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# THE PRIMALITY OF RAMANUJAN'S TAU-FUNCTION

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**Introduction.** The function $\tau(n)$ introduced by Ramanujan in 1916 [1] as a natural outgrowth of the functions $\sigma_k(n)$, the sum of the $k$th powers of the divisors of $n$, has been the subject of numerous investigations ever since. It is defined most simply as the coefficient of $X^n$ in the expansion of the product

$$X \prod_{m=1}^{\infty} (1 - X^m)^{24} = \sum_{n=1}^{\infty} \tau(n) X^n = X - 24X^2 + 252X^3 + \cdots.$$

Although a number of remarkable properties of $\tau(n)$ have been established, some of which are cited below, there remains a number of unsolved questions about $\tau(n)$; for example: What is the exact order of magnitude of $\tau(n)$ (see [2])? Is $\tau(n)=0$ for some $n>0$ (see [3])? In this note we address ourselves to the question: Is $\tau(n)$ ever a prime? We answer this question by

THEOREM A. *The integer $\tau(n)$ is composite for $2 \leqq n \leqq 63000$, but*

$$\tau(63001) = 80561663527802406257321747$$

*is a prime number.*

Since published tables of $\tau(n)$, [4], extend to $n=1000$ and unpublished tables to $n=10000$, [5], it is clear that to prove Theorem A requires the use of some of the known properties of $\tau(n)$, namely the formulas and congruence properties listed below. Numbers in square brackets give references to papers where these results are established. In what follows $p$ always designates a prime.

*Required Properties.*

(1) If $(a, b) = 1$, then $\tau(ab) = \tau(a)\tau(b)$. [6]

(2) $\tau(p^{\alpha+1}) = \tau(p)\tau(p^\alpha) - p^{11}\tau(p^{\alpha-1})$, $(\alpha > 0)$. [6]

As an immediate consequence of (1) and (2)

(3) If $p \mid \tau(p)$ then $p \mid \tau(np)$, $(n > 0)$.

(4) If $n$ is odd $\tau(n) \equiv \sigma(n) \pmod 8$. [7]

Setting $p=2$ in (3) and using (4) we easily derive

(5) $\tau(n)$ is odd if and only if $n$ is an odd square.

(6) If $n$ is odd, $\tau(n) \equiv \sigma_3(n) \pmod{32}$. [8]

(7) If $(n, 3) = 1$, $\tau(n) \equiv \sigma(n) \pmod 3$. [9]

(8) If $3p = u^2 + 23v^2$, $\tau(p) \equiv -1 \pmod{23}$. [10], [3]

(9) $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$. [2], [11]

*Proof of Theorem A.* We begin by assuming there exists a least integer $n_0 \leqq 63000$ for which $\tau(n_0)$ is a prime. If $n_0$ is not a power of a prime then $n_0 = ab$

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