Fourier Analysis: Curve 17a1 (Level 17) — Restricted to Primes

The Data: A Key Difference

You analyzed 400 values of "minimum moduli" from elliptic curve **17a1**, but with a crucial twist: the analysis uses only the coefficients a_p where p is **prime**. Specifically, the n-th data point corresponds to the coefficient at the n-th prime p_n (so $n = 1$ uses $p_1 = 2$, $n = 2$ uses $p_2 = 3$, etc., up to $n = 400$ using $p_{400} = 2741$).

This curve has conductor 17 (prime) and rank 0. The minimum moduli values span from 0.003272 to 3.181, analyzed over prime indices p_1 through p_{400}.

This is fundamentally different from all previous analyses, which used consecutive integers $n = 1, 2, 3, \ldots$. Here we're looking at behavior **only at primes**.

Main Finding: Two Dominant Periodicities

Your data shows **only two statistically significant repeating patterns**:

Rank	Period	Power	Relative Strength
1	3.81 indices	4.52×10^4	100% (strongest)
2	2.11 indices	4.63×10^3	10.2%

Note: Fewer harmonics detected than in previous curves, likely due to the restriction to primes.

What This Means

Period 1: The Dominant Cycle (3.81 indices)

This is **overwhelmingly** the strongest pattern. With power 4.52×10^4, it's roughly **9.8 times stronger** than the second pattern.

Interpretation: The minimum moduli at prime indices exhibit a very strong oscillation that repeats approximately every 3.81 primes.

Amplitude: This oscillation has amplitude 0.1388 in the log-transformed detrended data. This is substantial—about 13.9% of the standard deviation (1.0016) of the detrended signal.

KEY OBSERVATIONS:

- Period $\approx 3.81 \approx 4$: The pattern repeats roughly every 4 primes
- $3.81 \times 6 = 22.86 \approx 23$, but $17 \neq 23$
- $17/3.81 = 4.46 \approx 4$ or ≈ 4.5
- $3.81 \approx 4$ suggests behavior related to primes $\bmod 4$

Period 2: Possible Harmonic (2.11 indices)

The second pattern has period $2.11 \approx 2$, with power about 10.2% of the dominant pattern.

Key observation: Notice that $2.11 \times 1.8 \approx 3.8$. This isn't a clean harmonic (not 2:1 ratio). However, period ≈ 2 suggests alternating behavior every other prime.

Interpretation: This might reflect systematic differences between primes in different residue classes (e.g., primes $\equiv 1$ mod 4 vs $\equiv 3$ mod 4, or odd-indexed primes vs even-indexed primes).

Amplitude: Moderate at 0.0601, contributing about 6% of signal variation.

Comparison Across All Curves

Now including this special case:

Level	11	17	19	37	43	53
Index set	$\mathbb{Z}^+$	**Primes**	$\mathbb{Z}^+$	$\mathbb{Z}^+$	$\mathbb{Z}^+$	$\mathbb{Z}^+$
Period	11.76	3.81	2.84	6.56	7.30	6.35
k factor	1	4.5	6–7	6	6	8
Harmonics	4	2	2–3	3	3	4
Rank	0	0	0	1	1	1

Important: The 17a1 analysis is **not directly comparable** to others because it uses a different index set (primes vs all positive integers).

What the Prime Restriction Reveals

Period ≈ 4 in Prime Sequence

The dominant period of $3.81 \approx 4$ primes suggests the minimum moduli cycle through patterns based on groups of 4 primes. This could relate to:

1. **Primes modulo 4:** Since primes $p > 2$ satisfy either $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$, and these alternate somewhat regularly in the prime sequence, a period-4 pattern might capture this structure.

2. **Primes modulo 8 or 17:** More refined residue classes might create the observed periodicity:

   - Primes mod8: values in $\{1, 3, 5, 7\}$ (4 classes)

   - Primes mod17: many more classes, but systematic patterns exist

3. **Distribution in arithmetic progressions:** The minimum might systematically achieve its value in different residue classes as we move through the primes.

4. **Sato-Tate distribution:** For a fixed curve, the values $a_p/2\sqrt{p}$ are distributed according to Sato-Tate (semicircle law). The periodicity might reflect clustering in this distribution.

Why Fewer Harmonics?

The prime sequence has only 2 significant peaks versus 3–4 for integer sequences. Possible reasons:

1. **Irregular spacing:** Prime gaps are irregular (by the prime number theorem, $p_n \approx n \ln n$), which can suppress higher harmonics that rely on uniform spacing.
2. **Number-theoretic structure:** The coefficients a_p for prime p have special multiplicativity properties (Hecke relations) that don't apply to composite n. This different structure might yield simpler periodicity.
3. **Smaller effective sample:** Though we have 400 data points, they span primes up to 2741, so the "density" is lower than 400 consecutive integers, potentially reducing resolution.

Mathematical Implications for Level 17 at Primes

For curve 17a1 restricted to primes, the period ≈ 4 structure is fascinating:

1. **Level 17 properties:** The conductor 17 is prime. Some facts:
   - $17 = 16 + 1 = 2^4 + 1$ (Fermat number)
   - $17 \equiv 1 \pmod 4$ (and $\equiv 1 \pmod 8$)
   - $17 = 4 \times 4 + 1$

   The appearance of period ≈ 4 when $17 = 4^2 + 1$ might not be coincidental.
2. **Rank 0:** Like curve 11a1, this is rank 0. Both rank-0 curves show interesting behavior:
   - 11a1 (all n): Period $\approx 12 \approx$ Level, 4 harmonics
   - 17a1 (primes only): Period ≈ 4, 2 harmonics
3. **Unique newform:** Level 17 has only one newform, like 11, 19, 43, 53.
4. **Frobenius at primes:** The coefficient a_p equals $p+1-\#E(\mathbb{F}_p)$, the trace of Frobenius. The periodicity in minimum moduli might reflect systematic patterns in how the curve reduces mod p for different types of primes.
5. **Quadratic reciprocity:** Since $17 \equiv 1 \pmod 4$, quadratic reciprocity gives:

   $$\left(\frac{p}{17}\right) = \left(\frac{17}{p}\right)$$

   for odd primes p. The residue classes of p mod 17 might influence a_p systematically.
6. **Comparison with integer indices:** It would be extremely interesting to run the same analysis on curve 17a1 using **all integers** $n = 1, 2, 3, \ldots$ to see:
   - Does Period $\times 6 \approx 17$ hold (which would give period ≈ 2.8)?

- How does the periodicity differ between prime and composite indices?
- Are there harmonics present in all-n analysis that are absent in primes-only?

Period ≈ 4 and Primes Modulo 17

The primes $p \not\equiv 0 \pmod{17}$ fall into residue classes $1, 2, 3, \ldots, 16 \pmod{17}$. However, by quadratic reciprocity and multiplicative structure, these 16 classes can be grouped. For instance:

- Quadratic residues mod 17: There are 8 such classes
- Quadratic non-residues mod 17: The other 8 classes

But period 4 suggests an even coarser grouping. Possible explanations:

1. **Fourth power residues:** The primes might group by whether p is a fourth-power residue mod 17 (there are 4 such classes).
2. **Splitting in $\mathbb{Q}(i)$:** Since $17 = (4 - i)(4 + i)$ in $\mathbb{Z}[i]$, primes split/ramify/stay inert based on their residue class. This creates natural groupings.
3. **Cyclotomic fields:** The 17th roots of unity and related Galois structures might impose 4-fold symmetry on how primes behave.

Signal Reconstruction

The signal with only 2 components:

$$f(n) \approx 0.1388 \cos\left(\frac{2\pi n}{3.81}\right) + 0.0601 \cos\left(\frac{2\pi n}{2.11}\right)$$

This is much simpler than the 3–4 harmonic structures seen in curves 37a1, 43a1, 53a1, 11a1. The simplicity might be intrinsic to restricting to primes.

Statistical Significance

- Dominant pattern is $\approx 10\times$ stronger than the second
- Only 2 significant peaks (fewest of any curve)
- Power ratio similar to other curves (8–10:1 range)
- Clear structure, not random
- Detrended std dev (1.0016) is moderate, similar to levels 19 and 43

Comparative Analysis: Integer Indices vs Prime Indices

This analysis raises a fundamental question: **How do periodicities differ between a_n for all n versus a_p for prime p only?**

Property	All Integers n	Primes p Only
Multiplicativity	$a_{mn} = a_m a_n$ (coprime)	N/A (primes are atomic)
Hecke relations	Full structure	Simplified
Spacing	Uniform (gaps of 1)	Irregular (prime gaps)
Harmonics expected	Many (3–4)	Fewer (2)
Periodicity source	Composite + prime	Prime structure only

Hypothesis: The richer harmonic structure (3–4 harmonics) in all-n analyses comes from the interplay between prime and composite indices. When restricting to primes, this structure simplifies to 2 harmonics.

Bottom Line

Curve 17a1 data restricted to primes shows:

- Dominant period $\approx 3.81 \approx 4$ prime steps
- Only 2 significant harmonics (simplest structure yet)
- Power ratio $\approx 10 : 1$ (standard range)
- Period-4 suggests grouping by residue classes mod 4, mod 8, or fourth-power residues mod 17
- Much simpler than integer-indexed curves, likely due to prime restriction
- Rank 0, unique newform at level 17

Most important: This analysis **cannot be directly compared** to previous curves because it uses a different index set. To properly compare:

1. Run curve 17a1 with **all integer indices** to see if Period $\times 6 \approx 17$ holds
2. Run other curves (11a1, 37a1, 43a1, 53a1) with **prime indices only** to see if they also simplify to period $\approx$ 4–5 with 2 harmonics
3. Compare the two regimes to understand the role of composite indices in creating richer harmonic structure

Conjecture: The periodicity pattern may be fundamentally different between:

- Prime-indexed coefficients: Simple period ≈ 4, determined by prime residue classes
- All-integer coefficients: Complex harmonics, determined by Level/k for varying k

Testing this would require running both analyses on the same curve.