# Fourier Analysis of Eigenvalue Minimum Moduli: Ramanujan's Tau Function and Elliptic Curve Cusp Forms

Conversation between User and Claude

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**Abstract**

This document summarizes a technical discussion applying Fourier analysis to study the oscillatory behavior of minimum eigenvalue moduli arising from characteristic polynomials of matrices associated with Ramanujan's tau function and elliptic curve cusp forms. The analysis reveals distinct periodic structures that may provide insights into Lehmer's conjecture regarding zeros of the tau function.

## Contents

# 1 Introduction

## 1.1 Context and Motivation

The discussion began with analyzing a Jupyter notebook studying Ramanujan's tau function $\tau(p)$ evaluated at prime numbers. The work uses matrix identities from symmetric function theory, specifically examining eigenvalues of characteristic polynomials of matrices $J_n(j)$ where the determinant $|J_n(j)| = n! \cdot \tau(p_n)$.

The key connection to Lehmer's question about zeros of $\tau(p_n)$ is:

$$\begin{aligned} \tau(p_n) = 0 &\iff |J_n(j)| = 0 && (1)\\ &\iff \Pi_n(0) = 0 && (2)\\ &\iff \text{characteristic polynomial has a zero eigenvalue} && (3) \end{aligned}$$

## 1.2 The Deformation Framework

The matrices can be "deformed" by replacing the first entry with a parameter $c$, creating matrices $J_n^{(c)}(j)$. For fixed $n$, varying $c$ and computing determinants allows interpolation to obtain polynomials in $c$. The notebook studies the behavior of eigenvalues of these deformed characteristic polynomials.

# 2 Initial Discussion: Understanding the Data

## 2.1 Clarification of Objects

Initial confusion arose when Claude incorrectly referred to "j-invariant coefficients." The user clarified:

> The j-invariant is an object in the theory of elliptic curves and the theory of modular forms. The objects with coefficients in this notebook are polynomials over $\mathbb{Q}$. The plots depict behavior of the eigenvalues of the characteristic functions of these polynomials.

This established the correct framework: polynomials over the rationals with characteristic polynomials whose eigenvalues are being studied.

## 2.2 The Reference Paper

The user provided the paper "Ramanujan's Function on Small Primes" which describes:

- Matrix identities (Lemmas 2.1 and 2.2) governing additive convolution identities
- The matrices $J_n(j)$ and $H_n(h)$ with the relationship:

$$J_n(j) = \begin{pmatrix} j_1 & -1 & 0 & \cdots & 0 \\ j_2 & j_1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_n & j_{n-1} & j_{n-2} & \cdots & j_1 \end{pmatrix} \tag{4}$$

- The characteristic polynomial $\Pi_n(x) = \chi(J_n(j))(x)$
- The observation that minimum moduli of eigenvalues appear to oscillate in a nearly periodic fashion

# 3 Fourier Analysis Methodology

## 3.1 What Fourier Analysis Can Extract

Claude proposed that Fourier analysis could extract several types of information from the plots:

### 3.1.1 From Coefficient and Magnitude Plots

- **Growth rates and asymptotic behavior**: The Fourier transform reveals dominant frequency components in coefficient growth
- **Periodicity detection**: Identifies periodic patterns not obvious from raw numbers
- **Power spectrum analysis**: Shows which "frequencies" dominate oscillatory growth
- **Spectral exponent**: Reveals self-similar or fractal-like structure

### 3.1.2 From Complex Plane Plots

- **Angular frequency distribution**: Detects preferred directions or symmetries
- **Radial oscillations**: Periodic variations in distance from origin

### 3.1.3 From Minimum Moduli Plots

- **Oscillation frequencies**: Characteristic timescales of variation
- **Trend vs. fluctuation separation**: Low frequencies represent trend, high frequencies represent rapid fluctuations

## 3.2 Implementation Requirements

The user asked whether plots alone would suffice. Claude explained:

> The plots alone are insufficient for proper Fourier analysis. From plots I can only extract approximate values by reading pixel coordinates (low precision, ∼1-2 significant figures). For meaningful Fourier analysis I need the actual numerical sequences.

# 4 Implementation: Ramanujan Tau Function

## 4.1 The Analysis Code

A Python/SageMath analysis cell was developed with the following structure:

1. **Data Preparation**: Convert SageMath types to NumPy floats

```
indices = np.array([float(pair[0]) for pair in mins_list],
                   dtype=np.float64)
min_moduli = np.array([float(pair[1]) for pair in mins_list],
                      dtype=np.float64)
```

2. **Detrending**: Remove polynomial growth to isolate oscillations

```
poly_coeffs = np.polyfit(indices, log_min_moduli, deg=2)
poly_trend = np.polyval(poly_coeffs, indices)
log_detrended_poly = log_min_moduli - poly_trend
```

3. **FFT Computation**: Compute Fourier transform and power spectrum

```
fft_result = np.fft.fft(log_detrended_poly)
fft_frequencies = np.fft.fftfreq(len(indices), d=1.0)
power_spectrum = np.abs(fft_result)**2
```

4. **Peak Detection**: Identify dominant frequencies

```
peak_indices = signal.find_peaks(power,
                                 height=np.max(power)*0.1)[0]
```

5. **Visualization**: Six-panel plot including power spectrum and spectrogram

6. **Signal Reconstruction**: Rebuild signal using top frequencies only

### 4.2 Results: Ramanujan Tau Function

For the tau function data ($n = 2$ to 175, 174 data points):

| Rank | Period | Frequency | Power |
|---|---|---|---|
| 1 | 4.14 | 0.241379 | $6.77 \times 10^3$ |
| 2 | 2.05 | 0.488506 | $9.36 \times 10^2$ |
| 3 | 3.70 | 0.270115 | $7.32 \times 10^2$ |

Table 1: Top dominant periodicities for Ramanujan tau function

**Detrended data standard deviation**: 0.9308

### 4.3 Interpretation: The Period-4 Structure

The analysis revealed three main rhythms:

- **Dominant rhythm (Period $\approx$ 4.14)**: The strongest signal with power 6,770. This matches the observation of mod 4 residue patterns in the paper's Table 1, where residues form regular blocks.

- **Fast oscillation (Period $\approx$ 2.05)**: A quicker back-and-forth pattern, second strongest with power 936.

- **Medium rhythm (Period $\approx$ 3.70)**: Close to period 4, creating "beats" with the dominant rhythm, power 732.

The period-4 dominance strongly suggests a connection to modular arithmetic mod 4. The paper's Table 1 showed residues mod 4 of abscissas from the lower envelope forming regular blocks of sizes 10, 9, 9, ..., confirming the quasi-period of 36-40 indices ($\approx 4 \times 9$).

# 5 Implementation: Elliptic Curve Cusp Form

## 5.1 New Dataset

The user found a weight 2 cusp form from Cremona's database displaying similar oscillatory behavior. The notebook computed eigenvalues for matrices derived from an elliptic curve with conductor 11, with data extending to $n = 400$.

## 5.2 Results: Elliptic Curve Cusp Form

Data range: $n = 1$ to 400 (400 data points)

| Rank | Period | Frequency | Power |
|---|---|---|---|
| 1 | 11.76 | 0.085000 | $3.83 \times 10^4$ |
| 2 | 5.97 | 0.167500 | $4.15 \times 10^3$ |
| 3 | 3.96 | 0.252500 | $4.10 \times 10^3$ |
| 4 | 2.96 | 0.337500 | $2.40 \times 10^3$ |

Table 2: Top dominant periodicities for elliptic curve cusp form

**Detrended data standard deviation**: 1.0370

## 5.3 Interpretation: The Period-12 Harmonic Structure

The elliptic curve data revealed a dramatically different pattern:

- **Dominant period $\approx$ 12**: Extremely strong signal (power 38,300), about 10$\times$ stronger than any component in the tau function case
- **Harmonic series**: The detected periods are approximately 12, 6, 4, 3, which are related by simple fractions:
$$T_n \approx \frac{12}{n} \quad \text{for } n = 1, 2, 3, 4 \tag{5}$$
- **Clean spectrum**: Only 4 significant frequencies detected, indicating highly organized oscillation
- **Stable pattern**: The spectrogram showed these frequencies remain constant throughout the data range

# 6 Comparative Analysis

## 6.1 Key Differences

## 6.2 Mathematical Significance

### 6.2.1 Ramanujan Tau Function

- Period-4 structure suggests connection to $\mathbb{Z}/4\mathbb{Z}$
- The "tuning" effect of the deformation parameter $c$ organizes eigenvalues into these specific frequencies
- Predictable oscillation rather than chaotic approach to zero

| Property | Tau Function | EC Cusp Form |
|---|---|---|
| Dominant Period | $\approx 4$ | $\approx 12$ |
| Max Power | $6.77 \times 10^3$ | $3.83 \times 10^4$ |
| Signal Type | 3 mixed frequencies | Harmonic series |
| Data Points | 174 | 400 |
| Std Dev (detrended) | 0.93 | 1.04 |

Table 3: Comparison of tau function and elliptic curve results

#### 6.2.2 Elliptic Curve Cusp Form

- Period-12 may relate to the conductor or other arithmetic properties of the level-11 elliptic curve
- Harmonic structure $(12, 6, 4, 3)$ suggests deep mathematical symmetry
- Extremely clean signal: complex eigenvalue behavior reduces to 4 simple periodic functions
- The period-12 might be related to:
  - The conductor of the elliptic curve
  - Divisibility properties in the coefficient sequence
  - Modular symmetries at level 11

## 7 Technical Details

### 7.1 Detrending Rationale

The polynomial detrending with degree 2 is crucial because:

- Raw eigenvalue moduli grow (approximately) exponentially
- Taking logarithms converts exponential growth to polynomial growth
- A quadratic fit log(min modulus) $\approx an^2 + bn + c$ captures the overall trend
- Subtracting this trend isolates the oscillatory component
- Without detrending, the FFT would be dominated by low-frequency trend components

### 7.2 Power Spectrum Interpretation

The power spectrum $P(f) = |\mathrm{FFT}(x)|^2$ measures the "strength" of each frequency component:

- High power at frequency $f$ means strong oscillation with period $T = 1/f$
- Peaks in the power spectrum indicate dominant periodicities
- The ratio of peak powers indicates relative importance of different oscillations
- A "clean" spectrum (few sharp peaks) indicates organized behavior
- A "noisy" spectrum (many small peaks) indicates complex or chaotic behavior

### 7.3 Signal Reconstruction

For a signal decomposed via FFT, reconstruction using top $k$ frequencies:

$$\text{reconstructed}(n) = \sum_{i=1}^{k} A_i \cos(2\pi f_i n + \phi_i) \tag{6}$$

where:

- $A_i = |\text{FFT}[i]|/N$ is the amplitude
- $f_i$ is the frequency
- $\phi_i = \arg(\text{FFT}[i])$ is the phase
- $N$ is the number of data points

The quality of reconstruction (measured by residual) indicates how well a few simple periodic functions capture the complex behavior.

### 7.4 Spectrogram Analysis

The spectrogram uses Short-Time Fourier Transform (STFT):

$$\text{STFT}(t, f) = \int_{-\infty}^{\infty} x(\tau) w(\tau - t) e^{-2\pi i f \tau} d\tau \tag{7}$$

where $w(t)$ is a window function.

This reveals:

- Whether frequencies change over time ("chirp")
- When specific oscillations start or stop
- Stability of the periodic structure across the data range

For both datasets, the spectrogram showed stable frequencies, indicating the periodic structure persists throughout.

## 8 Implications and Future Directions

### 8.1 Connection to Lehmer's Conjecture

The Fourier analysis provides several insights relevant to whether $\tau(p) = 0$ for some prime $p$:

1. **Predictable approach to zero**: Eigenvalues approach zero following regular periodic patterns, not randomly
2. **Lower envelope structure**: The deepest approaches (smallest minimum moduli) occur at predictable intervals governed by the dominant period
3. **Quantifiable bounds**: The amplitude and phase information from Fourier analysis could potentially bound how close eigenvalues can get to zero
4. **Deformation behavior**: The dramatic difference between "tuned" (deformed, $c \neq 0$) and "untuned" ($c = 0$) cases suggests the deformation parameter organizes the spectrum

### 8.2 Open Questions

1. Why does the tau function exhibit period-4 structure while the elliptic curve cusp form shows period-12?
2. Can the observed periodicities be predicted from the arithmetic properties of the modular forms (conductor, level, character)?
3. How does the relationship between deformed and undeformed eigenvalues inform bounds on the minimum moduli?
4. Can the harmonic structure in the elliptic curve case be explained by representation theory or Galois symmetries?
5. Do other cusp forms exhibit similar periodic structures? Is there a classification?
6. Can the Fourier analysis provide rigorous lower bounds that rule out zero eigenvalues?

### 8.3 Methodological Extensions

Future work could explore:

- **Wavelet analysis**: Better time-frequency localization than STFT
- **Longer datasets**: Extend to larger $n$ to confirm periodicity persistence
- **Multiple cusp forms**: Systematic study across Cremona's database
- **Theoretical prediction**: Develop formulas predicting the periods from modular form properties
- **Phase analysis**: Study the phase relationships between different frequency components
- **Multitaper methods**: More robust spectral estimation for finite data

## 9 Conclusion

Fourier analysis revealed distinct periodic structures in the minimum eigenvalue moduli for both Ramanujan's tau function and an elliptic curve cusp form:

- **Tau function**: Period-4 dominance with mixed frequencies, suggesting connection to arithmetic mod 4
- **Elliptic curve**: Period-12 harmonic series (12, 6, 4, 3), indicating deep mathematical symmetry possibly related to the conductor
- **Both cases**: Clean, stable periodic structure rather than chaotic behavior, with the deformation parameter "tuning" the eigenvalue distribution

These findings provide a new quantitative framework for understanding the oscillatory behavior observed in the original plots and may offer new approaches to questions like Lehmer's conjecture about zeros of modular forms.

# Appendix: Complete Python Code

## Fourier Analysis Code for SageMath

Listing 1: Complete Fourier analysis implementation

```
# FOURIER ANALYSIS OF MINIMUM MODULI
# Performs spectral analysis on minimum moduli sequences
# to detect periodicities and dominant frequencies

import numpy as np
from scipy import signal
import matplotlib.pyplot as plt

# Data Preparation - Convert SageMath to NumPy
indices = np.array([float(pair[0]) for pair in mins_list],
                   dtype=np.float64)
min_moduli = np.array([float(pair[1]) for pair in mins_list],
                      dtype=np.float64)
log_min_moduli = np.log(min_moduli)

print(f"Data range: n = {int(indices[0])} to {int(indices[-1])}")
print(f"Number of data points: {len(indices)}")
print(f"Min modulus range: {min_moduli.min():.6f} to "
      f"{min_moduli.max():.6f}")

# Detrending - Remove polynomial growth
log_detrended_linear = signal.detrend(log_min_moduli, type='linear')
poly_coeffs = np.polyfit(indices, log_min_moduli, deg=2)
poly_trend = np.polyval(poly_coeffs, indices)
log_detrended_poly = log_min_moduli - poly_trend

print(f"\nTrend removed. Detrended std dev: "
      f"{np.std(log_detrended_poly):.4f}")

# Compute FFT
fft_result = np.fft.fft(log_detrended_poly)
fft_frequencies = np.fft.fftfreq(len(indices), d=1.0)
power_spectrum = np.abs(fft_result)**2

# Extract positive frequencies
positive_freq_mask = fft_frequencies > 0
frequencies = fft_frequencies[positive_freq_mask]
power = power_spectrum[positive_freq_mask]
periods = 1.0 / frequencies

# Identify dominant frequencies
peak_indices = signal.find_peaks(power,
                                 height=np.max(power)*0.1)[0]
peak_powers = power[peak_indices]
peak_periods = periods[peak_indices]
sorted_indices = np.argsort(peak_powers)[::-1]
top_peaks = sorted_indices[:min(10, len(sorted_indices))]

```

```python
# Print results
print("\n" + "="*70)
print("TOP DOMINANT PERIODICITIES")
print("="*70)
print(f"{'Rank':<6} {'Period':<12} {'Frequency':<12} {'Power':<15}")
print("-"*70)
for rank, idx in enumerate(top_peaks, 1):
    period = peak_periods[idx]
    freq = frequencies[peak_indices[idx]]
    pwr = peak_powers[idx]
    print(f"{rank:<6} {period:>10.2f}   {freq:>10.6f}   "
          f"{pwr:>12.2e}")

# [Visualization code follows - see full implementation above]
```