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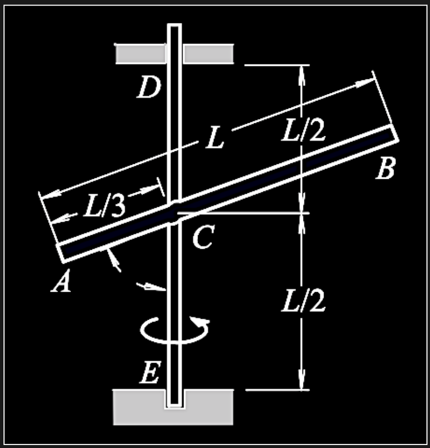
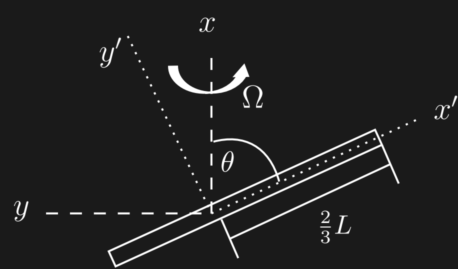
MODULE 6 — Assignment

EXERCISE 5.26

Thin bar ACB is welded to a shaft that rotates at the constant angular speed Ω, so the angle θ between the bar and the shaft is constant.
(a) Derive expressions for the angular momentum $\bar{H}_C$ and the kinetic energy of the bar. Draw a sketch of $\bar{H}_C$.
(b) Based on an analysis of the manner in which $\bar{H}_C$ in Part (a) rotates, derive an expression for $\frac{\partial}{\partial t}\bar{H}_C$.
(c) Use Eq. (5.3.4) to evaluate $\frac{\partial}{\partial t}\bar{H}_C$, and compare it with the result of Part (b).



The following sketch shows the two frames of the system:



The rotation matrix that converts the $\{xyz\}$ frame to the $\{x'y'z'\}$ frame is:

$$R = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Constructing the angular velocity vector:

$$\bar{\omega} = \Omega\ i$$

In the body frame is:

$$\bar{\omega} = R\ \Omega\ i = \begin{bmatrix} \Omega\ \cos(\theta)\ \ i' \\ \Omega\ \sin(\theta)\ \ j' \\ 0\ k' \end{bmatrix}$$

From textbook Appendix, the centroidal inertia mass properties of the shaft are:

$$I_{xx} = 0$$
$$I_{yy} = \frac{1}{12} m\, L^2$$
$$I_{zz} = \frac{1}{12} m\, L^2$$

This expressed as the inertia tensor is:

$$I_{x'y'z'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m\, L^2 & 0 \\ 0 & 0 & \frac{1}{12} m\, L^2 \end{bmatrix}$$

The distance from the body frame is:

$$d_{x'y'z'} = \begin{bmatrix} \frac{1}{6} L & 0 & 0 \end{bmatrix}$$

Using the parallel axis theorem to get the parallel axis transformation of inertia matrix relative to the center of the frame of reference in question:

$$I_{pat} = m \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{36}\,L^2 & 0 \\ 0 & 0 & \frac{1}{36}\,L^2 \end{bmatrix}$$

This makes the inertia tensor at point C:

$$I_C = I_{x'y'z'} + I_{pat} m \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{9}\,L^2 & 0 \\ 0 & 0 & \frac{1}{9}\,L^2 \end{pmatrix}$$

From this we can compute the angular momentum as:

$$\boldsymbol{\bar{H}_C = I_C\;\bar{\omega} = \frac{1}{9} m\,L^2\,\Omega, \sin\left(\theta\right)\;\; j'}$$

$\longrightarrow$ *Answer*

Since the bar rotates about the x axis, the rotation component in the terminal frame that is orthogonal to the j'-axis is $\Omega\cos(\theta)\ i'$, therefore:

$$\boldsymbol{\dot{\bar{H}}_C = \frac{1}{9} m\,L^2\,\Omega^2 \cos\left(\theta\right) \sin\left(\theta\right)\;\; k'}$$

$\longrightarrow$ *Answer*

If we use Eq. (5.3.4) to evaluate this, we get the same answer:

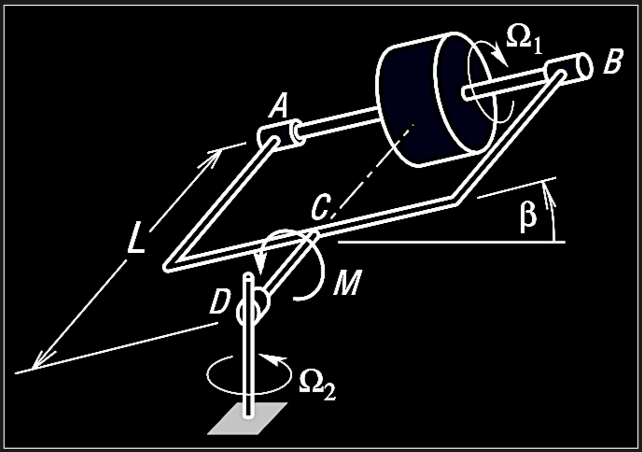
$$\dot{\bar{H}}_C = \cancelto{0}{\frac{\partial}{\partial t}\bar{H}_C} + \bar{\omega} \times \bar{H}_C$$

$$\begin{aligned} \boldsymbol{\dot{\bar{H}}_C} &= \begin{bmatrix} \boldsymbol{0\; i'} \\ \boldsymbol{0\; j'} \\ \boldsymbol{\frac{1}{9} m\,L^2\,\Omega^2 \cos\left(\theta\right)\; \sin\left(\theta\right)\;\; k'} \end{bmatrix} \\ &= \boldsymbol{\frac{1}{9} m\,L^2\,\Omega^2 \cos\left(\theta\right)\; \sin\left(\theta\right)\;\; k'} \end{aligned}$$

$\longrightarrow$ *Answer*

EXERCISE 6.8

The torque M acting on the gimbal of the gyroscopic turn indicator is exerted by a torsional spring, so $M = -k\beta$. The precession rate Ω_2 is a specified function of time, and the spin rate Ω_1 is held constant by a servomotor. Let I_1 denote the moment of inertia of the flywheel about axis AB, and let I_2 be the centroidal moment of inertia perpendicular to axis AB. Derive the differential equation of motion for β.



Submitted by Austin Barrilleaux on October 4, 2024.