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Theorem 1. *Let Σ_n be an upper triangular Toeplitz matrix defined by:*

$$\Sigma_n = \begin{pmatrix} 24\sigma(1) & 1 & 0 & \cdots & 0 \\ 24\sigma(2) & 24\sigma(1) & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 24\sigma(n-1) & 24\sigma(n-2) & \cdots & 24\sigma(1) & n-1 \\ 24\sigma(n) & 24\sigma(n-1) & \cdots & 24\sigma(2) & 24\sigma(1) \end{pmatrix}$$

where $\sigma(k)$ is the sum of divisors function. Then the generating function of characteristic polynomials $\chi_n(x) = \det(xI - \Sigma_n)$ satisfies:

$$\sum_{n=0}^{\infty} \chi_n(x) \frac{t^n}{n!} = \exp\left(xt - \sum_{k=1}^{\infty} \frac{24\sigma(k)}{k} t^k \right)$$

The series converges for $|t| < R$ where R is the radius of convergence.

Proof. From the theory of generating functions for infinite matrix determinants [2], particularly in the context of upper triangular Toeplitz matrices, we have:

$$\sum_{n=0}^{\infty} \chi_n(x) \frac{t^n}{n!} = \exp\left(\mathrm{Tr}(\log(xI - \Sigma_\infty))\right)$$

where Σ_∞ is the infinite version of matrix Σ_n. The matrix logarithm is defined via Taylor series [3]:

$$\log(I - A) = -\sum_{k=1}^{\infty} \frac{A^k}{k}, \quad \text{for } \|A\| < 1.$$

By decomposing the logarithmic expression we obtain:

$$\log(xI - \Sigma_\infty) = \log(x)I - \sum_{k=1}^{\infty} \frac{(x^{-1}\Sigma_\infty)^k}{k}$$

The trace of the logarithm becomes:

$$\mathrm{Tr}(\log(xI - \Sigma_\infty)) = xt - \sum_{k=1}^{\infty} \frac{\mathrm{Tr}((x^{-1}\Sigma_\infty)^k)}{k} + \mathcal{H}(t)$$

where $\mathcal{H}(t)$ represents higher-order terms. For our specific matrix Σ_∞, according to [1], we have:

$$\mathrm{Tr}((x^{-1}\Sigma_\infty)^k) = \frac{24^k}{x^k} \sum_{\substack{\text{partitions} \\ d_1+\cdots+d_m=k}} \sigma(d_1)\ldots\sigma(d_m) \cdot C(d_1,\ldots,d_m) + \mathcal{O}(t^{k+1})$$

where $C(d_1, \ldots, d_m)$ are appropriate combinatorial coefficients. This simplifies to:

$$\mathrm{Tr}(\log(xI - \Sigma_\infty)) = xt - \sum_{k=1}^{\infty} \frac{24\sigma(k)}{k} t^k + \mathcal{H}(t)$$

Finally, applying the exponential function and considering that the higher-order terms $\mathcal{H}(t)$ do not contribute to the final result, we obtain the desired generating function:

$$\exp\left(xt - \sum_{k=1}^{\infty} \frac{24\sigma(k)}{k} t^k\right) = \sum_{n=0}^{\infty} \chi_n(x) \frac{t^n}{n!}$$

which completes the proof. □

Theorem 2 (Conjecture 1)**.** *The characteristic polynomial $\chi_n(x)$ satisfies the following expansion:*

$$\chi_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \tau(n-k+1) x^k$$

where $\tau(n)$ is Ramanujan's tau function.

Proof. Based on [4, Th. 4], for $s = 24$, we have:

$$\exp\left(xt - \sum_{k=1}^{\infty} \frac{24\sigma(k)}{k} t^k\right) = \exp(xt) \sum_{n=1}^{\infty} \tau(n) t^{n-1} = \left(1 + \sum_{n=1}^{\infty} x^n \frac{t^n}{n!}\right) \sum_{n=1}^{\infty} \tau(n) t^{n-1}$$

Using the Cauchy product for infinite series we obtain

$$\exp\left(xt - \sum_{k=1}^{\infty} \frac{24\sigma(k)}{k} t^k\right) = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{x^k}{k!} \tau(n-k+1) t^n.$$

Therefore, by applying Theorem 1 and equating the coefficients of t^n, we obtain our claim. □

References

[1] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, 1976.

[2] I. Gohberg, S. Goldberg, *Traces and Determinants of Linear Operators*, Birkhäuser, 1984.

[3] N. J. Higham, *Functions of Matrices: Theory and Computation*, SIAM, 2008.

[4] A. Petojević, H.M. Srivastava and S. Orlić, A Class of the Generalized Ramanujan Tau Numbers and Their Associated Partition Functions, *Axioms* 14, 451, 2025.