An Explicit Formula for the Coefficients of Eta-Products

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Abstract

In this paper, we derive an explicit combinatorial formula for the Fourier coefficients of eta-products. We establish a general explicit formula for the coefficients of modular forms that are eta-products of the form $F(z) = \sum_{n=0}^{\infty} a_F(n)q^n$. As applications, we show how this general formula yields known expressions for the coefficients of Ramanujan's tau function and the partition function, thereby unifying and simplifying their derivation through a unified combinatorial approach based on the theory of eta-products.

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1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, and complex numbers, respectively. Let Dedekind's eta function be defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i z}.$$

An eta-product is a function of the form [5]

$$F(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where $r_\delta \in \mathbb{Z}$ and N is the level. It can be written as an Euler product

$$F(z) = q^A \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}, \tag{1.1}$$

where $A = \frac{1}{24} \sum_{\delta | N} r_\delta \delta$, and the coefficients $c(n)$ are defined by

$$c(n) = \sum_{\delta | N,\ \delta | n} r_\delta.$$

If we define an arithmetic function $C : \mathbb{N} \to \mathbb{C}$ by

$$C(m) = \sum_{d|m} c(d) \cdot d,$$

then, taking the logarithmic derivative of (1.1), we obtain the representation

$$F(z) = q^A \exp\left(- \sum_{m=1}^{\infty} \frac{C(m)}{m} q^m \right).$$

Denote the Fourier expansion (also called the q-expansion) of the eta-product $F(z)$ by

$$F(z) = \sum_{n=0}^{\infty} a_F(n) q^n, \quad q = e^{2\pi i z}.$$

For an arbitrary function

$$H(z) = \exp\left(\sum_{m=1}^{\infty} b_m \frac{z^m}{m!} \right) = \sum_{n=0}^{\infty} h(n) z^n,$$

it follows from [3, pp. 27–28, 137–142] that the coefficient $h(n)$ is given by the known formula

$$h(n) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{1\cdot k_1 + 2\cdot k_2 + \cdots + n\cdot k_n = n \\ k_1 + k_2 + \cdots + k_n = k}} \binom{k}{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \left(\frac{b_i}{i!} \right)^{k_i}.$$

Consequently, in the special case where the leading exponent A is an integer, we find that

$$a_F(m + A) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \sum_{\substack{1\cdot k_1 + 2\cdot k_2 + \cdots + m\cdot k_m = m \\ k_1 + k_2 + \cdots + k_m = k}} \binom{k}{k_1, k_2, \ldots, k_m} \prod_{i=1}^{m} \left(\frac{C(i)}{i} \right)^{k_i}. \tag{1.2}$$

2. **Special Cases**

First, B. Brent in [2, Th. 3.1] (see also [4]) proved the following decomposition for Ramanujan's tau function:

$$\tau(n+1) = \sum_{1\cdot k_1 + 2\cdot k_2 + \cdots + (n-1)\cdot k_n = n} (-24)^{k_1 + k_2 + \cdots + k_n} \frac{\sigma(1)^{k_1} \sigma(2)^{k_2} \cdots \sigma(n)^{k_n}}{1^{k_1} k_1! 2^{k_2} k_2! \cdots n^{k_n} k_n!},$$

where $\sigma(i) = \sum_{d|i} d$ is the sum of all positive divisors of i [6, A000203]. As a special case of Eq. (1.2), we recover Brent's formula for $\tau(n+1)$. Since $A = 1$, $c(n) = 24$ and $C(m) = 24\sigma(m)$, we have

$$\tau(m+1) = \sum_{k=0}^{m} \frac{(-24)^k}{k!} \sum_{\substack{1\cdot k_1 + 2\cdot k_2 + \cdots + m\cdot k_m = m \\ k_1 + k_2 + \cdots + k_m = k}} \binom{k}{k_1, k_2, \ldots, k_m} \prod_{i=1}^{m} \left(\frac{\sigma(i)}{i} \right)^{k_i}.$$

Note that the factor $(-1)^k$ in Eq. (1.2) becomes $(-24)^k$ in this special case since $C(i) = 24\sigma(i)$. These equivalent forms are obtained by recognizing that the multinomial sum over partitions $1 \cdot k_1 + \cdots + m \cdot k_m = m$ with $k_1 + \cdots + k_m = k$ corresponds to summing over all compositions $n_1 + \cdots + n_k = m$ with $n_i \geq 1$. Therefore, the following equivalent results hold:

$$\tau(m+1) = \sum_{k=0}^{m} \frac{(-24)^k}{k!} \sum_{\substack{n_1+\cdots+n_k=m \\ n_i \geq 1}} \prod_{i=1}^{k} \frac{\sigma(n_i)}{n_i},$$

$$\tau(m+1) = \sum_{k=0}^{m} \frac{(-24)^k}{k!} [t^m] \left(\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} t^n \right)^k,$$

where the second identity follows from the observation that the inner sum represents the coefficient extraction from the power of a generating function.

Second, as a consequence of Eq. (1.2), we obtain an explicit formula for the partition function $p(n)$ [6, A000041]. Considering the eta-product related to Euler's product formula for the generating function of partitions, $1/\eta(z) = \prod_{n=1}^{\infty}(1-q^n)^{-1}$, we have $A = 0$, $c(n) = -1$, and $C(m) = -\sigma(m)$. Although this formula is implicitly contained in the exponential representation of the generating function for partitions, its derivation from the theory of eta-products gives:

$$p(n) = \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{1\cdot k_1+\cdots+n\cdot k_n=n \\ k_1+\cdots+k_n=k}} \binom{k}{k_1, \ldots, k_n} \prod_{i=1}^{n} \left(\frac{\sigma(i)}{i} \right)^{k_i}, \quad n \geq 1.$$

Finally, as additional illustrations, we consider the following eta-products whose parameters are shown in Table 1.

Parameter	$F(z) = \eta(2z)^{24}$	$F(z) = \frac{\eta(4z)^8}{\eta(2z)^4}$
N	2	4
r_δ	$r_2 = 24$	$r_2 = -4$, $r_4 = 8$
A	$\frac{24\cdot 2}{24} = 2$	$\frac{-4\cdot 2+8\cdot 4}{24} = 1$
$c(n)$	$c(n) = \begin{cases} 24, & 2 \mid n \\ 0, & 2 \nmid n \end{cases}$	$c(n) = \begin{cases} 0, & 2 \nmid n \\ -4, & 2 \mid n,\ 4 \nmid n \\ 4, & 4 \mid n \end{cases}$
$C(m)$	$C(m) = \begin{cases} 0, & m \text{ odd} \\ 24 \sum_{\substack{d \mid m \\ 2 \mid d}} d, & m \text{ even} \end{cases}$	$C(m) = \begin{cases} 0, & m \text{ odd} \\ -8 \cdot \sigma(m/2), & m \equiv 2 \pmod 4 \\ 8 \cdot \sigma(m/4), & m \equiv 0 \pmod 4 \end{cases}$

Table 1. Parameters and structure of eta-products $F(z) = \eta(2z)^{24}$ and $F(z) = \eta(4z)^8/\eta(2z)^4$.

Conflict of Interest The authors declare that they have no conflict of interest.

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