统计学方法及其应用

Statistical Methods with Applications

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Random Variables

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Defining a probability function

Let $S = \{s_1, \ldots, s_n\}$ be a finit set. Let $\mathcal{B}$ be any sigma algbra of subsets of S. Let $p_1, \ldots, p_n$ be nonnegative numbers that sum to 1. For any $A \in \mathcal{B}$, define $P(A)$ by

$$P(A) = \sum_{\{i : s_i \in A\}} p_i.$$

Then P is a probability funciton on $\mathcal{B}$. This remains true if $S = \{s_1, s_2, \ldots\}$ is a countable set.

Tossing coins

- 扔一枚硬币，观察到正面的概率
  - $S = \{H, T\}$
  - $P(\text{正面}) = P(\{H\}) = 1/2$
- 扔一枚硬币三次，观察到两次正面的概率
  - $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$
  - $P(\text{两次正面}) = P(\{HHT, HTH, THH\}) = 3/8$
- 扔一枚硬币一百次，观察到十次正面的概率
  - $S = \{2^{100} \text{ elements}\}$
  - $P(\text{十次正面}) = $ Unable to count!
- 实际上正面出现的次数仅有101种可能

It is much easier to deal with a summary variable than with the original probability structure.

How to reduce the sample space?

- 定义计数函数
  - $X(s) = \#\{\mathrm{H}\}$
  - 定义域 $\mathcal{S}$ 包含 2^{100} 个元素
  - 值域 $[0, 100]$ 包含 101 个元素
- 观察到十次正面的次数
  - $P(\#\{\mathrm{H}\}=10)=P(X=10)=\mathrm{C}(100,10)\times 0.5^{10}\times 0.5^{90}\approx 1.37\times 10^{-17}$
- 扔任意硬币 n 次，观察到 x 次正面的次数
  - $P(X=x \mid n, p) = \mathrm{C}(n, k) \times p^k \times (1-p)^{n-k}$

Random variables

Random variable

A **random variable** is a function from a sample space $\mathcal{S}$ into the real numbers.

- 随机变量是定义在样本空间上的实值函数
- 随机变量用大写字母表示，例如X, Y, Z
- 随机变量的取值用对应的小写字母表示，例如x, y, z

Examples of random variables

- 掷一只骰子
  - $X =$ 观测到的点数
- 掷两只骰子
  - $X =$ 观测到的点数之和
  - $Y =$ 观测到的点数之差的绝对值
- 扔一枚硬币3次
  - $X =$ 观测到正面的次数
- 从一副扑克牌中任意抽取五张
  - $X =$ 抽到K的张数

随机变量的引入简化了研究的问题，

体现了统计学中**数据简约**的思想

随机变量的取值很重要，

但随机变量以什么概率取得这些值更重要

Define a probability on the domain

Let $S = \{s_1, \ldots, s_n\}$ be a finit set. Let $\mathcal{B}$ be any sigma algbra of subsets of S. Let $p_1, \ldots, p_n$ be nonnegative numbers that sum to 1. For any $A \in \mathcal{B}$, define $P(A)$ by

$$P(A) = \sum_{\{i: s_i \in A\}} p_i.$$

Then P is a probability funciton on $\mathcal{B}$. This remains true if $S = \{s_1, s_2, \ldots\}$ is a countable set.

Induce a probability on the range

Suppose that the range of X is also a finite set $\mathcal{X}$, we can then define

$$P_X(X = x_i) = P\left(\{s_j : s_j \in \mathcal{S}, X(s_j) = x_i\}\right)$$

Now, let the sigma algebra $\mathcal{B}$ be the collection of all subsets of $\mathcal{X}$,

Axiom 1 : for any set $A \in \mathcal{B}$,

$$\begin{aligned} P_X(A) &= P\left(\bigcup_{x_i \in A} \{s_j : s_j \in \mathcal{S}, X(s_j) = x_i\}\right) \\ &= \sum_{x_i \in A} P\left(\{s_j : s_j \in \mathcal{S}, X(s_j) = x_i\}\right) \\ &\geq 0 \end{aligned}$$

Axiom 2 : for the entire sample space $\mathcal{X}$,

$$P_X(\mathcal{X}) = P\left(\bigcup_{x_i \in \mathcal{X}} \{s_j : s_j \in \mathcal{S}, X(s_j) = x_i\}\right) = P(\mathcal{S}) = 1$$

Axiom 3 : for pairwise disjoint sets $A_i, A_2, \ldots,$

$$\begin{aligned} P_X\left(\bigcup_{k=1}^{\infty} A_k\right) &= P\left(\bigcup_{k=1}^{\infty} \left\{\bigcup_{x_i \in A_k} \{s_j : s_j \in \mathcal{S}, X(s_j) = x_i\}\right\}\right) \\ &= \sum_{k=1}^{\infty} P\left(\bigcup_{x_i \in A_k} \{s_j : s_j \in \mathcal{S}, X(s_j) = x_i\}\right) \\ &= \sum_{k=1}^{\infty} P_X(A_k) \end{aligned}$$

Change of the sample space

- 样本空间的转换
  - 在随机变量的定义域上 $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$
  - 在随机变量的值域上 $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$
  - 随机变量建立的映射 $X\colon \mathcal{S} \mapsto \mathcal{X}$
- 定义在随机变量定义域上的概率函数

$$\begin{aligned} P(s_j) &= p_j \\ P(A) &= \textstyle\sum_{s_j \in A} p_j \end{aligned}$$

- 定义在随机变量值域上的概率函数

$$P_X(X = x_i) = P(\{s_j \in \mathcal{S}\colon X(s_j) = x_i\})$$

Distributions of random variables

- 随机变量的所有可能取值及取得每一个值的概率
- 扔一枚硬币三次，观察出现正面的次数
  - $\mathcal{S} = \{\text{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}\}$
  - $\mathcal{X} = \{0, 1, 2, 3\}$
  - X: $\mathcal{S} \mapsto \mathcal{X}$

    $X(\text{HHH})=3$ $X(\text{HHT})=2$ $X(\text{HTH})=2$ $X(\text{THH})=2$

    $X(\text{TTH})=1$ $X(\text{THT})=1$ $X(\text{HTT})=1$ $X(\text{TTT})=0$
  - $P(X = 0) = 1/8$ $P(X = 1) = 3/8$

    $P(X = 2) = 3/8$ $P(X = 3) = 1/8$

Cumulative distribution function (cdf)

Distribution function

The **cumulative distribution function** (cdf) of a random variable X, denotd by $F_X(x)$, is defined by

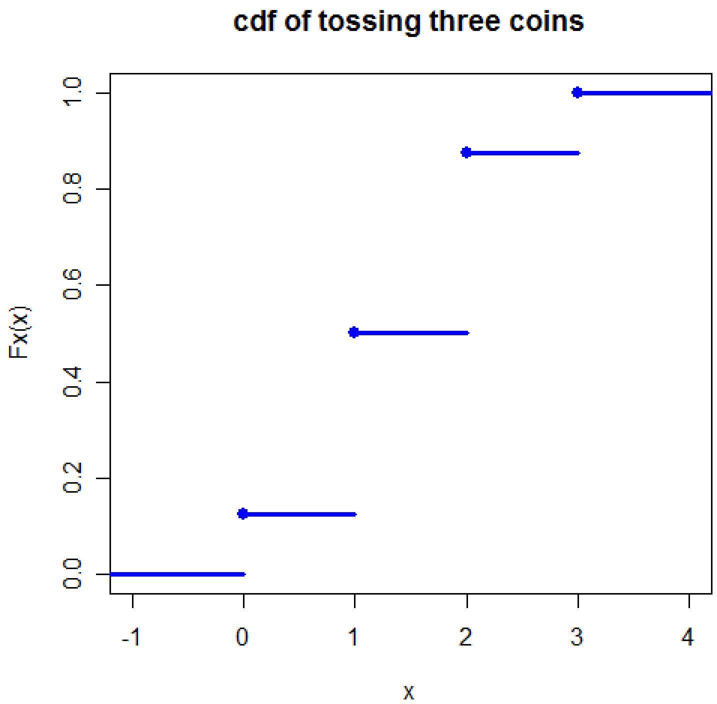
$$F_X(x) = P_X(X \leq x), \text{ for all } x.$$

At most

cdf

- 扔一枚硬币三次，观察出现正面的次数
  - $\mathcal{X} = \{0, 1, 2, 3\}$
  - $P(X=0) = 1/8$　　$P(X=1) = 3/8$
    $P(X=2) = 3/8$　　$P(X=3) = 1/8$
  - 分布函数

$$F_X(x) = \begin{cases} 0 & \text{if} \quad -\infty < x < 0; \\ {}^1\!/\!{}_8 & \text{if} \quad 0 \le x < 1; \\ {}^1\!/\!{}_2 & \text{if} \quad 1 \le x < 2; \\ {}^7\!/\!{}_8 & \text{if} \quad 2 \le x < 3; \\ 1 & \text{if} \quad 3 \le x < \infty. \end{cases}$$



Necessary and sufficient condition

Necessary and sufficient condition

The function $F(x)$ is a cdf if and only if the following **three conditions** hold:

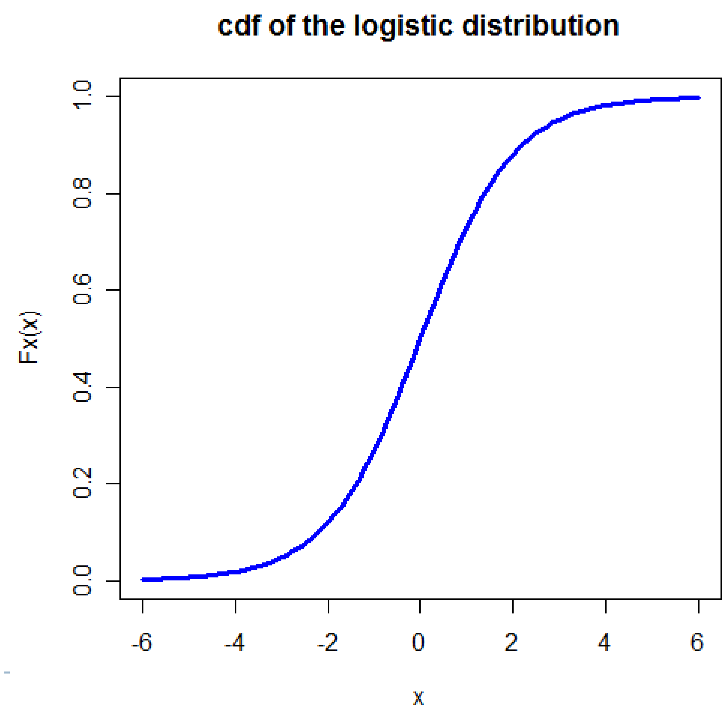
1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$;
2. $F(x)$ is a nondescreasing function of x;
3. $F(x)$ is right-continuous; that is, for every number x_0, $\lim_{x \to x_0^+} F(x) = F(x_0)$.

Logistic cdf

- Logistic distribution

$$F_X(x) = \frac{1}{1+e^{-x}}$$

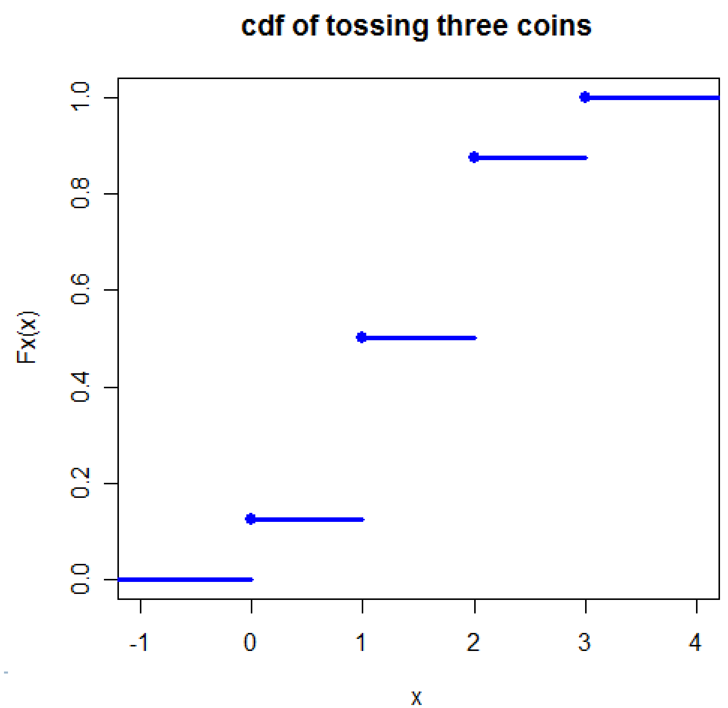
- 充要条件的满足性
  - 负无穷时为0
  - 正无穷时为1
  - 不减
  - 右连续



Discrete random variables

Discrete random variables

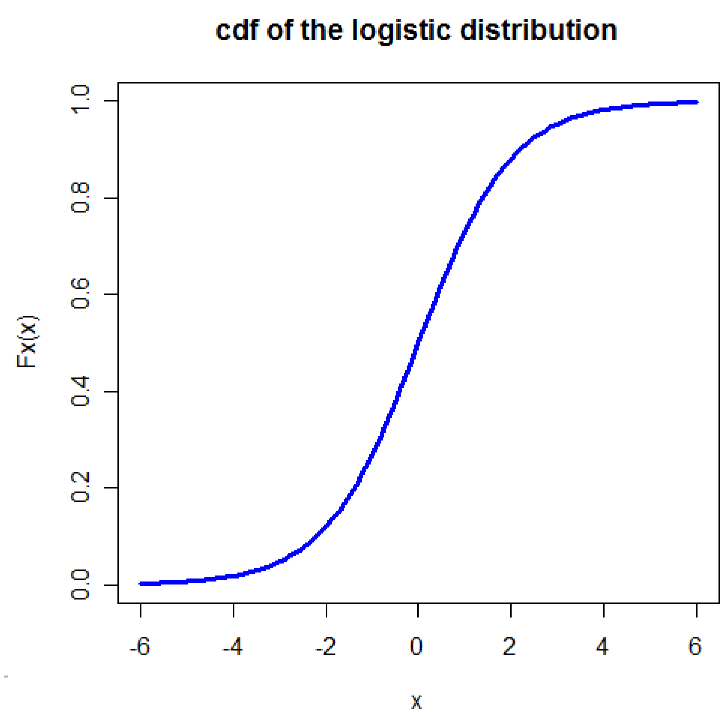
A random variable X is **discrete** if $F_X(x)$ is a step function of x.



Continuous random variables

Continuous random variables

A random variable X is **continuous** if $F_X(x)$ is a continuous function of x.



Identically distributed

Identically distributed

The random variables X and Y are **identically distributed** if, for every set $A \in \mathcal{B}^1$,

$$P(X \in A) = P(Y \in A).$$

1. $\mathcal{B}^1$ is the smallest sigma algebra containing all the intervals of real numbers of the form (a,b), $[a,\ b)$, $(a,b]$, and $[a,b]$.
2. Two identically distributed random variables are not necessarily equal.

Identically distributed

The following two statements are equivalant

1. Two random variables X and Y are identically distributed;
2. $F_X(x) = F_Y(x)$ for every x.

$$P(X \in A) = P(Y \in A) \text{ for any set } A \in \mathcal{B}^1 \quad \Rightarrow$$
$$P(X \in (-\infty, x]) = P(Y \in (-\infty, x]) \quad \Rightarrow$$
$$F_X(x) = F_Y(x)$$

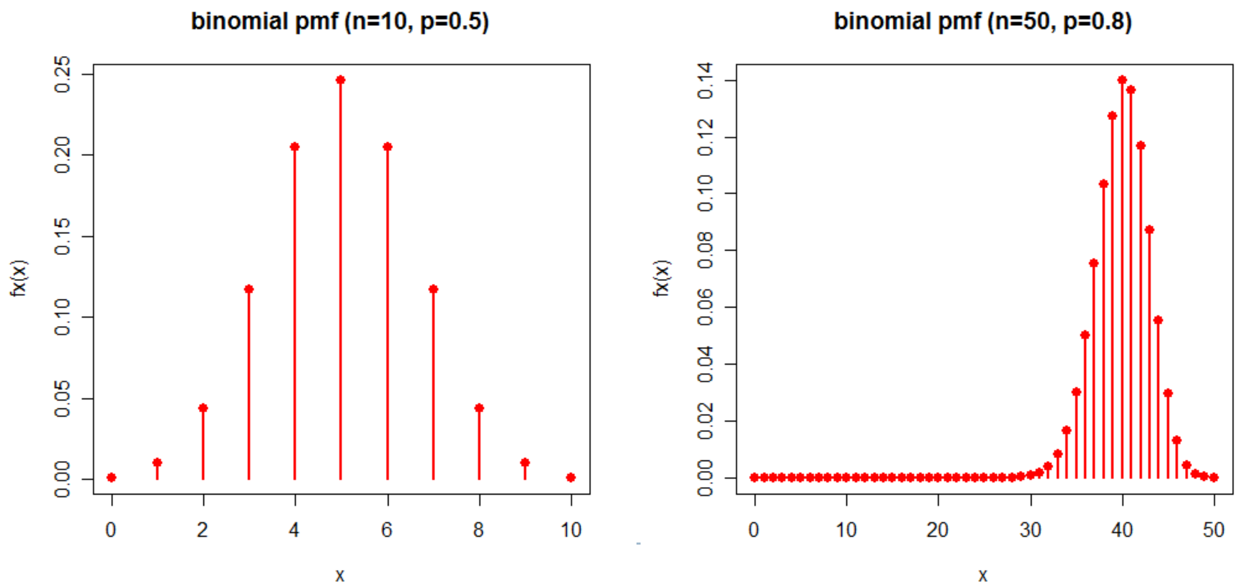
Probability mass functions

Probability mass function

The **probability mass function** (pmf) of a discrete random variable X, denoted by $f_X(x)$, is given by

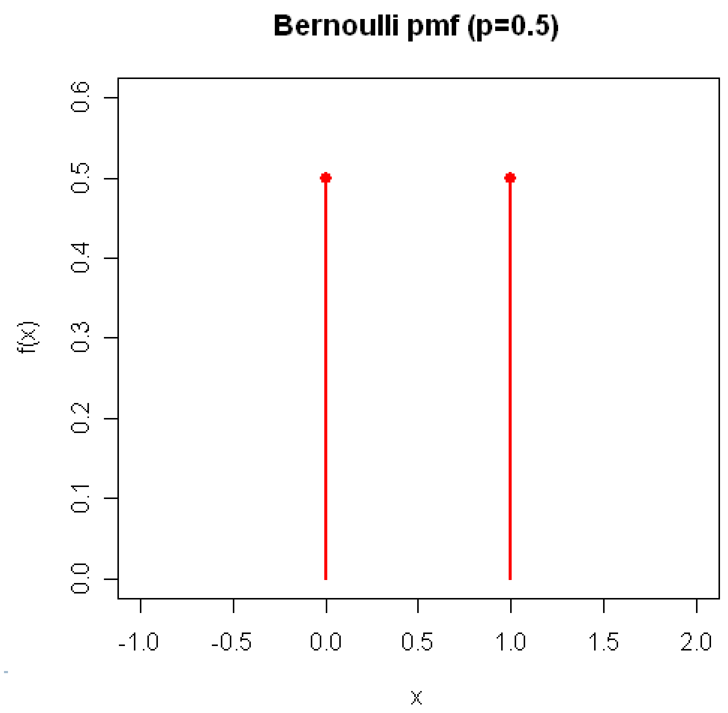
$$f_X(x) = P_X(X = x), \text{ for all } x.$$

Exact



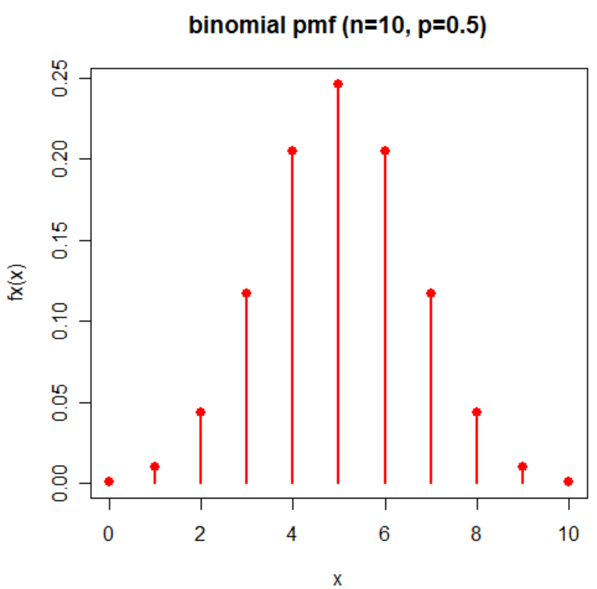
Bernoulli distribution

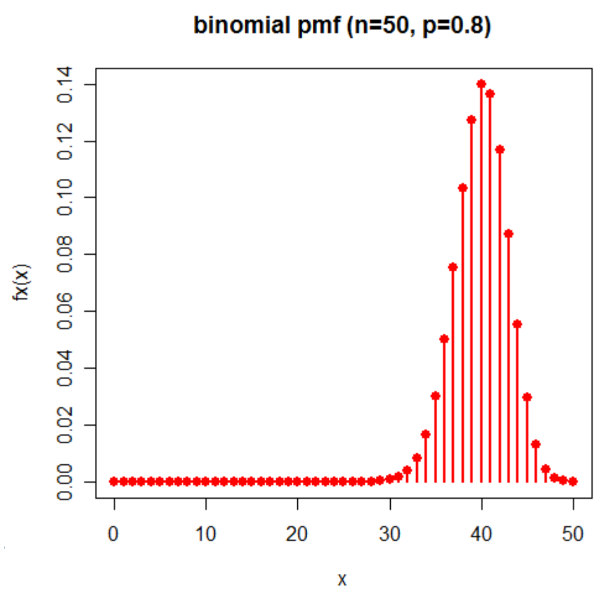
$$X = \begin{cases} 1 \ (\text{success}) & \text{with probability } p \\ 0 \ (\text{failure}) & \text{with probability } 1 - p \end{cases}$$

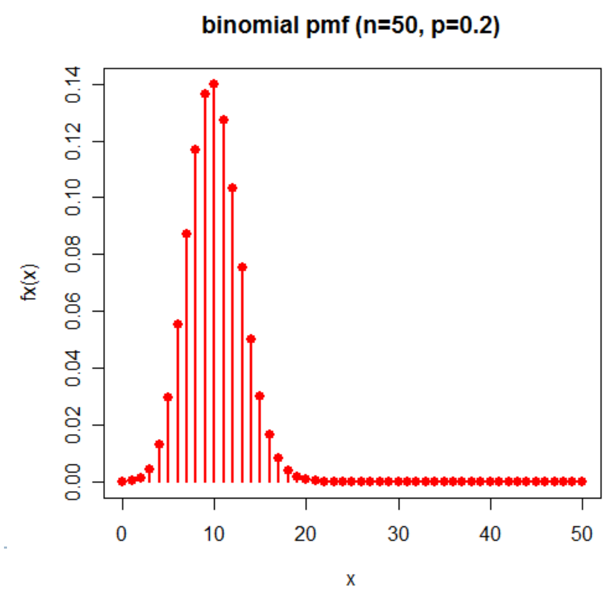


Binomial distribution

$$P(X = x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \ldots, n$$







Relation of cdfs and pmfs

$$P(a \le X \le b) = \sum_{k=a}^{b} f(k)$$

$$P(X \le b) = \sum_{k=-\infty}^{b} f(k)$$

$$P(X \ge b) = \sum_{k=b}^{\infty} f(k)$$

$$F(x) = P(X \le x) = \sum_{k=-\infty}^{x} f(k)$$

For a continuous random variable

$$P(X=x) = ?$$

- $\{X = x\} \quad \subset \{x-\varepsilon < X \leq x\}$ for any x and ε
- $$\begin{aligned} P\{X = x\} &\leq P\{x-\varepsilon < X \leq x\} \\ &= P\{X \leq x \cap X > x-\varepsilon\} \\ &= P\{X \leq x \cap (X \leq x-\varepsilon)^{\mathrm{c}}\} \\ &= P\{X \leq x\} - P(X \leq x \cap X \leq x-\varepsilon\} \qquad \textbf{Why?} \\ &= F_X(x) - F_X(x-\varepsilon) \end{aligned}$$
- $0 \leq P\{X = x\} \leq \lim_{\varepsilon \to 0} [F_X(x) - F_X(x-\varepsilon)] = 0$

$$\boldsymbol{P\{X=x\} = 0 \text{ for any } x}$$

$$P\{a<X<b\}=P\{a<X \leq b\}=P\{a\leq X<b\}=P\{a\leq X\leq b\} \text{ for any } x$$

Probability density functions

Probability density function

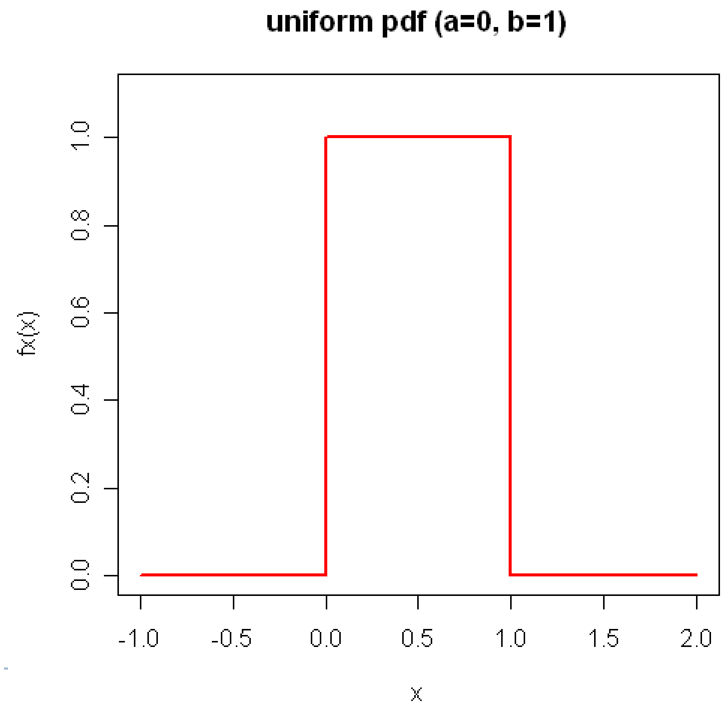
The **probability density function** (pdf), denoted by $f_X(x)$, of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt, \quad \text{for all } x.$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

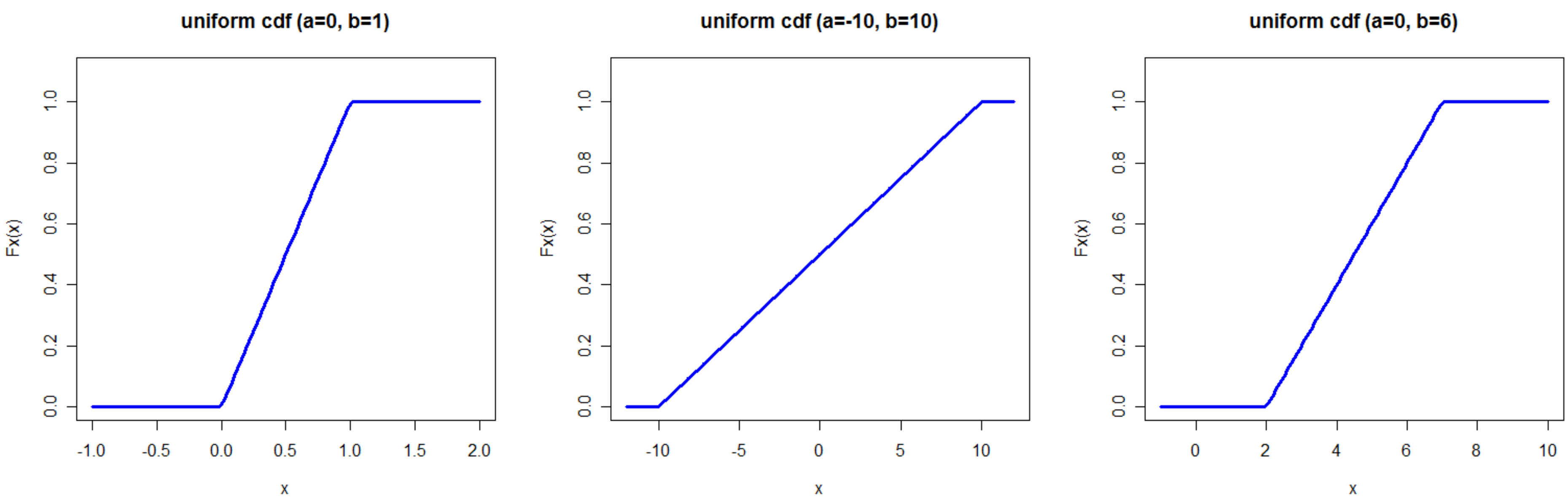
Uniform distribution

$$f_X(x \mid a,b) = \begin{cases} 0 & x < a \\ 1/(b-a) & a \le x \le b \\ 0 & x > b \end{cases}$$



Uniform distribution

$$F_X(x \mid a, b) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



Relation of cdfs and pdfs

$$P(a < X < b) = \int_a^b f(x)dx$$

$$P(X < x) = \int_{-\infty}^x f(t)dt$$

$$P(X > x) = \int_x^\infty f(t)dt = 1 - \int_{-\infty}^x f(t)dt$$

$$F(x) = P(X \le x) = \int_{-\infty}^x f(t)dt$$

Necessary and sufficient condition

Necessary and sufficient condition

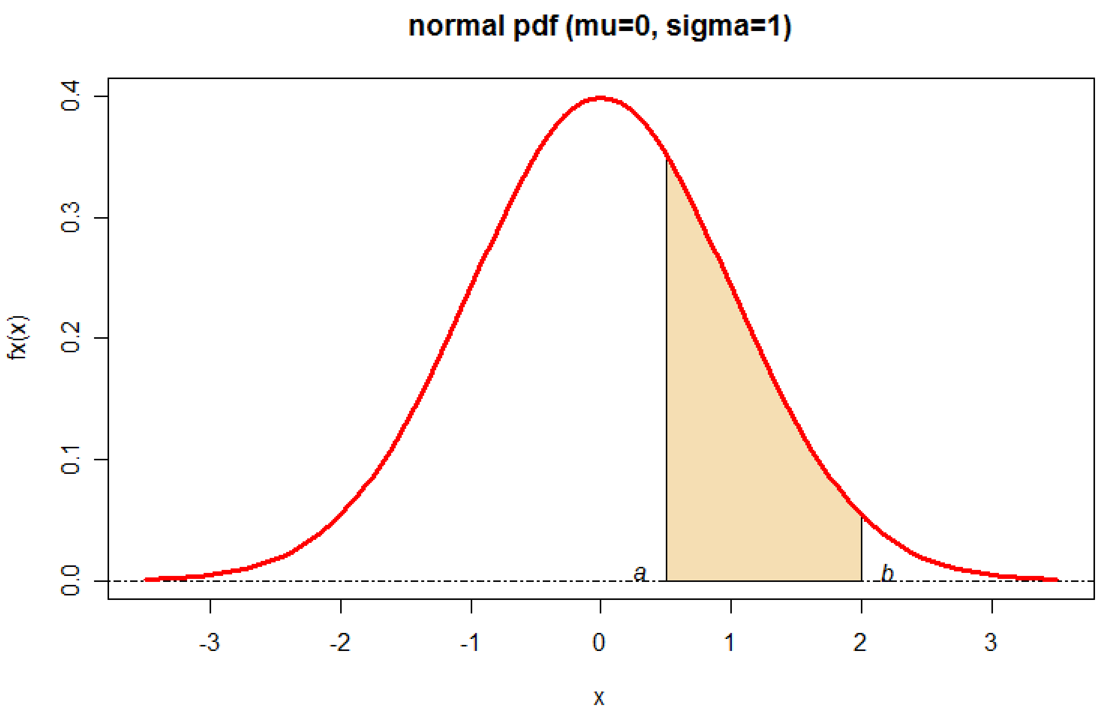
A function $f_X(x)$ is a pdf or pmf of a random variable X if and only if the following two conditions hold:

1. $f_X(x) \geq 0$ for all x;
2. $\sum_x f_X(x) = 1$ (pmf) or

   $\int_{-\infty}^{\infty} f_X(x) = 1$ (pdf).

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



$$\begin{aligned}
&P(a \leq x \leq b) \\
&= P(a < x \leq b) \\
&= P(a \leq x < b) \\
&= P(a < x < b) \\
&= F_X(b) - F_X(a) \\
&= \int_a^b f_X(x)dx
\end{aligned}$$

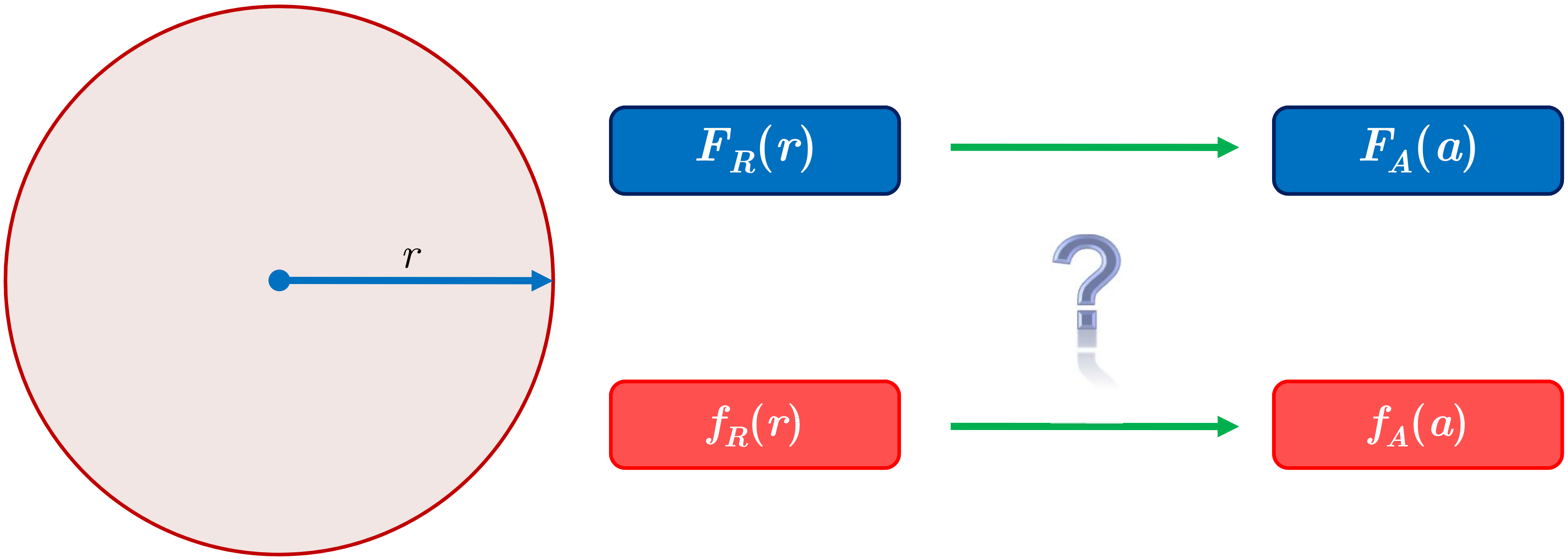
Transformations

统计学方法及其应用 > 统计学基础 > 随机变量的函数

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Why need functions of random variables

- 已知一些量的分布，而关心的是另一些量的分布
  - 半径 $r \sim \text{Uniform}(0, 1)$
  - 面积 $A \sim ?$

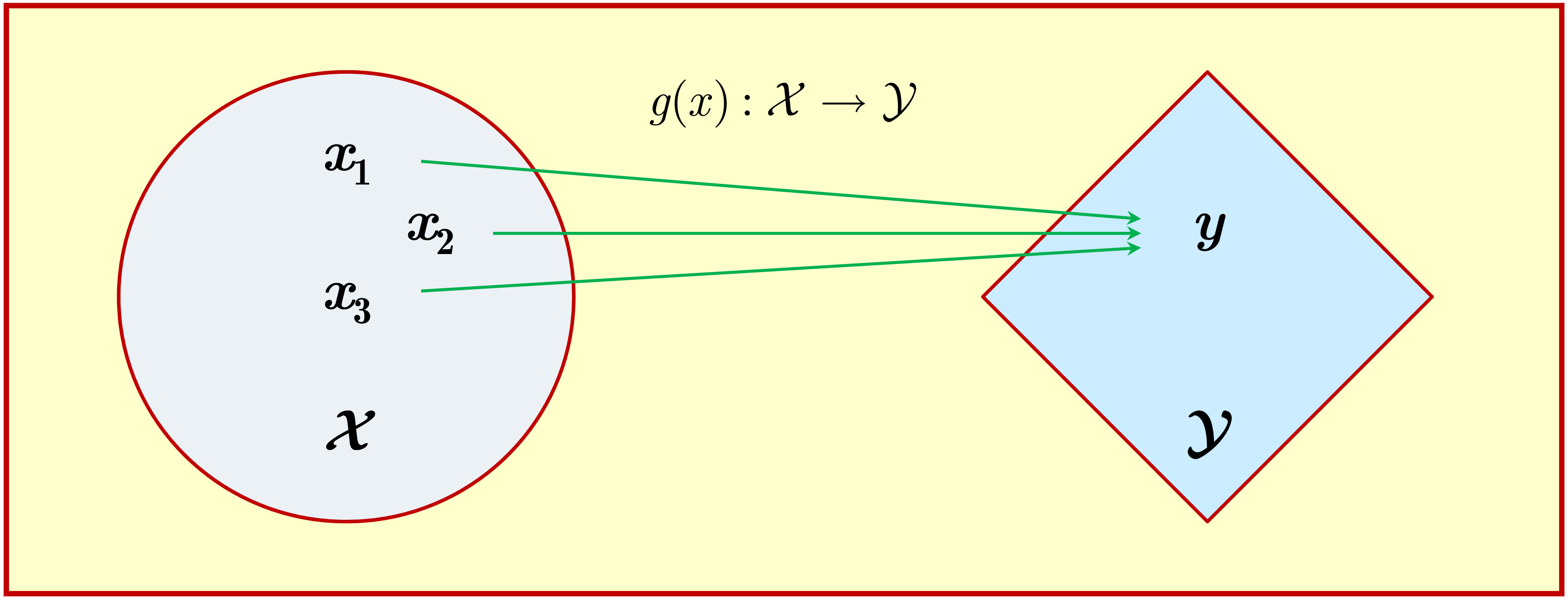


Function of a random variable

- If X is a random variable with cdf $F_X(x)$, then any function of X, say, $Y = g(X)$, is also a random variable
- The probability behavior of Y can be described using X

$$P(Y \in A) = P(g(X) \in A)$$

depending on the distribution of X and the function g



Transformation of a pmf

$$\begin{aligned} P(Y \in A) &= P(g(X) \in A) \\ &= P(x \in \mathcal{X} : g(x) \in A) \\ &= P(X \in g^{-1}(A)) \end{aligned}$$

If X is discrete, then Y is also discrete (because both $\mathcal{X}$ and $\mathcal{Y}$ are coutable)

If $y \in \mathcal{Y}$, then

$$\begin{aligned} f_Y(y) &= P(Y = y) \\ &= \sum_{x \in g^{-1}(y)} P(X = x) \\ &= \sum_{x \in g^{-1}(y)} f_X(x). \end{aligned}$$

If $y \notin \mathcal{Y}$, then

$$f_Y(y) = 0.$$

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}$$

$$g^{-1}(y) = \{x \in \mathcal{X} : g(x) = y\}$$

Binomial transformation

Suppose $f_X(x) = P(X = x \mid n, p) = \begin{pmatrix} n \\ x \end{pmatrix} p^x (1-p)^{n-x}, x = 0, 1, \ldots, n,$

that is, X a binomial distribution with parameters n and p, and

$Y = g(X) = n - X$ is a transformation. Then, $y = g(x) = n - x$,

and

$$\begin{aligned} f_Y(y \mid n, p) &= \sum_{x \in g^{-1}(y)} f_X(x \mid n, p) \\ &= f_X(n - y \mid n, p) \\ &= \begin{pmatrix} n \\ n-y \end{pmatrix} p^{n-y} (1-p)^{n-(n-y)} \\ &= \begin{pmatrix} n \\ y \end{pmatrix} (1-p)^y p^{n-y} \end{aligned}$$

is also a binomial distribution with parameters n and $1 - p$.

Transformation of a cdf

The cdf of $Y = g(X)$ is

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(g(X) \le y) \\ &= P(\{x \in \mathcal{X} : g(x) \le y\}) \\ &= \int_{\{x \in \mathcal{X} : g(x) \le y\}} f_X(x)dx \end{aligned}$$

How to obtain $\{x \in \mathcal{X} : g(x) \le y\}$?

1. The function $g(x)$ is monotone increasing
2. The function $g(x)$ is monotone decreasing
3. The function $g(x)$ is piecewise monotone

Monotone increasing

Let X have cdf $F_X(x)$. Let $Y = g(x)$. Let

$$\mathcal{X} = \{x : f_X(x) > 0\}, \text{ and}$$

$$\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

If $g(x)$ is monotone increasing, then the mapping $x \to g(x)$ is "one-to-one" and "onto", and $g^{-1}(y)$ is single-valued and also monotone increasing. Therefore,

$$\{x \in \mathcal{X} : g(x) \le y\} = \{x \in \mathcal{X} : x \le g^{-1}(y)\}$$

$$\begin{aligned} F_Y(y) &= \int_{\{x \in \mathcal{X} : g(x) \le y\}} f_X(x)dx \\ &= \int_{\{x \in \mathcal{X} : x \le g^{-1}(y)\}} f_X(x)dx \\ &= F_X(g^{-1}(y)) \end{aligned}$$

Monotone decreasing

Let X have cdf $F_X(x)$. Let $Y = g(x)$. Let

$$\mathcal{X} = \{x : f_X(x) > 0\}, \text{ and}$$
$$\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

If $g(x)$ is monotone decreasing, then the mapping $x \to g(x)$ is "one-to-one" and "onto", and $g^{-1}(y)$ is single-valued and also monotone increasing. Therefore,

$$\{x \in \mathcal{X} : g(x) \leq y\} = \{x \in \mathcal{X} : x \geq g^{-1}(y)\}$$

$$\begin{aligned} F_Y(y) &= \int_{\{x \in \mathcal{X} : g(x) \leq y\}} f_X(x)dx \\ &= \int_{\{x \in \mathcal{X} : x \geq g^{-1}(y)\}} f_X(x)dx \\ &= 1 - F_X(g^{-1}(y)) \end{aligned}$$

Transformation of a cdf

Let X have cdf $F_X(x)$, let $Y = g(x)$, and let

$$\mathcal{X} = \{x : f_X(x) > 0\}, \text{ and}$$

$$\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then

If g is an increasing function on $\mathcal{X}$,

$$F_Y(y) = F_X(g^{-1}(y)) \text{ for } y \in \mathcal{Y}.$$

If g is a decreasing function on $\mathcal{X}$,

$$F_Y(y) = 1 - F_X(g^{-1}(y)) \text{ for } y \in \mathcal{Y}.$$

Exponential distribution

Suppose that $f_X(x) = 1$ for $x \in (0,1)$ and 0 otherwise, that is, X has a uniform distribution, and $Y = g(X) = -\lambda \log X$ $(\lambda > 0)$ is a transformation. Then,

$$
\begin{aligned}
F_X(x) &= x \text{ for } x \in (0,1). \\
g(x) &= -\lambda \log x \text{ is descreasing on its support, and} \\
g^{-1}(y) &= \exp(-y / \lambda) \text{ is also decreasing on its domain } 0 < y < \infty.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F_Y(y) &= 1 - F_X(g^{-1}(y)) = 1 - F_X(\exp(-y / \lambda)) \\
&= 1 - \exp(-y / \lambda), 0 < y < \infty.
\end{aligned}
$$

Probability integral transformation

Let X have continuous cdf $F_X(x)$ and define a random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on $(0,1)$, that is,

$$P(Y \le y) = y,\ 0\text{<y<}1.$$

$$\begin{aligned}
P(Y \le y) &= P(F_X(X) \le y) \\
&= P(F_X^{-1}[F_X(X)] \le F_X^{-1}(y)) \\
&= P(X \le F_X^{-1}(y)) \\
&= F_X(F_X^{-1}(y)) \\
&= y
\end{aligned}$$

cdfs → pdfs

If g is an increasing function on $\mathcal{X}$,

$$F_Y(y) = F_X(g^{-1}(y)) \text{ for } y \in \mathcal{Y}.$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

On the other hand, if g is a decreasing function on $\mathcal{X}$,

$$F_Y(y) = 1 - F_X(g^{-1}(y)) \text{ for } y \in \mathcal{Y}.$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right] = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

Transformation of a pdf

Let X have pdf $f_X(x)$, let $Y = g(x)$, where g is a **monotone** function. Suppose $f_X(x)$ is continuous on $\mathcal{X} = \{x : f_X(x) > 0\}$ and $g^{-1}(y)$ has a continuous derivative on

$$\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \dfrac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise.} \end{cases}$$

Cauchy distribution

Suppose that $f_X(x) = 1/\pi$ for $x \in (-\pi/2, \pi/2)$ and 0 otherwise, that is, X has a uniform distribution, and $Y = g(X) = \tan X$ is a transformation. Then, in the interval $y \in (-\infty, \infty)$

$$\begin{aligned} x &= g^{-1}(y) &&= \arctan y, \\ \frac{d}{dy} g^{-1}(x) &= \frac{d}{dy} \arctan y &&= \frac{1}{1+y^2}, \end{aligned}$$

and
$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy} g^{-1}(x)\right| = \frac{1}{\pi}\frac{1}{1+y^2}, -\infty < y < \infty.$$

Non-monotone transformation

Suppose $f_X(x) = (\sqrt{2\pi})^{-1}\exp(-x^2/2)$ for $x \in (-\infty, \infty)$, that is, X has a standard normal distribution, and $Y = g(X) = X^2$ is a transformation. Then, in the interval $y \in (0, \infty)$

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) = P(X^2 \le y) \\
&= P(-\sqrt{y} \le X \le \sqrt{y}) \\
&= P(X \le \sqrt{y}) - P(X \le -\sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}} e^{-\frac{y}{2}}, 0 < y < \infty.
\end{aligned}
$$

Piecewise monotone

Let X have pdf $f_X(x)$, let $Y = g(x)$, and define the sample space $\mathcal{X}$ as the support set of X. Suppose there exists a partition, $A_0, A_1, \ldots, A_k$, of $\mathcal{X}$ such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i. Further, suppose there exist functions $g_1(x), \ldots, g_k(x)$, defined on $A_1, \ldots, A_k$, respectively, satisfying

1. $g(x) = g_i(x)$, for $x \in A_i$,
2. $g_i(x)$ is monotone on A_i,
3. the set $\mathcal{Y} = \{y : y = g_i(x)$ for some $x \in A_i\}$ is the same for each $i = 1, \ldots, k$,

Then

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise.} \end{cases}$$

Chi-squared distribution

Notice that $Y = g(X) = X^2$

$x \in (-\infty, 0), y = g_1(x) = x^2,\ h_1(y) = -\sqrt{y}, \text{decreasing;}$

$x \in (0, +\infty), y = g_2(x) = x^2,\ h_2(y) = \sqrt{y}, \text{increasing;}$

$x = 0$ (with probability 0).

Define $A_0 = \{0\}; A_1 = (-\infty, 0); A_2 = (0, \infty).$

Then $A_0 \cap A_1 \cap A_2 = \emptyset$ and $A_0 \cup A_1 \cup A_2 = (-\infty, \infty).$

In A_1, $f_1(y) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}} \left| -\dfrac{1}{2}\dfrac{1}{\sqrt{y}} \right| = \dfrac{1}{2}\dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\sqrt{y}} e^{-\frac{y}{2}}$

In A_2, $f_2(y) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \left| \dfrac{1}{2}\dfrac{1}{\sqrt{y}} \right| = \dfrac{1}{2}\dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\sqrt{y}} e^{-\frac{y}{2}}$

Then, $f(y) = f_1(y) + f_2(y) = \dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\sqrt{y}} e^{-\frac{y}{2}} \sim \chi_1^2$

Summary

$$\begin{tikzcd}[column sep=huge, row sep=huge]
F_X(x) \arrow[r, "{F_Y(y)=F_X(g^{-1}(y)),\ g(x)\ \text{单调增时}}", "{F_Y(y)=1-F_X(g^{-1}(y)),\ g(x)\ \text{单调减时}}"'] \arrow[d, shift left, "{f(t)=\frac{d}{dx}F(x)}"] & F_Y(y) \arrow[d, shift left, "{f(t)=\frac{d}{dx}F(x)}"] \\
f_X(x) \arrow[u, shift left, "{F(x)=\int_{-\infty}^{x} f(t)dt}"] \arrow[r, "{f_Y(y)=f_X(g^{-1}(y))\frac{d}{dx}g^{-1}(y),\ g(x)\ \text{单调增时}}", "{f_Y(y)=-f_X(g^{-1}(y))\frac{d}{dx}g^{-1}(y),\ g(x)\ \text{单调减时}}"'] & f_Y(y) \arrow[u, shift left, "{F(x)=\int_{-\infty}^{x} f(t)dt}"]
\end{tikzcd}$$

Location family

Suppose X is a random variable having pdf $f(x)$.

Consider the transform

$$Y = X + \mu.$$

Since

$$X = Y - \mu,$$

$$\frac{dx}{dy} = 1.$$

Therefore

$$g(y) = f(y - \mu)$$

The family of pdfs $f(x-\mu)$, indexed by the parameter $\mu, -\infty < \mu < \infty$, is called the **location family** with **standard pdf** $f(x)$, and μ is called the **location parameter** for the family.

Normal location family

Suppose X is a standard normal random variable. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Consider the transform

$$Y = X + \mu.$$

Since

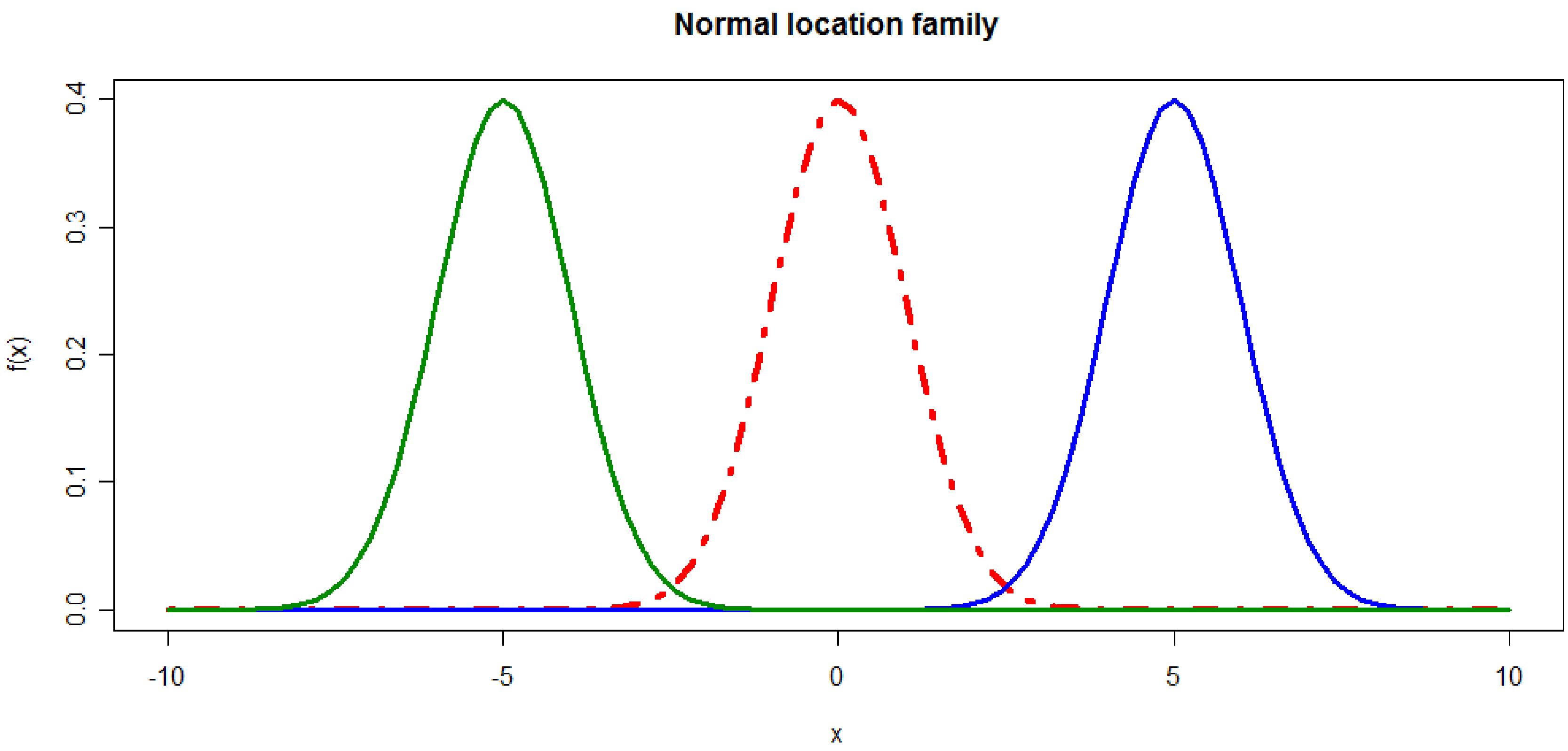
$$X = Y - \mu,$$

$$\frac{dx}{dy} = 1.$$

Therefore

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2}}$$

Normal location family



Scale family

Suppose X is a random variable having pdf $f(x)$.

Consider the transform

$$Y = \sigma X.$$

Since

$$X = Y \,/\, \sigma,$$

$$\frac{dx}{dy} = \frac{1}{\sigma}.$$

Therefore

$$g(y) = \frac{1}{\sigma} f\left(\frac{y}{\sigma}\right)$$

The family of pdfs $\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$, indexed by the parameter $\sigma, \sigma > 0$, is called the **scale family** with **standard pdf** $f(x)$; σ is called the **scale parameter** for the family.

Normal scale family

Suppose X is a standard normal random variable. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Consider the transform

$$Y = \sigma X.$$

Since

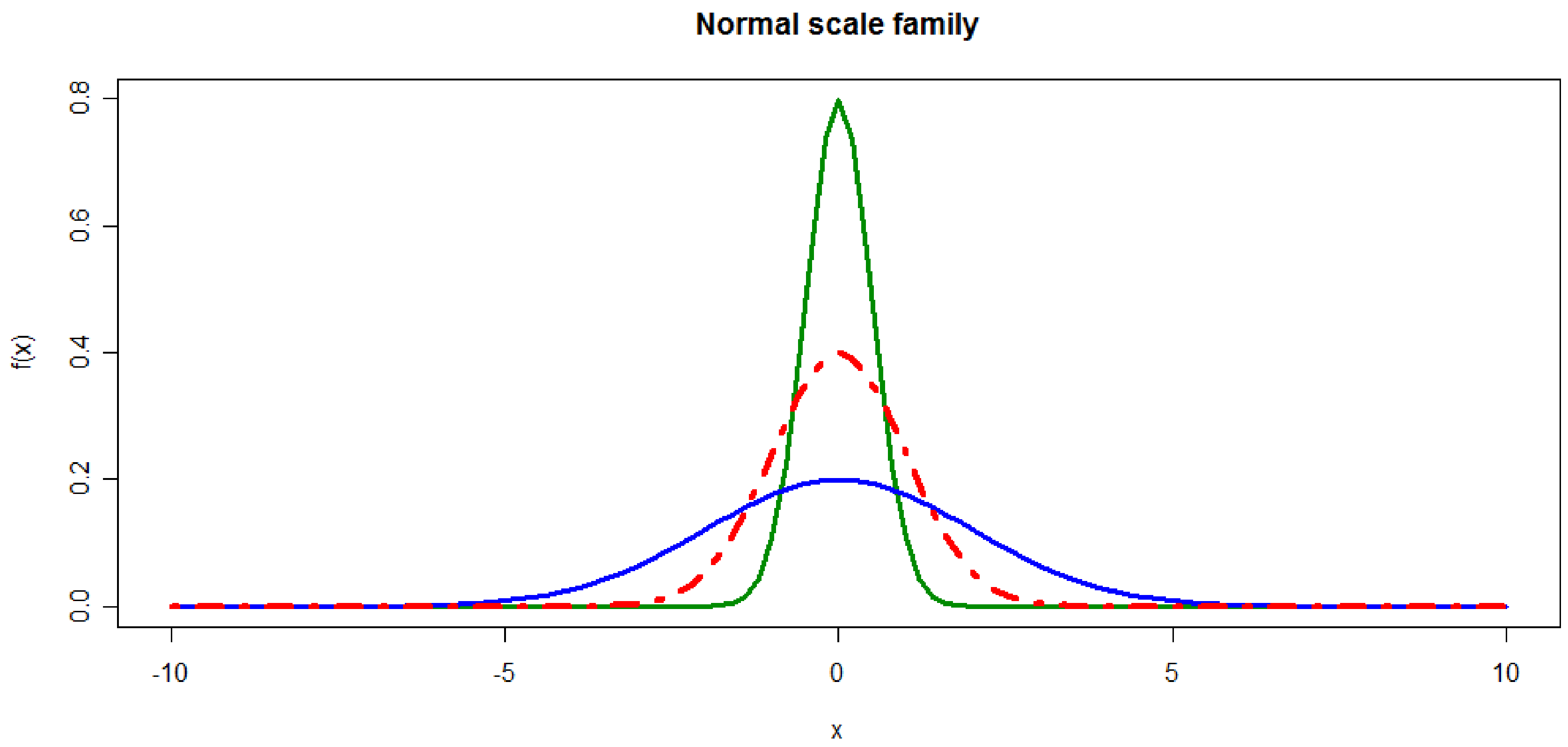
$$X = Y \,/\, \sigma,$$

$$\frac{dx}{dy} = 1 \,/\, \sigma.$$

Therefore

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

Normal scale family



Location-scale family

Suppose X is a random variable having pdf $f(x)$.

Consider the transform

$$Y = \sigma X + \mu.$$

Since

$$X = (Y - \mu) / \sigma,$$

$$\frac{dx}{dy} = \frac{1}{\sigma}.$$

Therefore

$$g(y) = \frac{1}{\sigma} f\left(\frac{y-\mu}{\sigma}\right)$$

The family of pdfs $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$, indexed by the parameters μ and σ $(\sigma > 0)$, is called the **location-scale family** with **standard pdf** $f(x)$; μ is called the location **parameter** for the family, and σ is called the **scale parameter** for the family.

Normal location-scale family

Suppose X is a standard normal random variable. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Consider the transform

$$Y = \sigma X + \mu.$$

Since

$$X = (Y - \mu) / \sigma,$$

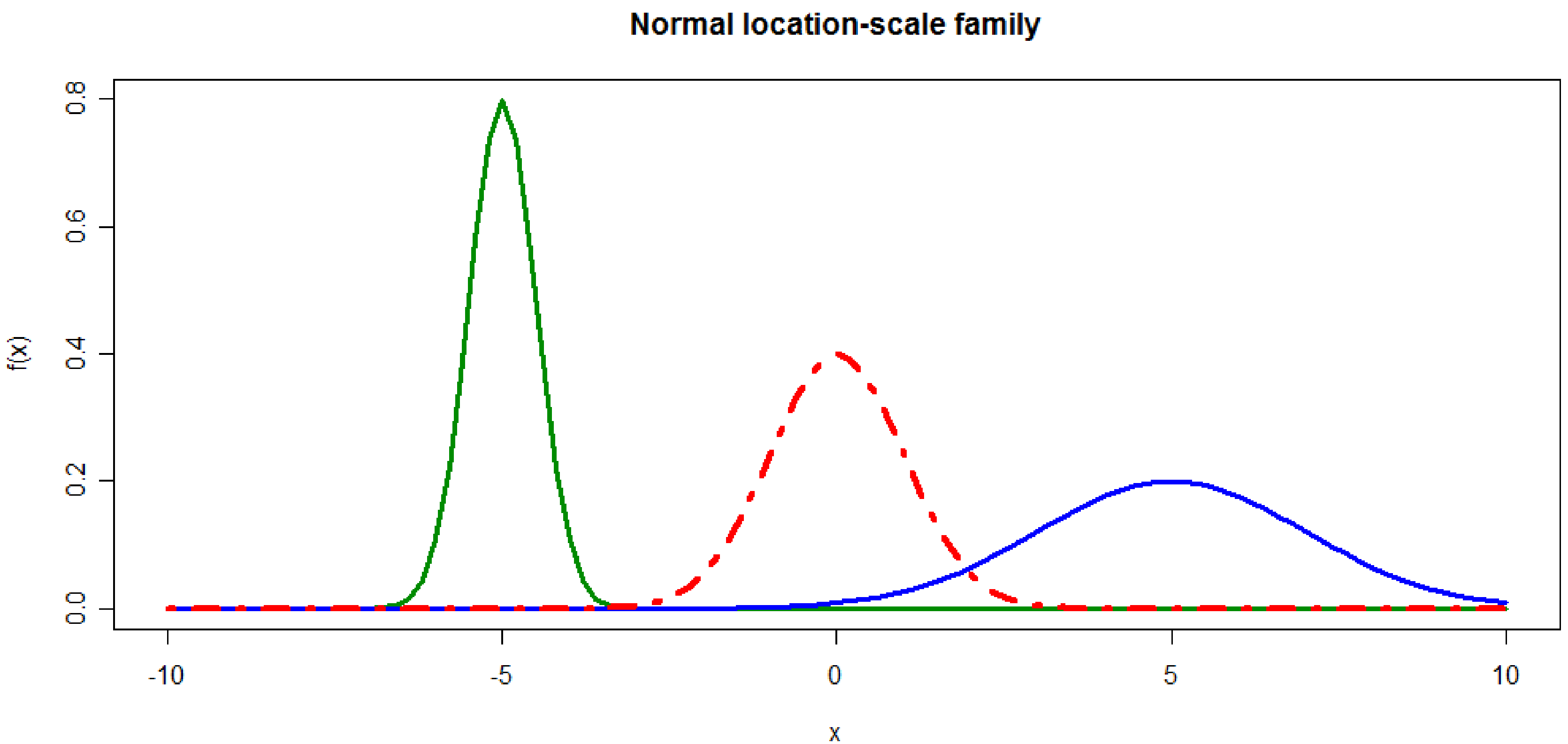
$$\frac{dx}{dy} = 1 / \sigma.$$

Therefore

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

This is the pdf of a **normal distribution**. In other words, $Y \sim N(\mu, \sigma^2)$.

Normal location-scale family



Exponential families

A family of pdfs or pmfs is called an *exponential family* if it can be expressed as

$$f(x;\theta) = h(x)c(\theta)\exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(x)\right),$$

where $h(x) \geq 0, c(\boldsymbol{\theta}) \geq 0, w_i(\theta)$ are real valued functions of the parameter $\boldsymbol{\theta}$, and $t_i(x)$ are real valued functions of the observation x.

Binomial exponential family

- Binomial pmf

$$
\begin{aligned}
f(x \mid p) &= \binom{n}{x} p^x (1-p)^{n-x} \\
&= \binom{n}{x} (1-p)^n \left(\frac{p}{1-p}\right)^x \\
&= \binom{n}{x} (1-p)^n \exp\left[\log\left(\frac{p}{1-p}\right) x\right]
\end{aligned}
$$

- Exponential family pmf

$$
f(x;\boldsymbol{\theta}) = h(x) c(\boldsymbol{\theta}) \exp\left(\sum_{\mathbf{i=1}}^{k} w_i(\theta) t_i(x)\right)
$$

$$
h(x) = \binom{n}{x} \qquad c(p) = (1-p)^n
$$

$$
w_1(p) = \log\left(\frac{p}{1-p}\right) \qquad t_1(x) = x
$$

Normal exponential family

- Normal pdf

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$= \underbrace{1}_{h(x)} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right)}_{c(\mu,\sigma^2)} \exp\left(\underbrace{\frac{1}{2\sigma^2}}_{w_1(\mu,\sigma^2)} \underbrace{(-x^2)}_{t_1(x)} + \underbrace{\frac{\mu}{\sigma^2}}_{w_2(\mu,\sigma^2)} \underbrace{x}_{t_1(x)}\right)$$

- $d = 2$ parameters, $k = 2$ items in the sum in the exponent
  - $d < k \mapsto$ ***curved*** exponential family, e.g., $N(\mu, \mu^2)$
  - $d = k \mapsto$ ***full*** exponential family, e.g., $N(\mu, \sigma^2)$

Expectations of Random Variables

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Mode

Mode

The **mode** of a random variable X is the value that occurs the most frequently in the probability distribution, corresponding to the maximum value in the pmf or pdf.

Median

Median

The **median** of a random variable X is a value m such that

$$P(X \leq m) \geq \frac{1}{2} \text{ and } P(X \geq m) \geq \frac{1}{2}$$

For a continuous random variable X, the median m satisfies

$$\int_{-\infty}^{m} f(x)dx = \int_{m}^{\infty} f(x)dx = \frac{1}{2}$$

Expectations

Expected value

The **expected value** or **mean** of a random variable $g(X)$, denoted by $\mathrm{E}g(x)$, is

$$\mathrm{E}g(x) = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx & \text{if } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} g(x) f_X(x) & \text{if } X \text{ is descrete} \end{cases}$$

provided that the integral or sum exists. If $\mathrm{E}\left|g(X)\right| = \infty$, we say that $\mathrm{E}g(X)$ does not exist.

Normal mode

Suppose X is a normal (μ,σ^2) random variable. Then,

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
\log f(x) &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{1}{2}(\sigma^2)(x-\mu)^2
\end{aligned}
$$

Obviously, the maximum value is obtained at $x = \mu$.

Therefore,

The mode of a normal distribution is its location parameter.

Normal median

Suppose X is a normal (μ,σ^2) random variable. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\int_{-\infty}^{\mu} f(x)dx = \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \xrightleftharpoons{y=x-\mu} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$\int_{\mu}^{\infty} f(x)dx = \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \xrightleftharpoons{y=-(x-\mu)} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

Obviously, these two integrals are equal

Therefore,

The median of a normal distribution is its location parameter.

Standard normal expectation

Suppose

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty \leq x < \infty$$

that is, X has an **standard normal distribution** $N(0,1)$.

Then,

$$\begin{aligned}
\mathrm{E}X &= \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \\
&= -\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} d\left(-\frac{x^2}{2}\right) \\
&= e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} \\
&= 0
\end{aligned}$$

Cauchy expectation

- Suppose

$$f(x \mid \lambda) = \frac{1}{\pi}\frac{1}{1+x^2}, -\infty < x < \infty,$$

that is, X has a Cauchy distribution, denoted as $X \sim$ Cauchy. Then,

$$\begin{aligned}
\mathrm{E}|X| &= \int_{-\infty}^{\infty} \frac{|x|}{\pi}\frac{1}{1+x^2}\,dx \\
&= \frac{2}{\pi}\int_0^{\infty} \frac{x}{1+x^2}\,dx \\
&= \lim_{M\to\infty} \frac{2}{\pi}\int_0^{M} \frac{x}{1+x^2}\,dx \\
&= \frac{1}{\pi}\lim_{M\to\infty} \log(1+M^2) \\
&= \infty
\end{aligned}$$

Properties of expectation

Properties of expectation

Let X be a random variable and let $a, b,$ and c be constants. Then for any functions $g_1(X)$ and $g_2(X)$ whose expectations exists,

1. $\mathrm{E}[ag_1(X) + bg_2(X) + c] = a\mathrm{E}g_1(X) + b\mathrm{E}g_2(X) + c;$
2. If $g_1(X) \geq 0$ for all x, then $\mathrm{E}g_1(X) \geq 0;$
3. If $g_1(X) \geq g_2(X)$ for all x, then $\mathrm{E}g_1(X) \geq \mathrm{E}g_2(X);$
4. If $a \leq g_1(X) \leq b$ for all x, then $a \leq \mathrm{E}g_1(X) \leq b.$

Moments of random variables

Moment

For eqch integer n, the ***n*th moment** of a random variable X, μ_n', is

$$\mu_n' = \mathrm{E}X^n.$$

The ***n*th central moment** of X, μ_n, is

$$\mu_n = \mathrm{E}(X - \mu)^n,$$

where $\mu = \mu_n' = \mathrm{E}X$.

Mean

Mean

The **mean** of a random variable X is its first moment

$$\mu = \mathrm{E}X.$$

Variance

Variance

The **variance** of a random variable X is its second central moment

$$\mathrm{Var}X = \mathrm{E}(X - \mathrm{E}X)^2.$$

The positive sequre root of $\mathrm{Var}X$ is the **standard deviation** of X.

Properties of variance

Properties of variances

If X is a random variable with finite variance, then for any constants a and b

$$\mathrm{Var}(aX+b) = a^2 \mathrm{Var} X.$$

$$\text{since} \quad \begin{aligned} \mathrm{Var}(aX+b) &= \mathrm{E}((aX+b) - E(aX+b))^2 \\ &= \mathrm{E}(aX - aEX)^2 \\ &= a^2 \mathrm{E}(X - \mathrm{E}X)^2 \\ &= a^2 \mathrm{Var}(X). \end{aligned}$$

$$\mathrm{Var} X = \mathrm{E}X^2 - (\mathrm{E}X)^2$$

$$\text{since} \quad \begin{aligned} \mathrm{Var} X &= \mathrm{E}(X - \mathrm{E}X)^2 \\ &= \mathrm{E}(X^2 - 2X\mathrm{E}X + (\mathrm{E}X)^2) \\ &= \mathrm{E}X^2 - 2(\mathrm{E}X)^2 + (\mathrm{E}X)^2 \\ &= \mathrm{E}X^2 - (\mathrm{E}X)^2. \end{aligned}$$

Bernoulli variance

- Suppose

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases},$$

that is, X has a Bernoulli distribution, denoted as $X \sim \text{Bernoulli}(p)$. Then,

$$\begin{aligned} \mathrm{E}X &= p \times 1 + (1-p) \times 0 = p \\ \mathrm{E}X^2 &= p \times 1 + (1-p) \times 0 = p \\ \mathrm{Var}X &= \mathrm{E}X^2 - (\mathrm{E}X)^2 \\ &= p - p^2 \\ &= p(1-p). \end{aligned}$$

Standard normal variance

Suppose X is a standard normal random variable. Then,

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
\mathrm{E}X &= \int_{-\infty}^{\infty} x(2\pi)^{-1/2} \exp(-x^2/2)dx \\
&= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-x^2/2)d(-x^2/2) \\
&= -(2\pi)^{-1/2} \exp(-x^2/2)\Big|_{-\infty}^{\infty} \\
&= 0 \\
\mathrm{E}X^2 &= \int_{-\infty}^{\infty} x^2(2\pi)^{-1/2} \exp(-x^2/2)dx \\
&= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} x d \exp(-x^2/2) \\
&= -(2\pi)^{-1/2} x \exp(-x^2/2)\Big|_{-\infty}^{\infty} + (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-x^2/2)dx \\
&= 1
\end{aligned}
$$

Therefore

$$\mathrm{Var}X = \mathrm{E}X^2 - (\mathrm{E}X)^2 = 1$$

$$\int_{-\infty}^{\infty} \exp(-p^2x^2 + qx)dx = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{p} \quad (p > 0)$$

Skewness

Skewness

The **skewness** of a random variable X is its third central moment over the cube of the standard deviation

$$\beta_s = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3}.$$

Standard normal skewness

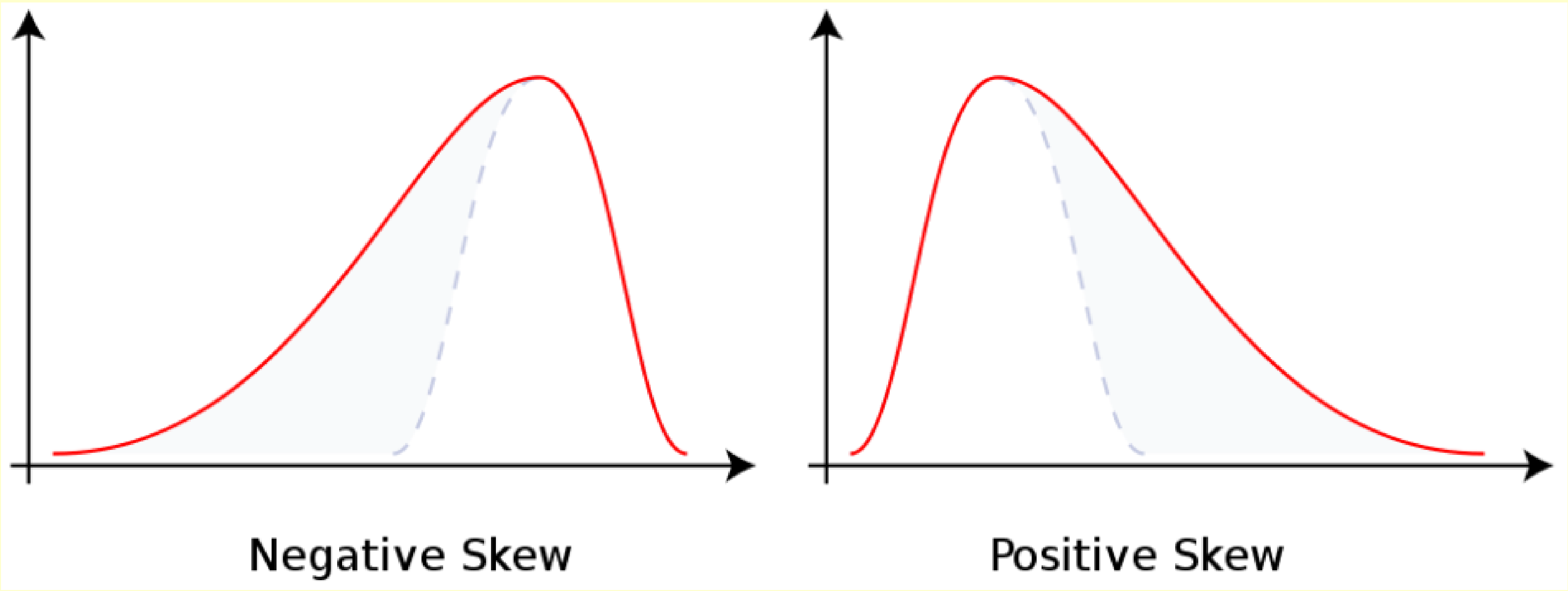
Suppose X is a standard normal random variable. Then,

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
\mathrm{E}X^3 &= \int_{-\infty}^{\infty} x^3 (2\pi)^{-1/2} \exp(-x^2 / 2) dx \\
&= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} x^2 d \exp(-x^2 / 2) \\
&= -(2\pi)^{-1/2} x^2 \exp(-x^2 / 2) \Big|_{-\infty}^{\infty} + (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-x^2 / 2) dx^2 \\
&= -2(2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-x^2 / 2) d(-x^2 / 2) \\
&= 0
\end{aligned}
$$

Therefore

$$\beta_s = 0$$

Skewness



Kurtosis

Kurtosis

The **Kurtosis** of a random variable X is its fourth central moment over the fourth power of the standard deviation

$$\beta_k = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma^4}.$$

Standard normal kurtosis

Suppose X is a standard normal random variable. Then,

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
\mathrm{E}X^4 &= \int_{-\infty}^{\infty} x^4 (2\pi)^{-1/2} \exp(-x^2 / 2) dx \\
&= -(2\pi)^{-1/2} \int_{-\infty}^{\infty} x^3 d \exp(-x^2 / 2) \\
&= -(2\pi)^{-1/2} x^3 \exp(-x^2 / 2) \Big|_{-\infty}^{\infty} + (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-x^2 / 2) dx^3 \\
&= 3 \left[(2\pi)^{-1/2} \int_{-\infty}^{\infty} x^2 \exp(-x^2 / 2) dx \right] \\
&= 3
\end{aligned}
$$

Therefore

$$\beta_k = 3$$

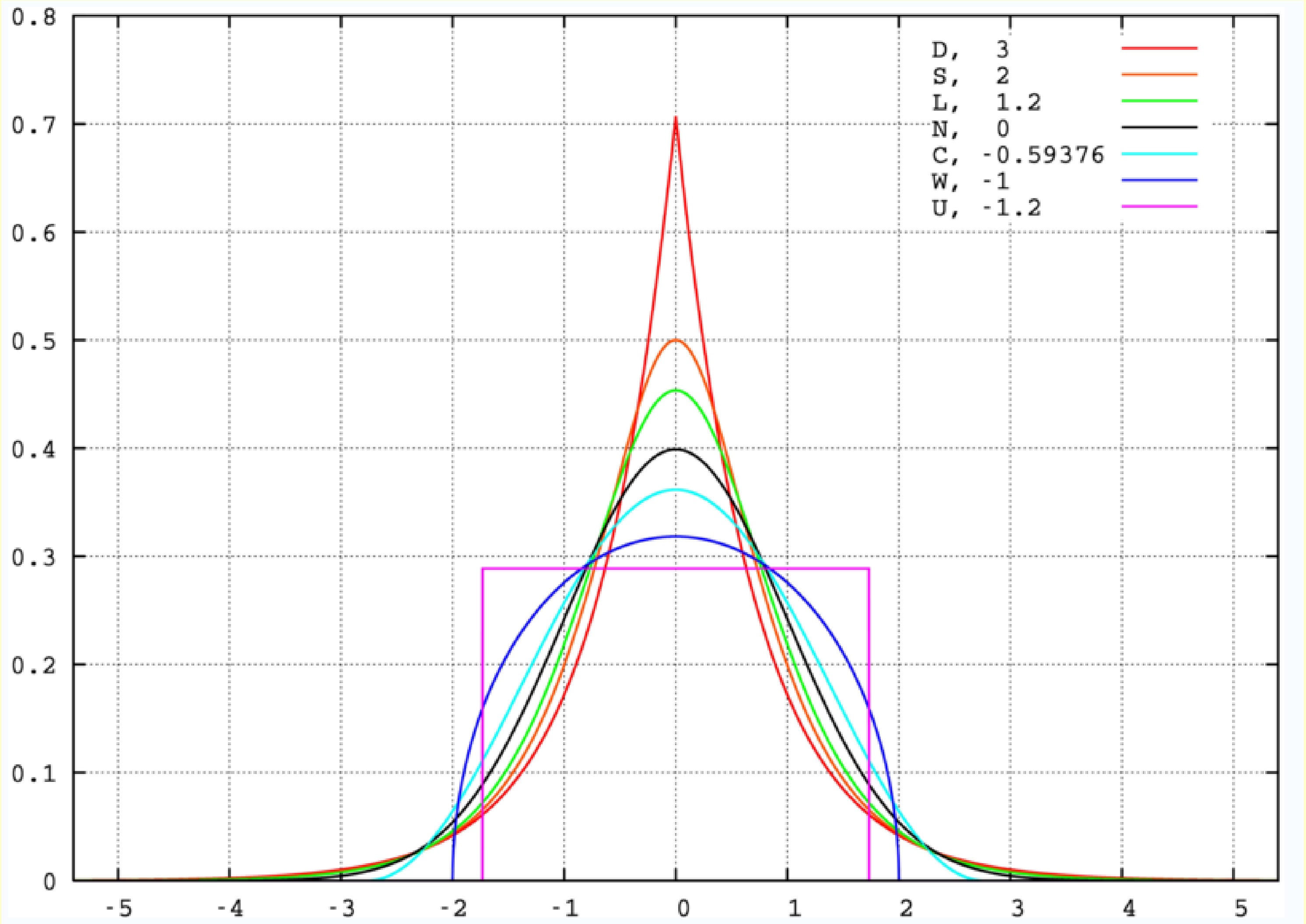
Excess kurtosis

Excess kurtosis

The **Excess Kurtosis** of a random variable X is its fourth central moment over the fourth power of the standard deviation minus 3

$$\beta_k = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] - 3 = \frac{\mu_4}{\sigma^4} - 3.$$

Excess kurtosis



Moment generating function

Moment generating function

Let X be a random variable with cdf $F_X(x)$. The **moment generating function (mgf)** of X, $M_X(t)$, is

$$M_X(t) = \mathrm{E}e^{tX},$$

provided that the expectation exists for t in some neighborhood of 0.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

$$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} P(X = x)$$

Normal moment generation function

$$
\begin{aligned}
M(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + tx\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2 - 2\mu x + \mu^2 + 2\sigma^2 tx}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2 - 2(\mu+\sigma^2 t)x + (\mu+\sigma^2 t)^2 + \mu^2 - (\mu+\sigma^2 t)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2}\left(x-(\mu+\sigma^2 t)\right)^2 + \left(\mu t + \frac{1}{2}\sigma^2 t^2\right)\right] dx \\
&= \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2}\left(x-(\mu+\sigma^2 t)\right)^2\right] dx}_{=1}
\end{aligned}
$$

Deriving moments from mgf

Deriving moments

If X has mgf $M_X(t)$, then

$$\mathrm{E}X^n = M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\bigg|_{t=0}.$$

That is, the n-th moment is equal to the n-th derivative of $M_X(t)$, evaluated at $t = 0$.

Standard normal moments

Standard normal mgf is

$$M(t) = \exp\left(\frac{t^2}{2}\right)$$

$$\frac{d}{dx}M(t) = t\exp\left(\frac{t^2}{2}\right) \Rightarrow \mu_1' = 0 \Rightarrow \mu = 0$$

$$\frac{d^2}{dx^2}M(t) = (t^2+1)\exp\left(\frac{t^2}{2}\right) \Rightarrow \mu_2' = 1 \Rightarrow \sigma^2 = 1$$

$$\frac{d^3}{dx^3}M(t) = (t^3+3t)\exp\left(\frac{t^2}{2}\right) \Rightarrow \mu_3' = 0 \Rightarrow \beta_s = 0$$

$$\frac{d^4}{dx^4}M(t) = (t^4+6t^2+3)\exp\left(\frac{t^2}{2}\right) \Rightarrow \mu_4' = 3 \Rightarrow \beta_k = 3$$

Distribution Functions of Random Variables

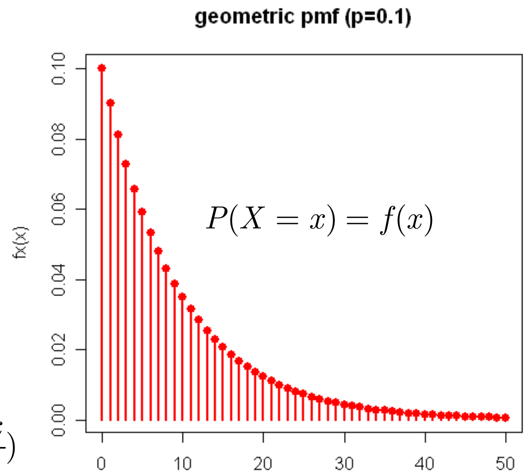
统计学方法及其应用 统计学基础 随机变量的函数

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

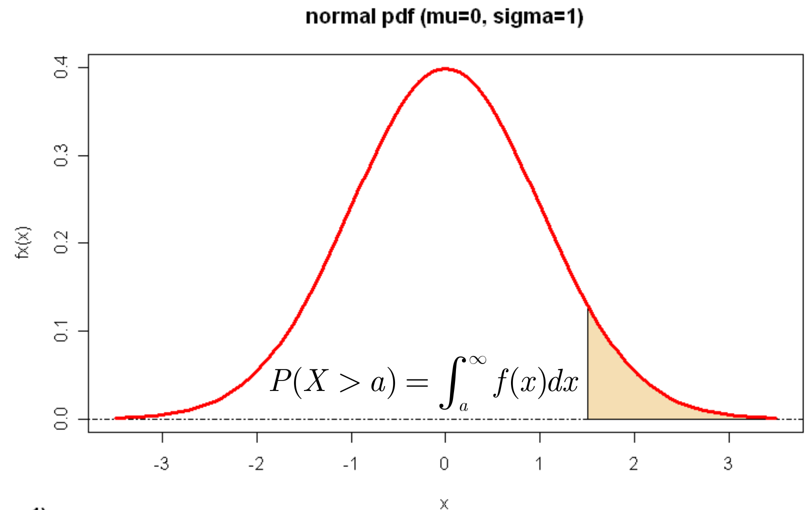
Distribution functions

- Probability mass (density) function (pmf, pdf)
  - Probability at or near a particular value
- Cumulative distribution function (cdf)
  - Probability less than or equal to a particular value
- Quantile function
  - The particular value corresponding to a probability, on the basis of the cdf
- Random numbers
  - Points distributed as the given distribution

Probability mass/density functions

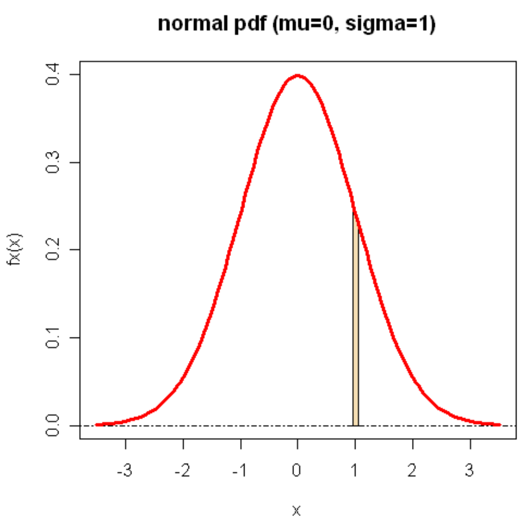


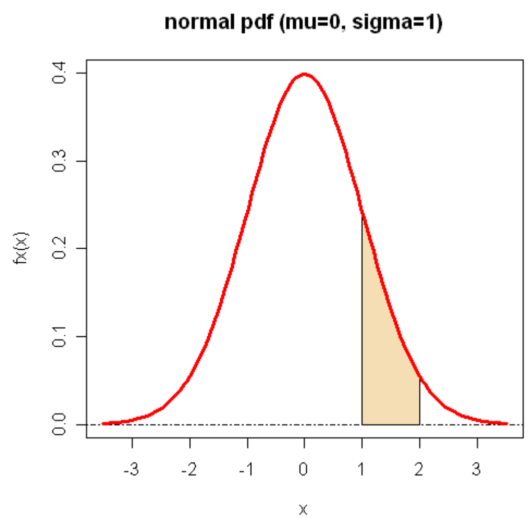
$$P(X = x) = f(x)$$



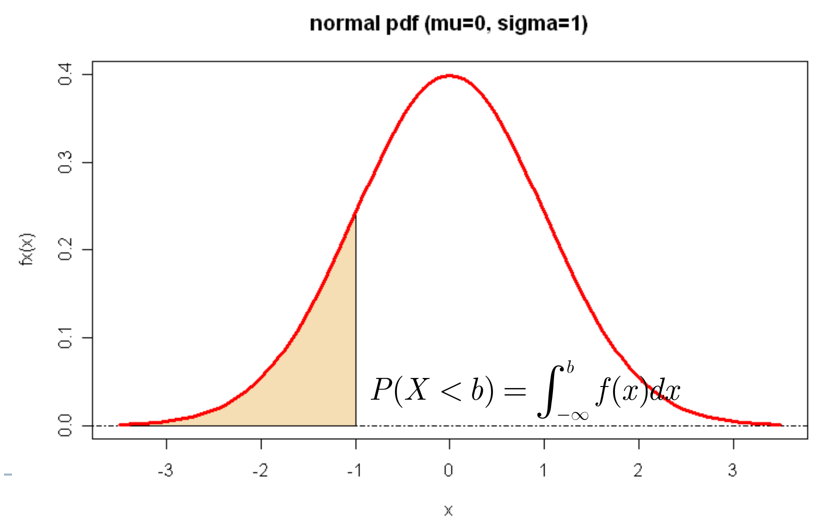
$$P(X > a) = \int_a^\infty f(x)dx$$

$$P(x - \frac{\Delta x}{2} \leq X \leq x + \frac{\Delta x}{2})$$
$$= \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(x)dx \approx f(x)\Delta x$$



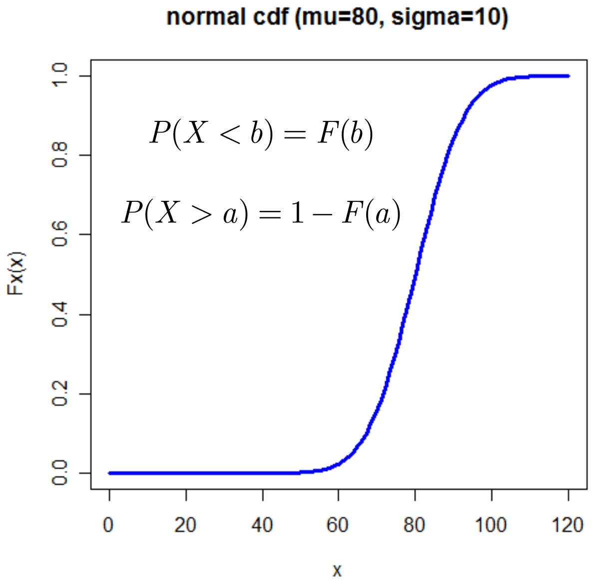


$$P(a < X < b) = \int_a^b f(x)dx$$



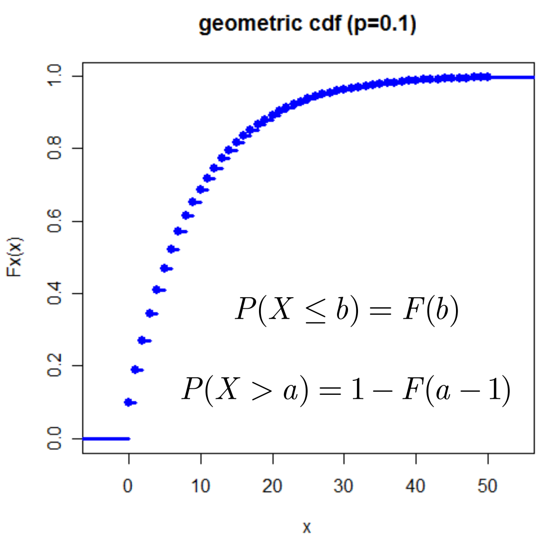
$$P(X < b) = \int_{-\infty}^b f(x)dx$$

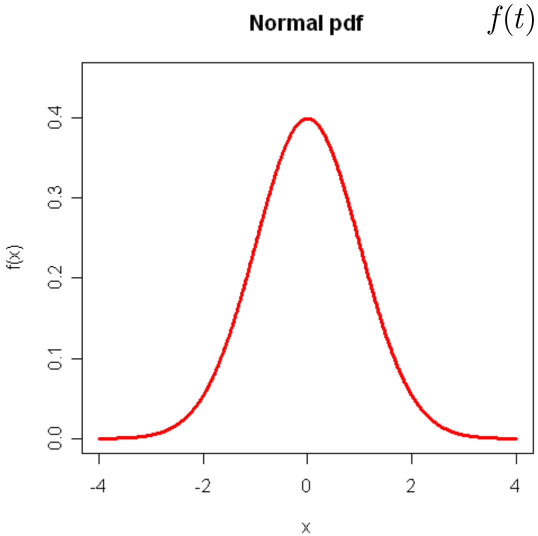
Cumulative distribution functions

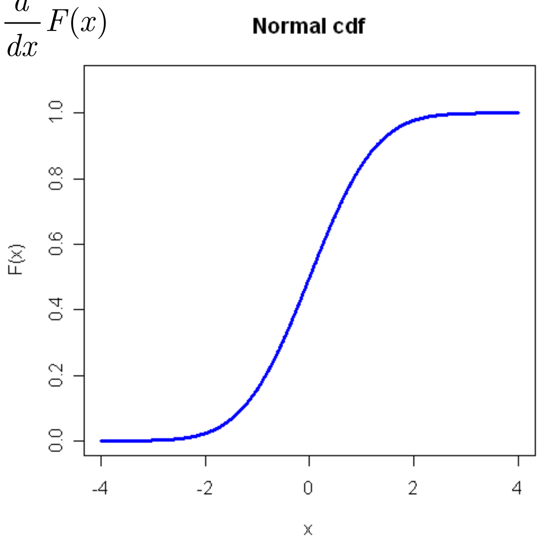


$$F(x) = \int_{-\infty}^{x} f(t)dt$$

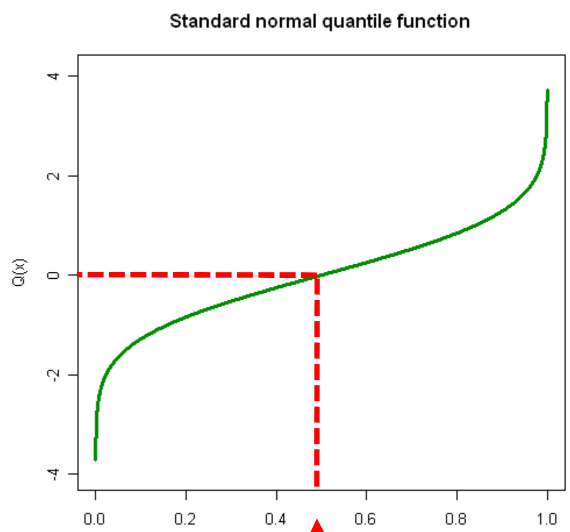
$$f(t) = \frac{d}{dx}F(x)$$



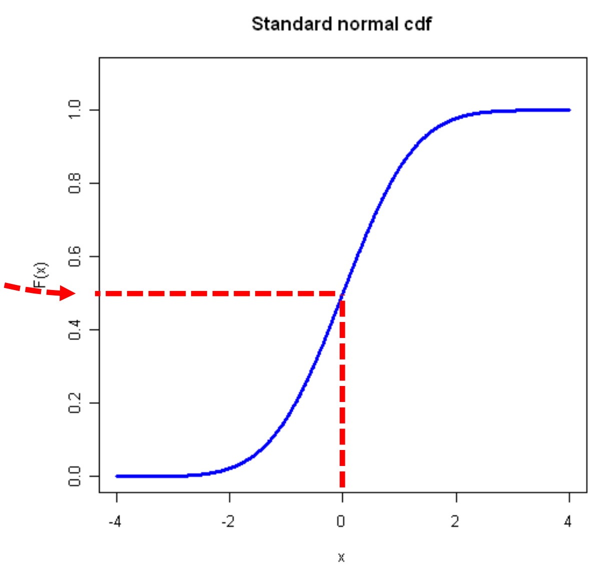




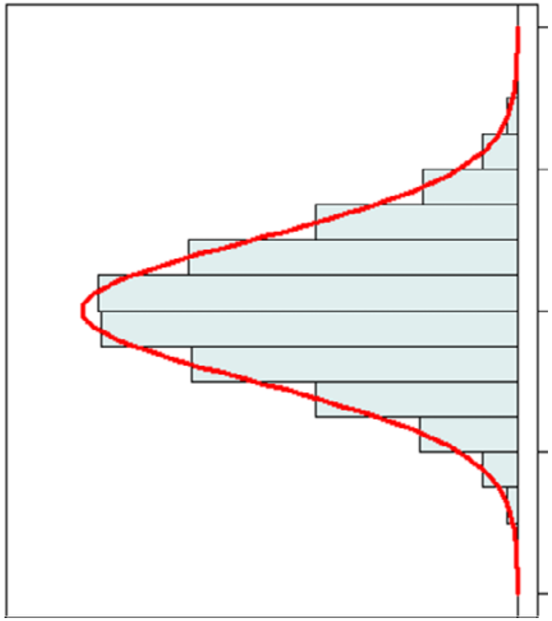
Quantile functions

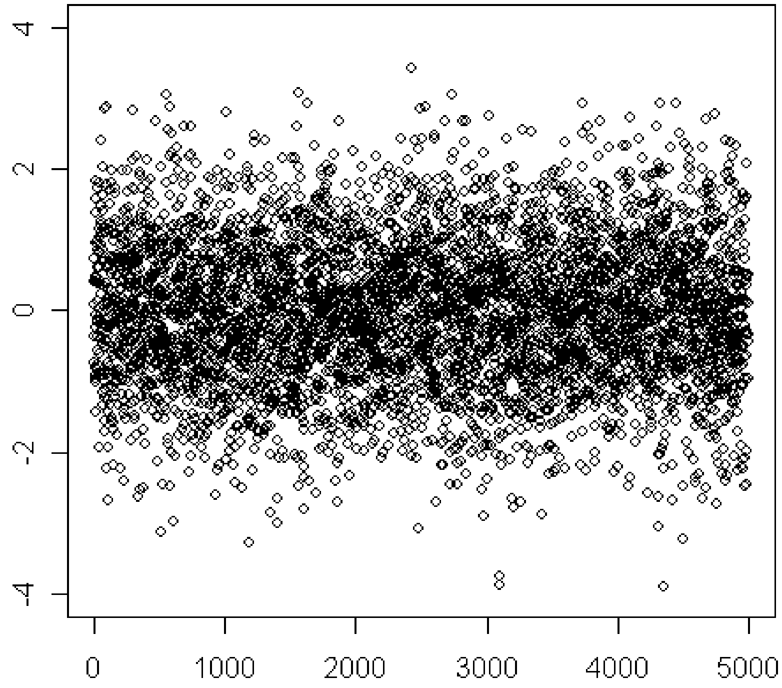


$$Q(t) = F^{-1}(x)$$



Random number generators





Distribution functions in R

- pmf or pdf
  $f_X(x)$ `dxxxx(x, parameters)`
- cdf
  $F_X(x)$ `pxxxx(q, parameters)`
- Quantile function
  ${F_X}^{-1}(p)$ `qxxxx(p, parameters)`
- Random numbers
  `rxxxx(n, parameters)`

Thank you very much