Fourier Analysis: Curve 53a1 (Level 53)

The Data

You analyzed 400 consecutive values of "minimum moduli" from elliptic curve **53a1**, the unique newform at level 53. This curve has conductor 53 (prime) and rank 1. The raw values span from 0.000056 to 5.313, analyzed over indices $n = 1$ to $n = 400$.

Main Finding: Four Dominant Periodicities

Your data shows **four statistically significant repeating patterns**:

Rank	Period	Power	Relative Strength
1	6.35 indices	4.00×10^4	100% (strongest)
2	2.13 indices	5.01×10^3	12.5%
3	3.20 indices	4.34×10^3	10.9%
4	2.68 indices	2.47×10^3	6.2%

What This Means

Period 1: The Dominant Cycle (6.35 indices)

This is **by far** the strongest pattern in your data. With power 4.00×10^4, it's roughly **8.0 times stronger** than the second pattern and **16.2 times stronger** than the fourth.

Interpretation: Your minimum moduli exhibit a very strong oscillation that repeats approximately every 6.35 steps.

Amplitude: This oscillation has amplitude 0.2219 in the log-transformed detrended data. This is substantial—about 19.5% of the standard deviation (1.1391) of the detrended signal.

KEY OBSERVATION: Notice that $6.35 \times 6 = 38.1$. But wait—that's nowhere near 53! Let's check: $53/6 = 8.83$. So this curve **does not follow** the Period $\times 6 \approx$ Level pattern seen in curves 19, 37, 43.

However, $53/8 = 6.625 \approx 6.35$. So we might have: **Period** $\times 8 \approx$ **Level**!

Period 2: Not a Simple Harmonic (2.13 indices)

The second pattern has period 2.13, with power about 12.5% of the dominant pattern.

Key observation: Notice that $2.13 \times 3 = 6.39 \approx 6.35$. This is **almost exactly one-third** the dominant period! So this is the **second harmonic** (not first).

Amplitude: Moderate at 0.0742, contributing about 6.5% of signal variation.

Period 3: First Harmonic (3.20 indices)

The third pattern has period 3.20, with power about 10.9% of the strongest.

Key observation: Notice that $3.20 \times 2 = 6.40 \approx 6.35$. This is **almost exactly half** the dominant period! This is the **first harmonic**.

Amplitude: Strong at 0.0956, about 8.4% of signal variation. Interestingly, this harmonic is **stronger** than the second harmonic (period 2.13), which is unusual.

Period 4: Third Harmonic (2.68 indices)

The fourth pattern has period 2.68, with power about 6.2% of the strongest.

Key observation: Notice that $2.68 \times 2.37 \approx 6.35$. This doesn't fit cleanly into the harmonic series. However, $53/20 = 2.65 \approx 2.68$, suggesting this might relate directly to the level.

Amplitude: Weaker at 0.0303, about 2.7% of signal variation.

Comparison Across All Five Curves

Now we can see an expanded pattern:

Property	Level 11	Level 19	Level 37	Level 43	Level 53
Dominant period	11.76	2.84	6.56	7.30	6.35
Period/Level	1.07	0.15	0.18	0.17	0.12
Level/Period	0.94	6.69	5.64	5.89	8.35
Nearest integer	1	7	6	6	8
k where $k\times$ Period $\approx$ Level	1	6–7	6	6	8
Power ratio (1st:2nd)	9.2:1	10.1:1	7.4:1	8.4:1	8.0:1
Harmonics	4	2–3	3	3	4
Rank	0	–	1	1	1

The Harmonic Structure: Irregular but Present

Curve 53a1 exhibits harmonics, but in an unusual order:

- Fundamental: 6.35
- 1st harmonic ($\div 2$): 3.20 (detected 3rd by power)
- 2nd harmonic ($\div 3$): 2.13 (detected 2nd by power)
- Possible other component: 2.68

Notable: The first harmonic (3.20) is actually **weaker** in power than the second harmonic (2.13), which is reversed from the usual pattern. This suggests a waveform with a particularly strong third harmonic.

The signal can be approximated as:

$$f(n) \approx 0.2219 \cos\left(\frac{2\pi n}{6.35}\right) + 0.0956 \cos\left(\frac{2\pi n}{3.20}\right) + 0.0742 \cos\left(\frac{2\pi n}{2.13}\right) + 0.0303 \cos\left(\frac{2\pi n}{2.68}\right)$$

Statistical Significance

The power spectrum shows:

- The dominant pattern is 8–16 times stronger than other components
- Four significant peaks detected
- Harmonics present but in unusual power ordering
- Strong periodicity, not random
- Highest detrended standard deviation (1.1391) of all curves analyzed—most volatile

Mathematical Implications for Level 53

For curve 53a1, the period structure reveals new patterns:

1. **Period** $\times 8 \approx$ **Level:** We have:

$$6.35 \times 8 = 50.8 \approx 53$$

   This suggests the relationship might be **Level**/Period $\approx k$ where k varies with level.

2. **Revised hypothesis:** Looking at all curves:

Level	Period	Level/Period
11	11.76	$0.94 \approx 1$
19	2.84	$6.69 \approx 6$–7
37	6.56	$5.64 \approx 6$
43	7.30	$5.89 \approx 6$
53	6.35	$8.35 \approx 8$

   The ratio Level/Period takes values approximately in $\{1, 6, 7, 8\}$. This is **not** a simple linear relationship!

3. **Level 53 properties:** The conductor 53 is prime. Some facts:
   - $53 = 8 \times 6 + 5$
   - $53 \equiv 5 \pmod 6$
   - $53 \equiv 5 \pmod 8$
   - $53 = 52 + 1 = 4 \times 13 + 1$

4. **Rank 1:** Like curves 37a1 and 43a1, this is rank 1. However, its periodicity factor (8) differs from theirs (6), so rank alone doesn't determine the pattern.

5. **Unique newform:** Level 53 has only one newform, like 11, 19, 43 (but unlike 37).

6. **Unusual harmonic ordering:** The fact that the second harmonic (period 2.13) has **higher power** than the first harmonic (period 3.20) is rare. This indicates:

- The waveform has particularly strong $\div 3$ components
- Systematic behavior every 2 steps
- Possible connection to residue classes mod3

Emerging Complexity: No Single Formula

The data now reveals that the relationship between Level and Period is **more complex** than initially thought:

1. **Small level (11):** Period $\approx$ Level (factor $k \approx 1$)
2. **Medium levels (19, 37, 43):** Period $\times 6 \approx$ Level (factor $k \approx 6$)
3. **Level 53:** Period $\times 8 \approx$ Level (factor $k \approx 8$)

Question: What determines k? Possibilities:

- Level mod6 or mod8?
- Number of newforms at that level?
- Rank of the curve?
- Genus of $X_0(N)$?
- Class number or other arithmetic invariant?

None of these perfectly predict k across all curves so far.

The k-Factor Pattern

Let's examine what might determine the factor k:

Level	k	Rank	Newforms	Genus($X_0(N)$)	N mod 6	N mod 8
11	1	0	1	1	5	3
19	6–7	0	1	1	1	3
37	6	1	2	2	1	5
43	6	1	1	3	1	3
53	8	1	1	4	5	5

Observation: When $N \equiv 1 \pmod 6$, we get $k \approx 6$ (levels 19, 37, 43). When $N \equiv 5 \pmod 6$, we get $k \in \{1, 8\}$ (levels 11, 53).

This suggests N mod 6 might be relevant, though the pattern isn't perfect.

What the Amplitude Pattern Tells You

The amplitude ordering for 53a1:

Component	Period	Amplitude
Fundamental	6.35	0.2219 (large)
1st harmonic	3.20	0.0956 (moderate)
2nd harmonic	2.13	0.0742 (moderate)
Other	2.68	0.0303 (small)

The second harmonic amplitude (0.0742) being only slightly smaller than the first (0.0956) suggests a waveform that's particularly rich in the $\div 3$ frequency component.

Bottom Line

Curve 53a1 data shows:

- Dominant period $6.35 \approx 53/8$ (factor $k = 8$, different from previous curves)
- Four harmonics detected, but in unusual power ordering
- Second harmonic **stronger** than first harmonic (rare)
- Highest volatility (std dev 1.1391) of all curves studied
- Confirms the relationship is **not simply** Period $\times 6 \approx$ Level
- Factor k (where Period $\times k \approx$ Level) varies: $k \in \{1, 6, 7, 8\}$

Major finding: The periodicity structure is **more intricate** than a simple linear formula. The factor k connecting Period to Level depends on arithmetic properties of the conductor, possibly related to N mod 6 or other modular invariants. Testing more curves (especially at levels $\equiv 5 \pmod 6$ and levels between 11 and 53) would help clarify the pattern.