\left(\theta_1\right) + \frac{1}{2} L\, \cos\left(\theta_2\right) \right) - \frac{1}{2} m\, g\, L \cos\left(\theta_1\right)$$

The system Lagrangian is:

$$\begin{aligned} L &= T - V \\ &= \frac{2}{3}\, L^2\, m\, \dot{\theta}_1^{\,2} + \frac{1}{6} L^2\, m\, \dot{\theta}_2^{\,2} + \frac{1}{2} L^2\, m\, \cos\left(\theta_1 - \theta_2\right)\, \dot{\theta}_2\, \dot{\theta}_1 \\ &\quad + m\, g \left(L\, \cos\left(\theta_1\right) + \frac{1}{2} L\, \cos\left(\theta_2\right) \right) + \frac{1}{2} m\, g\, L \cos\left(\theta_1\right) \end{aligned}$$

Solving for the equations of motion along the generalized coordinate, θ_1:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = Q_{\theta_1} + a_{11} \lambda_1$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta_1} = \frac{1}{2} L^2\, m\, \sin\left(\theta_2 - \theta_1\right)\, \dot{\theta}_2\, \dot{\theta}_1 - \frac{3}{2}\, L\, g\, m\, \sin\left(\theta_1\right)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{4}{3}\, L^2\, m\, \dot{\theta}_1 + \frac{1}{2} L^2\, m\, \cos\left(\theta_1 - \theta_2\right)\, \dot{\theta}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \frac{4}{3}\,L^2\,m\,\ddot{\theta}_1 + \frac{1}{2}L^2\,m\,\cos\left(\theta_1 - \theta_2\right)\,\ddot{\theta}_2 - \frac{1}{2}L^2\,m\,\sin\left(\theta_1 - \theta_2\right)\,\left(\dot{\theta}_1 - \dot{\theta}_2\right)\dot{\theta}_2$$

This results in the equation of motion, where $Q_{\theta_1} = M_i \frac{\partial \omega_{R_z}}{\partial \dot{\theta}_1} = M$:

$$\frac{4}{3}\,L^2\,m\,\ddot{\theta}_1 + \frac{1}{2}L^2\,m\,\sin\left(\theta_1 - \theta_2\right)\,\dot{\theta}_2{}^2 + \frac{1}{2}L^2\,m\,\cos\left(\theta_1 - \theta_2\right)\,\ddot{\theta}_2 + \frac{3}{2}\,L\,g\,m\,\sin\left(\theta_1\right) = M + \lambda_1$$

Solving for the equations of motion along the generalized coordinate, θ_2:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = Q_{\theta_2} + a_{12}\lambda_1$$

The component parts of this equation are:

$$\frac{\partial L}{\partial \theta_2} = \frac{1}{2}L^2\,m\,\sin\left(\theta_2 - \theta_1\right)\,\dot{\theta}_2\,\dot{\theta}_1 - \frac{3}{2}\,L\,g\,m\,\sin\left(\theta_1\right)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{3}L^2\,m\,\dot{\theta}_2 + \frac{1}{2}L^2\,m\,\cos\left(\theta_1 - \theta_2\right)\,\dot{\theta}_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{3}L^2\,m\,\ddot{\theta}_2 + \frac{1}{2}L^2\,m\,\cos\left(\theta_1 - \theta_2\right)\,\ddot{\theta}_1 - \frac{1}{2}L^2\,m\,\sin\left(\theta_1 - \theta_2\right)\,\left(\dot{\theta}_1 - \dot{\theta}_2\right)\dot{\theta}_1$$

This results in the equation of motion, where $Q_{\theta_1} = 0$:

$$\frac{1}{3}L^2\,m\,\ddot{\theta}_2 - \frac{1}{2}L^2\,m\,\sin\left(\theta_1 - \theta_2\right)\,\dot{\theta}_1{}^2 + \frac{1}{2}L^2\,m\,\cos\left(\theta_1 - \theta_2\right)\,\ddot{\theta}_1 + \frac{1}{2}L\,g\,m\,\sin\left(\theta_2\right) = \cos\left(\theta_1 - \theta_2\right)\lambda_1$$

The equations of motion for the system are:

$$\boldsymbol{\frac{4}{3}\,L^2\,m\,\ddot{\theta}_1 + \frac{1}{2}L^2\,m\,\sin\left(\theta_1 - \theta_2\right)\,\dot{\theta}_2{}^2 + \frac{1}{2}L^2\,m\,\cos\left(\theta_1 - \theta_2\right)\,\ddot{\theta}_2 + \frac{3}{2}\,L\,g\,m\,\sin\left(\theta_1\right) = M + \lambda_1}$$

$$\boldsymbol{\frac{1}{3}L^2\,m\,\ddot{\theta}_2 - \frac{1}{2}L^2\,m\,\sin\left(\theta_1 - \theta_2\right)\,\dot{\theta}_1{}^2 + \frac{1}{2}L^2\,m\,\cos\left(\theta_1 - \theta_2\right)\,\ddot{\theta}_1 + \frac{1}{2}L\,g\,m\,\sin\left(\theta_2\right) = \cos\left(\theta_1 - \theta_2\right)\lambda_1}$$

$\longrightarrow$ *Answer*

```
clc,clear
%[text] Variable setup:
syms t m L theta_1(t) theta_2(t)
assume(theta_1(t),'real');
assume(theta_2(t),'real');
sympref('FloatingPointOutput', false) %[output:8a093e18]
% Angle rates
theta_1_dot = diff(theta_1,t);
theta_2_dot = diff(theta_2,t);
%[text] Position:
p_cg_1 = (L/2)*[sin(theta_1);cos(theta_1)] %[output:54e5bab9]
p_b = L*[sin(theta_1);cos(theta_1)] %[output:02495b50]
p_cg_2 = (L/2)*[sin(theta_2);cos(theta_2)] + p_b %[output:493958b4]
p_c = (L)*[sin(theta_2);cos(theta_2)] + p_b %[output:76841511]
%[text] Velocity:
v_b = diff(p_b,t) %[output:4cf8b6f8]
v_c = diff(p_c,t) %[output:05c2599b]
v_cg_1 = diff(p_cg_1,t) %[output:3ac5ac17]
v_cg_2 = diff(p_cg_2,t) %[output:4de95351]
%[text] Centriodal Inertia Properties Of Bar:
I_1 = (1/12)*m*L^2;
I_2 = (1/12)*m*L^2;
%[text] Kinetic Energy:

T = (1/2)*m*(transpose(v_cg_1)*v_cg_1) +...
    (1/2)*I_1*theta_1_dot^2 + ...
    (1/2)*m*(transpose(v_cg_2)*v_cg_2) +...
    (1/2)*I_2*theta_2_dot^2;

T = simplify(T,1000) %[output:4b531595]
%[text] Potential Energy:
syms g
V = m*g*([0 -1]*p_cg_1)+m*g*([0 -1]*p_cg_2) %[output:6221af15]
%[text] Solve Lagranges Equations:
L = T-V %[output:6857e415]

% theta_1 coordinate derivatives
dL_dtheta_1     = diff(L,theta_1) %[output:0722c161]
dL_dtheta_1_dot = diff(L,theta_1_dot) %[output:9bb59a00]
dL_dtheta_1_dot_dt = simplify(diff(dL_dtheta_1_dot,t)) %[output:3b3555ba]
% theta_2 coordinate derivatives
dL_dtheta_2     = diff(L,theta_2) %[output:368c1aa0]
dL_dtheta_2_dot = diff(L,theta_2_dot) %[output:86dc97c0]
dL_dtheta_2_dot_dt = simplify(diff(dL_dtheta_2_dot,t)) %[output:27624ea1]
% Resulting equations of motion excluding external forces
```

```
EOM_1 = simplify(dL_dtheta_1_dot_dt-dL_dtheta_1) %[output:0c1ac215]
EOM_2 = simplify(dL_dtheta_2_dot_dt-dL_dtheta_2) %[output:7b04de84]
%[text] The constraint is:
r_B_A = [p_b,;0];
v_c = [v_c;0];
[0,0,1]*simplify(cross(v_c,r_B_A),1000) == 0 %[output:2d8b85d9]
```

Submitted by Austin Barrilleaux on November 4, 2024.