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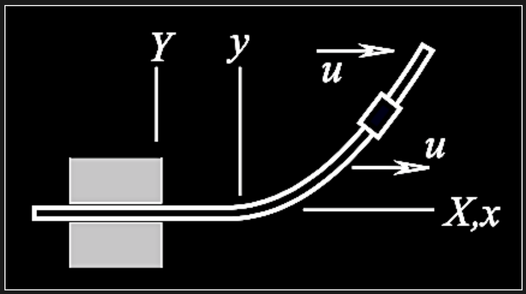
MODULE 7 — Assignment

EXERCISE 7.1

The slider descends along a curved guide as the guide translates to the right at the constant speed u. The shape of the guide bar in terms of a body-fixed set of coordinates is $y = \beta x^2$. Generalized coordinates selected for this system are the fixed X and Y coordinates of the collar. Independently derive the velocity and configuration constraint equations relating X and Y. Then show that integration of the velocity constraint yields the configuration constraint.



Given the graphic provided, we can write that the velocities in the generalized coordinates are:

$$\dot{X} = \dot{x} + u$$
$$\dot{Y} = \dot{y} = 2\beta x\dot{x}$$

The position of X is defined as:

$$X = ut + x \quad \rightarrow \quad x = X - ut$$

We can combine the above, and find that the velocity constraint for $\dot{Y}$ becomes:

$$\dot{Y} = 2\beta\left(X - ut\right)\left(\dot{X} - u\right)$$

This can be written as:

$$\boldsymbol{2\beta\left(X - ut\right)\dot{X} - \dot{Y} - 2\beta\left(X - ut\right)u = 0}$$

$\longrightarrow \mathcal{A}\textsf{nswer}$

The configuration constraint is:

$$Y = y = \beta x^2 = \beta\left(X - ut\right)^2$$

More simply:

$$\boldsymbol{Y = \beta\left(X - ut\right)^2}$$

$\longrightarrow \mathcal{A}\textsf{nswer}$

Looking at the velocity constraint above, we know by reversing the chain rule for derivatives:

$$\int f(g(x))\; g'(x)\; dx = F(g(x)) + C$$

Where for the velocity constraint:

$$\begin{aligned} f(g(x)) &= 2\left(X - ut\right) \\ g'(x) &= \left(\dot{X} - u\right) \\ F(g(x)) &= \left(X - ut\right)^2 \end{aligned}$$

Therefore:

$$Y = \beta\int\left(2\left(X - ut\right)\right)\left(\dot{X} - u\right)dt = \beta\int g(t)\; g'(t)\; dx = \beta\left(X - ut\right)^2 + C$$

We can integrate the velocity constraint to get the configuration constraint:

$$Y = \beta\left(X - ut\right)^2 + C$$

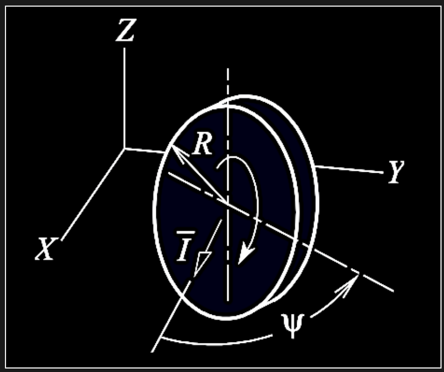
If the initial condition constant of integration, C is zero:

$$\boldsymbol{Y = \beta\,(X - ut)^2}$$

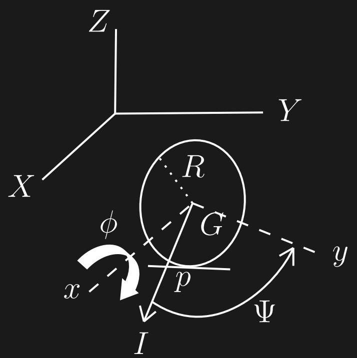
⟶ *Answer*

EXERCISE 7.12

The figure shows a disk that is constrained to roll without slipping on a horizontal XY plane, such that its plane remains vertical. Let the position coordinates X and Y of the geometric center, the heading angle Ψ, and the spin angle ϕ be generalized coordinates. Describe the velocity constraints between these generalized coordinates. From those results, determine the number of degrees of freedom, and whether the system is holonomic.



This system can be defined by the following sketch:



Constructing the angular velocity vector $\bar{\omega}$ of xyz by vectorially adding the simple rotation rates according to:

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2$$

This gives us:

$$\bar{\omega} = \phi\left(-\bar{i}\right) + \Psi\bar{k} = -\phi\bar{i} + \Psi\bar{k}$$

The velocity of the center of the disk at point G from the no-slip point, point p is defined as:

$$\bar{v}_G^{xyz} = \cancelto{0}{\bar{v}_p} + \bar{\omega} \times \bar{r}_{G/p}$$

This evaluates to:

$$\bar{v}_G^{xyz} = \bar{\omega} \times \bar{r}_{G/p} = \left(-\phi\bar{i} + \Psi\bar{k}\right) \times R\bar{k} = R\dot{\phi}\ \bar{j}$$

The rotation from the body frame to the (inertial) generalized coordinate frame is defined by the following sequence of rotations:

$$R_{\text{rot}} = \begin{bmatrix} \cos\left(-\frac{\pi}{2}\right) & -\sin\left(-\frac{\pi}{2}\right) & 0 \\ \sin\left(-\frac{\pi}{2}\right) & \cos\left(-\frac{\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\Psi\right) & -\sin\left(\Psi\right) & 0 \\ \sin\left(\Psi\right) & \cos\left(\Psi\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Or:

$$R_{\text{rot}} = \begin{bmatrix} \sin\left(\Psi\right) & \cos\left(\Psi\right) & 0 \\ -\cos\left(\Psi\right) & \sin\left(\Psi\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the velocity $\bar{v}_G$ can be expressed as:

$$\bar{v}_G^{XYZ} = R_{\text{rot}}\ \bar{v}_G^{xyz} = \begin{bmatrix} \sin\left(\Psi\right) & \cos\left(\Psi\right) & 0 \\ -\cos\left(\Psi\right) & \sin\left(\Psi\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0\ \bar{i} \\ R\dot{\phi}\ \bar{j} \\ 0\ \bar{k} \end{bmatrix} = \begin{bmatrix} R\ \dot{\phi}\cos\left(\Psi\right)\ \bar{I} \\ R\ \dot{\phi}\sin\left(\Psi\right)\ \bar{J} \\ 0\ \bar{K} \end{bmatrix}$$

This can be expressed as the following velocity constraints:

$$\boldsymbol{\dot{X} = R\,\dot{\phi}\ \cos(\Psi)}$$
$$\boldsymbol{\dot{Y} = R\,\dot{\phi}\ \sin(\Psi)}$$

$\longrightarrow$ *Answer*

Or:

$$\dot{X} - R\,\dot{\phi}\,\cos(\Psi) = 0$$
$$\dot{Y} - R\,\dot{\phi}\,\sin(\Psi) = 0$$

This system has **4** generalized coordinates, $\boldsymbol{q_i = \{X, Y, \Psi, \phi\}}$, and **2** velocity constraints. Subtracting the two gives us **2 degrees of freedom.**

$\longrightarrow$ *Answer*

To evaluate whether the system is holonomic, we can isolate the coefficients of each generalized velocity:

$$a_{11} = 1,\ a_{12} = 0,\ a_{13} = 0,\ a_{14} = -R\ \ \cos(\Psi),\ b_1 = 0$$
$$a_{21} = 0,\ a_{22} = 1,\ a_{23} = 0,\ a_{24} = -R\ \ \sin(\Psi),\ b_2 = 0$$

Looking at the terms a_{13} and a_{14}:

$$\frac{\partial f_1}{\partial \Psi} = g_1(X, Y, \Psi, \phi)\ \ 0 = 0$$
$$\frac{\partial f_1}{\partial \phi} = g_1(X, Y, \Psi, \phi)\left(-R\ \ \cos(\Psi)\,\dot{\phi}\right)$$

The second equation implies that the derivative of the function f_1 with respect to ϕ, will contain Ψ, however the second implies that the derivative of the function with respect to Ψ will be zero. This is inconsistent. We see the same thing evaluating a_{23} and a_{24}. There is no integrating factor g_1 that can satisfy both of these, therefore the system is not integrable.

Since the velocity constraints are not integrable, the system is **non-holonomic**.

$\longrightarrow$ *Answer*

Submitted by Austin Barrilleaux on October 15, 2024.