Exercise 2

Random Variables

Deadline: October 7, 2016.

1. Reading.

   (a) Lecture notes 2

   (b) Chpaters 2 and 3 of the book "Statistical Inference".

2. In each of the following show that the given function is a cdf and find $F_X^{-1}(y)$.

   (a) $F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$

   (b) $F_X(x) = \begin{cases} e^x/2 & \text{if } x < 0 \\ 1/2 & \text{if } 0 \leq x < 1 \\ 1 - (e^{1-x}/2) & \text{if } 1 \leq x \end{cases}$

   (c) $F_X(x) = \begin{cases} e^x/4 & \text{if } x < 0 \\ 1 - (e^{-x}/4) & \text{if } x \geq 0 \end{cases}$

   Note that, in part (c), $F_X(x)$ is discontinuous but $F_X^{-1}(y) = inf\{x : F_X(x) \geq y\}$ is still the appropriate definition of $F_X^{-1}(y)$.

3. Let X have the pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, 0 < x < \infty, \beta > 0$$

(a) Verify that $f(x)$ is a pdf.

(b) Find EX and $VarX$

4. (a) Let X be a continuous, nonnegative random variable $[f(x) = 0$ for $x < 0]$. Show that

$$EX = \int_0^\infty [1 - F_X(x)]dx$$

where $F_X(x)$ if the cdf of X.

(b) Let X be a discrete random variable whose range is the nonnegative integers. Show that

$$EX = \sum_{k=0}^{\infty}(1 - F_X(k))$$

where $F_X(k) = P(X \leq k)$. Compare this with part(a).

5. Let $f(x)$ be a pdf and let a be a number such that, for all $\epsilon > 0, f(a + \epsilon) = f(a - \epsilon)$. Such a pdf is said to be symmetric about the point a.

(a) Give three examples of symmetric pdfs.

(b) Show that if $X \sim f(x)$, symmetric, then the median of X is the number a.

(c) Show that if $X \sim f(x)$, symmetric, and EX exists, then $EX = a$.

(d) Show that $f(x) = e^{-x}, x \geq 0$, is not a symmetric pdf.

(e) Show that for the pdf in part (d), the median is less than the mean.

6. Find the moment generating function corresponding to

(a) $f(x) = \frac{1}{c}, 0 < x < c$

(b) $f(x) = \frac{2x}{c^2}, 0 < x < c$

(c) $f(x) = \frac{1}{2\beta}e^{-|x-\alpha|/\beta}, -\infty < x < \infty, -\infty < \alpha < \infty, \beta > 0$

7. In each of the following, find the pdf of Y.

(a) $Y = X^2$ and $f_X(x) = 1, 0 < x < 1$

(b) $Y = -logX$ and X has pdf

$$f_X(x) = \frac{(m+n+1)!}{n!m!} x^n (1-x)^m, 0 < x < 1, m, n \text{ positive integers}$$

(c) $Y = e^X$ and X has pdf

$$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2}, 0 < x < \infty, \sigma^2 \text{ positive constant}$$

8. A random variable X is said to have a Gamma distribution if its pdf is

$$f(x|\text{shape} = \alpha, \text{scale} = \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta},$$

where $x \in [0, \infty), \alpha > 0, \theta > 0$.

(a) Verify $f(x|\text{shape} = \alpha, \text{scale} = \theta)$ is a valid pdf;

(b) Find the **mode** of a Gamma random variable (for $\alpha > 1$);

(c) Find the **moment generating function** $M(t)$ of a Gamma random variable;

(d) Find the **mean**, the **variance**, the **skewness** and the **kurtosis** of a Gamma random variable;

(e) Let $Y = 1/X$. What is the pdf of Y ? (Y is said to have an inverse Gamma distribution)

(f) Find the **mean**, the **variance**, the **skewness** and the **kurtosis** of a inverse Gamma random variable;

9. A random variable X is said to have a Possion distribution if its pdf is

$$f(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, ...$$

(a) Verify $f(X = k)$ is a valid pdf;

(b) Find the **moment generating function** $M(t)$ of a Poisson random variable;

(c) Find the **mean**, the **variance**, the **skewness** and the **kurtosis** of a Poisson random variable;

10. Show that

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}, \alpha = 1, 2, 3, ...$$

(Hint:Use integration by parts.) Express this formula as a probabilistic relationship between Possion and gamma random variables.