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**MODULE 11 — Assignment**

## Problem 1: Solve Ginsberg 9.28

**The absolute velocity of a particle may be represented by the components $v_x$, $v_y$, and $v_z$ relative to the axes of a moving reference system $xyz$. Suppose that the angular velocity $\bar{\omega}$ of $xyz$ and the velocity $\bar{v}_O$ of the origin of $xyz$ are known as functions of time. Derive the Gibbs-Appell equations of motion relating the quasi-velocities $\dot{\gamma}_1 = v_x$, $\dot{\gamma}_2 = v_y$, and $\dot{\gamma}_3 = v_z$ to the resultant force acting on the particle.**

Where:

$$\bar{\omega} = \left\langle \begin{array}{c} \omega_x \\ \omega_y \\ \omega_z \end{array} \right\rangle$$

Given that:

$$\bar{v} = \left\langle \begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right\rangle = \left\langle \begin{array}{c} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{array} \right\rangle$$

Solving for acceleration:

$$\begin{aligned} \bar{a} &= \frac{\partial \bar{v}}{\partial t} + \bar{\omega} \times \bar{v} \\ &= \frac{\partial \dot{\gamma}}{\partial t} + \bar{\omega} \times \dot{\gamma} \\ &= \left\langle \begin{array}{c} \ddot{\gamma}_1 - \dot{\gamma}_2\,\omega_z + \dot{\gamma}_3\,\omega_y \\ \ddot{\gamma}_2 + \dot{\gamma}_1\,\omega_z - \dot{\gamma}_3\,\omega_x \\ \ddot{\gamma}_3 - \dot{\gamma}_1\,\omega_y + \dot{\gamma}_2\,\omega_x \end{array} \right\rangle \end{aligned}$$

Given that the Gibbs-Appell function for a system of particles is:

$$S = \sum_p \frac{1}{2} m \bar{a}_p \cdot \bar{a}_p$$

For this single particle case:

$$\begin{aligned} S &= \frac{1}{2} m \left(\bar{a} \cdot \bar{a}\right) \\ &= \frac{1}{2} m \left[ \left(\ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x\right)^2 + \left(\ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x\right)^2 + \left(\ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y\right)^2 \right] \end{aligned}$$

Where the equations of motion are calculated as:

$$\frac{\partial S}{\partial \ddot{\gamma}_j} = \Gamma_j = \Gamma_1$$

The virtual work associated with the forces applied to the particle is:

$$\delta W = \sum \bar{F} \cdot \delta \bar{x} = \sum_{j1}^{K} \Gamma_j \; \delta \gamma_j = \sum \bar{F} \cdot \left\langle \begin{array}{c} \delta\gamma_1 \\ \delta\gamma_2 \\ \delta\gamma_3 \end{array} \right\rangle$$

The equation of motion is solved for as:

$$\frac{\partial S}{\partial \ddot{\gamma}} = m \left\langle \begin{array}{c} \ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y \\ \ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x \\ \ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x \end{array} \right\rangle = \Gamma_1 = \left\langle \begin{array}{c} \sum F_x \\ \sum F_y \\ \sum F_z \end{array} \right\rangle = \left\langle \begin{array}{c} F_x \\ F_y \\ F_z \end{array} \right\rangle$$

Or:

$$\boldsymbol{\begin{aligned} m \left(\ddot{\gamma}_1 - \dot{\gamma}_2 \, \omega_z + \dot{\gamma}_3 \, \omega_y\right) &= F_x \\ m \left(\ddot{\gamma}_2 + \dot{\gamma}_1 \, \omega_z - \dot{\gamma}_3 \, \omega_x\right) &= F_y \\ m \left(\ddot{\gamma}_3 - \dot{\gamma}_1 \, \omega_y + \dot{\gamma}_2 \, \omega_x\right) &= F_z \end{aligned}}$$

$\longrightarrow \mathcal{A}\text{nswer}$

Where:

$$\dot{\gamma} = \left\langle \begin{array}{c} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{array} \right\rangle = v = \left\langle \begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right\rangle = v_0 + \bar{\omega} \times \bar{r} = \left\langle \begin{array}{c} v_{0_x} - \omega_z \, y + \omega_y \, z \\ v_{0_y} + \omega_z \, x - \omega_x \, z \\ v_{0_z} - \omega_y \, x + \omega_x \, y \end{array} \right\rangle$$

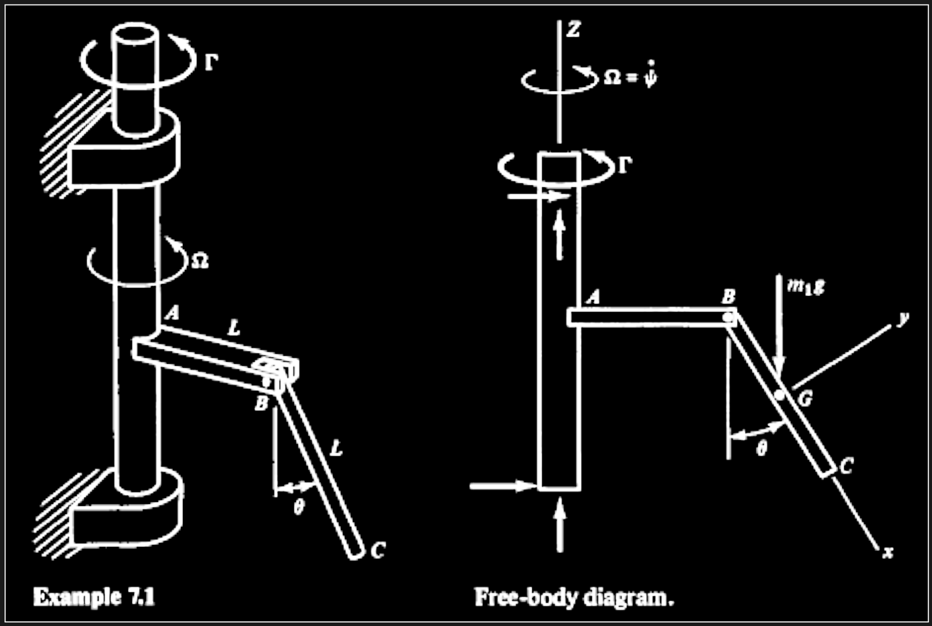
The following MATLAB script was used to solve this problem:

```
%[text] Problem 1: Solve Ginsberg 9\.28
%[text]
syms t m
syms v_x(t) v_y(t) v_z(t)
syms v_x_o(t) v_y_o(t) v_z_o(t)
syms x(t) y(t) z(t)
syms omega_x(t) omega_y(t) omega_z(t)

omega_bar = [omega_x;omega_y;omega_z];
v_o = [v_x_o;v_y_o;v_z_o];
v_bar = [v_x;v_y;v_z];
r_p_o = [x;y;z];

v_p = v_o + cross(omega_bar,r_p_o) % gamma_dot %[output:3f34eeb3]

syms gamma_dot_1(t) ...
     gamma_dot_2(t) ...
     gamma_dot_3(t)

gamma_dot = [gamma_dot_1;
             gamma_dot_2;
             gamma_dot_3];

a_p = diff(gamma_dot,t) + cross(omega_bar,gamma_dot) %[output:6931f860]

S = simplify(0.5*m*(transpose(a_p)*a_p)) %[output:10e7d569]

Gamma_lhs = simplify([ ... %[output:group:8b609d04] %[output:92d1caac]
    diff(S,diff(gamma_dot_1,t)); %[output:92d1caac]
    diff(S,diff(gamma_dot_2,t)); %[output:92d1caac]
    diff(S,diff(gamma_dot_3,t))]) %[output:group:8b609d04] %[output:92d1caac]
syms F_x F_y F_z
F =[F_x F_y F_z] %[output:89f6e948]
Gamma_rhs = [[F_x 0 0]*gamma_dot; %[output:group:02f1af22] %[output:7788bb0f]
             [0 F_y 0]*gamma_dot; %[output:7788bb0f]
             [0 0 F_z]*gamma_dot] %[output:group:02f1af22] %[output:7788bb0f]

collect(Gamma_lhs,m) == Gamma_rhs %[output:585416fa]
```

## Problem 2:

**Use the Gibbs-Appell approach to find the equations of motion for this problem.**

**A torque $\Gamma$ applied to the vertical shaft of the T-bar causes the rotation rate $\Omega$ about the vertical axis to increase in proportion to the angle $\theta$ by which bar BC swings outward, that is, $\Omega = c\theta$. The mass of bar BC is $m_1$ and the moment of inertia of the T-bar about its axis of rotation is $I_2$. Determine the equations of motion for the system, and for the torque $\Gamma$.**




For the purpose of this problem, as was recommended in the office hour, we will replace the $y$-axis of the body frame with the $z$-axis. This will simplify the inertial to body rotation we perform later in the solution.

First we will determine location of point $G$ in the inertial frame. By inspection this is:

$$\bar{r}_G = \left\langle \begin{array}{c} \cos(\psi)\left(L + \frac{1}{2}L\sin(\theta)\right) \\ \sin(\psi)\left(L + \frac{1}{2}L\sin(\theta)\right) \\ -\frac{1}{2}L\cos(\theta) \end{array} \right\rangle$$

From this we can compute the velocity at point $G$:

$$\bar{v}_G = \left\langle \begin{matrix} \frac{1}{2}L\,\cos\left(\psi\right)\,\cos\left(\theta\right)\,\frac{\partial}{\partial t}\theta - \sin\left(\psi\right)\,\left(L + \frac{1}{2}L\,\sin\left(\theta\right)\right)\,\frac{\partial}{\partial t}\psi \\ \frac{1}{2}L\,\cos\left(\theta\right)\,\sin\left(\psi\right)\,\frac{\partial}{\partial t}\theta + \cos\left(\psi\right)\,\left(L + \frac{1}{2}L\,\sin\left(\theta\right)\right)\,\frac{\partial}{\partial t}\psi \\ \frac{1}{2}L\,\sin\left(\theta\right)\,\frac{\partial}{\partial t}\theta \end{matrix} \right\rangle$$

Which we can then use to compute acceleration at point $G$:

$$\bar{a}_G = \left\langle \begin{matrix} \frac{L\,\cos(\psi)\,\cos(\theta)\,\frac{\partial^2}{\partial t^2}\theta}{2} - \frac{L\,\cos(\psi)\,\sin(\theta)\,\left(\frac{\partial}{\partial t}\theta\right)^2}{2} - \frac{L\,\cos(\psi)\,\left(\sin(\theta)+2\right)\,\left(\frac{\partial}{\partial t}\psi\right)^2}{2} - \frac{L\,\sin(\psi)\,\left(\sin(\theta)+2\right)\,\frac{\partial^2}{\partial t^2}\psi}{2} - L\,\cos\left(\theta\right)\,\sin\left(\psi\right)\,\frac{\partial}{\partial t}\theta\,\frac{\partial}{\partial t}\psi \\ \frac{L\,\cos(\theta)\,\sin(\psi)\,\frac{\partial^2}{\partial t^2}\theta}{2} - \frac{L\,\sin(\psi)\,\sin(\theta)\,\left(\frac{\partial}{\partial t}\theta\right)^2}{2} - \frac{L\,\sin(\psi)\,\left(\sin(\theta)+2\right)\,\left(\frac{\partial}{\partial t}\psi\right)^2}{2} + \frac{L\,\cos(\psi)\,\left(\sin(\theta)+2\right)\,\frac{\partial^2}{\partial t^2}\psi}{2} + L\,\cos\left(\psi\right)\,\cos\left(\theta\right)\,\frac{\partial}{\partial t}\theta\,\frac{\partial}{\partial t}\psi \\ \frac{L\,\sin(\theta)\,\frac{\partial^2}{\partial t^2}\theta}{2} + \frac{L\,\cos(\theta)\,\left(\frac{\partial}{\partial t}\theta\right)^2}{2} \end{matrix} \right\rangle$$

Replacing $\dot{\theta}$ and $\dot{\psi}$ with the quasi-velocity terms gives us:

$$\bar{a}_G = \left\langle \begin{matrix} \frac{L\,\cos(\psi)\,\cos(\theta)\,\ddot{\gamma_1}}{2} - \frac{L\,\cos(\psi)\,\sin(\theta)\,(\dot{\gamma_1})^2}{2} - \frac{L\,\cos(\psi)\,(\dot{\gamma_2})^2\,\left(\sin(\theta)+2\right)}{2} - \frac{L\,\sin(\psi)\,\left(\sin(\theta)+2\right)\,\ddot{\gamma_2}}{2} - L\,\cos\left(\theta\right)\,\sin\left(\psi\right)\,\dot{\gamma_2}\,\dot{\gamma_1} \\ \frac{L\,\cos(\theta)\,\sin(\psi)\,\ddot{\gamma_1}}{2} - \frac{L\,\sin(\psi)\,\sin(\theta)\,(\dot{\gamma_1})^2}{2} - \frac{L\,\sin(\psi)\,(\dot{\gamma_2})^2\,\left(\sin(\theta)+2\right)}{2} + \frac{L\,\cos(\psi)\,\left(\sin(\theta)+2\right)\,\ddot{\gamma_2}}{2} + L\,\cos\left(\psi\right)\,\cos\left(\theta\right)\,\dot{\gamma_2}\,\dot{\gamma_1} \\ \frac{L\,\sin(\theta)\,\ddot{\gamma_1}}{2} + \frac{L\,\cos(\theta)\,(\dot{\gamma_1})^2}{2} \end{matrix} \right\rangle$$

To convert this acceleration into the body frame, we will define the following inertial to body rotation:

$$R_{xyz}^{XYZ} = R_z(\psi)R_y(\theta) = \begin{bmatrix} \cos\left(\psi\right) & -\sin\left(\psi\right) & 0 \\ \sin\left(\psi\right) & \cos\left(\psi\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\theta\right) & 0 & \sin\left(\theta\right) \\ 0 & 1 & 0 \\ -\sin\left(\theta\right) & 0 & \cos\left(\theta\right) \end{bmatrix}$$

Which simplifies to:

$$R_{xyz}^{XYZ} = \begin{bmatrix} \cos\left(\psi\right)\,\cos\left(\theta\right) & -\sin\left(\psi\right) & \cos\left(\psi\right)\,\sin\left(\theta\right) \\ \cos\left(\theta\right)\,\sin\left(\psi\right) & \cos\left(\psi\right) & \sin\left(\psi\right)\,\sin\left(\theta\right) \\ -\sin\left(\theta\right) & 0 & \cos\left(\theta\right) \end{bmatrix}$$

The transpose of this gives us the inertial to body transform:

$$R_{XYZ}^{xyz} = \begin{bmatrix} \cos\left(\psi\right)\,\cos\left(\theta\right) & \cos\left(\theta\right)\,\sin\left(\psi\right) & -\sin\left(\theta\right) \\ -\sin\left(\psi\right) & \cos\left(\psi\right) & 0 \\ \cos\left(\psi\right)\,\sin\left(\theta\right) & \sin\left(\psi\right)\,\sin\left(\theta\right) & \cos\left(\theta\right) \end{bmatrix}$$

We can compute the acceleration term as:

$$\bar{a}_G = R_{XYZ}^{xyz}\bar{a}_G = \left\langle \begin{array}{c} \frac{L\,\cos(2\,\theta)\,\ddot{\gamma}_1}{2} - \frac{L\,\sin(2\,\theta)\,(\dot{\gamma}_1)^2}{2} - \frac{L\,\sin(2\,\theta)\,(\dot{\gamma}_2)^2}{4} - L\,\cos(\theta)\,\left(\dot{\gamma}_2\right)^2 \\ L\,\ddot{\gamma}_2 + \frac{L\,\sin(\theta)\,\ddot{\gamma}_2}{2} + L\,\cos(\theta)\,\dot{\gamma}_2\,\dot{\gamma}_1 \\ \frac{L\,\left(2\cos(\theta)^2-1\right)\,(\dot{\gamma}_1)^2}{2} - L\,\sin(\theta)\,\left(\dot{\gamma}_2\right)^2 - \frac{L\,(\dot{\gamma}_2)^2}{4} + \frac{L\,\left(2\cos(\theta)^2-1\right)\,(\dot{\gamma}_2)^2}{4} + L\,\cos(\theta)\,\sin(\theta)\,\ddot{\gamma}_1 \end{array} \right\rangle$$

The Gibbs-Appell function for the system is given by:

$$S = \frac{1}{2}m\left(\bar{a}_G\cdot\bar{a}_G\right) + \frac{1}{2}\bar{\alpha}\cdot\frac{\partial \bar{H}_G}{\partial t} + \bar{\alpha}\cdot\left(\bar{\omega}\times\bar{H}_G\right) + \frac{1}{2}I_2\ddot{\psi}^2$$

The first term of $S$ is computed as:

$$\begin{aligned}
\frac{1}{2}m\left(\bar{a}_G\cdot\bar{a}_G\right) = \frac{1}{2}m&\left(L\,\ddot{\gamma}_2 + \frac{L\,\sin(\theta)\,\ddot{\gamma}_2}{2} + L\,\cos(\theta)\,\dot{\gamma}_2\,\dot{\gamma}_1\right)^2 \\
&+ \frac{1}{2}m\left(L\,\cos(\theta)\,\left(\dot{\gamma}_2\right)^2 + \frac{L\,\sin(2\,\theta)\,\left(\dot{\gamma}_1\right)^2}{2} + \frac{L\,\sin(2\,\theta)\,\left(\dot{\gamma}_2\right)^2}{4} - \frac{L\,\cos(2\,\theta)\,\ddot{\gamma}_1}{2}\right)^2 \\
&+ \frac{1}{2}m\left(\frac{L\,\left(2\cos(\theta)^2-1\right)\,\left(\dot{\gamma}_1\right)^2}{2} - L\,\sin(\theta)\,\left(\dot{\gamma}_2\right)^2 - \frac{L\,\left(\dot{\gamma}_2\right)^2}{4}\right. \\
&\qquad\left. + \frac{L\,\left(2\cos(\theta)^2-1\right)\,\left(\dot{\gamma}_2\right)^2}{4} + L\,\cos(\theta)\,\sin(\theta)\,\ddot{\gamma}_1\right)^2
\end{aligned}$$

The angular velocity vector of the system is in the body frame is, by inspection:

$$\bar{\omega} = \left\langle \begin{array}{c} -\cos(\gamma_1)\,\dot{\gamma}_2 \\ -\dot{\gamma}_1 \\ \sin(\gamma_1)\,\dot{\gamma}_2 \end{array} \right\rangle$$

The angular acceleration vector is therefore:

$$\bar{\alpha} = \frac{\partial\bar{\omega}}{\partial t} + \left(\bar{\omega}\times\bar{\omega}\right) = \left\langle \begin{array}{c} \sin(\gamma_1)\,\dot{\gamma}_2\,\dot{\gamma}_1 - \cos(\gamma_1)\,\ddot{\gamma}_2 \\ -\ddot{\gamma}_1 \\ \sin(\gamma_1)\,\ddot{\gamma}_2 + \cos(\gamma_1)\,\dot{\gamma}_2\,\dot{\gamma}_1 \end{array} \right\rangle$$

The inertia tensor of the bar centered at $G$ is:

$$I = \frac{1}{12} m L^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

From this we can compute the angular momentum as:

$$\bar{H}_G = I\bar{\omega} = \left\langle \begin{matrix} 0 \\ -\frac{L^2\, m\, \dot{\gamma}_1}{12} \\ \frac{L^2\, m\, \sin(\gamma_1)\, \dot{\gamma}_2}{12} \end{matrix} \right\rangle$$

The second term in the Gibbs-Appell function is computed as:

$$\frac{1}{2}\bar{\alpha}\cdot\frac{\partial \bar{H}_G}{\partial t} = \left(\frac{\sin(\gamma_1)\ \ddot{\gamma}_2}{2} + \frac{\cos(\gamma_1)\ \dot{\gamma}_2\, \dot{\gamma}_1}{2}\right)\left(\frac{L^2\, m\, \sin(\gamma_1)\ \ddot{\gamma}_2}{12} + \frac{L^2\, m\, \cos(\gamma_1)\ \dot{\gamma}_2\, \dot{\gamma}_1}{12}\right) + \frac{L^2\, m\, (\ddot{\gamma}_1)^2}{24}$$

The third term is:

$$\bar{\alpha}\cdot\left(\bar{\omega} \times \bar{H}_G\right) = \frac{L^2\, m\, \cos(\gamma_1)\ \left(\sin(\gamma_1)\ \ddot{\gamma}_2 + \cos(\gamma_1)\ \dot{\gamma}_2\, \dot{\gamma}_1\right)\ \dot{\gamma}_2\, \dot{\gamma}_1}{12} - \frac{L^2\, m\, \cos(\gamma_1)\ \sin(\gamma_1)\ (\dot{\gamma}_2)^2\, \ddot{\gamma}_1}{12}$$

The third term can be rewritten as:

$$\frac{1}{2} I_2 \ddot{\psi}^2 = \frac{I_2\, (\ddot{\gamma}_2)^2}{2}$$

All together, we can compute the following derivatives that form the Gibbs-Appell equations as:

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = \frac{L^2\, m\, \ddot{\gamma}_1}{3} - \frac{L^2\, m\, \sin(2\,\gamma_1)\ (\dot{\gamma}_2)^2}{24} - \frac{L^2\, m\, \sin(2\,\theta)\ (\dot{\gamma}_2)^2}{8} - \frac{L^2\, m\, \cos(\theta)\ (\dot{\gamma}_2)^2}{2}$$

$$\begin{aligned} \frac{\partial S}{\partial \ddot{\gamma}_2} = {} & I_2\, \ddot{\gamma}_2 + \frac{4\, L^2\, m\, \ddot{\gamma}_2}{3} - \frac{L^2\, m \cos(\gamma_1)^2\, \ddot{\gamma}_2}{12} - \frac{L^2\, m \cos(\theta)^2\, \ddot{\gamma}_2}{4} + L^2\, m\, \sin(\theta)\ \ddot{\gamma}_2 + L^2\, m\, \cos(\theta)\ \dot{\gamma}_2\, \dot{\gamma}_1 \\ & + \frac{L^2\, m\, \sin(2\,\gamma_1)\ \dot{\gamma}_2\, \dot{\gamma}_1}{12} + \frac{L^2\, m\, \cos(\theta)\ \sin(\theta)\ \dot{\gamma}_2\, \dot{\gamma}_1}{2} \end{aligned}$$

Imposing the constraint that $\dot{\gamma}_2 = c\ \theta$ and $\ddot{\gamma}_2 = c\ \dot{\theta} = c\dot{\gamma}_1$:

$$
\begin{aligned}
\frac{\partial S}{\partial \ddot{\gamma}_1} &= \frac{1}{3}L^2\, m\, \ddot{\gamma}_1 - \frac{1}{3}L^2\, c^2\, m\, \sin\left(\theta\right) cos\left(\theta\right)\, {\gamma_1}^2 - \frac{1}{2}L^2\, c^2\, m\, \cos\left(\theta\right)\, {\gamma_1}^2 \\
\frac{\partial S}{\partial \ddot{\gamma}_2} &= I_2\, c\, \dot{\gamma}_1 + \frac{4}{3}\, L^2\, c\, m\, \dot{\gamma}_1 - \frac{1}{3}L^2\, c\, m \cos\left(\theta\right)^2 \dot{\gamma}_1 + L^2\, c\, m\, \sin\left(\theta\right)\, \dot{\gamma}_1 \\
&\qquad + \frac{2}{3}L^2\, c\, m\, \sin\left(\theta\right) cos\left(\theta\right)\, \gamma_1\, \dot{\gamma}_1 + L^2\, c\, m\, \cos\left(\theta\right)\, \gamma_1\, \dot{\gamma}_1
\end{aligned}
$$

Further simplifyng:

$$
\begin{aligned}
\frac{\partial S}{\partial \ddot{\gamma}_1} &= \frac{1}{3}L^2\, m\, \ddot{\gamma}_1 - L^2\, c^2\, m\, \cos\left(\theta\right)\, {\gamma_1}^2 \left(\frac{1}{2} + \frac{1}{3}\sin\left(\theta\right)\right) = \Gamma_1 \\
\frac{\partial S}{\partial \ddot{\gamma}_2} &= I_2\, c\, \dot{\gamma}_1 +\, L^2\, c\, m\, \left(1 + \sin\left(\theta\right) + \frac{1}{3}\sin\left(\theta\right)^2\right) \dot{\gamma}_1 + L^2\, c\, m\, \cos\left(\theta\right) \left(1 + \frac{2}{3}cos\left(\theta\right)\right)\, \gamma_1\, \dot{\gamma}_1 = \Gamma_2
\end{aligned}
$$

Solving for $\Gamma_j$ based on the generalized forces:

$$
\delta W = \bar{\Gamma} \cdot \delta\bar{\gamma} = \begin{bmatrix} -m\, g\, \frac{L}{2}\sin(\theta)\ \delta\gamma_1 \\ \Gamma\ \delta\gamma_2 \end{bmatrix}
$$

Therefore, the equations of motion for the system are:

$$
\begin{aligned}
\frac{1}{3}L^2\, m\, \ddot{\gamma}_1 - L^2\, c^2\, m\, \cos\left(\theta\right)\, {\gamma_1}^2 \left(\frac{1}{2} + \frac{1}{3}\sin\left(\theta\right)\right) &= -m\, g\, \frac{L}{2}\sin(\theta) \\
I_2\, c\, \dot{\gamma}_1 +\, L^2\, c\, m\, \left(1 + \sin\left(\theta\right) + \frac{1}{3}\sin\left(\theta\right)^2\right) \dot{\gamma}_1 + L^2\, c\, m\, \cos\left(\theta\right) \left(1 + \frac{2}{3}\cos\left(\theta\right)\right)\, \gamma_1\, \dot{\gamma}_1 &= \Gamma
\end{aligned}
$$

The equations of motion after simplification are:

$$
\boldsymbol{
\begin{aligned}
\frac{1}{3}\ddot{\gamma}_1 -\ c^2\ \cos\left(\theta\right)\, {\gamma_1}^2 \left(\frac{1}{2} + \frac{1}{3}\sin\left(\theta\right)\right) &= -\frac{g}{2L}\sin(\theta) \\
\frac{1}{L^2\, m} I_2 \dot{\gamma}_1 + \left(1 + \sin\left(\theta\right) + \frac{1}{3}\sin\left(\theta\right)^2\right) \dot{\gamma}_1 + \cos\left(\theta\right) \left(1 + \frac{2}{3}\cos\left(\theta\right)\right)\, \gamma_1\, \dot{\gamma}_1 &= \frac{1}{L^2\, c\, m}\Gamma
\end{aligned}
}
$$

$\longrightarrow \mathcal{A}$nswer

The following MATLAB script was used to solve this problem:

```
clc,clear
syms t psi(t) theta(t) L c m
syms gamma_1(t) gamma_2(t)



r_G = [(L+sin(theta)*L/2)*cos(psi); %[output:group:7db8fe06] %[output:1a1aebde]
       (L+sin(theta)*L/2)*sin(psi); %[output:1a1aebde]
       -cos(theta)*L/2]  %[output:group:7db8fe06] %[output:1a1aebde]
v_G = diff(r_G,t) %[output:30bffc96]
a_G = simplify(diff(v_G,t))  %[output:42414e71]

a_G = subs(a_G,... %[output:group:8507d802] %[output:456fd1b3]
     [diff(theta,t) diff(psi,t)],[diff(gamma_1,t),diff(gamma_2,t)]) %[output:group:8507d80


R = transpose(zRot(psi)*yRot(theta)) % transpose ? %[output:6d81fc71]
a_G = simplify(R*a_G) %[output:39ba844a]

S1 = (1/2)*m*simplify(transpose(a_G)*a_G) %[output:6860e3fb]

omega_bar = [-diff(gamma_2,t)*cos(gamma_1); %[output:group:4c308172] %[output:927f17bf]
             -diff(gamma_1,t); %[output:927f17bf]
              diff(gamma_2,t)*sin(gamma_1)] %[output:group:4c308172] %[output:927f17bf]

alpha_bar = diff(omega_bar,t)+cross(omega_bar,omega_bar) %[output:1d7cb1ea]

I = (m*L^2/12)*diag([0,1,1]) %[output:743b6c5e]

H_bar_G = I*omega_bar %[output:04de197a]

S2 = (1/2)*transpose(alpha_bar)*... %[output:group:2b48c35b] %[output:0a548ac1]
    (diff(H_bar_G,t))%+cross(omega_bar,H_bar_G)) %[output:group:2b48c35b] %[output:0a548ac

syms I_2 c

S3 = transpose(alpha_bar)*cross(omega_bar,H_bar_G) %[output:7e3a6cfa]


ST = (1/2)*I_2*diff(gamma_2,t,t)^2 %[output:0af23cf5]

S = S1 + S2 + S3 + ST %[output:73d236de]

EOM_1 = simplify(diff(S,diff(gamma_1,t,t)),1000) %[output:5ddf0e27]
EOM_2 = simplify(diff(S,diff(gamma_2,t,t)),1000) %[output:5ae9317d]
```

```
EOM_1 = simplify(subs(EOM_1,diff(gamma_2,t),c*gamma_1) %[output:7378a275]
EOM_2 = subs(EOM_2,diff(gamma_2,t),c*gamma_1) %[output:2c497ac6]


function R = yRot(ang)
    R = [ cos(ang) 0 sin(ang);
                  0 1        0;
         -sin(ang) 0 cos(ang)];
end

function R = zRot(ang)
    R = [ cos(ang) -sin(ang) 0;
          sin(ang) cos(ang) 0;
                  0         0 1];
end
```

*Submitted by Austin Barrilleaux on November 13, 2024.*