Polynomial interpolation of modular forms for Hecke groups

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Abstract

For $m = 3, 4, ...,$ let $\lambda_m = 2\cos\pi/m$ and let $J_m(m = 3, 4, ...)$ be triangle functions for the Hecke groups $G(\lambda_m)$ with Fourier expansions $J_m(z) = \sum_{n=-1}^{\infty} a_n(m)q_m^n$, where $q_m(z) = \exp 2\pi iz/\lambda_m$. (When normalized appropriately, J_3 becomes Klein's j-invariant $j(z) = 1/e^{2\pi iz} + 744 + ...$) For $n = -1, 0, 1, 2$ and 3, Raleigh [25] gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $m = 3, 4, ...,$ and conjectured that similar relations hold for all positive integers n. Akiyama proved this conjecture in [1]. We apply work of Hecke [4] to study experimentally similar polynomial interpolations of the J_m Fourier coefficents and the Fourier coefficients of other, positive weight, modular forms for $G(\lambda_m)$. We connect these polynomials (again, only empirically) with variants of Dedekind's eta function, with the Fourier expansions of some standard Hauptmoduln, and, in the case of analogues of Eisenstein series for $SL(2, \mathbb{Z})$, with certain divisor sums.

1 Introduction

Let $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ and $\mathbb{H}$ denote, respectively, the set of rational integers, the set of rational numbers, the set of complex numbers, and the set of complex numbers with positive imaginary parts. We write $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$, and we equip $\mathbb{H}^*$ with the Poincaré metric. Figures T made by three geodesics of $\mathbb{H}^*$ are called hyperbolic or circular-arc triangles. Let $\lambda_m = 2\cos\pi/m$. For $m = 3, 4, ...,$ we define the Hecke group $G(\lambda_m)$ as the discrete group generated by the maps $z \to -1/z$ and $z \to z + \lambda_m$. The full modular group $SL(2, \mathbb{Z})$ is identical to $G(\lambda_3)$.

To define modular forms for the Hecke groups, we preview a definition from Berndt [4], which we will quote again in a later section. (We depart occasionally from Berndt's choices of variable to avoid clashes with some of our other notation.)

We say that f belongs to the space $M(\lambda, k, \gamma)$ if

1.
$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda},$$
where $\lambda > 0$ and $\tau \in \mathbb{H}$, and

2. $f(-1/\tau) = \gamma(\tau/i)^k f(\tau)$, where $k > 0$ and $\gamma = \pm 1$.

We say that f belongs to the space $M_0(\lambda, k, \gamma)$ if f satisfies conditions 1 and 2 and if $a_n = O(n^c)$ for some real number c, as n tends to ∞.

Members of $M(\lambda, k, \gamma)$ are known as modular forms for $G(\lambda)$ of weight k. Condition 1 tells us that they are invariant under translations $\tau \mapsto \tau + \lambda$. Next we preview Berndt's definition of cusp forms for Hecke groups. If $f \in M(\lambda, k, \gamma)$ and $f(i\infty) = 0$, then we call f a cusp form of weight k and multiplier γ with respect to $G(\lambda)$. For cusp forms, the constant terms of condition 1 vanish. We denote by $C(\lambda, k, \gamma)$ the vector space of all cusp forms of this kind.

For our purposes, Schwarz triangles T are hyperbolic triangles in $\mathbb{H}^*$ with certain restrictions on the angles at the vertices. From a Euclidean point of view, their sides are vertical rays, segments of vertical rays, semicircles orthogonal to the real axis and meeting it at points $(r, 0)$ with r rational, or arcs of such semicircles. We choose λ, μ and ν, all non-negative, such that $\lambda + \mu + \nu < 1$; then the angles of T are $\lambda\pi, \mu\pi$, and $\nu\pi$. By reflecting T across one of its edges, we get another Schwarz triangle. The reflection between two triangles in $\mathbb{H}^*$ is effected by a Möbius transformation, so the orbit of T under repeated reflections is associated to a collection of Möbius transformations. The group generated by these transformations is a triangle group. By the Riemann Mapping Theorem there is a conformal, onto map $\phi : T \mapsto \mathbb{H}^*$ called a triangle function.

Hecke groups [17] are triangle groups H that act properly discontinuously on $\mathbb{H}$. This means that for compact $K \subset \mathbb{H}$, the set $\{\mu \in H \text{ s.t.} K \cap \mu(K) \neq \varnothing\}$ is finite. Recall that $G(\lambda_m)$ is the Hecke group generated by the maps $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + \lambda_m$. Apparently it was Hecke who, also in [17], established that $G(\lambda_m)$ has the structure of a free product of cyclic groups $C_2 * C_m$, generalizing the relation [29, 8] $SL(2, \mathbb{Z}) = C_2 * C_3$.

Let $\rho = -\exp(-\pi i/m) = -\cos(\pi/m) + i\sin(\pi/m)$, and let $T_m \subset \mathbb{H}^*$ denote the hyperbolic triangle with vertices ρ, i, and $i\infty$. The corresponding angles are $\pi/m, \pi/2$ and 0 respectively. Let ϕ_{λ_m} be a triangle function for T_m. The function ϕ_{λ_m} has a pole at $i\infty$ and period λ_m. For $P, Q \in \mathbb{H}^*$, let us us write $P \equiv_H Q$ when $\mu \in H$ and $Q = \mu(P)$. Then ϕ_{λ_m} extends to a function $J_m : \mathbb{H}^* \to \mathbb{H}^*$ by declaring that $J_m(P) = J_m(Q)$ if and only if $P \equiv_H Q$. J_m is a modular function for $G(\lambda_m)$.

Schwarz [28], Lehner [20], Raleigh [25] and others studied Schwarz triangle functions, which map hyperbolic triangles T in the extended upper half z-plane onto the extended upper half w-plane. For certain $T = T_m$, a triangle function $\phi_{\lambda_m} : T \to \mathbb{H}^*$ extends to a map $J_m : \mathbb{H}^* \to \mathbb{H}^*$ invariant under modular transformations from $G(\lambda_m)$. Suitably normalized, the J_m become analogues j_m of the normalized Klein's modular invariant

$$j(z) = -1/q + 744 + 196884q + ...$$

where $q = q(z) = \exp(2\pi iz)$ and $j_3(z) = j(z)$.

With $\lambda_m = 2\cos\pi/m$ and $q_m(z) = \exp(2\pi iz/\lambda_m)$, the original J_m have Fourier series $J_m(z) = \sum_{n\geqslant -1} a_n(m) q_m(z)^n$. For $n = -1, 0, 1, 2$ and 3, Raleigh [25] gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $m = 3, 4, ...,$ and conjectured that similar relations hold for all positive integers n. Akiyama [1] proved Raleigh's conjecture in the passage after his (Akiyama's) equation (6).

Hecke ([4], [17]) built families of modular forms f_m for $G(\lambda_m)$ sharing particular properties, such as their weight; we will describe his construction later. Earlier authors, whose work we will also describe, had already built modular functions (meromorphic functions invariant under the action of $G(\lambda)$, thus, of weight zero) from triangle functions.

The plan of the article is as follows. We sketch the theory of Schwartz triangles; then, the construction of their triangle functions; from the triangle functions, the modular functions for the Hecke groups; and, from them, modular forms for these groups. To this point, the article is just a summary of background material. By methods familiar from the classical case, we then construct modular forms with rational Fourier expansions, describe our modifications of them, and describe computer experiments on the modifications. We documented our experiments in a series of *Mathematica* notebooks and accompanying data files. A notebook containing a searchable library of definitions of functions mentioned in the other ones is here ([5],"mf23.nb".) We link to the other supporting documents as they come up. The experiments indicate that the Fourier expansions of several Hecke modular forms are interpolated in the sense of Raleigh by polynomials. We describe connections among these polynomials, certain divisor sums, and Hauptmoduln discussed in the Moonshine literature. We offer only conjectures; we prove no theorems of our own. All theorems that we mention are part of the existing literature.

The earliest computer code we located for calculating Fourier expansions of triangle functions for Hecke groups is that of Leo; it is based on Lehner's construction. Leo also calculates the Fourier coefficients of weight 4 and weight 6 Hecke-analogues of classical Eisenstein series in Chapter 4 of [21]. Our code for triangle functions, which is based on Leo's, comes from the papers of Lehner and Raleigh. J. Jermann's package [18] is also concerned with modular forms of triangle groups for Hecke groups, but we did not use his code in our experiments. For definiteness, we transcribe our code defining triangle functions and modular forms in an appendix.

2 A glossary

Several special functions in this list are so closely related that one might wish to know why we have included all of them. It is for the convenience of both the reader and ourselves, because differing forms (basically just different notations) are used by Lehner and Raleigh, and we hope in this way to discuss both of them while avoiding confusion.

1. The digamma function $\psi(z) := \Gamma'(z)/\Gamma(z)$.

2. The Schwarzian derivative ([10], p. 130, equation 370.8)

$$\{w, z\} = \frac{2w'w''' - 3w''^2}{2w'^2} \tag{1}$$

for $w = w(z)$.

3. The Pochhammer symbol

$(a)^0 := 1$ and, for $n \geqslant 1, (a)^n := a(a+1)...(a+n-1) = \Gamma(a+n)/\Gamma(a)$.

4. The function c_ν given by ([10], p. 138, equation 377.3)

$$c_\nu = c_\nu(\alpha, \beta, \gamma) := \frac{(\alpha)^\nu(\beta)^\nu}{\nu!(\gamma)^\nu}, \nu \geqslant 0.$$

To facilitate comparison with Raleigh's equation (9^1) [25], we remark that

$$c_\nu = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)} \cdot \frac{\Gamma(1)}{\Gamma(1+\nu)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+\nu)}. \tag{2}$$

In the terms of this article's Theorem 1 below, Raleigh is treating the case $\lambda = 0$, for which (equation (6) below) $\gamma = 1$ and the expression on the right side of (2) becomes, as in Raleigh,

$$\frac{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}{\Gamma(\alpha)\Gamma(\beta)(\nu!)^2}.$$

5. The function e_ν given by ([25], equation 9^1)

$$e_\nu = e_\nu(\alpha, \beta) := \sum_{p=0}^{\nu-1} \left(\frac{1}{\alpha+p} + \frac{1}{\beta+p} - \frac{2}{1+p} \right).$$

Here, we are dealing with the same ambiguity present in the definition of c_ν: this is a specialization to the case $\gamma = 1$ of the e_ν for $\nu \geqslant 1$ given by ([10], p. 153, equation 387.5)

$$e_\nu = e_\nu(\alpha, \beta, \gamma) := \sum_{p=0}^{n-1} \left(\frac{1}{\alpha+p} + \frac{1}{\beta+p} - \frac{2}{\gamma+p} \right).$$

Unless it is explicitly indicated to be otherwise, we intend the former (Raleigh's) definition.

6. (a) Gauss's hypergeometric series given by [10], p. 138, equation (377.4),

$$F(\alpha, \beta, \gamma; \tau) := \sum_{\nu=0}^{\infty} c_\nu(\alpha, \beta, \gamma)\tau^\nu$$

   F is occasionally written in [10] as ϕ_1 (for example, on, p. 152.)

   (b) As defined in the first line of [10], p. 142:

$$F_1(\alpha, \beta, \gamma; \tau) := F(\alpha, \beta, \gamma + 1; \tau).$$

   (c) Alternatively, dropping γ, as defined in [21], equation (3.5):

$$F_1(\alpha, \beta; z) := \sum_{\nu=1}^{\infty} \frac{(\alpha)_k(\beta)_\nu}{(\nu!)^2} e_\nu(\alpha, \beta).$$

   It is in the latter form, defined more cryptically in [20], p. 244, that we will use F_1; to establish his series for the triangle functions, which we will apply below, Lehner uses this definition of F_1, as well as certain theorems from Fricke [14]. Referring to item 4, we see that

$$F_1(\alpha, \beta; z) = \sum_{\nu=1}^{\infty} c_\nu(\alpha, \beta, 1)e_\nu(\alpha, \beta).$$

   We will derive another form of $F_1(\alpha, \beta; z)$ in item 7.

7. With $F = F(\alpha, \beta, \gamma; \tau)$, a special function

$$F^*(\alpha, \beta, \gamma; \tau) := \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \beta} + 2\frac{\partial F}{\partial \gamma};$$

   by [10], equation (387.4) on p. 153, F^* may be written

$$F^*(\alpha, \beta, \gamma; \tau) = \sum_{\nu=1}^{\infty} c_\nu(\alpha, \beta, \gamma)e_\nu(\alpha, \beta, \gamma)\tau^\nu.$$

   It follows that $F^*(\alpha, \beta, 1; \tau) = F_1(\alpha, \beta; z)$.

8. A special function $\phi_2^*(\tau)$ is defined as a certain limit ([10], p. 152, equation 386.2) but is immediately (equation 386.3) reduced to

$$\phi_2^*(\tau) = F(\alpha, \beta, 1; \tau)\log\tau + F^*(\alpha, \beta, 1; \tau).$$

9. The set $\mathscr{Q} = \{2, 5, 6, 8, 10, 11, 14, 15, 17, 18, 20, 22, 23, ...\}$ of positive integers not represented by the quadratic form $x^2 + xy + y^2$ [30]. B. Cloitre asserts, on the same O.E.I.S. page, that $\mathscr{Q}$ is also the set of non-negative integers n such that $\delta(n)$ is non-zero, where η is Dedekind's eta function and $\sum_n \delta(n)x^n = \eta(x^3)/\eta(x)^3$.

10. The McKay-Thompson series of class 4A, $\{1, 24, 276, 2048, ...\}$, which is the sequence of coefficients in the q-series of a certain hauptmodul discussed in [22]. McKay relates the hauptmoduln treated in that paper to Dedekind's eta function. We identified the class 4A McKay Thompson series with the $\overline{a}_n$ of our conjecture 1 after finding it in the *On-line Encyclopedia of Integer Sequnces* [35].

3 Calculation of Schwarz's inverse triangle function

Schwarz proved

Theorem 1. *([10], §374)*

1. *Let the half-plane $\Im z > 0$ be mapped conformally onto an arbitrary circular-arc triangle whose angles at its vertices A, B, and C are $\pi\lambda, \pi\mu$, and $\pi\nu$, and let the vertices A, B, C be the images of the points $z = 0, 1, \infty$, respectively. Then the mapping function $w(z)$ must be a solution of the third-order differential equation*

$$\{w, z\} = \frac{1-\lambda^2}{2z^2} + \frac{1-\mu^2}{2(1-z^2)} + \frac{1-\lambda^2-\mu^2+\nu^2}{2z(1-z)}. \tag{3}$$

2. *If $w_0(z)$ is any solution of equation (3) that satisfies $w_0'(z) \neq 0$ at all interior points of the half-plane, then the function*

$$w(z) = \frac{aw_0(z)+b}{cw_0(z)+d} \qquad (ad - bc \neq 0)$$

   is likewise a solution of equation 3.

3. *Also, every solution of equation (3) that is regular and non-constant in the half-plane $\Im z > 0$ represents a mapping of this half-plane onto a circular-arc triangle with angles $\pi\lambda, \pi\mu$, and $\pi\nu$.*

In Carathéodory's lexicon ([9] p. 124), a regular function is one that is differentiable on an open connected set. Carathéodory writes the left side of (3) as "$\{w, z\} = \frac{w'w''' - 3w''^2}{w'^2} = ...$", but this is not the case. We infer that the Schwarzian derivitave $\{w, z\}$ is intended from the automorphy property of clause 2.

Let us write

$$\alpha = \frac{1}{2}(1 - \lambda - \mu + \nu), \tag{4}$$

$$\beta = \frac{1}{2}(1 - \lambda - \mu - \nu), \tag{5}$$

and

$$\gamma = 1 - \lambda. \tag{6}$$

The solutions w of (3) are inverse to triangle functions; they are quotients of arbitrary solutions of

$$u'' + p(z)u' + q(z)u = 0 \tag{7}$$

when ([10], p. 136, equation (376.4))

$$p = \frac{1-\lambda}{z} - \frac{1-\mu}{1-z}$$

and

$$q = -\frac{\alpha\beta}{z(1-z)}.$$

Equation (7) reduces ([10], p. 137, equations 376.5-7) to the hypergeometric differential equation

$$z(1-z)u'' + (\gamma - (\alpha+\beta+1)z)u' - \alpha\beta u = 0. \tag{8}$$

As long as γ is not a non-positive integer, $u = F(\alpha, \beta, \gamma; z)$ is a solution of (8); it is the only solution regular at $z = 0$, and it satisfies $F(\alpha, \beta, \gamma; 0) = 1$ (final paragraph of [10], §377, p. 138.)

In [10], §§386-388 (pp. 151-155), we find that when $\gamma = 1$ and $\lambda = 0$, another, linearly independent, solution of equation (7) is $\phi_2^*(z)$. [10], Section 394, pp. 165 - 167 is devoted to the case $\lambda = 0$. The mapping function w of Theorem 1 satisfies ([10], p. 166, equation 394.4)

$$w = \frac{1}{\pi i}\left[\frac{\phi_2^*}{\phi_1} - (2\psi(1) - \psi(1-\alpha) - \psi(1-\beta))\right] + i\frac{\sin \pi\mu}{\cos \pi\mu + \cos \pi\nu}. \tag{9}$$

4 Inversion of Schwarz's inverse triangle function

Following Lehner and Raleigh, we consider the Schwarz triangle T_m with vertices at $\rho = -\exp(-\pi i/m), i$, and $i\infty$. In terms of Theorem 1, T_m has $\lambda = 0$ (an angle 0 at the vertex $i\infty$), $\mu = 1/2$ (an angle $\pi/2$ at i), and $\nu = 1/m$ (an angle π/m at ρ.) In this situation, $\gamma = 1$.

Let J_m be automorphic for $G(\lambda_m)$ with $J_m(\rho) = 0, J_m(i) = 1$, and $J_m(i\infty) = \infty$. In terms of Theorem 1, w and J_m are inverse functions. We are going to write down the Fourier expansion $\sum_{n=-1}^{\infty} a_n q_m(z)^n$ of J_m.

By clause 2 of Theorem 1, if w satisfies equations (3) and (9), so does $\tau = \tau(z) = \lambda_m w(z)/2$, and therefore

$$2\pi i\tau/\lambda_m = \frac{\phi_2^*}{\phi_1} - (2\psi(1) - \psi(1-\alpha) - \psi(1-\beta)) - \pi\sec(\pi/m).$$

Let us write $\log A_m = -2\psi(1) + \psi(1-\alpha) + \psi(1-\beta) - \pi\sec(\pi/m)$. In general, $A_m = a_{-1}(m)$ ([25], for example two lines below equation (12).) Recalling the definitions of ϕ_1 and ϕ_2^* from our glossary items 6 and 8, we find (abbreviating $J_m(\tau)$ as J_m) that

$$2\pi i\tau/\lambda_m = -\log J_m + \frac{F^*(\alpha,\beta,1;1/J_m)}{F(\alpha,\beta,1;1/J_m)} + \log A_m. \tag{10}$$

Equation (10) is equation (6) of [25], but Raleigh suppresses the subscripts. He also writes $\exp 2\pi i\tau/\lambda_m$ as x_m, so that (in our earlier notation) $x_m = q_m(\tau)$.

In Raleigh's notation, after taking exponentials,

$$x_m/A_m = \frac{1}{J_m}\exp\frac{F^*(\alpha,\beta,1;1/J_m)}{F(\alpha,\beta,1;1/J_m)}, \tag{11}$$

the right side of which has a power series in J_m with rational coefficients. Writing $X_m = x_m/A_m$ we can regard $X_m = X_m(J_m)$ as a power series in J_m. Following [20] and [25], we inverted this power series to obtain one for the modular function J_m. Let $\mathscr{I}$ be a formal operation taking a power series $\sigma(v)$ to its inverse; that is, if $u = \sigma(v)$ then $v = \mathscr{I}(\sigma)(u)$. Let $Y_m(J)$ be a power series such that

$$Y_m(J_m) = J_m \exp\frac{F^*(\alpha,\beta,1;J_m)}{F(\alpha,\beta,1;J_m)} = X_m\,(1/J_m)$$

and hence

$$Y_m(1/J_m) = \frac{1}{J_m}\exp_m\frac{F^*(\alpha,\beta,1;1/J_m)}{F(\alpha,\beta,1;1/J_m)} = X_m(J_m),$$

so that $\mathscr{I}(Y_m)(X_m(J)) = 1/J_m$ and, therefore, $J_m = 1/\mathscr{I}(Y_m)(X_m)$.

We noticed several typos in [25]. Four of Raleigh's equations—(I), (10), and the two equations on p. 109 that begin "$a_{-1}(q) = \ldots$" (where his q is our m)—are pairwise contradictory. From the second paragraph on Raleigh's p. 110, we expect that $A_3 = a_{-1}(3) = 1/1728, A_4 = a_{-1}(4) = 1/256$, and $A_6 = a_{-1}(6) = 1/108$. These values are consistent with Raleigh's equation (10), but not with the others. We infer that all of them except (10) are incorrect. Thus, following Raleigh by writing ψ for the digamma function, $\alpha(m)$ for $(1/2 - 1/m)/2$, and $\beta(m)$ for $(1/2 + 1/m)/2$,

$$a_{-1}(m) = \exp\left(-2\psi(1) + \psi(1-\alpha(m)) + \psi(1-\beta(m)) - \pi\sec(\pi/m)\right). \tag{12}$$

5 Modular forms from modular functions

When the w-image of $\mathbb{H}^*$ is T_m, the inverse of w is ϕ_{λ_m}. J_m, the extension by modularity of ϕ_{λ_m} to $\mathbb{H}^*$, is periodic with period λ_m and maps ρ to 0, i to 1, and $i\infty$ to ∞ ([20], equation (2).) These mapping properties allow us, following

Berndt's exposition [4] of Hecke, to construct positive weight modular forms for $G(\lambda_m)$ from J_m. This section describes results of Hecke that are perhaps most easily accessible for the classical case $m = 3$ in Schoeneberg [27] and, for the general case, in Berndt [4].

5.1 The case $m = 3$.

By keeping track of the weights, zeros and poles of the constituent factors in the numerator and denominator of the fraction defining

$$f_{a,b,c} = \frac{J'^a}{J^b(J-1)^c},$$

Schoeneberg [27] (Theorem 16, p.45) demonstrates that $f_{a,b,c}$ is an entire modular form of weight $2a$ for $SL(2,\mathbb{Z})$ if $a \geqslant 2, 3c \leqslant a, 3b \leqslant 2a$, $b + c \geqslant a$ and a, b, c are integers. (Schoeneberg speaks of "dimension $-2a$.") Thus he is able to write down a weight 4 entire modular form $E_4^* = f_{2,1,1}$ for $SL(2,\mathbb{Z})$ with a zero of order $\frac{1}{3}$ at $\rho = e^{2\pi i/3}$ and a weight 6 entire modular form $E_6^* = f_{3,2,1}$ for $SL(2,\mathbb{Z})$ with a zero of order $\frac{1}{2}$ at i. (Schoeneberg writes G_4^*, G_6^*.) It is well known that the (vector space) dimension of the spaces of weight 4 and 6 entire modular forms for $SL(2,\mathbb{Z})$ is equal to one, so E_4^* and E_6^* may be identified with the usual weight 4 and weight 6 Eisenstein series, up to a normalization. Finally, Schoeneberg defines the weight 12 cusp form $\Delta^* = E_4^{*3} - E_6^{*2}$ with a zero of order 1 at $i\infty$. It is a multiple of Δ.

5.2 The case $m \geqslant 3$.

We quote statements from Berndt [4], which is an exposition of Hecke's [16] and other writings. We depart occasionally from Berndt's choices of variable to avoid clashes with our earlier notation.

Definition 1. *([4], Definition 2.2) We say that f belongs to the space $M(\lambda, k, \gamma)$ if*

1.
$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda},$$
where $\lambda > 0$ and $\tau \in \mathbb{H}$, and

2. *$f(-1/\tau) = \gamma \cdot (\tau/i)^k f(\tau)$, where $k > 0$ and $\gamma = \pm 1$.*

We say that f belongs to the space $M_0(\lambda, k, \gamma)$ if f satisfies conditions 1 and 2, and if $a_n = O(n^c)$ for some real number c, as n tends to ∞.

After defining the notion of a fundamental region in the usual way and defining as $G(\lambda)$ the group of linear fractional transformations generated by $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + \lambda$, Berndt states (for $\tau = x + iy$)

Theorem 2. *([4], Theorem 3.1) Let $B(\lambda) = \{\tau \in \mathbb{H} : x < \lambda/2, |\tau| > 1\}$. Then if $\lambda \geqslant 2$ or if $\lambda = 2\cos(\pi/m)$, where $m \geqslant 3$ is an integer, $B(\lambda)$ is a fundamental region for $G(\lambda)$.*

Definition 2. *([4], Definition 3.4) Let $T_A = \{\lambda : \lambda = 2\cos(\pi/m), m \geqslant 3, m \in \mathbb{Z}\}$.*

Berndt states in his Theorem 5.4 that $G(\lambda)$ is discrete if and only if λ belongs to T_A. This discreteness is the premise of the theory of automorphic functions generally. He embeds within the proof of his Lemma 3.1 (which we omit), the

Definition 3. *τ_λ denotes the intersection in $\mathbb{H}$ of the line $x = -\lambda/2$ and the unit circle $|\tau| = 1$.*

(Berndt remarks at the top of page 35 that τ_λ is the lower left corner of $B(\lambda)$). and that $\pi\theta = \pi - \arg(\tau_\lambda)$, so that $\cos(\pi\theta) = \lambda/2$.)

To characterize Eisenstein series, we need to keep track of some analytical properties. The next definition summarizes the second paragraph of Berndt's Chapter 5. (Throughout his Chapter 5, $\lambda < 2$.)

Definition 4. *Let $f \in M(\lambda, k, \gamma), f$ not identically zero.*

1. *$N = N_f$ counts the zeros of f on $\overline{B(\lambda)}$ with multiplicities.*
2. *N_f does not count zeros at τ_λ, at $\tau_\lambda + \lambda$, at i, or at $i\infty$.*
3. *If $\tau_0 \in \overline{B(\lambda)}, f(\tau_0) = 0$ and $\Re(\tau_0) = -\lambda/2$, then $f(\tau_0 + \lambda) = 0$ and N_f counts only one of the two zeros.*
4. *If $\tau_0 \in \overline{B(\lambda)}, f(\tau_0) = 0$, and $|\tau_0| = 1$, then, $f(-1/\tau_0) = 0$, and N_f counts only one of these two zeros.*
5. *The numbers n_λ, n_i, and n_∞ are the orders of the zeros of f at τ_λ, i and $i\infty$, repectively. The order n_∞ is measured in terms of $\exp(2\pi i\tau/\lambda)$.*

The multiplier γ is given by

Theorem 3. *([4], Corollary 5.2) Let $f \in M(\lambda, k, \gamma)$ and let n_i be the order of the zero of f at $\tau = i$. Then*
$$\gamma = (-1)^{n_i}.$$

The next two results tell us that the only nontrivial case in this theory is the one that we are interested in.

Theorem 4. *([4], Lemma 5.1) If $\dim M(\lambda, k, \gamma) \neq 0$,*
$$N_f + n_\infty + \frac{1}{2}n_i + \frac{n_\lambda}{m} = \frac{1}{2}k\left(\frac{1}{2} - \theta\right).$$

By Berndt's equation (5.16), if $m \geqslant 3$ then the right side can be written as $k(m-2)/4m$.

Theorem 5. *([4], Theorem 5.2) If* $\dim M(\lambda, k, \gamma) \neq 0$, *then* $\theta = 1/m$ *where* $m \geqslant 3$ *and* $m \in \mathbb{Z}$.

We are concerned with $\lambda \in T_A$. This makes $\lambda < 2$ as in all the results of Berndt's Chapter 5.

One estimate for $\dim M(\lambda, k, \gamma)$ is

Theorem 6. *([4], Theorem 5.6) If* $\lambda \notin T_A$, *then* $\dim M(\lambda, k, \gamma) = 0$. *If* $\lambda = 2\cos(\pi/m) \in T_A$, *then for nontrivial* $f \in M(\lambda, k, \gamma)$, *the weight* k *has the form*

$$k = \frac{4h}{m-2} + 1 - \gamma,$$

where $h \geqslant 1$ *is an integer. Furthermore,*

$$\dim M(\lambda, k, \gamma) = 1 + \left\lfloor \frac{h + (\gamma - 1)/2}{m} \right\rfloor.$$

Eliminating h, we find that

$$\dim M(\lambda, k, \gamma) = 1 + \left\lfloor k\left(\frac{1}{4} - \frac{1}{2m}\right) + \frac{\gamma}{4} - \frac{1}{4} \right\rfloor. \tag{13}$$

Berndt ([4], Remark 5.3) proves that the dimension formula above holds also when $h = 0$.

The existence of certain modular forms is provided by

Theorem 7. *([4], Theorem 5.5) Let* $\lambda \in T_A$. *Then there exist functions* f_λ, f_i, *and* $f_\infty \in M(\lambda, k, \gamma)$ *such that each has a simple zero at* τ_λ, i, *and* $i\infty$, *respectively, and no other zeros. Here,* γ *is given by [Theorem 3 (this article)], and* k *is determined in each case from [Theorem 4 (this article)]. Thus,* $f_\lambda \in M(\lambda, 4/(m-2), 1), f_i \in M(\lambda, 2m/(m-2), -1)$, *and* $f_\infty \in M(\lambda, 4m/(m-2), 1)$.

Remark 1. *([4], pages 47-48) By the Riemann mapping theorem there exists a function* $g(\tau)$ *that maps the simply connected region* $B(\lambda)$ *one-to-one and conformally onto* $\mathbb{H}$. *If we require that* $g(\tau_\lambda) = 0, g(i) = 1$, *and* $g(i\infty) = \infty$, *then* g *is determined uniquely.*

Now we can write down f_λ, f_i, and f_∞ explicitly. The next theorem is extracted from the proof of Theorem 7. f_λ and f_i correspond to Eisenstein series and f_∞ to a cusp form. In our code, we take g to be a normalized form of J_m.

Theorem 8. *([4], page 50)*

$$f_\lambda(\tau) = \left\{ \frac{g'(\tau)^2}{g(\tau)(g(\tau)-1)} \right\}^{1/(m-2)},$$

$$f_i(\tau) = \left\{ \frac{g'(\tau)^m}{g(\tau)^{m-1}(g(\tau)-1)} \right\}^{1/(m-2)},$$

and

$$f_\infty(\tau) = \left\{ \frac{g'(\tau)^{2m}}{g(\tau)^{2m-2}(g(\tau)-1)^m} \right\}^{1/(m-2)}.$$

In our applications to Lehmer's problem, we will be interested in the dimensions of the weight 12 cusp spaces for $\lambda = \lambda_m = 2\cos\pi/m$.

Definition 5. *([4], Definition 5.2) If $f \in M(\lambda, k, \gamma)$ and $f(i\infty) = 0$, then we call f a cusp form of weight k and multiplier γ with respect to $G(\lambda)$. We denote by $C(\lambda, k, \gamma)$ the vector space of all cusp forms of this kind.*

Remark 2. *([4], equation (5.25))*

$$\dim C(\lambda, k, \gamma) \geqslant \dim M(\lambda, k, \gamma) - 1.$$

Remark 3. *In view of (i) Theorem 6, (ii) equation (12), (iii) Remark 2, and (iv) the fact that $\gamma = \pm 1$, we see that* $\dim C(\lambda_m, 12, \gamma) > 1$ *when m is greater than or equal to* 12.

6 The objects in our experiments

In this section, we write down versions of the functions from Theorem 8 such that, at $m = 3$, they reduce to corresponding functions in the classical theory. The classical objects (in Serre's notation [29]), are Klein's j-invariant, the weight four Eisenstein series E_2, the weight six Eisenstein series E_3, and the generating function of Ramanujan's tau function, namely the normalized weight twelve cusp form Δ. Our definitions come in pairs because we find that, in this way, we can extract a little more information about the interpolating polynomials in some situations. The second definition in each pair replaces 1 by $2^6 m^3$, but (when it matters) they can be cast into the original form up to a multiplicative constant depending upon m.

Definition 6. *Writing the Fourier expansion of $J_m(z)$ as $\sum_{n=-1}^{\infty} a_n(m) q_m(z)^n$, we set $j_m(\tau) := \sum_{n=-1}^{\infty} a_n(m)(2^6 m^3 q_m(z))^n$.*

(Inspection indicates that j_3 is the classical j function, and similar remarks apply to the definitions below.) Corresponding to f_λ,

Definition 7. *1. $H_{\lambda,m}(\tau)$:=*

$$\left\{ \frac{J'_m(\tau)^2}{J_m(\tau)(J_m(\tau)-1)} \right\}^{1/(m-2)}.$$

2. $H_{\lambda,4,m}(\tau) := H_{\lambda,m}(\tau)^{m-2}$.

3. Writing the Fourier expansion of $H_{\lambda,4,m}(\tau)$ as $\sum_{n=0}^{\infty} h_{4,n}(m) q_m(z)^n$, we set $H^*_{\lambda,4,m}(\tau) := \sum_{n=-1}^{\infty} h_{4,n}(m)(2^6 m^3 q_m(z))^n$.

4. $H_{\lambda,6,m}(\tau) := H_{\lambda,m}(\tau)^{\frac{3}{2}(m-2)}$.
5. $\widetilde{H}_{\lambda,m}(\tau)$:=
$$\left\{\frac{j_m'(\tau)^2}{j_m(\tau)(j_m(\tau)-2^6m^3)}\right\}^{1/(m-2)}.$$
6. $\widetilde{H}_{\lambda,4,m}(\tau) := \widetilde{H}_{\lambda,m}(\tau)^{m-2}$.
7. $\widetilde{H}_{\lambda,6,m}(\tau) := \widetilde{H}_{\lambda,m}(\tau)^{\frac{3}{2}(m-2)}$.

Corresponding to f_i, we state

Definition 8. *1.*
$$H_{i,m}(\tau) := \left\{\frac{J_m'(\tau)^m}{J_m(\tau)^{m-1}(J_m(\tau)-1)}\right\}^{1/(m-2)}.$$

2. $H_{i,4,m}(\tau) := H_{i,m}(\tau)^{2(m-2)/m}$.
3. $H_{i,6,m}(\tau) := H_{i,m}(\tau)^{3(m-2)/m}$.
4. *Let $\epsilon(m) = e^{-i\pi/(m-2)}$ or one, depending on whether m is odd or even, respectively.*
$$\widetilde{H}_{i,m}(\tau) := \epsilon(m)\left\{\frac{j_m'(\tau)^m}{j_m(\tau)^{m-1}(j_m(\tau)-2^6m^3)}\right\}^{1/(m-2)}.$$
5. $\widetilde{H}_{i,4,m}(\tau) := \widetilde{H}_{i,m}(\tau)^{2(m-2)/m}$.
6. $\widetilde{H}_{i,6,m}(\tau) := \widetilde{H}_{i,m}(\tau)^{3(m-2)/m}$.

Remark 4. *In view of Berndt's theorem 7 above, the functions defined in items 2, 3 and 6 of definition 7 and items 2 and 5 of definition 8 all have weight* 4*. Similarly, the functions defined in items 4 and 7 of definition 7 and items 3 and 6 of definition 8 have weight* 6*.*

In our *Mathematica* code, Hecke modular forms and functions are expressed as power series in x_m, so that *Mathematica* is performing a purely formal series operation when the exponent $1/(m-2)$ is applied in the series above. Furthermore, in order to obtain a series with rational coefficients for $\widetilde{H}_{i,m}$, we found we had to use the *Mathematica* command `Refine[x>0]` at this stage. The result agrees at $m = 3$ with the weight six Eisenstein series $E_3 = 1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n$ on page 93 of [29].

Definition 9. *1. Corresponding to f_∞, we have*
$$\Delta_{\infty,m}(\tau) := \left\{\frac{J_m'(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau)-1)^m}\right\}^{1/(m-2)}.$$

2. $\Delta^{\dagger}_m := H^3_{\lambda,m} - H^2_{i,m}$.

3. $\Delta^{\diamond}_m := H^3_{\lambda,m}/J_m$.

4. Writing the Fourier expansion of $\Delta^{\diamond}_m$ *as* $\sum_{n=-1}^{\infty} \tau^{\diamond}_m q_m(z)^n$*, we set* $\Delta^{\diamond\diamond}_m(\tau) := \sum_{n=-1}^{\infty} \tau^{\diamond}_m (2^6 m^3 q_m(z))^n$.

5.

$$\widetilde{\Delta}_{\infty,m}(\tau) := \left\{ \frac{j'_m(\tau)^{2m}}{j_m(\tau)^{2m-2}(j_m(\tau) - 2^6 m^3)^m} \right\}^{1/(m-2)}.$$

6. $\widetilde{\Delta}^{\dagger}_m := \widetilde{H}^3_{\lambda,m} - \widetilde{H}^2_{i,m}$.

7. $\widetilde{\Delta}^{\diamond}_m := \widetilde{H}^3_{\lambda,m}/j_m$.

Remark 5. *By definition 1,*f^2 *belongs to* $M(\lambda, 2k, 1)$ *if* f *belongs to* $M(\lambda, k, -1)$. *Therefore, all of the functions defined above belong to* $M(\lambda, 12/(m-2), 1)$.

7 Interpolation by polynomials of the Fourier coefficients of modified modular forms and functions for Hecke groups

In this section, we state conjectures about polynomials interpolating coefficients of modular forms for Hecke groups. Conjectures 6 and 7 bear on Lehmer's question about the existence of zeros of Ramanujan's tau function.

Let

$$J_m(z) = \sum_{n=-1}^{\infty} a_n(m) q_m(z)^n.$$

For integers $m \geqslant 3$ Raleigh showed that, for $n = 0, 1, 2, 3$, $a_n(m) = m^{-2n-2} a_{-1}(m)^{-n} P_n(m)$ where $P_n(x)$ is a polynomial with rational coefficients and degree $2n+2$, such that the coefficient of x^n is zero when n is odd. As we said in the introduction, he conjectured, and Akiyama proved, that similar relations exist among the a_n for all positive n.

Raleigh writes $a_n(q)$ for our $a_n(m)$. Another one of our choices that may pose a risk of confusion is that we use the notation $a_n(m)$ in conjecture 1 below, but the meaning is different. Our $a_n(m)$ names a Fourier coefficient of j_m, not of J_m.)

Berndt's (Hecke's) theorems 7 and 8 above make it clear that Akiyama's result extends in some way to the modular forms defined in the previous section. Our experiments were done to explore the details; the observations are summarized in the conjectures below.

Documentation for the conjectures is here [5]. The link takes the reader to a clickable table of contents on *Github*; we have tried to make the remarks, clauses of conjectures, *etc.,* findable by reading the file names. Data files are named less transparently, but they are defined within the documents containing the routines that generated them.

Conjecture 1. *Let the Fourier expansion of j_m be*

$$j_m = 1/q_m + \sum_{n \geqslant 0} \alpha_n(m) q_m^n.$$

1. *For each integer n greater than -2, there exists a polynomial $A_n(x) \in \mathbb{Q}[x]$ that satisfies the relation $\alpha_n(m) = A_n(m)$ for $m = 3, 4, \ldots$, with $A_{-1}(x) \equiv 1, A_0(x) = 24x(x^2 + 4/3)$, and $A_1(x) = 276x^2(x^4 - 8x^2/69 - 16/23)$.*
2. *Let n be larger than one. Then*
   (a) *$A_n(x) = \overline{a}_n(x-2)(x+2)x^{n+1}a_n(x)$ where $\overline{a}_n$ is a rational number and $a_n(x)$ an irreducible monic polynomial in $\mathbb{Q}[x]$.*
   (b) *$\{\overline{a}_n\}$ is the McKay-Thompson series of class 4A (item 10 in our glossary).*
   (c) *The coefficient of x^k in a_n is non-zero if and only if k is even and not greater than $2n$.*
   (d) *All of the complex roots of $a_n(x)$ lie in the disk with center zero and radius $n/\log(n)$.*
3. *For n greater than one, let G_n be the Galois group of $a_n(x)$ over the rationals. The size of G_n is $2^n n!$ and (if n is greater than two) G_n is isomorphic to a permutation group on $2n$ elements $\{e_1, \ldots, e_{2n}\}$ with three generators: a transposition (e_j, e_k), a product $(e_j, e_{j'})(e_k, e_{k'})$, and a product of disjoint cycles C_1C_2, each of length n, such that C_1 sends e_j to $e_{j'}$ and C_2 sends e_k to $e_{k'}$.*

Clause 2 implies that, for m greater than or equal to three, $\alpha_n(m)$ is nonzero. It is already known that, for all integers $n \geqslant -1$, the n^{th} Fourier coefficient of $j = j_3$, namely $c(n) = \alpha_n(3)$, is positive; see, for example, page 199 in [26].

We tested clause 2(d) in several ways. We approximated the roots of the $a_n(x)$ with root-finding routines and compared their complex moduli with $n/\log(n)$. We used the argument principle to count the zeros in central disks of radius $n/\log(n)$. We superimposed plots of the roots of $a_n(x)$ against plots of circles with radius $n/\log(n)$ and center at the origin. An example is depicted in Figure 1. (We produced this plot in our *Mathematica* notebook "conjecture1clause2d.nb". The words in the caption "length of data file" refer to the number of values of m that we tested to arrive at the interpolating polynomial.) We tested the a_n for $n \leqslant 24$. For clause 3, we computed the Galois groups in *Magma*.

Remark 6. *Like G_n in clause 3, the index-n hyperoctahedral group has size $2^n n!$ [12, 23, 15]. It seems natural to ask whether or not they are isomorphic.*

We found the sequence $\{c_n^*\}$ mentioned below on page A008547 of [30].

Conjecture 2. *Let the Fourier expansion of $H^*_{\lambda,4,m}$ be $\sum_{n=0}^{\infty} h^*_{4,n}(m) q_m(z)^n$.*

1. *For each n there is a polynomial $B_n^*(x)$ with rational coefficients such that $h^*_{4,n}(m) = B_n^*(m)$ for $m = 3, 4, \ldots$.*
2. *If n is positive, then the degree of $B_n^*(x)$ is $6n$.*
3. *$B_0^*(x) \equiv 1$ and, if n is positive, then*
$$B_n^*(x) = c_n^*(x^2-4)x^{4n} b_n^*(x),$$
where $c_n^* = 16 \sum^*_{d|n} (-1)^{n-d} d^3$ and $b_n^*(x)$ is a monic irreducible polynomial in $\mathbb{Q}[x]$.

Conjecture 3. *Let the Fourier expansion of $\widetilde{H}_{\lambda,m}$ be*
$$\widetilde{H}_{\lambda,m} = \sum_{n=0}^{\infty} \beta_n(m) q_m(z)^n.$$

1. *For each n there is a polynomial $B_n(x)$ with rational coefficients such that $\beta_n(m) = B_n(m)$ for $m = 3, 4, \ldots$.*
2. *If n is positive, then the degree of $B_n(x)$ is $3n-1$.*
3. *$B_0(x) \equiv 1$ and $B_1(x) = 16x(x+2)$.*
4. *Let $\mathscr{Q}$ be as in item 9 of our glossary. If $n \in \mathscr{Q}$, then*
$$B_n(x) = \bar{b}_n(x^2-4)(x-6)x^n b_n(x),$$
where $\bar{b}_n$ is rational and b_n is a monic irreducible polynomial. Otherwise (for n greater than one) $B_n(x) = \bar{b}_n(x^2-4)x^n b_n(x)$ where again $\bar{b}_n$ is rational and b_n is a monic irreducible polynomial in $\mathbb{Q}[x]$. In both cases, $\bar{b}_n = 16(-1)^{n+1}\sum^*_{d|n} 1/d$ where the asterisk means that the sum is taken over the odd positive divisors of n.

(We identified the $\bar{b}_n$ by looking at pages A034020 and A098985 in [30].)

Thus, in the range of our observations ($3 \leqslant m \leqslant 302, 0 \leqslant n \leqslant 100$), the only integer value of m such that $\widetilde{H}_{\lambda,m}$ has any vanishing coefficients is six, and $\beta_n(6)$ is zero just if n is in $\mathscr{Q}$.

Conjecture 4. *Let the Fourier expansion of $\widetilde{H}_{i,m}$ be*
$$\widetilde{H}_{i,m}(z) = \sum_{n=0}^{\infty} \gamma_n(m) q_m(z)^n.$$

1. *For each non-negative integer n, there is a polynomial $C_n(x)$ in $\mathbb{Q}[x]$ such that*
   (a) *The degree of C_n is $3n$.*
   (b) *$C_n(x) =$ a rational number $\overline{d}_n \times$ a product of monic irreducible polynomials.*
   (c) *$\overline{d}_0 = 1$ and, for n a positive integer, $\overline{d}_n = 24(-1)^n \sum^*_{d|n} d$. Again, the asterisk means that the sum is taken over the odd positive divisors of n.*
2. *$\gamma_n(m) = C_n(m)$ for $m = 3, 4, ...$*
3. *$C_0(x) \equiv 1$ identically, $C_1(x) = -24(x - 2/3)x^2$, and $C_2(x) = 24(x - 2/3)(x - 2)x^3(x - 14)$.*
4. *For n larger than two, $C_n(x) = \overline{d}_n(x - 2)(x - 2/3)x^{n+1}d_n(x)$ where $d_n(x)$ is a monic irreducible polynomial in $\mathbb{Q}[x]$.*

Let Δ be usual normalized discriminant, a weight 12 cusp form for $SL(2, \mathbb{Z}) = G(\lambda_3)$ with integer coefficients. Its Fourier expansion is written

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$$

where $q = e^{2\pi iz}$ and $\tau(n)$ is Ramanujan's function. Whether or not the equation $\tau(n) = 0$ has any solutions is (notoriously) an open question [19]. (Recently, Balakrishnan, Craig, and Ono [3] ruled out $\pm1, \pm3, \pm5, \pm7$, and ±691 from membership in the image of tau.)

Conjecture 5. *Let $\Delta_m = \Delta_m^{\diamond\diamond}, \widetilde{\Delta}_{\infty,m}, \widetilde{\Delta}_m^{\dagger}$, or $\widetilde{\Delta}_m^{\diamond}$, and let the Fourier expansion of $\Delta_m(z)$ be*

$$\Delta_m(z) = \sum_{n=1}^{\infty} \tilde{\tau}_m(n)q_m^n.$$

where $\tau_m = \tau_m^{\diamond\diamond}, \tilde{\tau}_{\infty,m}, \tilde{\tau}_m^{\dagger}$, or $\tilde{\tau}_m^{\diamond}$, respectively. Let τ denotes Ramanujan's tau function.

1. *There is a set of polynomials $T_n(x) = T_n^{\diamond\diamond}(x), \widetilde{T}_{\infty,n}(x), \widetilde{T}_n^{\dagger}(x)$, or $\widetilde{T}_n^{\diamond}(x)$, respectively, with coefficients in $\mathbb{Q}$ such that $\tau_m(n) = T_n(m)$ for each $m = 3, 4, ...$*
2. *$T_n^{\diamond\diamond}(x) = (-1)^{n+1}64\tau(n)x^{n+3}t^{\diamond\diamond}(x)$, where $t^{\diamond\diamond}(x)$ is a monic irreducible polynomial over $\mathbb{Q}$ of degree $2n - 2$.*
3. *$\widetilde{T}_{\infty,1}(x) \equiv 1$ identically, and, if n is greater than one,*
   (a) *$\widetilde{T}_{\infty,n}(x) = \tilde{s}_{\infty,n}(x - 2)^2x^{n-1}\tilde{t}_{\infty,n}(x)$, where $\tilde{t}_{\infty,n}(x)$ is a monic irreducible polynomial over $\mathbb{Q}$ of degree $2n - 4$ and*

(b) $\tilde{s}_{\infty,n}$ *is (in the notation of [11], Chapter 7, Theorem 7) the coefficient of* q^n *in the Fourier expansion of* $\Delta_8(z)$ *[31].*

(c) *Also ([32]),* $\tilde{s}_{\infty,n} = (-1)^{n+1}\times$ *the sum of the cubes of the divisors* d *of* n *such that* n/d *is odd; this sum is the coefficient of* q^n *in the Fourier expansion of* $E_{\infty,4}$*, the unique normalized weight-4 modular form for* $\Gamma_0(2)$ *with simple zeros at* $i\infty$ *([6], equation (2-3)) and the number of representations of* $n-1$ *as a sum of* 8 *triangular numbers ([24], theorem 5.)*

(d) *Finally,* $\tilde{s}_{\infty,n}$ *is the coefficient of* q^n *in the expansion of* $\eta(2z)^{16}/\eta(z)^{-8}$ *where* $\eta(z)$ *is Dedekind's function ([6], equation (2-16).)*

4. (a) $\widetilde{T}_1^{\dagger}(x) = 16x(3x^2 + x + 6)$.

   (b) $\widetilde{T}_2^{\dagger}(x) = -16x^2(39x^4 - 95x^3 + 66x^2 - 260x - 120)$.

   (c) $\widetilde{T}_3^{\dagger}(x) =$

$$64x^3(189x^6 - 3021x^5 + 9574x^4 - 12520x^3 + 19136x^2 - 2960x - 2208)/9.$$

   (d) *If* n *is greater than* 3*, then*

      i. $\widetilde{T}_n^{\dagger}(x) = \tilde{s}_n^{\dagger}(x-2)^2x^n t_n^{\dagger}(x)$*, where* $\tilde{t}_n^{\dagger}(x)$ *is a monic polynomial, irreducible over* $\mathbb{Q}$*, of degree* $2n-1$ *and*

      ii. $\tilde{s}_n^{\dagger}$ *is (in the notation of [11], Chapter 7, Section 2) the coefficient of* q^n *for the theta series of the direct sum of two copies of the* D_4 *lattice in powers of* q^2 *[33].*

      iii. *Also (proposition 7.5 of [22])* $\tilde{s}_n^{\dagger} = (-1)^n\times$ *the coefficient of* q^{2n} *in the Fourier expansion of* $\{f_8, z\}$ *where* $f_8(z) = \eta(4z)^{12}/(\eta(2z)^4\eta(8z)^8$*. Here* f_8 *is the normalized Hauptmodul for the* $SL(2,\mathbb{Z})$ *subgroup* $\Gamma_0(8)$ *and the braces stand for the Schwarzian derivative (item 2 in our glossary.)*

5. (a) $\widetilde{T}_1^{\diamond}(x), \widetilde{T}_2^{\diamond}(x)$*, and* $\widetilde{T}_3^{\diamond}(x)$ *are irreducible polynomials over* $\mathbb{Q}$ *of degrees* $3, 6,$ *and* 9*, respectively.*

   (b) *If* n *is greater than* 2*,* $\widetilde{T}_1^{\diamond}(x) = \tilde{s}_n^{\diamond}(x-2)x^{n-1}\tilde{t}_n^{\diamond}(x)$*, where* $\tilde{t}_n^{\diamond}(x)$ *is a monic polynomial, irreducible over* $\mathbb{Q}$*, of degree* $2n-3$*, and* $\tilde{s}_n^{\diamond}$ *can be written in terms of Dedekind's eta function in several ways. First,*

$$\sum_{n=0}^{\infty} \tilde{s}_n^{\diamond} q^n = \prod_{n\,odd} (1-q^n)^{24} \times \prod_{n\equiv 2(4)} (1-q^n)^{-24} = \eta^{24}(z)\eta^{24}(4z)\eta^{-48}(2z).$$

   Also, $\tilde{s}_n^{\diamond} = (-1)^{n+1}\times$ *the coefficient of* q^n *in* $(\eta(2z)/\eta(z))^{24}$.

The product decomposition in clause 5(b) above is a guess based on 43 terms of the series using Euler's method [13]; also, [2], Theorem 14.8. We find that it also appears in [36]. The second decomposition appears in [34].

Remark 7. *We connect the polynomials T in conjecture 5 to Lehmer's question on the existence of zeros for Ramanujan's tau function as follows. First (from clause 2), for all $m = 3, 4, ...,$ $\tau_m^{\diamond\diamond}(n) = 0$ if and only if $\tau(n) = 0$. Second, let us write $S(x) := \sum \tau(n)x^n$, and regard this as a formal power series. Let us also write down the formal power series*

1. $\widetilde{S}_\infty(x) := \sum \tilde{\tau}_{\infty,3}(n)x^n$.
2. $\widetilde{S}^\dagger(x) := \sum \tilde{\tau}_3^\dagger(n)x^n$.
3. $\widetilde{S}^\diamond(x) := \sum \widetilde{\tau}_3^\diamond(n)x^n$.
4. $S^{\diamond\diamond}(x) := \sum \tau_3^{\diamond\diamond}(n)x^n$.

Then we have (empirically) $S(x) = \widetilde{S}_\infty(x) = \widetilde{S}^\dagger(x) = \widetilde{S}^\diamond(x) = A_3 S^{\diamond\diamond}(x/A_3) = S^{\diamond\diamond}(1728x)/1728$. Consequently, for $T_n(x) = \widetilde{T}_{\infty,n}(x), \widetilde{T}_n^\dagger(x), \widetilde{T}_n^\diamond(x)$, or $T_n^{\diamond\diamond}(x)$, if $T_n(3)$ is not equal to zero, neither is $\tau(n)$.

Conjecture 5 implies

Conjecture 6. *For $T_n(x)$ and τ_m as above, none of the $T_n(x)$ takes an integer greater than two to zero; consequently, none of the τ_m vanishes for $m = 3, 4, ...$*

For T_n as above, for each positive integer n, and for each integer m greater than two, let the minimum distance from m of any root of T_n be denoted as $d(m, n)$. Conjecture 6 can be restated by saying that $d(m, n)$ is positive. Our experiments also indicate

Conjecture 7. *That*

1. *for fixed m, $d(m, n)$ decays exponentially as n increases, and,*
2. *if m is greater than three, then $d(m, n) > d(3, n)$.*

(But $\widetilde{T}_{\infty,1}$ is identically equal to one, and the corresponding root set is empty.)

Conjectures 1 and 5 lead us to the following speculation about the objects of conjectures 3 and 4, based on nothing more than a bias towards symmetry and the example of the classical Eisenstein series:

Conjecture 8. *There are objects $g_4(z)$ and $g_6(z)$ that are modular for the full modular group $SL(2, \mathbb{Z})$ with Fourier expansions $g_4(z) = \sum_{n=-1}^{\infty} \tilde{b}_n q^n$ and $g_6(z) = \sum_{n=-1}^{\infty} \tilde{d}_n q^n$ and such that (in the notation of conjecture 3 and 4) , for n positive, $\tilde{b}_n = \bar{b}_n$ and $\tilde{d}_n = \bar{d}_n$.*

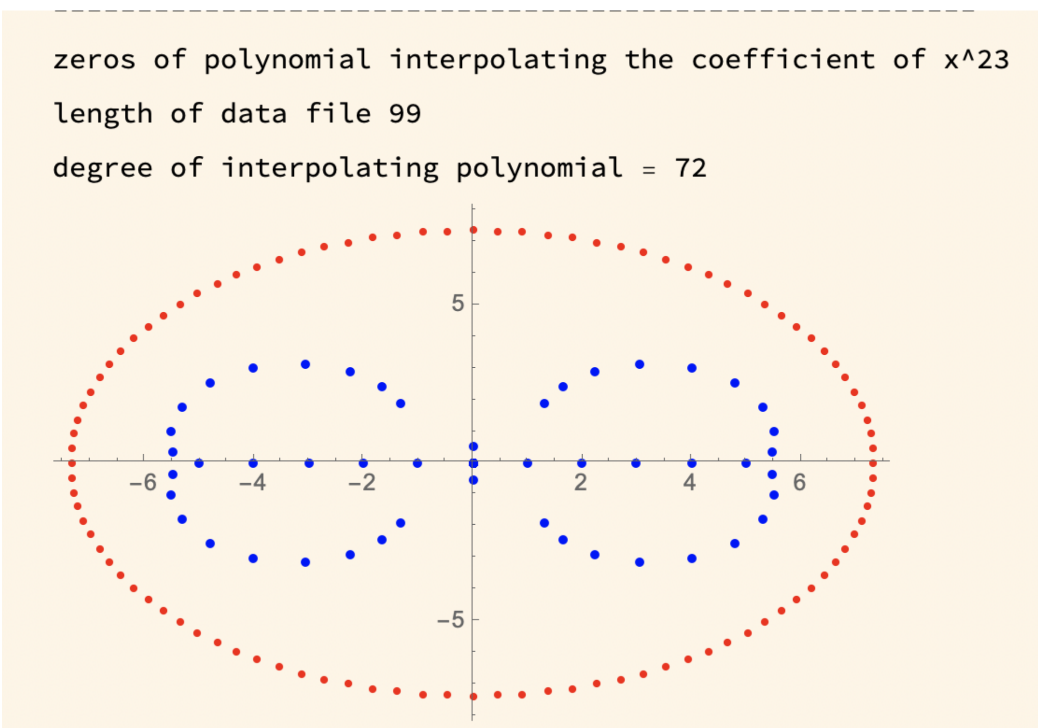
Remark 8. *While checking our calculations, we compared the Fourier expansion of $H_{\lambda,4}(x/A_4)$ (abusing notation in the obvious way) with Leo's ([21], p.54) expansion of the weight 4 Eisenstein series at $m = 4$. (Recall that $A_4 = 1/256$.) In the range of our observations, they do coincide. The expansions (in our own notation) begin*

$$1 + 48q_4 + 624q_4^2 + 1344q_4^3 + ...$$

We also encountered the O.E.I.S page [33], which mentions our expansion ([7], equation (2-1), p, 260): $E_{\gamma,2} = 1 + 24\sum_{n=1}^{\infty}\sigma^{\text{odd}}(n)q^n =$ (in the notation of clause 1(c) of conjecture 4 in the present article)
$1+24\sum_{n=1}^{\infty}(\sum_{d|n}^{*} d)q^n$. *Sloane comments that the sequence $\{1, 48, 624, ...\}$ is the same as that of the coefficients of $E_{\gamma,2}^2$. It happens that $E_{\gamma,2}^2$ is a weight 4, level 2 modular form, that is, a weight 4 modular form for the $SL(2,\mathbb{Z})$ subgroup $\Gamma_0(2)$. While it is too vague to state as a conjecture, we venture a guess that, behind this observation and that of clause 3(c) in conjecture 5, there is a theory connecting level N classical modular forms with Hecke modular forms.*

8 Figures

8.1 Figure 1.



ROOTS OF POLYNOMIAL INTERPOLATING THE COEFFICIENT OF q_m^{23} IN THE FOURIER EXPANSION OF j_m.

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