Pattern Recognition and Machine Learning: Homework 2, Zhengzuo Liu

Problem 1

Answer:

The distance of point $\boldsymbol{v}$ to H is

$$d(\boldsymbol{v}, H) = \frac{|\boldsymbol{w}^T \boldsymbol{v} + b|}{||\boldsymbol{w}||_2}$$

Problem 2

Answer:

$$L(f(x_i; w, b), y_i) = \frac{1}{2} \sum (w^T x + b - y)^2$$

$$\frac{\partial L}{\partial w} = 0 \Rightarrow \sum [(w^T x + b - y) \cdot x] = 0. \Leftrightarrow \sum [x \cdot (x^T w + b - y)] = 0 \quad ①.$$

$$\frac{\partial L}{\partial b} = 0 \Rightarrow \sum (w^T x + b - y) = 0, \text{ since } \sum y = N_1 \cdot (\frac{N}{N_1}) + N_2 \cdot (-\frac{N}{N_2}) = 0$$

$$\Rightarrow b = -\frac{1}{N} \sum w^T x = -\frac{1}{N} \sum x^T w = -\frac{1}{N} (N_1 m_1^T + N_2 m_2^T) w.$$

$$① \Leftrightarrow \sum (x \cdot x^T - \frac{x}{N} \sum x^T) w = \sum x \cdot y = N_1 m_1 (\frac{N}{N_1}) + N_2 \cdot m_2 \cdot (-\frac{N}{N_2}) = N(m_1 - m_2).$$

$$S_w = \sum (x - m)(x - m)^T = \sum_{x \in X_1} (x - m_1)(x - m_1)^T + \sum_{x \in X_2} (x - m_2)(x - m_2)^T$$

$$= \sum x x^T - N_1 m_1 m_1^T - N_2 m_2 m_2^T$$

$$① \text{ Left} = [\sum x x^T - \sum \frac{x}{N} (N_1 m_1^T + N_2 m_2^T)] w$$

$$= [\sum x x^T - \frac{1}{N} (N_1 m_1 + N_2 m_2)(N_1 m_1 + N_2 m_2)^T] w$$

$$= [\sum x x^T - \frac{1}{N} (N_1^2 m_1 m_1^T + N_2^2 m_2 m_2^T + N_1 N_2 (m_1 m_2^T + m_2 m_1^T))] w$$

$$= [\sum x x^T - \frac{1}{N} (N_1^2 m_1 m_1^T + N_1 N_2 m_1 m_1^T + N_1 N_2 m_2 m_2^T + N_2^2 m_2 m_2^T$$

$$- N_1 N_2 (m_1 - m_2)(m_1 - m_2)^T)] w$$

$$= [S_w + \frac{N_1 N_2}{N} (m_1 - m_2)(m_1 - m_2)^T] w = N(m_1 - m_2).$$

$$\Leftrightarrow S_w \cdot w = [N - \frac{N_1 N_2}{N} (m_1 - m_2)^T w] (m_1 - m_2).$$

given that $[N - \frac{N_1 N_2}{N} (m_1 - m_2)^T w]$ is scalar. and we only care about the direction of w. we get:

$$w^* = S_w^{-1} (m_1 - m_2). = w^* \quad \square.$$

Problem 3

Answer:

- $\sigma(x) + \sigma(-x) = 1$

  $\sigma(x) + \sigma(-x) = \frac{1}{1+e^{-x}} + \frac{1}{1+e^{x}} = \frac{e^{x}}{e^{x}+1} + \frac{1}{1+e^{x}} = 1.$

- $\sigma'(x) = \sigma(x)(1-\sigma(x)).$

  $\sigma'(x) = \frac{-1}{(1+e^{-x})^2} \cdot (-e^{-x}) = \frac{1}{(1+e^{-x})} \cdot \frac{e^{-x}}{1+e^{-x}} = \sigma(x)(1-\sigma(x)).$

- $\tanh(x) = 2\sigma(2x) - 1.$

  $\tanh(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{2}{1+e^{-2x}} - 1 = 2\sigma(2x) - 1.$

$$\frac{\partial L}{\partial w} = \frac{\partial((\sigma(w^Tx+b)-\hat{y})^2)}{\partial w}$$
$$= 2(\sigma(w^Tx+b)-\hat{y}) \cdot \frac{\partial \sigma(w^Tx+b)}{\partial w}$$
$$= 2(\sigma(w^Tx+b)-\hat{y}) \cdot \sigma(w^Tx+b)\,\sigma(-w^Tx-b) \cdot x$$
$$\frac{\partial L}{\partial b} = 2(\sigma(w^Tx+b)-\hat{y}) \cdot \frac{\partial \sigma(w^Tx+b)}{\partial b}$$
$$= 2(\sigma(w^Tx+b)-\hat{y}) \cdot \sigma(w^Tx+b)\,\sigma(-w^Tx-b)$$

Problem 4

Answer:

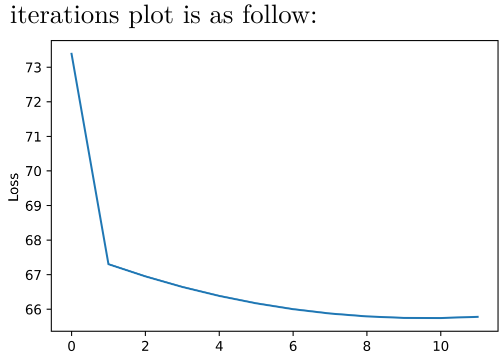
1

$$w^* = [-0.0013, -0.0009, -0.0006, -0.0003, -0.0004, -0.0019, -0.0008 - 0.0008, -0.00005]^T$$

The classification accuracy on the dataset is 96.33%.
Program see ”PRML_H2_4.1.py”.

2

For all processes and results see ”PRML_H2_4.2.py”
The the loss value against iterations plot is as follow:



Final accuracy is 100% (tested on the last 100 samples, trained on others), with a modified threshold value 0.64.

3

The cosine between two w in section 1 and 2 is -0.91. It means that the two w are almost in the exact reverse direction—very similar as far as we concern. The explanation should be similar to that of Problem 2.

The sixth component of both w has the largest absolute value. Hence, single epithelial cell size the most indicative feature that implies one gets breast cancer.