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MODULE 11 — Assignment

Problem 1: Solve Ginsberg 9.28

The absolute velocity of a particle may be represented by the components v_x, v_y, and v_z relative to the axes of a moving reference system xyz. Suppose that the angular velocity $\bar{\omega}$ of xyz and the velocity $\bar{v}_O$ of the origin of xyz are known as functions of time. Derive the Gibbs-Appell equations of motion relating the quasi-velocities $\dot{\gamma}_1 = v_x$, $\dot{\gamma}_2 = v_y$, and $\dot{\gamma}_3 = v_z$ to the resultant force acting on the particle.

Where:

$$\bar{\omega} = \left\langle \begin{array}{c} \omega_x \\ \omega_y \\ \omega_z \end{array} \right\rangle$$

Given that:

$$\bar{v} = \left\langle \begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right\rangle = \left\langle \begin{array}{c} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{array} \right\rangle$$

Solving for acceleration:

$$\begin{aligned} \bar{a} &= \frac{\partial \bar{v}}{\partial t} + \bar{\omega} \times \bar{v} \\ &= \frac{\partial \dot{\gamma}}{\partial t} + \bar{\omega} \times \dot{\gamma} \\ &= \left\langle \begin{array}{c} \ddot{\gamma}_1 - \dot{\gamma}_2\,\omega_z + \dot{\gamma}_3\,\omega_y \\ \ddot{\gamma}_2 + \dot{\gamma}_1\,\omega_z - \dot{\gamma}_3\,\omega_x \\ \ddot{\gamma}_3 - \dot{\gamma}_1\,\omega_y + \dot{\gamma}_2\,\omega_x \end{array} \right\rangle \end{aligned}$$

Given that the Gibbs-Appell function for a system of particles is:

$$S = \sum_p \frac{1}{2} m \bar{a}_p \cdot \bar{a}_p$$

For this single particle case:

$$\begin{aligned} S &= \frac{1}{2} m \left(\bar{a} \cdot \bar{a} \right) \\ &= \frac{1}{2} m \left[\left(\ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x \right)^2 + \left(\ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x \right)^2 + \left(\ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y \right)^2 \right] \end{aligned}$$

Where the equations of motion are calculated as:

$$\frac{\partial S}{\partial \ddot{\gamma}_j} = \Gamma_j = \Gamma_1$$

The virtual work associated with the forces applied to the particle is:

$$\delta W = \sum \bar{F} \cdot \delta \bar{x} = \sum_{j1}^{K} \Gamma_j \; \delta \gamma_j = \sum \bar{F} \cdot \left\langle \begin{matrix} \delta \gamma_1 \\ \delta \gamma_2 \\ \delta \gamma_3 \end{matrix} \right\rangle$$

The equation of motion is solved for as:

$$\frac{\partial S}{\partial \ddot{\gamma}} = m \left\langle \begin{matrix} \ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y \\ \ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x \\ \ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x \end{matrix} \right\rangle = \Gamma_1 = \left\langle \begin{matrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{matrix} \right\rangle = \left\langle \begin{matrix} F_x \\ F_y \\ F_z \end{matrix} \right\rangle$$

Or:

$$\boldsymbol{\begin{aligned} m \left(\ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y \right) &= F_x \\ m \left(\ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x \right) &= F_y \\ m \left(\ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x \right) &= F_z \end{aligned}}$$

$\longrightarrow \mathcal{A}$nswer

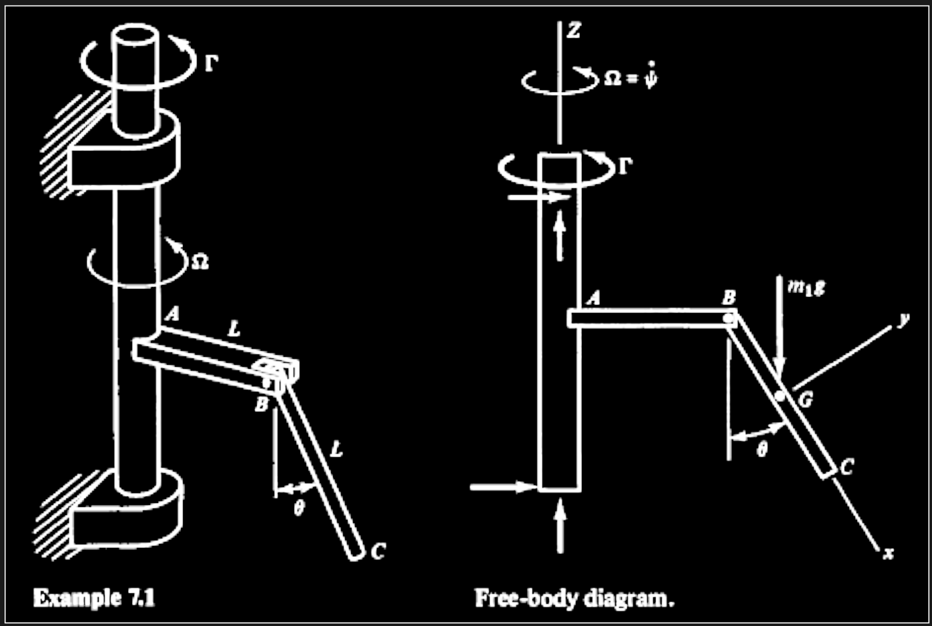
Where:

$$\dot{\gamma} = \left\langle \begin{matrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{matrix} \right\rangle = v = \left\langle \begin{matrix} v_x \\ v_y \\ v_z \end{matrix} \right\rangle = v_0 + \bar{\omega} \times \bar{r} = \left\langle \begin{matrix} v_{0_x} - \omega_z \, y + \omega_y \, z \\ v_{0_y} + \omega_z \, x - \omega_x \, z \\ v_{0_z} - \omega_y \, x + \omega_x \, y \end{matrix} \right\rangle$$

Problem 2:

Use the Gibbs-Appell approach to find the equations of motion for this problem.

A torque Γ applied to the vertical shaft of the T-bar causes the rotation rate Ω about the vertical axis to increase in proportion to the angle θ by which bar BC swings outward, that is, $\Omega = c\theta$. The mass of bar BC is m_1 and the moment of inertia of the T-bar about its axis of rotation is I_2. Determine the equations of motion for the system, and for the torque Γ.



For the purpose of this problem, as was recommended in the office hour, we will replace the y-axis of the body frame with the z-axis. This will simplify the inertial to body rotation we perform later in the solution.

First we will determine location of point G in the inertial frame. By inspection this is:

$$\bar{r}_G = \left\langle \begin{array}{c} \cos(\psi)\,\left(L + \frac{1}{2}L\,\sin(\theta)\right) \\ \sin(\psi)\,\left(L + \frac{1}{2}L\,\sin(\theta)\right) \\ -\frac{1}{2}L\,\cos(\theta) \end{array} \right\rangle$$

From this we can compute the velocity at point G:

$$\bar{v}_G = \left\langle \begin{matrix} \frac{1}{2}L\,\cos(\psi)\,\cos(\theta)\,\frac{\partial}{\partial t}\theta - \sin(\psi)\,\left(L + \frac{1}{2}L\,\sin(\theta)\right)\,\frac{\partial}{\partial t}\psi \\ \frac{1}{2}L\,\cos(\theta)\,\sin(\psi)\,\frac{\partial}{\partial t}\theta + \cos(\psi)\,\left(L + \frac{1}{2}L\,\sin(\theta)\right)\,\frac{\partial}{\partial t}\psi \\ \frac{1}{2}L\,\sin(\theta)\,\frac{\partial}{\partial t}\theta \end{matrix} \right\rangle$$

Which we can then use to compute acceleration at point G:

$$\bar{a}_G = \left\langle \begin{matrix} \frac{L\,\cos(\psi)\,\cos(\theta)\,\frac{\partial^2}{\partial t^2}\theta}{2} - \frac{L\,\cos(\psi)\,\sin(\theta)\,\left(\frac{\partial}{\partial t}\theta\right)^2}{2} - \frac{L\,\cos(\psi)\,\left(\sin(\theta)+2\right)\,\left(\frac{\partial}{\partial t}\psi\right)^2}{2} - \frac{L\,\sin(\psi)\,\left(\sin(\theta)+2\right)\,\frac{\partial^2}{\partial t^2}\psi}{2} - L\,\cos(\theta)\,\sin \\ \frac{L\,\cos(\theta)\,\sin(\psi)\,\frac{\partial^2}{\partial t^2}\theta}{2} - \frac{L\,\sin(\psi)\,\sin(\theta)\,\left(\frac{\partial}{\partial t}\theta\right)^2}{2} - \frac{L\,\sin(\psi)\,\left(\sin(\theta)+2\right)\,\left(\frac{\partial}{\partial t}\psi\right)^2}{2} + \frac{L\,\cos(\psi)\,\left(\sin(\theta)+2\right)\,\frac{\partial^2}{\partial t^2}\psi}{2} + L\,\cos(\psi)\,\cos \\ \frac{L\,\sin(\theta)\,\frac{\partial^2}{\partial t^2}\theta}{2} + \frac{L\,\cos(\theta)\,\left(\frac{\partial}{\partial t}\theta\right)^2}{2} \end{matrix}$$

Replacing $\dot{\theta}$ and $\dot{\psi}$ with the quasi-velocity terms gives us:

$$\bar{a}_G = \left\langle \begin{matrix} \frac{L\,\cos(\psi)\,\cos(\theta)\,\ddot{\gamma}_1}{2} - \frac{L\,\cos(\psi)\,\sin(\theta)\,(\dot{\gamma}_1)^2}{2} - \frac{L\,\cos(\psi)\,(\dot{\gamma}_2)^2\,\left(\sin(\theta)+2\right)}{2} - \frac{L\,\sin(\psi)\,\left(\sin(\theta)+2\right)\,\ddot{\gamma}_2}{2} - L\,\cos(\theta)\,\sin(\psi)\,\dot{\gamma}_2\,\dot{\gamma} \\ \frac{L\,\cos(\theta)\,\sin(\psi)\,\ddot{\gamma}_1}{2} - \frac{L\,\sin(\psi)\,\sin(\theta)\,(\dot{\gamma}_1)^2}{2} - \frac{L\,\sin(\psi)\,(\dot{\gamma}_2)^2\,\left(\sin(\theta)+2\right)}{2} + \frac{L\,\cos(\psi)\,\left(\sin(\theta)+2\right)\,\ddot{\gamma}_2}{2} + L\,\cos(\psi)\,\cos(\theta)\,\dot{\gamma}_2\,\dot{\gamma} \\ \frac{L\,\sin(\theta)\,\ddot{\gamma}_1}{2} + \frac{L\,\cos(\theta)\,(\dot{\gamma}_1)^2}{2} \end{matrix}$$

To convert this acceleration into the body frame, we will define the following inertial to body rotation:

$$R_{xyz}^{XYZ} = R_z(\psi)R_y(\theta) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Which simplifies to:

$$R_{xyz}^{XYZ} = \begin{bmatrix} \cos(\psi)\,\cos(\theta) & -\sin(\psi) & \cos(\psi)\,\sin(\theta) \\ \cos(\theta)\,\sin(\psi) & \cos(\psi) & \sin(\psi)\,\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

The transpose of this gives us the inertial to body transform:

$$R_{XYZ}^{xyz} = \begin{bmatrix} \cos(\psi)\,\cos(\theta) & \cos(\theta)\,\sin(\psi) & -\sin(\theta) \\ -\sin(\psi) & \cos(\psi) & 0 \\ \cos(\psi)\,\sin(\theta) & \sin(\psi)\,\sin(\theta) & \cos(\theta) \end{bmatrix}$$

We can compute the acceleration term as:

$$\bar{a}_G = R_{XYZ}^{xyz}\bar{a}_G = \left\langle \begin{array}{c} \frac{L\,\cos(2\,\theta)\,\ddot{\gamma}_1}{2} - \frac{L\,\sin(2\,\theta)\,(\dot{\gamma}_1)^2}{2} - \frac{L\,\sin(2\,\theta)\,(\dot{\gamma}_2)^2}{4} - L\,\cos(\theta)\,\left(\dot{\gamma}_2\right)^2 \\ L\,\ddot{\gamma}_2 + \frac{L\,\sin(\theta)\,\ddot{\gamma}_2}{2} + L\,\cos(\theta)\,\dot{\gamma}_2\,\dot{\gamma}_1 \\ \frac{L\left(2\cos(\theta)^2-1\right)(\dot{\gamma}_1)^2}{2} - L\,\sin(\theta)\,\left(\dot{\gamma}_2\right)^2 - \frac{L\,(\dot{\gamma}_2)^2}{4} + \frac{L\left(2\cos(\theta)^2-1\right)(\dot{\gamma}_2)^2}{4} + L\,\cos(\theta)\,\sin(\theta)\,\ddot{\gamma}_1 \end{array} \right\rangle$$

The Gibbs-Appell function for the system is given by:

$$S = \frac{1}{2}m\left(\bar{a}_G \cdot \bar{a}_G\right) + \frac{1}{2}\bar{\alpha} \cdot \frac{\partial \bar{H}_G}{\partial t} + \bar{\alpha} \cdot \left(\bar{\omega} \times \bar{H}_G\right) + \frac{1}{2}I_2\ddot{\psi}^2$$

The first term of S is computed as:

$$\frac{1}{2}m\left(\bar{a}_G \cdot \bar{a}_G\right) = \frac{1}{2}m\left[\left(L\,\ddot{\gamma}_2 + \frac{L\,\sin(\theta)\,\ddot{\gamma}_2}{2} + L\,\cos(\theta)\,\dot{\gamma}_2\,\dot{\gamma}_1\right)^2 + \left(L\,\cos(\theta)\,\left(\dot{\gamma}_2\right)^2 + \frac{L\,\sin(2\,\theta)\,\left(\dot{\gamma}_1\right)^2}{2} + \frac{L\,\mathrm{s}}{}\right.\right.$$

The angular velocity vector of the system is in the body frame is, by inspection:

$$\bar{\omega} = \left\langle \begin{array}{c} -\cos(\gamma_1)\,\dot{\gamma}_2 \\ -\dot{\gamma}_1 \\ \sin(\gamma_1)\,\dot{\gamma}_2 \end{array} \right\rangle$$

The angular acceleration vector is therefore:

$$\bar{\alpha} = \frac{\partial \bar{\omega}}{\partial t} + \left(\bar{\omega} \times \bar{\omega}\right) = \left\langle \begin{array}{c} \sin(\gamma_1)\,\dot{\gamma}_2\,\dot{\gamma}_1 - \cos(\gamma_1)\,\ddot{\gamma}_2 \\ -\ddot{\gamma}_1 \\ \sin(\gamma_1)\,\ddot{\gamma}_2 + \cos(\gamma_1)\,\dot{\gamma}_2\,\dot{\gamma}_1 \end{array} \right\rangle$$

The inertia tensor of the bar centered at G is:

$$I = \frac{1}{12}mL^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

From this we can compute the angular momentum as:

$$\bar{H}_G = I\bar{\omega} = \left\langle \begin{array}{c} 0 \\ -\frac{L^2\,m\,\dot{\gamma}_1}{12} \\ \frac{L^2\,m\,\sin(\gamma_1)\,\dot{\gamma}_2}{12} \end{array} \right\rangle$$

The second term in the Gibbs-Appell function is computed as:

$$\frac{1}{2}\bar{\alpha}\cdot\frac{\partial \bar{H}_G}{\partial t} = \left(\frac{\sin\left(\gamma_1\right)\ \ddot{\gamma}_2}{2} + \frac{\cos\left(\gamma_1\right)\ \dot{\gamma}_2\,\dot{\gamma}_1}{2}\right)\left(\frac{L^2\,m\ \sin\left(\gamma_1\right)\ \ddot{\gamma}_2}{12} + \frac{L^2\,m\ \cos\left(\gamma_1\right)\ \dot{\gamma}_2\,\dot{\gamma}_1}{12}\right) + \frac{L^2\,m\left(\ddot{\gamma}_1\right)^2}{24}$$

The third term is:

$$\bar{\alpha}\cdot\left(\bar{\omega}\times\bar{H}_G\right) = \frac{L^2\,m\ \cos\left(\gamma_1\right)\ \left(\sin\left(\gamma_1\right)\ \ddot{\gamma}_2 + \cos\left(\gamma_1\right)\ \dot{\gamma}_2\,\dot{\gamma}_1\right)\ \dot{\gamma}_2\,\dot{\gamma}_1}{12} - \frac{L^2\,m\ \cos\left(\gamma_1\right)\ \sin\left(\gamma_1\right)\ \left(\dot{\gamma}_2\right)^2\,\ddot{\gamma}_1}{12}$$

The third term can be rewritten as:

$$\frac{1}{2}I_2\ddot{\psi}^2 = \frac{I_2\left(\ddot{\gamma}_2\right)^2}{2}$$

All together, we can compute the following derivatives that form the Gibbs-Appell equations as:

$$\begin{aligned}
\frac{\partial S}{\partial \ddot{\gamma}_1} &= \frac{L^2\,m\,\ddot{\gamma}_1}{3} - \frac{L^2\,m\ \sin\left(2\,\gamma_1\right)\ \left(\dot{\gamma}_2\right)^2}{24} - \frac{L^2\,m\ \sin\left(2\,\theta\right)\ \left(\dot{\gamma}_2\right)^2}{8} - \frac{L^2\,m\ \cos\left(\theta\right)\ \left(\dot{\gamma}_2\right)^2}{2} \\
\frac{\partial S}{\partial \ddot{\gamma}_2} &= I_2\,\ddot{\gamma}_2 + \frac{4\,L^2\,m\,\ddot{\gamma}_2}{3} - \frac{L^2\,m\cos\left(\gamma_1\right)^2\,\ddot{\gamma}_2}{12} - \frac{L^2\,m\cos\left(\theta\right)^2\,\ddot{\gamma}_2}{4} + L^2\,m\ \sin\left(\theta\right)\ \ddot{\gamma}_2 + L^2\,m\ \cos\left(\theta\right)\ \dot{\gamma}_2\,\dot{\gamma}_1 \\
&\quad + \frac{L^2\,m\ \sin\left(2\,\gamma_1\right)\ \dot{\gamma}_2\,\dot{\gamma}_1}{12} + \frac{L^2\,m\ \cos\left(\theta\right)\ \sin\left(\theta\right)\ \dot{\gamma}_2\,\dot{\gamma}_1}{2}
\end{aligned}$$

Imposing the constraint that $\dot{\gamma}_2 = c\ \theta$ and $\ddot{\gamma}_2 = c\ \dot{\theta} = c\dot{\gamma}_1$:

$$\begin{aligned}
\frac{\partial S}{\partial \ddot{\gamma}_1} &= \frac{L^2\,m\,\ddot{\gamma}_1}{3} - \frac{L^2\,c^2\,m\ \sin\left(\theta\right)cos\left(\theta\right)\ {\gamma_1}^2}{3} - \frac{L^2\,c^2\,m\ \cos\left(\theta\right)\ {\gamma_1}^2}{2} \\
\frac{\partial S}{\partial \ddot{\gamma}_2} &= I_2\,c\,\dot{\gamma}_1 + \frac{4\,L^2\,c\,m\,\dot{\gamma}_1}{3} - \frac{L^2\,c\,m\cos\left(\theta\right)^2\,\dot{\gamma}_1}{3} + L^2\,c\,m\ \sin\left(\theta\right)\ \dot{\gamma}_1 \\
&\quad + \frac{2L^2\,c\,m\ \sin\left(\theta\right)cos\left(\theta\right)\ \gamma_1\,\dot{\gamma}_1}{3} + L^2\,c\,m\ \cos\left(\theta\right)\ \gamma_1\,\dot{\gamma}_1
\end{aligned}$$

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