LEMMAS FOR "RAMANUJAN'S FUNCTION ON SMALL PRIMES".

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1. Lemmas

The first lemma appears in (we believe) an erroneous form on page 119 of Adams' reference work [A22]. Entry 0.313 of Gradshteyn's and Ryzhik's reference work [GR88] propagates Adams' version. Neither book provides a proof or a citation to one. Both give the denominator of the fraction c_n as a_0^n, instead of what we think is the correct a_0^{n+1}. (We note that there is another formulation of the coefficient c_n in terms of a different determinant in Gholami's article [Gh11].)

Lemma 1.1 (corrected version of Adams [A22], entry 6.360). *Let $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ be power series with $a_0 \neq 0$. If*

$$\left(\sum_{k=0}^{\infty} b_k x^k\right) \Big/ \left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} c_k x^k,$$

then $c_n = (-1)^n |M_n| / a_0^{n+1}$, where M_n is the $n \times n$ matrix

$$M_n = \begin{pmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \cdots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & \cdots & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix}.$$

Proof. From the hypothesis,

$$\sum_{k=0}^{j} a_k c_{j-k} = b_j \tag{1}$$

for each j. Since $c_0 = b_0/a_0$,

$$a_0 c_j + a_1 c_{j-1} + \cdots + a_{j-1} c_1 = (a_0 b_j - a_j b_0)/a_0,$$

so, for $n \geq 1$,

$$\begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = - \begin{pmatrix} (a_1 b_0 - a_0 b_1)/a_0 \\ (a_2 b_0 - a_0 b_2)/a_0 \\ (a_3 b_0 - a_0 b_3)/a_0 \\ \vdots \\ (a_n b_0 - a_0 b_n)/a_0 \end{pmatrix}. \tag{2}$$

Let

$$A_n = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix} \tag{3}$$

and let

$$B_n = \begin{pmatrix} a_0 & 0 & 0 & \cdots & -(a_1b_0 - a_0b_1)/a_0 \\ a_1 & a_0 & 0 & \cdots & -(a_2b_0 - a_0b_2)/a_0 \\ a_2 & a_1 & a_0 & \cdots & -(a_3b_0 - a_0b_3)/a_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & -(a_nb_0 - a_0b_n)/a_0 \end{pmatrix}, \tag{4}$$

so that

$$c_n = \frac{|B_n|}{|A_n|} = \frac{|B_n|}{a_0^n}. \tag{5}$$

Let

$$D_n = \begin{pmatrix} -(a_1b_0 - a_0b_1)/a_0 & a_0 & 0 & \cdots & 0 \\ -(a_2b_0 - a_0b_2)/a_0 & a_1 & a_0 & \cdots & 0 \\ -(a_3b_0 - a_0b_3)/a_0 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(a_nb_0 - a_0b_n)/a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix},$$

so $|D_n| = -|M_n|/a_0$ and $|B_n| = (-1)^{n-1}|D_n|$. Finally,

$$c_n = \frac{|B_n|}{a_0^n} = \frac{(-1)^{n-1}|D_n|}{a_0^n} = \frac{(-1)^{n-1}(-|M_n|/a_0)}{a_0^n} = \frac{(-1)^n}{a_0^{n+1}}\,|M_n|\,. \tag{6}$$

□

Lemma 1.2. *The equations below are equivalent. (We will refer to both of them as equation (D) in the sequel.) Let $h_0 = 1$ and*

$$nh_n = \sum_{r=1}^{n} j_r h_{n-r} \tag{D1}$$

for $n \geq 1$. With $H(t) = \sum_{n=0}^{\infty} h_n t^n$ and $J(t) = \sum_{r=1}^{\infty} j_r t^r$,

$$t\frac{d}{dt}H(t) = H(t)J(t). \tag{D2}$$

Claim: Equation (D1) implies equation (D2).

Proof. From equation (D1),

$$t\frac{d}{dt}H(t) = \sum_{n\geq 1} nh_n t^n = \sum_{n\geq 1}\left(\sum_{r=1}^{n} j_r h_{n-r}\right)t^n,$$

$$\text{while } H(t)J(t) = \left(\sum_{k=0}^{\infty} h_k t^k\right)\left(\sum_{r=1}^{\infty} j_r t^r\right) = \sum_{n=1}^{\infty}\left(\sum_{\substack{k=n-r\\k\geq 0\\r\geq 1}} h_k j_r\right) t^n =$$

$$\sum_{n=1}^{\infty}\left(\sum_{\substack{n-r\geq 0\\r\geq 1}} h_{n-r} j_r\right) t^n = \sum_{n=1}^{\infty}\left(\sum_{1\leq r\leq n} h_{n-r} j_r\right) t^n.$$

□

<u>Claim</u>: (D2) implies (D1).

Proof. Re-ordering some of the equations from the proof of the first claim, we have

$$\sum_{n\geq 1} n h_n t^n = t\frac{d}{dt}H(t) = H(t)J(t) = \sum_{n=1}^{\infty}\left(\sum_{1\leq r\leq n} h_{n-r} j_r\right) t^n,$$

and the claim follows by equating coefficients. □

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