Notes on BIOMEDE 211, or: Circuits, Systems, & Signals in Biomedical Engineering

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April 15, 2019

Contents

0.1 How can I print off and use this document?

Frankly, in just about any way thats useful to you. I am going to try something here, where I will try to make more or less the entirety of the notes associated with the Winter 2019 semester of BIOMEDE 211, “Circuits, Systems, and Signals in Biomedical Engineering”, to you, dear reader.

Please dont plagiarize this. If you were raised right, you ought to know what that is. If youd like my judgment on any sort of action, my opinions can be laid bare.

The first assignment I am giving you (worth 4% of your grade and which must be completed by the end of the semester) is to figure out where this document is located online, joining the Github, and making at least one contribution to this repository. Failure to contribute to this living document by the end of the semester for those in this class will result in a loss of up to 4% of one’s total grade outright.

0.2 How can I contribute to GitHub?

Follow these general steps to propose a change to this online document:

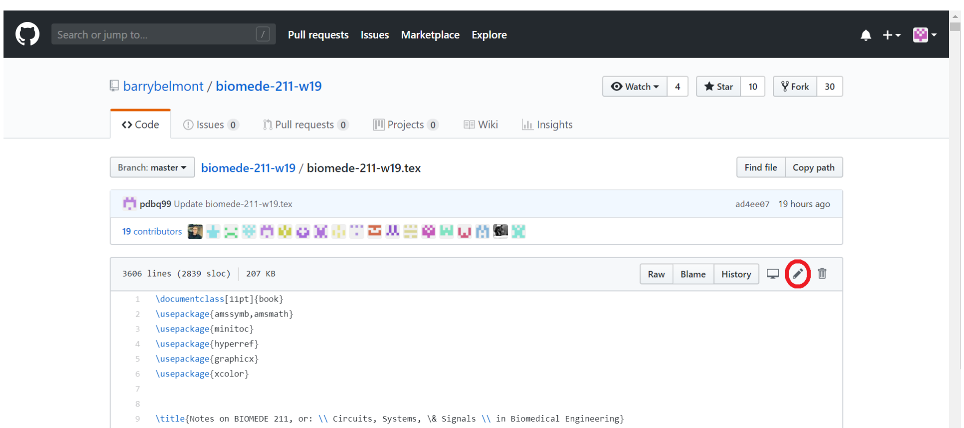
1. Create a GitHub account

   This should be rather self-explanatory. Use your e-mail account and verify it to be able to edit. You should proceed with the following steps while logged onto your account.

2. Find Dr. Belmont's GitHub page and go to the biomede-211-w19 repository ("repo"). Then click on the biomede-211-w19.tex file.

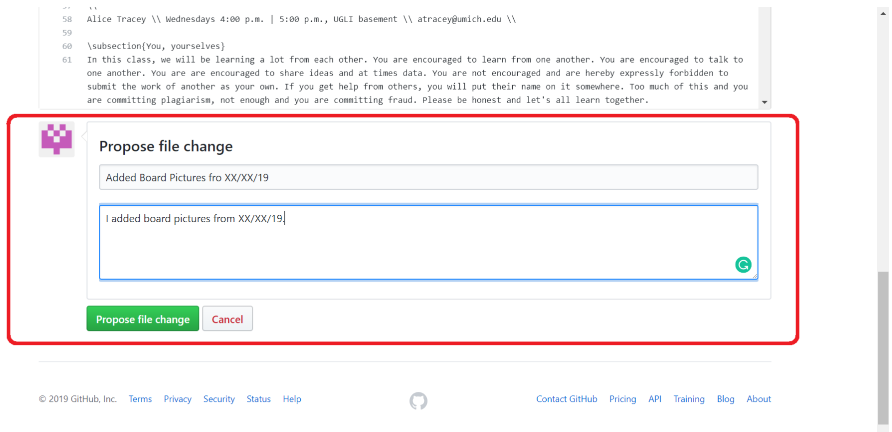
3. Edit the file

   You will find a small pencil icon on the right side of the page. Click on this to create your own branch ("forking"), and edit the file as you wish.



4. Propose file change

   After making your changes, you should scroll to the bottom of the page, find the message box that says, 'Propose file change', and fill it out. The first line should say what you have updated and can be explained in the description.

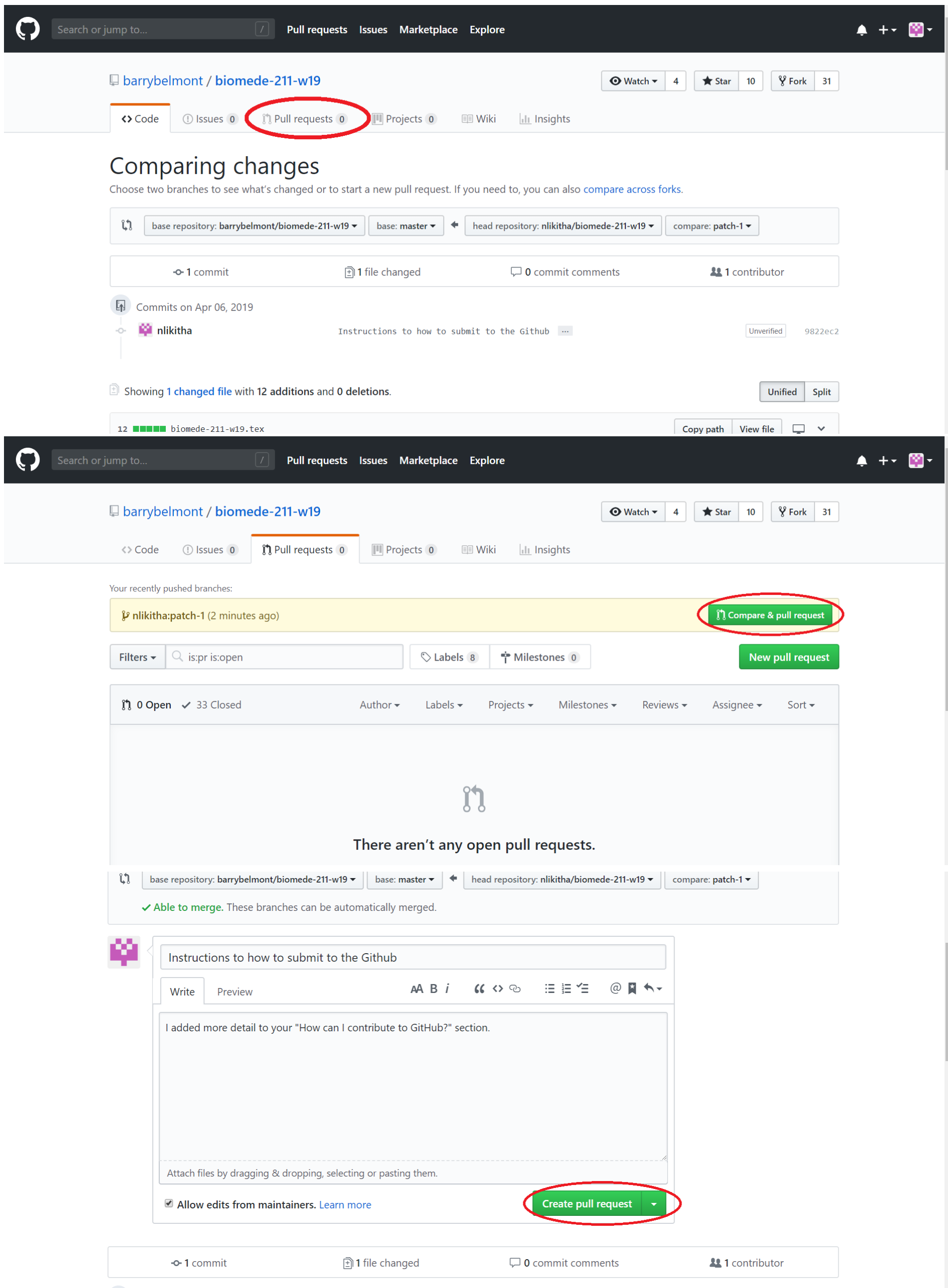


For those of you that are trying to figure out how to add your board pictures to the Github, the code to add images is as follows. To use the following code, you need to make sure the images are uploaded to the "figures" folder in the Github. Also, you will need to replace “INSERT_FILE_NAME_HERE” with the name of the image file and “FILE_EXTENSION” with the extension of the time (e.g. png or jpeg). You will need to copy and paste this code for each image you want to include.

```
\includegraphics[width=textwidth]{figures/INSERT_FILE_NAME_HERE.FILE_EXTENSI
```

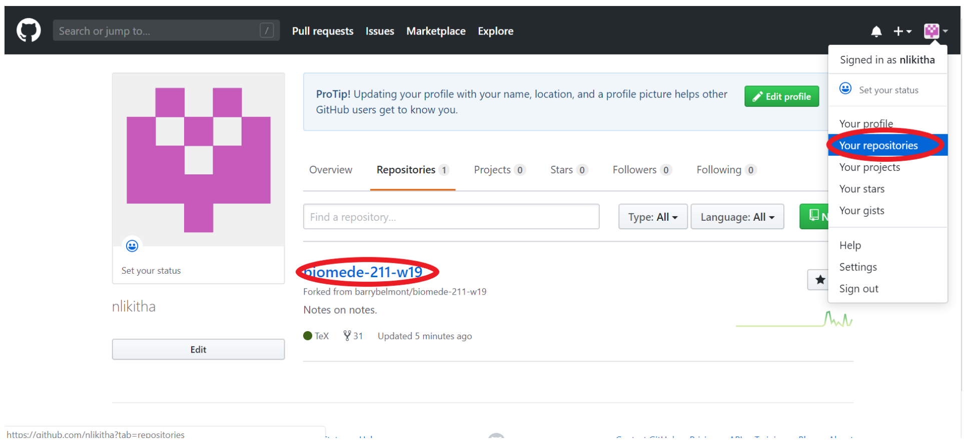
5. Create pull request

After finishing your file, you will be brought to a page that displays what you have modified on the original document. Press the green ‘Create pull request’ button to let Dr. Belmont know that you want to create a change. Once he has approved via his own GitHub account, your changes should now be in the updated master branch!

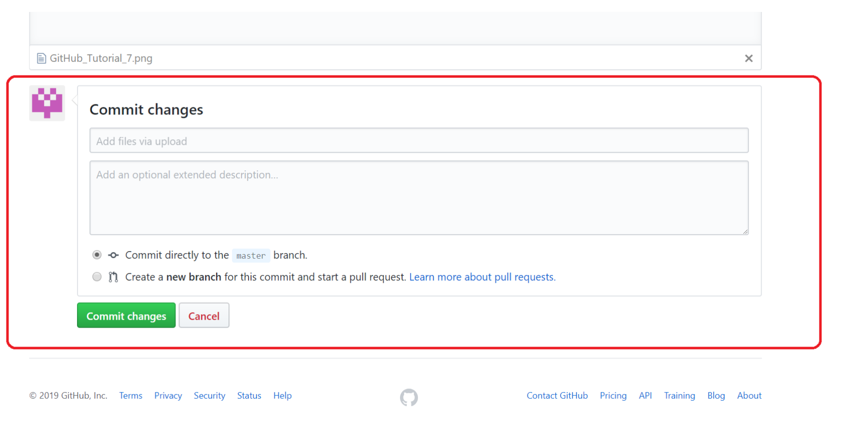


6. Uploading Images

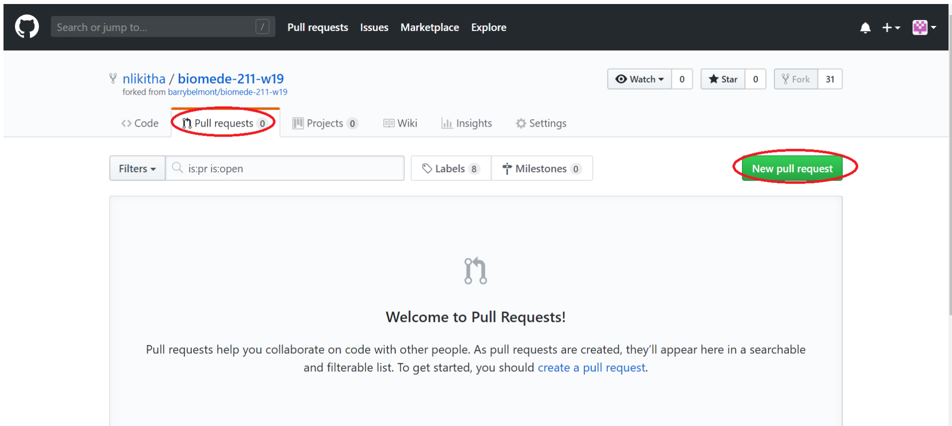
To upload images to the Figures folder, you will need to locate your "fork" of the "biomede-211-w19" repo and commit changes there. To get there, click on your profile in the upper right corner and click on "Your Repositories". Then click on " biomede-211-w19" in the center of the screen.



Then, you'll want to click on the Figures folder and then click on Upload files in the top right. Upload your image files, scroll down, and "commit" your changes.



Then, click on "Pull Request" at the top and create a new pull request. Submit that pull request the same way that you submitted the other one, and you're good to go!


Search or jump to...
Pull requests
Issues
Marketplace
Explore
nlikitha / biomede-211-w19
forked from barrybelmont/biomede-211-w19
Watch 0
Star 0
Fork 31
Code
Pull requests 0
Projects 0
Wiki
Insights
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Filters
is:pr is:open
Labels 8
Milestones 0
New pull request
Welcome to Pull Requests!
Pull requests help you collaborate on code with other people. As pull requests are created, they'll appear here in a searchable and filterable list. To get started, you should create a pull request.

0.3 Who comprises this class and how can they be reached?

0.3.1 The Captain at the helm

Barry Belmont
Wednesdays 11:00 a.m. — 1:00 p.m., 2130 LBME
belmont@umich.edu

0.3.2 The A-Team

Annabelle St. Pierre
Wednesdays 5:00 p.m. — 6:30 p.m., UGLI basement
astpierr@umich.edu

Alice Tracey
Wednesdays 4:00 p.m. — 5:00 p.m., UGLI basement
atracey@umich.edu

0.3.3 You, yourselves

In this class, we will be learning a lot from each other. You are encouraged to learn from one another. You are encouraged to talk to one another. You are are encouraged to share ideas and at times data. You are not encouraged and are hereby expressly forbidden to submit the work of another as your own. If you get help from others, you will put their name on it somewhere. Too much of this and you are committing plagiarism, not enough and you are committing fraud. Please be honest and let's all learn together.

1. Kristian A.
2. Matt A.
3. Rayna B.
4. Ellianna B.
5. Megan B.
6. Ashley B.

7. Dawn C.
8. Weihong C.
9. Caila C.
10. Jenna D.
11. Sandra D.
12. James G.
13. Natalie H.
14. Yazmin H.
15. Felicia H.
16. Isabel H.
17. Danica J.
18. Surabhi J.
19. David K.
20. Erica K.
21. Eunjeong L.
22. Annie L.
23. Ian M.
24. Madelynn M.
25. Devak N.
26. Likitha N.
27. Bree P.
28. Neelay P.
29. Matei P.
30. Shaunak P.

31. Francesca Q.
32. Raahul R.
33. Abigail R.
34. Alexander R.
35. Andrew R.
36. Elizabeth R.
37. Kiana S.
38. Ryan S.
39. Sydney S.
40. Hao S.
41. Andrew S.
42. Elijah S.
43. Cara S.
44. Aparna S.
45. Alec S.
46. Madison W.
47. Jordyn W.

0.4 What is this class about?

From the course catalog, "Students learn circuits and linear systems concepts necessary for analysis and design of biomedical systems. Theory is motivated by examples from biomedical engineering. Topics covered include electrical circuit fundamentals, operational amplifiers, frequency response, electrical transients, impulse response, transfer functions and convolution, all motivated by circuit and biomedical examples. Elements of continuous time-domain and frequency-domain analytical techniques are developed."

From your instructor's heart, "Youre going to learn the fundamental basis of electricity and its specific applicability to biomedical engineering, broadly. With this knowledge, you will have a command over a vaster swath of this world than most as youll know something important about how it works: namely, how human beings push electrons around the world to do some quite interesting things and how those electrons push back. This has relevance to a great many disparate situations: from the swelling of potential as your heart dances in your chest to nearly every source of light youve ever seen sans the sun (and even then...). All this to say, what we do here matters. And well aim to do it well."

0.5 When and where does this class meet?

Tuesdays and Thursdays, 12:30 – 2:30 p.m., 1006 DOW

0.6 What is required for this class?

0.6.1 Writing materials

A good pen and some sheets of paper ought to do.

0.6.2 Access to the internet periodically

A good bulk of our classs administration will be done through Canvas. Check it regularly.

0.6.3 A textbook of some kind

Getting more than one perspective on a topic helps most people learn better. As such, in addition to the textbook which you are currently reading, I recommend consulting another electric circuits text. We have, in the past, used *Fundamentals of Electric Circuits*, 6th edition, by Charles Alexander and Matthew Sadiku, which is also used by our EECS brothers and sisters. Digital or hardcopy is fine. As are older editions or equivalents.

0.7 What determines the grade?

0.7.1 The abject material of the the thing

Grades are determined essentially through three distinct types of assessment.

- The first and foremost of these are out-of-class based assignments (henceforth, “homework”), individually worth 8 percent, and collectively worth 48 percent of the overall grade.
- The second and secondmost of these are three in-class based assignments (henceforth, “glorious quizzes/exams’), which are worth 12, 12, and 20 percent of the overall grade, respectively.
- The final type is the evaluation of a certain in-class/out-of-class *je-nais se quoi* quality of the participatory student. “Participation” is something I take seriously in my classes and in this class will consist of (1) at least one in-class demonstration of your solutions to problems posed during the lecture period and (2) contribution to this specific document through the GitHub. Each of these forms of participation are worth 4 percent of the grade.

0.7.2 The largely arbitrary, but nevertheless significant scale by which the grade will be measured by the instructors

As this is my second time teaching this course, some calibration (on a per assignment and/or class- wide basis) may be required. That said, I plan on grading this class just as you might expect any other engineering class to do so against the following scale:

$$\begin{array}{c} \text{A+} \geq 97;\ \text{A} \geq 94;\ \text{A-} \geq 90; \\ \text{B+} \geq 87;\ \text{B} \geq 84;\ \text{B-} \geq 80; \\ \text{C+} \geq 77;\ \text{C} \geq 74;\ \text{C-} \geq 70; \\ \text{D} \geq 60;\ \text{F} \geq 50 \end{array}$$

0.7.3 The policy regarding missed chances, do overs, and dishonesty

- **If you will be absent** for some key assessment (a homework, an exam, participation), let the instructor know ahead of time and we can work something out.
- **If you feel the assessment of your work was wrong, misguided, or unfair**, let the instructor know immediately and specifically (within 48 hours of the assessment being returned) and we will work something out.

- **If you insist at any point on swindling yourself out of an honest education**, please reconsider your decisions to do so (perhaps by discussing the matter with your instructor), otherwise this whole engineering thing for you is not going to work out. You are encouraged to read the Honor Code of the institution to which you are responsible.

0.8 What are the objectives of this class?

1. To generate physical understanding of fundamental circuit and systems concepts.
2. To relate classroom material to real-world applications including selected biomedical systems.
3. To teach students basic circuit and linear systems including transient, frequency response, impulse response, and transfer functions.
4. To introduce mathematical concepts necessary to accomplish the above including convolution, Laplace transforms, and Fourier transforms.

0.9 What are the outcomes I can expect of myself from this class?

1. Learn basic circuit and systems techniques necessary for understanding biomedical instrumentation systems, bioelectrical systems, and medical imaging systems.
2. Develop an insight into systems analysis techniques motivated by development of electrical circuits concepts.
3. Develop an insight into operational amplifier analysis and design techniques as motivated by a series of practical biomedical amplifier designs.
4. Understand mathematical tools necessary for linear systems and circuit analysis/design.

Part I

Circuits

Chapter 1

I. Potential, current, energy, conservation

01/10/2019

Contents

1.1 What is electricity?

1. A form of energy resulting from the existence of charged particles
2. The physical phenomena arising from the existence, presence, and motion of charged particles
3. Rather ill-defined in common vernacular we will generally avoid its use

1.2 Charge

1. Charge is the property of matter that causes it to experience a force when placed in an electromagnetic field; measured in coulombs (C)
2. Charges are found in nature in discrete, integral multiples of electronic charge: e = -1.602 x 10^{-19} C (the charge of one electron)
3. **How many electrons are needed to form one coulomb?** (What is the weight of all those electrons?)
4. One byte is eight bits. Bits are essentially a single electron stored in a transistor. **If we were to take all the electrons from one terabyte of well distributed information (equal number of ones and zeros), how many coulombs would we have?**

1.3 Current

1. The time rate of change of charge charges (charged particles) in motion; measured in amperes; defined mathematically as

$$i := dq/dt \tag{1.1}$$

where i is current, q is charge, and t is time

2. Conversely, the total charge transferred over time can be expressed as

$$Q := \int_{t_0}^{t} idt \tag{1.2}$$

3. 1 ampere is equal to 1 coulomb/second
4. Direct current, “DC”, is current that remains constant with time
5. Alternating current, “AC”, is current that varies sinusoidally with time

1.3.1 The directionality of current

Ultimately, the direction in which we say "current" flows is largely arbitrary. As arbitrary as choosing one type of charge and calling it "positive" and another "negative". The reason it doesn't matter is that the only consequence of having chosen a "wrong direction" for the current in a given analysis is that we have to switch the sign of the value. Thus, 3 amps in one direction is *the exact same thing* as -3 in the opposite direction.

1. Thanks to Benjamin Franklin we say that current is

    i. **Positive in the direction in which positively charged particles flow** and

    ii. **Negative in the direction in which negatively charged particles**

    iii. We also now know that current results primarily from the movement of negatively charged particles (electrons) and therefore our convention is wrong in one sense, though convenient and entrenched enough that were not liable to change it in our life time (besides, the math comes out the same, and the actual flow of electrons will only matter to us in a few special circumstances, diodes)

1.3.2 The at times deadly serious nature of current

Much of the point of learning this material here is its eventual application by our hands or by the hands of those we work with. Before we put any of this stuff in our hands, we should probably know what is and is not safe.

1. 1 mA, you will feel
2. 10 mA, you will really feel
3. 100 mA, you will likely die
4. 1000 mA, you will definitely die

1.3.3 The "speed" of current

A possible misconception is that the electrons inside a wire travels at the speed of light. The speed of current is actually relatively slow. If one were to imagine an electron starting at the wire next to a light switch in an average classroom, it would take a very long period of time for it to travel to the light itself. The light's immediate reaction to a switch is due to a "hose effect"; the electrons inside the wire push other electrons in the direction opposite to the [conventional] current. This cascade of electrons is what happens close to the speed of light, not the electron movement itself.

1. **The *signal* of electrical current (that is electromagnetic radiation) travels anywhere between about 50-99% the speed of light** (dependent on a number of conditions) depending upon the material through which it travels (based on a dielectric behavior known as permittivity)

2. **The drift velocity of electrons** within a copper wire is 25 μm/s, so how does anything ever turn on?

3. **The hose effect** - The electrons at the light switch will almost certainly never pass through a light bulb, but they will move around and bump into their neighbors which bump into their neighbors which bump into their neighbors, etc., until it causes the electrons nearest the light to pass through. This is how water at a spigot is able to push water at the end of a hose.

4. **How current and drift veolcity related?** - Mathematically, it's represented as follow:
$$I = \frac{Q}{t} = \frac{neAd}{d/v_d} \tag{1.3}$$
Imagine there is a cylindrical wire with length d and cross-sectional area A. Suppose there are n electrons per unit volume and each with a charge of e. Then in the whole cylinderical wire, the total charge is derived by multuplying the total volume Ad, the number of electron per unit volume n, and the elementary charge e. This is the numerator part of the story. On the denominator is the time component. Assume all electrons are moving in one direction and with drift veolcity. It would take t amount of time for the last electron to move a distance of d from one end to the other and exit the cylinderical wire. In fact, this is how we, in a microscopic sense, perceive the concept of current, which is defined by the total charge that passes through a single reference cross-section over a given amount of time.

1.4 Potential (difference)

1. The amount of work needed to move a unit of (positive) charge from a reference point to another point [without producing an acceleration]).

2. Potential is measured in “volts” and is often called “voltage”. In this class we will endeavor to avoid such a term as it can be very confusing to talk about potential as if there were such a *thing* as voltage.

3. Defined as
$$v := \frac{dw}{dq} \tag{1.4}$$

4. Potential describes the *potential* to do something. Increasing the potential is akin to increasing the height of a cliff. The height does not *do* anything other than increase what can be done on the drop. If potential is the cliff’s height, charge would be pebbles you’d drop off the side, and current describes how fast those pebble fall.

5. In this class, and for the vast vast majority of electrical engineering work, we care about the *difference* in potential. One element held at 100 billion volts and another held at 100 billion + 1 volts has a *potential difference* of 1 V, which is less than a single AA battery.

6. Voltage can also be thought of as how badly the current ”wants” to flow, while current is the actual flow of charges per second. Since charge flows but not the voltage, voltage can exist without current - a single charge induces a voltage. On the other hand, current can’t exist without voltage since having a current means that charges are flowing, and if charges are flowing there is a potential difference across the charges.

7. Some typical voltages to be aware of

   Consumer level batteries (AA, AAA): 1.5 V (DC); 9 V (DC)

   Car batteries: 12 V (DC)

   The “mains” (levels provided by power companies to consumers): 110-120 V (AC) and 220-240 V (AC) in America

   Power transmission lines: 110-1200 kV (AC), transformers are used to step up and down the potential before used by consumers

1.5 Power

1. The time rate of expending or absorbing energy.

2. Quantifies the rate of energy transfer.

3. Mathematically:
$$p = \frac{dw}{dt} = \frac{dw}{dq} \cdot \frac{dq}{dt} = v \cdot i \tag{1.5}$$

4. Measured in watts: $1\ \mathrm{W} = 1\ \frac{\mathrm{J}}{\mathrm{s}} = 1\ \frac{\mathrm{N \cdot m}}{\mathrm{s}} = 1\ \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^3}} = 1\ \mathrm{V} \cdot 1\ \mathrm{A}$

5. **Passive sign convention**: If current enters through the positive terminal of an element, $p = +vi$; if current enters through the negative terminal of an element, $p = -vi$.

1.6 Energy

1. The capacity to do work.

2. Measured in joules.

3. $E = \int \frac{dw}{dt} dt \rightarrow$ power x time

4. $\mathrm{J} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}} = \mathrm{N \cdot m} = \mathrm{Pa \cdot m^3} = \mathrm{W \cdot s} = \mathrm{C \cdot V}$

1.7 Conservation

Here, as elsewhere, things will be conserved. In electrical circuits there are two laws of conservation that will matter most for us:

1. **The Conservation of Mass.** The conservation of mass means that no mass can be added to or removed from a circuit without being accounted for. Put differently, in a closed system (the type we will concern ourselves with here) no mass is added or removed.

   In electrical circuits, the mass we care the most about are the charges whipping around. Thus, for us, *the amount of charge within a circuit must remain constant.*

2. **The Conservation of Energy.** The conservation of energy means that no energy can be added to or removed from a circuit without being accounted for. Put differently, in a closed system (the type we will concern ourselves with here) no energy is added or removed.

   In electrical circuits, the energy we care the most about is the potential provided by sources and depleted by other elements in the circuits. Thus, for us, *the sum of potentials within a circuit must equal zero.*

In evaluating circuits, the main focus of the first third of this class, it will be the application of these two conservative laws that will enable us to "solve" them. That is, by understanding (1) how energy is generated and used and (2) how charges move around in closed loops ("circuits") we will be able to predict the behavior of the myriad electrical systems which may cross our paths.

1.8 Worksheet

1.8.1 A constant charge through a cross-section

How much charge passes through a cross-section of a conductor in 60 seconds if a DC current value is measured at 0.1 mA? **Solution**

1.8.2 An arbitrary charge through a cross-section

Determine the total charge entering a terminal between $t = 0$ seconds and $t = 10$ seconds if the current (in amps) passing through is

$$i(t) = \frac{1}{\sqrt{5t + 2}}. \tag{1.6}$$

Solution

1.8.3 A "tera"ble puzzle

Approximately how much current is necessary to transmit one terabyte of information in an hour? **Solution**

1.8.4 A pacemrker's power requirements

A cardiac pacemaker will provide approximately 5,000 J of energy over 5 years. Determine the capacity of a 5 V lithium battery necessary to drive this pacing such that only 40% of its energy is spent over that time. **Solution**

1.8.5 A neuron's excitation energy

A colleague of yours has been in their lab ginning up new neurons. You, as their resident electrical expert, are tasked with determining the energy consumed by the cell. If the current and voltage variations are found to be functions of time ($t \geq 0$)

$$i(t) = 3t \tag{1.7}$$

$$v(t) = 10e^{6t} \tag{1.8}$$

determine the energy consumed between 0 and 2 ms. **Solution**

1.8.6 A thump to the chest

(a) A typical defibrillator delivers 200-1000 V in less than 10 ms. How much current is needed to deliver 120, 240, and 360 Joules?
(b) A human heart ways about 300 grams. From approximately how high of a cliff would one have to drop a heart such that the impact was equivalent to the energy delivered to someone's chest from a defibrillator? **Solution**

Chapter 2

An introduction: II. Circuit elements

Contents

01/15/2019

2.1 Active v. passive

1. Active elements are capable of generating energy while passive components cannot
2. **Active**: generators, batteries, operational amplifiers, "sources"
3. **Passive**: resistors, capacitors, inductors, i.e., most circuit elements

2.2 Ohm's Law and what it means

Ohm's Law is concerned with the relationship between voltage, or potential difference, and current across a conductor. The potential difference across a conductor is proportional to the current flowing thorugh the conductor with the proportionality constant being denoted as R, or resistance. This can be expressed as:

$$V := iR \tag{2.1}$$

This essentially states that the drop in potential across the conductor, or resistor, is equivalent to the current flowing through the conductor and its resistance. When considering impedance, the equation can be modified to state:

$$V := iZ \tag{2.2}$$

2.3 Sources

1. **An ideal independent source** is an active element that provides a specified value of potential or current, regardless of other circuit elements.

   Batteries and power supplies may be approximated as ideal potential sources.

2. **An ideal dependent (or controlled) source** is an active element in which the source quantity is controlled by another quantity (such as potential, current, temperature, measured resistance, etc.).

3. **An ideal potential source** will produce any current required to ensure that the terminal voltage stated is satisfied.

4. **An ideal current source** will produce any voltage required to ensure that the terminal current as stated is satisfied

5. Symbols

   Voltage-controlled voltage source, VCVS

   Current-controlled voltage source, CCVS

   Voltage-controlled current source, VCCS

   Current-controlled current source, CCCS

2.4 Resistors

Resistors are electrical (circuit) elements that resist the flow of electric charge (current); passive two-terminal components that implement a defined/"constant" resistance; meant to reduce current flow and change potential

2.4.1 Resistance, R

1. **Resistance** is the physical property describing an element's ability to resist current and is most often represented by R

2. Resistance is measured in "ohms", Ω, which is equivalent to 1 V/A

3. Resistance is one half of a broader physical phenomenon known as **"impedance"** - the property describing an element's ability to *impede*

current. Impedance is typically represented by Z, which we'll explore more thorough in a bit.

2.4.2 Resistivity, ρ

1. The resistance of an element (such as a resistor) depends on three things:

    Resistivity, ρ, of the material comprising the element, which is the *material*'s ability to resist the flow of charges; measured in ohm-meters

    Length, l, of the element; measured in meters

    Area, A, of the cross-section of the element; measured in m^2

    Such that $R = \rho \frac{l}{A}$

    What units are we left with?

    What are the effects of length and area?

2. **Materials with low resistivity** are generally called (and treated as) "conductors" as they are able to more effectively *conduct* the motion of electrical charges than materials with high resistivity

3. **Materials with very high resistivity** are generally used as "insulators" as they prevent the flow of current through them and thus *insulate* the current within prescribed bounds, such as with a copper wire with plastic wrapped around it.

Here is a link to a video that further explains the concepts of resistivity and resistance: <https://www.youtube.com/watch?v=4rsswT_Rv1M>.

2.4.3 Conductance

1. The inverse of resistance is conductance, G, which describes the ability of an element to conduct current

2. Measured in Siemens

3. Allows us express Ohm's law slightly differently, $i = Gv$, which says that the current generated through an element by a potential is directly proportional to some constant, namely conductance.

4. The material specific property **conductivity**, σ is measured in S/m

2.5 Capacitors

1. Passive two-terminal components that store energy in an electric field; introduces capacitance to a circuit.

2. Can be thought of as two conductive plates sandwiching a "dielectric" material. Essentially it is two "conductors" separated by a "non-conductive region".

3. When a capacitor is attached across a source, an electric field develops across the dielectric causing a net positive charge to collect on one conductor and a net negative charge to collect on the other.

4. We can define the capacitance of an element mathematically as

$$C = Q/V \tag{2.3}$$

   where C is capacitance in farads, Q is positive or negative charge on each conductor, and V is the potential between them

5. We can also represent capacitance by the voltage-based rate of charge accumulation: $C = dQ/dV$.

2.5.1 Its time varying behavior

Unlike resistors, capacitors have a *time-varying* element to that. That is, since $C = Q/V$, $V = Q/C$.

If we then recall Equation 1.2, we can write the time-dependent potential relationship of a capacitor

$$V(t) = \frac{Q(t)}{C} = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + V(t_0) \tag{2.4}$$

We can also recall[1] Equation 1.1, and represent the time-dependent current relationship as

$$I(t) = \frac{dQ(t)}{dt} = C\frac{dV(t)}{dt} \tag{2.5}$$

[1]We could also take the derivative of the equation preceding this one and do a little rearrangement. As it turns out, these physical relationships are rather codified and thus can be gotten out by any number of means.

2.5.2 Charge accumulation

1. While charges accumulate on a capacitor, no current flows *through* the capacitor.
2. **Well, then why use them? After awhile won't the current just stop?** Yes, indeed it will – in a DC circuit!
3. The capacitor will become "charged" over time, eventually reaching the same potential as that established across it, e.g., by a source. Since potential only ever travels down potential gradients, if the capacitor and the source (say, a battery) are at the same potential, no current will flow.
4. Thus, a fully charged capacitor will act as an "open" circuit, while an uncharged capacitor will act as a "short" circuit.

2.5.3 A simple example

If we consider Ohm's law for a simple RC circuit (one in which a source, a resistor, and a capacitor are in series), we can describe the system by

$$V_0 = v_R(t) + v_C(t) \tag{2.6}$$

$$V_0 = i(t)R + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau \tag{2.7}$$

Taking the derivative of both sides:

$$0 = R\frac{di(t)}{dt} + \frac{1}{C}i(t) \tag{2.8}$$

$$0 = RC\frac{di(t)}{dt} + i(t) \tag{2.9}$$

$$i(t) = \frac{V_0}{R} \cdot e^{-t/RC} \tag{2.10}$$

$$v(t) = V_0\left(1 - e^{-t/RC}\right) \tag{2.11}$$

$$Q(t) = C \cdot V_0\left(1 - e^{-t/RC}\right) \tag{2.12}$$

2.6 Inductors

1. Passive two-terminal components that store energy in a magnetic field

2. Can be thought of as an insulated wire wound into a coil around a core (which may either be filled with a material or left open to the environment)

3. Behavior can be modeled as $L = \frac{\Phi}{I}$, where L is the inductance, Φ is the magnetic flux generated by a current, I.

4. By Faraday's law of induction, voltage induced by a change in magnetic flux through a circuit is
$$v = \frac{d\Phi}{dt} \tag{2.13}$$
which we can rewrite as
$$v = \frac{d}{dt}(Li) = L\frac{di}{dt} \tag{2.14}$$

5. In this class, at this level, and for most biomedical applications you're liable to experience in your tenure, you will not work extensively with inductors. However, you should be able to recall at least this much at a moment's notice to be able to ascertain a system's behavior.

2.7 Impedance

1. The measure of opposition a circuit element presents to a current when a potential is applied. (It is measured in ohms.)

2. It is "complex" in two sense of the term. First, the actual phenomenon itself comprises complex numbers; that is, there is both a "real" and an "imaginary" component.

   The real component is known as resistance, R

   The imaginary component is known as reactance, X

   Impedance can be represented as a combination of either

   Resistance and reactance: $\mathbf{Z} = R + \jmath X$, where $\mathbf{Z}$ is impedance, R is resistance, and X is reactance, or

   Magnitude and phase: $\mathbf{Z} = |Z|e^{\jmath\theta}$, where $|Z|$ is the magnitude of the impedance vector, $\mathbf{Z}$, and θ is the phase of said vector (i.e., the delay between current and potential). Phase, θ is equivalent to $\tan^{-1}(X/R)$

3. Impedance is also complex in the sense that it is complicated. The impedance of an object is a factor of many parameters including permittivity, geometry, quantum states, thermal stability, etc. Let us not view this sort of complexity as an impediment to our understanding of impedance.

4. The inverse of impedance is **admittance**, Y, and comprises a real component, **conductance**, G, (which is the inverse of resistance) and an imaginary component, **susceptance**, B (which is the inverse of reactance). (It is measured in Siemens.)

   $\mathbf{Y} = G + \jmath B$

2.7.1 A quick note on "imaginary" numbers

The term "imaginary" is an unfortunate name for an excellent mathematical tool. All the imaginary operator – in this class represented by $\jmath = \sqrt{-1}$ – is is a type of number "orthogonal" to our "real" numbers. Imaginary numbers are no less "real" than real numbers. Unfortunately, they aren't necessarily the most intuitive to our little mammalian brains and thus we must be trained to work with them. However, as we will see in this class, they can be quite useful.

2.8 Equivalent impedance

1. It will often be more convenient to think about the impedance which a component burdens a system with (or the conductance which it affords) rather than its resistance. **Therefore, we need to begin to think in terms of equivalent impedances as we start to evaluate circuits.**

2. Recall Ohm's law

   Resistors, $v = iR \quad \rightarrow Z_{eq,R} = R$

   Capacitors, $v = \frac{1}{C}\int i dt \quad \rightarrow Z_{eq,C} = \frac{1}{\jmath\omega C}$

   Inductors, $v = L\frac{di}{dt} \quad \rightarrow \jmath\omega L$

   I want to plant a flag here for you to notice the relationship between the $\jmath\omega$ terms from the capacitor and inductor and the corresponding derivative and integral forms of current in the Ohm's law representation. This will become very important once we get into the Laplace and Fourier transforms.

3. We must also recognize that few will be the circuits comprising but a single element. As such, we should know how to find the equivalent impedance of many elements.

2.8.1 Impedances in general

Series

$$Z_{eq,series} = Z_1 + Z_2 + Z_3 + ... \tag{2.15}$$

Parallel

$$\frac{1}{Z_{eq,parallel}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + ... \tag{2.16}$$

A special case to remember. When dealing with only two elements:

$$Z_{eq} = \frac{Z_1 \cdot Z_2}{Z_1 + Z_2} \tag{2.17}$$

2.8.2 Resistors

Series

$$R_{eq,series} = R_1 + R_2 + R_3 + ... \tag{2.18}$$

Parallel

$$\frac{1}{R_{eq,parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ... \tag{2.19}$$

2.8.3 Capacitors

Series

$$\frac{1}{C_{eq,series}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ... \tag{2.20}$$

Parallel

$$C_{eq,parallel} = C_1 + C_2 + C_3 + ... \tag{2.21}$$

2.8.4 Delta-Wye (Δ-Y) transformations

Going from Delta to Wye

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \tag{2.22}$$
$$Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c} \tag{2.23}$$
$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} \tag{2.24}$$

Going from Wye to Delta

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_1} \tag{2.25}$$
$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_2} \tag{2.26}$$
$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3} \tag{2.27}$$
$$\tag{2.28}$$

2.8.5 A few examples

Example 1 Find the equivalent resistance, if a resistor $R_1 = 10$ kΩ is connected in parallel to $R_2 = 3.3$ kΩ.

Solution. $R_{eq} = \frac{R_1 \cdot R_2}{R_1 + R_2} = \frac{(10)(3.3)}{10+3.3} = 2.48$ kΩ

Example 2 Find the equivalent resistance of three parallel-connected resistors of equal value. If $R = R_1 = R_2 = R_3 = 10$ kΩ, what's R_{eq}?

Solution. Recall, Equation 2.19

$$\frac{1}{R_{eq}} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} \rightarrow 3R_{eq} = R \rightarrow R_{eq} = \frac{R}{3} \rightarrow R_{eq} = \frac{10k}{3} = 3.33k\Omega \tag{2.29}$$

Example 3 Four resistors are connected in parallel. $R_1 = 10$ kΩ, $R_2 = 1$ kΩ, $R_3 = 5$ kΩ, and $R_4 = 3$ kΩ. Calculate their equivalent resistance.

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \tag{2.30}$$
$$\frac{1}{R_{eq}} = \frac{1}{10k} + \frac{1}{1k} + \frac{1}{5k} + \frac{1}{3k} \tag{2.31}$$
$$= 612.3\ \Omega \tag{2.32}$$

2.9 Grounds

1. A reference point in an electrical circuit from which potentials are measured
2. A common return path within a circuit

2.10 Conductors

1. Allow from the transmission of electrical energy
2. Serve to connect circuit elements
3. Also known as wires and traces
4. Within circuit schematics we must be mindful of “junctions” and “jumps” in conductors

2.11 Operational amplifiers (”Op-amps”)

1. Active components that deliver the amplified difference between its inverting and non-inverting terminals
2. Will be discussed at length in the next class and along with resistors, capacitors, and sources, will be among the primary circuit components we work with
3. Allow us to model mathematical functions; any mathematical function that can be represented by a differential equation can be replicated with an op-amp

2.12 Diodes

Two-terminal circuit elements that allow current to flow only in one direction

2.13 Switches

Make/break/change circuit paths (thereby diverting current or removing potential)

1. Single pole, single throw, SPST

2. Single pole, double throw, SPDT
3. Double pole, single throw, DPST
4. Double pole, double throw, DPDT

2.14 Transistors

2.15 Transformers

1. Transfer electrical energy between circuits using induction
2. Allows for the effective transmission of power and the stepping up/down of potential
3. Crucial for the transmission, distribution, and utilization of AC

2.16 Worksheet

2.16.1 Problem 1, expressing power in ohms

Utilizing Ohm's law, express units of power to include ohms.

Solution

2.16.2 Problem 2, a couple toaster based problems

A toaster draws 2 A at 120 V. What is its resistance?

Solution

How much current is drawn by a toaster with a resistance of 10 Ω at 110 V?

Solution

2.16.3 Problem 3, currently conducting power

In the circuit shown, calculate the current, i, the conductance, G, and the power, p.

Solution

2.16.4 Problem 4, conductance of a sodium channel

Conductance (G/A) of a sodium channel of a cell membrane at a specific time is 10 mS/cm^2. If the channel length as 100 nm, what is its conductivity?

Solution

2.16.5 Problem 5, resistance of a simple tissue

Determine the resistance of a homogenous and isotropic tissue with a cross-sectional area which can be described by the functions $y = 8 - x^2$ from $x = -2$ cm to $x = +2$ cm, a length of 10 cm (parallel to the z-axis), and a resistivity of 80 Ωm.

Solution

Chapter 3

An introduction: III. Operational amplifiers

01/17/2019

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3.1 Some details

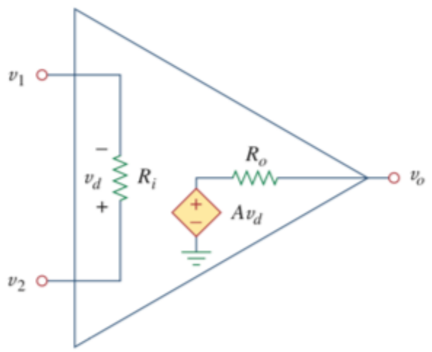


Figure 5.4
The equivalent circuit of the nonideal op amp.

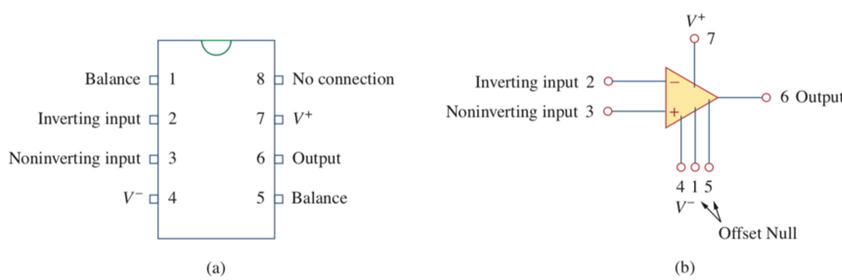


Figure 5.2
A typical op amp: (a) pin configuration, (b) circuit symbol.

1. Behaves like a voltage-controlled voltage source
2. They can amplify, sum, subtract, multiply, differentiate, integrate
3. They are active circuit elements
4. Though they have somewhat more complicated internal workings, we typically represent them in electrical circuits as a triangle with three (sometimes five) very important terminals:

   An inverting input (sign, typically represented up top for convenience, but it need not be)

   A non-inverting input (+ sign, typically on bottom)

   An output

3.2 Some rules

There are **three important features of ideal operational amplifiers** that we must understand thoroughly. These are things worth stamping in your brain.

1. **Infinite open-loop gain.** The "A" of the gain is infinitely large such that any difference in voltages V_1 and V_2 causes an enormously large output voltage. As much as is being supplied. (The real value of gain in most operational amplifiers is between 10^5 and 10^8.)

2. **Infinite input impedance.** Current cannot travel between the inverting and non-inverting terminals. (Really, the impedance is between 10^5 and $10^1 3$ ohms and is often signal dependent.)

3. **Zero output impedance.** There is no loss transmitting a voltage difference to the output. (Really is about 10-100 ohms and is chip dependent.)

3.3 Some conveniences

1. With infinite input impedance, no current can flow into or out of the terminals and hence i_1 and i_2 are equal to 0.

2. Since no current flows across the terminals, the terminals are at equal potential. Hence "$v_1 = v_2$".

Some extra facts with operational amplifiers are that they can be combined with a capacitors to create different filters. Adding a capacitor in series with the input resistor creates a "high-pass filter" amplifier, where it passes signals with frequency higher than a specific cutoff frequency and attenutates signals lower than the cutoff. Adding a capacitor in parallel with the feedback resistor creates a "low-pass filter" amplifier, where it passes signals with frequency lower than the cutoff and attenuates signals higher than it. The corner frequency for the cutoff may be caulcated with f=1/(2piRC). Having both capacitors would create a band-pass filter which attenuates signals lower than the lower cutoff and higher than the upper cutoff frequencies.

3.4 Some examples

3.4.1 Inverting amplifier

We will apply the conservation of mass at this point to solve our equations. This is among the simplest and most effective ways to add gain to a circuit. So much so that you will use it again and again and again in life and especially in labs

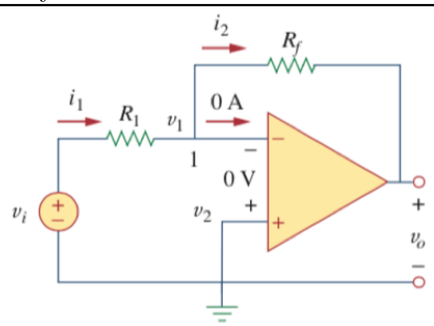


Figure 5.10
The inverting amplifier.

A key feature of the inverting amplifier is that both the input signal and the feedback are applied at the inverting terminal of the op amp.

We apply KCL at the node for v1

1. I1 i2 i3 = 0
2. I3 = 0
3. I1 i2 = 0
4. I1 = i2
5. I1 = (vi v1)/R1
6. I2 = (v1 - Vo)/Rf
7. (vi v1)/R1 = (v1 - Vo)/Rf
8. V1 = V2 = 0
9. Vi/R1 = -Vo/Rf

10. Vo = -Rf/R1 * Vi

11. R2/R1 is our gain, gain factor.

3.4.2 Non-inverting amplifier

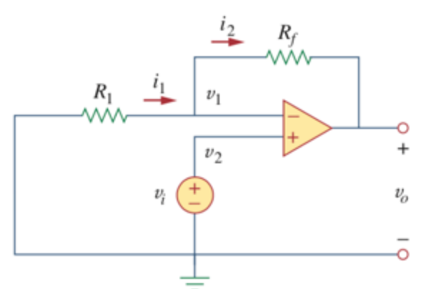


Figure 5.16
The noninverting amplifier.

Again, the name might imply what it does. It will amplify our input signal without inverting it.

We can again perform Nodal analysis.

1. I1 i2 i3 = 0

2. I3 = 0, since no current enters the non-inverting input

3. I1 = i2

4. (Vg v2)/R1 = (v2 Vo)/Rf

5. v2/R1 = (vi Vo)/Rf

6. Vo = (1 + Rf/R1) * Vi

3.4.3 Voltage follower

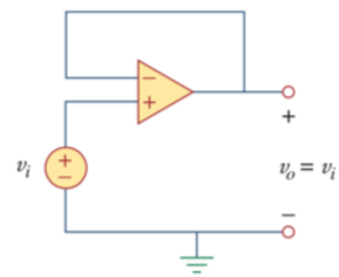


Figure 5.17
The voltage follower.

What if we didnt have any resistors? → Vi = v2 = v1 = Vo → $Vi = Vo$

3.4.4 Summing amplifier

3.4.5 Differential amplifier (as homework)

Chapter 4

Circuit analysis: I. Nodal analysis

01/22/2019

Contents

4.1 Nodes and branches

1. **A branch** is any two-terminal element. (examples: Resistor, Capacitor, Wire, etc.)

   A branch *is any two-terminal element. What are some two-terminal elements weve learned?*

2. **A node** (junction) is a point of connection between two or more branches.

   A node *is a point of connection between two or more branches. Often indicated by a dot. What else have we called a node? A junction.*

3. A loop is **independent** if at least one branch is not part of any other independent loop.

   A loop *is any closed path within a circuit. A closed path formed by starting at a node, passing through a set of nodes, returning to the starting node without passing through any node more than once.*

4.1.1 The Seven Bridges of Konigsberg

The Knigsberg bridge problem asks if the seven bridges of the city of Knigsberg (left figure; Kraitchik 1942), formerly in Germany but now known as Kaliningrad and part of Russia, over the river Preger can all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began.

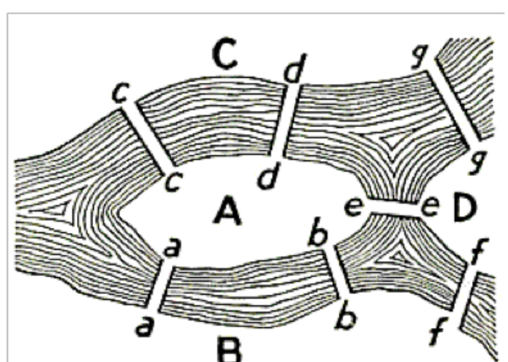


FIGURE 98. *Geographic Map: The Königsberg Bridges.*

4.1.2 Independence

A loop is independent if it contains at least one branch which is not a part of any other independent loop

1. Each of the loops in the circuit at right are independent
2. Independent loops lend themselves to sets of equations to be solved!

4.1.3 Fundamental theorem of network topology

Fundamental theorem of network topology states that the number of branches, b, must equal the sum of the independent loops, l and nodes, m minus one, that is

$$b = l + n - 1 \tag{4.1}$$

With this fundamental theorem we can also redefine/refine our definition of series and parallel.

- **Series** — two or more elements share a single node and thereby carry the same current
- **Parallel** — connected to the same two nodes and thereby have the same voltage across them (potential difference)

Practice Problem 2.4

How many branches and nodes does the circuit in Fig. 2.14 have? Identify the elements that are in series and in parallel.

Answer: Five branches and three nodes are identified in Fig. 2.15. The 1-Ω and 2-Ω resistors are in parallel. The 4-Ω resistor and 10-V source are also in parallel.

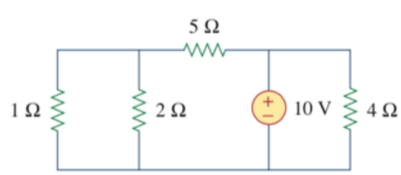


Figure 2.14
For Practice Prob. 2.4.

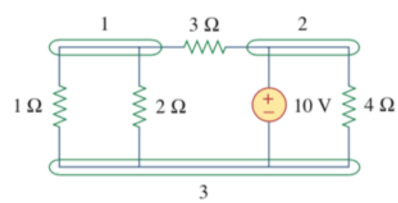


Figure 2.15
Answer for Practice Prob. 2.4.

4.2 Kirchhoff's Laws

I am not personally a fan named laws of nature, especially ones which are mere recapitulations of already perfectly good laws. Thus do we introduce Kirchhoffs laws, known as

- Kirchhoffs current law ("KCL")
- Kirchhoffs voltage law ("KVL")

These are, as far as Im concerned, mere restatements of *the conservation of mass* and *the conservation of energy*, respectively.

4.2.1 Kirchhoff's Current Law

Kirchhoff's current law states that any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node.

- Put differently, the algebraic sum of currents entering a node (or any closed boundary) is zero
- For those mathematically inclined among us, that is: $\sum_{x=1}^{n} i_x = 0$
- Another way this often gets stated is by saying that the algebraic sum of charges within a system cannot change and is thus sometimes referred to as "the conservation of charge".
- Well since all of our charge carriers are merely particles (electrons, protons, ions, etc.), this is just another layer over the top of the underlying law which is that mass cannot be created or destroyed.
- However, Kirchhoff's formulation of this law (conserving charge, mass) is useful in circuits as it gives us a great tool, being able to say that current going in is equal to current going out
- For the figure $i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0 \rightarrow i_1 + i_3 + i_4 = i_2 + i_5$
- KCL forms the basis of a technique well spend the next couple of lectures on known as nodal analysis because we evaluate the current going into and coming out of nodes

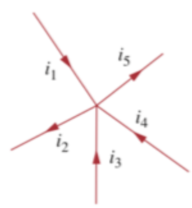


Figure 2.16
Currents at a node illustrating KCL.

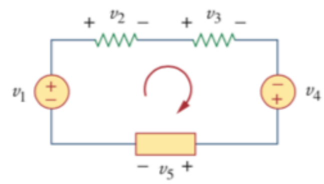


Figure 2.19
A single-loop circuit illustrating KVL.

4.2.2 Kirchhoff's Voltage Law

Kirchhoff's voltage law states that the sum of electrical potential differences (voltage) around any closed network is zero

- That is, for any closed path (loop), the sum of voltages is zero.
- Mathematically: $\sum_{m=1}^{M} v_m = 0$, where M is the number of voltage drops (caused by circuit elements) in the loop and vm is the mth voltage drop
- The sum of voltage rises = the sum of voltage drops

  $v_1 + (-v_2) + (-v_3) + v_4 + (-v_5) = 0$

  $v_1 + v_4 = v_2 + v_3 + v_5$

4.2.3 A few examples

1. Simple 1 Vs 1 R circuit; Vs = 10, R = 1 kohm, I = 0.01 A / 10 mA
2. Simple 1 Vs 2 R in series; Vs = 10, R = 1 kohm, I = 0.005 A / 5 mA

   What does KCL tell us? (Current is the same through resistors).
3. Simple 10 mA source, 2 R (1 kohm) in parallel; What is the voltage?

   $i_1 + (-i_2) + (-i_3) = 0] \rightarrow i_1 = i_2 + i_3 \rightarrow i_1 = V/R_1 + V/R_2$

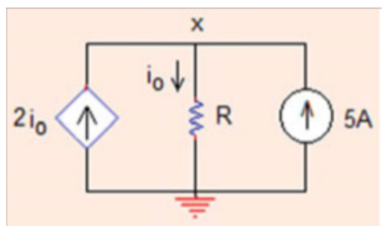
   $0.01 = V/1000 + V/1000 \rightarrow 0.01 = 2V/1000 \rightarrow 0.010/2 * 1000 = 5$ V

   Nodes are at the same voltage!

4. 10 mA up, 5 mA down, 2 R (1 kohm), Rl (500 ohm), all in parallel. What is the current through load Rl? [have someone come to the board and solve]

   When current sources are in parallel they add together

5. For the circuit shown below, use KCL to find the remaining branch currents



2.13 For the circuit in Fig. 2.77, use KCL to find the branch currents I_1 to I_4.

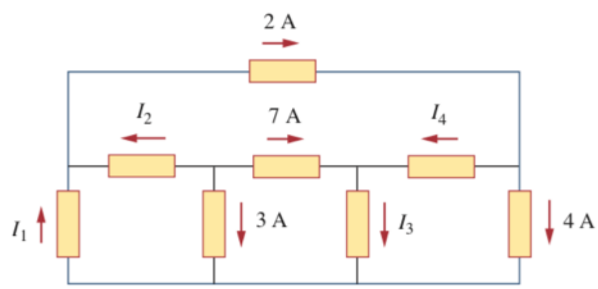


Figure 2.77
For Prob. 2.13.

For Figure 2.13, you know to use KCL because the circuit contains two current sources. Ideal current sources will deliver whatever potential is necessary to obtain the desired current so we don't know how potential behaves through it. If we look at node x and arbitrarily set i_1 as entering the node from the left, i_2 as exiting the node down the branch containing the resistor, and i_3 as entering the node from the right, then:

$i_1 - i_2 + i_3 = 0$

$2i_0 - i_0 + 5 = 0$

$i_0 = -5A$ The negative sign just means that the arbitrary direction we chose for our analysis is opposite of the actual direction of current flow

For Figure 2.77, start at a node with only one unknown current such as the upper right node.

$2 - i_4 - 4 = 0$

$i_4 = -2A$ Repeat the process for remaining nodes

$7 + i_4 - i_3 = 0$

$i_3 = 7 + (-2) = 5A$

$-i_2 - 3 - 7 = 0$

$i_2 = -10A$

$i_1 + i_2 - 2 = 0$

$i_1 = 2 - (-10) = 12A$

We could check our work by performing nodal analysis at the last node. However, it is not a necessary step as we have already found the desired unknowns.

4.3 Nodal analysis

Nodal analysis — a general circuit analysis technique in which we try to determine the potential difference between nodes by applying KCL and KVL (in my experience, usually focusing a bit more on KCL)

Your textbook offers the following description of the technique which seems pretty good to me:

1. Select a node as a reference. Assign voltages $(v_1, v_2, ..., v_{n-1})$ for the remaining n-1 nodes, all of which will be referenced with respect to the reference node.

2. Apply KCL to each of the n-1 nonreference nodes. Use Ohms law to express the branch currents in terms of node voltages.

3. Solve the resulting simultaneous equations (system of equations) for each unknown node voltage.

4. It's as easy as that! But, well, actually, it can get a little hairy once you start to apply it in earnest.

4.3.1 The procedure

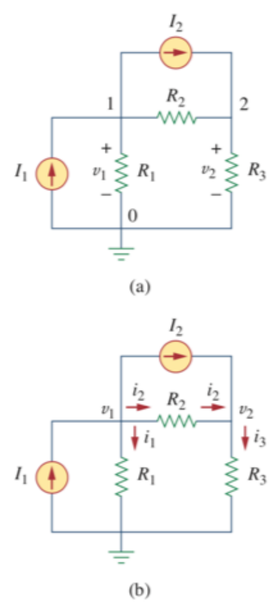


Figure 3.2
Typical circuit for nodal analysis.

1. Begin by putting a reference, usually a "ground"

   This ground can be one of two sorts: (1) **Earth ground** in which ultimately the whole earth is used as the reference point; or **Chassis ground** in which the case of the device in which the circuit is in will act as a reference as it will presumably be sufficiently large as to serve fine [this is also partly the reason why you can "feel" a MacBook charge up its charger does not utilize a traditional "earth ground"

   Either will suffice for our purposes here

2. Next we label all the nodes.

   How many branches, nodes, and loops?

   5 branches, 3 nodes, 3 loops. Satisfies our network condition.

   You can give them any label you want, but I find working your way up from the ground in a clockwise manner and numbering them sequentially is a good habit to get into.

   Keep in mind that we typically set our reference node to have a voltage of 0. We can actually set it to be anything wed like, but the math is often easier if we just make it 0.

3. Then we apply KCL to each nonreference node in the circuit.

   At node 1, $IA - i1 - i2 - IB \rightarrow IA = IB + i1 + i2$

   At node 2, $IB + i2 - i3IB + i2 = i3$

   Once weve got that, now its a matter of applying Ohms law. Thought typically written as V = iR, it is perhaps more helpful to write its full extension here and note that (Va Vb) = iR

   $I = (Va - Vb)/R$

   Thus we can state

   $I1 = (v1 - v0)/R1 \rightarrow I1 = G1(v1 - v0)$

   $I2 = (v1 - v2)/R2 \rightarrow I2 = G2(v1 - v2)$

4.4 Solving simultaneous equations

4.4.1 Cramer's Rule

When given a system of linear equations, Cramer's Rule allows us to solve directly for the specific variable whose value we are looking for.

To use Cramer's Rule, the system of equations must satisfy two conditions:

1. There must be the same number of equations as variables (the coefficient matrix must be a square).

2. The determinant of the coefficient must be non-zero.

Use the following steps to apply Cramer's Rule:

1. Write the coefficient matrix of the system (call this matrix A). Make sure this is a square matrix; otherwise Cramer's Rule is not applicable.

2. Compute the determinant of matrix A. Make sure that this value is non-zero; otherwise Cramer's Rule is not applicable here.

3. Suppose the first variable of the system is x. Write the matrix Ax by placing the column of numbers to the right of the equals sign as the first column of Ax and using the non-x coefficients of matrix A as the remaining columns.

4. The value of x is the determinant of Ax divided by the determinant of A.

5. Repeat steps 3 and 4 to solve for any other variable as needed.

Example implementing Cramer's Rule: We will use the following system of equations to demonstrate how to use Cramer's Rule to solve for the value of x:

x + y + z = 4 -x + 2y = 1 -y + z = 1

1. Write the coefficient matrix A and check that it is a square matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

2. Solve for the determinant of A. (—A— = 4). The determinant does not equal zero.

3. Write the matrix Ax:

$$Ax = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

4. Solve for the determinant of Ax. (—Ax— = 4).

5. Solve for the value of x:

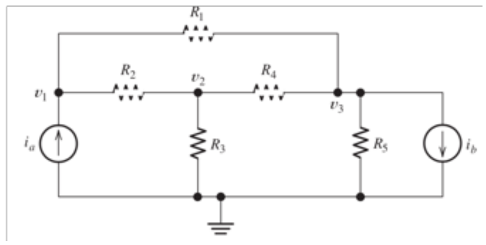
$$\text{x} = \frac{|Ax|}{|A|} \text{ x} = 1$$

Reference: Explanation and example inspired by ”Solving System of Linear Equations: (lesson 4 of 5), Cramers Rule”, MathPortal: https://www.mathportal.org/algebra/solving-system-of-linear-equations/cramers-rule.php. [Accessed 23 January, 2019].

4.5 Worksheet

4.5.1 Problem 1, KCL at a few nodes

Use KCL to write equations at each node.



Solution

4.5.2 Problem 2, matrix notation

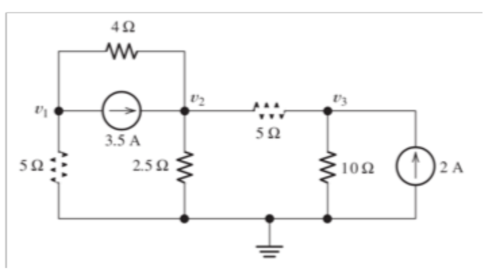
Write the matrix form of the equations written above.
Solution

4.5.3 Problem 3, Cramer's rule

Using Cramers rule on the matrix equations above, what are the results?
Solution

4.5.4 Problem 4, Straight to the matrix

Write the node-voltage equations to the circuit at right in the matrix form.



Solution

Chapter 5

Circuit analysis: II. Mesh analysis

01/24/2019

Contents

5.1 Clarifications on the homework

5.2 Mesh analysis

- A mesh is a loop which does not contain any other loops within it.
- ABEF and BCDE are meshes. ABCDEF is not.
- We apply KVL to find the mesh currents within a given circuit (this can be particularly helpful / convenient when dealing with parallel circuits).

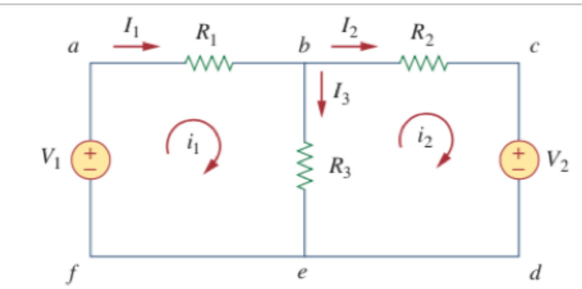


Figure 3.17
A circuit with two meshes.

5.3 Steps of mesh analysis

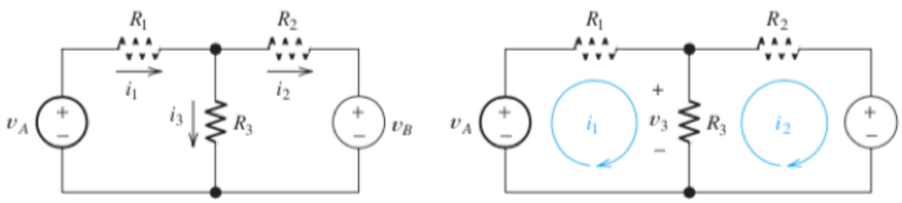
1. **For each of your n meshes assign a mesh current (i_1, i_2, ..., i_n).** You may draw them in any direction you want, but generally a nice clockwise direction will help with consistency.
2. **Apply KVL to each of the n meshes.** Use Ohms law to express the voltages in terms of the mesh currents
3. **Solve the resulting n simultaneous equations to get the mesh currents.**

5.3.1 Important condition for use

Unlike nodal analysis, mesh analysis only works for "planar networks" — that is one which can be written in two-dimensions without any branches.[1]

[1] Nonplanar networks can still be solved using nodal analysis.

5.4 An example



(a) Circuit with branch currents (b) Circuit with mesh currents

For the example seen above, we might at this point in our lives be tempted to solve the circuit with branch currents (the current that actually flows into and out of every branch. That is certainly a valid way by which to look at the problem. But now we're going to try another. We're going to try to understand "mesh currents".

Those branch currents can be formed into a pair of coupled equations

$$V_A - R_1 i_1 - R_3 i_3 = 0 \tag{5.1}$$
$$V_B - R_2 i_2 - R_3 i_3 = 0 \tag{5.2}$$

We can also apply KCL at the node between the three resistors and recognize a third bounding equation: $i_1 + i_2 - i_3 = 0$. We can then solve these three equations through any number of means (substitution, elimination, etc.).

But mesh analysis allows us to model the situation more simply and often more efficiently.

We begin by defining two arbitrary mesh currents, i_1 and i_2. Then we apply KVL as we do a "walk" around the loop defined by the mesh current.

$$V_A - R_1 i_1 - R_3(i_1 - i_2) = 0 \tag{5.3}$$
$$-R_3(-i_1 + i_2) - R_2 i_2 - V_B = 0 \tag{5.4}$$

This given us two equations with two unknowns (the arbitrary mesh currents, i_1 and i_2, we defined).

5.5 Another example

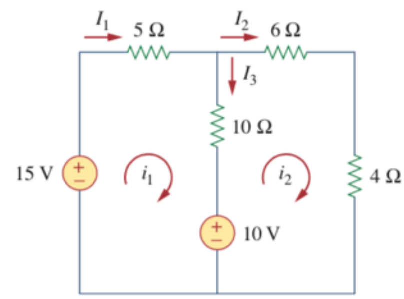


Figure 3.18
For Example 3.5.

For the circuit above, let's try to find the branch currents by performing mesh analysis (i.e., by. first finding the mesh currents).

Solution.

Obtain mesh current using KVL. For mesh 1:

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0 \tag{5.5}$$

which simplifies to

$$3i_1 - 2i_2 = 1 \tag{5.6}$$

For mesh 2:

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0 \tag{5.7}$$

simplifying to

$$i_1 = 2i_2 - 1 \tag{5.8}$$

5.5.1 Solving it one way

Perhaps the technique we'd most likely reach for if we knew nothing else about a situation is substitution. In this case, let's substitute equation 5.8 into equation 5.6:

$$6i_2 - 3 - 2i_2 = 1 \rightarrow i_2 = 1 \text{ A} \tag{5.9}$$

And plugging that result back in

$$i_1 = 2i_2 - 1 = 2 - 1 = 1 \text{ A} \tag{5.10}$$

And thus I_3 being equal to the difference in current between i_1 and i_2 is equal t zero.

5.5.2 Solving it another

Another way of doing this is by recognizing some of the techniques afforded to us by linear algebra. If we squint at the equations with a little bit of good brain wrinkling, we can see that

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.11}$$

is just as valid of way of putting it as any other.

In this form we can use a few techniques to solve for the unknowns. One technique to be aware of given its deployability in those times in which you might need to write out your work is *Cramer's rule*.

Cramer's rule says that

$$i_1 = \frac{\Delta_1}{\Delta} \tag{5.12}$$

$$i_2 = \frac{\Delta_2}{\Delta} \tag{5.13}$$

where

$$\Delta = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = (3)(2) - (-1)(-2) = 4 \tag{5.14}$$

$$\Delta_1 = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = (1)(2) - (1)(-2) = 4 \tag{5.15}$$

$$\Delta_2 = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = (3)(1) - (-1)(1) = 4 \tag{5.16}$$

$$\tag{5.17}$$

Thus

$$i_1 = \frac{4}{4} = 1 \tag{5.18}$$

$$i_2 = \frac{4}{4} = 1 \tag{5.19}$$

which agrees with the results we obtained before.

5.6 Yet another example

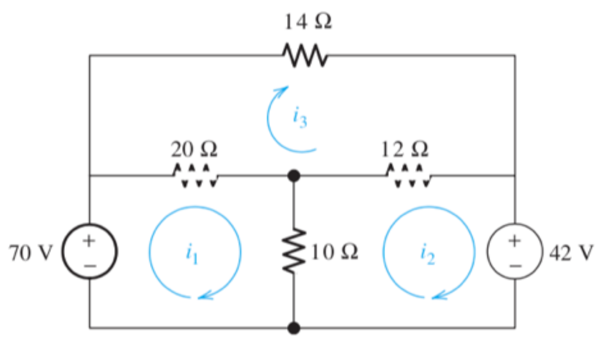


Figure 2.34 Circuit of Example 2.13.

Let's begin by defining the mesh currents (flowing clockwise around each mesh)

Mesh 1

$$20(i_1 - i_3) + 10(i_1 - i_2) - 70 = 0 \tag{5.20}$$

Mesh 2

$$10(i_2 - i1) + 12(i_2 - i_3) + 42 = 0 \tag{5.21}$$

Mesh 3

$$20(i_3 - i_1) + 14i_3 + 12(i_3 - i_2 = 0 \tag{5.22}$$

Putting the equations into standard form:

$$30i_1 - 10i_2 - 20i_3 = 70 \tag{5.23}$$
$$-10i_2 + 22i_2 - 12i_3 = -42 \tag{5.24}$$
$$-20i_1 - 12i_2 + 46i_3 = 0 \tag{5.25}$$

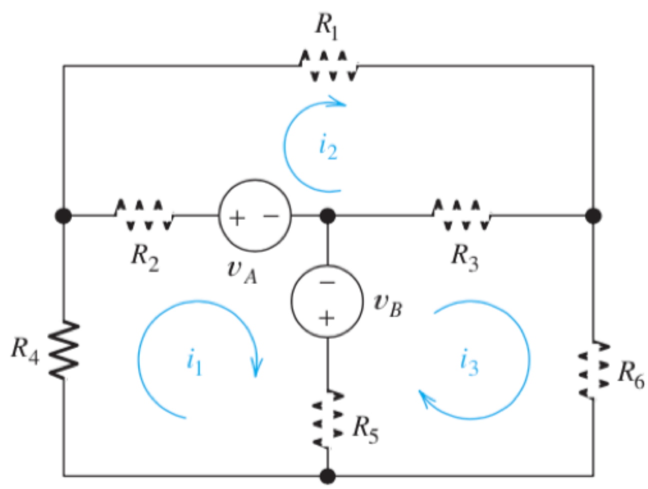
In matrix notation that becomes:

$$\begin{bmatrix} 30 & -10 & -20 \\ -10 & 22 & -12 \\ -20 & -12 & 46 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 70 \\ -42 \\ 0 \end{bmatrix} \tag{5.26}$$

While this can be solved any number of ways, in MATLAB one can simply write `I = R/V`, which should tell you something about what is "going on" mathematically in these operations.

5.7 Writing mesh equations directly in matrix form

Let's take a look at yet one more example.



We might, by now, begin to appreciate that one of our matrices is simply the resistances (impedances) of a mesh bundled up in one matrix, the currents bundled up in another, and the potentials bundled up in another, shown below.

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{5.27}$$

To properly assemble these matrices there's just a few simple rules.

First, let's take a loop around each mesh and determine the equivalent impedance (of that mesh!)

- **Mesh 1.** $R_2 + R_5 + R_6$
- **Mesh 2.** $R_2 + R_1 + R_3$
- **Mesh 3.** $R_3 + R_6 + R_5$

These will comprise values along the diagonal of the resistance (impedance) matrix (r_{11}. r_{22}, and r_{33}).

Next, let's look for elements that are shared among meshes

- **Mesh 1 & 2** share R_2, so r_{21} and r_{12} both become $-R_2$
- **Mesh 2 & 3** share R_3, so r_{23} and r_{32} both become $-R_3$
- **Mesh 3 & 1** share R_5, so r_{13} and r_{31} both become $-R_5$

Thus, our entire resistance matrix becomes

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} (R_2 + R_5 + R_6) & -R_2 & -R_5 \\ -R_2 & (R_2 + R_1 + R_3) & -R_3 \\ -R_5 & -R_3 & (R_3 + R_6 + R_5) \end{bmatrix} \tag{5.28}$$

The matrix it buts up against is easy enough to construct, it is simply the mesh current of each mesh

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \tag{5.29}$$

And the potential matrix is the sum of sources in the direction of the mesh current

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_A + v_B \\ v_A \\ -v_B \end{bmatrix} \tag{5.30}$$

Giving rise to a final form which can be solved efficiently by any computation engine we put before us.

$$\begin{bmatrix} (R_2 + R_5 + R_6) & -R_2 & -R_5 \\ -R_2 & (R_2 + R_1 + R_3) & -R_3 \\ -R_5 & -R_3 & (R_3 + R_6 + R_5) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -v_A + v_B \\ v_A \\ -v_B \end{bmatrix} \tag{5.31}$$

Chapter 6

Circuit analysis: III. Supernodes and supermeshes

01/29/2019 Lecture 6.

Contents

6.1 A review of nodal and mesh analysis

6.1.1 Nodal analysis

1. Select a node as a reference. Assign voltages (v_1, v_2, ..., v_{n-1}) for the remaining n-1 nodes, all of which will be referenced with respect to the reference node.

2. Apply KCL to each of the n-1 nonreference nodes. Use Ohms law to express the branch currents in terms of node voltages.

3. Solve the resulting simultaneous equations (system of equations) for each unknown node voltage.

4. (Its as easy as that! But, well, actually, it can get a little hairy once you start to apply it in earnest.)

6.2 Nodal analysis with an independent current source

Let's begin by analyzing the following circuit.

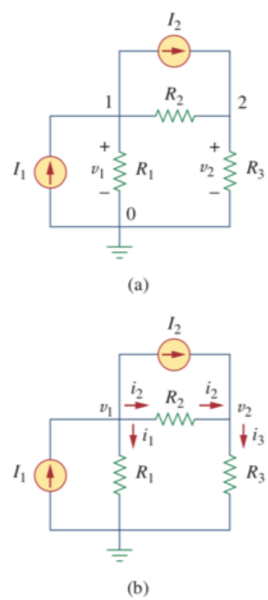


Figure 3.2
Typical circuit for nodal analysis.

1. Begin by putting a reference, usually a "ground".

2. Next we label all the nodes.

   How many branches, nodes, and loops? (5 branches, 3 nodes, 3 loops. Satisfies our network condition.)

   You can give them any label you want, but I find working your way up from the ground in a clockwise manner and numbering them sequentially is a good habit to get into.

   Keep in mind that we typically set our reference node to have a voltage of 0. We can actually set it to be anything wed like, but the math is often easier if we just make it 0.

3. Then we apply KCL to each nonreference node in the circuit.

   At node 1, $IA - i1 - i2 - IB \rightarrow IA = IB + i1 + i2$

At node 2, $IB + i2 - i3 - IB + i2 = i3$

Once weve got that, now its a matter of applying Ohms law. Thought typically written as $V = iR$, it is perhaps more helpful to write its full extension here and note that $(VaVb) = iR$

4. $I = (Va - Vb)/R$, Thus we can state

   - $I1 = (v1 - v0)/R1 \rightarrow I1 = G1(v1 - v0)$
   - $I2 = (v1 - v2)/R2 \rightarrow I2 = G2(v1 - v2)$
   - $I3 = (v2 - v0)/R3 \rightarrow I3 = G3(v2 - v0)$

5. Now we can substitute these relationships into our previous equations

6.3 A brief review of Cramer's rule

system of equations	coefficient matrix's determinant	answer column	D_x: coefficient determinant with answer-column values in x-column
$2x+1y+1z=3$ $1x-1y-1z=0$ $1x+2y+1z=0$	$D=\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$	$D_x=\begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix}$

Similarly, D_y and D_z would then be:

$$D_y=\begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} \qquad D_z=\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

Evaluating each determinant, we get:

$$D=\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}=(-2)+(-1)+(2)-(-1)-(-4)-(1)=3$$

$$D_x=\begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix}=(-3)+(0)+(0)-(0)-(-6)-(0)=-3+6=3$$

$$D_y=\begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix}=(0)+(-3)+(0)-(0)-(0)-(3)=-3-3=-6$$

$$D_z=\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}=(0)+(0)+(6)-(-3)-(0)-(0)=6+3=9$$

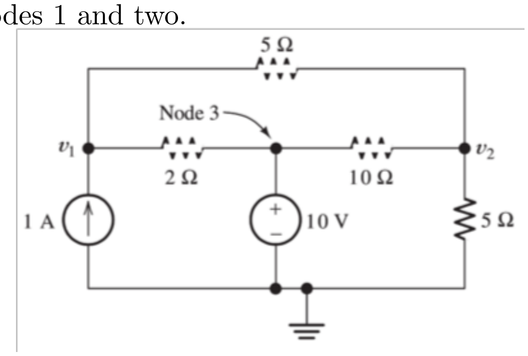
Cramer's Rule says that $x=D_x \div D$, $y=D_y \div D$, and $z=D_z \div D$. That is:

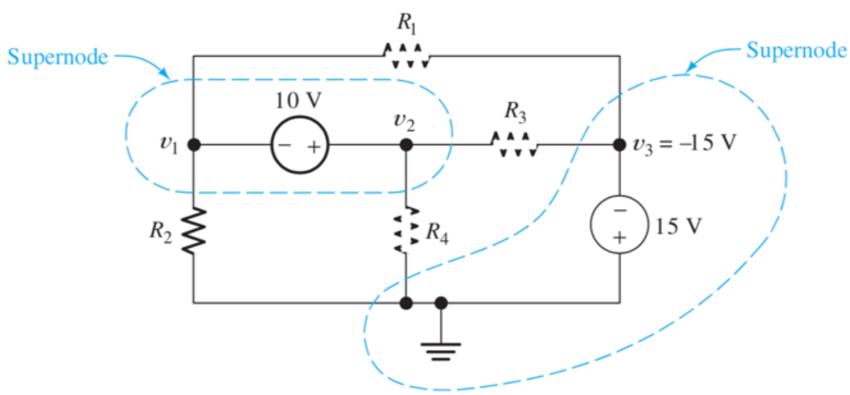
$$x={}^{3}/_{3}=1,\ y={}^{-6}/_{3}=-2,\ \text{and}\ z={}^{9}/_{3}=3$$

6.4 Nodal analysis with voltage sources, Supernodes

We often pick the reference node at one end of the source (and then we have one less unknown node voltage to solve.

Typical examples can be typically solved. Write current equations at nodes 1 and two.





If we try to write a current equation at node 1, we must include a term for current through the 10V source. We could assign an unknown, or...

Or we can create a "supernode" which we do by drawing a dashed line around several nodes, including the elements between them, and apply KCL more broadly

- Recall that KCL says that the net current flowing through any closed surface must be equal to zero. Thus, we apply KCL to the supernode.

$$\frac{v_1}{R_2} + \frac{}{v_1 - (-15)} R_! + \frac{v_2}{R_4} + \frac{v_2 = (-15)}{R_3} = 0 \tag{6.1}$$

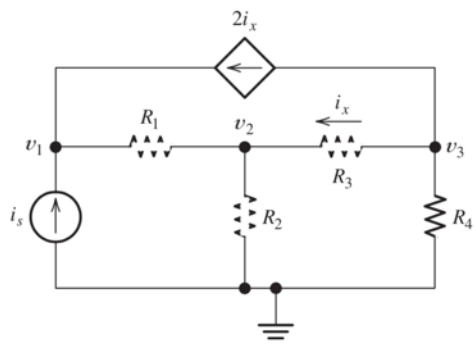
Now you might be tempeted to write another current equation for the other super node, however, youd find that the equation is equivalent to the one we just found. If we tried to solve by substitution, all the terms would drop out and wed see that the matrix was singular (the determinant is 0, it won't tell you anything new.

Instead we apply KVL, noting that $-v_1 - 10 + v_2 = 0$

Next, we find an expression for the controlling variable ix in terms of the node voltages. Notice that since in this case ix is the current flowing away from node 3 through R3, we can say ix = (v3 v2)/R3.

6.5 Nodal analysis with controlled sources

We approach it the same way, but we're mindful of the dependence



Write KCL equations at each node, including the current of the controlled source, just as if it were an ordinary current source

$$\frac{v_1 - v_2}{R_1} = i_s + 2i_x$$

$$\frac{v_2 - v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_2 - v_3}{R_3} = 0$$
$$\frac{v_3 - v_2}{R_3} + \frac{v_3}{R_4} + 2i_x = 0$$

Next, we find an expression for the controlling variable ix in terms of the node voltages. Notice that since in this case ix is the current flowing away from node 3 through R3, we can say $ix = (v3 - v2)/R3$.

$$\frac{v_1 - v_2}{R_1} = i_s + 2\frac{v_3 - v_2}{R_3}$$
$$\frac{v_2 - v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_2 - v_3}{R_3} = 0$$
$$\frac{v_3 - v_2}{R_3} + \frac{v_3}{R_4} + 2\frac{v_3 - v_2}{R_3} = 0$$

So long as only three of those variable are unknown (say the voltages), then it can be solved by whatever method youd like.

6.6 Mesh analysis with current sources

6.7 Mesh analysis with controlled sources, Supermeshes

Chapter 7

Circuit analysis: IV. Circuit theorems

02/05/2019 Lecture 7.

Contents

7.1 Circuit theorems

Circuit theorems are ways of quickly describing, summarizing, analyzing circuit, based on some mathematical tricks, two of which well review today: Thevenin and Norton equivalent circuits.

- **Thevenin** — taking a complex circuit seen between two terminals and representing it as a resistor in series with a voltage source
- **Norton** — taking a complex bit of circuitry between two terminals and representing it as a current source in parallel with a current source

1. These only apply to "linear circuits" that is, circuits made of (ideal) resistors, capacitors, inductors, op-amps, filters
2. Non-linear elements include things like diodes, transistors, and some of the digital logic stuff well get into later in the semester.
3. It might pay us dividends for us to consider what we mean by linearity at this point in the semester as it will become very relevant in our upcoming analyses

7.2 Linearity

Comprises two separate yet equally important properties

1. Homogeneity — If the input (the excitation) is multiplied by a constant, the output (the response) is multiplied by the same constant

   $v = iR \rightarrow kv = kiR$

2. Additivity — The response to a sum of excitations is equal to the responses to each individual excitation

   $v_1 = i_1 R; v_2 = i_2 R; \rightarrow v = (i_1 + i_2)R = i_1 R_1 + i_2 R = v_1 + v_2$

Much of what we will do in this class is linear and much in life, given sufficient approximating, can be considered linear. Such relationships are useful, if not always strictly true (recall that even wires have resistivities and resistors have frequency ranges)

Note that since power is $i^2 R = v^2/R$, the relationship is a quadratic and thus our necessary assumptions of linearity are not applicable.

7.3 Superposition

- Until now, weve been considering everything in a circuit always “on”, but one of the things the linearity of the system allows us to do is to selectively turn on and off sources while we solve, so long as we sum them up in the end.

- This principle — **superposition** — states that the voltage (or current) through an element (in a linear circuit) is the algebraic sum of the voltages across (or current through) that element due to each independent source acting alone.

- To apply the superposition principle

  1. Turn off all independent source but one and find the output (voltage or current) due to that source [using any of the techniques youve previously learned]
  2. Do this for each independent source
  3. Find the total contribution [for any given element, or indeed for all of them] by adding all the contributions due to the independent sources
  4. (Leave in all dependent sources since they are controlled by circuit variables)

- To turn off a source:

  Voltage sources get replaced by a 0V source / a short circuit — **Turn voltage sources into wires**

  Current sources get replaced by a 0A source / an open circuit — **Cut current sources out**

Example 4.5

For the circuit in Fig. 4.12, use the superposition theorem to find i.

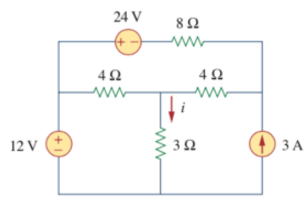


Figure 4.12
For Example 4.5.

Solution:
In this case, we have three sources. Let

$$i = i_1 + i_2 + i_3$$

where i_1, i_2, and i_3 are due to the 12-V, 24-V, and 3-A sources respectively. To get i_1, consider the circuit in Fig. 4.13(a). Combining 4 Ω (on the right-hand side) in series with 8 Ω gives 12 Ω. The 12 Ω in parallel with 4 Ω gives $12 \times 4/16 = 3$ Ω. Thus,

$$i_1 = \frac{12}{6} = 2 \text{ A}$$

To get i_2, consider the circuit in Fig. 4.13(b). Applying mesh analysis gives

$$16i_a - 4i_b + 24 = 0 \quad \Rightarrow \quad 4i_a - i_b = -6 \tag{4.5.1}$$

$$7i_b - 4i_a = 0 \quad \Rightarrow \quad i_a = \frac{7}{4}i_b \tag{4.5.2}$$

Substituting Eq. (4.5.2) into Eq. (4.5.1) gives

$$i_2 = i_b = -1$$

To get i_3, consider the circuit in Fig. 4.13(c). Using nodal analysis gives

$$3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4} \quad \Rightarrow \quad 24 = 3v_2 - 2v_1 \tag{4.5.3}$$

$$\frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3} \quad \Rightarrow \quad v_2 = \frac{10}{3}v_1 \tag{4.5.4}$$

Substituting Eq. (4.5.4) into Eq. (4.5.3) leads to $v_1 = 3$ and

$$i_3 = \frac{v_1}{3} = 1 \text{ A}$$

Thus,

$$i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2 \text{ A}$$

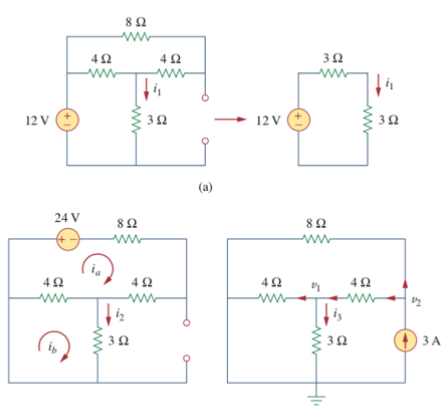


Figure 4.13
For Example 4.5.

7.4 Source transformation

Source transformation is the process of replacing a voltage source v_s in series with a resistor R by a current source i_s in parallel with a resistor R or

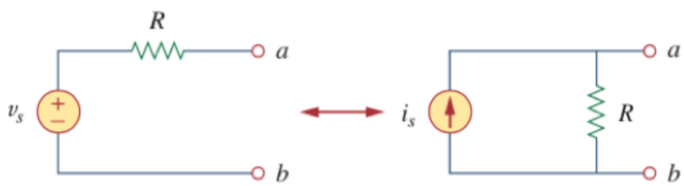


Figure 4.15
Transformation of independent sources.

vice versa.

- We can prove these two are equivalent between a-b by short circuiting the two-terminals.
- If the sources are turned off, the resistance at terminal a-b are the same (R)
- If the terminals are shorted, current flowing from a to be is $i_{ab} = Vs/R$ and $i_{ab} = i_s$. Thus $Vs/R = i_s$
- You can also replace dependent sources this way, provided youre careful, but well skip that analysis here and I wont encourage its use unless you personally feel comfortable with the material

7.5 Thevenin equivalents

- A linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source, V_{th}, in series with a resistor, R_{th}

  V_{th} is equal to open-circuit voltage of original network, $V_{th} = V_{oc}$

- If we were to short the connection across terminals a and b, we can see (as we did previously), that $i_{sc} = V_{th}/R_{th}$

  Thus the short-circuit current is equal to the current for the original circuit and the Thevenin equivalent.

- These two facts allow us to say something rather interesting, namely that the **Thevenin resistance** of a circuit is equal to the ratio of its open-circuit voltage and its short-circuit current. $R_{th} = V_{oc}/I_{sc}$

- Thus, **to determine the Thevenin equivalent circuit**, we start by analyzing the original network for its open-circuit voltage and its short circuit current. [A more robust derivation of this is shown in your book in chapter 4.7]

- **To find the thevenin equivalent resistance**:

  If there are ***no*** *dependent sources*, turn off all independent sources in the network. R_{th} is the input resistance to the network between the two terminals of interest

  If ***there are*** *dependent sources*, still turn off all independent sources and apply superposition.

  Apply a voltage at the terminals and determine the resulting current. Then R_{th} is the ratio of applied voltage and elicited current v_o/i_o

  Or apply a current and determine the resulting voltage. Use which ever you feel comfortable with.

7.5.1 An example

Create thevenin equivalent circuit between a and b [R_L is a potentiometer and thus its value can change. We want to simplify our understanding of how all the complex stuff before it will act.]

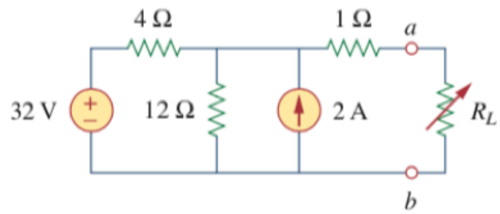


Figure 4.27
For Example 4.8.

1. Start by turning off the independent sources (32 V and 2 A), replacing them with a short- and an open-circuit respectively
2. This will yield our resistance as a 4 —— 12 + 1 = (4*12)/16 + 1 = 4 ohms
3. Next, identify what our V_{Th} would be
4. To find V_{Th} we can apply mesh analysis

$$324i1 - 12i1 + 12i2 = 0 \tag{7.1}$$
$$i2 = 2 \tag{7.2}$$
$$i1 = 0.5A \tag{7.3}$$
$$\tag{7.4}$$

then plug it in for the voltage across the 12 ohm

$$V_{th} = 1_2(i_1 - i_1) = 30 \tag{7.5}$$

5. Or could use nodal analysis

$$(32V_{th})/4 + 2 = V_{Th}/12 \rightarrow V_{Th} = 30 \tag{7.6}$$

7.6 Norton equivalents

- A linear two-terminal circuit can be replaced by an equivalent circuit comprising a current source, I_N in parallel with a resistor R_N

  I_N is equal to the short-circuit current through the terminals

- To find the Norton equivalent resistance, we start the same way we did for R_Th — that is, short our voltage sources, open our current sources and find the resistance.

  Well then this should suggest to us that R_N is equal to R_Th, and indeed they are equal.

- The Norton current, I_N, is equal to the short-circuit current.

  So we merely find the current that would travel through the short

7.6.1 An example

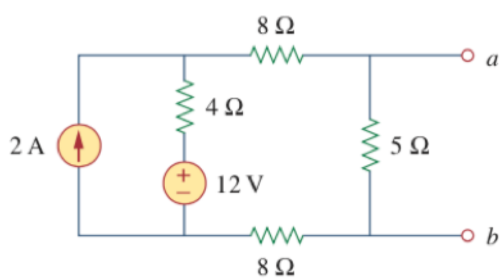


Figure 4.39
For Example 4.11.

1. We begin by zeroing all of our independent sources

   $R_N = 5||(8+4+8[+0]) = 5||20 = (5*20)/(5+20) = 4$ ohms

2. We can apply mesh analysis to find I_N

   We can ignore the 5-ohm resistor as its in parallel with a short and current will always seek the path of least resistance.

   Recall our bioimpedance example where with increasing frequency caused the current to flow down a different branch, because with greater frequency the capacitor went from acting like an open circuit (at low frequencies) to a short circuit (at high frequencies), thus bypassing the other branch altogether.

$$I_1 = 2 \tag{7.7}$$
$$I_2 = 124i_2 + 4i_1 8i_2 8i_2 \rightarrow 4(2)20(i_2) = -12 \tag{7.8}$$
$$I_2 = 1 = i_{sc} = I_N \tag{7.9}$$

3. Alternatively, we could have found the Thevinin voltage [meshing it up]

$$i3 = 2 \tag{7.10}$$
$$12 - 4(i4) + 4(i1)8(i4) - 5(i4)8(i4) \rightarrow 4(2)25(i4) = -12 \tag{7.11}$$
$$I4 = 0.8A \tag{7.12}$$

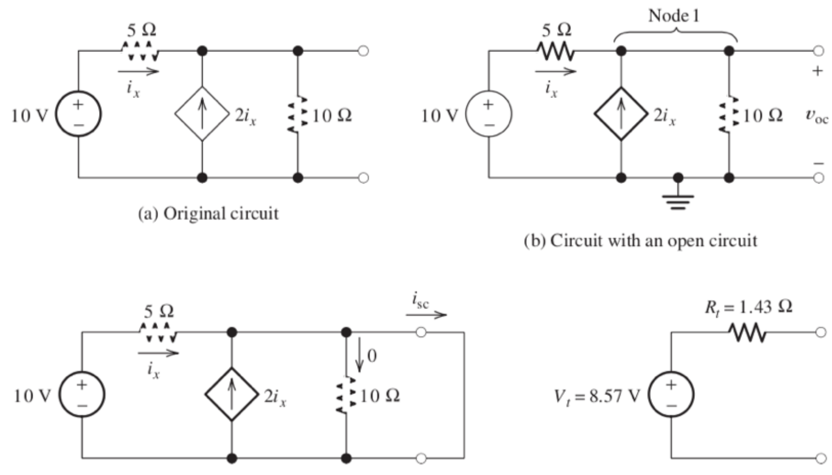
   Since we know 0.8 A goes through the 5 ohm resistor, and since we know that that 5 ohm resistor experiences the same voltage drop as the open circuit, we simple multiple 5*0.8 to give us, 4 V

4. Well since we know $R_N = R_T h$ is 4 and $V_T h$ is 4: $I_N = V_T h / R_T h = 1A$

5. Thus Norton and Thevenin circuits are two sides of the same source transformation coin

7.7 Equivalents with dependents

When weve got a dependent source, we cant just remove the sources and combine impedances. Instead, weve got to analyze the circuit to find the open-circuit voltage and the short-circuit current (and indeed such an approach will work in all cases, its just a little bit more work on our parts)



(a) Original circuit

(b) Circuit with an open circuit

(c) Circuit with a short circuit

(d) Thévenin equivalent

- Im personally a fan of finding the Thevenin open voltage because it just clicks with my brain a little better, but if you like the Norton short-circuit current, feel to take that approach
- So I apply nodal analysis

$$I_x + 2i_x V_{oc}/10 = 0 \rightarrow 3i_x = V_{oc}/10 \quad (7.13)$$

$$\text{Next we write an expression for our controlling variable, ix} \quad (7.14)$$

$$10V_{oc} = i_x(5) \rightarrow ix = (10V_{oc})/5 \quad (7.15)$$

$$\text{We can substitute this into our previous equation} \quad (7.16)$$

$$3 * (10 - V_{oc})/5 = Voc/10 \rightarrow \mathbf{V_{oc}} = \mathbf{V_{Th}} = \mathbf{8.57} \quad (7.17)$$

- Now we can consider the short-circuit conditions.

$$I_x + 2i_x i_{sc} = 0 \rightarrow 3i_x = i_{sc} \quad (7.18)$$

$$I_x = 10/5 = 2 \rightarrow \mathbf{I_{sc}} = \mathbf{3i_x} = \mathbf{I_N} = \mathbf{6} \quad (7.19)$$

- From this, we can take the ratio of the open circuit voltage and the short circuit current and find the equivalent resistance of the network via Ohm's law: 8.57 V / 6 A = 1.43 ohms

7.8 A step-by-step procedure

1. Perform two of these:

   Determine the open-circuit voltage $V_t = v_{oc}$

   Determine the short-circuit current $I_n = i_{sc}$

   Zero the independent sources and find the Thevenin resistance R_t looking back into the terminals. Do not zero dependent sources

2. Use the equations $V_t = R_t In$ to compute the remaining value

3. The Thevenin equivalent consists of a voltage source V_t in series with R_t

4. The Norton equivalent consists of a current source I_n in parallel with R_t

Chapter 8

A review of the material so far

02/07/2019

8.1 A set of exercises in preparation of the first Glorifed Quiz

1. Find the equivalent resistance of the following ["+" means "in series with", "||" means "in parallel with"]:

    a. R1 + R2||(R3 + R4)

    b. (R1 + R2)||(R3 + R4)

    c. R1 + (R2 + R3||(R4 + R5)——R6)

    d. (R1||R2||R3)||(R4||R5||R6)——(R7 + R8||R9)

2. To use superposition, one must turn off sources. A voltage source is replaced by what? A current source is replaced by what?

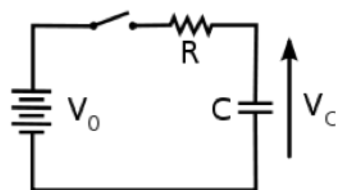
3. A 5 F capacitor has accumulated a total charge of 1250 C. What is the voltage across the capacitor?

4. Name each of the four types of dependent power sources and explain their functioning a bit beyond their name.

5. The unit of current is the ampere, A. What are some equivalent units (for instance, what is its relationship to charge)?

6. Does potential change instantaneously over the terminals of a capacitor?

7. An RC circuit (in which "R + C", seen below) has a time constant, $\tau = RC$. Prove the following at a time t_0 when the switch is thrown to complete the circuit.



   a. $I(t) = \frac{V_0}{R} \cdot e^{-t/\tau}$
   b. $V(t) = V_0 \left(1 - e^{-t/\tau}\right)$
   c. $Q(t) = C \cdot V_0 \left(1 - e^{-t/\tau}\right)$

8. What is the difference between active and passive circuit elements?

9. What can an ideal current source do what a real one cannot?

10. What can ideal voltage source do what a real one cannot?

11. Draw the symbol for a polarized capacitor.

12. Draw the symbol for a potentiometer.

13. Draw an op-amp, label its terminals, and explain what each does.

14. Whats another name for a conductor?

15. Which way does current run in a diode?

16. Describe what Ohms law tells us in your own words.

17. What is conductance? State Ohms law using conductance. What does it tell us about eh current passing through conductive materials?

18. What is the equation for resistance in terms of resistivity? If resistivity goes up, what happens to resistance? If the cross-sectional areas decreases, what results? If a resistor keeps the exact same cross-sectional area over a length L, yet we contorted it into a very weird shape (say a balloon-animal poodle), will its resistance remain the same according to the aforementioned equation. Does that agree with your intuition?

19. A capacitor has a time variance of current according to the follow formulation $I(t) = \frac{dQ(t)}{dt} = C\frac{dV(t)}{dt}$. That might be useful to know. See if you can prove that relationship to yourself knowing that C = Q/V.

20. What is an inductor and how does it work?

21. What is impedance? What is impeded? What does the impeding?

22. What is the real and imaginary component of impedance.

23. How would I represent impedance by a magnitude and phase?

24. What is impedances inverse? What are its real and imaginary components?

25. What is the impedance of a resistor, capacitor, and inductor?

26. Find the equivalent impedance of the following, being sure to remove imaginary numbers from the denominator ["+" means "in series with", "||" means "in parallel with"]:
    a. Z1||(Z2 + Z3)
    b. (R1 + C1)||C2||(R2 + R3 + C3)
    c. (R1 + L1)||(C1 + L1)||(C1 + R1)

27. Convert a Wye circuit, with all R = 10 ohms, to a Delta circuit.

28. Find the equivalent impedance of a Delta circuit of capacitors.

29. Given the following map, can you plan a parade rate that crosses each bridge exactly once? Why or why not?

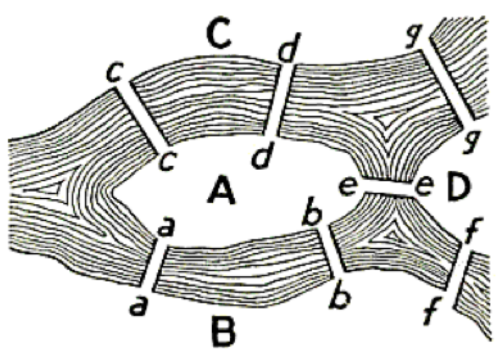


FIGURE 98. *Geographic Map: The Königsberg Bridges.*

30. 32. Prove the fundamental theorem of network topology given any circuit (for instance, those aforementioned).

31. Kirchhoffs current law is a mere recitation of what other law of physics?

32. Kirchhoffs voltage law is a mere recitation of what other law of physics?

33. When do we most use KCL? When do we must use KVL? Which do you prefer in general and why?

34. Draw a complicated circuit (at least four branches) and solve for the current going through each component.

35. Analyze the aforementioned complicated circuit using nodal analysis. Analyze said circuit with mesh analysis. Prove the validity of the fundamental theorem of network topology using said circuit.

36. What is the difference between earth ground and chassis ground?

37. Draw a circuit using each of the following, R1 = 10 ohms, R2 = 20 ohms, R3 = 30 ohms, R4 = 40 ohms, V1 = 20 V, V2 = 10 V, and I1 = 2 A. Solve using nodal analysis if you can (do you need to make a supernode?). Solve using mesh analysis if you can (do you need to make a supermesh?).

38. Can you imagine modeling some actions in terms of circuits? For instance, if you were asked to model an equivalent circuit based on the operations of a shop floor, could you do it? If you were asked to model some aspect of physiology could you? The resistance of blood to flow? The capacitance of the lungs? The duration of our blinks?

39. What is Cramers rule and how do we use it?

40. Draw a circuit with at least five resistors and two current sources. Put that circuit into matrix formation for nodal voltages from inspection alone. What values should be along the diagonal, what values should be on the off-diagonal?

41. To solve the above circuit in MATLAB, what would you type in?

42. For any circuit stated previously, add a voltage source to one of the branches and solve using nodal analysis. What new technique did you need to employ and why did you need to? What does that technique do to our matrix?

43. To any previously mentioned circuit, add a dependent current source and let it be dependent on the current going through R1. How does this change your nodal voltages?

44. Given a jolt of 240 J from an AED, provide an energy equivalent. For example, how many little soft kitten jumps on the chest (1 kg, 5cm) is it equivalent to? It would be like falling from a height of how much? [Hint: the ground comes up at you.]

45. A hospital consumes 500 kW in 60 seconds. How many people running on treadmills would it take to power it for a minute?

46. A nurse starts their shift turning on medical doohickey that consumes 50 W at 220 V. How many electrons worth of charge pass through the doohickey before the nurse turns it off at the end of their shift, 10 hours later?

47. Explain how mesh analysis works.

48. Draw a mesh circuit with at least 6 unique elements. Analyze it.

49. Draw another mesh circuit with at least four loops. Analyze it.

50. For the two above-mentioned mesh circuits, put them directly into matrix form. What values go along the diagonal? What values go on the off-diagonals?

51. For one of the mesh circuits you drew in either, add a dependent current source and solve using mesh analysis.

52. How do we measure current and potential using actual ammeters and voltmeters? Why must we break the circuit to measure current? How do we measure potential across an element?

53. Say, in as much detail as you can muster, what you believe the Thevenin circuit theorem is about.

54. Say, in as much detail as you can muster, what you believe the Norton circuit theorem is about.

55. Changing a potential source to a current source is an example of what technique to analyze circuits?

56. What are two attributes necessary for things to be "linear"?

57. What is superposition and how do we use it to analyze a circuit?

58. How do you turn off a voltage source in a circuit diagram? How do you turn off a current source?

59. To one of the above dozen or so circuits you have drawn, find the Thevenin equivalent.

60. To one of the above dozen or so circuits you have drawn, find the Norton equivalent.

61. To the Thevenin in 62, convert it to its Norton equivalent. To the Norton in 63, convert it to its Thevenin equivalent.

62. What is an open circuit? What is a short circuit? How are they relevant to Thevenin and Norton equivalents?

63. Draw an LM741 and label all 8 pins.

64. What is an operational amplifier? What does it behave like? What can it do for us? How are they different from resistors, capacitors, and their ilk?

65. What are the three key features we need to know to analyze op-amp circuits?

66. Draw an inverting amplifier with R1 = 20 ohms, and R2 = 100 ohms. What is the gain on the input signal?

67. Draw an inverting amplifier with Z1 = R1 = (Ra||Rb) and Z2 = R2 = (Rc + Rb). Solve for the output voltage in terms of the input voltage.

68. Draw a non-inverting amplifier with R1 = 100 kohms and R2 = 100 kohms.

69. Draw the two circuits you drew above in series with one another. What is the output?

70. What is a voltage follower and why is it important?

71. If its so important you should be able to draw one and analyze its operation! Lets do that. Draw a voltage follower and demonstrate its behavior in as much detail as you would need to convince another engineer you know what youre talking about.

72. Why might we want to use a voltage follower with an electrocardiogram?

73. What is a summing amplifier?

74. A summing amplifier can actually act as a logic converter is set up right. Can you imagine using a set of summing amplifiers to determine a winner in a simple tic-tac-toe game in which button presses turned on new voltage sources? Briefly describe how you could determine a winner.

75. If I have a summing amplifier in which V1 = 10 V, R1 = 10 ohms, V2 = 10 V, R2 = 10 ohms, and Rf = 20 ohms, what is my output voltage?

76. What is a differential amplifier and why is it important?

77. Draw a differential amplifier. Analyze said differential amplifier. What is the output voltage in terms of the input voltage?

78. What is a transfer function?

79. You know how to go from a triangle (Delta) to a y (Wye) circuit for equivalence. How would you find the equivalent of going from a square to a cross?

80. Draw a blood vessel. Find the electrical equivalent across the vessel and through the vessel.

81. Draw the meanest circuit you think Id ever give you on an exam. Then add a voltage dependent voltage source to one branch of it (its dependence rests on you). Solve it using nodal analysis, mesh analysis, superposition, and just plain looking at it.

A Glorified Quiz I

Given on February 12, 2019. All problems worth 10 points. Do not spend an inordinate amount of time on any given problem. Try your best.

Contents

8.2 Problem 1 on basic impedance

Find the equivalent impedance of the following ("+" means "in series with" and "||" means "in parallel with") being sure that you end up with no imaginary terms in denominators. It may be helpful to draw these circuits as you understand them.

1.
2.
3.

8.3 Problem 2 on basic knowledge

1.
2.
3.
4.
5.

8.4 Problem 3 on current and potential

A current, $i(t)$, enters a device, causing a potential drop, $v(t)$. The current and the potential can be described mathematically as

$$i(t) = \begin{cases} 0 \text{ A}, & t < 0 \\ f(t) \text{ mA}, & t \geq 0 \end{cases}$$

$$v(t) = \begin{cases} 0 \text{ V}, & t < 0 \\ g(t) \text{ V}, & t \geq 0 \end{cases}$$

1.
2.
3.

8.5 Problem 4 on transresistance amplification or bioamplification

There are at least two kinds of current-to-voltage converters (also known as *transresistance amplifiers).*

8.6 Problem 5 on equivalent impedance

Choose one of the following (or do both for bonus points).

1. Determine the equivalent impedance seen between terminals a and b.

2. Determine the equivalent impedance of a three-dimensional network of resistors, each of value R, as measured from the two farthest corners.

8.7 Problem 6 on differential amplification

1. What is the potential difference between points a and b?

2. What is the gain seen between points a and b?

3. What is the current running through R2?

4. What is the current running through R4?

5. If the operational amplifier (LM741) were only supplied 10 V, what is the largest V1 we could input before our output was saturated?

8.8 Problem 7 on counting cells

A Coulter counter (an electronic means by which to count the number of cells passing by a channel) is essentially a bridge circuit that determines the presence of a cell by tracking a change in due to a change in resistance from the cell. Referring to the circuit below, $R_1 = R_2 = R_3 = R_4 = R$ at rest (no cells present) and every time a cell passes through the channel R2 increases its resistance by ΔR.

1.

2.

3.

8.9 Problem 8 on your own

1. Set up the series of simultaneous equations to **find the mesh currents**.
2. Determine **the matrix form** of the simultaneous equations.
3. Determine **the power dissipated** by three of your resistors.
4. Find **the equivalent resistance** of the system as measured from the two most distant corners of your circuit drawing.

8.10 Problem 9 also on your own

1. Determine whether your circuit **satisfies the fundamental theorem of network topology**.
2. **Determine the nodal voltages** (*at the very least set them up as a solvable set of simultaneous equations* and try to solve them).
3. **Determine the current** going through three of your resistors.

8.11 Problem 10 on miscellanea

1.
2.
3.

8.12 Useful equations in alphabetical order

$$
\begin{aligned}
b &= l + n - 1 \\
C &= \frac{Q}{V} \rightarrow i(t)\frac{dQ(t)}{dt} = C\frac{dV(t)}{dt} \\
\text{Cramer's Rule: } x &= \frac{D_1}{D}, y = \frac{D_1}{D} \text{ for } a_1x + b_1y = c_1, a_2x + b_2y = c_2 \\
\text{where D} &= \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, D_1 = \begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}, D_2 = \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \\
\mathbf{E} &= \rho\mathbf{J} \\
E &= pt \\
e &= -1.602 \cdot 10^{-19} \text{ C} \\
i &= \frac{dq}{dt} \\
i_N &= i_{sc} = V_{Th}/R_{Th} \\
\text{KCL: } \sum_{x=1}^{n} i_x &= 0 \\
\text{KVL: } \sum_{x=1}^{n} v_x &= 0 \\
L &= \frac{\Phi}{I} \rightarrow v(t) = \frac{d\Phi}{dt} \\
p &= \frac{dw}{dt} \\
Q &= \int_{t_1}^{t_2} i(t)dt \\
R &= \rho\frac{l}{A} \\
v &= \frac{dw}{dq} \\
v &= iR \\
V_{Th} = V_{oc} & \\
\mathbf{Z} &= R + \jmath X = |\mathbf{Z}|e^{\jmath\theta} \\
\text{Parallel: } \frac{1}{Z_{eq}} &= \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + ... \\
\text{Series: } Z_{eq} &= Z_1 + Z_2 + Z_3 + ...
\end{aligned}
$$

Part II

Systems

Chapter 9

The Laplace Transform: I. What it is and why it is important

02/14/2019 Lecture 9.

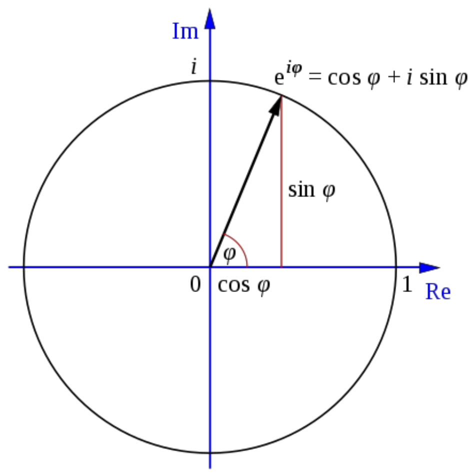
Contents

9.1 Euler's formula, Euler's identity

You must all know, from this moment on, that Euler's formula is thus

$$e^{\jmath x} = \cos x + \jmath \sin x \tag{9.1}$$

where e is Euler's number, $\jmath$ is the imaginary unit, and cos and sin are trigonometric functions cosine and sine respectively. This is a very unique relationship in the history of mathematics because it combines so much of our tools in mathematics. It has an exponential, it has a variable, it's got all of trigonometry, and it's even got this orthogonal reality of the "imaginary" domain. The formula describes how a vector on the complex manifests as a function of the variable x alone. The implications of this will be appreciated with a time and effort on our parts.



There is a special case of this formula known as Euler's identity and regardless of the number of times I prove it to myself, I just can't believe it all works out like this. If $x = \pi$ then

$$e^{\jmath\pi} = -1 \tag{9.2}$$

which has just about every important mathematical thing in it: exponentials, imaginary numbers, constants of the universe, negative numbers, the single unit. In fact, add the one back to the other side and you'll see a zero appear from the depths. All this to say, recognizing the interrelatedness of exponential functions to just about everything else in mathematics and engineering will help us here and elsewhere immensely.

9.2 The Laplace transform

Mathematically, the Laplace transform is simple to describe. First, get yourself a curly L, looks like this

$$\mathcal{L} \tag{9.3}$$

Next to that $\mathcal{L}$, put some curly brackets, like so

$$\mathcal{L}\{\} \tag{9.4}$$

Within those brackets put some function, say one of time, $f(t)$

$$\mathcal{L}\{f(t)\} = \text{The Laplace transform of } f(t) \tag{9.5}$$

We may also, sometimes, take a function's name, say $f(t)$, capitalize it, $F(t)$, and call that the Laplace transform. I am not personally the biggest fan of this notation as it can become quite distracting and confounding when combined with other transforms you might want to one day use, such as Fourier transforms, Hilbert transforms, and so on.

I prefer to designate my Laplace transforms with a squiggly sign over a capitalized version of the function name, like so

$$\mathcal{L}\{f(t)\} = \tilde{F}(s) \tag{9.6}$$

That squiggly line, and many of those that follow, help explain what the Laplace transform is kind of doing to the reality around you. It's seeing relationships outside of time, cast in a plane of infinite exponential sine waves, with bottomless zeros and infinities you can draw a small circle around. The Laplace transform, should be, I believe, a transformative way of viewing the world around you. The power of the Laplace transform (and its ilk) is to reveal useful realities to us and our machines. Turns out there's more to life than just the time ahead of us. And we can usefully describe it.

Some of those astute among you will notice I tried to slip something by you in the above definition. (Those of you glancing over the mathematics, get yourself back in the habit of *reading* mathematics, out loud if need be. Make sure you understand what I'm saying before agreeing to it.) What I tried to sneak into this definition is a new variable s that somehow our function, $f(t)$, became in terms of when transformed. We will get into the more mystical nature of the s-plane and its variable s, but for now, let's just call it some dummy variable. To you, at this point, it is just some stranger you've met in the street. You don't know them at all. First thing you've got to do, is say, Hello.

Before any of that, though, we must get to the definition of the definition of the Laplace transform. It is what we do to a function $f(t)$ to yield another equation $\tilde{F}(s)$. In that sense, the Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{9.7}$$

which is to say it is the integral over all of time of the product of our function $f(t)$ with an exponential function raised to a power of time, t, scaled negatively with some dummy variable, s. Essentially, if I were to multiple my function, $f(t)$ at every point in time with a probing function e^{-st} and summed it all up, what would I get? And more importantly to all of this, what would it tell me?

Another way of thinking about what the Laplace transform is "doing" is to think about what the mathematics itself is proposing to do. It's taking some function, $f(t)$, and wrapping it along a real and an imaginary exponential, producing both sines and cosines.

9.3 The Laplace transform of 1

Let me start off with the simplest number I can think of, sophist that I am. What is the Laplace transform of the number 1. If I can't solve this integral for the number 1, what the heck can I do? So let's convince ourselves that we can solve at least that much.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(1)\} = \int_0^\infty e^{-st}(1) dt \tag{9.8}$$

Some, at this point, might be concerned with that ∞ up there in this integral. It creates what is sometimes called an improper integral, that is, one which has either (or both) limits of infinity or an integrand that approaches infinity at one or more points in the range of integration[1]. To get around this improperness, we create for ourselves yet another dummy variable, let's call this one A. Thus we can also write the Laplace transform as

$$\mathcal{L}\{(1)\} = \int_0^\infty e^{-st}(1) dt = \lim_{A \to \infty} \int_0^A e^{-st}(1) dt \tag{9.9}$$

[1]This latter fact forms much of the basis of our understanding of control theory.

which is the integral from 0 to A – which we can do – evaluated, ultimately, as A goes infinitely. We take the antiderivative, bounded by our limits, to be

$$\lim_{A\to\infty}\int_0^A e^{-st}(1)dt = \lim_{A\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^A \tag{9.10}$$

$$= \lim_{A\to\infty}\left[\left(-\frac{1}{s}e^{-sA}\right)-\left(-\frac{1}{s}\right)\right] \tag{9.11}$$

$$= \lim_{A\to\infty}\left[-\frac{1}{s}e^{-sA}+\frac{1}{s}\right] \tag{9.12}$$

Let us think about what our limit does. In this case the only factor with an A involved is $-\frac{1}{s}e^{-sA}$. Here, if A becomes huge, it knocks against that s, getting even bigger, and knocks against that negative sign to become very very negative. Having a very very negative negative number in our exponential function causes it to go screaming down to 0. In this case, the case where the function $f(t) = 1$, upper infinite limit has obliterated the term associated with it, whereas the lower limit has preserved the term associated with it. This leaves us with the final result:

$$\mathcal{L}\{f(t)\} = \frac{1}{s} \tag{9.13}$$

9.4 The s-plane

Where some see time, others see cycles, decay. The s-plane allows us to describe the physical world around us and much of the activity that occurs therein as a bunch of sines and exponentials, which is a quite clever way of squinting at a world that's always got us looking across time. Instead of charting a function or describing some behavior (perhaps of some system) as an amplitude changing over time, let us consider what such a function might look like if on one axis we had a variable describing how fast it decays over time and on the other axis we described how hard the system rattles at each frequency. In plotting such a function, one could see how a system will react under a myriad of conditions which you literally do not have the time to do if you confine yourself to just this slice of reality.

And so we must turn orthogonally to the only world we've ever known and stare into the s-domain.

We can start by taking a peak at our variable, s. For our first use of the Laplace transform, we didn't even need to know what s was to work with it and to get an answer (of some sort). Indeed I took to calling it a dummy variable since the mechanical evaluation of the mathematics does not require it. It is important to make sure you can always do the math if you are ever called upon to do so. For much of what follows clear and clever shorthand will help us to view signals in a multitude of ways, including our bundling up of terms within s. As with much in life and in this class, s comprises a real component, σ and an imaginary component, ω:

$$s = \sigma + \jmath\omega \tag{9.14}$$

By themselves these two parameters are underwhelming. When raised as the exponent to an exponential and they start to describe a whole other world. Whether it has been stressed to you in your differential equations class or not, every single differential equation in the world has as at least one of its solutions the combination of exponential functions with and sine (or cosine) waves. The reasons for this range from the philosophic (we are trapped between repetition and degradation) to the mechanical (the derivatives of exponentials and sines and just scaled exponentials and sines). Let us take a look at the form s takes itself for the Laplace transform:

$$e^{-st} = e^{-(\sigma+\jmath\omega)t} \tag{9.15}$$

If were were to expand it out, we could perhaps convince ourselves that two separate yet equally important components emerge. One describes a function which dampens a response with time and one which oscillates back and forth with time.

$$e^{-(\sigma+\jmath\omega)t} = \underbrace{e^{-\sigma t}}_{\text{damping}} + \underbrace{e^{-\jmath\omega t}}_{\text{sinusoids}} \tag{9.16}$$

The first of these behaviors we can probably prove to ourselves by inspection. If $s > 0$, and let us say that it is, then as t gets really large, the exponential function will die down to zero, under the influence of that negative infinity. Therefore, we are right to call $e^{-\sigma t}$ a damping term, since it will eventually smooth everything away to nothingness.

The second of these behaviors requires a brief detour back to our Eulerian days[2].

A benefit to this whole situation is that it reduces the mathematical complexity of the reality of the situation (represented by the mathematical operation of convolution, which we will get to in a couple weeks) to algebra.

[2]One should get use to thinking in both Eulerian and Laplacian terms.

9.5 The linearity of the Laplace transform

The Laplace transform has at least one property that we must address here, its linearity. As the transform satisfies the two conditions of linearity previously established, homogeneity (a response is proportional to an excitation) and additivity (a response is the sum of excitations), we can use it with near reckless abandon for linear systems, i.e., those we will concern ourselves with here.

The linearity of the transform means that the transform of a sum is that same as the sum of individual transforms. Moreover, scaling constants are not affected by the transform and remain constant throughout. Mathematically this may all be stated as:

$$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \tilde{F}_1(s) + a_2 \tilde{F}_2(s) \tag{9.17}$$

This result is convenient for more involved analyses and is one we must keep in mind.

9.6 The Laplace transform of e^{at}

To apply the Laplace transform, we take our $f(t)$, multiple it by our probing function, e^{-st}, and integrate over all time (for the moment, let us stipulate $s > a$):

$$\begin{aligned}
\tilde{F}(s) = \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} e^{at} dt && (9.18)\\
&= \int_0^\infty e^{(a-s)t} dt && (9.19)\\
&= \int_0^\infty e^{-(s-a)t} dt && (9.20)\\
&= \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^\infty && (9.21)\\
&= \left(-\frac{e^{-(s-a)\infty}}{s-a} \right) - \left(-\frac{e^0}{s-a} \right) && (9.22)\\
&= 0 + \frac{1}{s-a} && (9.23)\\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a} && (9.24)
\end{aligned}$$

From this we can perhaps see a pattern or two. Our Laplace transform took the exponent of the exponential and dropped it down in the denominator with s as a subtractive component. Thus, one might just as well commit to memory that that Laplace transform of e^{4t} is $\frac{1}{s-4}$ as look it up in a table. From this simple derivation we now know that $e^{-0.25t}$ transforms to $\frac{1}{4s+1}$ and could even convince ourselves that $3e^{-2t}$ becomes $\frac{3}{s+2}$. In fact, hopefully we can recognize that our previous derivation of the Laplace transform of the number 1 is just the case where $a = 0$ when $f(t) = e^{at}$.

You might wonder, what do I care about having a new fraction? What good does that do me? Already it has told you something you likely didn't know: as the values of a and s approach equality, the fraction blows up to infinity. We have what we call a "pole" at $s = a$.

9.7 The Laplace transform of dx/dt

Generally one does not spend a lot of time working through the mathematical conjuring of taking the Laplace of a derivative. The reason for this is because its final form is easy to learn, memorize, and deploy readily.

Let us begin with a simple, perhaps much too simple, example.

$$\frac{dx}{dt} - ax = 0 \tag{9.25}$$

Leveraging what we've learned of the linearity of our technique, let's find the Laplace transforms each portion of that equation.

$$\text{Laplace transform 1} = \int_0^\infty e^{-st}\frac{dx}{dt}dt \tag{9.26}$$

$$\text{Laplace transform 2} = \int_0^\infty e^{-st}x dt \tag{9.27}$$

The latter equation, "Laplace transform 2" is just another way of saying $\mathcal{L}\{x(t)\}$, so we can feel comfortable simply labeling that as $\tilde{X}(s)$.

The former equation requires us to integrate by parts. I will be assumed here that some calculus teacher somewhere once taught you the importance of integration by parts. It will be further assumed here that you know how to do it or could figure out how to do it if called upon to do so. As such, I will not belabor its implementation here and instead present my formulation of the logic of the situation for you to follow:

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \int_0^{\infty} e^{-st}\frac{dx}{dt}dt \tag{9.28}$$
$$= -\int_0^{\infty} x(t)\left(-se^{-st}dt\right) + \left[xe^{-st}\right]_0^{\infty} \tag{9.29}$$
$$= s\int_0^{\infty} x(t)\left(e^{-st}dt\right) + \left[\left(xe^{-\infty}\right) - \left(xe^{0}\right)\right] \tag{9.30}$$
$$= s\tilde{X}(s) - x(0) \tag{9.31}$$

What this end result suggests to us, is that the Laplace of the derivative of some function $x(t)$ is the Laplace of the function itself, $\tilde{X}(s)$, multiplied by s to which an initial condition, $x(0)$ is subtracted.[3]

Combining these two results and rewriting:

$$\mathcal{L}\left\{\frac{dx}{dt} - ax = 0\right\} \rightarrow \mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\left\{ax\right\} = \mathcal{L}\left\{0\right\} \tag{9.37}$$
$$s\tilde{X}(s) - x(0) - a\tilde{X}(s) = 0 \tag{9.38}$$
$$\tilde{X}(s)\left(s-a\right) = x(0) \tag{9.39}$$
$$\tilde{X}(s) = \frac{x(0)}{s-a} \tag{9.40}$$

Without too much squinting, we can probably see that such a form is quite close to what we found for our exponential function's transformation, $\frac{1}{s-a}$. In fact, as will become clearer and more comfortable to you over time, is that indeed we can begin to quickly map a solution to a differential equation by first transforming it over the s-domain. In this particular case, we can

[3] In fact, we can expand this to n-th order derivatives. It will be of the form:

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = s\tilde{X} - x(0) \tag{9.32}$$
$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} = s^2\tilde{X} - sx(0) - \frac{dx(0)}{dt} \tag{9.33}$$
$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} = s^2\tilde{X} - sx(0) - x'(0) \tag{9.34}$$
$$\mathcal{L}\left\{\frac{d^3x}{dt^3}\right\} = s^3\tilde{X} - s^2x(0) - sx'(0) - x''(0) \tag{9.35}$$
$$\mathcal{L}\left\{\frac{d^4x}{dt^4}\right\} = s^4\tilde{X} - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0) \tag{9.36}$$

hazard a guess that a solution to the differential equation $\frac{dx}{dt} - ax = 0$ is the function $x(t) = x(0)e^{at}$, which indeed from inspection appears to work.

It you didn't catch it there, let me point it out here: you have just learned a very neat mathematical trick for an engineer. This might be the quickest way to solve a differential equation that you will ever know of.

We've taken a differential equation and reduced it to simple algebra.

9.8 The Laplace transform of sin

There are many ways of going about proving this to yourself. This is the one I prefer. It forces us to begin, as we should, with Euler's formula.

$$\mathcal{L}\{\sin at\} = \mathcal{L}\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\} \tag{9.41}$$

$$= \frac{1}{2i}\left(\mathcal{L}\{e^{iat}\} - \mathcal{L}\{e^{-iat}\}\right) \tag{9.42}$$

$$= \frac{1}{2i}\left(\frac{1}{s - ia} - \frac{1}{s + ia}\right) \tag{9.43}$$

$$= \frac{1}{2i}\left(\frac{s + ia - s + ia}{s^2 + a^2}\right) \tag{9.44}$$

$$= \frac{1}{2i}\left(\frac{2ia}{s^2 + a^2}\right) \tag{9.45}$$

$$= \frac{a}{s^2 + a^2} \tag{9.46}$$

9.9 The Laplace transform of $\int_0^t x(u)du$

We have shown that taking the derivative of a function in the time domain is analogous to multiplying its Laplace transform by s in the frequency domain. Using integration by parts, we can similarly show that taking the integral of a function is analogous to dividing its Laplace transform by s. (Source: https://www.youtube.com/watch?v=IYOzTt-gB8A)

9.10 The Laplace transform in RLC circuits

9.10.1 Resistors

To find the voltage drop across a resistor we apply Ohm's law, $v = iR$. To go from the time-domain to the Laplace-domain for is notationally simple:

$$v(t) \to \tilde{V}(s) \tag{9.47}$$
$$i(t) \to \tilde{I}(s) \tag{9.48}$$
$$v(t) = i(t)R \to \tilde{V}(s) = \tilde{I}(s)R \tag{9.49}$$

That is, the voltage drop in the s-domain is equal to the current in the s-domain, scaled to the resistance, R.

9.10.2 Inductors

$$v(t) \to \tilde{V}(s) \tag{9.50}$$
$$i(t) \to \tilde{I}(s) \tag{9.51}$$
$$v(t) = L\frac{di(t)}{dt} \tag{9.52}$$

Recall from a previous section that the Laplace of a derivative, $\mathcal{L}\{\frac{dx}{dt}\}$ is equal to the product of the Laplace of the original function and s (with the initial conditions subtracted): $s\tilde{X}(s) - x(0)$. We can apply this same logic to solve the above problem, yielding the inductor's behavior:

$$v(t) = L\frac{di(t)}{dt} \tag{9.53}$$
$$\tilde{V}(s) = L(s\tilde{I}(s) - i(0)) \tag{9.54}$$
$$\tilde{V}(s) = sL\tilde{I}(s) - Li(0) \tag{9.55}$$

We can even redraw the circuit to see it as we would from the s-domain. If I've got a circuit, that starts with a horizontal wire with a current $\tilde{I}(s)$ going through it, I've currently got a potential of $\tilde{V}(s)$ at the rightmost end of that wire. Connecting that wire vertically, we can walk across our equation and see what it's telling us. Our first element is going to affect the current by $sL\tilde{I}(s)$. That's our inductor. We continue our KVL-inspired walk and see that we have a negative drop (a rise) in potential equal to $Li(0)$, the current at our initial condition scaled to our inductor's size. This is a potential source. From this, we have a new vantage point to survey circuits from. Where before we thought of voltage sources as blackboxes, there's a curious response that gets induced in them.

If we were to solve for $\tilde{I}(s)$ by rearranging the equation

$$\tilde{I}(s) = \frac{\tilde{V}(s)}{sL} + \frac{i(0)}{s} \tag{9.56}$$

and we can redraw this as a parallel circuit. Or essentially, we are losing $\tilde{I}(s)$ through a node with two branches (see the two components on the right hand side). So for our rightmost branch we have a current source, scaled to $1/s$, and then we have our inductor, with an inductance of sL.

This technique is extensible to other electronic components. And is in fact another way of transforming our circuits, should the need or desire arise, more specifically source transformation with voltage sources in series with an inductor and current source in parallel with an inductor.

9.10.3 Capacitors

Imagine you've got current going into a capacitor and it's forming a potential on either side of it. we can describe this as

$$i = C\frac{dv}{dt} \tag{9.57}$$

We can take the Laplace of both sides

$$\mathcal{L}\{i\} = \mathcal{L}\{C\frac{dv}{dt}\} \tag{9.58}$$

$$\tilde{I}(s) = C(s\tilde{V}(s) - v(0)) \tag{9.59}$$

$$\tilde{I}(s) = Cs\tilde{V}(s) - Cv(0) \tag{9.60}$$

which is to say we have a current, $\tilde{I}(s)$, being divvied up at a node between two branches. The rightmost branch is a source, $v(0)$, scaled by a capacitance, C. (And because it's negative, it turns out this branch is actually entering the node just like $\tilde{I}(s)$). The other branch is a capacitor, $1/C$, multiplied by s.

Just as before with the inductor we can also imagine transforming this source. The simplest way I can think to prove that to ourselves is to rearrange our equation

$$\tilde{V}(s) = \frac{1}{sC}\tilde{I}(s) + \frac{v(0)}{s} \tag{9.61}$$

From this we can see that our potential drop across a series connected components, $\tilde{V}(s)$ is first dropped by a capacitor $1/sC$, and then dropped by a source, $v(0)$ scaled to s.

$$v(t) = \frac{1}{C}\int_{t_1}^{t_2} \tag{9.62}$$

9.10.4 RLC circuits

Combining these results and recognizing the linearity of each, in a series RLC circuit, we can merely add these all up to yield:

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{\tau=t} i(\tau)d\tau = v(t) \tag{9.63}$$

An example

What is the Laplace transform of a series RLC circuit with $L = 1$, $R = 7$, and $C = 10$.

Chapter 10

The Laplace Transform: II. How to use it

02/19/2019

Contents

10.1 A brief review

Returning for a moment to our series RLC circuit from last time, once converted with the Laplace transform, the system is described via a second-order polynomial.

$$v_L = L\frac{di_L(t)}{dt} \tag{10.1}$$
$$v_R = i_R(t)R \tag{10.2}$$
$$v_C = \frac{1}{C}\int_0^t i(\tau)d\tau + V_c(0) \tag{10.3}$$

When all elements are placed in series, a simple KVL walk will suggest to us that

$$L\frac{di_L(t)}{dt} + i_R(t)R + \frac{1}{C}\int_0^t i(\tau)d\tau + V_c(0) = 0 \tag{10.4}$$

Transforming via Laplace[1]:

$$L(s\tilde{I} - I_0) + R\tilde{I} + \frac{\tilde{I}}{sC} + \frac{V_0}{s} = 0 \tag{10.5}$$
$$s^2\tilde{I} - sI_0 + s\frac{R}{L}\tilde{I} + \frac{\tilde{I}}{C} + V_0 = 0 \tag{10.6}$$

Rearranging and rewriting

$$\tilde{I} = \frac{sI_0 - \frac{V_0}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{10.7}$$

If we give to our RLC circuit the values of $L = 1$, $R = 7$, and $C = 10$ and initial conditions V_0 =10 and $I_0 = 1$ we can see, without too much effort that the equation becomes

$$\tilde{I} = \frac{s - \frac{10}{1}}{s^2 + \frac{7}{1}s + \frac{1}{1\cdot 10}} \tag{10.8}$$
$$\tilde{I} = \frac{s - 10}{(s+2)(s+5)} \tag{10.9}$$

In this form, we now see laid bare before us two important places in the s-domain: poles and zeros.

[1]It may do you some good to remember that in the Laplace domain, differentiation is achieved by multiplying by s and integration is accomplished by dividing by s.

10.2 Two important places, zeros and poles

It may help us to recall that a point in the s-domain characterizes a system's response as a combination of decay, growth, and oscillations. Thus, since s is a coefficient of a exponential function, any value thereof will tell us whether the function

- will growth with respect to time (s > 0);
- will decay with respect to time (s < 0);
- will oscillate over time (s has an imaginary component); or
- will remain constant (s = 0).

A system can be characterized by its combination of these behaviors by finding the roots of s for the system as described in the s-domain. These can either be a "zero" or a "pole".

In our systems, as the value of s approaches

- a **zero**, the numerator of the transfer function approaches 0 and thus so too does the whole system and
- when approaching a **pole**, the denominator of the transfer function approaches 0, causing the system to approach the infinite.

Recalling our RLC circuit:

$$\tilde{I} = \frac{s - 10}{(s + 2)(s + 5)} \tag{10.10}$$

For this system, we have three special points.

1. $s \to 10$ causes the numerator to approach 0 (**and is thus a zero**) and so too will the system;
2. $s \to -2$ causes the denominator to approach 0 (**and is thus a pole**), causing the system response to be quite large; and
3. $s \to -5$ also causes the denominator to approach 0 (**and is thus a pole**), causing the system response to be quite large.

10.3 The general form of just about all systems

As it turns out, most systems on earth and else can be expressed by a generalized second-order equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = f(t) \tag{10.11}$$

where ζ is the damping coefficient and ω_n is the natural frequency of the system, and $f(t)$ is some "forcing" function.

In our case, $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$[2] and $\omega_n = \sqrt{\frac{1}{LC}}$

- **The natural frequency** is the frequency at which a system tends to oscillate in the absence of any driving or damping forces. It's just what wants to happen.

- **The damping coefficient**, or the damping ratio, is a dimensionless measure of decay. It describes how oscillations in a system die down after a disturbance.

For the moment, we will consider systems for which $s < 0$. These systems are guaranteed to be "stable" in the sense that they will not grow exponentially large. Rather, given the fullness of time, we are assured that each of these types of systems will eventually decay away to a stable point.

10.3.1 Undamped systems

When $\zeta = 0$, a system comprises a single harmonic oscillator, as is the case of a mass suspended by a perfect spring. This is what we would expect with no "real" components ($e^{\jmath\omega_n t}$)

- Two imaginary poles at $\pm\jmath\omega_n$

- Undamped sinusoid with frequency equal to the imaginary part of the pole.

- Solution: $f(t) = A\cos(\omega_n t - \phi)$

[2]It may be more intuitive to think of this terms instead as $\zeta = \alpha/\omega_n$, where $\alpha = R/2L$ represents an attenuation coefficient (of units nepers/second).

10.3.2 Underdamped systems

When $0 < \zeta < 1$, a system will initially oscillate in response to a disturbance before dying down to zero. Can be described by the function $e^{\jmath\omega_n\sqrt{1-\zeta^2}t}$

- Two complex poles at $-\sigma \pm \jmath\omega_d$
- Damped sinuosoid with exponential envelope. Time constant is equal to the reciprocal of the pole's real part.
- Solution: $f(t) = Ae^{-\sigma t}\cos(\omega t - \phi)$

10.3.3 Critically damped systems

When $\zeta = 1$, a system decays away as fast as possible with no oscillations or "overshoots". As you might expect, this is a desirable outcome for many many cases of engineering design (e.g., the rate at which a door opens or closes)

- Two real poles at $-\sigma$.
- Response: the sum of two scaled exponentials of the same time constant, one scaled to time.
- Solution: $f(t) = k_1 e^{-\sigma t} + k_2 t e^{-\sigma t}$

10.3.4 Overdamped systems

When $\zeta > 1$, a system in does not oscillate and decays at a slower rate than the critical case.

- Two real poles at $-\sigma_1$ and $-\sigma_2$.
- Response: sum of two exponentials with different time constant
- Solution: $f(t) = k_1 e^{-\sigma_1 t} + k_2 e^{-\sigma_2 t}$

10.4 "Inverting" the Laplace transform

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{f(t)\}(s) = \tilde{F}(s) \tag{10.12}$$

Mellin's inverse formula An integral formula for the inverse Laplace transform, called the Mellin's inverse formula, the Bromwich integral, or the FourierMellin integral, is given by the line integral:

$$f(t) = \mathcal{L}^{-1}\{\tilde{F}(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st}\tilde{F}(s)\, ds, \tag{10.13}$$

where the integration is done along the vertical line Re(s) $= \gamma$ in the complex plane such that γ is greater than the real part of all singularities of F(s) and F(s) is bounded on the line, for example if contour path is in the region of convergence. If all singularities are in the left half-plane, or F(s) is an entire function , then γ can be set to zero and the above inverse integral formula becomes identical to the inverse Fourier transform.

In practice, computing the complex integral can be done by using the Cauchy residue theorem. Or simply by recognizing common forms.

10.4.1 A convenient way of doing it

To find solutions to the equations left behind after we have transformed them via Laplace, I personally prefer to use the "cover-up method".

Returning to our function describing current throughout the circuit

$$\tilde{I} = \frac{s-10}{(s+2)(s+5)} \tag{10.14}$$

10.5 Worksheet

10.5.1 Problem 1

Find the Laplace transform of the following:

- $f(t) = 9$
- $f(t) = \delta(t)$ the Dirac-delta function
- $f(t) = e^{-3t/2}$
- $f(t) = \sin \omega t$

Solution

10.5.2 Problem 2

What is the differential equation describing an inductor, a resistor, and a capacitor in series?

Solution

10.5.3 Problem 3

What is the differential equation describing an inductor, a resistor, and a capacitor in series? What is a solution to that differential equation if L = 1 H, R = 1 Ω, and C = 100 mF? Take the Laplace transform of the differential equation. What are its poles and zeros?

Solution

10.5.4 Problem 4

Draw an s-plane. Label the axes. Plot the poles and zeros you found. What can you say of the behavior of the system? If the poles had an imaginary component to them (that is, if they were pushed along the vertical axis), how would that affect our system?

10.5.5 Problem 5

Draw an s-plane. Label the axes. Plot these poles – (-2,0) and (-5,0) – and this zero – (0,0). Plot what the signal would look over time at each of these poles and zeros.

Chapter 11

Circuits as ODEs: I. First-order

02/21/2019 Lecture 12.

11.1 Source-free RC circuits

11.1.1 One resistor, one capacitor

This circuit is comprised of one resistor and one capacitor in series with no source. To find the function for the voltage $V(t)$ of this source-free circuit, we start by doing KCL. Using the node in between the branch containing the resistor and the branch containing the the capacitor, we get that the current coming into the node from the resistor is $(Vg - Va)/R$, in which Va is just the voltage at the node and Vg is the ground voltage, which would be zero. The equation for the current going out of the node through the capacitor is $C(dV/dt)$, which can be found from the equation $Q = CV$. From these, we get the equation $(0 - Va)/R - C(dVa/dt) = 0$, which simplifies to $dVa/dt + Va/RC = 0$. We now take the LaPlace transform which gives us: $s\tilde{V} - V(0) + (1/RC)Vtilde = 0$ Simplifying gives us: $\tilde{V} = V(0)/(s + 1/RC)$. The pole is s $= -1/RC$. Taking the inverse LaPlace to get it back into the t-domain gives us: $V(t) = V(0)e^(-t/RC)$ As seen in this equation and with the negative pole, the solution to the circuit decays, making it stable. This is what we'd expect for a circuit, for it must be stable otherwise we would not see it in real life.

11.1.2 Two or more resistors and/or capacitors

11.2 Source-free "active" circuits

11.3 First-order systems with sources

11.4 Several singular functions

11.4.1 Unit step function, $u(t - t_0) = 1, t > t_0$

The Laplace transform of the unit step function

11.4.2 Unit impulse function, $\delta(t) = du(t)/dt$

Its "sifting" abilities

The Laplace transform of the unit impulse function

11.4.3 Unit ramp function, $r(t) = \int u(t)dt$

The Laplace transform of the unit impulse function

Chapter 12

Circuits as ODEs: II. Second-order

02/26/2019 Lecture 13.

12.1 A series RLC circuit

Chapter 13

System response: I. Convolution

02/28/2019

Contents

13.1 An introduction to thinking in systems

Completely characterizing the input v. output properties of a system by exhaustive measurement of all parameters involved is usually not possible. As

13.2 Pulse and impulse

We have previously noted that the unit impulse function

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{13.1}$$

But another useful way of conceiving of the impulse signal is as a limiting case of the "pulse signal", $\delta_\Delta(t)$:

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & \text{if } 0 < t < \Delta \\ 0, & \text{otherwise} \end{cases} \tag{13.2}$$

It can be seen without much effort that the impulse signal is equal to the pulse signal when the pulse gets infinitely short.

It may also be helpful to consider integration at this point as a limiting case of summation:

$$\int_{t=-\infty}^{\infty} x(t)dt = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta \tag{13.3}$$

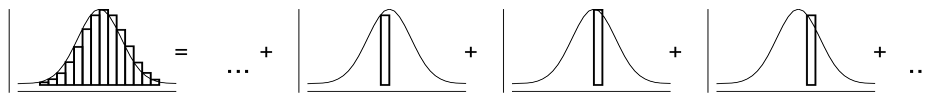
This even allows us to perhaps more easily reckon that the unit step signal can be obtained by integrating the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{13.4}$$

Well, as it turns out, any signal can be expressed as a sum of scaled and shift unit impulses. Indeed, thanks to our digital world, this is indeed how it mostly works out there in the real world, wherein discrete samples from an original signal may be approximated more or less exactly as pulse signals scaled to the amplitude of the sample. Mathematically,

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta \tag{13.5}$$

Graphically,



As we let $\Delta \to 0$, our approximation of $x(t)$ hones in on reality with the summation approaching an integral and the pulse approach the unit impulse:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{13.6}$$

Thus, we can represent any signal as an infinite sum of shifted and scaled unit impulse. This is indeed that best way we human beings have thus far figured out how to store information. A train of 1s and 0s.

13.3 Convolution

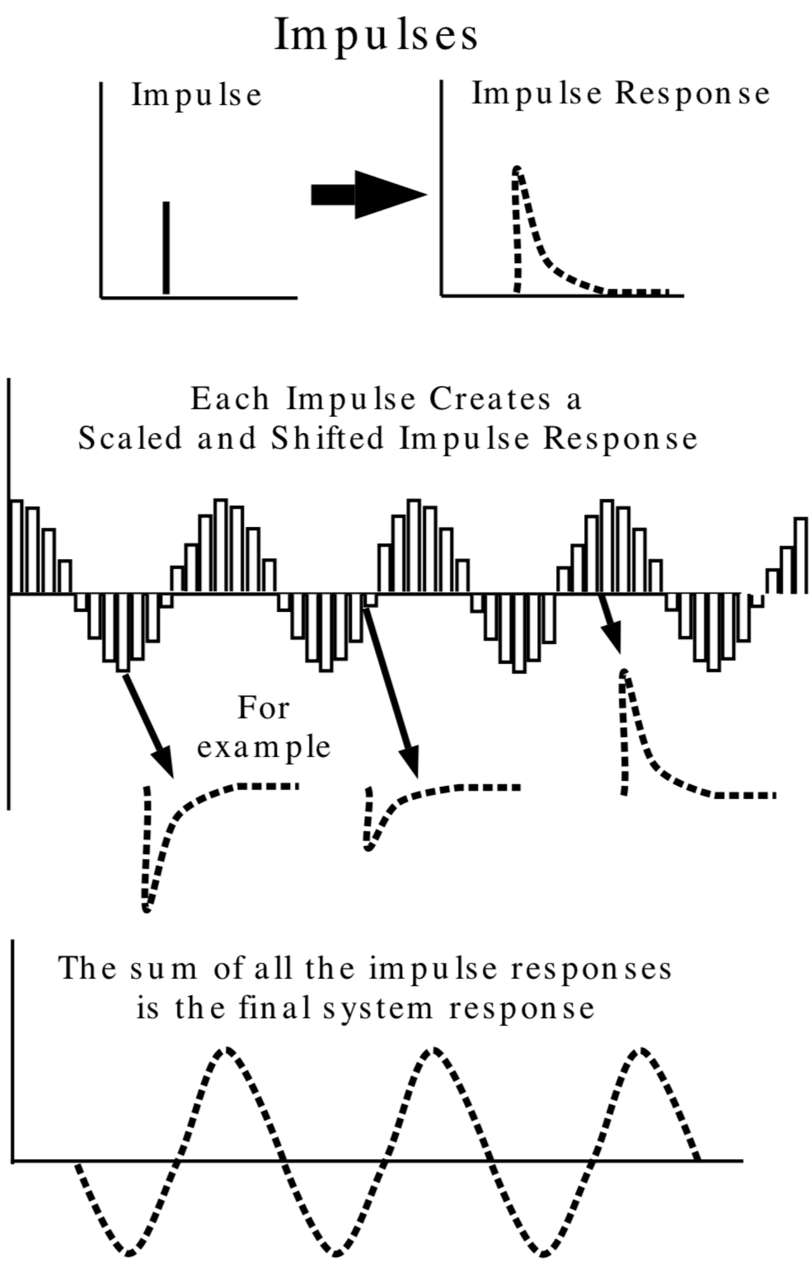
Recall, the two principles of linearity inherent in our studies here: **homogeneity** which enables the scaling of linear systems and **additivity** which allows us to sum up two or more linear systems. From these we arrive at the emergent phenomenon of **superposition** which states that $T(ax_1 + bx_2) = aT(x_1) + bT(x_2)$.

There is also a facet of these systems known as **shift-invariance** in which we say that so long as we don't (cause long-term) damage (to) a system by inputting some signal, then whether we input that signal at a time t or a time $t + \tau$ some while later will not significantly alter the response of the system. Such systems are said to be shift-invariant, and we will consider their behaviors here.[1]

To characterize a linear, shift-invariant, one need only measure how a system responds to a unit impulse. This response is known as the **impulse response function** and in principle it allows us to predict how a system will respond to any other possible stimulus.

Graphically, this may be easy to understand. We can imagine that an impulse given to some system cause it to rapidly rise, then decay (exponentially) away. If we do this for ever point in time for a system of interest, we could fully characterize the total response of the system.

[1] A shift-invariant system is the discrete equivalent of the time-invariant systems we have heretofore concerned ourselves with.



Mathematically, we can call upon our two principles of linearity to guide our way. Let us define some output, $y(t)$, of a system, characterized by transfer function[2], T, and an input signal, $x(t)$

$$y(t) = T[x(t)] = T\left(\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right) \tag{13.7}$$

$$= T\left(\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta\right) \tag{13.8}$$

From additivity,

$$y(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} T\left(x(k\Delta)\delta_\Delta(t-k\Delta)\Delta\right) \tag{13.9}$$

Taking the limit,

$$y(t) = \int_{-\infty}^{\infty} T(x(\tau)\delta(t-\tau)d\tau) \tag{13.10}$$

Homogeneity suggests:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)T(\delta(t-\tau))d\tau \tag{13.11}$$

If we then define for ourselves the response of T to the unshifted unit response $h(t) = T[\delta(t)]$, then via shift-invariance we are left with

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{13.12}$$

This last result is worth sitting down and thinking about. Its consequence is that for any shift-invariant system, T, once we known its impulse response, we know how it will respond to every other system, since every other signal is merely a combination of scaled and shifted impulses.

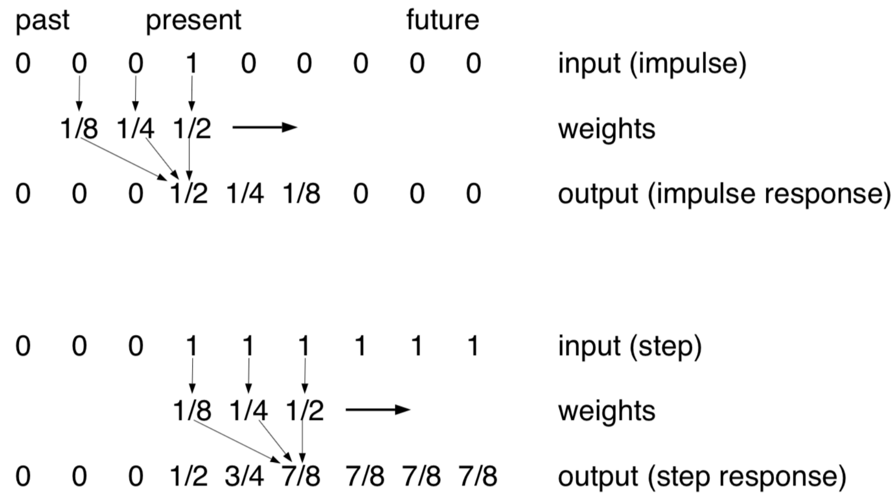
This way of combining signals is so widespread and useful that it has its own mathematical shorthand, "$*$", known as convolution. For any two signals, x and y, an output, $z(t)$, may be found by convolving x with y

$$z(t) = x * y = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \tag{13.13}$$

One way of thinking about this operation is it is as if you took two signals, left one where it was, put there at one side of eternity and found the piecewise multiplication and sum of two signals as you moved the other signal to the other side of eternity.

[2]It has helped some students to think of a "transfer" function merely as a "transform" function such that given an input, $x(t)$, there exists an output, $y(t)$, which is mapped from $x(t)$ via the transform, T.

13.3.1 Convolution as a series of weighted sums



The output of convolution may be thought of as a series of weight sums. More specifically, a linear, shift-invariant system will output a response that is a weighted sum of the inputs. That *weighting function* which helps maps the inputs to the output is very closely related to the impulse response of the system. Indeed, the impulse response and weighting function are time-reversed copies if each other (as shown in the top part of the graph).

13.3.2 A few properties of convolution

- Commutative: $x * y = y * x$
- Associative: $(x * y) * z = x * (y * z)$
- Distributive: $(x * z) + (y * z) = (x + y) * z$

13.3.3 The Laplace transform of a convolution

To editorialize a bit, perhaps the single most important thing to get out of this chapter is that convolution in the time-domain is the same as multiplication in the s-domain. That is

$$\mathcal{L}(f * g) = \tilde{F}(s) \cdot \tilde{G}(s) = \tilde{G}(s) \cdot \tilde{F}(s) = \mathcal{L}(g * f) \tag{13.14}$$

Let us prove this to ourselves.[3]

13.4 A few examples

13.4.1 Convolution of an exponential and a sin function

Convolution of two functions.

Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau$.

Integrate by parts twice: $\int_0^t e^{-\tau} \sin(t - \tau)\, d\tau =$

$$\Big[e^{-\tau} \cos(t - \tau)\Big]\Big|_0^t - \Big[e^{-\tau} \sin(t - \tau)\Big]\Big|_0^t - \int_0^t e^{-\tau} \sin(t - \tau)\, d\tau,$$

$$2\int_0^t e^{-\tau} \sin(t - \tau)\, d\tau = \Big[e^{-\tau} \cos(t - \tau)\Big]\Big|_0^t - \Big[e^{-\tau} \sin(t - \tau)\Big]\Big|_0^t,$$

$$2(f * g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).$$

We conclude: $(f * g)(t) = \frac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big]$. $\triangleleft$

[3] 0^- and t^+ will be avoided in this derivation for convenience, but the interested reader is encouraged to redo this proof with their inclusion. It should come out the same

Chapter 14

System response: II. Stability

03/12/2019

Contents

14.1 Bound inputs, bound outputs

Systems may be thought of as stable or unstable. That is, they either approach some known, finite value, or they approach some infinite value.

One of the more convenient ways to think about stability, is to concern ourselves with systems whose outputs are based on its inputs. With such an approach, we may say of a stable system that the output should be *bounded* for each *bounded* input for each and every time point. Such systems are **bounded-input, bounded-output stable**, or exhibit **BIBO stability**.

What "bounded" means here is that we are confining the values being input (and thus output) to some *range* of values. Another way of thinking about it is that from time of $-\infty$ to $+\infty$, the amplitude of the input is never infinite. For a BIBO stable system, the amplitude of the output will never be infinite during all that time. For a BIBO unstable system, that is not true for at least one point in time.

Those really looking for formality here may use the following definition of boundedness: a signal is bounded if there exists a finite value $B > 0$ such that the magnitude of the signal never exceeds, $\|y(t)| \leq B, \forall t \in \mathbb{R}$.

14.1.1 A few standard bounded signals

- **DC signals**, e.g., $y(t) = 2$ — consider that at all points of t, the amplitude will equal a constant
- **Sines and cosines**, e.g., $y(t) = \sin(t)$ — for all points t, the amplitude is bounded In other words, to -1 and +1
- **The unit step function**[1], e.g., $y(t) = u(t)$ — for all points t, the amplitude is either 0 or 1.

[1]But what about the unit impulse function? Doesn't this violate our "never infinite" rule? Only kind of! First, it will help to realize that the impulse function, $\delta(t)$, is an *idealization* of a signal that is

1. very large near $t = 0$,
2. very small away from $t = 0$, and
3. has an integral of 1.

That is, it is a signal whose width is ϵ and whose amplitude is $1/\epsilon$, where ϵ is very small. In fact, the impulse function is formally defined by the property that $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$. Hence, $\delta(t)$ is not *really* defined for any t, only its behavior within an integral. For a simple demonstration of this conundrum, consider Problem 14.7.3.5.

14.1.2 A few example bounded signals

In the examples which follow, we can an imagine an input signal, $x(t)$, going into a system ("being convolved with"), such that it produces an output, $y(t)$. Somewhat graphically, $x(t) \to \text{system} \to y(t)$.

1. $y(t) = t \cdot x(t)$ — Let's choose a simple input, $x(t) = u(t)$, such that $y(t) = t \cdot u(t) = r(t)$. *Note here that the product of a signal with time is equal to its integral.*[2]

   However! This is not a stable system. Simply because an input is bounded does not mean a system will be stable. Such boundedness merely gives us a specific means by which to evaluate stability in systems. In this case a bounded input produces an unbounded output.

2. $y(t) = x(t) + 5$ — Here we see the effects of a possible DC offset (the "+5" term) on a system. Let's input another simple bounded signal, $x(t) = -3.3$. Then, $-3.3 \to \text{system} \to y(t) = -3.3 + 5 = 1.7$.

   Here we have a bounded input which produces a bounded output. Hence, we have a stable system!

3. $y(t) = x(t) \cdot \sin(t)$

14.2 Stability at poles and zeroes

Recall our series RLC circuit. It produced a transfer function with two general points of interest, poles and zeroes. Pondering still further what the Laplace transform tells us, we are ultimately taking the integral sum of an exponential function whose exponential is determined by either a pole or a zero.

- In the case of **zeros**, summation is to zero. It is equivalent to taking the integral of a sinusoidal function centered at 0 over time. It sums to zero. Hence the name.

- The other case that we care about is that of the **poles** in which the summation is just barely infinite. In this case, you can imagine a sinusoid shifted such that its minimum intersects the x-axis. In this

[2]It should also further be recalled here that multiplication by s is the equivalent of taking the derivative and division by s its integral. Further note that under appropriate circumstances $t = 1/s$. From this, the result reported above should follow.

scenario, the area under the curve as time approached infinity would be just barely infinite.

It would become unbounded if its amplitude slightly increased with time and it would be finite if its amplitude decreased with time. The sweet spot representing this boundary is our "pole".

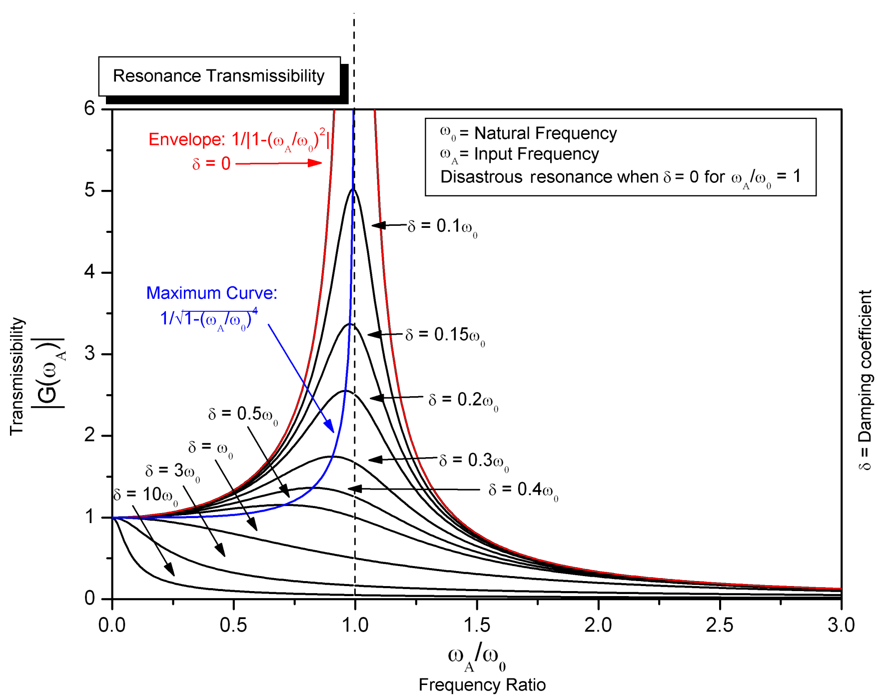
It is within those just barely infinite realms that we will meet our destiny.

14.3 Resonance

Resonance can be both useful and annoying and it is a prime example of marginal stability. Electrical resonance occurs in a electric system at a particular resonant frequency when the impedance of the circuit is

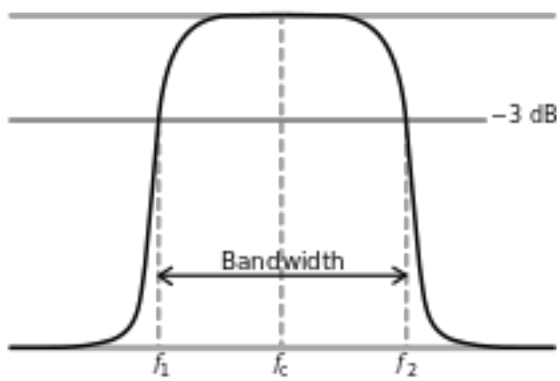
- at a minimum in a series circuit or
- at maximum in a parallel circuit.

That is, resonance usually happens when the transfer function peaks in absolute value.



14.4 Q factor

- The quality factor ("Q factor") is a dimensionless parameter that describes how damped an oscillator or resonator is.
- It characterizes a resonator's bandwidth relative to its center frequency.
- A higher Q indicates a lower rate of energy loss relative to the stored energy of the oscillator, i.e., the oscillations die out more slowly.
- For example, a pendulum suspended from a high-quality bearing, oscillating in air, would have a high Q, while a pendulum immersed in oil would have a low Q.



The bandwidth $\Delta f = f_2 - f_1$ of a damped oscillator is shown on a graph of energy versus frequency. The Q factor of the damped oscillator, or filter, is $f_c/\Delta f$. The higher the Q, the narrower and sharper the peak is.

The resonant frequency f_r, has a resonance width or **full width at half maximum (FWHM)** of Δf and can be used to defined the Q factor as

$$Q = \frac{f_r}{\Delta f} \tag{14.1}$$

14.5 A few examples

14.5.1 What is $\tilde{Z}(s)$ (i.e., $\tilde{V}(s)/\tilde{I}(s)$) of a resistor and a capacitor in parallel?

1. Show the *frequency response* (i.e., $\tilde{Z}(s)$ v. frequency) of such a system. If it helps to ascribe values, assume that the resistor is 1 ohm and the capacitor is 10 farad.

2. Comment on the behavior. Is it like anything you have seen before?

3. (*The kind of question that might be on a glorified quiz.*) If the current through the circuit is 1 amp when $t \geq 0$, what will be the voltage response?

14.5.2 What is the voltage time response of a system with a given impedance?

Given that

$$\tilde{Z}(s) = \frac{s^3 + 9}{(s+1)(s+3)} \tag{14.2}$$

1. What is the *time response* of potential to a unit step of current?

2. (*The kind of question that might be on a glorified quiz.*) Which exponential term will most significantly affect the signal?

14.5.3 Is the system stable?

1. A system, $y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$, when $x(t) = \cos(t)$.

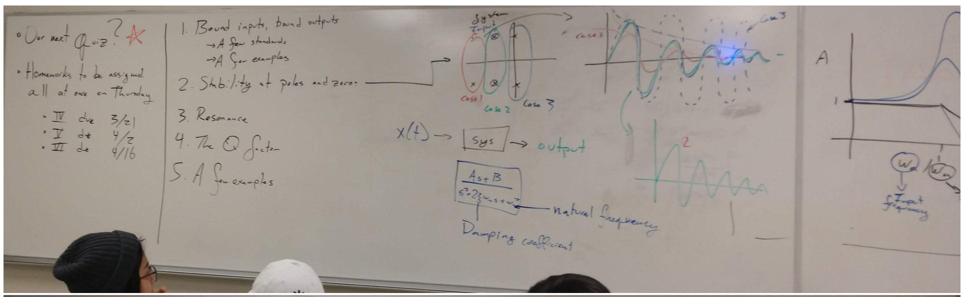
2. A system, $y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$, when $x(t) = u(t)$.

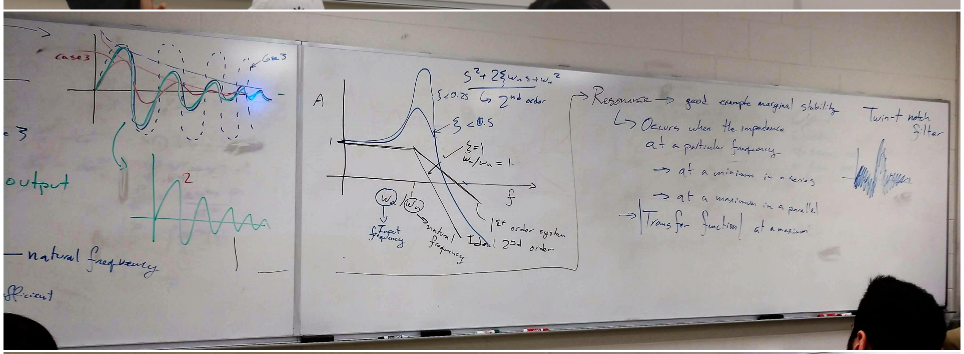
3. A system, $y(t) = x(t)/t$, when $x(t) = 2$.

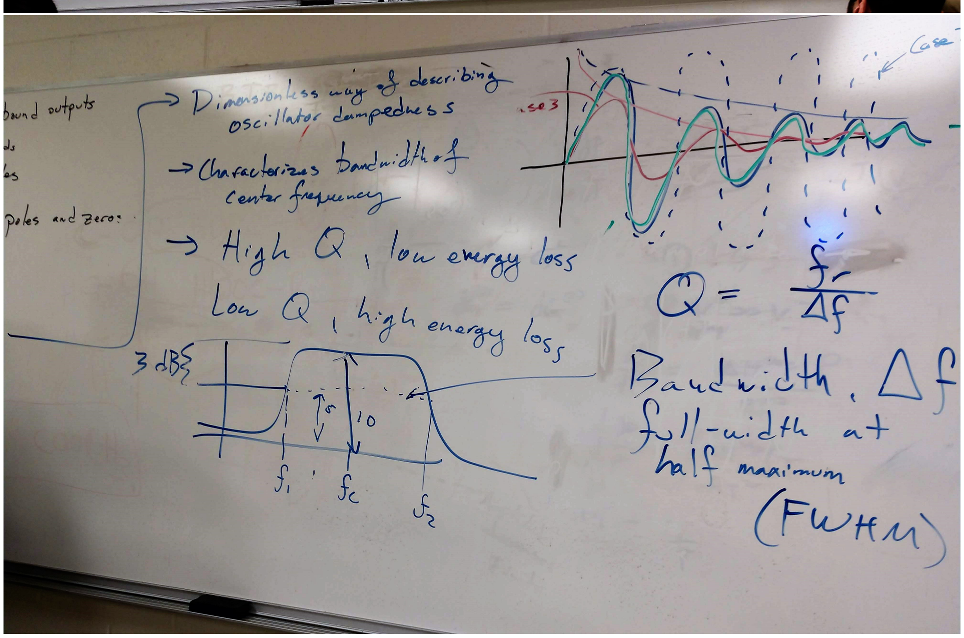
4. A system, $y(t) = dx(t)/dt$, when $x = 1$.

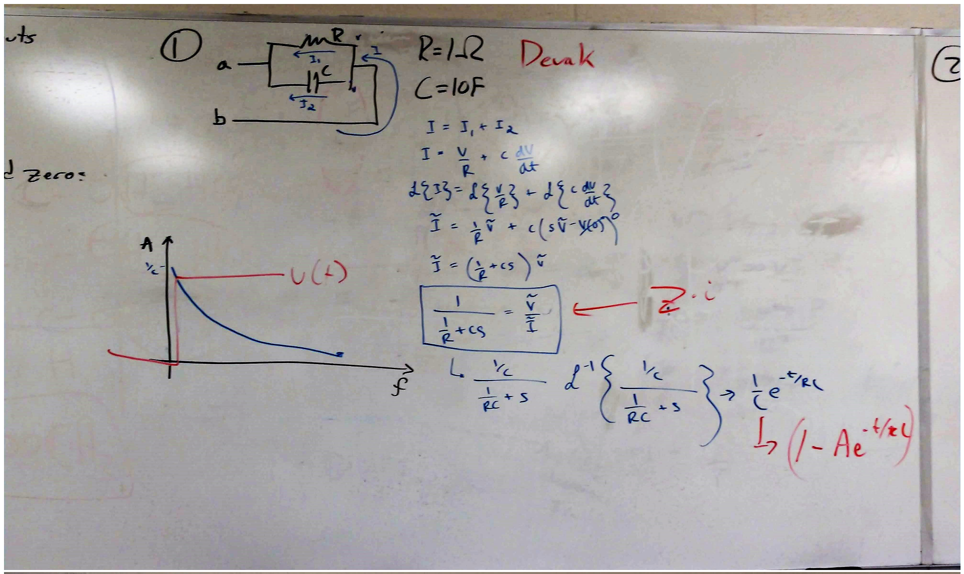
5. A system, $y(t) = dx(t)/dt$, when $u(t)$.

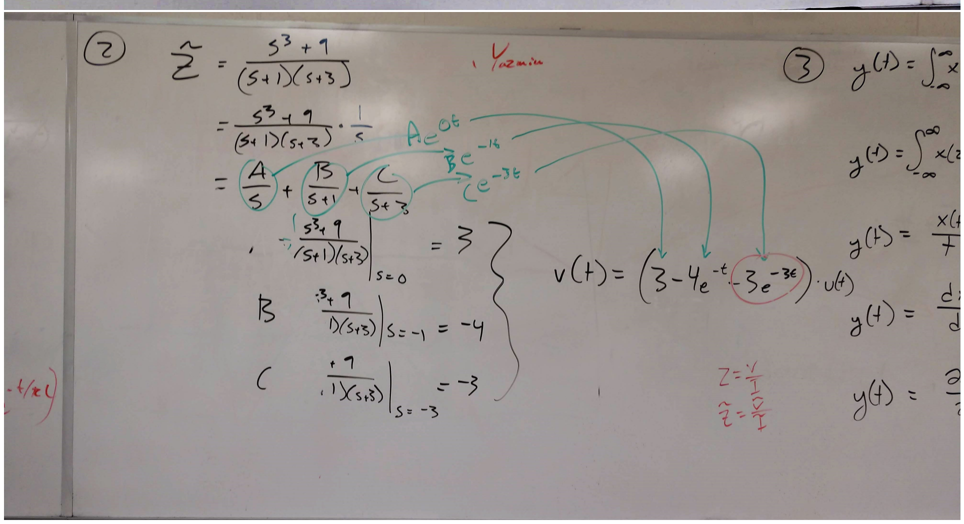
14.6 Lecture 14 Board Pictures (3/12/19)

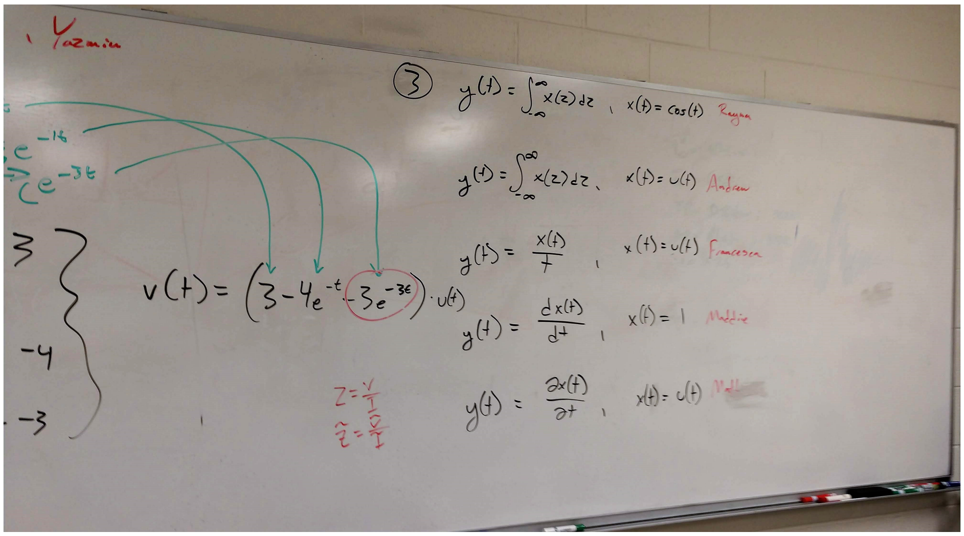







R=1Ω
C=10F
Devak
I = I₁ + I₂
v(t)


Yazmin
v(t) = (3 − 4e^{−t} − 3e^{−3t}) · u(t)



Part III

& Signals

Chapter 15

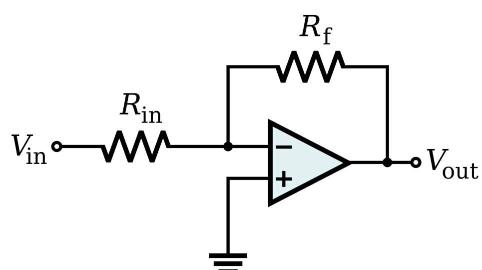
System response: III. Active filters

03/14/2019

15.1 Recall, an inverting amplifier

Given an inverting amplifier (as reported below), we know that the ratio of the output to the input is

$$\frac{V_o}{V_i} = -\frac{R_f}{R_{in}} \tag{15.1}$$



This can be more generally thought of in terms of a ratio of the feedback impedance, Z_f, and the input impedance, Z_{in}

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}} \tag{15.2}$$

With this in mind, we can start to analyze any number of impedance combinations to determine four very useful types of active filters.

15.2 An ideal inverting integrator

The first combination of impedances we will consider is a resistor, R_{in}, at the input and a capacitor, C_F in the feedback loop.

This will produce a (voltage gain) transfer function, V_o/V_{in}:

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}} = -\frac{1/sC_F}{R_{in}} = -\frac{1}{sR_{in}C_F} \tag{15.3}$$

We may remember that division by s in the s-domain is equivalent to integration in the time domain. Thus, the time-varying output is

$$V_o(t) = -\frac{1}{R_{in}C_F}\int_{-\infty}^{t} V_{in}(\tau)d\tau \tag{15.4}$$

Hence, this system will produce an output which integrates the input with respect to time and scales it to $1/R_{in}C_F$ An **integrator** produces an output proportional to the time integral of its input.

Question to ponder: what frequency wouldn't produce an output voltage?

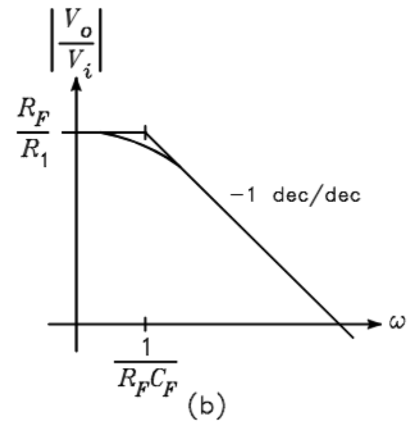
15.2.1 A modification yielding a low pass filter

Because at 0 Hz, C_F produces an open circuit, the feedback loop is no longer connected and hence no output is produced, we can include a resistor in combination with that feedback capacitor, so as to prevent this. This leads to our second combination: **a resistor, R_{in}, at the input and a capacitor, C_F, in parallel with a resistor, R_F, in the feedback loop.**

This will yield a transfer function of

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}} = -\frac{R_F||(1/sC_F)}{R_{in}} = -\frac{1}{R_{in}C_F}\cdot\frac{sR_FC_F}{1+sR_FC_F} \tag{15.5}$$

And let us consider its response with respect to frequency. To help matters, let us consider resistors of value 1 Ω and a capacitor of 10 mF. (Let's consider the output for frequencies of 1 Hz, 10 Hz, 100 Hz, 1 kHz, and 10 kHz.)



15.3 An ideal differentiator

For further reference, here is some more info on low pass filters that encorporates the corner frequency and other possible modifications:

Passive Low Pass: https://www.electronics-tutorials.ws/filter/filter_2.html
Active Low Pass: https://www.electronics-tutorials.ws/filter/filter_5.html

The third combination of impedances we will consider is a capacitor, C_{in}, at the input and a resistor, R_F in the feedback loop.

This produces a voltage gain response of

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}} = -\frac{R_F}{1/sC_{in}} = -sR_F C_{in} \tag{15.6}$$

Since multiplication by s in the s-domain is equivalent to differentiation in the time-domain, then the time-domain output potential may be written as

$$V_o = -R_f C_{in} \frac{dV_{in}(t)}{dt} \tag{15.7}$$

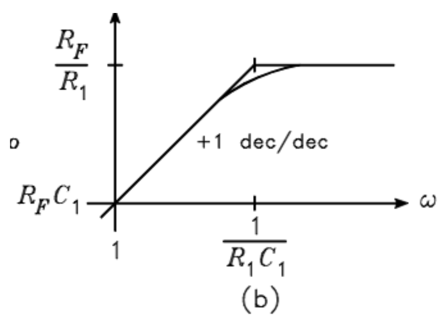
Hence, in this case, the circuit has a transfer function which takes the time-derivative of the input voltage, scales it by the product of the resistor and capacitor, and inverts.

15.3.1 A modification yielding a high pass filter

At very low frequencies, the input capacitor begins to introduce some effects that me undesirable (e.g., at low frequencies, no current flows in the input branch.) If we were to add a resistor in series with the capacitor at the input to prevent the non-flowing of current we would produce our forth combination: **a capacitor, C_{in}, in series with a resistor, R_{in} at the input and a resistor, R_F in the feedback loop**.

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}} = -\frac{R_F}{R_{in} + (1/sC_F)} = -R_F C_{in} \cdot \frac{1}{1 + sR_1 C_1} \tag{15.8}$$

And let us consider its response with respect to frequency. To help matters, let us consider resistors of value 1 Ω and a capacitor of 10 mF. (Let's consider the output for frequencies of 1 Hz, 10 Hz, 100 Hz, 1 kHz, and 10 kHz.)

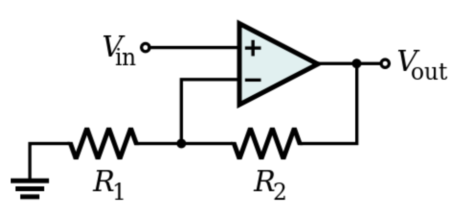


For further reference, here is some more info on high pass filters that encorporates the corner frequency and other possible modifications:

Passive High Pass: https://www.electronics-tutorials.ws/filter/filter_3.html
Active High Pass: https://www.electronics-tutorials.ws/filter/filter_6.html

15.4 Non-inverting variants

For each of these types of active filters, there are non-inverting versions which produce similar responses.



Such a circuit will produce a gain transfer function of

$$\frac{V_o}{V_{in}} = 1 + \frac{R_2}{R_1} \tag{15.9}$$

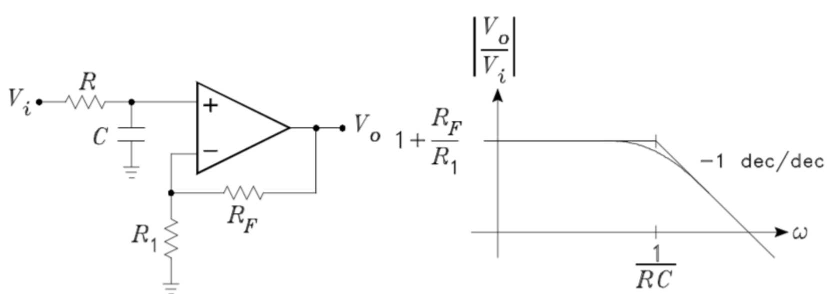
A more generalized version of this relationship may be thought of the ratio of the "feedback" impedance, Z_F, (R_2 in the figure above) to the "ground" impedance, Z_G (R_1 in the figure above) plus 1. Hence,

$$\frac{V_o}{V_{in}} = 1 + \frac{Z_F}{Z_G} \tag{15.10}$$

15.4.1 A non-inverting low pass filter

Let us approach this case by viewing the two inputs of our operational amplifier as having undergone voltage division.

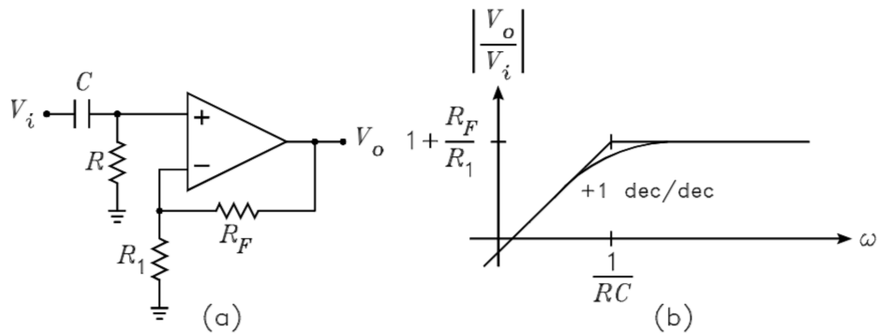
$$\frac{V_o}{V_{in}} = \frac{V_+}{V_{in}} \cdot \frac{V_o}{V_+} = \left(\frac{1/sC}{R + 1/sC}\right) \cdot \left(1 + \frac{R_F}{R_G}\right) = \left(\frac{1}{1 + sRC}\right) \cdot \left(1 + \frac{R_F}{R_G}\right) \tag{15.11}$$



15.4.2 A non-inverting high pass filter

Let us again approach this by viewing the two inputs of our operational amplifier as having undergone voltage division.

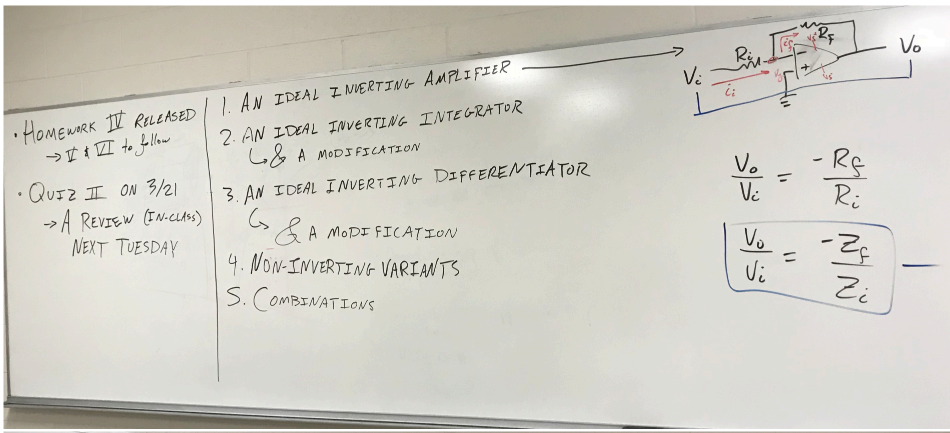
$$\frac{V_o}{V_{in}} = \frac{V_+}{V_{in}} \cdot \frac{V_o}{V_+} = \left(\frac{R}{R + 1/sC}\right) \cdot \left(1 + \frac{R_F}{R_G}\right) = \left(\frac{sRC}{1 + sRC}\right) \cdot \left(1 + \frac{R_F}{R_G}\right) \tag{15.12}$$

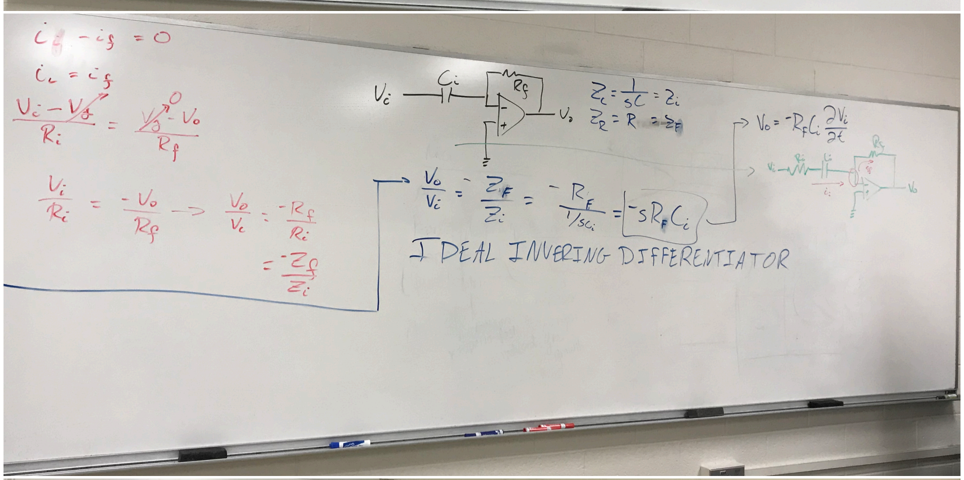


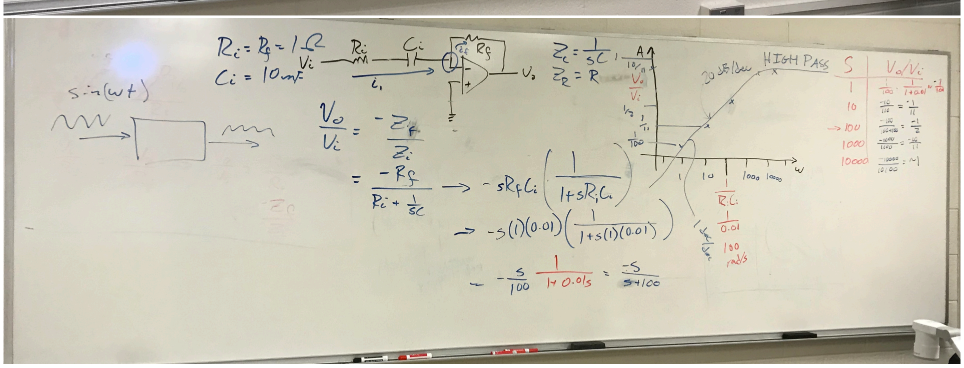
C
V_i
R
V_o
R_F
R_1
(a)
$\left|\frac{V_o}{V_i}\right|$
$1+\frac{R_F}{R_1}$
+1 dec/dec
ω
$\frac{1}{RC}$
(b)

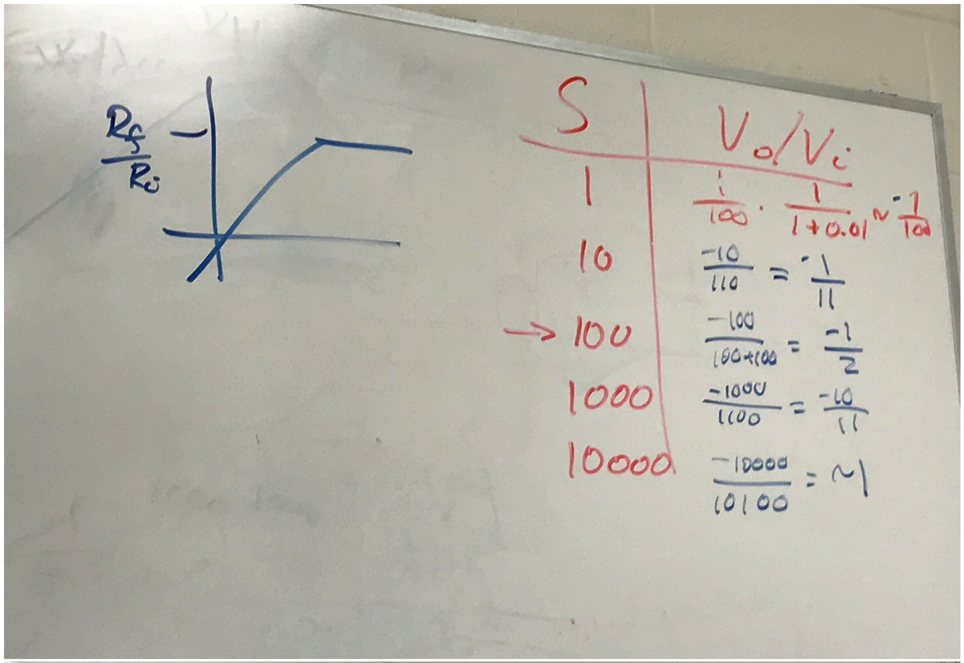
15.5 Combining active filters

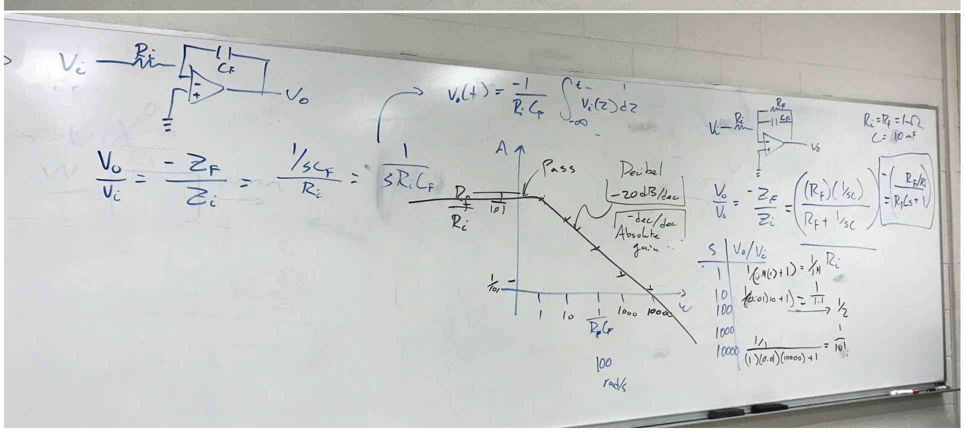
15.6 Lecture 15 Board Pictures (3/14/19)

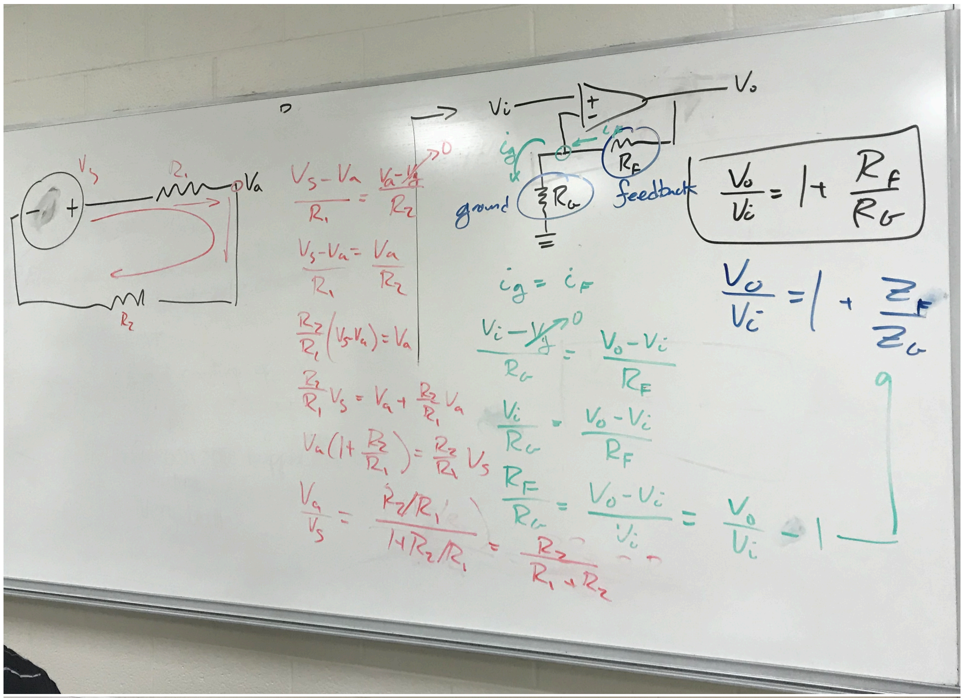


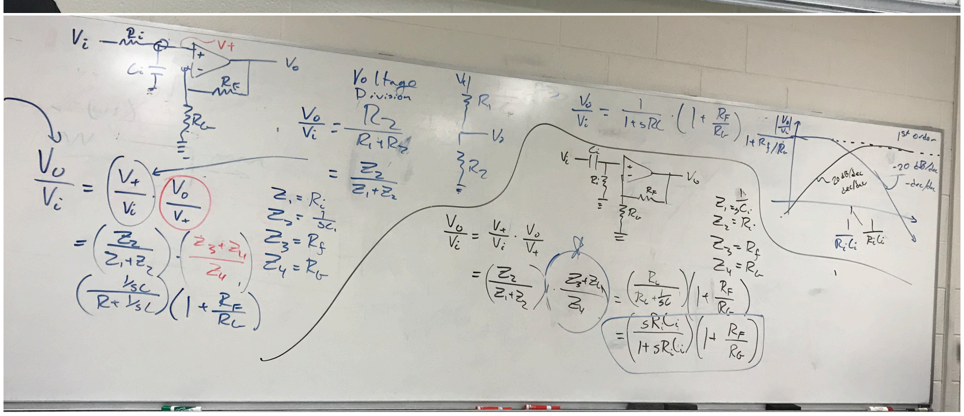





S
Vo/Vi
1
10
100
1000
10000


Pass
Decibel
-20dB/dec
Absolute gain
100
rad/s


ground
feedback
$\frac{V_o}{V_i} = 1 + \frac{R_F}{R_G}$
$\frac{V_o}{V_i} = 1 + \frac{Z_F}{Z_G}$
$i_g = i_F$


Voltage Division
1st order
-20 dB/dec

Chapter 16

System response: IV. Feedback and moving forward

03/28/2019

Contents

16.1 An introduction

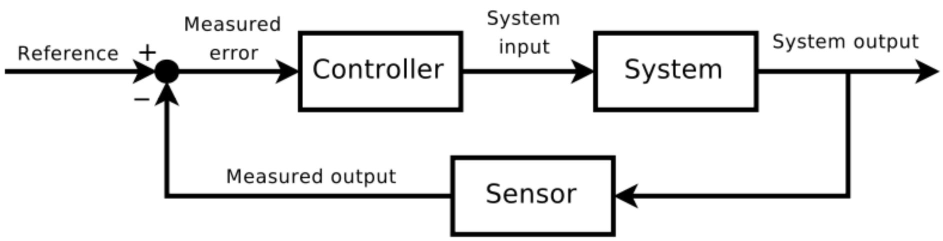
- **A system** is a thing or a group of things whose behavior can be described.
- **A control system** is a thing or group of things whose behavior can be *controlled*; in this sense a control system is able to alter the future state of a system based on a desired state.
- **Control theory** is the strategy to control systems by designing the systems or the inputs with which they are convolved in time.

16.2 To this point, an open loop

- Until now we have dealt with "open-loop" systems – systems whose input does not depend on a system's output
- This is good for systems with consistent and generally predictable states (e.g., lights being on or off, the surface potential of your heart's dipole, the temperature of water during a shower)
- In the case of a car, for example, the input that is delivered by a driver to accelerate or decelerate the vehicle is the angle they push the gas pedal to. We can imagine then, a system which describes the relationship between the θ of that pedal input and convert it to the velocity of the car moving forward. Such a system is sometimes called a "plant"
- In such open loop designs, the relationship between the input and the output is pretty straightforward: the system, convolved with the input, produces an output

16.3 Onward, a closed loop

- We will now start to consider systems where at least some portion of the output is used to influence the system input, thereby changing the output, etc.
- In this sense, the output is "fedback" to the input.
- Moreover, this "closes the loop" between input and output and helps to hone in on the desired output.



- **Reference**, $r(t)$ is the value we wish the system to ultimate produce. This might be an intensity of light, a temperature of a room, a velocity of a car, etc.

- **System output**, $y(t)$, is the output the system produces in response to its system input (itself a function of the measured error's control strategy when passed to the controller)

- Measured error, $e(t)$, the difference between the reference value and the measured output value (which is the system output value that the sensor measures)

- **Controller**, C, uses the error between the reference and the output to change the system inputs, $u(t)$[1] to the system, P

- **System input**, $u(t)$ is the input delivered directly to the system/plant

- **System**, P is the system's transfer function. That is, this is the block in the diagram which represents the behavior of the system alone, absent any control strategies.

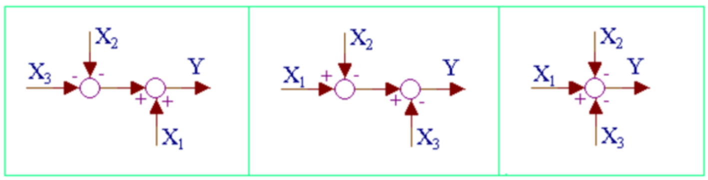
- **Sensor**, F, is used to measure the system output, $y(t)$ to compare the actual output to the desired output.

[1]Please note that the $u(t)$ here is not the same as the unit step function.

16.4 Block diagram algebra

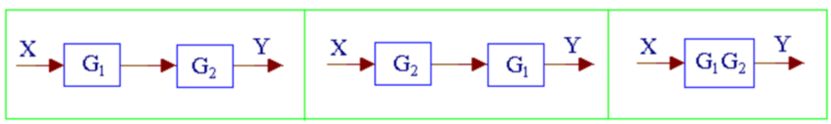
Rule:1

$$Y = X_1 - X_2 - X_3$$



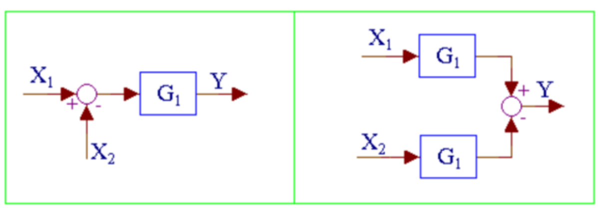
Rule: 2 (Associative and Commutative Properties)

$$Y = G_1 G_2 X = G_2 G_1 X$$



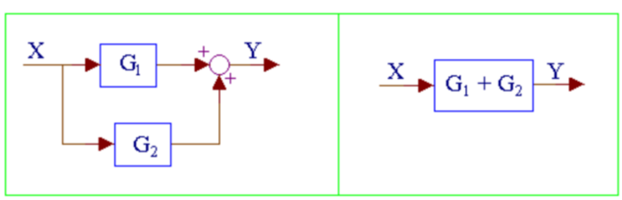
Rule: 3 (Distributive Property)

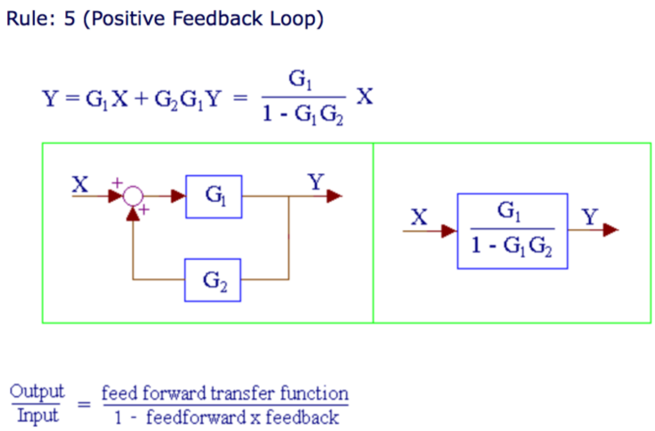
$$Y = G_1(X_1 - X_2) = G_1 X_1 - G_1 X_2$$

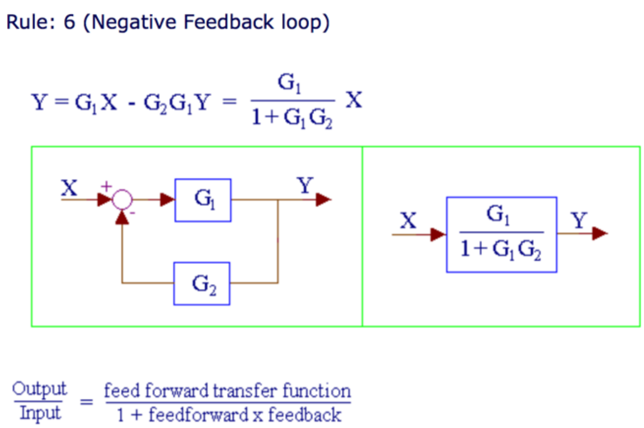


Rule: 4 (Blocks in Parallel)

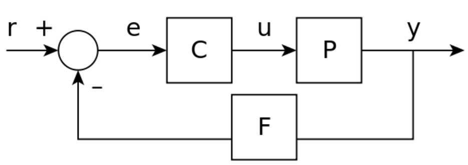
$$Y = X(G_1 + G_2) = G_1 X + G_2 X$$







16.5 A closed-loop transfer function



1. The output, $Y(s) = P(s)U(s)$
2. The system input, $U(s) = C(s)E(s)$
3. The (reference) error, $E(s) = R(s) - F(s)Y(s)$

$$Y(s) = \left(\frac{P(s)C(s)}{1 + P(s)C(s)F(s)} \right) R(s) = H(s)R(s) \tag{16.1}$$

The expression

$$H(s) = \frac{P(s)C(s)}{1 + F(s)P(s)C(s)} \tag{16.2}$$

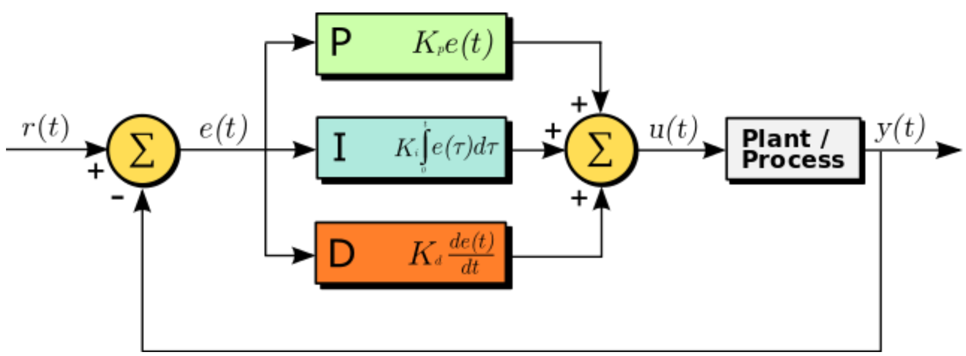
is referred to as the closed-loop transfer function.

- The **numerator** is the forward (open-loop) gain from r to y.
- The **denominator** is one plus the gain in going around the feedback loop, the “loop gain”.

16.6 PID feedback control

A **proportional-integral-derivative controller** (“PID controller”) is a feedback-control technique widely used to control systems.

A PID controller continuously calculates an error value $e(t)$ as the difference between a desired setpoint and a measured process variable and applies a correction based on proportional, integral, and derivative terms.



If u(t) is the control signal sent to the system, $y(t)$ is the measured output and $r(t)$ is the desired output, and $e(t) = r(t) - y(t)$ is the tracking error, a PID controller has the general form

$$u(t) = K_P e(t) + K_I \int e(\tau) \mathrm{d}\tau + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t} \tag{16.3}$$

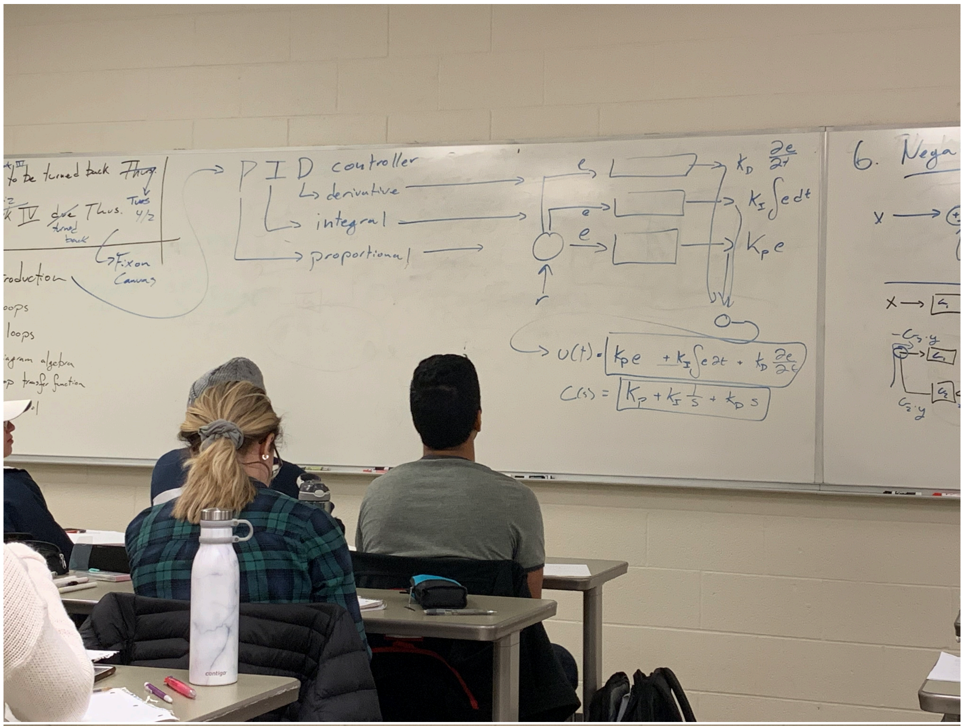
where K_P, K_I, and K_D are constants which scale the effects of each form of control. Generally these are chosen to achieve some effect (minimize overshoot, increase response time, minimize prolonged error, etc.).

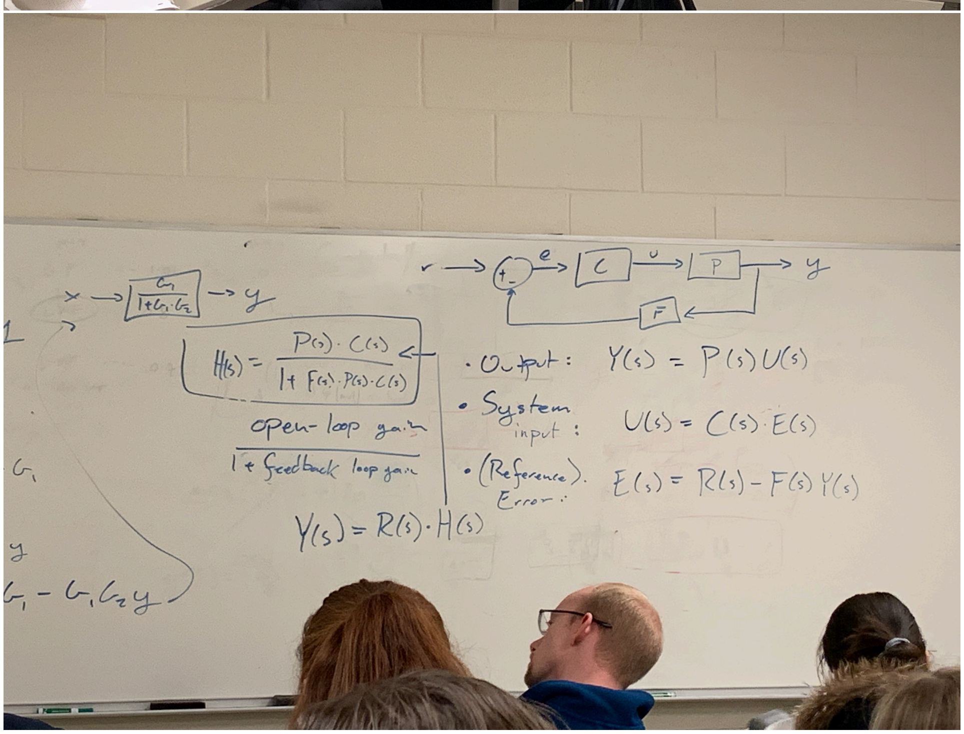
Without much erffort we can see the the PID controller transfer function would be of the general form

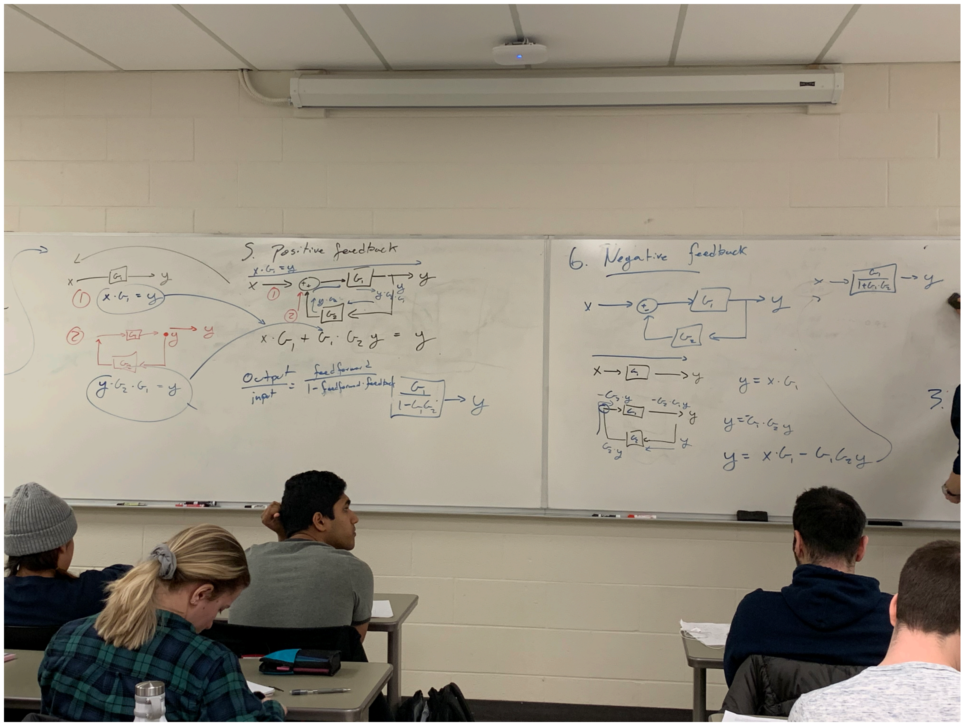
$$C(s) = \left(K_P + K_I \frac{1}{s} + K_D s \right) \tag{16.4}$$

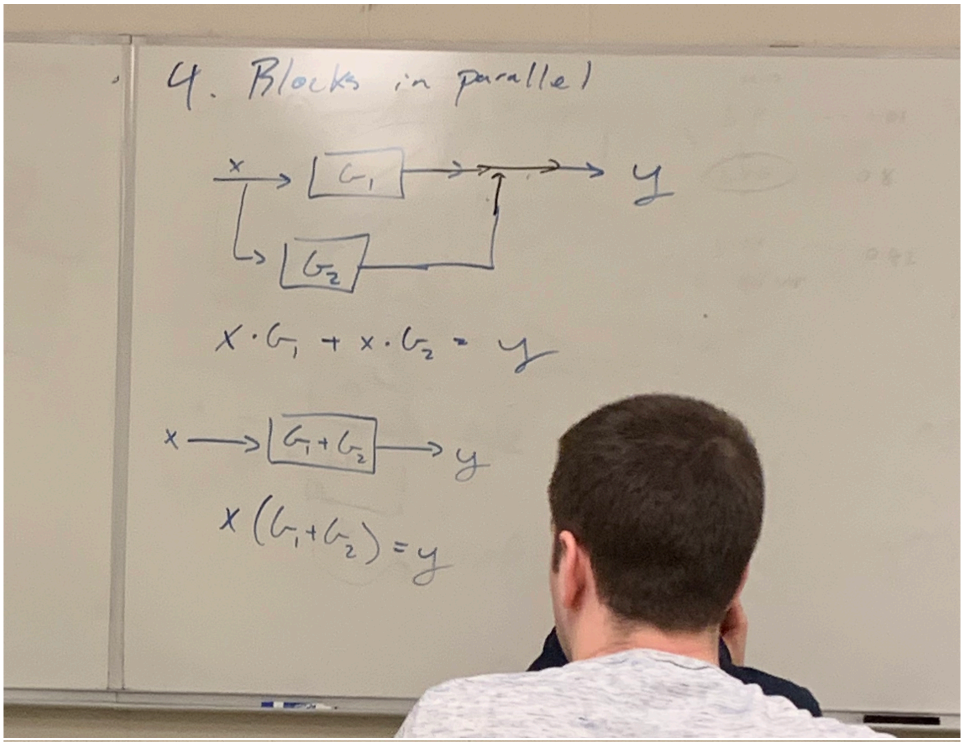
The goal here would be to use these terms to try to compensate for terms present in the plant. For instance, a system in which the output completely and flawless tracked the input would have a transfer function of $H(s) = 1$.

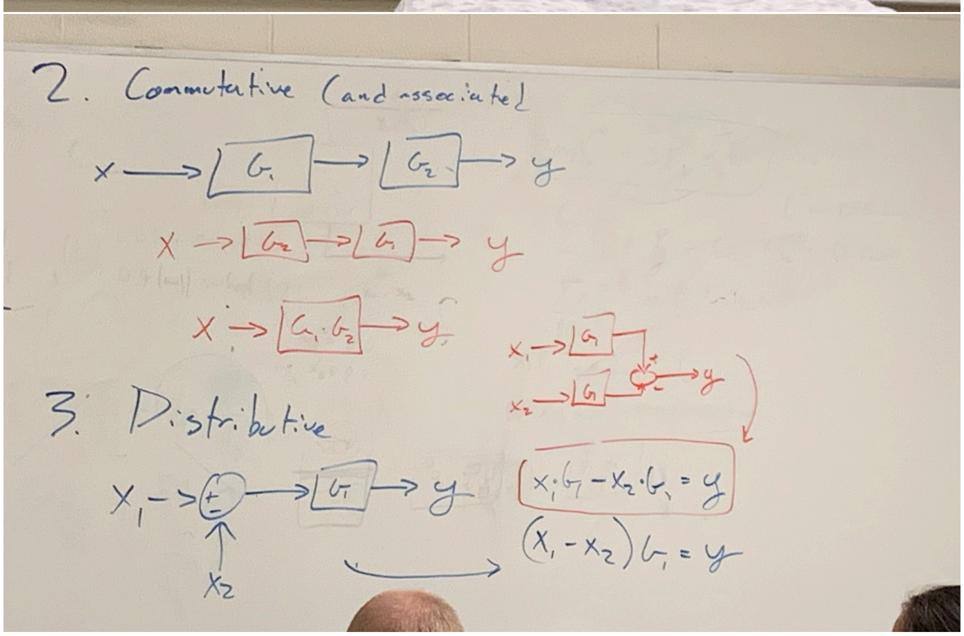
16.7 Board Pictures (3/26/19)

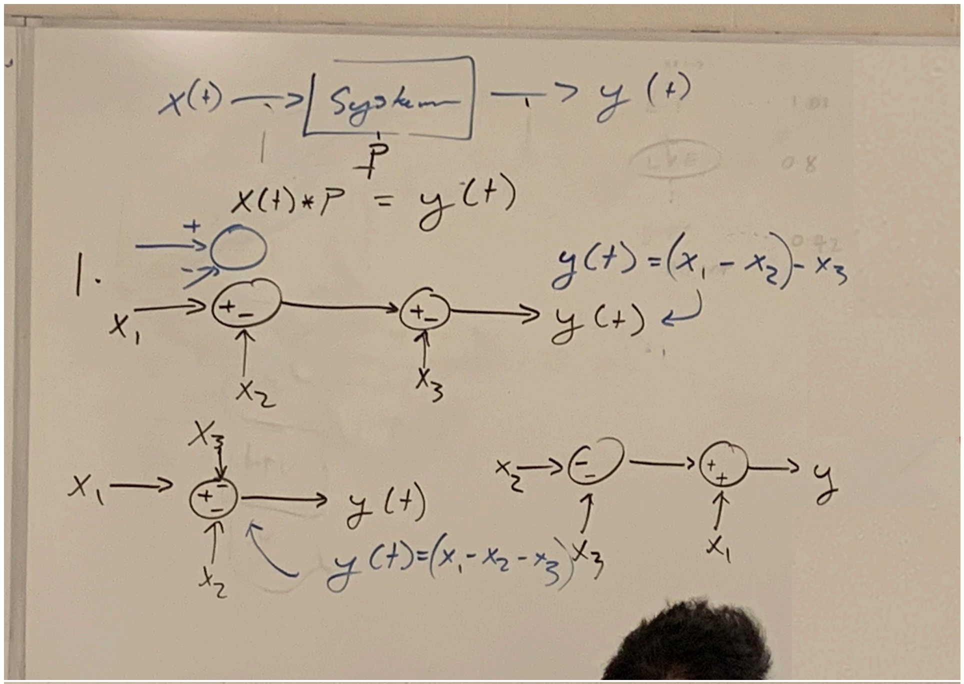


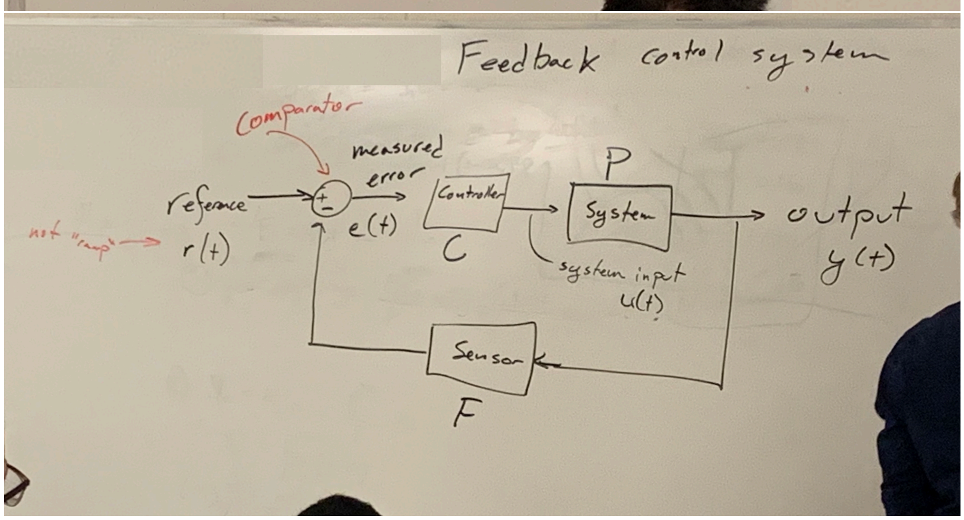


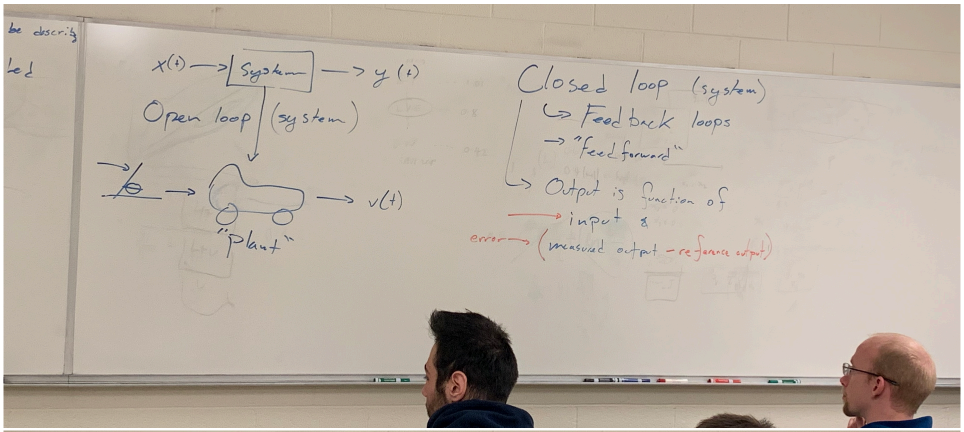

5. Positive feedback
6. Negative feedback

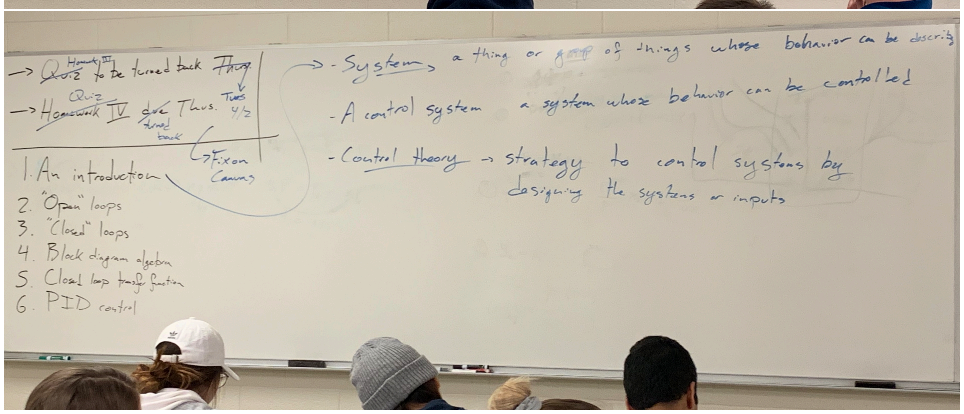

4. Blocks in parallel
x
G1
G2
y
x·G1 + x·G2 = y
x → G1+G2 → y
x(G1+G2) = y


2. Commutative (and associated)
x → G1 → G2 → y
X → G2 → G1 → y
X → G1·G2 → y
3. Distributive
X1 → ± → G1 → y
X2
x1·G1 − x2·G1 = y
(X1 − X2) G1 = y


x(t)
System
y(t)
P
x(t) * P = y(t)
1.
x1
x2
x3
y(t) = (x1 - x2) - x3
y(t)
y(t) = (x1 - x2 - x3)
y


Feedback control system
comparator
measured error
reference
r(t)
not "ramp"
e(t)
Controller
C
P
System
system input
u(t)
output
y(t)
Sensor
F


x(t) → System → y(t)
Open loop (system)
v(t)
"Plant"
Closed loop (system)
Feedback loops
"feed forward"
Output is function of
input &
error
(measured output - reference output)


Homework IV due Thus.
1. An introduction
2. "Open" loops
3. "Closed" loops
4. Block diagram algebra
5. Closed loop transfer function
6. PID control
- System, a thing or group of things whose behavior can be describ
- A control system a system whose behavior can be controlled
- Control theory → strategy to control systems by designing the systems or inputs

A Glorified Quiz II

03/19/2019, a review held 03/21/2019, a quiz given

A set of exercises in preparation of the second glorified quiz.

1. Find the transfer function (as V/I, I/V, Vo/Vin, etc.) of the following ("+" means "in series with", "||" means in parallel with"):

   R1 + R2||(R3 + C1)

   (R1 + R2)||(R3 + C1 + L1)

   C1 + (C2 + R1||(R2 + C3)||L1)

   (R1||R2||R3)||(R4||R5||R6)||(R7 + R8||R9)

2. Label and describe the pins of an operational amplifier.

3. Draw and derive an inverting amplifier.

4. Draw and derive a non-inverting amplifier.

5. Draw and derive a differential amplifier.

6. Draw and derive an operational amplifier.

7. What is the Laplace transform? What is its definition? What does it do for us? How does it do that? Why do we care? When can you use it? When can you not? When should you?

8. What is the s-domain? What are its axes? What do those axes represent? When is looking at the s-domain useful for us?

9. The s of the s-domain has two components. What are they and what do they individually and collectively represent?

10. What is at least one condition (w/r/t s) that must exist for a system to be stable?

11. What is the Laplace transform of a constant (i.e., of 1)?
12. What is the Laplace transform of an exponential function (i.e., of e^{at})?
13. What is the Laplace transform of a derivative (i.e., of dx/dt)? First, second, third, fourth, etc.?
14. Is the Laplace transform linear? And if so, what does that mean?
15. What is Eulers formula? What is Eulers identity? When might it be useful (especially in this class)?
16. What is the differential equation of a series RLC circuit? [I will take series RLC circuit here to mean a resistor, an inductor, and a capacitor in series.]
17. What is the differential equation of a parallel RLC circuit? [I will take parallel RLC circuit here to mean a resistor, an inductor, and a capacitor in parallel.]
18. What is the differential equation of a series RLC circuit in parallel with another series RLC circuit?
19. What the inverse Laplace transform?
20. Make sure you can invert a simple Laplace transformation (say of an RLC circuit) to yield the time-domain equation. For example:

    $1/(s^2 + 5s + 6)$

    $(s + 2)/(s + 3)(s + 4)$

    $s^2/(s(s^2 + 4s + 4))$

    $(s + 1)/(s^1 + 1)$
21. For the s-domain representations show above, what sort of frequency dependent behavior do you expect each to have? How do you know?
22. We have learned the general form the denominator (the poles) that many transfer functions take is: $s^2 + 2\zeta\omega_n s + \omega_n^2$. What is ζ? What is ω_n?
23. Describe the behavior of a system when ζ is (1) zero, (2) between zero and one, (3) one, and (4) greater than one. Include graphs. Include solutions to the differential equations thus described. Be sure you could identify such behavior if given a graphical representation.

24. A charged capacitor (charge it to whatever value youd like) is dropped in series with a resistor at time t = 0. Use KCL to find first-order differential equation. What is its solution?

25. The solution to the above example has a time constant, τ. What is it (in terms of the resistor and capacitor terms)? What is the amplitude of the response when t = 1τ? When t = 2τ? 3τ? 4τ? 5τ? 10τ?

26. If one were to take the derivative of the solution to 24 at t = 0, the slope of the line would intersect time axis at what point? Why might this be useful?

27. If there is no excitation signal (as in 24), but there are initial conditions, what kind of response is this, ZIR or ZSR?

28. BTW, whats ZIR & ZSR?

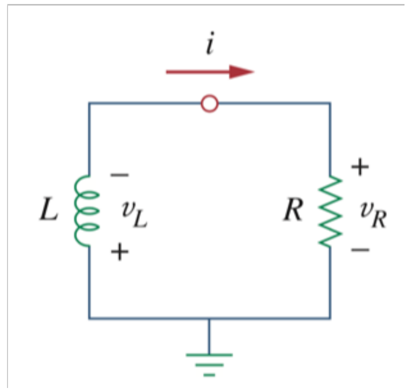
29. A voltage source, a resistor, and a switch are in series with (R2 + C1)||R3. What is the differential equation of the circuit? What is its Laplace transformation? What is its solution?

30. 30. What is partial fraction decomposition and why should I know it?

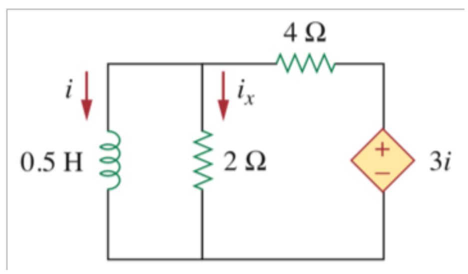
31. For the circuit at right, assume the initial condition of the inductors current is i(0) = I0. Apply KVL around the loop to find the first-order differential equation. Solve said equation. What is the time constant?

32. If instead of charging the inductor in the above example, we removed the charge and put in a time varying voltage signal and measured the voltage drop across the resistor, would we record be recording the zero state response, the zero input response, or some combination of the two?

33. For the circuit below, assuming i(0) = 10 A, calculate i(t) and ix(t).



34. Below is whats called a gate function, which is like a step function that switches on at one value of t and switches off at another. Express this voltage pulse (seen at right) in terms of the unit step. Calculate its derivative.



35. What is the definition of a unit step function?

36. What is the definition of a unit impulse function? How is it related to the unit step function?

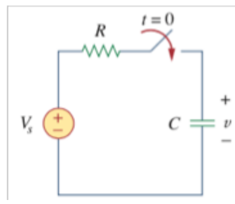
37. What is the definition of a unit ramp function? How is it related to the unit step function?

38. For 35-37, draw a graphical representation of the functions.

39. If you wanted to shift a unit step/impulse/ramp function by an amount t0 in time, how does this alter your definition of the function as reported above?

40. If you multiplied a function f(t) by an impulse function that has been time shifted (by an amount t0), what is the result?

41. If the switch of the circuit shown below is closed at t = 0, find the time-dependent voltage function for the circuit for both t < 0 and t ≥ 0.



42. The above example has two responses, a natural response (using stored energy) and a forced response (using an independent source). Describe these in your own words.

43. One could also decompose the responses by a transient response and a steady-state response. Describe these in your own words.

44. Describe convolution in your own words.

45. What is the formal definition of convolution?

46. Draw a (simple) system function and a (simple) input function. What is the convolution of your input with your system?

47. Convolution with the impulse response produces what?

48. What are the commutative, associative, and distributive properties of convolution?

49. What is the relationship between convolution and the Laplace transform? (Similarly, what is the relationship between convolution the Fourier transform?)

50. Draw a passive low-pass circuit. What is the transfer function between Vo/Vin? What is the cutoff frequency?

51. Draw a passive high-pass circuit. What is the transfer function between Vo/Vin? What is the cutoff frequency?

52. Draw a bode plot for a first-order passive low-pass filter. What is the pass band? What is the stop band? What is the slope of the tracing within the stop band? At the corner frequency, what is the amplitude in dB? Draw a corresponding phase diagram. What is the phase shift

one order of magnitude below the cutoff frequency? What is the phase shift one order of magnitude above the cutoff frequency?

53. Draw a bode plot for a first-order passive high-pass filter. List all the ways it differs from the first-order passive low-pass filter of 50.

54. Draw a bode plot and phase diagram for the series combination of a first-order passive low-pass filter with a cutoff frequency of 50 Hz and a first-order passive low-pass filter with a cutoff frequency of 500 Hz.

55. Redo the above questions but using "second-order" instead of "first-order" filters. How does this change your results?

56. Why do we care about the phase of a signal?

57. Draw the bode plot and phase diagram of the following transfer functions:

    $100(s+1)/(s+10)(s+100)$

    $(s^2+27s+100)/(s^2+4s+4)$

    $(s(s+100))/(s(s+1)(s+10))$

    $s^2/(s^3+2s^2+3s+4)$

58. What are the general forms of the transfer functions for low-pass, high-pass, and band-pass filters?

59. What is an integrator? How would you make one with an operational amplifier? What is the Laplace transform of an integrator / integration? What is one advantage and disadvantage to using an ideal inverting integrator?

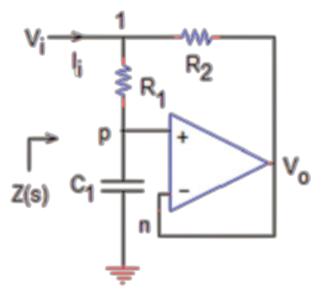
60. We can modify the ideal inverting integrator by placing a resistor in the feedback loop of the op-amp. What does this produce? Sketch a bode plot relating Vo/Vin for such a modified inverting integrator.

61. Given real values for the modified inverting integrator, could you determine the frequency at which you would expect the amplitude of a signal in the stop-band to be half the value one would expect to see in the pass-band?

62. What is a differentiator? How could you make one with an operational amplifier? What is the Laplace transform of a differentiator / differentiation? What is one advantage and disadvantage to using an ideal inverting differentiator?

63. We can modify the ideal inverting differentiator by inserting an input resistor (generally before the capacitor). What does this produce? Sketch a bode plot relating Vo/Vin for such a modified inverting differentiator.

64. What is the input impedance of the following system? (Note, this is from an actual bit of implantable electronics.)



65. What is a plant? What is a controller? What is a system?

66. What is open-loop control? What is closed-loop control?

67. Given a simple block diagram with F = 1 / (s + a), G = s / (s + b), and H = 1 / s; what is the overall system response (output / input)?

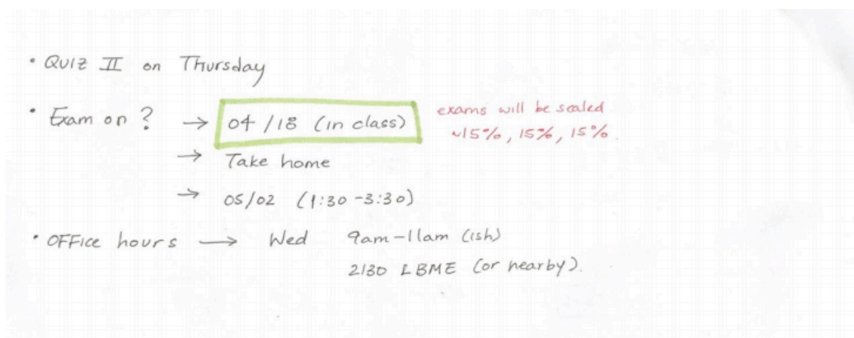
68. What is proportional, integral, and derivative control?

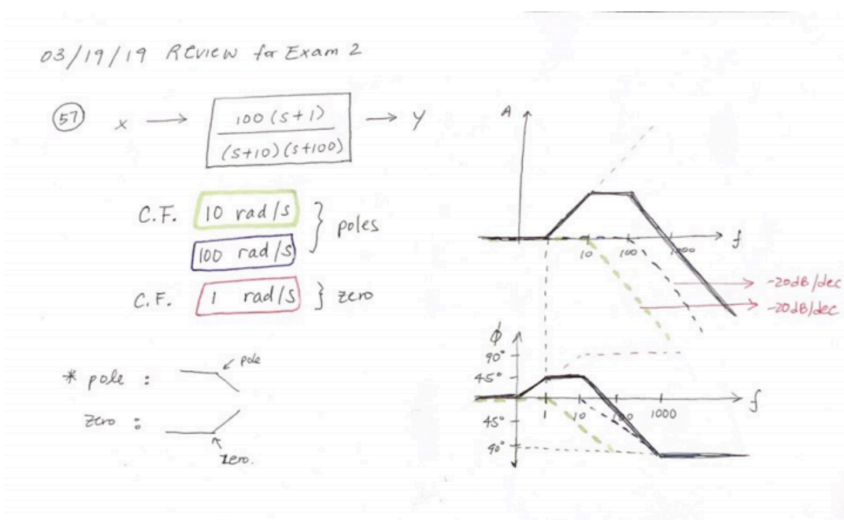
69. What is the s-domain representation of P, I, and D control?

70. How can we make for a stable arms race?

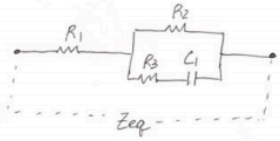
16.8 Exam 2 Review Board Pictures (3/19/19)

The glares made parts of the board difficult to see. All the notes below are hand renderings of board pictures taken today.





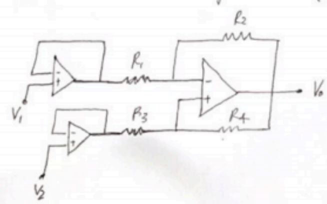
(1) $R_1 + R_2 \parallel (R_3 + C_1)$ transfer function $\Rightarrow$ equivalent impedance



$$Z_1 = R_1$$
$$Z_2 = R_2$$
$$Z_3 = R_3 + \frac{1}{j\omega C} = R_3 + \frac{1}{sC}$$

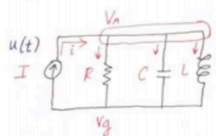
$$Z_{eq} = Z_1 + \frac{Z_2 \cdot Z_3}{(Z_2 + Z_3)} = R_1 + \left(\frac{R_2\left(R_3 + \frac{1}{sC}\right)}{R_2 + R_3 + \frac{1}{sC}}\right)$$

– Instrumentation Amplifier (Voltage follower + Differential Amplifier)



* does not allow current to pass.

(17)



$$I - I_R - I_C - I_L = 0$$
$$I = \frac{V_a - V_g}{R} + C\frac{\partial V_a}{\partial t} + \frac{1}{L}\int_{-\infty}^{t} V(\tau)\,d\tau$$
$$\frac{\partial I_s}{\partial t} = \frac{\partial V_a}{\partial t}\cdot\frac{1}{R} + C\frac{\partial^2 V_a}{\partial t^2} + \frac{1}{L}V$$

$\tilde{I}s - I(0)$

$I(0) = 0 \quad I = u(s) \to \frac{1}{s}$

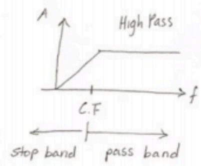
$$\longrightarrow \frac{\hat{\tilde{I}}s - \cancel{I(0)}}{s} = C\left(\tilde{V}s^2 - \cancel{v(0)}s - \cancel{v'(0)}\right) + \frac{1}{R}\left(\tilde{V}s - \cancel{V(0)}\right) + \frac{1}{L}\tilde{V}$$

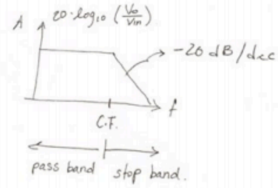
$$\frac{\hat{\tilde{I}}\cancel{s}}{\cancel{s}} = \tilde{V}\left(Cs^2 + \frac{1}{R}s + \frac{1}{L}\right)$$

$$\boxed{Z_{eq} = \frac{\hat{V}}{\hat{I}} = \frac{1}{Cs^2 + \frac{1}{R}s + \frac{1}{L}}}$$

- High Pass → High frequency input allowed to pass
  Low frequency input is attenuated (Reduced.

  Low Pass → Low frequency input allowed to pass
  High frequency input is attenuated.





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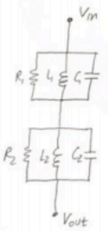
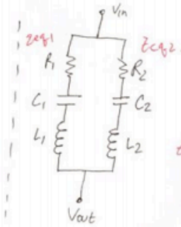


2 parallel RLC circuits in series

$$\left(\frac{1}{z^2+2\zeta\omega_n s+\omega_n^2}\right)\cdot\left(\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$



$$Z_{eq} = \frac{Z_{eq1}\cdot Z_{eq2}}{Z_{eq1}+Z_{eq2}}$$

- Mellin's Inverse Formula

$$f(t) = \mathcal{L}^{-1}\{\tilde{F}\} = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{\gamma-jT}^{\gamma+jT} e^{sT}\tilde{F}\,\partial s$$

Know for exam:
$$\boxed{\begin{array}{ll} \mathcal{L}\{1\} = \frac{1}{s} & \mathcal{L}\left\{\frac{\partial x}{\partial t}\right\} = \tilde{x}s - x(0) \\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a} & \mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2} \end{array}}$$

- $I \cdot u(t) \longrightarrow \boxed{\overset{Z_{eq}}{\frac{1}{s^2+5s+6}}} \longrightarrow V(t) = Ae^{-2t} + Be^{-3t} = \boxed{e^{-2t} - e^{-3t}}$

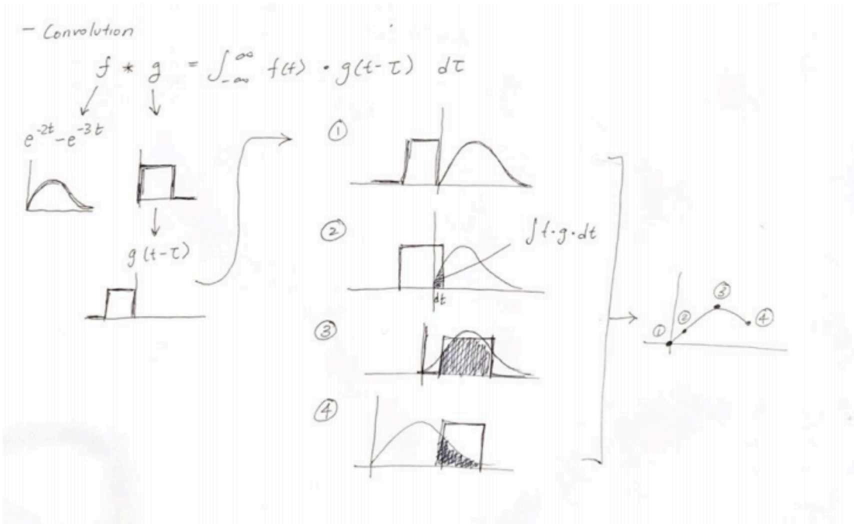
$\downarrow$

$\frac{1}{(s+2)(s+3)}$

<u>cover-up method</u>

poles: $-2 \longrightarrow A = \frac{1}{s+3}\Big|_{s=-2} = 1$

$-3 \longrightarrow B = \frac{1}{s+2}\Big|_{s=-3} = -1$



- Convolution
f * g = ∫_{-∞}^{∞} f(t) · g(t-τ) dτ
e^{-2t} - e^{-3t}
g(t-τ)
①
②
∫f·g·dt
dt
③
④
①
②
③
④

Part IV

in Biomedical Engineering

Chapter 17

Bioelectricity: I. Passive properties

04/02/2019

Contents

17.1 A quick review of some of our basic impedance knowledge

17.1.1 Impedance as a vector

$$Z = R + \jmath X = |Z|\angle\theta = |Z|e^{\jmath\theta} \tag{17.1}$$

17.1.2 Impedance as admittance

$$\frac{1}{Z} = Y = G + jB \tag{17.2}$$

17.1.3 Impedance equivalents

$$Z_{eq,R} = R \tag{17.3}$$

$$Z_{eq,C} = \frac{1}{\jmath\omega C} \tag{17.4}$$

$$Z_{eq,L} = \jmath\omega L \tag{17.5}$$

17.1.4 Impedance of networks

$$Z_{eq,series} = Z_1 + Z_2 + Z_3 + ... \tag{17.6}$$

$$Z_{eq,parallel} = \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + ...\right)^{-1} \tag{17.7}$$

17.2 A long derivation of a simple model

Draw for yourself a simple picture a cell in a bath and while doing so ask yourself the deep questions regarding the universe. If you're looking for one to munch on consider this: what separates life from non-life?

The cell membrane.

And it acts like a capacitor. Cell membranes are made of a phospholipid bilayer, which, if you squint at it, is kind of like two conductive plates sandwiching a dielectric. (Close enough for government work.)

On either side of that cell membrane is what we might call an *intra*cellular and *extra*cellular component. These, at their most basic level, must put up some sort of fight to an induced voltage, to scale the resistance of current. So let's say they are both resistors.

With this, we may begin our first modeling foray into the wonderful world of bioimpedance (a model whose use first began to see prominence in the early 1900s:

$$R_1 + R_2||C. \tag{17.8}$$

This yields an initial model of

$$\begin{align}
Z_{eq} &= Z_1 + \frac{Z_2 \cdot Z_3}{Z_2 + Z_3} \tag{17.9}\\
&= R_1 + \frac{R_2 \cdot (1/\jmath\omega C)}{R_2 + (1/\jmath\omega C)} \tag{17.10}\\
&= R_1 + \frac{R_2}{1 + \jmath\omega R_2 C}. \tag{17.11}
\end{align}$$

To remove that bottomr denominator we apply the complex conjugate

$$\begin{align}
Z_{eq} &= R_1 + \frac{R_2}{1 + \jmath\omega R_2 C} \cdot \frac{1 - \jmath\omega C}{1 - \jmath\omega C} \tag{17.12}\\
&= R_1 + \frac{R_2}{1 - \jmath\omega R_2 C + \jmath\omega R_2 C - \jmath^2\omega^2 R_2^2 C^2} \tag{17.13}\\
&= R_1 + \frac{R_2 - \jmath\omega R_2^2 C}{1 + \omega^2 R_2^2 C^2}. \tag{17.14}
\end{align}$$

At this point it might be useful to define a time constant, since it appears that the cell related terms (the cell membrane, C, and the intracellular component, R_2). Let's say

$$\tau = R_2 C. \tag{17.15}$$

Rewriting the previous equation:

$$Z_{eq} = R_1 + \frac{R_2 - \jmath\omega\tau R_2}{1 + \omega^2\tau^2}. \tag{17.16}$$

Separating the real and imaginary terms so that we might consider resistance and reactance separately:

$$Z_{eq} = \left(R_1 + \frac{R_2}{1 + \omega^2\tau^2}\right) - \jmath\left(\frac{\omega\tau R_2}{1 + \omega^2\tau^2}\right). \tag{17.17}$$

Hence,

$$\begin{align}
R &= \left(R_1 + \frac{R_2}{1 + \omega^2\tau^2}\right) \tag{17.18}\\
X &= -\left(\frac{\omega\tau R_2}{1 + \omega^2\tau^2}\right). \tag{17.19}
\end{align}$$

It's at a point like this that I like to consider a few extremes in a problem. The reason for that is because it is a very easy measurement to take with the equipment you will have. It will always be easy to send in a 0 and send in way more than you want. So, it being easy to find in lab, in the world, and in life, let use familiarize ourselves with their nature.

At very very very low frequencies ($\omega \to 0$), resistance tends to become equal to the sum of R_1 and R_2. At very very high frequencies ($\omega \to \infty$), resistance tends to become equal to R_1. From this I think we can all agree to the following:

$$R_0 = R_1 + R_2 \tag{17.20}$$
$$R_\infty = R_1 \tag{17.21}$$
$$R_1 = R_\infty \tag{17.22}$$
$$R_2 = R_0 - R_\infty. \tag{17.23}$$

We can perform a similar analysis for reactance. At very very very low frequencies ($\omega \to 0$), reactance tends to become 0. At very very high frequencies ($\omega \to \infty$), reactance tends to become equal to 0. The combination of these two points represent the extremes of our resistance reactance plane. And they are measurable!

But before we measure, we often want to have in mind some notion of the result to be gotten. Let's consider, analytically, what the signal will end up looking like on the resistance-reactance plane.

Converting our equations from the model-based parameters to the measured ones, yields

$$R = \left(R_\infty + \frac{R_0 - R_\infty}{1 + \omega^2\tau^2} \right) \tag{17.24}$$
$$X = -\left(\frac{\omega\tau(R_0 - R_\infty)}{1 + \omega^2\tau^2} \right). \tag{17.25}$$

From this, we can attempt to solve the equations by looking for shared terms. In this case, let us try to isolate the $\omega\tau$ found in both the resistance and reactance terms.

$$R = R_\infty + \frac{R_0 - R_\infty}{1+\omega^2\tau^2} \tag{17.26}$$

$$R - R_\infty = \frac{R_0 - R_\infty}{1+\omega^2\tau^2} \tag{17.27}$$

$$(R - R_\infty)(1+\omega^2\tau^2) = R_0 - R_\infty \tag{17.28}$$

$$1+\omega^2\tau^2 = \frac{R_0 - R_\infty}{R - R_\infty} \tag{17.29}$$

$$\omega^2\tau^2 = \frac{R_0 - R_\infty}{R - R_\infty} - 1 \tag{17.30}$$

$$\omega\tau = \sqrt{\frac{R_0 - R_\infty}{R - R_\infty} - 1}. \tag{17.31}$$

We can substitute this into our reactance equation

$$X = -\left(\frac{\omega\tau(R_0 - R_\infty)}{1+(\omega\tau)^2}\right) \tag{17.32}$$

$$X = -\left(\frac{\left(\sqrt{\frac{R_0-R_\infty}{R-R_\infty}} - 1\right)(R_0 - R_\infty)}{1+\left(\sqrt{\frac{R_0-R_\infty}{R-R_\infty}} - 1\right)^2}\right) \tag{17.33}$$

$$X = -\left(\frac{\left(\sqrt{\frac{R_0-R_\infty}{R-R_\infty}} - 1\right)(R_0 - R_\infty)}{1+\frac{R_0-R_\infty}{R-R_\infty} - 1}\right) \tag{17.34}$$

$$X = -\left(\frac{\left(\sqrt{\frac{R_0-R_\infty}{R-R_\infty}} - 1\right)(R_0 - R_\infty)}{\frac{R_0-R_\infty}{R-R_\infty}}\right) \tag{17.35}$$

$$X = -\left(\left(\sqrt{\frac{R_0 - R_\infty}{R - R_\infty}} - 1\right)(R - R_\infty)\right) \tag{17.36}$$

$$X^2 = \left(\frac{R_0 - R_\infty}{R - R_\infty} - 1\right)(R - R_\infty)^2 \tag{17.37}$$

$$X^2 = \left(\frac{R_0 - R_\infty - R + R_\infty}{R - R_\infty}\right)(R - R_\infty)^2 \tag{17.38}$$

$$X^2 = \left(\frac{R_0 - R}{R - R_\infty}\right)(R - R_\infty)^2 \tag{17.39}$$

$$X^2 = (R_0 - R)(R - R_\infty) \tag{17.40}$$

$$X^2 = R_0 R - R_0 R_\infty - R^2 + R R_\infty. \tag{17.41}$$

It is with the appearance of two squared axes terms (X^2 and R^2) in that equation that put me in a mind go go looking for circles. With that focus, I will endeavor to put this into a format consistent with a semi-circle.

$$X^2 = R_0R - R_0R_\infty - R^2 + RR_\infty \tag{17.42}$$

$$X^2 + R^2 - R_0R - RR_\infty = -R_0R_\infty \tag{17.43}$$

$$X^2 + R^2 - R(R_0 + R_\infty) = -R_0R_\infty \tag{17.44}$$

$$X^2 + R^2 - R(R_0 + R_\infty) + \left(\frac{R_0 + R_\infty}{2}\right)^2 = -R_0R_\infty + \left(\frac{R_0 + R_\infty}{2}\right)^2 \tag{17.45}$$

$$X^2 + \left(R - \frac{R_0 + R_\infty}{2}\right)^2 = -R_0R_\infty + \frac{R_0^2}{4} + \frac{2R_0R_\infty}{4} + \frac{R_\infty^2}{4} \tag{17.46}$$

$$X^2 + \left(R - \frac{R_0 + R_\infty}{2}\right)^2 = \frac{R_0^2}{4} - \frac{R_0R_\infty}{2} + \frac{R_\infty^2}{4} \tag{17.47}$$

$$X^2 + \left(R - \frac{R_0 + R_\infty}{2}\right)^2 = \left(\frac{R_0 - R_\infty}{2}\right)^2. \tag{17.48}$$

This final result demonstrates that we would expect the impedance vector to carve out a semi-circle in the resistance-reactance plane, centered at $((R_0 + R_\infty)/2, 0)$ and with a radius of $(R_0 - R_\infty)/2$.

Such a tracing on the resistance-reactance plane is sometimes called an Argand diagram, a Wessel diagram, or a Cole-Cole plot. The latter is a woefully entrenched incorrect way of describing it and the former two describe a complex plane more generally. I prefer to simply call it the result on the resistance-reactance plane.

17.3 Implications of the model

17.3.1 What happens if R_1 goes up?

In the case in which there is less conductive material in our extracellular space, for instance, when there is less water within our blood vessels (i.e., when we become dehydrated), what happens?

Recall, $R_0 = R_1 + R_2$ and $R_\infty = R_1$, so it would shift the curve rightward, elongating our impedance vector and making our phase angle a little more shallow.

17.3.2 What happens if R_1 goes down?

In this case there is more conductive fluid in our extracellular space, as is what happens in many renal disorders, such as end-stage renal disease, one of the final consequences of diabetes. This shifts the curve leftward. This also makes our phase angle a more sensitive measure and thus if we were to try to characterize the disorder, we could use it. (As is the case of those working with "bioreactance".)

17.3.3 What happens if R_2 goes up?

R_2 may increase, for example, when cells are lysing/dying, being invaded by parasites, etc. Such an increase would expands the circle, but leaves the leftmost point the same, which makes sense since the extracellular has not been altered. Thus as cells begin to die, their impedance goes up. This is a phenomenon that is seen in electrosurgery and is one of the challenges to ensuring an optimal amount of current delivery to heat the tissue.

17.4 Now, that's what I call a darn good question

In class, I was thoroughly stumped. I was asked, "why is the reactance negative?" We had just derived a basic bioimpedance model and the reactance, as you may well know dear reader, is negative, such that the impedance vector carves out a shifted semi-circle along the *negative* reactance portion of the resistance-reactance plane.[1] There I stood dumbfounded for eternity until a student pointed out that there was a negative sign in our math from the get go!

It answered the question, but not thoroughly enough for my like. I report below a more thorough attempt at an answer.

The impedance of a capacitor is

$$Z_C = \frac{1}{\jmath\omega C}. \tag{17.49}$$

Let's play with some algebra. Recall that we define $\jmath = \sqrt{-1}$. I think that you would agree that it's fair to say that the following is true

$$\frac{1}{\jmath} = \frac{1}{\jmath}. \tag{17.50}$$

[1] Yes, this would be a model without a constant phase element.

No tricks so far. Let's multiply the right-hand-side by $\frac{\jmath}{\jmath}$, such that

$$\frac{1}{\jmath} = \frac{1}{\jmath} \cdot \frac{\jmath}{\jmath}. \tag{17.51}$$

This I think you'll agree leads to the following logic:

$$\begin{align}
\frac{1}{\jmath} &= \frac{1}{\jmath} \tag{17.52}\\
&= \frac{1}{\jmath} \cdot \frac{\jmath}{\jmath} \tag{17.53}\\
&= \frac{\jmath}{\jmath^2} \tag{17.54}\\
&= \frac{\jmath}{-1} \tag{17.55}\\
&= -\jmath \tag{17.56}\\
\frac{1}{\jmath} &= -\jmath. \tag{17.57}
\end{align}$$

Thus,

$$Z_C = \frac{1}{\jmath} \cdot \frac{1}{\omega C} = -\frac{1}{\omega C}. \tag{17.58}$$

Therefore, the *reactance that is produced by capacitance* will be negative. Conversely, reactance produced by inductance will be positive. I leave it to the interested reader to prove this to themselves.

Really, what the “$-\jmath$” result represents in our math is a 90 degree phase shift between the capacitor's voltage and its current. That is, if one were to try to force a current into a capacitor, a potential would develop in a manner that in which its intensity is 90 degrees out of alignment. Put simply, the current ”leads” the voltage (by about 90 degrees)[2].

17.5 A summary of “The theory and fundamentals of bioimpedance analysis in clinical status monitoring and diagnosis of diseases”

17.5.1 Article details

The Theory and Fundamentals of Bioimpedance Analysis in Clinical Status Monitoring and Diagnosis of Diseases. Source: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4118362/

[2]Again, we are not considering other sorts of “constant phase elements”, though the interested reader is encouraged to consider their consequences.

17.5.2 Advantages of bioimpedance measurements

- low cost
- portable systems
- wide range of utilizations, non-invasive

17.5.3 Definition of Bioimpedance

The ability of biological tissue to impede electric current

17.5.4 Active vs. Passive Response

- **Active:** Biological tissues provoke electricity on their own from ionic activities inside cells (for example, electrocardiogram signals from the heart)
- **Passive:** Biological tissues are stimulated through an external electrical current source; an elicited reaction to a stimulus.

17.5.5 Methods for measuring whole-body impedance using different placements and types of electrodes

1. Hand to foot method: gel-filled electrodes made to minimize gap
2. Foot to foot method: pressure-contact foot-pad electrode
3. Hand to hand method: handheld impedance meter
4. Segmental bioimpedance analysis: detects the fluctuation in extracellular fluid due to differences in posture. This method measures the body in 5 segments treated as cylinders (both arm and leg limbs plus the torso).

17.5.6 Using Bioimpedance to measure human health status

estimating the hydration status of individuals using height indexed resistance and reactance data (an R-Xc graph) from bioimpedance measurements. The data allows creation of 50%, 75%, and 95% tolerance ellipses that determine increasing and decreasing body mass if the minor vector falls in the left and right half of the 50% ellipse, along with increasing and decreasing hydration ratio if the major vector falls in the lower and upper half of the 50% ellipse.

17.5.7 Clinical Bio-impedance

Read about how our own professor developed a technique called “DRIVE”, also known as dynamic respiratory impedance volume evaluation which “uses the bioimpedance of the upper limb to predict shifts in blood volume in response to cardiac and respiratory phenomenon” : https://belmont.bme.umich.edu/research/

Using this technique, a wearable sensor smaller than a credit card was designed to measure heart rate, respiratory rate, and body temperature and volume status.

17.5.8 Body Composition

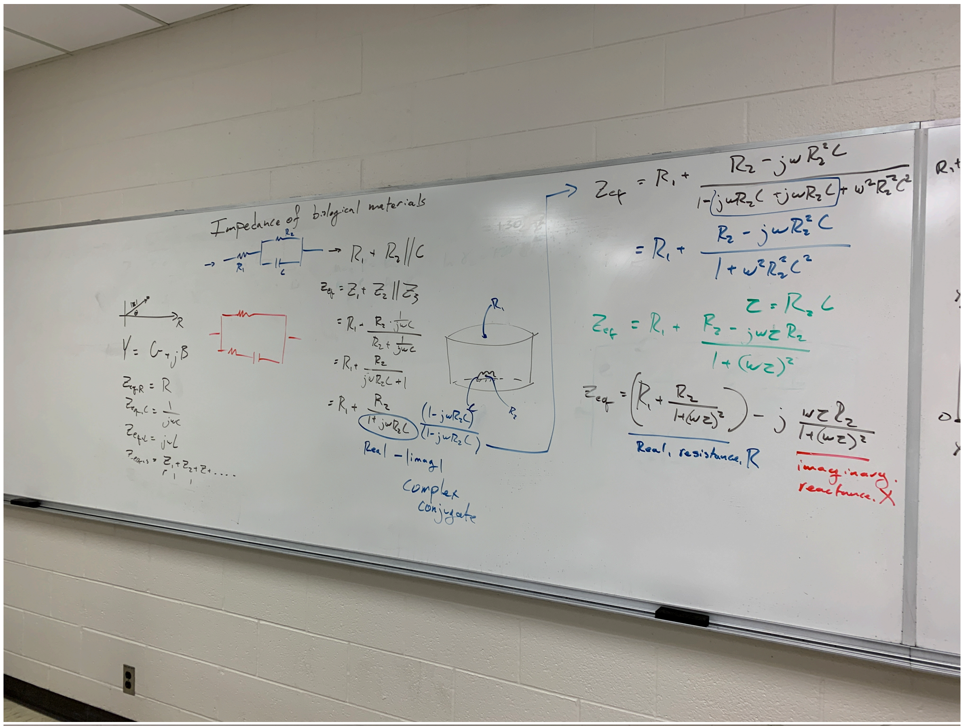
Impedance used to measure Fat mass and Fat-free mass, as well as create body fluids estimations.

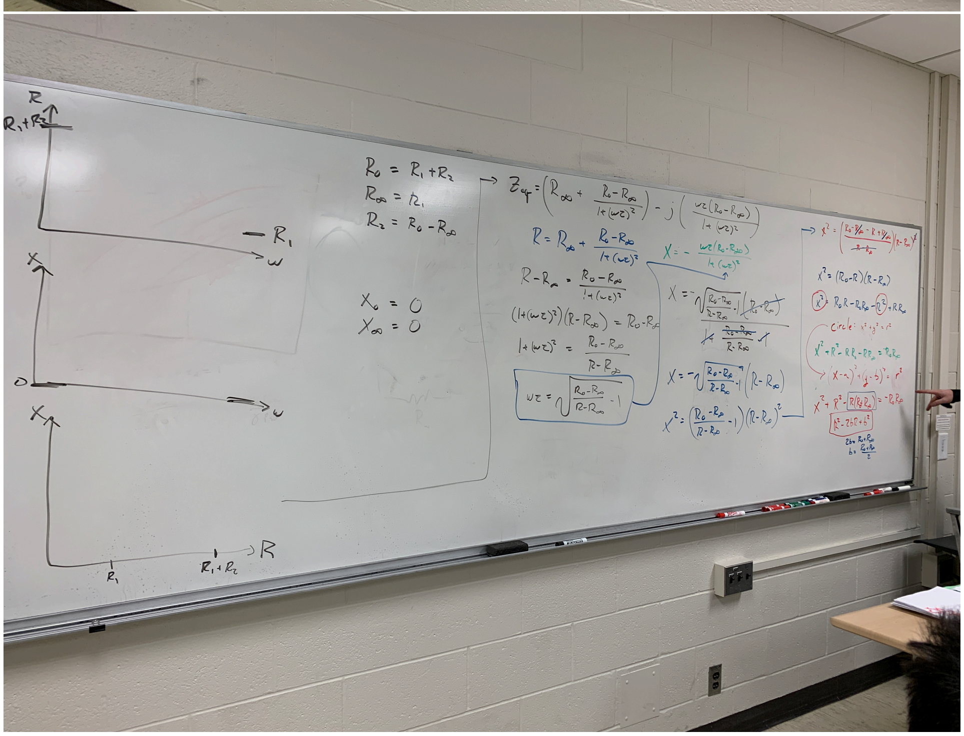
Your body is made of fat mass and fat free mass. Fat free mass is composed of bone mass and body cell mass, of which in body cell mass is composed of 73.2% water and the other 26.8% proteins. Of the total body water weight, 29% is made from extracellular fluids and 44% is made of intracellular fluids.

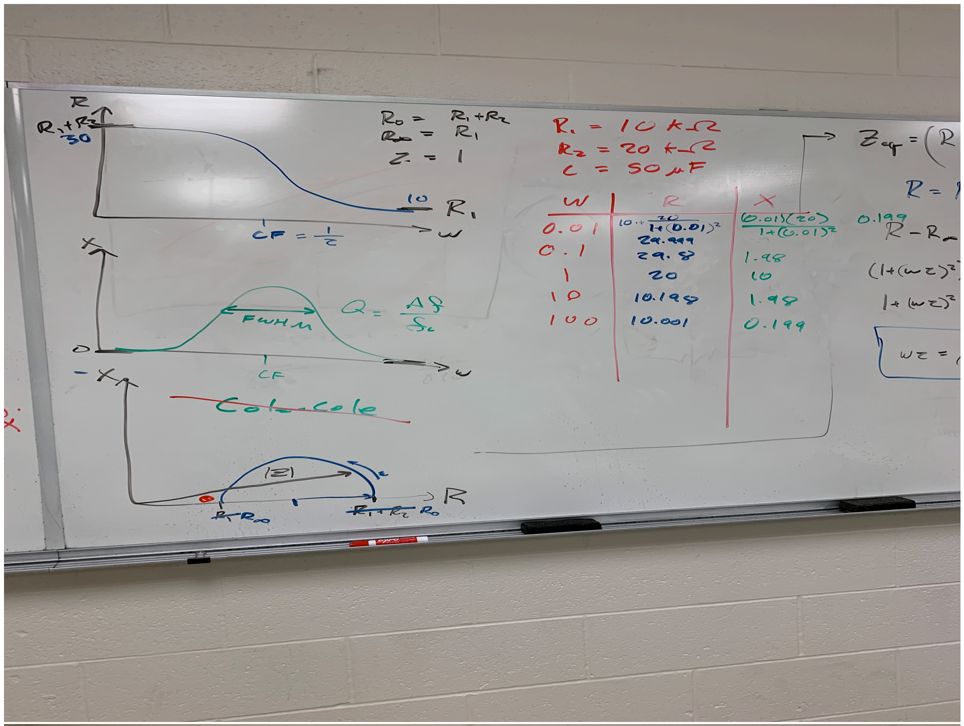
17.5.9 Potential future applications

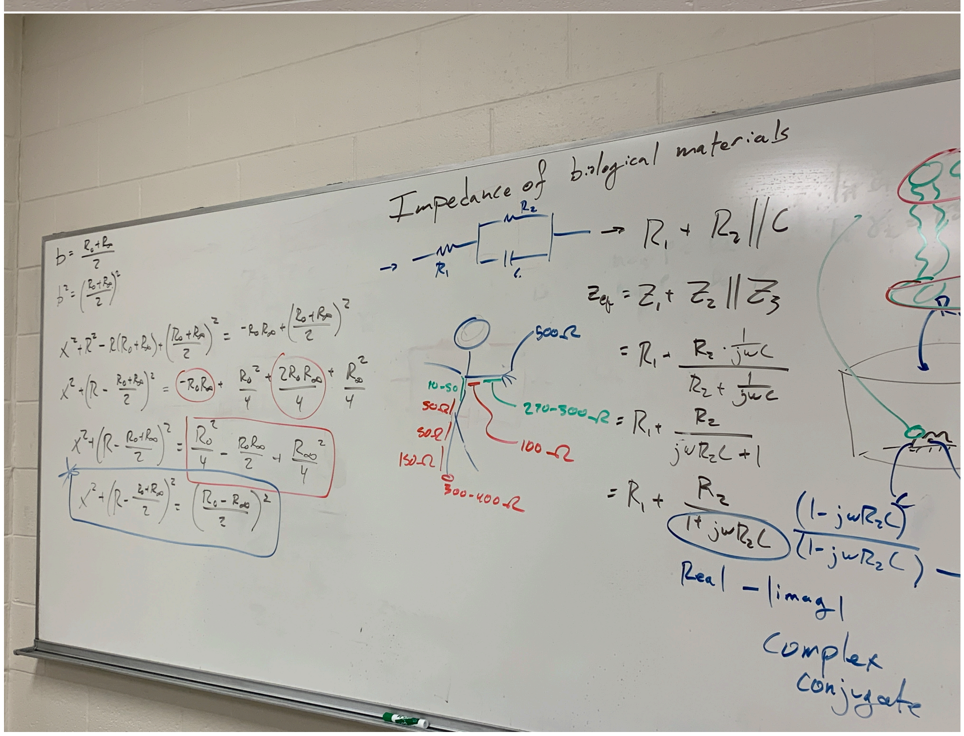
Measuring the bio impedance of the blood and using a control feedback loop to to supply the correct amount of blood back during blood loss. Could potentially be used for dialysis to take the correct amount of waste out of the body.

17.6 Board Pictures (4/2/19)






R₀ = R₁ + R₂
R∞ = R₁
τ = 1
R₁ = 10 kΩ
R₂ = 20 kΩ
C = 50 μF
CF = 1/τ
FWHM
Cole-Cole
|Z|


Impedance of biological materials
R₁ + R₂ || C
Z_eq = Z₁ + Z₂ || Z₃
500 Ω
100 Ω
150 Ω
300-400 Ω
Real − Imag
Complex
Conjugate

Chapter 18

Bioelectricity: II. Active properties

04/04/2019

Contents

18.1 How electrical potential arises in a cell

As we know, cells are far from stagnant. There is a constant movement of molecules and ions in and out of the cell depending on that cell's needs. When an uneven distribution of charges occur inside and outside the cell, an ion gradient is formed, which in turn creates a potential across the membrane of the cell.

18.1.1 Sodium-potassium pump

- Active element
- Only pump we have that consumes energy in the form of ATP (from food)
- The pump utilizes an ATPase
- Pumps 3 sodium ions out of the cell and 2 potassium ions into the cell
  - This aids in creating a net negative charge within the cell and a net postive charge outside the cell. This charge separation creates a negative transmembrane potential.

The purpose of the sodium-potassium pump in many cells is to transport these ions across the membrane in order to reach some kind of threshhold required by the cell to move other larger particles in and out. This is an active pump, meaning it requires the input of energy to function (i.e. food). This pump accomplishes the movement of 3 Na+ ions out of the cell for every 2 K+ ions moved in, creating an overall negative charge inside the cell and a relatively positive one outside, creating a potential across the membrane.

18.1.2 Membrane capacitance

- Passive element
- approximately 1 μF/cm^2
- High capacitance is needed for energy storage across membranes
- Determines temporal voltage response (Activation)

The cell membrane itself has a capacitance of about 1 microFarad per centimeter squared. This is a relatively high capacitance, required in order to store energy across the membrane. The capacitance determines the time response of voltage across the cell membrane.

18.1.3 Potassium ion channel

- Only allows potassium through it
- Opens/Closes dynamically and stochastically (random with respect to time)
- Contains 4 gating elements (all N gates)
  - Open when there is a positive membrane potential

This channel, as you may guess, transports K+ ions. It has dynamically opens and closes stochastically, which in layman's terms means at random intervals of time. This pump contains four "N gates" that control the opening and closing of the channel, typically opening at times when the transmembrane potential is positive.

18.1.4 Sodium ion channel

- Only allows sodium through it
- Mostly closed at rest
- Opens/Closes based on time
- Contains 4 gating elements (3 M and 1 H gate)

Another surprise, the sodium ion channel transports sodium ions across the cell membrane. In their resting and inactive positions, they allow no ions through. This state is reached over time, after a trigger causes the gate to become activated. It is only in their activated state that Na+ is allowed through the channel. There are again four gates, 3 "M gates" and 1 "H gate".

18.1.5 Leakage channels

- Small channels that allow the passage of small particles
  - Typically chloride
- Stabilizes voltage changes

This subsection of channels includes all other channels that transport small charged particles across the membrane (such as chloride, for example). It is thanks to these "others" that quick changes in potential between the inside and outside of a cell can be readily stabilized.

18.2 Determining resting potential from a Kirchhoffian perspective

18.3 Electrocardiography

Each of the previously discussed channels separate ions across the membrane, and have a gradient across them, which in turn creates an ionic current. The membrane current as a whole can be described as:

$$I_m = I_k + I_N a + I_l eakage + I_c + I_p \tag{18.1}$$

Each of these individual terms are of the form:

$$I_k = g_k * (V_m - E_k) \tag{18.2}$$

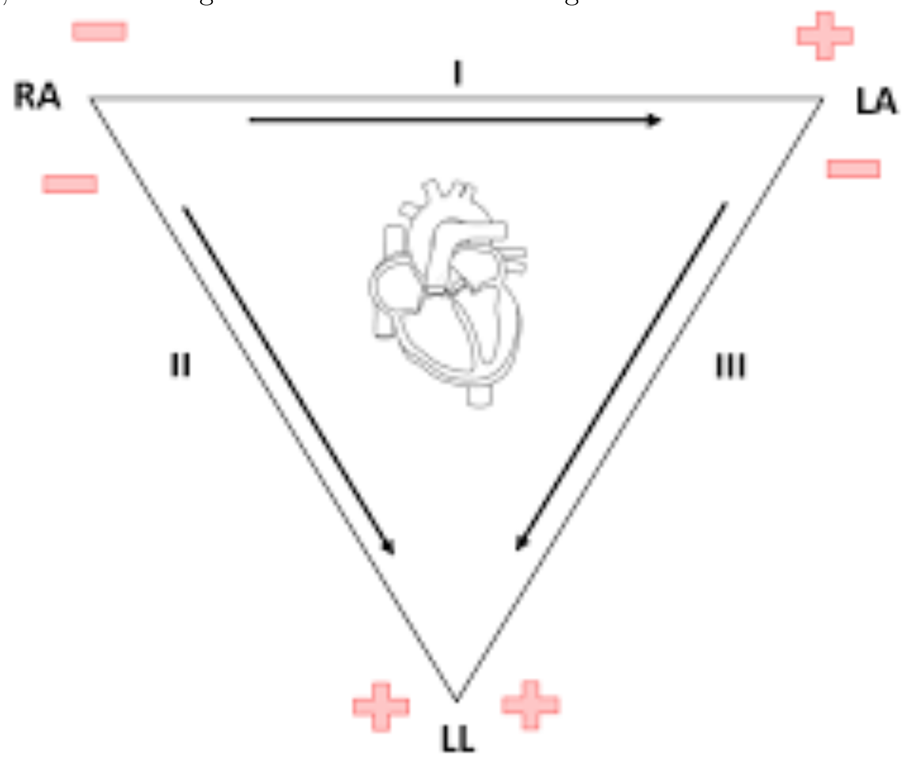
where g_k is the conductance of potassium, V_m is the membrane voltage at rest, and E_k is the Nernst potential. In order to find the membrane potential using all of the channels known to be present in the cell, use the following equation:

$$V_m = \frac{g_k * E_k + g_N a * E_N a + g_l eakage * E_l eakage...}{g_k + g_N a + g_l eakage...} \tag{18.3}$$

These equations can be applied to the human heart. Cardiomyocyte contractions are caused by the changes in membrane potential. Due to the uneven distribution of charges, a dipole arises, and we are able to measure this via leads and Einthoven's triangle.

18.3.1 Leads and Einthoven's triangle

Einthoven's triangle is an imaginary triangle made with three limb leads, the right arm, left arm, and the left leg. It forms an inverted triangle centered



around the heart.

- Lead I goes from right shoulder to left shoulder with negative electrode on right shoulder and positive on the left one. Lead I = left arm - right arm

- Lead II goes from the right arm and reaches to the leg on the left side with negative electrode on the shoulder and positive electrode on the leg. Lead II = left leg - right arm

- Lead III goes from the left shoulder, having a negative electrode, to the left leg with a positive electrode. Lead III = left leg - left arm

Only two leads are needed to characterize a vector. The heart at the center will produce a zero potential when the voltages are summed. According to Kirchoff's Voltage Law, I+III-II = 0.

18.3.2 The P-wave

The P-wave indicates atrial depolarization. This occurs when the sinoatrial node initiates an action potential over the atria. The Na+ channels in atrial cells open and become depolarized. The P-wave becomes visible on lead II.

18.3.3 The Q-wave

The Q-wave occurs next, as the impulse travels through the arterioventricular bundle and its branches. The impulse goes to the purkinje fibers. It indicates ventricular depolarization, which travels "up" the heart, making the Q-wave negative and create a pulse downwards.

18.3.4 The R-wave

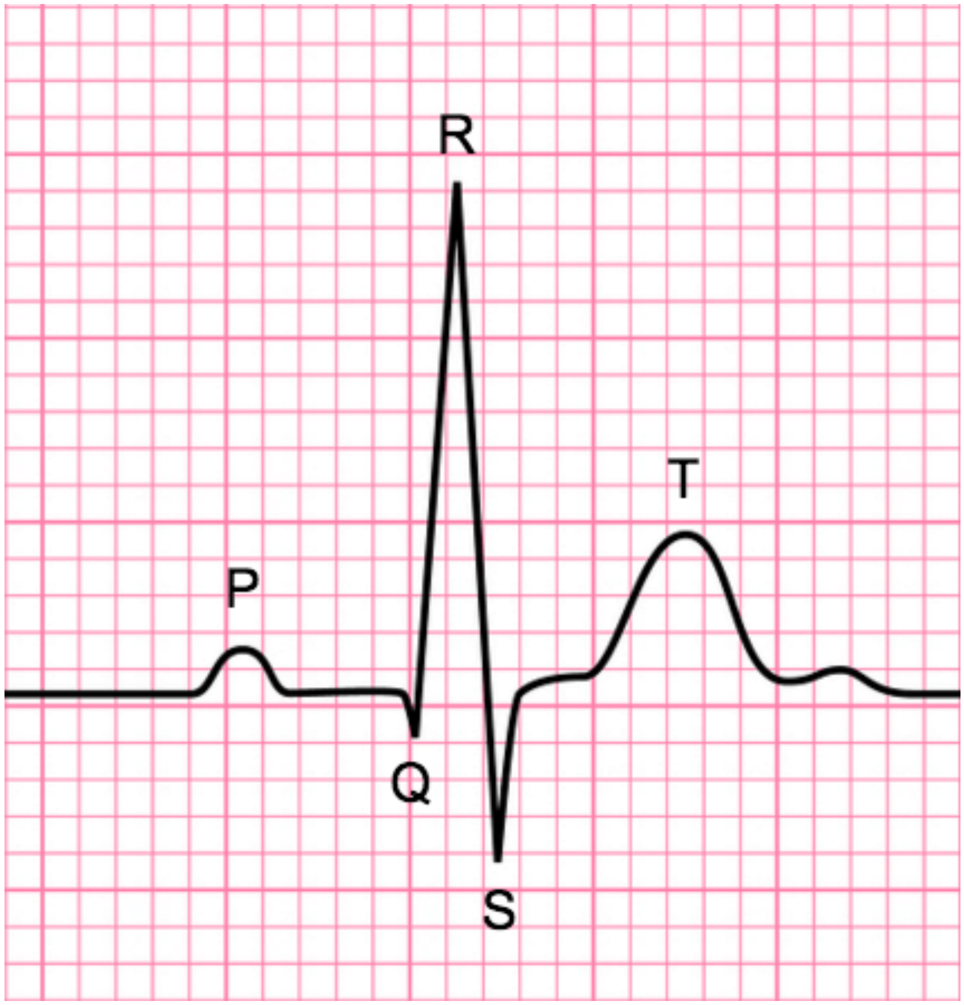
The R-wave is the largest wave in the complex. It represents the electrical stimulus as it passes through the main portion of the ventricular walls. This occurs when impulse spreads to the contricle fibers of the ventricles, and potassium starts to flow into the ventricle cells. This is the first phase of action potential.

18.3.5 The S-wave

The S-wave is the downward pulse that follows the R-wave, but may not be present in all ECG leads on a given patient. The S-wave represents depolarization in the purkinje fibers, and the ventricles contract. This is the second and third phase of action potential. The S-wave is visible on lead II.

18.3.6 The T-wave

The T-wave represents the repolarization of the ventricles. It occurs when ventricles contract, like for the S-wave, and the impulse spreads through the heart. This causes a three-dimensional shift of the heart's dipole. The T-wave is also visible on lead II.



18.4 Basic hardware engineer's concerns

1. Sensing electrodes
   - Interface between subject and the system (usually made with Ag/AgCl- Gel coupled materials)
2. Protection circuit
   - Protects the device from high voltage Example: Voltage follower
3. Lead selector

- Compares different sets of leads
- Can be used to scrutinize a particular lead

4. Calibration Signal
   - Send in a small impulse to figure out the transfer function of the system
   - Transfer function may vary with time, so this is essential to calibrate the system
5. Preamplifier
   - Amplifies ECG signal
   - High input impedance
   - High common-mode-rejection ratio
6. Driver amplifier
   - Filters ECG signal using a bandpass filter (to get rid of low frequency drift and high frequency noise)
7. Driven-reference circuit
   - Gives a reference point (ground) on the human body (usually the right leg is used)
8. Isolation circuit
   - Protects the patient from dangerous currents
   - Galvanic Isolation used in this class
9. Power supply

18.5 Basic software engineer's concerns

1. Analog digital conversion
2. Memory
3. Microcomputer/microcontroller
4. Control program
5. ECG analysis program

6. Recorder
7. Display
8. User input

18.6 Modeling a neuron

BME 211 Hodgkin Huxley Simulation

April 12, 2019

1 Analyzing Voltage Channels with the Hodgkin-Huxley Model

1.0.1 Abstract

Neurons, the cells in our brain that transmit nerve impulses, use the principles of Voltage, Conductance, Current, and Capacitance to connect our nervous system to our bodies, allowing for instantaneous responses to the environment. Each neuron cell is composed of dendrites, axons, and synapses where a "signal" is received, transported down the cell, and transmitted to neighboring cells, respectively. These "signals" are actually just depolarization and repolarization of the cell membrane from the movement of potassium and sodium ions, which causes a potential difference across the inner and outer spaces; the depolarization and repolarization represent a rise and fall in voltage which is called the Action Potential. The 1963 Nobel Prize in Physiology or Medicine was awarded to Alan Hodgkin and Andrew Huxley, who modeled Action Potentials by relating biological components to circuit components, allowing for a deeper understanding of how neurons work.

1.0.2 Background

The Neuron The human body is an evolutionary marvel, and its functionality is a culmination of the natural sciences. One of the most important features of the human body is its ability to communicate with itself in response to change, which is primarily governed by the Nervous System. The Nervous system is a network of fibers that transmits a signal from one part of the body to another, and is made up of the brain and spinal cord. The cells that do the transmitting are called neurons, and they transmit signals by sending electrical impulses, called Action Potentials, to a desired location. There are three functional parts of the neuron that are integral to understanding how they work: the cell body, dendrites and the axon (Figure 1). The cell body is the control center of the neuron where all inputs are gathered processed. Dendrites are an extension of the cell body that gather input from other neurons while the axon sends signals to other neighboring neurons. When a signal arrives at the dendrites on the left side of the neuron, the stimuli given to the dendrites are integrated at the cell body, which can then generate a nerve impulse (Action Potential). The impulse travels down the axon to the right side of the neuron, where it "jumps" to another set of dendrites and the process repeats.

Resource: https://www.khanacademy.org/test-prep/mcat/organ-systems/neuron-membrane-potentials/a/neuron-action-potentials-the-creation-of-a-brain-signal

Action Potential Neurons, like all cells, have a membrane that separates the inside of neuron from the outside. Potentials are able to be measured between the inside and outside because of a concentration gradient, which is created when there is a difference in the amount of ions inside the neuron than outside. Concentration gradients in the body often involve a difference in concentration of sodium and potassium. The Action Potential is simply a shift in the neuron's membrane potential (from negative to positive) that occurs when ions start flowing in or out of the cell through "gates," known as ion channels. The gates depend on voltage, and will open and close accordingly in response to the potential of the cell. Before any signal is received, the cell sits at a constant potential called the Resting Potential (experimentally measured to be -70 mV). Under typical situations, the Action Potential propagates as follows: 1. A signal is received from a cell, and the cell becomes depolarized, meaning positive charge flows from the outside of the cell to the inside of the cell. Once a certain potential is reached (experimentally measured to be -55mV), an Action Potential will begin to fire, and a gate will activate letting a larger amount of positive charge flow in (Sodium) 2. When another positive threshold is reached, the first gate is closed and another gate will open causing positive charge (Potassium) to flow out of the cell in a process called polarization 3. After another threshold, the final gate will close and the cell will regress to its Resting Potential Again

It is important to note that the Action Potential only fires when the specific thresholds are met, known as the "All or Nothing" principle.

Resource: https://en.wikipedia.org/wiki/Hodgkin%E2%80%93Huxley_model https://neuronaldynamics.epfl.ch/online/Ch2.S2.html

Hodgkin and Huxley Alan Hodgkin and Andrew Huxley, two British born scientists, authored a series of five papers describing nonlinear ordinary differential equations that model how action potentials can be initiated and propagated through an axon. The papers detailed their research into the squid giant axon, which has an abnormally large axon (1mm in diameter) big enough to conduct experiments on. This ground-breaking work has a wide range of applications on many organisms and won Hodgkin and Huxley the Nobel Prize in Physiology in 1963.

Resource: https://www.swarthmore.edu/NatSci/echeeve1/Ref/HH/HHmain.htm

1.0.3 Setup

In order to describe the way Action Potentials propagate, Hodgkin and Huxley related the membrane of the neuron to a circuit (Figure 2). The membrane has a capacitance C_m, and accounts for all of the gates that were described previously. The Sodium and Potassium gates are voltage dependent, and are represented in terms of their conductances. To find the total current of ions flowing from inside to outside as a function of time, Kirchhoff's rules and the Ohm's Law were used. The capacitance of the membrane is a constant, but by virtue of ions moving, the voltage is not. Using the relationship of capacitance and charge, the membrane current can be solved as such:

$$Q = C_m * V_m$$

$$\frac{dQ}{dt} = C_m * \frac{dV}{dt}$$

Where V_m is the intracellular voltage

For the gates, Ohm's Law describes how the conductance (which is $\frac{1}{R}$) and voltage relate to current passing in and out of the membrane. Because the sodium and potassium ions themselves have voltages (represented by batteries in each branch), the voltage term in the Ohm's Law equation is the voltage of the membrane (V_m) - the Nernst potentials of the ion (E_{Na}, E_K). The third gate is known as a leak channel, and is a gate that allows sodium and potassium to pass in order to maintain the resting potential, and is not voltage activated. The values of the conductance for sodium and potassium are dependent on the membrane voltage by formulas derived by Hodgkin and Huxley through their experiments. The equations are:

$$I_{Na} = g_{Na} * m^3 * h * (V_m - E_{Na})$$
$$I_K = g_K * n^4 * (V_m - E_K)$$
$$I_L = g_L * (V_m - E_L)$$

where:

$$\frac{dn}{dt} = \alpha_n(V_m) * (1 - n) - \beta(V_m) * n$$

$$\frac{dm}{dt} = \alpha_m(V_m) * (1 - m) - \beta(V_m) * m$$

$$\frac{dh}{dt} = \alpha_h(V_m) * (1 - h) - \beta(V_m) * h$$

The values of n, m, and h describe the probability that a gate will be open at a given time. Note that the Leak current isnt bound by any of these probabilities because it is not voltage gated. The values of n, m, and h are values between zero and one. α and β are described by the equations:

$$\alpha_n(V) = \frac{.01 * (10 - V)}{e^{\frac{10-V}{10}} - 1}$$

$$\beta_n(V) = .125 * e^{\frac{-V}{80}}$$

$$\alpha_m(V) = \frac{.01 * (25 - V)}{e^{\frac{25-V}{10}} - 1}$$

$$\beta_m(V) = 4 * e^{\frac{-V}{18}}$$

$$\alpha_h(V) = .07 * e^{\frac{-V}{20}}$$

$$\beta_h(V) = \frac{1}{e^{\frac{30-V}{10}} + 1}$$

Finally, because Kirchoff's Loop rule states that the sum of current in must equal the sum of the current out, the total current is equal to:

$$I_{ext} = I_{Na} + I_K + I_L + C_m * \frac{dV}{dt}$$

My model uses Eulers method to solve the nonlinear differential equations that describe the current flow of sodium and potassium. First, I created variables to describe the simulation time of my Action Potential. Second, I created a cell filled with tests of different currents which I control to see what impact the external current has on the Action Potential. Next, I used a for loop that goes through each time step in my simulation and calculates the value of each equation, and updates the array at the given i value. Finally, I graph the conductances of sodium and potassium, and the Action Potential. I repeated the process for each testing current and observed how the graphs changed with different magnitudes of current, and different deliveries.

To judge the relative success/failure of the tests, it is important to note what constitutes a "correct" graph. If the plotted graph has a sharp spike that goes up, and then down past the resting potential, and exhibits repeating behavior, that means it follows a typical action potential and is a success.

Resource: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC1392413/pdf/jphysiol01442-0106.pdf

In [228]:
```
## Import usual libraries
import numpy as np                          ## numpy is a library that inclues most of
import matplotlib.pyplot as plt             ## this is the library we use to plot
from IPython.display import Image           ## this is to import images from a file
```

In [88]:
```
# Create a time sufficient enough to deliver the impulse across the system
sim_time = 100
dt = .01
times = np.arange(0,sim_time,dt)
Ntimes = len(times)
```

In [299]:
```
## Set variable for the membrane current equal to values to simulate current inside of
IextArray = np.zeros(Ntimes)
current_1 = 20 # A current value that is within range of normal neuron Action Potentia
current_2 = 3 # A current that is much less than normal  neuron Action Potential condi
current_3 = 100 # A current that is much higher than the normal neuron Action Potentia

## Current Test 1: Constant current with each of the different testing values
#IextArray[0:Ntimes] = current_1
#IextArray[0:Ntimes] = current_2
#IextArray[0:Ntimes] = current_3

## Current Test 2: Step current with steps the size of the testing values
#IextArray[0:500] = current_1
#IextArray[501:2000] = 0
#IextArray[2001:Ntimes] = current_1

#IextArray[0:500] = current_2
#IextArray[501:2000] = 0
#IextArray[2001:Ntimes] = current_2

#IextArray[0:500] = current_3
#IextArray[501:2000] = 0
#IextArray[2001:Ntimes] = current_3


## Current Test 3: Sinusoidal current with amplitude the size of the testing values
#IextArray = current_1*np.sin(times)
```

```
#IextArray = current_2*np.sin(times)
IextArray = current_3*np.sin(times)

plt.plot(times, IextArray)
plt.xlabel('Time (ms)')
plt.ylabel('Current (mA)')
```

Out[299]: Text(0,0.5,'Current (mA)')

In [300]:
```
## These constants are used in the formulas above and were calculated from the researc
gbar_K = 36 ## conductance of the potassium channel
gbar_Na = 120 ## conductance of the sodium channel
g_L = .3 ## conductance of the leak channel
E_K = -12 ## Voltage associated with potassium
E_Na = 115 ## Voltage associated with sodium
E_L = 10.6 ## Voltage associated with the leak
C = 1 ## Membrane capacitance
```

In [301]:
```
## These are the arrays necessary to hold the values from the for loop
VArray = np.zeros(Ntimes)
I_NaArray = np.zeros(Ntimes)
I_KArray = np.zeros(Ntimes)
I_LArray = np.zeros(Ntimes)
I_totalArray = np.zeros(Ntimes)
a_nArray = np.zeros(Ntimes)
b_nArray = np.zeros(Ntimes)
a_mArray = np.zeros(Ntimes)
b_mArray = np.zeros(Ntimes)
a_hArray = np.zeros(Ntimes)
b_hArray = np.zeros(Ntimes)
nArray = np.zeros(Ntimes)
mArray = np.zeros(Ntimes)
hArray = np.zeros(Ntimes)
```

In [302]:
```
## Create a loop to simulate the Action Potential
VArray[0] = 0 ## Initialize the Voltage to 0. This will be brought to its experimental
a_nArray[0] = .01*((10-VArray[0])/(np.exp((10-VArray[0])/10)-1)) ## Formula for alpha 
b_nArray[0] = .125*np.exp(-VArray[0]/80) ## Formula for beta n with initial value for 
a_mArray[0] = .1*((25-VArray[0])/(np.exp((25-VArray[0])/10)-1)) ## Formula for alpha m
```

```
b_mArray[0] = 4*np.exp(-VArray[0]/18) ## Formula for beta m with initial value for vol
a_hArray[0] = .07*np.exp(-VArray[0]/20) ## Formula for alpha h with initial value for
b_hArray[0] = 1/(np.exp((30-VArray[0])/10)+1) ## Formula for beta h with initial value
nArray[0] = a_nArray[0]/(a_nArray[0] + b_nArray[0]) ## Formula for initial n
mArray[0] = a_mArray[0]/(a_mArray[0] + b_mArray[0]) ## Formula for initial m
hArray[0] = a_hArray[0]/(a_hArray[0] + b_hArray[0]) ## Formula for initial h

for i in range(Ntimes-1):
    a_nArray[i+1] = .01*((10-VArray[i])/(np.exp((10-VArray[i])/10)-1)) ## The followin
    b_nArray[i+1] = .125*np.exp(-VArray[i]/80)                          ## in the form
    a_mArray[i+1] = .1*((25-VArray[i])/(np.exp((25-VArray[i])/10)-1))
    b_mArray[i+1] = 4*np.exp(-VArray[i]/18)
    a_hArray[i+1] = .07*np.exp(-VArray[i]/20)
    b_hArray[i+1] = 1/(np.exp((30-VArray[i])/10)+1)

    I_NaArray[i+1] = (mArray[i]**3)*gbar_Na*hArray[i]*(VArray[i] - E_Na) ## Use the eq
    I_KArray[i+1] = (nArray[i]**4)*gbar_K*(VArray[i]-E_K)                ## Individual
    I_LArray[i+1] = g_L*(VArray[i]-E_L)                                  ## the total
    I_totalArray[i+1] = IextArray[i] - I_KArray[i] - I_NaArray[i] - I_LArray[i]

    VArray[i+1] = VArray[i] + dt*I_totalArray[i]/C                             ##
    nArray[i+1] = nArray[i] + dt*(a_nArray[i]*(1-nArray[i])-b_nArray[i]*nArray[i])  ##
    mArray[i+1] = mArray[i] + dt*(a_mArray[i]*(1-mArray[i])-b_mArray[i]*mArray[i])  ##
    hArray[i+1] = hArray[i] + dt*(a_hArray[i]*(1-hArray[i])-b_hArray[i]*hArray[i])
VArray = VArray - 70 ## Set the voltage equal to the resting potential
print(np.max(VArray)) ## See how action potential peak changes
```

```
41.30030671753612
```

In [303]:
```
## Plot the conductances for Potassium and Sodium
plt.plot(times,gbar_K*nArray**4)
plt.plot(times, gbar_Na*mArray**3*hArray)
plt.xlabel('Time(ms)')
plt.ylabel('Conductance(mS)')
plt.title('Potassium and Sodium Conductances')
plt.legend('K')
```

Out[303]: <matplotlib.legend.Legend at 0x23b953959b0>

```
In [304]: ## Plot the Action Potential
          plt.plot(times, VArray)
          plt.xlabel('Time(ms)')
          plt.ylabel('Voltage(mV)')

Out[304]: Text(0,0.5,'Voltage(mV)')
```

1.0.4 Results & Conclusion

Current Test 1: Constant Current Current_1: The graphs that resuted from this case exhibited Action Potential behavior. The spikes in potential occured relatively at the same time as the spikes in the conductances, meaning that the peak potential was occuring when the conductances were activated. This is consistent with the biology of the model because the Action Potential occurs when the gates open, allowing ions to flow in.

Current_2: The graphs that resulted did not exhibit the Action Potential behavior. Because the external current is so small, the voltage variable in the code was affected and thus the rest of the code since the alpha and beta values depended on voltage. The current allows for one Action Potential to fire, but as the voltage lowers, it will not reach the "all or nothing" threshold so the gates will not open and the Action Potential cannot propagate any futher.

Current_3: The graphs that resulted from this case exhibited Action Potential behavior. Like in current_1, the peaks in conductance and potential occur almost simulataneously, but now the amplitudes are different. The current_3 amplitudes are smaller than the current_1 because the alpha and beta values exhibit different behaviors as V becomes increasing large (which again, is proportional to external current getting larger). That is also why this is the only graph where the conductance of Potassium is greater than that of sodium.

It is important to note that the each graph has an initial peak that is large compared to the peaks that it converges to. This happens because initially the voltage is responding to an initial Action Potential that passes the "All or Nothing" threshold but as time passes the values of potential, alpha, and beta respond to the changes. The simulation time has a dt of .01, so change isnt apparent immediately.

Current Test 2: Step Current Current_1: The graphs that resulted from this case exhibited Action Potential behavior. Like current_1 from the previous test, the peaks occur at the same times, but with one important distinction: There are two peaks that do not match the peaks after some time has passed (Reason for that is explained below).

Current_2: The graphs that resulted from this case exhibited Action Potential behavior for a few seconds, but ulitmately reverted to non-Action Potential behavior. The fact that two peaks still occur for both conductances and potentials is a consequence of the timing and type of change of the current. Because the step down from high current to 0 current occurs at the polarization state, it does not matter that the original external current is small, another Action Potential will fire regardless. This type of external current allows Action Potentials to be fired at least once regardless of small current which has applications in studying how signals transfer down the axon.

Current_3: The graphs that resulted from this case exhibited Action Potential behavior. The relationship between current_1 and current_3 for this test is the same as in the previous test.

In this test current is supplied for a very short amount of time, decreased to 0 for a while, then allowed to return back to the original supply. The small application of current leads to an Action Potential, then at 0 it is allowed to come back to resting and spike again, and after the value is constant allowing for the smaller peaks to propogate. For the conductances, the gates are opening and closing very quickly and it makes sense that there would also be two peaks because the gates opening is a response to the potential.

Current Test 3: Sinusoidal Current Current_1: The graphs that resulted from this case exhibited Action Potential Behavior. This time around, the peaks on the graph are all the same and it maintains the relative shape that is expected. The sodium conductance is much larger than the potassium and again the peaks seem to in phase (used this terminology now that sin is involved)

Current_2: The graphs that result from this case, by my definition, displays Action Potential Behavior, but in terms of the biology of the model would not be a plausible Action Potential. The graphs have lost their Action Potential shape and resorted to essentially the same shape as the current itself. Interestingly, the potential graph shows the potential alternating between -67 and -73 which is 3 plus and 3- the resting potential, respectively. A possible explanation to this is that average of sin is 0, so the average value of the conductances are 0 since they exhibit the same sinusoidal behavior. This means the total current is just Iext-Ileak which causes the behavior that is shown (Ileak is a constant, remember).

Current_3: The graphs that result from this case, by my definition, displays Action Potential Behavior, but in terms of the biology of the model would not be a plausible Action Potential. The graph is similar to the current_1 graph as we have seen in the previous tests, with one big difference: the inability to return to resting potential. What the graph does is repeat immediately after diping below the resting potential, which could be because the negative side portion of the sin curve makes the max current negative, and the potential cannot recover to the resting potential before reaching the threshold again.

The usuage of a current that is driven like this is unrealistic because the constantly changing value of I creates a constantly changing value in alpha and beta which means the conductances and Action Potentials will not be normal. I included this current to make two points 1. Unlike the circuits in Physics 260, which seem to be more efficient operating at alternating current, this model cannot work under these conditions. The known biology of Action Potentials does not account for such a current so it makes sense the model wouldnt either 2. Systems like this may not be within the scope of normal connditions, but could possibly describe a mutation of some sort. For example, mutations in signaling could cause the ions to flow in/out at such a current which would cause problems in transmission, in which case this model would not be very helpful since assumptions are made strictly in ideal conditions

1.0.5 Figures

```
In [229]: ## Figure 1
          Image(filename = "img/neuron.png", width = 400, height = 400)
```

Out[229]:

In [230]:
```
## Figure 2
Image(filename = "img/hodgkin.png", width = 400, height = 400)
```

Out[230]:

Chapter 19

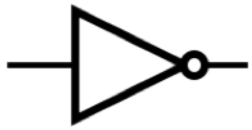
Digital circuits: I. Logic and digital components

04/09/2019

Contents

19.1 Digital components, "logic gates"

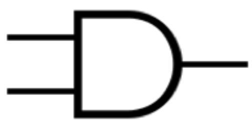
19.1.1 NOT



$$\overline{A} \text{ or } \neg A \tag{19.1}$$

Input	**Output**
A	¬ A
1	0
0	1

19.1.2 AND



$$A \cdot B \tag{19.2}$$

Input 1	**Input 2**	**Output**
A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

19.1.3 OR

$$A + B \tag{19.3}$$

Input 1	**Input 2**	**Output**
A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

19.1.4 NAND

$$\overline{A \cdot B} \text{ or } A \uparrow B \tag{19.4}$$

Input 1	**Input 2**	**Output**
A	B	A NAN B
0	0	1
0	1	1
1	0	1
1	1	0

19.1.5 NOR

$$\overline{A + B} \text{ or } A \downarrow B \tag{19.5}$$

Input 1	**Input 2**	**Output**
A	B	A NOR B
0	0	1
0	1	0
1	0	0
1	1	0

19.1.6 XOR

$$A \oplus B \tag{19.6}$$

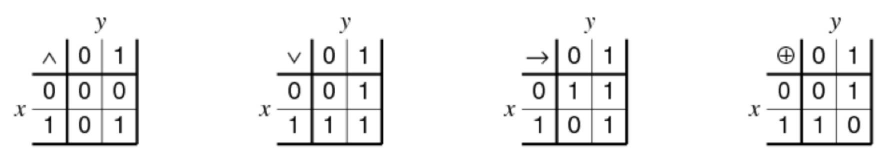
Input 1	**Input 2**	**Output**
A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

19.1.7 XNOR

$$\overline{A \oplus B} \text{ or} A \odot B \tag{19.7}$$

Input 1	**Input 2**	**Output**
A	B	A OR B
0	0	1
0	1	0
1	0	0
1	1	1

19.2 Representing logic graphically



$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\rightarrow$	0	1
0	1	1
1	0	1

$\oplus$	0	1
0	0	1
1	1	0

Figure 1. Truth tables

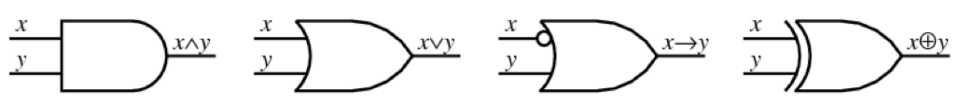


Figure 2. Logic gates

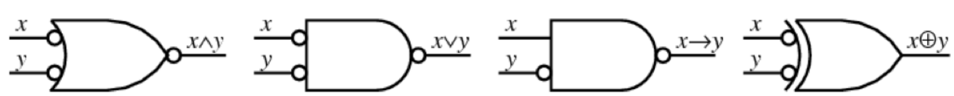


Figure 3. De Morgan equivalents

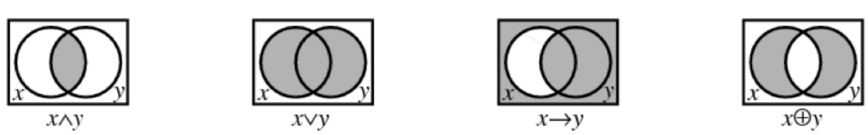
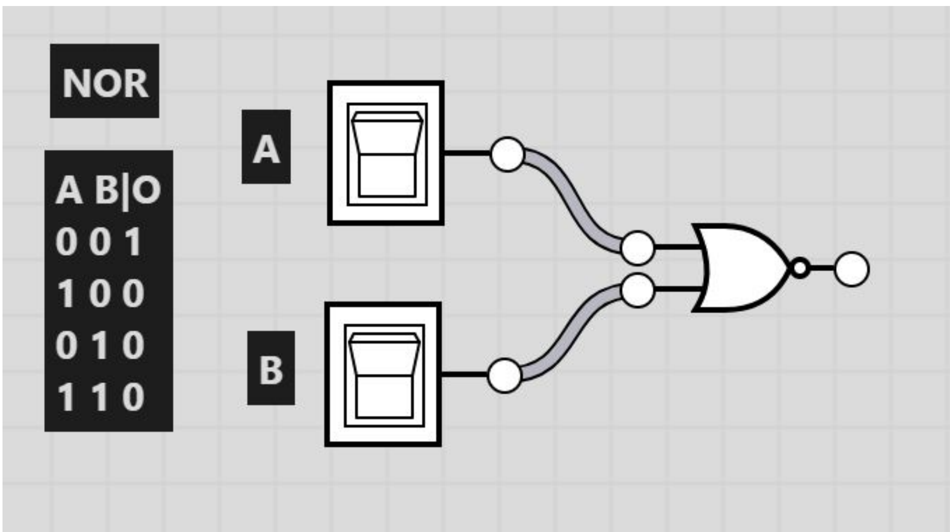


Figure 4. Venn diagrams

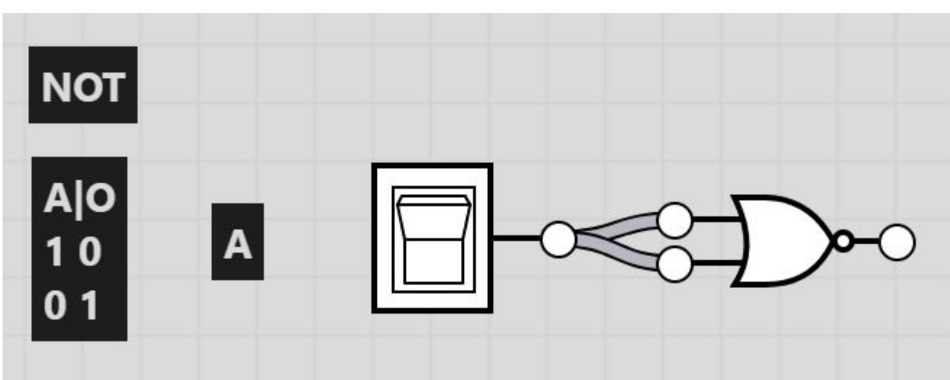
19.3 Functional completeness

Or, how to make everything from NORs.

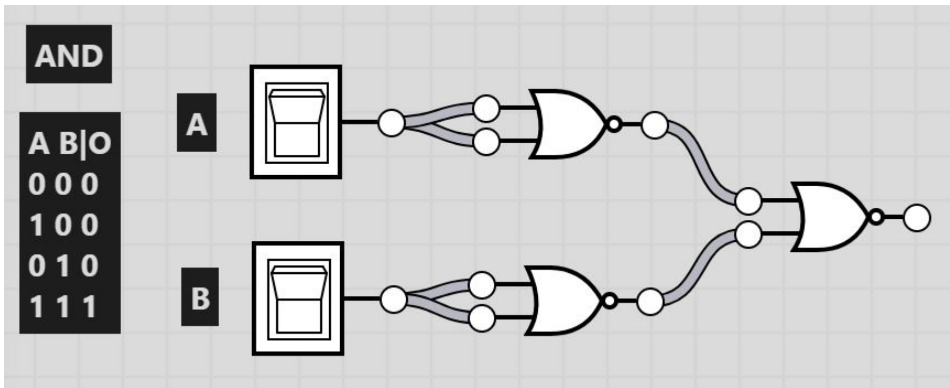
19.3.1 NOR from NOR



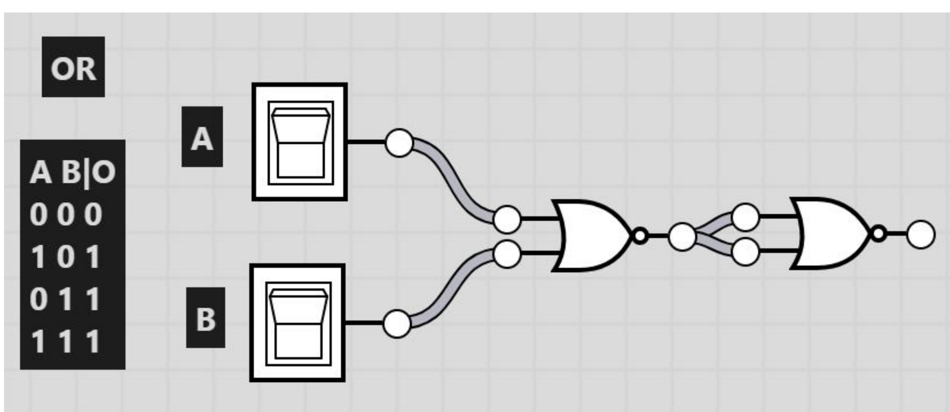
19.3.2 NOT from NOR



19.3.3 AND from NOR

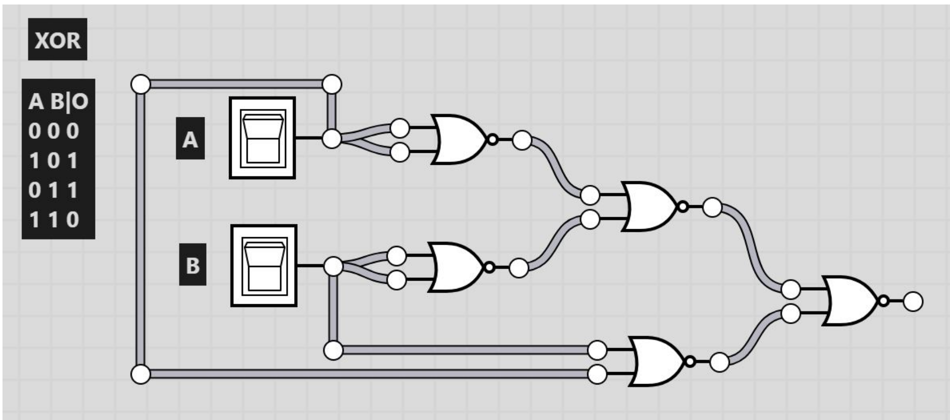


19.3.4 OR from NOR

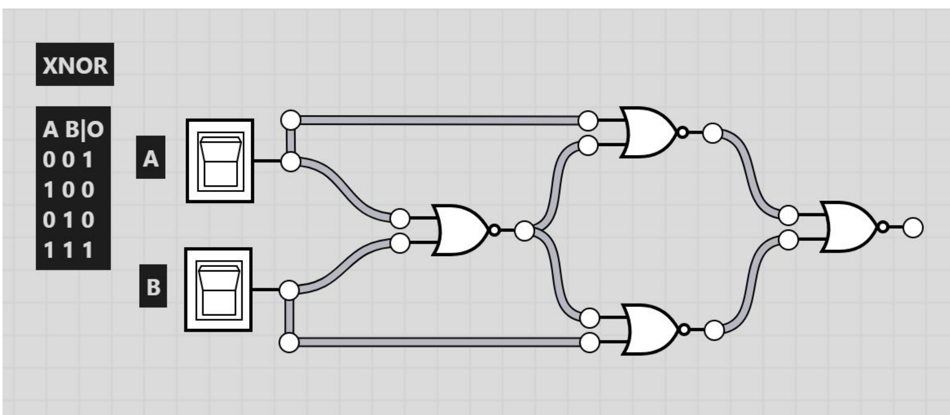


19.3.5 NAND from NOR

19.3.6 XOR from NOR



19.3.7 XNOR from NOR



19.4 Digital components

19.4.1 Why silicon?

- Given the lattice structure of silicon atoms, a silicon atom can bond to its four (4) neighbors.

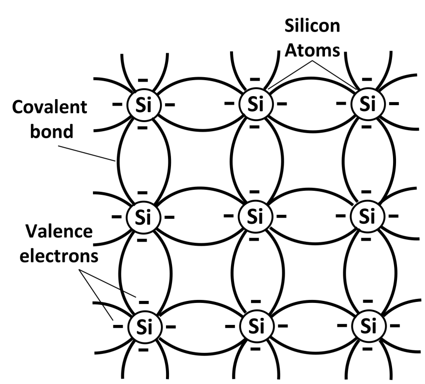
- Silicon has four (4) electrons in its valence shell.

- That enables the silicon atom to bind itself four (4) partners, covalently.

- If bound covalently to those neighbors, each of the silicon atom's electrons will exist in its valence band.

- In this case, the only electrons that can move around are free electrons (i.e., electrons that have absorbed some energy) and they would have a hard time at that. It therefore follows that crystalline silicon (typically) has low conductivity. *Pure silicon acts nearly as an insulator.*

- **Doping** is used to increase the performance of silicon, by mixing in an "impurity". These impurities interrupt the crystal and make conduction possible. Hence the term "semiconductors".



19.4.2 Doping

There are two types of doping:

N-type doping

Lets say you mix in a little bit of phosphorous with five (5) valence electrons. Now youve got one electron free to move around the system. Only a small amount (<< 1%) of impurity (e.g., phosphorus, arsenic) is needed to allow electric current to flow through the silicon. **N-type: electrons (a "1") can move around**

P-type doping

Let's now mix in a little boron or gallium (each having three (3) valence electrons. When mixed into the silicon lattice, they form "holes" where silicon has nothing to bond with. Holes have the effect of a positive charge and help conduct current by accepting electrons from a neighbor (thus moving the hole). **P-type: hole (a "0") can move around**

19.4.3 Diodes

- What would happen if we combined n-type silicon with p-type silicon
- We'd get a diode!
- How does a diode work?
  1. The abundant electrons present on the n-type semiconductor will tend to migrate over the boundary into the hole rich p-type semiconductor.

     (1s from n) → (0s in p)
  2. This will tend to make the p-side slightly more negative and it will make the n-type slightly more positive.

     (p ↓, n↑)
  3. Given sufficient time, a potential gradient will form sufficiently large to prevent any further natural migration of electrons

19.4.4 Transistors

- The electronics inn your pockets/bags contain *billions* of transistors
- Act as a switch with no moving parts
- Amplifies weak signals

- In its most basic sense, a transistor is just two of these diodes put back to back. (no matter how you flip the power source, looks the same.) This is known as a **bipolar junction transistor**.
- Different kinds of transistor (e.g., things like field-effect transistors) have different properties.

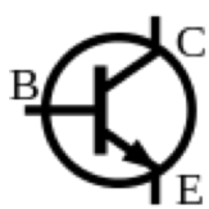
NPN

Let's begin by sandwiching a p-type semiconnductor between two n-types.

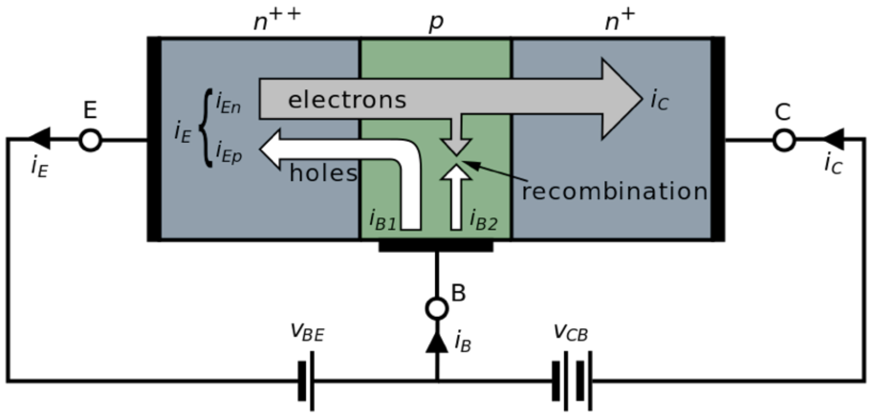
1. Let's connect a power supply through the whole thing
2. If we do that well have a reverse biasing circuit in either direction.
3. Let's also place another power supply between an N and a P.
4. This gives us a forward biasing diode! (positive to positive)
5. This will move electrons from the n region to the holes in the p region, this will cause some of the occupied holes electrons to be moved and circulate.
6. Not all electrons will flow through the one circuit, now they will also allow for the migration from the other (perhaps much larger) circuit
7. This gives us three important components that I can never really truly remember unless Im actually using the thing. But they are thus:

   You have a **small base current, B**, that can be amplified to a high **collector current, C**, by an **emitter, E**

   As a circuit element that looks like (NPN)

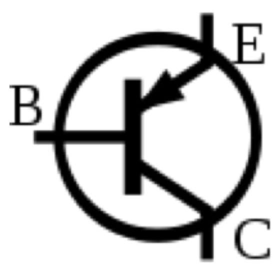
   

   You have a small base amplified to produce a large collector and emitter current. This will occur when there is a positive potential difference measured between the base to the emitter.



PNP

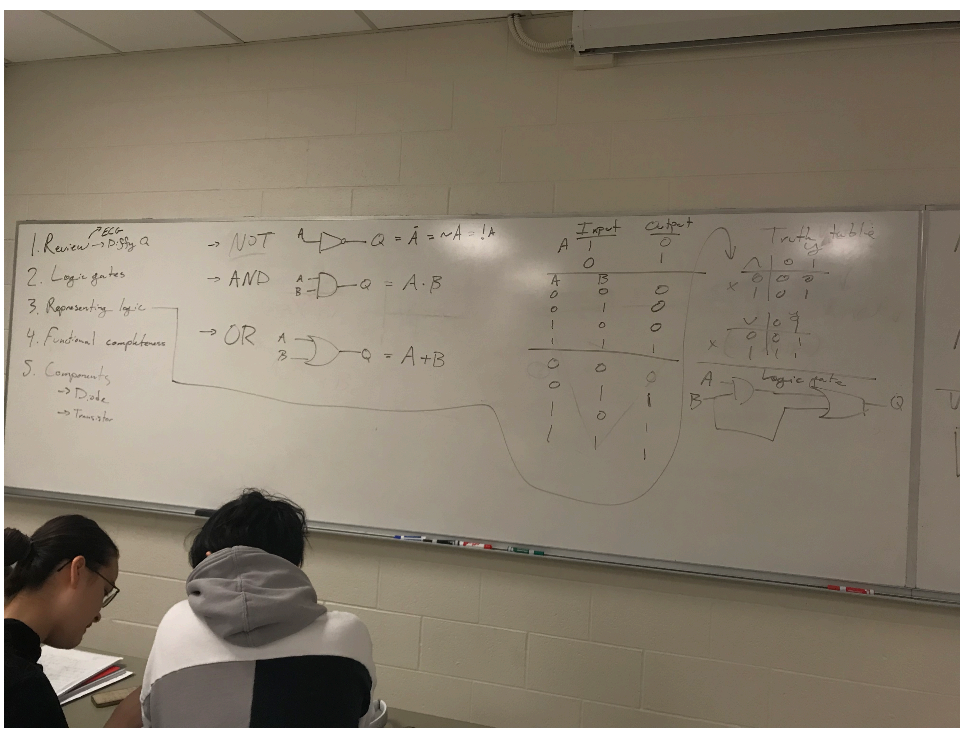
1. An n-type layer sandwiched between two layers of p-type material.
2. A small current leaving the base is amplified in the collector output.
3. A PNP transistor is on when the base is pulled low relative to the emitter
4. Holes are injected into the base as the minority carriers
5. Circuitly

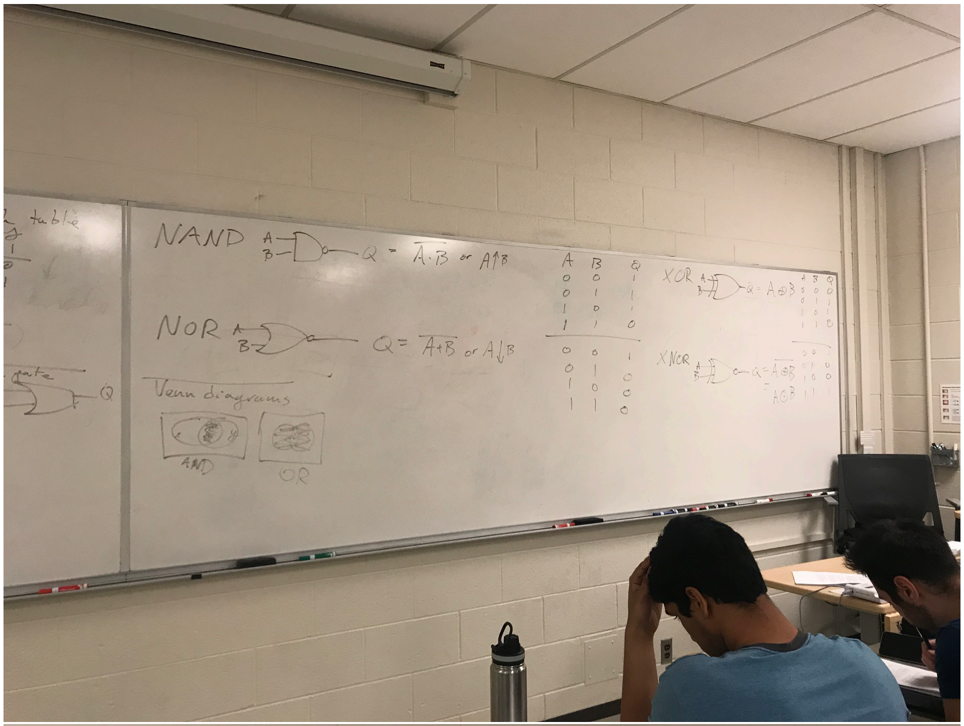


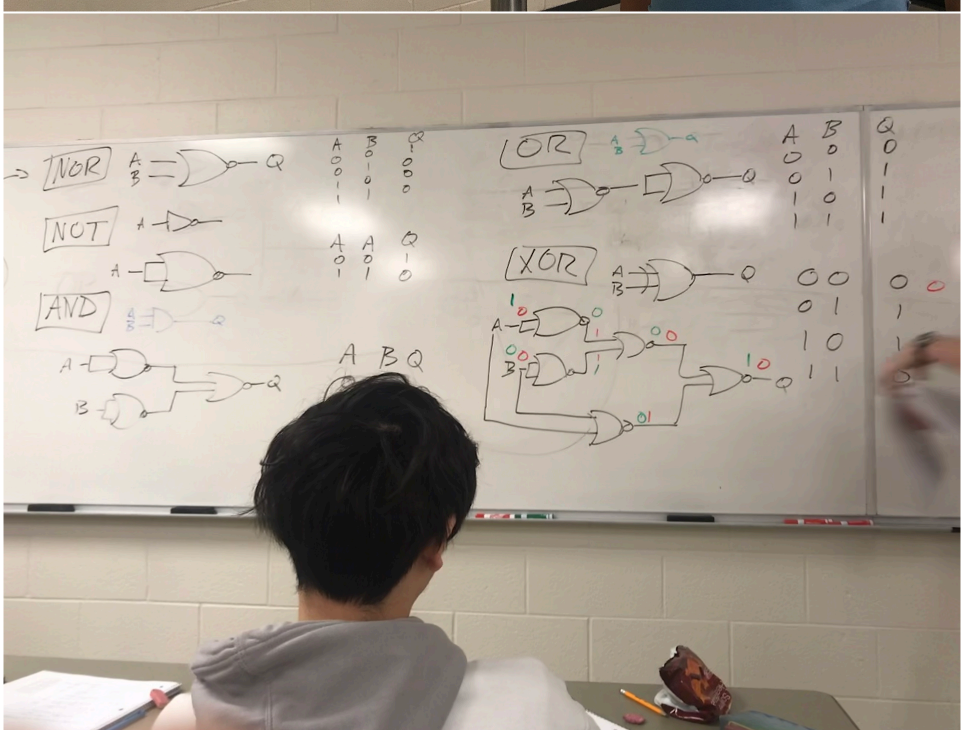
6. The arrows in the symbols indicate the PN junction between the base and the emitter
7. When the transistor is in the "forward saturated" / "active" / "forward active" mode, the arrow tells you which way the current is or very well ought to be going.

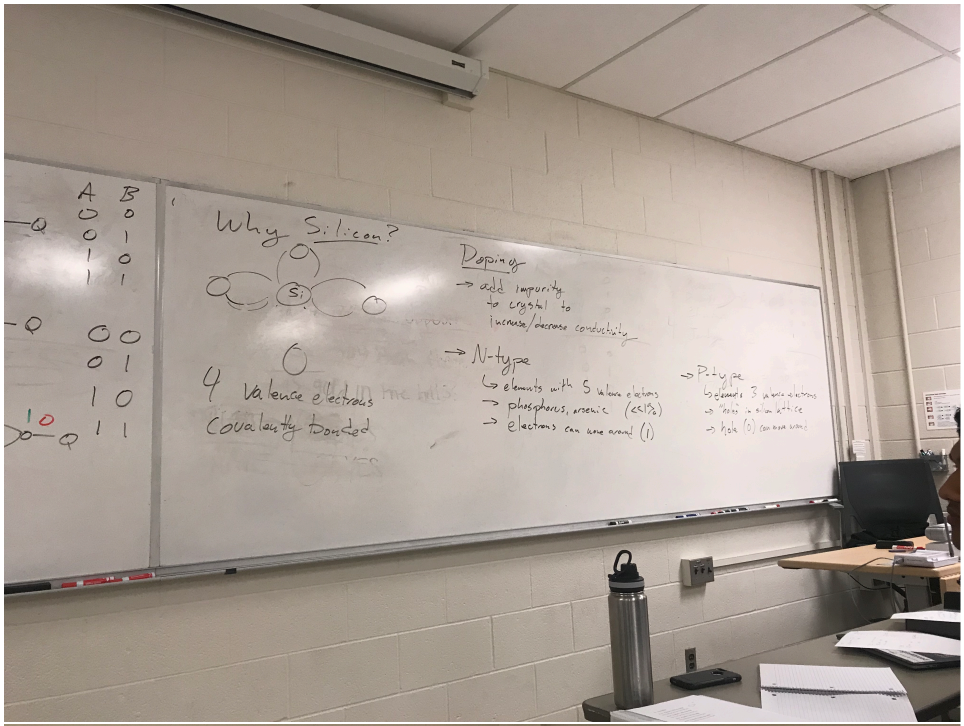
19.5 Board Pictures (4/9/19)

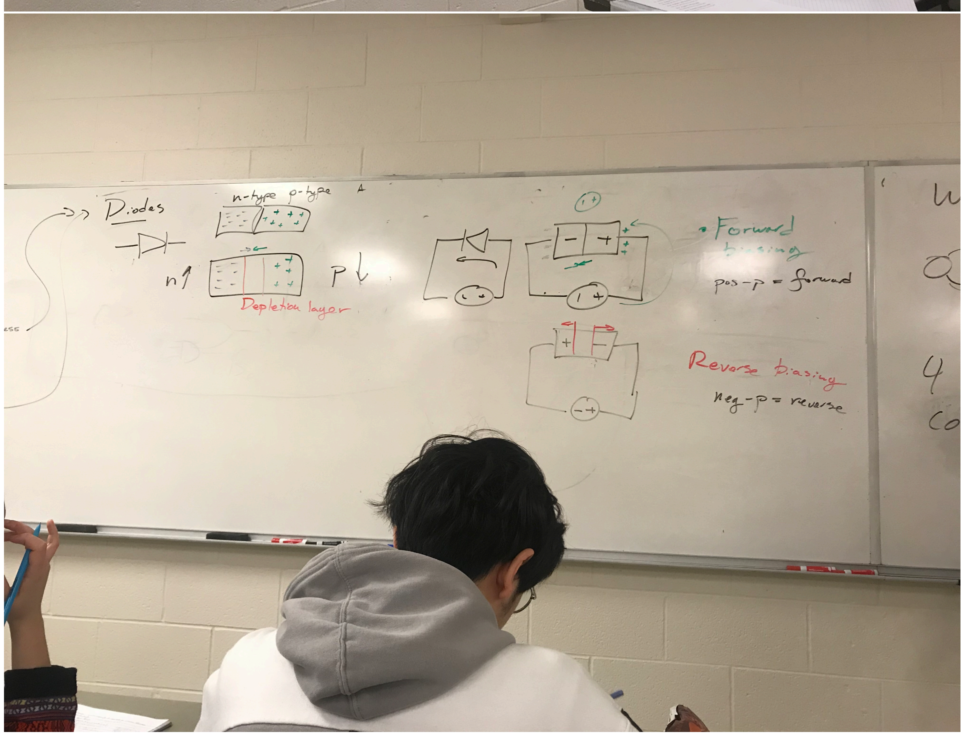
I also included my notes as some of the board pictures are not as clear as I had hoped.

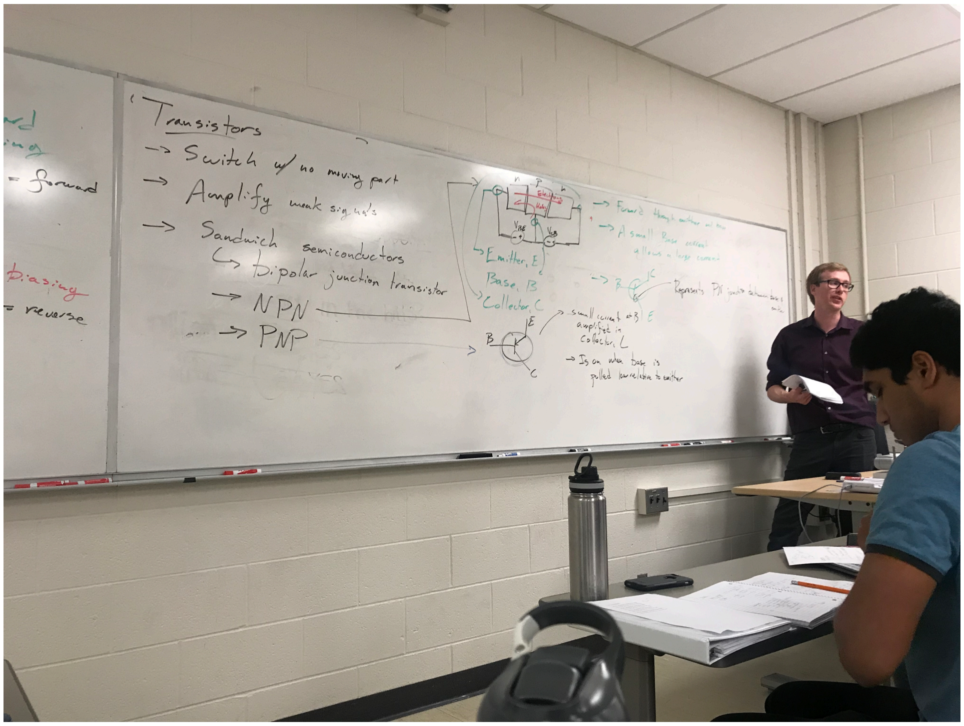



NAND
NOR
Venn diagrams
AND
OR
XOR
XNOR

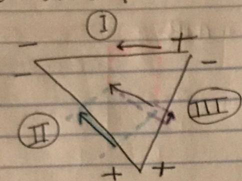

NOR
NOT
AND
OR
XOR

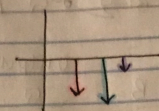

Why Silicon?
Si
4 valence electrons
Covalently bonded
Doping
→ add impurity to crystal to increase/decrease conductivity
→ N-type
→ phosphorus, arsenic
P-type


Diodes
n-type p-type
Depletion layer
Forward biasing
pos-p = forward
Reverse biasing
neg-p = reverse


Transistors
→ Switch w/ no moving part
→ Amplify weak signals
→ Sandwich semiconductors
Bipolar junction transistor
→ NPN
→ PNP
Emitter, E
Base, B
Collector, C

Lecture 19





Logic gates

NOT A → Q = $\bar{A}$ = ~A = !A
"not A"

input	output
1	0
0	1

AND A, B → Q = A·B

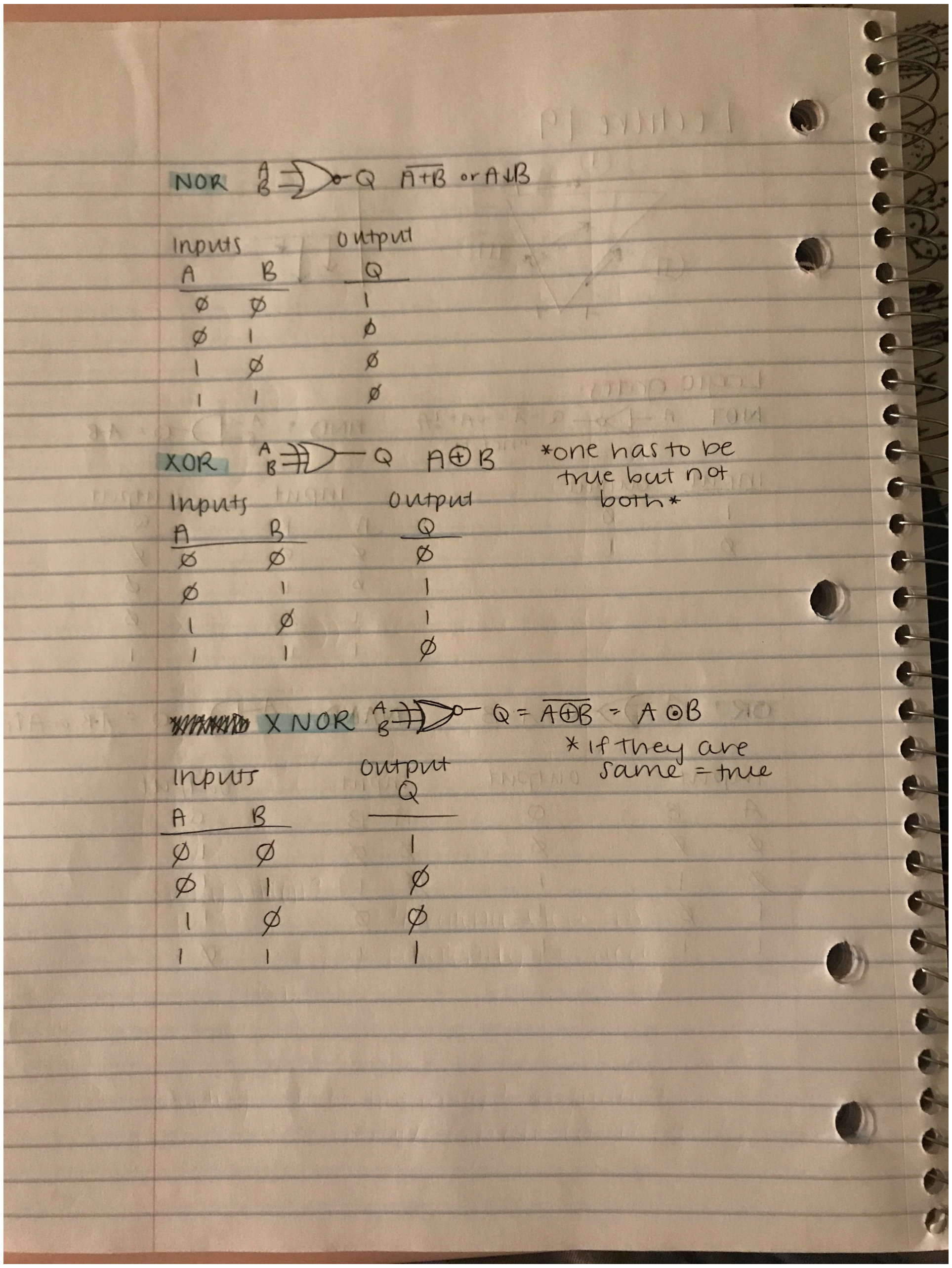
input A	input B	output Q
0	0	0
0	1	0
1	0	0
1	1	1

OR A, B → Q = A+B

input A	input B	output Q
0	0	0
0	1	1
1	0	1
1	1	1

NAND A, B → Q = $\overline{A\cdot B}$ or A↑B

input A	input B	output Q
0	0	1
0	1	1
1	0	1
1	1	0



NOR Q $\overline{A+B}$ or $A \downarrow B$

Inputs / Output

A	B	Q
Ø	Ø	1
Ø	1	Ø
1	Ø	Ø
1	1	Ø

XOR Q $A \oplus B$

one has to be true but not both

Inputs / Output

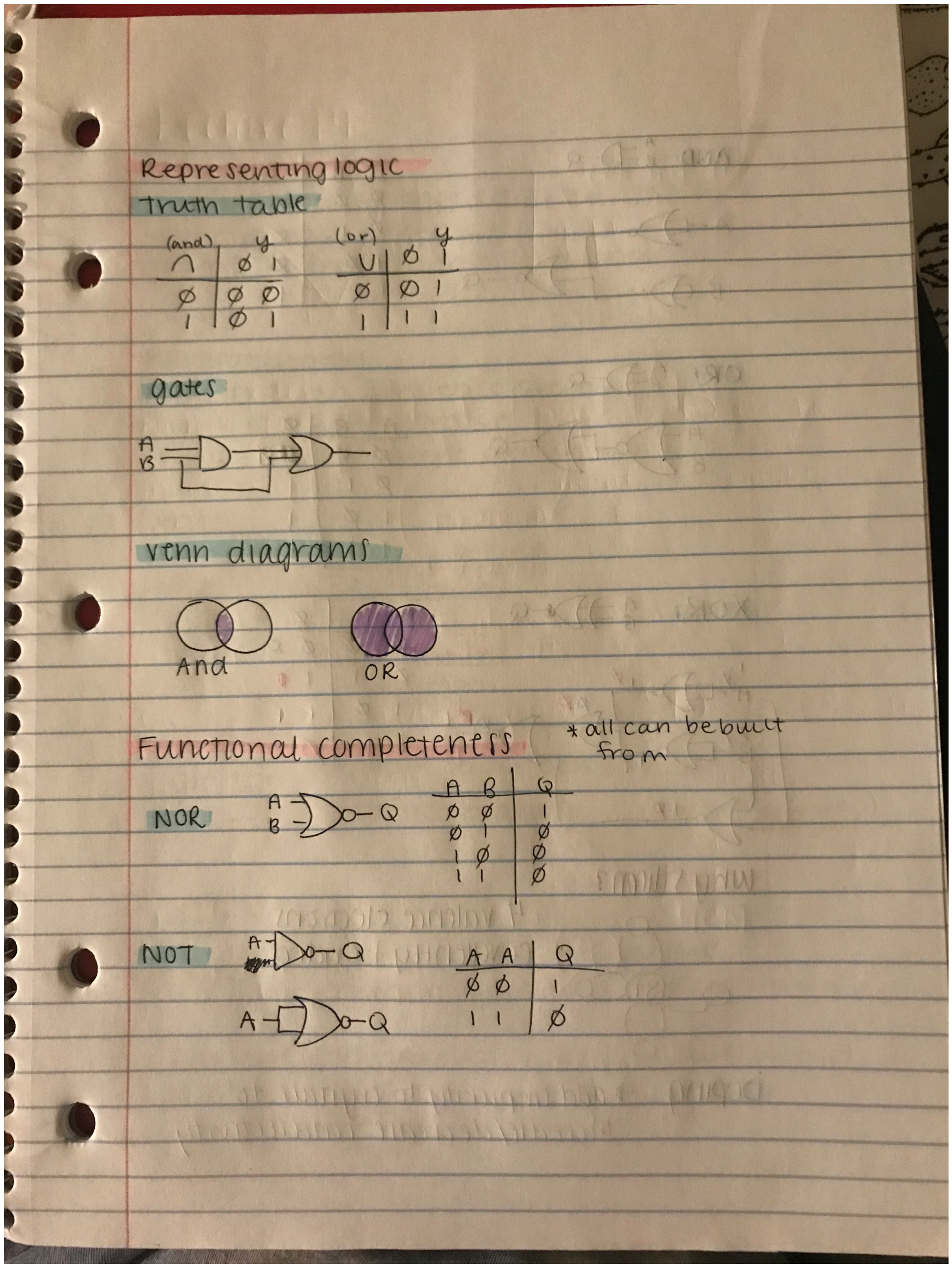
A	B	Q
Ø	Ø	Ø
Ø	1	1
1	Ø	1
1	1	Ø

XNOR $Q = \overline{A \oplus B} = A \odot B$

*if they are same = true

Inputs / Output

A	B	Q
Ø	Ø	1
Ø	1	Ø
1	Ø	Ø
1	1	1



Representing logic

Truth table

(and) $\cap$	y: Ø	1
Ø	Ø	Ø
1	Ø	1

(or) $\cup$	y: Ø	1
Ø	Ø	1
1	1	1

gates

venn diagrams

Functional completeness

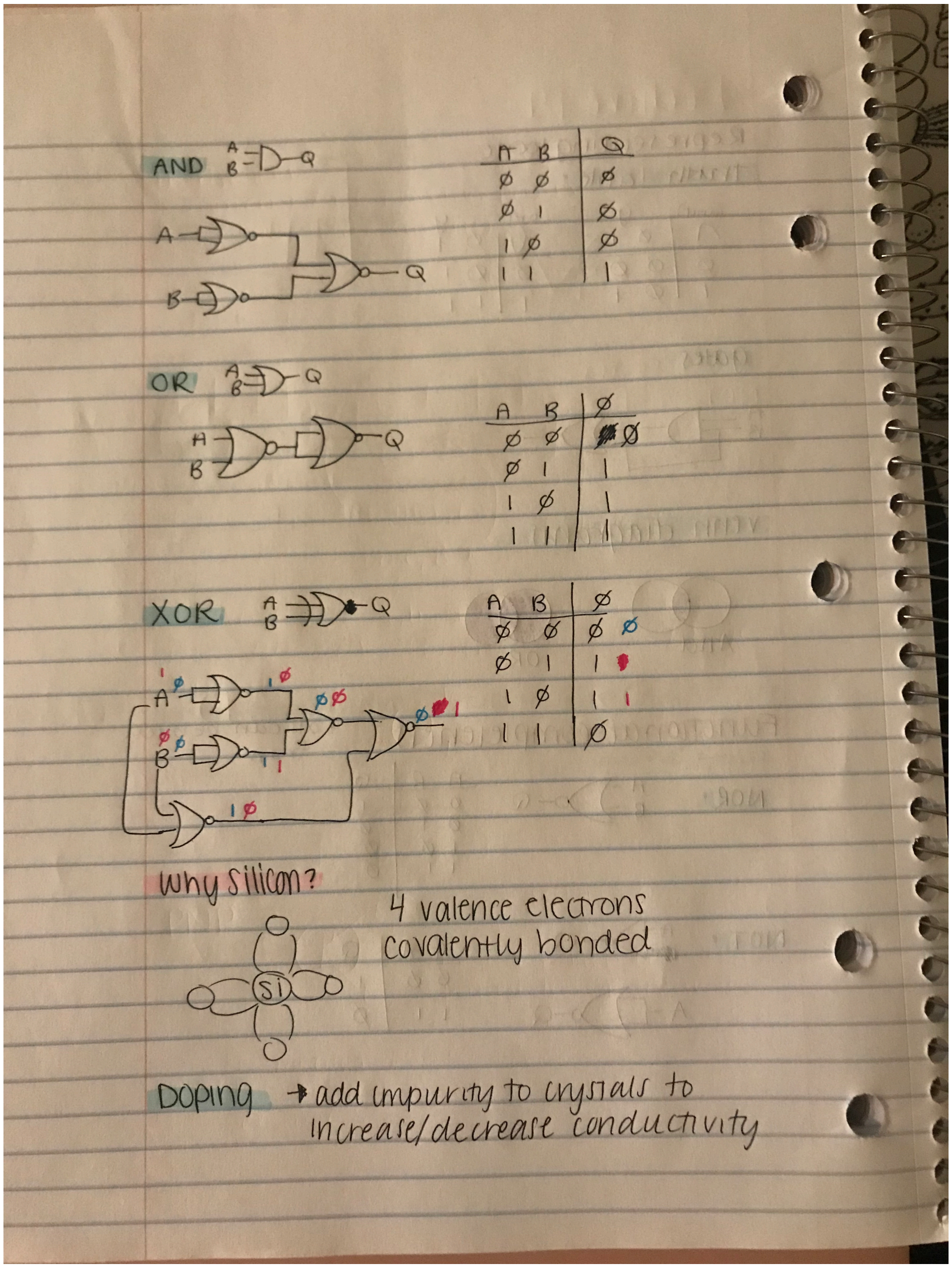
* all can be built from

NOR

A	B	Q
Ø	Ø	1
Ø	1	Ø
1	Ø	Ø
1	1	Ø

NOT

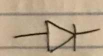
A	A	Q
Ø	Ø	1
1	1	Ø

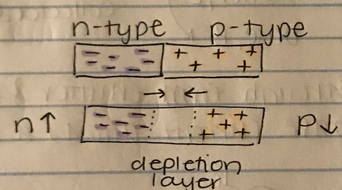

AND
OR
XOR
Why silicon?
4 valence electrons
covalently bonded
Si
Doping → add impurity to crystals to
increase/decrease conductivity

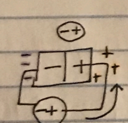
→ N-type doping
· elements w/ 5 valence electrons $\ll 1\%$
↳ phosphorus, arsenic
allows electrons to move around
↳ think of as (1)

→ P-type doping
· elements w/ 3 valence electrons
· "holes" in silicon lattice
· hole (0) can move around

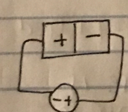
Diodes





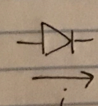


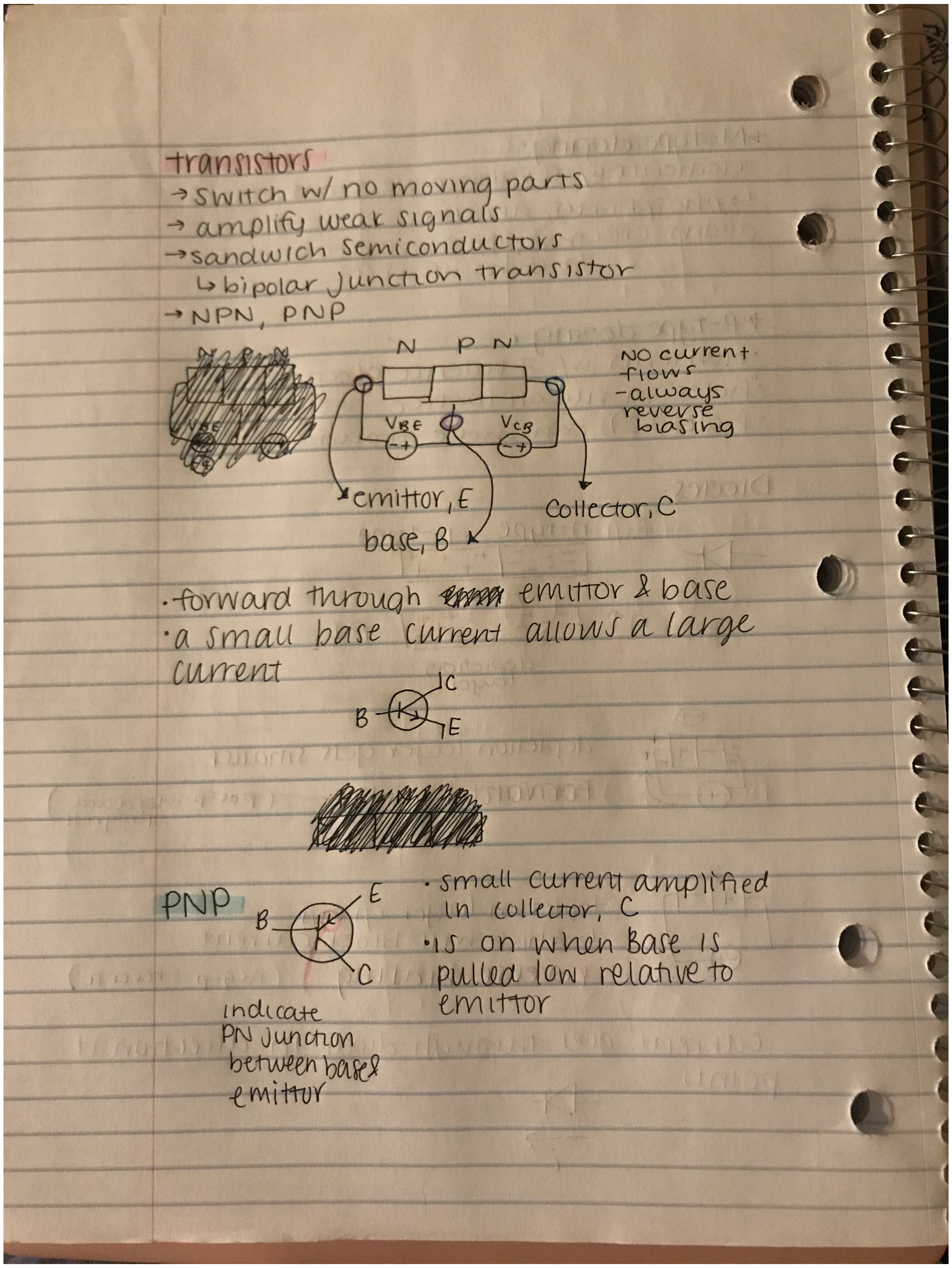
depletion layer gets smaller
Forward biasing (pos-p = forward progress)



depletion layer grows
↳ which will block current
reverse biasing (neg-p = reverse)

current goes through diode in direction it points





transistors
→ switch w/ no moving parts
→ amplify weak signals
→ sandwich semiconductors
↳ bipolar junction transistor
→ NPN, PNP

N P N
V_{BE} V_{CB}
no current flows - always reverse biasing
emittor, E
base, B
collector, C

· forward through emittor & base
· a small base current allows a large current

B C E

PNP
B E C
indicate PN junction between base & emittor

· small current amplified in collector, C
· is on when base is pulled low relative to emittor

19.6 Worksheet

In your extensive travels in this world you stumble unto a land inhabited only by knights and knaves. Knights always tell the truth and knaves always lie.

19.6.1 Problem 1, Pedro and Apollonia

You come across three inhabitants and ask the first, Pedro, Are you a knight or a knave? Pedro answers, but so quietly you cant hear him. You ask Apollonia What did he say? to which she responds Pedro said he was a knave. Upon hearing this, Peter piped up and said Dont believe that; its a lie. Is Peter a knight or a knave? [Further, is it possible to know what Pedro is?]

19.6.2 Problem 2, Roger and Oedipa

Shortly after that you meet two inhabitants, Roger Mexico and Oedipa Maas. Roger claims, Both of us are knaves. What are Roger and Oedipa?

19.6.3 Problem 3, Yes and No

Suppose youve heard a rumor that theres gold buried nearby. You meet a local and want to know whether there really is gold in them thar hills, but you dont know whether the person is a knight or a knave. If you are only allowed to ask only one question answerable by “yes” or “no”, what do you ask?

19.6.4 Problem 4, Lisa and Louise

Lisa and Louise are twins indistinguishable in appearance. One always lies, the other always tells the truth. You dont know which is which. You meet one of them and may ask one question to determine which twin is truth. What do you ask and what does it tell you?

19.6.5 Problem 5, NOT from NOR

Make a NOT gate from one or more NOR gates.

19.6.6 Problem 6, AND from NOR

Make an AND gate from one or more NOR gates.

19.6.7 Problem 7, OR from NOR

Make an OR gate from one or more NOR gates.

19.6.8 Problem 8, NAND from NOR

Make a NAND gate from one or more NOR gates.

Chapter 20

Digital circuits: II. Discretization and acquisition

04/11/2019 Here are some videos to help get started on building circuits: (1) How to use a Breadboard: https://youtu.be/6WReFkfrUIk This video describes how a breadboard is connected and what it means to connect circuit elements to the different holes in each row. It also builds a simple LED circuit.

(2) RC Filters: https://youtu.be/kc02HWprTo8 This video goes through the math behind RC filters and shows the resulting behavior of these filters on an oscilloscope, briefly showing the design of these circuits on a breadboard.

Chapter 21

Happenstance & Circumstance: A few BME specific situations and standards

04/16/2019

Contents

21.1 What is a "medical device"?

In the United States, medical devices are currently (ultimately) governed at the federal level where the term medical "device" is defined in 21 U.S.C. §321(h) as

> an instrument, apparatus, implement, machine, contrivance, implant, in vitro reagent, or other similar or related article, including any component, part, or accessory, which is–
>
> (1) recognized in the official National Formulary, or the United States Pharmacopeia, or any supplement to them,
>
> (2) intended for use in the diagnosis of disease or other conditions, or in the cure, mitigation, treatment, or prevention of disease, in man or other animals, or
>
> (3) intended to affect the structure or any function of the body of man or other animals, and
>
> which does not achieve its primary intended purposes through chemical action within or on the body of man or other animals and which is not dependent upon being metabolized for the achievement of its primary intended purposes.

Any "device" to which that definition applies[1] is regulated in this country by a suite of standards, ranging from the entirely voluntary to the strictly mandatory.

21.2 What is a "standard" and (how) is it enforced?

As good a definition as I've yet come across, W.D. Rowe in "Design and performance standards" from *Medical Devices: Measurements, Quality Assurance and Standards*, has it as

[1] With the exception of the term's use in 21 U.S.C. §331(i), §343(f), §352(c), and §362(c). Its currently debated relationship to software functions excluded in 21 U.S.C. §360(j)(o) is the subject of another discussion for another day.

> A standard is a multi-party agreement for establishing an arbitrary criterion for reference.

Each word in that sentence has a specific legal/functional meaning:

- **Multi-party** means that more than one person, party, organization, agency, agent, individual, and/or government may be involved;
- **Agreement** indicates that the parties involved come to some mutually agreed upon understanding of the matters involved and of ways to resolve them; generally this understanding has been arrived at through unanimity, consensus, ballot, or some other method;
- **Establishing** defines the purpose of the agreement (to create a standard and carry forth its provisions);
- **Arbitrary** underscores that there is no absolute criterion undergirding the standard;
- **Criterion**, criteria, are those features, facets, aspects, and/or conditions which the parties have agreed will be the basis of the standard;
- **Reference** represents the ideal towards which the standards sets one; it is what is desired and what reality will be measured against.

21.3 A hierarchy of standards

21.3.1 Local and/or proprietary standards

21.3.2 Common interest standards

21.3.3 Consensus standards

21.3.4 Regulatory standards

21.4 Legislation

A chronological list of significant medical device legislation in the United States. Much of the content in this section comes from the U.S. Food and Drug Administration's own website onn the history of the organization found **here**.

21.4.1 The "Pure Food and Drugs Act of 1906"

- Established a precursor to todays FDA ("Bureau of Chemistry")l
- Prohibited interstate commerce of misbranded and adulterated food, and drugsl

21.4.2 The "Federal Food, Drug, and Cosmetic Act of 1938"

- Primary statute that authorizes the FDAs regulation and oversight of medical products
- Granted authority for factory inspections
- Extended prohibition of interstate commerce to misbranded and adulterated cosmetics and therapeutic medical devices

21.4.3 The "Public Health Service Act of 1944"

- Established certification of laboratories.
- Expanded oversight of biologics.

21.4.4 The "Radiation Control for Health and Safety Act of 1968"

- Intended to minimize exposure to electronic product radiation and intense magnetic fields.
- Created performance standards for radiation-emitting products, such as diagnostic x-ray machines, MRIs, microwave, ultrasound or diathermy devices, UV devices and laser devices.

21.4.5 The "Medical Device Amendments of 1976"

- The first attempt at a federal regulatory mechanism for medical devices designed, developed, tested, and sold in this country.
- Perhaps best known for introducing our current medical device classification based on risk to humans when used.
- Established the regulatory pathways for new medical devices (devices that were not on the market prior to May 28, 1976, or had been significantly modified) to get to market: **Premarket Approval (PMA)** and **premarket notification (510(k))**.
- Created the regulatory pathway for new investigational medical devices to be studied in patients (Investigational Device Exemption (IDE)).
- Established several key postmarket requirements: (1) registration of establishments and listing of devices with the FDA, (2) Good Manufacturing Practices (GMPs), and (3) reporting of adverse events involving medical devices.
- Authorized the FDA to ban devices.

A classification of medical devices

Regardless of classification, there exist general controls which apply to all medical devices bought, sold, manufactured, and used in the United States. Such controls include

1. Prohibition against the adulteration or misbranding of devices;
2. Requirement that domestic device manufacturers and (initial) distributors register their establishments and list their devices;
3. The FDA has authority to audit/ban medical devices;
4. Notification requirements of risks, repairs, replacements, and refunds for customers and end-users.
5. Restriction of the sale, distribution, and/or use of devices; and
6. Maintenance of "good manufacturing practices" (GMPs), records, reports, and inspections.

- **Class I (General Controls)** devices are considered as low risks for human use. For a Class I device, there already exists sufficient information/evidence to provide reasonable assurance of device safety and effectiveness. Such devices are

  1. not to be for a use in supporting or sustaining human life or for a use which is of substantial importance in preventing impairment of human health, and
  2. does not present a potential unreasonable risk of illness or injury.

- **Class II (Performance Standards)** devices are considered moderate risks for human use. Class II devices are those "for which there is sufficient information to establish a performance standard to provide [] assurance" that it would work as intended "and for which it is therefore necessary to establish [...] a performance standard [...] to provide reasonable assurance of its safety and effectiveness."

- **Class III (Premarket Approval)** devices are considered high risks for human use. To be a Class III device a device

  1. "cannot be classified as a class I device",
  2. "cannot be classified as a class II device",
  3. "is purported or represented to be for a use in supporting or sustaining human life or for a use which is of substantial importance in preventing impairment of human health or presents a potential unreasonable risk of illness or injury"

21.4.6 The "Safe Medical Devices Act of 1990"

- Improved postmarket surveillance of devices by (1) requiring user facilities (such as hospitals and nursing homes) to report adverse events involving medical devices and (2) authorizing the FDA to require manufacturers to perform postmarket surveillance on permanently implanted devices if permanent harm or death could result from device failure.

- Authorized the FDA to order device recalls and to impose civil penalties for violations of the The "Federal Food, Drug, and Cosmetic Act of 1938".

- Defined substantial equivalence (the standard for marketing a device through the 510(k) program).

- Modified procedures for the establishment, amendment, or revocation of performance standards.
- Created the Humanitarian Use Device (HUD)/Humanitarian Device Exemption (HDE) programs to encourage development of devices targeting rare diseases.

21.4.7 The "Medical Device Amendments of 1992"

- Amended certain provisions (section 519 of the "Federal Food, Drug, and Cosmetic Act of 1938") that relate to the reporting of adverse events.
- The primary impact of the 1992 Amendments on medical device reporting was to define certain terms and to establish a single reporting standard for user facilities, manufacturers and distributors.

21.4.8 The "FDA Reform and Enhancement Act of 1996"

- Deals primarily with exports and establishing the basic requirements and procedures for exporting human drugs (also drug components) and biologics that may not be sold or distributed in the United States.
- Also lists requirements for exporting drugs that are approved for marketing in the United States, but which are being exported for an unapproved use.

21.4.9 The "Food and Drug Administration Modernization Act of 1997"

- Created the "least burdensome" provisions for premarket review.
- Created the option of accredited third parties to conduct initial premarket reviews for certain devices.
- Permitted the use of data from studies of earlier versions of a device in premarket submissions for new versions of the device.
- Provided for expanded access to investigational devices.
- Established the *De Novo* program through which novel low-to-moderate risk devices could be classified into Class I or II instead of automatically classifying them into Class III.

21.4.10 The "Medical Device User Fee and Modernization Act of 2002"

- Granted the FDA the authority to collect user fees for select medical device premarket submissions to help the FDA improve efficiency, quality, and predictability of medical device submission reviews.
- Enacted the Small Business Determination (SBD) program to permit reduced premarket approval fees for qualifying small businesses.
- Created FDA performance goals for decisions on certain premarket submissions.
- Established new regulatory requirements for reprocessed devices.
- Authorized electronic registration of medical device firms.
- Established the Office of Combination Products.

21.4.11 The "Food and Drug Administration Amendments Act of 2007"

- Reauthorized the medical device user fee ("Medical Device User Fee and Modernization Act of 2002", MDUFMA II), including improvements to premarket review times.
- Required that all registration and listing be performed electronically.
- Required the FDA to establish a unique device identification (UDI) system for medical devices to require device labels to bear a unique identifier.

21.4.12 The "Food and Drug Administration Safety and Innovation Act of 2012"

- Reauthorized the medical device user fee program (MDUFA III), including improvements to premarket review times and added shared outcome goals with industry.
- Created direct *De Novo* pathway, permitting the classification of novel, low-to-moderate risk devices into Class I or II (rather than Class III) without first having to submit a 510(k).
- Changed the standards associated with disapproval of an IDE.

- Permitted the FDA to work with foreign governments to harmonize regulatory requirements
- Required FDA to provide a Substantive Summary when requested by the holder of the submission for significant decisions
- Expanded the application of the "least burdensome" principles in premarket reviews.

21.4.13 The "21st Century Cures Act of 2016"

- Codified into law the FDAs expedited review program for breakthrough devices.
- Expanded the application of the "least burdensome" principles in premarket reviews.
- Streamlined processes for exempting devices from the premarket notification (510(k)) requirement.
- Increased the population estimate required to qualify for Humanitarian Use Device (HUD) designation from "fewer than 4,000" to "not more than 8,000" patients in the U.S. per year.
- Permitted the use of central Institutional Review Board (IRB) oversight rather than requiring only local IRBs for IDE and HDE activities.
- Required the FDA to revise the regulation of combination products.
- Codified into law a process for submitting requests for recognition/non-recognition of a standard.
- Clarified how certain digital health products can be regulated by defining the categories of medical software that can and cannot be regulated as devices.

21.4.14 The "Food and Drug Administration Reauthorization Act of 2017"

- Reauthorized the medical device user fee program (MDUFA IV), including improvements to premarket review times and investments in strategic initiatives like the National Evaluation System for health Technology (NEST) and patient input.

- Authorized risk-based inspection scheduling for device establishments and prescribed other process improvements related to device establishment inspections.
- Decoupled accessory classification from classification of the parent device.
- Required the FDA to conduct at least one pilot project to explore how real-world evidence can improve postmarket surveillance.

21.5 What does the FDA regulate?

While not an exhaustive list, the following categories of products fall under the FDA's regulatory concerns:

- Foods, including:

  dietary supplements

  bottled water

  food additives

  infant formulas

  other food products (although the U.S. Department of Agriculture plays a lead role in regulating aspects of some meat, poultry, and egg products)

- Drugs, including:

  prescription drugs (both brand-name and generic)

  non-prescription (over-the-counter) drugs

- Biologics, including:

  vaccines

  blood and blood products

  cellular and gene therapy products

  tissue and tissue products

  allergenics

- Medical Devices, including:

  simple items like tongue depressors and bedpans

complex technologies such as heart pacemakers
dental devices
surgical implants and prosthetics

- Electronic Products that give off radiation, including:
  microwave ovens
  x-ray equipment
  laser products
  ultrasonic therapy equipment
  mercury vapor lamps
  sunlamps
- Cosmetics, including:
  color additives found in makeup and other personal care products
  skin moisturizers and cleansers
  nail polish and perfume
- Veterinary Products, including:
  livestock feeds
  pet foods
  veterinary drugs and devices
- Tobacco Products, including:
  cigarettes
  cigarette tobacco
  roll-your-own tobacco
  smokeless tobacco

21.6 Some quick facts about the Food and Drug Administration

From the U.S. FDA's website:

1. FDA is responsible for the oversight of more than $2.5 trillion in consumption of food, medical products, and tobacco.

2. FDA-regulated products account for about 20 cents of every dollar spent by U.S. consumers.
3. FDA regulates about 75 percent of the U.S. food supply. This includes everything we eat except for meat, poultry, and some egg products.
4. There are over 19,000 prescription drug products approved for marketing.
5. FDA oversees over 6,000 different medical device product categories.
6. There are over 1,600 FDA-approved animal drug products.
7. There are about 340 FDA-licensed biologics products.
8. FDA regulations cover about 35,000 produce farms, 300,000 restaurant chain establishments, and 10,500 vending machine operators.
9. FDA products are manufactured or handled at nearly 270,000 registered facilities, more than half of which are overseas.
10. **About 80 percent of active pharmaceutical ingredients manufacturers are located outside of the U.S.**
11. FDA-regulated products account for 12 percent of U.S. imports and 16 percent of U.S. exports.
12. About 35 percent of medical devices used in this country are imports.
13. The FDA budget for FY 2018 was $5.4 billion.
14. About 55 percent, or $3 billion, of FDAs budget is provided by federal budget authorization. The remaining 45 percent, or $2.4 billion, is paid for by industry user fees.
15. The FDA budget is equivalent to $9.11 per American per year.
16. The FDA budget includes 17,803 full time equivalents (FTEs).
17. Human Drugs regulatory activities account for 30 percent of FDAs budget; 69 percent of these activities are paid for by industry user fees.
18. Devices and Radiological Health regulatory activities account for 10 percent of FDAs budget; 35 percent of these activities are paid for by industry user fees.

19. Foods regulatory activities account for 20 percent of FDAs budget; 1 percent of these activities are paid for by industry user fees.

20. Biologics regulatory activities account for 7 percent of FDAs budget; 40 percent of these activities are paid for by industry user fees.

21. Animal Drugs and Feeds regulatory activities account for 4 percent of FDAs budget; 13 percent of these activities are paid for by industry user fees.

Chapter 22

A philosophy of circuits, systems, and signals

04/23/2019

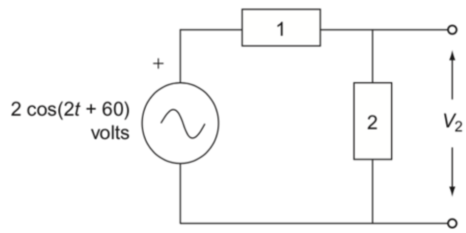
A Glorified Quiz III

04/18/2019, in class. Below is set of exercises in preparation of the third glorified quiz.

Contents

22.1 Basic analysis

In the circuit below, the voltage across element 1 is 2 cos($2t$ + 60). What is the voltage, V_2, across element 2?



22.2 A wound wire or two

1. A resistor is constructed of thin copper wire wound into a coil (a "wire-wound" resistor). The wire has a diameter of 1 mm.

a. How long is the wire required to be in order to make a resistor of 12 Ω?

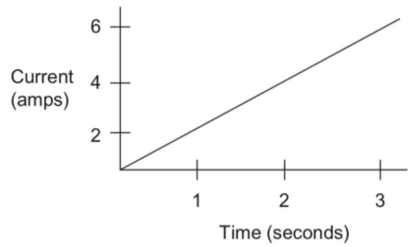
b. If this resistor is connected to a 5-volt source, how much power will it dissipate as heat?

2. **a.** A length of size #12 copper wire has a resistance of 0.05 Ω. It is replaced by #16 (AWG) wire. What is the resistance of this new wire?

b. Assuming both wires carry 2 amps of current, what is the power lost in the two wires?

22.3 Potential across an inductor

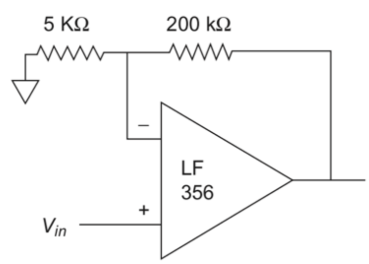
3. The figure below shows the current passing through a 2 h inductor.
a. What is the voltage drop across the inductor?
b. What is the energy stored in the inductor after 2 seconds?



4. The voltage drop across a 10 h inductor is measured as 10 cos(20*t*) volts. What is the current through the inductor?

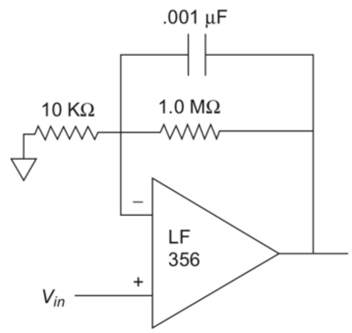
22.4 An operational amplifier's bandwidth

What is the bandwidth of the noninverting amplifier below? If the same feedback network were used to design an inverting amplifier (by switching ground and V_{in}), what would be the bandwidth of this circuit?



22.5 The phase shift of an amplifier

A 0.001 µf capacitor is added to the feedback circuit of the inverting op amp circuit shown below. You can assume that before the capacitor was added the phase shift due to the amplifier when $Av\beta = 1$ was 120 degrees. (The criterion for stability is that the phase shift induced by the op amp and the feedback network must be less than 180 degrees when $Av\beta = 1$, Equation 12.14.) After the capacitor is added, what is the phase shift of the op amp plus feedback network at the frequency where $Av\beta = 1$? Follow the example given in Section 12.5.1.2 on stability.

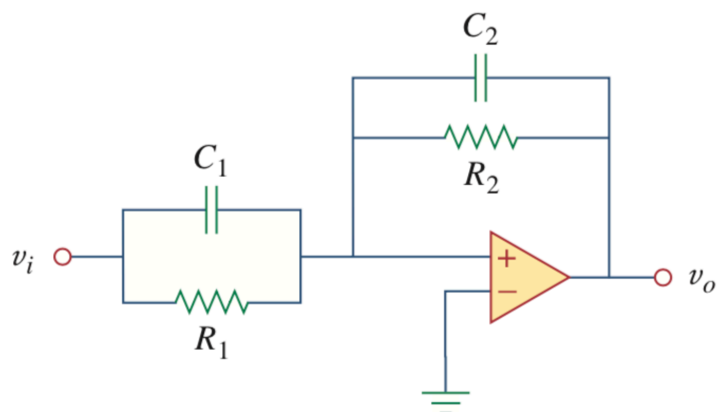


22.6 Designing an arbitrary op amp circuit

16.57 Design an op amp circuit, using Fig. 16.77, that wi realize the following transfer function:

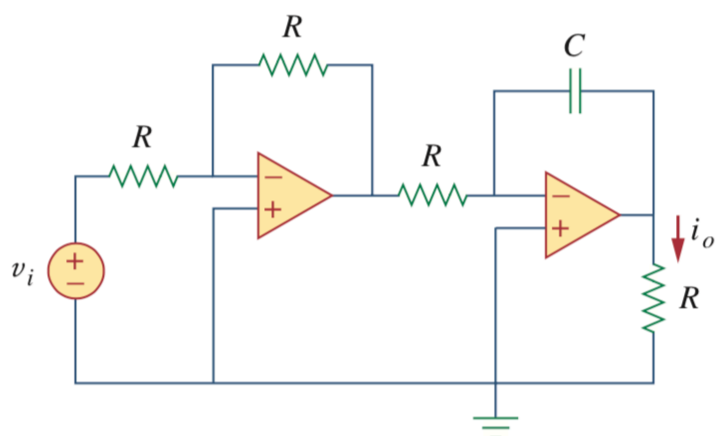
$$\frac{V_o(s)}{V_i(s)} = -\frac{s + 1000}{2(s + 4000)}$$

Choose $C_1 = 10\ \mu\text{F}$; determine R_1, R_2, and C_2.



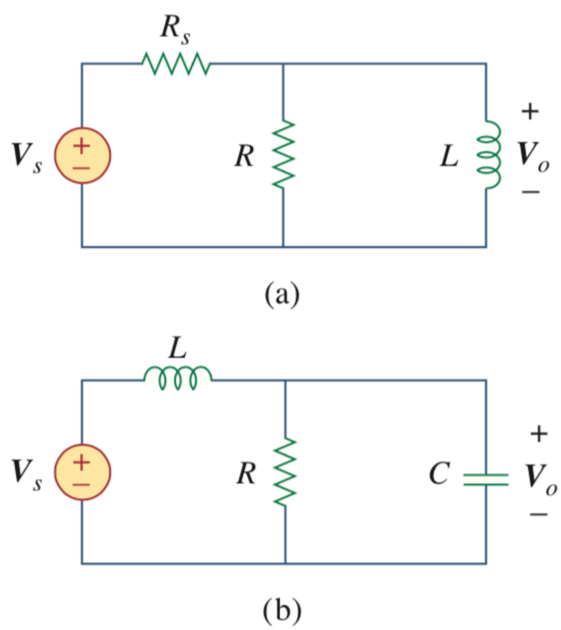
22.7 A gyrator

16.62 A gyrator is a device for simulating an inductor in a network. A basic gyrator circuit is shown in Fig. 16.81. By finding $V_i(s)/I_o(s)$, show that the inductance produced by the gyrator is $L = CR^2$.



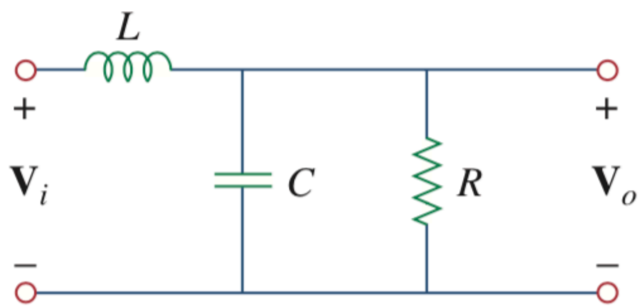
22.8 Transfer function from a circuit, voltage

14.5 For each of the circuits shown in Fig. 14.72, find $\mathbf{H}(s) = \mathbf{V}_o(s)/\mathbf{V}_s(s)$.

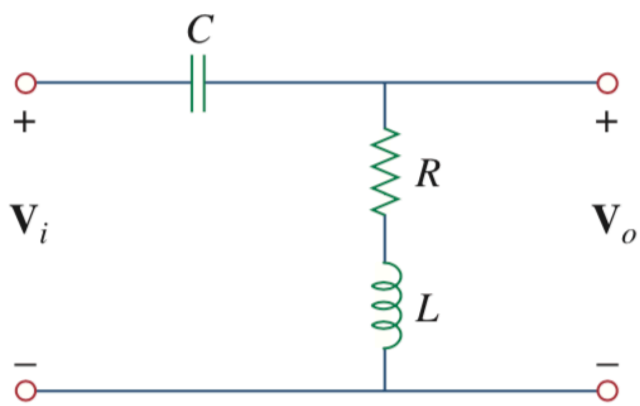


22.9 Transfer function from a circuit, voltage, again

Find the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o/\mathbf{V}_i$ of the circuits shown in Fig. 14.71.



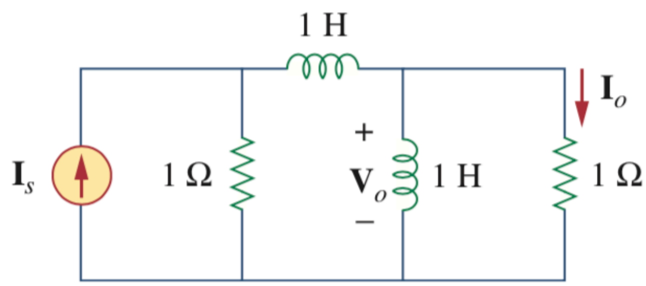
(a)



(b)

22.10 Transfer function from a circuit, current

14.6 For the circuit shown in Fig. 14.73, find $\mathbf{H}(s) = \mathbf{I}_o(s)/\mathbf{I}_s(s)$.

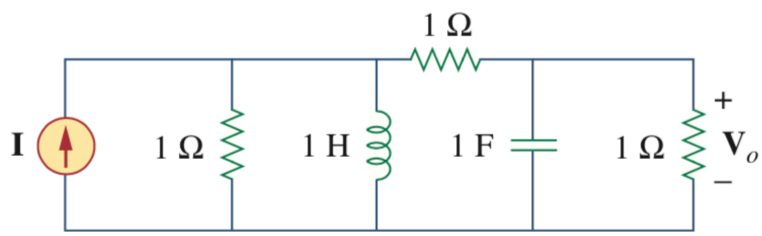


22.11 Transfer function from a circuit, impedance

14.46 For the network illustrated in Fig. 14.85, find

(a) the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o(\omega)/\mathbf{I}(\omega)$,

(b) the magnitude of $\mathbf{H}$ at $\omega_0 = 1$ rad/s.



22.12 Even more transfer functions, emphasis on the *fun*

For the following transfer functions

a. $\frac{6s-5}{14s^2+7s+1}$

b. $\frac{s^3-2s+3}{8s^2-2}$

c. $\frac{s}{s^4+5s+10}$

i. Write down the governing differential equations.

ii. Draw the location of the zeros and poles n the s-plane. You may use Matlab.

22.13 Sketching Bode plots

14.11 Sketch the Bode plots for

$$\mathbf{H}(\omega) = \frac{10 + j\omega}{j\omega(2 + j\omega)}$$

14.12 A transfer function is given by

$$T(s) = \frac{s + 1}{s(s + 10)}$$

Sketch the magnitude and phase Bode plots.

14.13 Construct the Bode plots for

$$G(s) = \frac{s + 1}{s^2(s + 10)}, \qquad s = j\omega$$

14.14 Draw the Bode plots for

$$\mathbf{H}(\omega) = \frac{50(j\omega + 1)}{j\omega(-\omega^2 + 10j\omega + 25)}$$

14.15 Construct the Bode magnitude and phase plots for

$$H(s) = \frac{40(s + 1)}{(s + 2)(s + 10)}, \qquad s = j\omega$$

14.16 Sketch Bode magnitude and phase plots for

$$H(s) = \frac{10}{s(s^2 + s + 16)}, \quad s = j\omega$$

14.17 Sketch the Bode plots for

$$G(s) = \frac{s}{(s + 2)^2(s + 1)}, \qquad s = j\omega$$

22.14 Ordinary differential equations

For the differential equations below

a. $0.3\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = 0.8\frac{d^3u}{dt^3} + 2\frac{d^2u}{dt^2} + u$

b. $\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 3\frac{dy}{dt} = 0.8\frac{d^2u}{dt^2} + 6\frac{du}{dt}$

c. $\frac{d^4y}{dt^4} + 2.5\frac{d^3y}{dt^3} - 7\frac{dy}{dt} - 4y = \frac{d^3u}{dt^3} - 1.541\frac{d^2u}{dt^2}$

d. $\frac{d^4y}{dt^4} + 2.5\frac{d^3y}{dt^3} - 7\frac{dy}{dt} - 4y = \frac{d^3u}{dt^3} - 3\frac{d^2u}{dt^2}$

e. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 13.25y = 2\frac{du}{dt} - 5u$

i. Write down the transfer function in the s-domain $\frac{y(s)}{u(s)}$, assuming zero initial conditions.

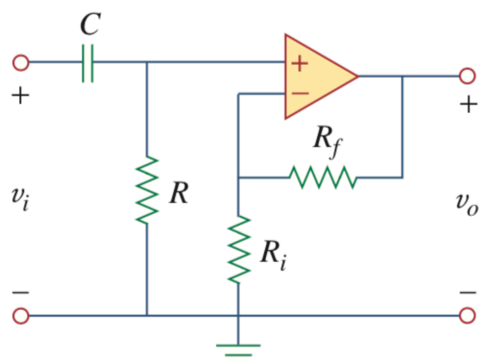
ii. Report the zeros and poles. You may use Matlab.

iii. Report if the system is stable, unstable or marginally stable.

22.15 A high pass

14.65 A highpass filter is shown in Fig. 14.92. Show that the transfer function is

$$\mathbf{H}(\omega) = \left(1 + \frac{R_f}{R_i}\right) \frac{j\omega RC}{1 + j\omega RC}$$

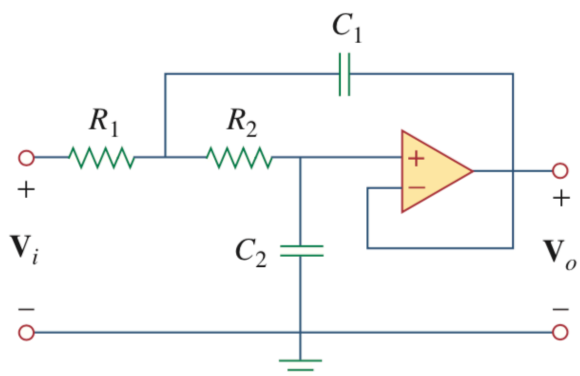


22.16 A low pass

***14.70** A second-order active filter known as a Butterworth filter is shown in Fig. 14.95.

(a) Find the transfer function $\mathbf{V}_o/\mathbf{V}_i$.

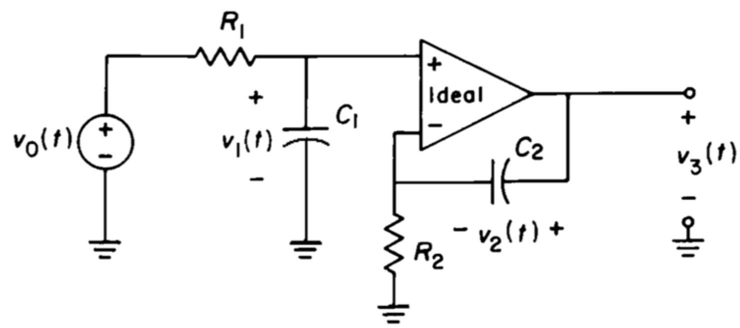
(b) Show that it is a lowpass filter.



22.17 An inverting integrator

Problem 1.3

With appropriate parameter values, the circuit shown below functions as a *non-inverting integrator*.



a) Write state equations for this circuit in terms of the capacitor voltages $v_1(t)$ and $v_2(t)$.

b) Show that the input-output differential equation relating $v_3(t)$ and $v_0(t)$ has the form

$$\frac{dv_3(t)}{dt} = Kv_0(t)$$

provided that $R_1C_1 = R_2C_2$. Find the magnitude and sign of K.

22.18 Playing with a given transfer function

Slate and Sheppard (1982) used a simplified dynamic LTI model to describe the transient reduction in mean arterial pressure following an IV infusion of sodium nitroprusside (SNP). Their transfer function is:

$$\frac{\Delta MAP}{U}(s) = \frac{-K_p \exp(-s\delta_p)}{(\tau_p s + 1)} = G_p(s)$$

where ΔMAP is the SNP-induced change in MAP; U is the specific IV infusion rate of SNP in (mg/kg patient weight)/min; K_p is the individual patient's steady-state response constant (units are mmHg/[(mg/kg patient weight)/min]; δ_p is the plant's dead time in minutes (also known as transport lag before a change in $u(t)$ affects the ΔMAP); τ_p is the plant's time constant in minutes. Assume that $K_p = 1.0$, $\tau_p = 0.75$ min, $\delta_p = 0.5$ min.

(a) Plot and dimension the plant's impulse response to ($u(t) = \delta(t)$).

(b) Plot and dimension the plant's response to a unit step of SNP infusion rate.

(c) Plot and dimension the plant's steady-state, sinusoidal frequency response for an input rate of the form: $u(t) = [1 + \sin(2\pi f t)]$ for 0.01 cycles/min $\leq f \leq$ 100 cycles/min. Note that the "1" in $u(t)$ is so $u(t)$ is nonnegative. Make a Bode plot: $20 \log|\mathbf{G}_p(2\pi f)|$ versus f on a log scale, and $\angle \mathbf{G}_p(2\pi f)$ versus f on the same scale. Note that $f = \omega/2\pi$.

22.19 Playing with another given transfer function

Consider the transfer function in which $K = 5.263$, and the natural frequencies are in r/min:

$$H(s) = \frac{K(s+1.9)}{(s+10)(s+1)}$$

(a) Find $h(t)$.

(b) Put the transfer function in time-constant form, then plot and dimension the complete Bode plot for the system over the range $0.01 \leq f \leq 100$ cycles/min.

22.20 Playing with yet another given transfer function

The transfer function of a system is

$$H_5(s) = 0.1\frac{s+100}{s+10}$$

(b) Find, plot, and dimension the system's steady-state, sinusoidal frequency response (Bode plot), dB magnitude, and phase for $0.1 \leq f \leq 10^3$ Hz.

22.21 Poles and zeros

The Laplace transform of a causal, LTI system's output is:

$$Y(s) = \frac{2}{s(s^2 + 3s + 2)}$$

(a) Factor the system's denominator to find its real poles.
(b) Use the Laplace final value theorem to find $y(\infty)$.
(c) Find $y(t)$ by partial fraction expansion and verify part (b). Also find $y(0)$.

22.22 System response from poles alone

Given the following poles, compare the time response (qualitatively) to a system with poles at -1 and $-1 \pm j$. You should compare any oscillations, decay or growth. Please explain your answers.

a. $-3, -5$

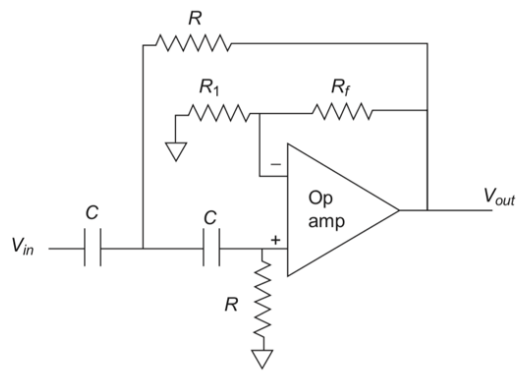
b. $-1, -1 \pm -2j$

c. $\pm 3j$

d. $-4, 2, \pm 3j$

22.23 Design, design, design, and design

Design a 1-pole low-pass filter with a bandwidth of 1 kHz. Assume you have capacitor values of 0.001 μF, 0.01 μF, 0.05 μF, and 0.1 μF, and a wide range of resistors.
Design a 2-pole low-pass filter with a cutoff frequency of 500 Hz and a damping factor of 0.8. Assume the same component availability as in Problem 12.
Design a 2-pole high-pass filter with a cutoff frequency of 10 kHz and a damping factor of 0.707. (The circuit for a high-pass filter is the same as for a low-pass filter except that the capacitors and resistors are reversed, as shown in the figure below.)



Design an instrumentation amplifier with a switchable gain of 10, 100, and 1000. [*Hint*: Switch the necessary resistors in or out of the circuit as needed.]

22.24 Bridges and amplifiers

22.24.1 Example 1

Problem 10.1.44 In Fig. 10.58, input of an instrumentation amplifier with the gain $G = 22$ is connected to (a, b) terminals of a bridge circuit.

(a) $R_1 = 12\,\mathrm{k\Omega}, \quad R_2 = 20\,\mathrm{k\Omega}, \quad R_3 = 2\,\mathrm{k\Omega}, \quad R_4 = 10\,\mathrm{k\Omega}$. $V_o/V_i = ?$

(b) Determine the value of R_G if AD620 type Instrumentation amplifier IC is used.

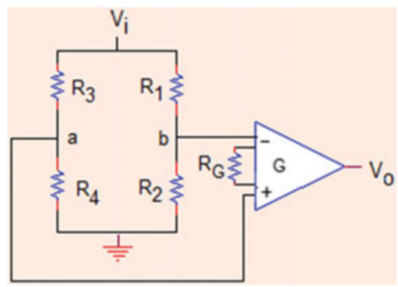


Fig. 10.58 The circuit for Problem 10.1.44

22.24.2 Example 2

Problem 10.1.45 Design a Wheatstone bridge to measure the voltage over a Pt 100 type temperature sensor placed in one of its grounded arms using a commercially available instrumentation amplifier (IA) with overall gain of 10 V/V. Run a simulation using SPICE, and plot the voltage at the output of IA against resistance of the sensor (wheatstone2.cir, ad620.cir).

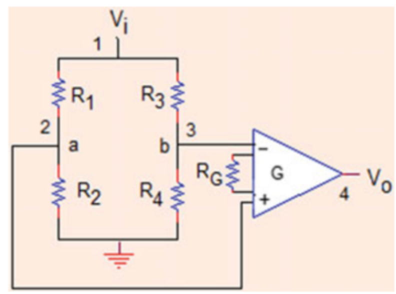


Fig. 10.59 Wheatstone bridge and instrumentation amplifier

22.24.3 Example 3

Problem 10.1.47

(a) Determine the voltage at the output of the circuit shown in Fig. 10.62.
(b) Comment on the form of relationship between the output voltage and the change of sensor resistance [6].
(c) Calculate the output voltage at 100 °C, if R_3 is a Pt 100 type sensor (use first-order approximation), $R_5 = 10\ \text{k}\Omega$, $R_1 = R_4 = 1\ \text{k}\Omega$, U = 15 V, $\alpha = 3.9 \times 10^{-3}$

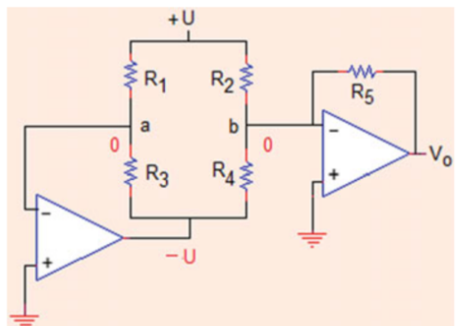


Fig. 10.62 The circuit for Problem 10.1.47

22.25 Bumping the jams

14.96 The crossover circuit in Fig. 14.108 is a lowpass filter that is connected to a woofer. Find the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o(\omega)/\mathbf{V}_i(\omega)$.

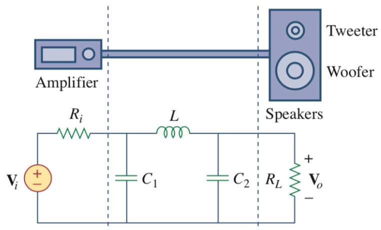


Figure 14.108
For Prob. 14.96.

14.97 The crossover circuit in Fig. 14.109 is a highpass filter that is connected to a tweeter. Determine the transfer function $\mathbf{H}(\omega) = \mathbf{V}_o(\omega)/\mathbf{V}_i(\omega)$.

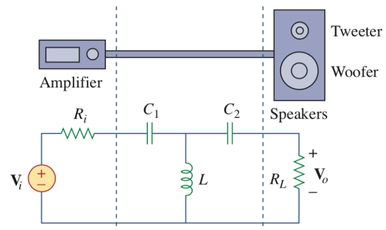
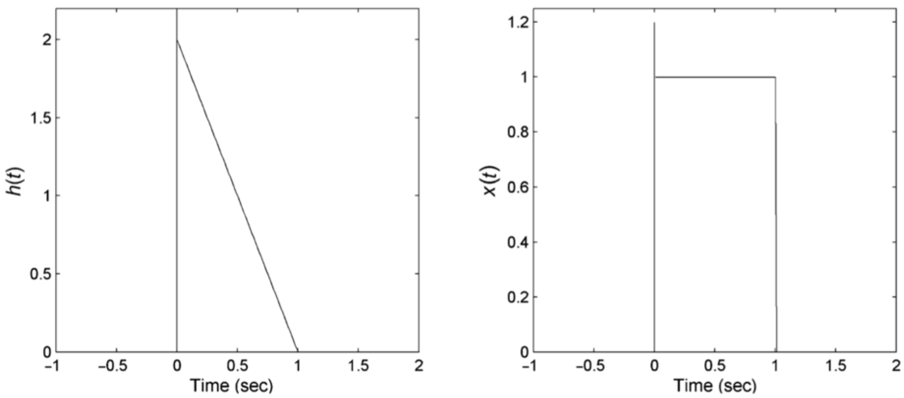


Figure 14.109
For Prob. 14.97.

22.26 Convolution

2. Use the basic convolution equation (Equations 7.1 and 7.2) to find the output of a system with an impulse response, $h(t) = 2(1 - t)$, to a 1-sec pulse having an amplitude of 1.



22.27 A pulse into a circuit

18.44 If the rectangular pulse in Fig. 18.46(a) is applied to the circuit in Fig. 18.46(b), find v_o at $t = 1$ s.

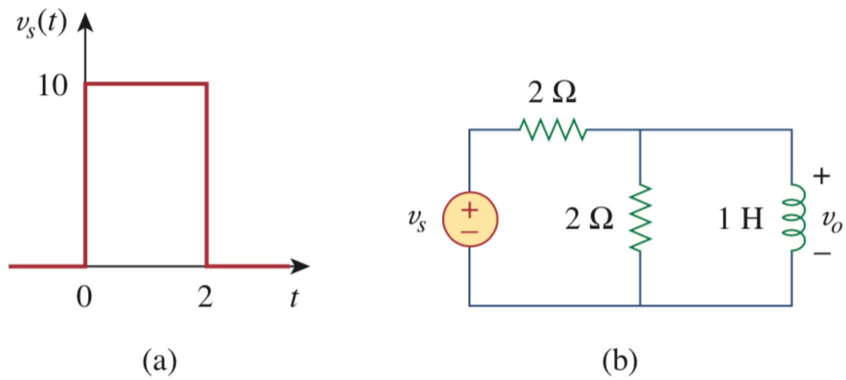


Figure 18.46
For Prob. 18.44.

22.28 "Silent" knights and knaves

We now visit another knight/knave island on which, like on the first one, all knights tell the truth and all knaves lie. But now there is another complication! For some reason, the natives refuse to speak to strangers, but they are willing to answer yes/no questions using a *secret* sign language that works like this:

Each native carries two cards on his person; one is red and the other is black. One of them means *yes* and the other means *no*, but you are not told which color means what. If you ask a yes/no question, the native will flash one of the two cards, but unfortunately, you will not know whether the card means *yes* or *no*!

Problem 3.1. Abercrombie, who knew the rules of this island, decided to pay it a visit. He met a native and asked him: "Does a red card signify *yes*?" The native then showed him a red card.

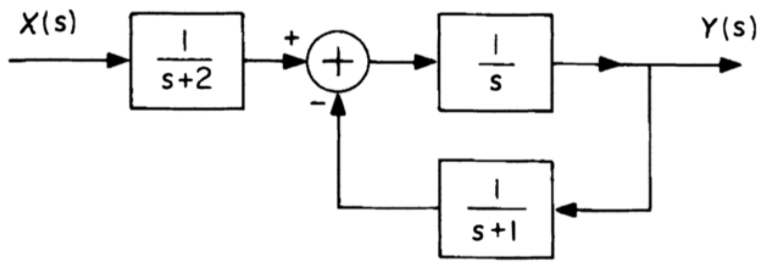
From this, is it possible to deduce what a red card signifies? Is it possible to deduce whether the native was a knight or a knave?

Problem 3.2. Suppose one wishes to find out whether it is a red card or a black card that signifies *yes*. What simple yes/no question should one ask?

Problem 3.3. Suppose, instead, one wishes to find out whether the native is a knight or a knave. What yes/no question should one ask?

22.29 A block diagram

Exercise 5.1



Show that the system function of the block diagram above is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^3 + 3s^2 + 3s + 2}.$$

22.30 The heart of an electrocardiogram

Problem 10.1.27 Design an instrumentation amplifier of Fig. 10.30, with voltage gain $\times 210$.

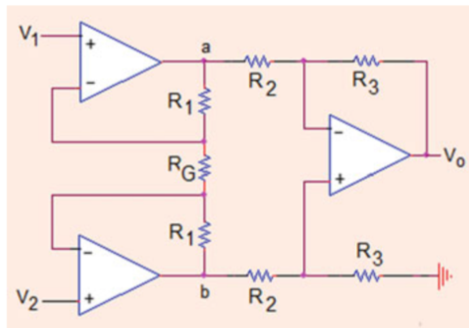


Fig. 10.30 Instrumentation amplifier

22.31 Current through a cell

The passive membrane of any cell can be modeled (Figure 3.10) as a capacitor (C_m) in parallel with a resistor (R_m) and battery (V_{rest}). The voltage across the membrane is simply the difference between the potential outside the cell (0V or ground) and the potential inside the cell (V_m). Assume that current is flowing in the intracellular space from cell 1 to cell 2 to cell 3. The current flows from cell to cell through gap junctions that may be modeled as resistors (R_i).

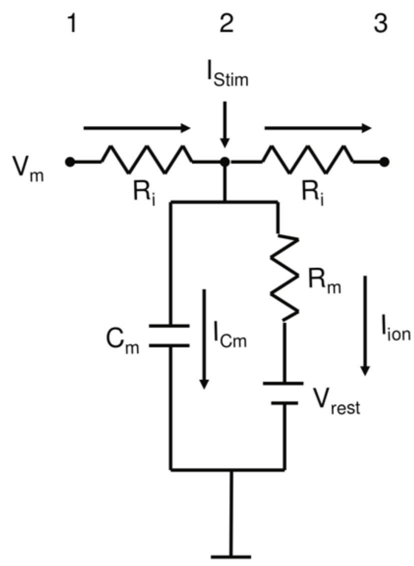


Figure 3.10: Passive circuit model for a cell membrane model.

a. Write an equation for the capacitive current (I_{Cm}) at node 2.
b. Write an equation for the ionic current (I_{ion}) at node 2.
c. Derive an equation to describe how Vm at node 2 changes over time (e.g., $\frac{dVm}{dt} =$). Your expression should contain only voltages, C_m, R_m, R_i, and I_{stim}.

Part V

Assignments

Homework I

Assigned January 22, 2019. Due January 31, 2019 as *both* a hard copy in class and a pdf on Canvas. Each problem worth 10 points.

1.1 On shocking the heart

(15 points) A typical automated external defibrillator (AED) delivers 200-1000 V in less than 10 ms.

1. If the AED in front us delivers a pulse of 600 V, how much current is needed to deliver 120, 240, and 360 Joules?

2. Assuming two human hands have a mass of approximately 1 kg, how many chest compressions would be needed to deliver an equivalent amount of energy if each compression had a depth of 5 cm and was delivered at a constant speed of 0.5 m/s every 2 seconds?

3. If you had the option of having your heart jump-started would you choose an AED or the bare hands of a stranger? Justify your answer.

1.2 On energy within batteries

(5 points) A 3.3 V battery you are considering for a wearable you are designing has a total charge of 300 mAh. How many joules is this battery capable of delivering?

1.3 On predicting energy use

(5 points) A fellow engineer bought a 12 V battery rated for 60 Ah. The experiment you both have in mind will draw 2 A over a 1 kΩ load. How long can your experiment last?

1.4 On charges through equipment

(5 points) A piece of medical equipment supplies 135 W at 220 V. How much electrical charge flows through the device in the 10 hours a nurse is on-call using it? And how many electrons does this charge correspond to?

1.5 *In vitro*

(10 points) During an in vitro (petri dish) experiment, the peak electric power that a group of stem cells can tolerate without some serious functional consequence is known to have a threshold of about 1 mW. If the power delivered to this group of cells is defined as $p(t) = 2e^{-t} \sin 5t$ [mW], will the cell be harmed and if so how long can power be delivered before the cells are harmed?

1.6 On equivalent resistance

(10 points) Find the equivalent resistances. (Show your work.)

- 1 square of resistors (R = 10 kΩ) measured across corners.
- 2 resistors (R1 = 10 kΩ, R2 = 4 kΩ) in series.
- 3 resistors (R1 = R2 = R3 = 10 kΩ) in parallel.
- 4 resistors (R1 = 1 kΩ, R1 = 2 kΩ, R1 = 3 kΩ, R1 = 4 kΩ) in parallel.
- 5 resistors (R = 10 kΩ) in a pentagon, measured across each resistor.

1.7 On uninterruptible power

(10 points) Hospitals often employ what is known as a dynamic uninterruptible power system (D-UPS) comprising a diesel generator, a synchronous machine, and a kinetic energy unit. By way of example, if the energy to a particular hospital fails, the kinetic energy unit continues to feed 600 kW to the hospital for 20 seconds, allowing the generator and synchronous machine to take over and feed the load.

1. What is the total energy capacity of this kinetic energy unit?
2. If the hospital were operating at 240 kW (instead of the aforementioned 600 kW), how much longer can the unit feed the load?

1.8 On modeling current and voltage

(10 points) Current passing through an electrical element in an AC situation can generally be defined as $i(t) = A \sin \omega t$.

1. Determine the energy on this element if the voltage across it is $v(t) = B \cos \omega t$.
2. Plot current, voltage, and energy as functions of time using MATLAB (or an equivalent software package). You may use any values of A, B, and ω that youd like, but unit values (=1) might make your life/work easier. (Present the code you used to arrive at your plot.)

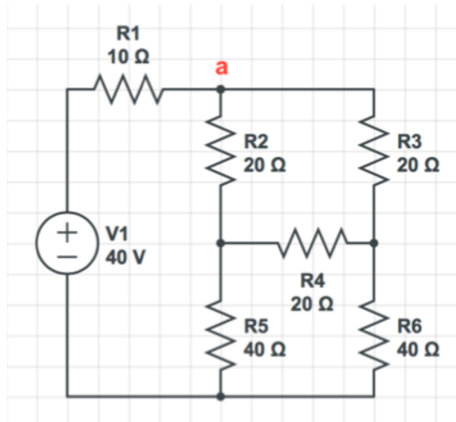
1.9 On the materiality of the human body

(10 points) The materiality of the human body ensures that it puts up some resistance to the flow of current. Find some values of impedance for the following human body parts. Be sure to cite your source(s) and explain why you trust it. (Given that there is a frequency dependence to these values, you may present any value of impedance found between 1 — 100 kHz.)

1. The whole body, from head to toe.
2. A single limb (such as an arm or a leg).
3. Blood.
4. Muscle.
5. Fat.

1.10 On circuit analysis

(20 points) For the circuit shown below.



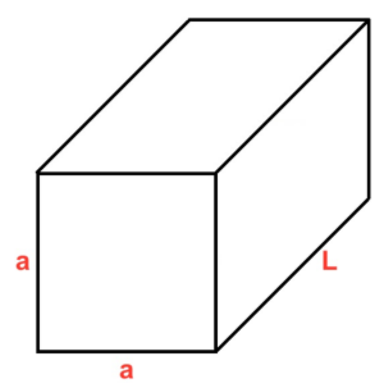
1. Use the delta-wye transformation rule to determine the power dissipated by R1.
2. Find the voltage at node a (just after R1).
3. If R4 were changed to 100 Ω, would the results to (10.1) and/or (10.2) change? How do you know?
4. How many nodes, branches, and loops are there?
5. What is the voltage at each node, the current through each branch?

Homework II

Assigned February 8, 2019. Due February 14, 2019 as *both* a hard copy in class and a pdf on Canvas. Each problem worth 10 points.

2.1 On the directionality of resistors

Though we rarely think about it, resistors have a directionality to them. To prove this to ourselves, let us consider a rectangular prism with a square cross-sectional area with a width of a on its square face, a length of L, and a resistivity of ρ. An example of such a conductor may be seen in the figure below.

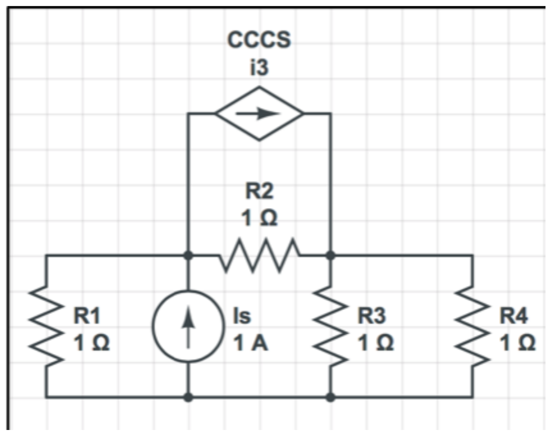


1. Determine the resistance between two parallel square faces, two parallel rectangular faces, and determine the ratio of these two resistances.

2. How would you go about performing a similar analysis for a resistor with a circular cross-sectional area?

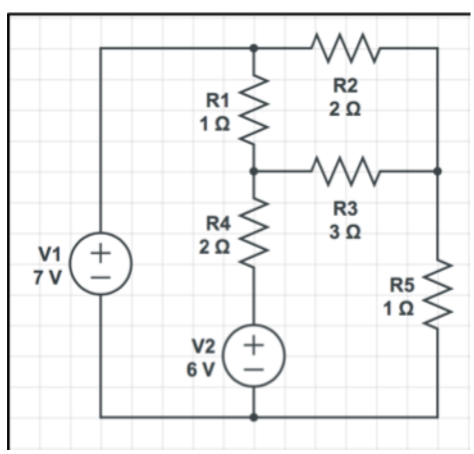
2.2 On nodal analysis

For the circuit shown below, find the current flowing though $R3$ ($i3$) and the current flowing through $R4$ ($i4$).



2.3 On mesh analysis

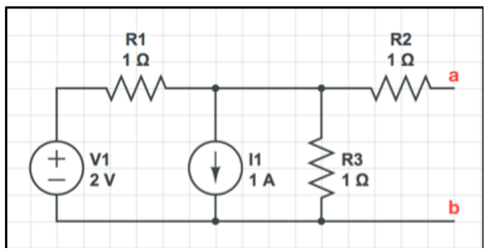
For the circuit shown below.



1. Perform a mesh analysis and determine the mesh currents in each loop. (You may solve it any way youd like, just walk me through how you solve it.)
2. Determine the current through R3. How you would actually measure the actual current through R3 in an actual circuit?

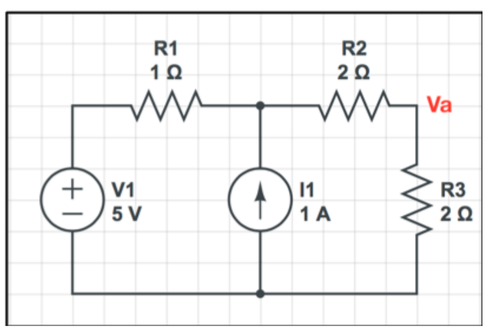
2.4 On Thevenin and Norton equivalent circuits

For the circuit shown below, determine the Thevenin and Norton equivalent circuits between terminals a and b.



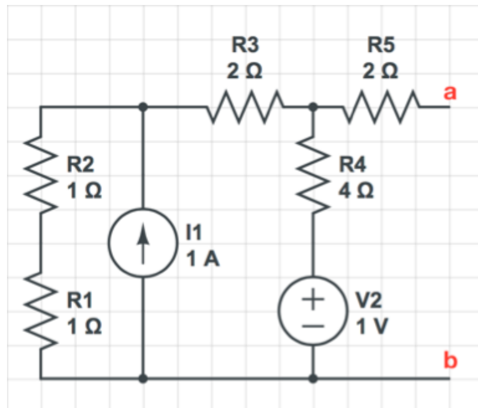
2.5 On superposition

Using the superposition theorem, find the voltage **Va** in in the circuit shown below. Be sure to show all your steps.



2.6 On Thevenin and Norton equivalent circuits, again

For the circuit shown below, determine the Thevenin equivalent circuit that would be found between the **a** and **b** terminals.

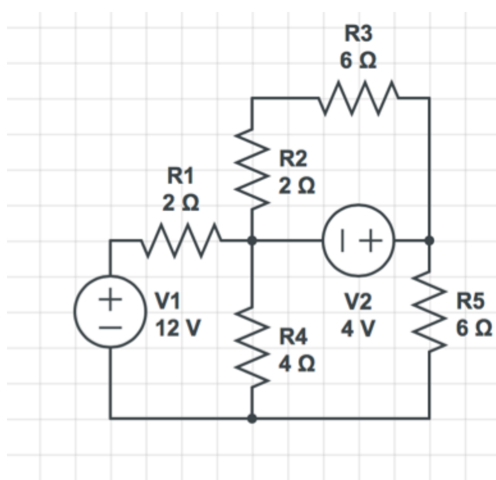


2.7 On input impedance

Electrocardiographic (ECG) signals tend to be about 1 mV before being measured by a monitoring system (that is, it is about 1 mV just before it gets to our skin). If skin resistance is about 100 kΩ, what is the voltage as measured by our system if our system has an input impedance of 500 kΩ, 1 MΩ, and 2MΩ. You may model the entire situation as a series combination of elements. What effect do you think sweating at the area of measurement will have on the voltages you just calculated?

2.8 On current through elements

For the circuit below, find the current passing through each element.



2.9 On blood vessel conductivity

Our blood vessels are something of glorified hollow conductive cylinders. Let us assume that the resistivity of the vessel, ρ, is constant throughout our length of interest, L. If we were to apply a potential difference, V, between the inner surface of the blood vessel (with radius a) and the outer surface of the blood vessel (with radius b), what would be the resistance we measure? (Hint: the differential resistance in this case is equal to $dR = \rho(dL/A)$.)

2.10 On personal experience

To the extent you feel comfortable sharing this information, have you ever had to interact with a piece of biomedical equipment that utilized electronic circuitry? (This may include anything from a Fitbit to an ECG and beyond.) If so, what aspects of that experience relating to the equipment did you find fascinating, and which aspects of it would you personally liked to have seen improved? [If you do not wish to share a personal experience or perhaps do not have one to share, consider the question in the abstract: *knowing what you know, what might you wish to see improved regarding current medical instruments*?]

Homework III

Assigned February 22, 2019. Due February 28, 2019 as *both* a hard copy in class (at the beginning thereof) and a pdf on Canvas (by 11:59pm). Each problem worth 20 points.

3.1 On mathematical proficiency

Please find the Laplace transforms for the following functions. Be sure to show as much of your work as you can. And please also resist the urge to use computational means of finding these answers. Try it yourself! It's good exercise.

1. $a(t) = \sin(\omega t)$
2. $b(t) = \cos(\omega t)$
3. $c(t) = 1 - \cos(\omega t)$
4. $d(t) = t$
5. $f(t) = t^n$

3.2 On poles and zeros, exponentially

Find the poles and zeroes of the following functions. Again, show as much of your work as you can and resist those computational urges.

1. $f(t) = e^{-3t}$
2. $f(t) = e^{-3t-2t+1t} \cdot e^{-t}$
3. $f(t) = e^{-3t} \cdot \sin(3t)$
4. $f(t) = e^{-3t} \cdot \cos(4t - 3t)$
5. $f(t) = e^{3t} \cdot e^{-2t} \cdot \sin(t) \cdot cos(-t)$

3.3 On the stability of geopolitical systems

"Stability" is a property we seek in many systems. Representing something of a "back of the envelope" approximation of the time course of geopolitical entities, a model was first proposed by a fellow by the name of Lewis Fry Richardson to track the development of weapons used/accrued by the nations of the earth. Richardson's Arms race, expressed as the coupled differential equation below, relates the rate at which one country (Country 1) develops its arms, dx/dt to the rate at which another country (Country 2) develops its arms dy/dt:

$$\frac{dx}{dt} = Ay - mx \tag{3.1}$$

$$\frac{dy}{dt} = Bx - ny \tag{3.2}$$

One can see that these rates are proportional to the the amount of arms the other country has minus the amount of arms the country has. If we know that $s < 0$ for a system to be stable, determine the conditions of stability in this system. (Assume the world starts peacefully with no weapons in either country.) Comment on the results.

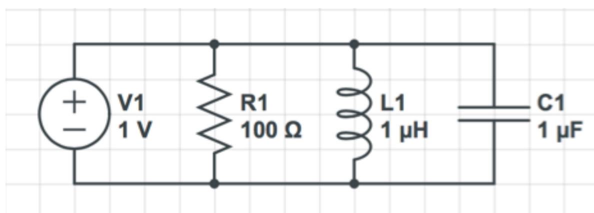
3.4 On a circuit as a differential equation

For circuit below

1. Write its differential equation and take its Laplace transform (assume all initial conditions are equal to zero).

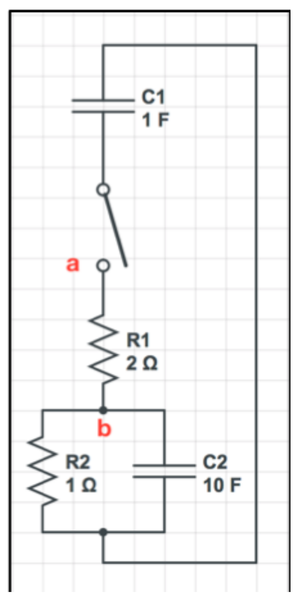
2. Find its transfer function (the ratio of voltage to current).

3. Choose **five** triplets of values (different from those printed on the figure). Plot the output of each transfer function as *either* a function of time (t) or frequency ($s = \jmath\omega$).



3.5 On poles and zeros, applicably

Believe it or not, this Laplace and circuitry stuff can actually be useful to those more biologically inclined among us. Indeed, it can be used to describe how drugs perfuse into our bodies. Let us consider taking some drugs.[1] We may model the drug as a charged capacitor, C1, with a potential of 100 V across it and a value of 1 F. Recalling that the tissues within our bodies may be modeled as the combination of a resistor, R1, in series with the parallel combination of a resistor, R2, and a capacitor, C2, we may model the perfusion of the drug within the body as the time rate of change of potential within the circuit shown at right. Once weve taken the drug (t = 0), the switch is closed. Using nodal analysis, determine the potential at point b in the s-domain. What are the poles of this model?



[1]Don't do drugs, kids.

Homework IV

Assigned March 14, 2019. Due March 28, 2019 as *both* a hard copy in class (at the beginning thereof) and a pdf on Canvas (by 11:59pm). Each problem worth 20 points.

4.1 On equivalent differential equations for transfer functions

For each of the transfer functions, $\tilde{H}(s)$, below, find a time-domain differential equation, $h(t)$, that would produce such a transfer function. (In this problem and those that follow from it, you may choose any initial conditions you would like, but I suggest setting them to 0 as a first attempt.)

1. $\tilde{H}(s) = \frac{4}{s^2+5s+6}$
2. $\tilde{H}(s) = \frac{4}{s^2+4s+4}$
3. $\tilde{H}(s) = \frac{4}{s^2+5s+4}$
4. $\tilde{H}(s) = \frac{4}{s^2-4}$
5. $\tilde{H}(s) = \frac{4}{s^2-5s-6}$

4.2 On designing circuitry

Let us treat the transfer functions reported in Problem 1 as system impedance, $\tilde{Z}(s)$. For each of the differential equations you found from Problem 1, draw a circuit which whose output, $v(t)$, represents the voltage time response of an input of a unit step of current, $u(t)$ applied at time, $t = 0$. (That is, design a circuit which when subjected to an input $u(t)$ would produces an output of $v(t) = z(t) * u(t)$.)

4.3 On determining output

Given the transfer functions reported in Problem 1, please answer the questions below.

1. Is the system stable when subjected to a unit step function? (In other words, is it BIBO stable?)

2. How damped is the system?

3. Where are the poles and zeros of the transfer function in the s-domain?

4. How long would it take for each system to achieve 95% of its total response? (For systems to which this does not apply, please state why.)

4.4 On designing filters

Design the following filters (using an input impedance, R1 = 1 kΩ) and produce a Bode plot demonstrating your results. (You may plot your results via computational means, i.e., a computer program, or by hand, by using what we learned in this class.)

1. A first order high-pass filter with a corner frequency at 0.5 Hz and a magnitude of gain of 10.

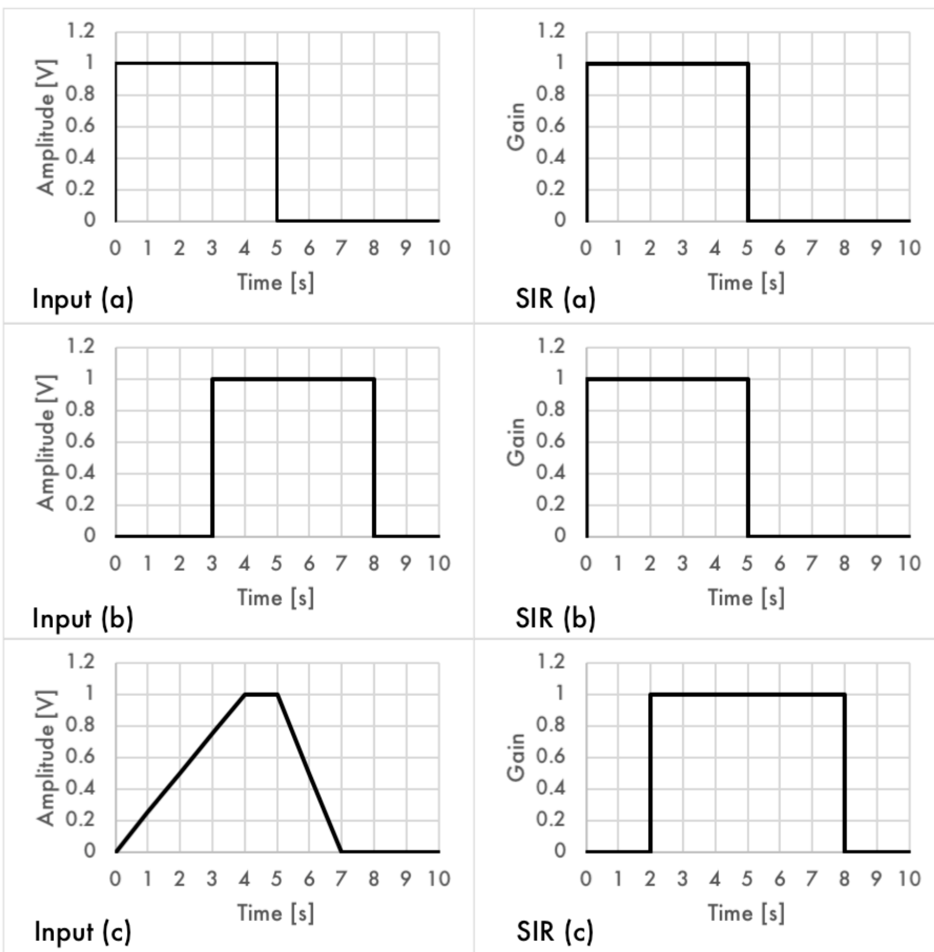
2. A second low-pass filter with a corner frequency of 40 Hz and a magnitude gain of 50.

3. A bandpass filter with filter with a lower corner frequency of 0.05 Hz and a higher corner frequency of 150 Hz.

4.5 On convolving

Given an input signal (seen on the left) and a system impulse response, SIR (seen on the right), determine the output signal that would arise. That is, convolve the input with the SIR. You may solve by hand (so long as your work is clear) and/or MATLAB, Excel, or some equivalent software to arrive at your answers. Please include all code used.

1. What is the output (the convolution) of “Input (a)” and “SIR (a)”?
2. What is the output of “Input (b)” and “SIR (b)”?
3. What is the output of “Input (c)” and “SIR (c)”?

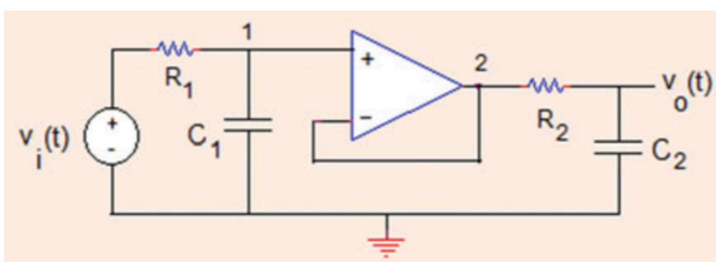


Homework V

Assigned March 14, 2019. Due April 11, 2019 as *both* a hard copy in class (at the beginning thereof) and a pdf on Canvas (by 11:59pm). Each problem worth 20 points.

5.1 On something new from a few familiar things

Two RC circuit sections (with capacitors grounded) are separated by a voltage buffer. An input signal $v_i(t)$ is applied to first R_1C_1 section, producing the signal $v_i(t)$ at node 1, it is buffered by the op-amp to the input of the second R_2C_2 section (node 2), the output of which is $v_o(t)$.



1. Derive the differential equation relating the circuit output, $v_o(t)$, to its input, $v_i(t)$.

2. Determine its characteristic equation (i.e., of the form $s^2 + 2\zeta\omega_n s +$ $omega_n^2$) and find its roots.

3. Determine whether this circuit could produce an underdamped response.

5.2 On reviewing the fundamentals

Recall a series circuit of a resistor, a capacitor, and an inductor. Let us hold that C = 200 nF and L = 50 mH.

1. Over what range of resistors values is the system underdamped, critically damped, and overdamped?

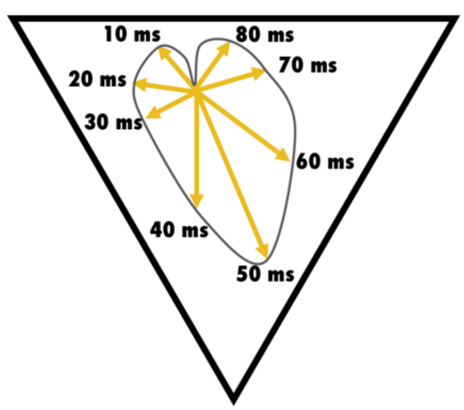
2. From within the range found above, select one representative value for the underdamped, critically damped, and overdamped case. Plot each cases voltage response over time.

5.3 On bioimpedance

In class we derived the analytical solution of the equivalent impedance, Z_{eq} of a simple bioimpedance model, $R_1 + R_2||C$ as seen from the resistance reactance plane. Find the analytical solution to the equivalent *admittance* of a model in which $R_1||(R_2 + C)$. (Hint: it might be more helpful to think in terms of conductances!) Interpret your results.

5.4 On the body electric

Given Einthoven's triangle below of the QRS complex, draw (by hand) the shape of the resulting electrocardiographic signal for lead I, lead II, and lead III. [Please use the conventional arrangement of Einthovens triangle when referring to your leads, unless you explicitly state you do otherwise.]



5.5 On identifying real-world instrumentation

As it turns out, there are hundreds and thousands of medical devices out there that use electric circuits. We noted five separate electrophysiological modalities in class electrocardiography, electroencephalography, electrogastrography, electromyography, and electrooculaography. Find at least three commercial examples of medical devices for each listed modality (i.e., a total of 15 devices). List (1) who makes it, (2) how the product generally works, and (3) if you can find it, how much it costs. (You may also comment on whether you believe that price to be fair/reasonable.)

Homework VI

Assigned March 14, 2019. Due April 23, 2019 as *both* a hard copy in class (at the beginning thereof) and a pdf on Canvas (by 11:59pm). Each problem worth X points.

6.1 On functional completeness

In class we discussed the fact that all possible computational relationships (NOT, AND, OR, NAND, NOR, XOR, and XNOR) can be represented by a combination of NOR gates. During class we were only able to prove it for NOT, AND, and OR. Prove it for all cases. That is, make NAND, NOR, and XOR gates (dont do XNOR) from NOR gates.

6.2 On knights, knaves, truths, and lies

Recall the land inhabited by knights and knaves. Knights always tell the truth and knaves always lie. For each of the following give an answer and explain your reasoning as best you can. It will often help to draw a truth table or a logic/decision tree. Please use your own methods or feel free to ask me what I might do.

Please note that I will make use of the singular form of "they" and "them" to describe the individuals below. If that leads to any consternation, you may replace with any singular pronoun of your choice.

1. You come across three inhabitants and ask the first, A, "Are you a knight or a knave?" A answers, but so quietly you cant hear them. You ask B "What did A say? to which B responds A said they were a knave." Upon hearing this, C piped up and said "Dont believe that; its a lie." Is C a knight or a knave? (Further, is it possible to know what A is?)

2. Instead of asking if A was a knight or a knave, you could have asked how many of the three of them were knaves. So, upon meeting the next three inhabitants (D, E, and F), D answers indistinctly, so you ask E what D had said. E said that D had said that exactly two of them were knaves. F says E is lying. Is it possible to know what D is? (Further, what are E and F?)

3. Meeting with two inhabitants, G and H, G says, "Both of us are knaves." What is G? What is H?

4. Meeting two other inhabitants, I and J, I says, "At least one of us is a knave." What are I and J?

5. Meeting yet two more, K and L, K says, "We are the same type we are either both knights or both knaves." What are K and L?

6. Meeting another two, M and N, you ask M if they are a knight and get an answer. You then ask N if M is a knight and get an answer. Are M and Ns answers the same or different?

6.3 On logic gates

Did you know that each of the above knight and knave situations can be represented by a combination of logic gates? For example, a simple single knave can be represented by a NOT gate since they always lie. An easy way to work in this matter is to assign 0s and 1s to lies and truth (and thereby liars and truth-tellers and thereby knaves and knights) and construct a truth table. From that truth table and from the logic you used in deriving the solution, put together a sequence of logic gates for each of the situations reported in Problem 2.

Chapter 7

Collected exercises

Contents

7.1 A constant charge through a cross-section

How much charge passes through a cross-section of a conductor in 60 seconds if a DC current value is measured at 0.1 mA?

7.2 An arbitrary charge through a cross-section

Determine the total charge entering a terminal between $t = 0$ seconds and $t = 10$ seconds if the current (in amps) passing through is

$$i(t) = \frac{1}{\sqrt{5t+2}}. \tag{7.1}$$

7.3 A “tera”ble puzzle

Approximately how much current is necessary to transmit one terabyte of information in an hour?

7.4 A pacemrker’s power requirements

A cardiac pacemaker will provide approximately 5,000 J of energy over 5 years. Determine the capacity of a 5 V lithium battery necessary to drive this pacing such that only 40% of its energy is spent over that time.

7.5 A neuron's excitation energy

A colleague of yours has been in their lab ginning up new neurons. You, as their resident electrical expert, are tasked with determining the energy consumed by the cell. If the current and voltage variations are found to be functions of time ($t \geq 0$)

$$i(t) = 3t \tag{7.2}$$

$$v(t) = 10e^{6t} \tag{7.3}$$

determine the energy consumed between 0 and 2 ms.

7.6 A thump to the chest

(a) A typical defibrillator delivers 200-1000 V in less than 10 ms. How much current is needed to deliver 120, 240, and 360 Joules?
(b) A human heart ways about 300 grams. From approximately how high of a cliff would one have to drop a heart such that the impact was equivalent to the energy delivered to someone's chest from a defibrillator?

7.7 An ohmic power expression

Utilizing Ohm's law, express units of power to include ohms.

7.8 A toaster based problem

A toaster draws 2 A at 120 V. What is its resistance?

7.9 Another toaster based problem

How much current is drawn by a toaster with a resistance of 10 Ω at 110 V?

7.10 A current power

In the circuit shown, calculate the current, i, the conductance, G, and the power, p.

7.11 A sodium channel's conductance

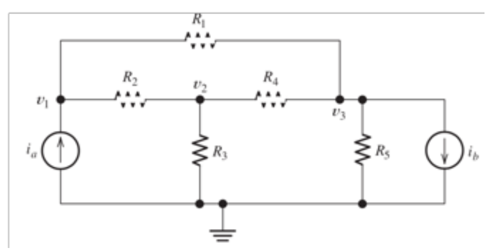
Conductance (G/A) of a sodium channel of a cell membrane at a specific time is 10 mS/cm^2. If the channel length as 100 nm, what is its conductivity?

7.12 A simple tissue's resistance

Determine the resistance of a homogenous and isotropic tissue with a cross-sectional area which can be described by the functions $y = 8 - x^2$ from $x = -2$ cm to $x = +2$ cm, a length of 10 cm (parallel to the z-axis), and a resistivity of 80 Ωm.

7.13 A few nodes of KCL

Use KCL to write equations at each node.



7.14 Matrix notation

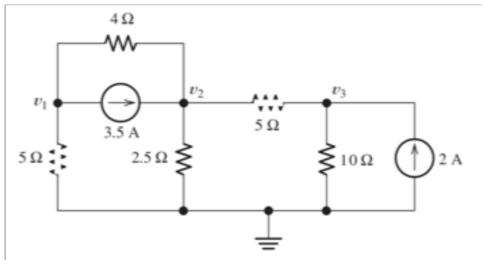
Write the matrix form of the equations written above.

7.15 Cramer's rule

Using Cramers rule on the matrix equations above, what are the results?

7.16 Straight to the matrix

Write the node-voltage equations to the circuit at right in the matrix form.



7.17 Laplace transformations

Find the Laplace transform of the following:

- $f(t) = 9$
- $f(t) = \delta(t)$ the Dirac-delta function
- $f(t) = e^{-3t/2}$
- $f(t) = \sin \omega t$

7.18 Differential equation of a series RLC circuit

What is the differential equation describing an inductor, a resistor, and a capacitor in series?

7.19 Differential equation of a series RLC circuit with numbers

What is the differential equation describing an inductor, a resistor, and a capacitor in series? What is a solution to that differential equation if L = 1 H, R = 1 Ω, and C = 100 mF? Take the Laplace transform of the differential equation. What are its poles and zeros?

7.20 An s-plane with found zeros

Draw an s-plane. Label the axes. Plot the poles and zeros you found. What can you say of the behavior of the system? If the poles had an imaginary component to them (that is, if they were pushed along the vertical axis), how would that affect our system?

7.21 An s-plane with arbitrary zeros

Draw an s-plane. Label the axes. Plot these poles – (-2,0) and (-5,0) – and this zero – (0,0). Plot what the signal would look over time at each of these poles and zeros.

7.22 $\tilde{Z}(s)$ (i.e., $\tilde{V}(s)/\tilde{I}(s)$) of a resistor and a capacitor in parallel

1. Show the *frequency response* (i.e., $\tilde{Z}(s)$ v. frequency) of such a system. If it helps to ascribe values, assume that the resistor is 1 ohm and the capacitor is 10 farad.
2. Comment on the behavior. Is it like anything you have seen before?
3. (*The kind of question that might be on a glorified quiz.*) If the current through the circuit is 1 amp when $t \geq 0$, what will be the voltage response?

7.23 A voltage time response of a system with a given impedance

Given that

$$\tilde{Z}(s) = \frac{s^3 + 9}{(s+1)(s+3)} \tag{7.4}$$

1. What is the *time response* of potential to a unit step of current?
2. (*The kind of question that might be on a glorified quiz.*) Which exponential term will most significantly affect the signal?

7.24 A stable system?

1. A system, $y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$, when $x(t) = \cos(t)$.
2. A system, $y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$, when $x(t) = u(t)$.
3. A system, $y(t) = x(t)/t$, when $x(t) = 2$.
4. A system, $y(t) = dx(t)/dt$, when $x = 1$.
5. A system, $y(t) = dx(t)/dt$, when $u(t)$.

7.25 Pedro, Apollonia, & Peter

You come across three inhabitants and ask the first, Pedro, "Are you a knight or a knave?" Pedro answers, but so quietly you can't hear him. You ask Apollonia "What did he say?" to which she responds "Pedro said he was a knave." Upon hearing this, Peter piped up and said "Don't believe that; its a lie." Is Peter a knight or a knave? (Further, is it possible to know what Pedro is?)

7.26 Roger & Oedipa

Shortly after that you meet two inhabitants, Roger Mexico and Oedipa Maas. Roger claims, "Both of us are knaves." What are Roger and Oedipa?

7.27 Yes & No

Suppose youve heard a rumor that theres gold buried nearby. You meet a local and want to know whether there really is gold in them thar hills, but you dont know whether the person is a knight or a knave. If you are only allowed to ask only one question answerable by "yes" or "no", what do you ask?

7.28 Lisa & Louise

Lisa and Louise are twins indistinguishable in appearance. One always lies, the other always tells the truth. You dont know which is which. You meet one of them and may ask one question to determine which twin is truth. What do you ask and what does it tell you?

7.29 NOT $\rightarrow$ NOR

Make a NOT gate from one or more NOR gates.

7.30 PAND $\rightarrow$ NOR

Make an AND gate from one or more NOR gates.

7.31 OR → NOR

Make an OR gate from one or more NOR gates.

7.32 NAND → NOR

Make a NAND gate from one or more NOR gates.