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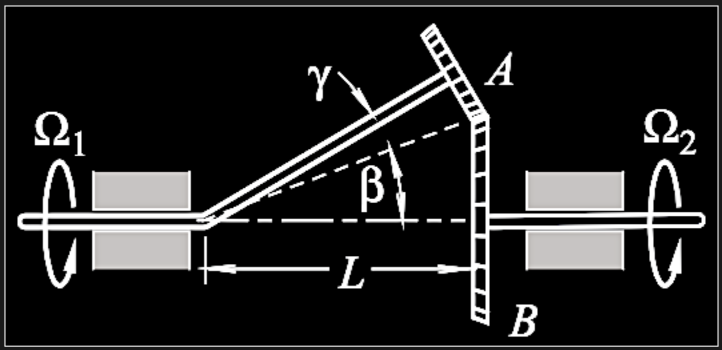
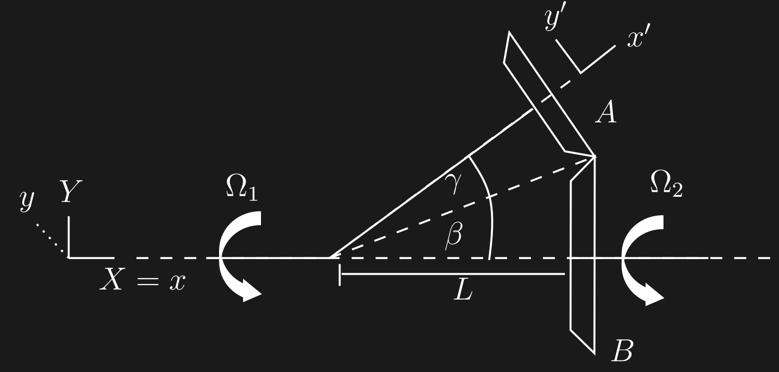
MODULE 4 — Assignment

EXERCISE 4.39

Gear A spins relative to its shaft, which rotates at variable rate Ω_1 about the horizontal axis. Gear B rotates at the variable rate Ω_2. Determine the angular velocity and angular acceleration of gear A.



In the following sketch, we will define two coordinate frames, $\{XYZ\}$, $\{xyz\}$ which rotates with Ω_1, and $\{x'y'z'\}$ which is rotated to align with the tilted disk from $\{xyz\}$ and rotates about the x-axis with Ω_1:



We can write the angular velocity $\bar{\omega}$ as:

$$\bar{\omega} = \Omega_1 \bar{i} + \Omega_A \bar{i}'$$

Where Ω_A is the rotation about the horizontal bar.

We will confront this problem in the rotating $\{x'y'z\}$ frame. The rotation from $\{xyz\}$ to $\{x'y'z'\}$ is defined by the z axis rotation as:

$$R = \begin{bmatrix} \cos(\beta+\gamma) & \sin(\beta+\gamma) & 0 \\ -\sin(\beta+\gamma) & \cos(\beta+\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which we can use to define:

$$\begin{aligned} \bar{i} = R\,\bar{i}' &= \begin{bmatrix} \cos(\beta+\gamma)\,\bar{i}' \\ -\sin(\beta+\gamma)\,\bar{j}' \end{bmatrix} \\ \bar{j} = R\,\bar{j}' &= \begin{bmatrix} \sin(\beta+\gamma)\,\bar{i}' \\ \cos(\beta+\gamma)\,\bar{j}' \end{bmatrix} \end{aligned}$$

This makes:

$$\bar{\omega} = \begin{bmatrix} \left\{\Omega_A + \cos(\beta+\gamma)\ \Omega_1\right\}\bar{i}' \\ \left\{-\sin(\beta+\gamma)\ \Omega_1\right\}\bar{j}' \end{bmatrix}$$

We can solve for Ω_A, by first solving for the velocity of $v_{B_{\{ijk\}}}$ relative to the rotation rate Ω_1. The rotation of Ω_2 relative to Ω_1 is :

$$\Omega_{2/1} = \Omega_2 - \Omega_1$$

Therefore:

$$\begin{aligned} v_{B_{\{ijk\}}} &= \left(\Omega_{2/1}\bar{i} \times L\tan(\beta)\bar{j}\right) \\ &= \left\{-L\tan(\beta)\ (\Omega_1 - \Omega_2)\ \cos(\beta+\gamma)^2 - L\tan(\beta)\ (\Omega_1 - \Omega_2)\ \sin(\beta+\gamma)^2\right\}\bar{k}' \end{aligned}$$

We can also define $v_{A_{\{ijk\}}}$ relative to the rotation rate Ω_1

$$
\begin{aligned}
v_{A_{\{ijk\}}} &= \Omega_A \bar{i}' \times L\frac{\sin(\gamma)}{\cos(\beta)}(-\bar{j}') \\
&= \left\{ -\frac{L\,\Omega_A\,\sin(\gamma)}{\cos(\beta)} \right\} \bar{k}'
\end{aligned}
$$

If we equate these two velocities as they are equal doe to the no slip condition, we can solve for Ω_A:

$$
v_{A_{\{ijk\}}} = v_{B_{\{ijk\}}} \quad \rightarrow \quad \Omega_A = \frac{\sin(\beta)}{\sin(\gamma)}\left(\Omega_1 - \Omega_2\right)
$$

This makes the angular velocity $\bar{\omega}$:

$$
\bar{\boldsymbol{\omega}} = \begin{bmatrix} \left\{ \frac{\sin(\beta)}{\sin(\gamma)}\left(\boldsymbol{\Omega_1} - \boldsymbol{\Omega_2}\right) + \cos(\boldsymbol{\beta}+\boldsymbol{\gamma})\ \boldsymbol{\Omega_1} \right\} \bar{\boldsymbol{i}}' \\ \left\{ -\sin(\boldsymbol{\beta}+\boldsymbol{\gamma})\ \boldsymbol{\Omega_1} \right\} \bar{\boldsymbol{j}}' \end{bmatrix}
$$

$\longrightarrow$ *Answer*

From this, the angular acceleration $\bar{\alpha}$ is solved for as:

$$
\begin{aligned}
\bar{\alpha} &= \sum_{2} \left(\dot{\omega}_n \bar{e}_n + \bar{\Omega}_n \times \omega_n \bar{e}_n \right) \\
&= \dot{\Omega}_1 \bar{i} + \overset{0}{\cancel{\Omega_1 \bar{i} \times \Omega_1 \bar{i}}} + \dot{\Omega}_A \bar{i}' + \bar{\omega} \times \Omega_A \bar{j} \\
&= \dot{\Omega}_1 \bar{i} + \frac{\sin(\beta)}{\sin(\gamma)} \left(\dot{\Omega}_1 - \dot{\Omega}_2 \right) \bar{i}' + \bar{\omega} \times \frac{\sin(\beta)}{\sin(\gamma)} \left(\Omega_1 - \Omega_2 \right) \bar{j}
\end{aligned}
$$

This evaluated to:

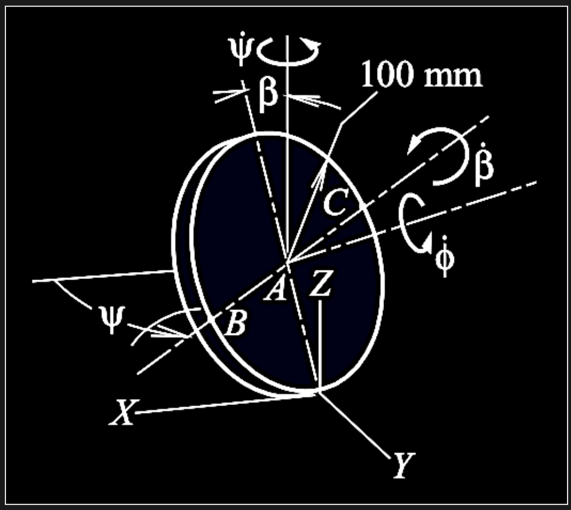
$$
\bar{\boldsymbol{\alpha}} = \begin{bmatrix} \left\{ \cos(\boldsymbol{\beta}+\boldsymbol{\gamma})\ \dot{\boldsymbol{\Omega}}_1 + \left(\dot{\boldsymbol{\Omega}}_1 - \dot{\boldsymbol{\Omega}}_2 \right) \frac{\sin(\beta)}{\sin(\gamma)} \right\} \bar{\boldsymbol{i}} \\ \left\{ -\sin(\boldsymbol{\beta}+\boldsymbol{\gamma})\ \dot{\boldsymbol{\Omega}}_1 \right\} \bar{\boldsymbol{j}} \\ \left\{ \sin(\boldsymbol{\beta}+\boldsymbol{\gamma}) \left(\boldsymbol{\Omega_1} - \boldsymbol{\Omega_2} \right) \boldsymbol{\Omega_1} \frac{\sin(\beta)}{\sin(\gamma)} \right\} \bar{\boldsymbol{k}} \end{bmatrix}
$$

$\longrightarrow$ *Answer*

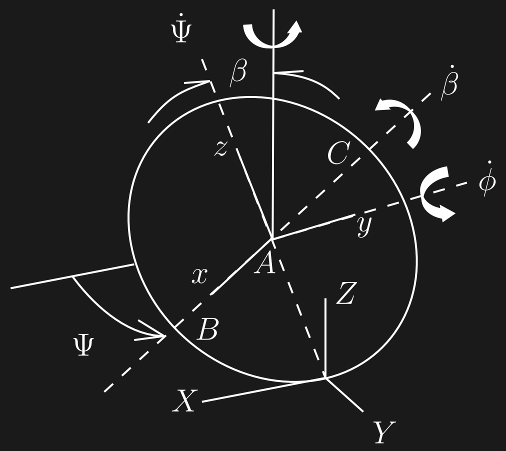
MATLAB's symbolic toolbox was leveraged in solving this problem. The script used to solve it is provided with the assignment submission as `Q4_39.m`.

EXERCISE 4.44

The disk rolls without slipping over the horizontal XY plane. At the instant when $\beta = 36.87°$, the X and Y components of the velocity of point B on the horizontal diameter of the disk are 8 m/s and -4 m/s, respectively, and the corresponding velocity components of center A at this instant are 4 m/s and 2 m/s. Determine the precession angle Ψ between the horizontal diameter BAC and the X axis, and also evaluate the precession, nutation, and spin rates.



In the following sketch, we will define two coordinate frames, $\{XYZ\}$ and $\{xyz\}$:



We can write the angular velocity $\bar{\omega}$ as:

$$\bar{\omega} = \dot{\Psi}\bar{K} + \dot{\beta}\bar{i} + \dot{\phi}\bar{j}$$

The transformation to convert $\{XYZ\}$ to $\{xyz\}$ is:

$$\begin{aligned} R &= \begin{bmatrix} \cos(\Psi) & -\sin(\Psi) & 0 \\ \sin(\Psi) & \cos(\Psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\Psi) & -\sin(\Psi)\cos(\beta) & \sin(\Psi)\sin(\beta) \\ \sin(\Psi) & \cos(\Psi)\cos(\beta) & -\cos(\Psi)\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{bmatrix} \end{aligned}$$

From this we see that:

$$\bar{i} = \begin{bmatrix} \cos(\Psi)\bar{I} \\ \sin(\Psi)\bar{J} \end{bmatrix}$$

$$\bar{j} = \begin{bmatrix} -\sin(\Psi)\cos(\beta)\bar{I} \\ \cos(\Psi)\cos(\beta)\bar{J} \\ \sin(\beta)\bar{K} \end{bmatrix}$$

$$\bar{k} = \begin{bmatrix} \sin(\Psi)\sin(\beta)\bar{I} \\ -\cos(\Psi)\sin(\beta)\bar{J} \\ \cos(\beta)\bar{K} \end{bmatrix}$$

Using these, we can write the angular velocity in terms of $\{XYZ\}$:

$$\bar{\omega} = \begin{bmatrix} \left\{\cos(\Psi)\dot{\beta} - \sin(\Psi)\cos(\beta)\dot{\phi}\right\}\bar{I} \\ \left\{\sin(\Psi)\dot{\beta} + \cos(\Psi)\cos(\beta)\dot{\phi}\right\}\bar{J} \\ \left\{\sin(\beta)\dot{\phi} + \dot{\Psi}\right\}\bar{K} \end{bmatrix}$$

From inspection of the sketch, we can infer that:

$$
\begin{aligned}
r_{A/D} &= 0.1\bar{k} \\
&= \begin{bmatrix} \left\{0.1\,\sin(\Psi)\,\sin(\beta)\right\}\bar{I} \\ \left\{-0.1\,\cos(\Psi)\,\sin(\beta)\right\}\bar{J} \\ \left\{0.1\,\cos(\beta)\right\}\bar{K} \end{bmatrix} \\
r_{B/D} &= 0.1\left(\bar{k}+\bar{i}\right) \\
&= \begin{bmatrix} \left\{0.1\,\cos(\Psi)+0.1\,\sin(\Psi)\,\sin(\beta)\right\}\bar{I} \\ \left\{0.1\,\sin(\Psi)-0.1\,\cos(\Psi)\,\sin(\beta)\right\}\bar{J} \\ \left\{0.1\,\cos(\beta)\right\}\bar{K} \end{bmatrix}
\end{aligned}
$$

Solving for the velocities at points A and B:

$$
\begin{aligned}
\bar{v}_A &= \cancelto{0}{\bar{v}_D} + \bar{\omega}\times\bar{r}_{A/D} \\
&= \begin{bmatrix} \left\{0.1\,\cos(\beta)\,\left(\dot{\beta}\,\sin(\Psi)+\dot{\phi}\,\cos(\Psi)\,\cos(\beta)\right)+0.1\,\cos(\Psi)\,\sin(\beta)\,\left(\dot{\Psi}+\dot{\phi}\,\sin(\beta)\right)\right\}\bar{I} \\ \left\{0.1\,\sin(\Psi)\,\sin(\beta)\,\left(\dot{\Psi}+\dot{\phi}\,\sin(\beta)\right)-0.1\,\cos(\beta)\,\left(\dot{\beta}\,\cos(\Psi)-\dot{\phi}\,\sin(\Psi)\,\cos(\beta)\right)\right\}\bar{J} \\ \left\{-0.1\,\dot{\beta}\,\sin(\beta)\right\}\bar{K} \end{bmatrix} \\
\bar{v}_B &= \cancelto{0}{\bar{v}_D} + \bar{\omega}\times\bar{r}_{B/D} \\
&= \begin{bmatrix} \left\{0.1\,\cos(\beta)\,\left(\dot{\beta}\,\sin(\Psi)+\dot{\phi}\,\cos(\Psi)\,\cos(\beta)\right)-\left(0.1\,\sin(\Psi)-0.1\,\cos(\Psi)\,\sin(\beta)\right)\,\left(\dot{\Psi}+\dot{\phi}\,\sin(\beta)\right)\right\}\bar{I} \\ \left\{\left(0.1\,\cos(\Psi)+0.1\,\sin(\Psi)\,\sin(\beta)\right)\,\left(\dot{\Psi}+\dot{\phi}\,\sin(\beta)\right)-0.1\,\cos(\beta)\,\left(\dot{\beta}\,\cos(\Psi)-\dot{\phi}\,\sin(\Psi)\,\cos(\beta)\right)\right\}\bar{J} \\ \left\{-0.1\,\dot{\phi}\,\cos(\beta)-0.1\,\dot{\beta}\,\sin(\beta)\right\}\bar{K} \end{bmatrix}
\end{aligned}
$$

Given this we can set the $\bar{I}$ and $\bar{J}$ components of both velocities base on the information from the problem, as well as $\beta = 36.87°$:

$$0.4 = \left\{0.0640\,\dot{\phi}\,\cos(\Psi) + 0.08\,\dot{\beta}\,\sin(\Psi) + 0.06\,\cos(\Psi)\,\left(\dot{\Psi} + 0.6\,\dot{\phi}\right)\right\}\bar{I}$$
$$0.2 = \left\{0.0640\,\dot{\phi}\,\sin(\Psi) - 0.08\,\dot{\beta}\,\cos(\Psi) + 0.06\,\sin(\Psi)\,\left(\dot{\Psi} + 0.6\,\dot{\phi}\right)\right\}\bar{J}$$
$$0.8 = \left\{0.0640\,\dot{\phi}\,\cos(\Psi) + 0.08\,\dot{\beta}\,\sin(\Psi) + \big(0.06\,\cos(\Psi) - 0.1\,\sin(\Psi)\big)\,\left(\dot{\Psi} + 0.6\,\dot{\phi}\right)\right\}\bar{I}$$
$$-0.4 = \left\{0.0640\,\dot{\phi}\,\sin(\Psi) - 0.08\,\dot{\beta}\,\cos(\Psi) + \big(0.1\,\cos(\Psi) + 0.06\,\sin(\Psi)\big)\,\left(\dot{\Psi} + 0.6\,\dot{\phi}\right)\right\}\bar{J}$$

Solving these simultaneous equations in MATLAB, the system resolves to two different answers, one of which matches the answer from the textbook:

$$\boldsymbol{\Psi = 33.6901^\circ}$$
$$\boldsymbol{\dot{\Psi} = -15.4277}\ \text{rad/s}$$
$$\boldsymbol{\dot{\beta} = 0.6934}\ \text{rad/s}$$
$$\boldsymbol{\dot{\phi} = 13.6942}\ \text{rad/s}$$

$\longrightarrow$ Answer

MATLAB's symbolic toolbox was leveraged heavily in solving this problem. The script used to solve it is provided with the assignment submission as `Q4_44.m`.

Submitted by Austin Barrilleaux on September 24, 2024.