Exercise 3

Random Vectors

Deadline: October 12, 2016.

1. Reading.

   (a) Lecture notes 3

   (b) Chpater 4 of the book "Statistical Inference".

2. The joint distribution of X and Y is

$$f(x,y) = \begin{cases} cx^2y, & x^2 \le y \le 1 \\ 0, & other \end{cases}$$

   (a) Calculate the constant c.

   (b) Find the marginal distribution of X and Y.

   (c) Calculate $f_{X|Y}(x,y)$. Write the conditional distribution of X when $Y = 1/2$.

   (d) Calculate the conditional distribution $P(Y \ge 1/8|X = 1/2)$

3. Show that if $(X,Y) \sim$ bivariate normal$(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$, then the following are true.

   (a) The marginal distribution of X is $N(\mu_X, \sigma_X^2)$ and the marginal distribution of Y is $N(\mu_Y, \sigma_Y^2)$.

   (b) The conditional distribution of Y given $X = x$ is

$$N(\mu_Y + \rho(\sigma_Y^2/\sigma_X^2)(x - \mu_X), \sigma_Y^2(1 - \rho^2))$$

(c) For any constants a and b, the distribution of $aX + bY$ is

$$N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$$

4. Let Z_1, Z_2 be independent $N(0, 1)$ random variables, and define new random variables X and Y by

$$X = a_X Z_1 + b_X Z_2 + c_X, \quad Y = a_Y Z_1 + b_Y Z_2 + c_Y$$

where $a_X, b_X, c_X, a_Y, b_Y, c_Y$ are constants.

   (a) Calculate $EX, EY, VarX, VarY, Cov(X, Y)$.

   (b) Find the joint distribution of (X, Y).

5. Suppose $X_1 \sim \Gamma(\alpha_1, \lambda), X_2 \sim \Gamma(\alpha_2, \lambda)$ and X_1, X_2 are independent,

   (a) Show that $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$ are independent, and calculate the pdf of Y_1, Y_2.

   (b) Show that $Y_1 = X_1 + X_2$ and $Y_3 = X_1/X_2$ are independent, and calculate the pdf of Y_3.

6. If X, Y are any two random variables, prove:

   (a) $EX = E(E(X|Y))$

   (b) $VarX = E(Var(X|Y)) + Var(E(X|Y))$

7. (Student's t distribution) Let $U \sim N(0, 1)$ and $V \sim \chi_p^2$ be two independent random variables. Define a transformation $T = \frac{U}{\sqrt{V/p}}$ and $W = V$.

   (a) Show the pdf of a χ_p^2 random variables.

   (b) Show the joint pdf of T and W.

   (c) Show the marginal pdf of T.

8. Let $(X_1, ..., X_n)$ have a multinomial distribution with m trails and cell probabilities $p_1, ..., p_n$ (see Definition 4.6.2). Show that, for every i and j,

$$X_i|X_j = x_j \sim binomial(m - x_j, \frac{p_i}{1 - p_j})$$

$$X_j \sim binomial(m, p_j)$$

and that $Cov(X_i, X_j) = -mp_ip_j$.

9. (Dirichlet distribution) Let $U_i \sim Gamma(\alpha_i, scale = \theta), i = 1, ..., m$ be a set of m independent Gamma random variables. Consider the transform

$$Y = \sum_{i=1}^{m} U_i, \quad X_k = \frac{U_k}{Y}, k = 1, ..., m-1$$

   (a) Derive the joint distribution of $(X_1, ..., X_{m-1})$.

   (b) Calculate $EX_i, VarX_i, Cov(X_i, X_j), i \neq j, \quad i, j = 1, ..., m-1$

10. (Conjugacy) Let $(X_1, ..., X_m) \sim$ Multinomial$(n, \theta_1, ..., \theta_m)$ and $(\theta_1, ..., \theta_m) \sim$ Dirichlet$(a_1, ..., a_m)$

   (a) (Prior distribution) Write the pdf of Dirichlet distribution, say, $p(\theta_1, ..., \theta_m)$.

   (b) (Likelihood function) Derive the conditional distribution $p(x_1, ..., x_m|\theta_1, ..., \theta_m)$.

   (c) (Posterior distribution) Derive the conditional distribution $p(\theta_1, ..., \theta_m|x_1, ..., x_m)$.

   (d) (Marginal likelihood) Derive the distribution $p(x_1, ..., x_m)$.

11. (Negative Binomial Distribution Derivation)

   (a) For the hierarchical model

$$Y|\Lambda \sim Poisson(\Lambda), \quad \Lambda \sim gamma(\alpha, \beta)$$

   find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

   (b) Show that three-stage model

$$Y|N \sim binomial(N, p), \quad N|\Lambda \sim Poisson(\Lambda), \quad \Lambda \sim gamma(\alpha, \beta)$$

   leads to the same marginal (unconditional) distribution of Y.