统计学方法及其应用

Statistical Methods with Applications

Rui Jiang, PhD

Associate Professor

Ministry of Education Key Laboratory of Bioinformatics
Bioinformatics Division, TNLIST/Department of Automation
Tsinghua University, Beijing 100084, China

Random Variables

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Defining a probability function

%&'(% = "#!)***)#„ $(+&(, (-./.'(0&'*(%&'($\mathcal{B}$(+&(, /1(0.2 3 , (, 42+5,
6-(07+0&'0(6-(%*(%&'(&!)***) &„ (+&(/6/ /&2, '.8&(/7 3 +&50('9, '
07 3 ('6(!*(: 65(, /1($ $\in \mathcal{B}$)(; &-./&(' <$=(+1

$$' <\$= = \sum_{"!\#_! \in \$\$} \&_!*$$

>9&/(' (.0(, (?56+, +.4.'1(-7/@.'6/(6/($\mathcal{B}$*(>9.0(5&3 , ./0('57&(.-
% = "#!)#A)***$(.0(, (@67/' , +4&(0&'*

Tossing coins

- 扔一枚硬币，观察到正面的概率
  - $S = \{H, T\}$
  - $P(\text{正面}) = P(\{H\}) = 1/2$
- 扔一枚硬币三次，观察到两次正面的概率
  - $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$
  - $P(\text{两次正面}) = P(\{HHT, HTH, THH\}) = 3/8$
- 扔一枚硬币一百次，观察到十次正面的概率
  - $S = \{2^{100} \text{ elements}\}$
  - $P(\text{十次正面})$ = Unable to count!
- 实际上正面出现的次数仅有101种可能

It is much easier to deal with a summary variable than with the original probability structure.

How to reduce the sample space?

- 定义计数函数
  - (<#=(B(C"D$
  - 定义域 $\mathcal{S}$ 包含 A!EE 个元素
  - 值域 FE)(!EEG 包含 !E! 个元素
- 观察到十次正面的次数
  - '<C"D$B!E=B '<(B!E=BH<!EE)!E=×E*I !E×E*I JE≈!*KL×!E M!L
- 扔任意硬币 " 次，观察到) 次正面的次数
  - '<(B)*+*")(&=(B(H<")(,=(× &, × <! —&="—,

Random variables

Random variable

N(!"#$%&'("!)"*+,(.0(,(-7/@'.6/(-563(,(0,3?4&(0?,@&($\mathcal{S}$(./'6 '9&(5&,4(/73+&50*

- 随机变量是定义在样本空间上的实值函数
- 随机变量用大写字母表示，例如() -) .
- 随机变量的取值用对应的小写字母表示，例如)) /) 0

Examples of random variables

- 掷一只骰子
  - (*B(观测到的点数
- 掷两只骰子
  - (*B(观测到的点数之和
  - - *B(观测到的点数之差的绝对值
- 扔一枚硬币K次
  - (*B(观测到正面的次数
- 从一副扑克牌中任意抽取五张
  - (*B(抽到K的张数

随机变量的引入简化了研究的问题，
体现了统计学中**数据简约**的思想
随机变量的取值很重要，
但随机变量以什么概率取得这些值更重要

Define a probability on the domain

%&'(% = "#!)***)#„$(+&(,(-./.'(0&'*(%&'(ℬ(+&(,/1(0.23 ,(,42+5,
6-(07+0&'0(6-(%*(%&'(&!)***)&„(+&(/6/ /&2,'.8&(/73+&50('9,'
073('6(!*(:65(,/1($ ∈ ℬ)(;&-./&('<$=(+1

'<$= = ∑ &!*
"!#! ∈ $$

>9&/(' (.0(,(?56+,+.4.'1(-7/@.'6/(6/(ℬ*(>9.0(5&3 ,./0('57&(.-
% = "#!)#A)***$(.0(,(@67/',+4&(0&'*

Induce a probability on the range

O7??60&('9 , '('9&(5 , / 2&(6-(((.0(, 406(, (-. / . '&(0&'($\mathcal{X}$)(P&(@ , / ('9& / (; &-. / &

$$'_{(}<(\;=\;)_{!}= \;=\; '\left("\#_{1}\ \#\#_{1} \in \mathcal{S})\ (\ <\#_{1} = \;=\;)_{!}\$ \right)$$

Q6P)(4&'('9&(0.2 3 , (, 42&+5 , ($\mathcal{B}$(+&('9&(@644&@'.6 / (6-(, 44(07+0&'0(6-($\mathcal{X}$)

- .)%&'/ 0'-65(, / 1(0&'($\$ \in \mathcal{B}$)

$$'_{(}<\$= \;=\; '\left(\bigcup\nolimits_{)_{!}\in\$} "\#_{1}\ \#\#_{1} \in \mathcal{S})\ (\ <\#_{1} = \;=\;)_{!}\$\right)$$

$$= \sum\nolimits_{)_{!}\in\$} '\left("\#_{1}\ \#\#_{1} \in \mathcal{S})\ (\ <\#_{1} = \;=\;)_{!}\$\right)$$

$$\geq \mathrm{E}$$

- .)%&'1 0'-65('9&(& / '.5&(0 , 3 ?4&(0? , @&($\mathcal{X}$)

$$'_{(}<\mathcal{X}= \;=\; '\left(\bigcup\nolimits_{)_{!}\in\mathcal{X}} "\#_{1}\ \#\#_{1} \in \mathcal{S})\ (\ <\#_{1} = \;=\;)_{!}\$\right) = \ '<\mathcal{S}= \;=\; !$$

- .)%&'2 0'-65(? , .5P.0&(; .0R6. / '(0&'0($\$_{!}$) $\$_{A}$)***)

$$'_{(}\left(\bigcup\nolimits_{,=!}^{\infty} \$_{,}\right) = \ '\left(\bigcup\nolimits_{,=!}^{\infty}\left\{\bigcup\nolimits_{)_{!}\in\$_{,}} "\#_{1}\ \#\#_{1} \in \mathcal{S})\ (\ <\#_{1} = \;=\;)_{!}\$\right\}\right)$$

$$= \sum\nolimits_{,=!}^{\infty} '\left(\bigcup\nolimits_{)_{!}\in\$_{,}} "\#_{1}\ \#\#_{1} \in \mathcal{S})\ (\ <\#_{1} = \;=\;)_{!}\$\right)$$

$$= \sum\nolimits_{,=!}^{\infty} '_{(}<\$_{,}=$$

Change of the sample space

- 样本空间的转换
  - 在随机变量的定义域上 　　$\mathcal{S}$ B("#$_!$)(#$_A$)(S)(#$_{"}$$
  - 在随机变量的值域上 　　$\mathcal{X}$ B(")$_!$)()$_A$)(S)() $_2$$
  - 随机变量建立的映射 　　(#($\mathcal{S} \mapsto \mathcal{X}$
- 定义在随机变量定义域上的概率函数

' <#$_1$= 　　B 　　&$_1$

' <$= 　　B 　　$\sum$$_{\#_1 \in \$}$ &$_1$

- 定义在随机变量值域上的概率函数

' $_($< (*B()$_!$= 　　B 　　' <"#$_1$ $\in$ $\mathcal{S}$# (<#$_1$= B)$_!$$=

Distributions of random variables

- 随机变量的所有可能取值及取得每一个值的概率
- 扔一枚硬币三次，观察出现正面的次数
  - $\mathcal{S}$ B "DDD)(DD>)(D>D)(>DD)(>>D)(>D>)(D>>)(>>>$
  - $\mathcal{X}$ B "E)(!)(A)(K$
  - (#($\mathcal{S} \mapsto \mathcal{X}$
(<DDD=BK((((<DD>=BA(((<D>D=BA((((<>DD=BA
(<>>D=B!(((((<>D>=B!((((<D>>=B!(((((<>>>=BE
  - '<(*B(E=(B(!TU '<(*B(!=(B(KTU
'<(*B(A=(B(KTU '<(*B(K=(B(!TU

Cumulative distribution function (cdf)

Distribution function

>9&(34&4+"5)(,'$)65!)*45)%#'74#35)%#(<345=(6-(,(5,/;63

8,5.,+4&(()(;&/6';(+1(6(<) =)(.0(;&-./&;(+1

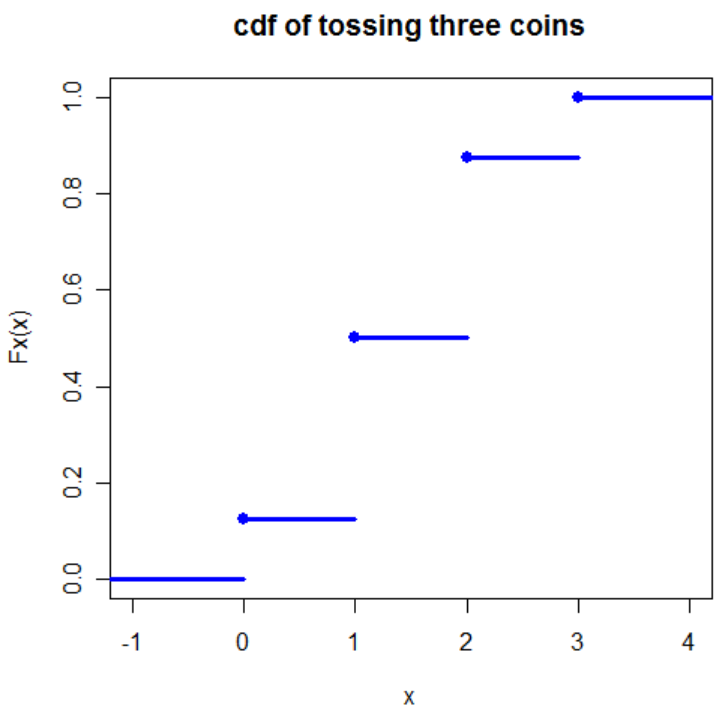
6(<)= = ' (<(≤) =)(-65(,44()*

At most

cdf

- 扔一枚硬币三次，观察出现正面的次数
  - $\mathcal{X}$ B "E)(!)(A)(K$
  - '<(*B(E=(B(!TU　　　'<(*B(!=(B(KTU
    '<(*B(A=(B(KTU　　　'<(*B(K=(B(!TU
  - 分布函数

$$6_{(}<)= = \begin{cases} E & .-(\quad -\infty <) < EV \\ !/U & .-(\quad E \le) < !V \\ !/A & .-(\quad ! \le) < AV \\ L/U & .-(\quad A \le) < KV \\ ! & .-(\quad K \le) < \infty * \end{cases}$$



Necessary and sufficient condition

Necessary and sufficient condition

>9&(-7/@'.6/(6<)=(.0(,(@;-(.-(,/;(6/41(.-('9&(-6446P./2(

58!,,(3%#$)5)%#6(964;#

!* 4.3 $_{)\to-\infty}$ 6<)= $=$ E(,/;(4.3 $_{)\to\infty}$ 6<)= $=$!V

A* 6<)=(.0(,(/6/;&0@5&,0./2(-7/@'.6/(6-()V

K* 6<)=(.0(5.29'M@6/'./7670V('9,'(.0)(-65(&8&51

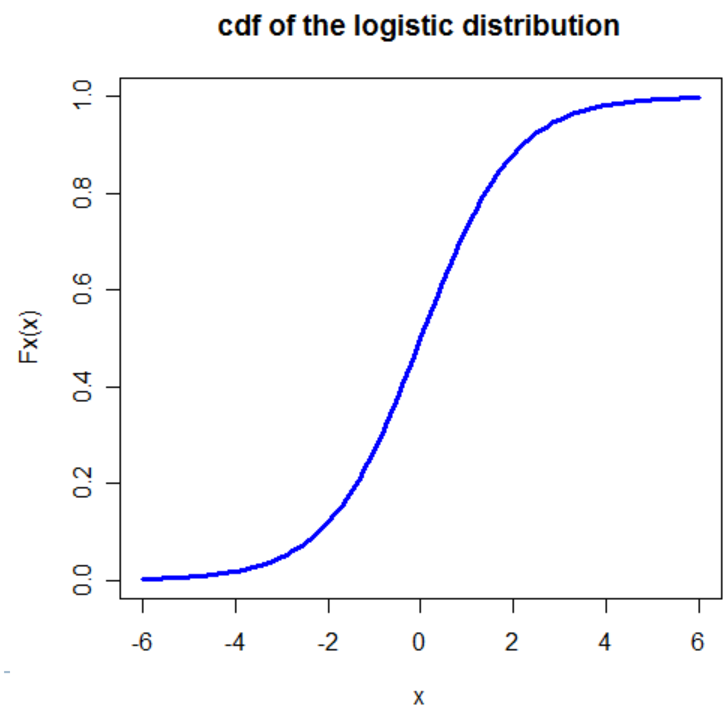
/73+&5() $_{E}$)(4.3 $_{)\to)_{E}^{+}}$ 6<)= $=$ 6<) $_{E}$=*

Logistic cdf

- Logistic distribution

$$6_{(}<)=\;=\frac{!}{!+7^{-)}}$$

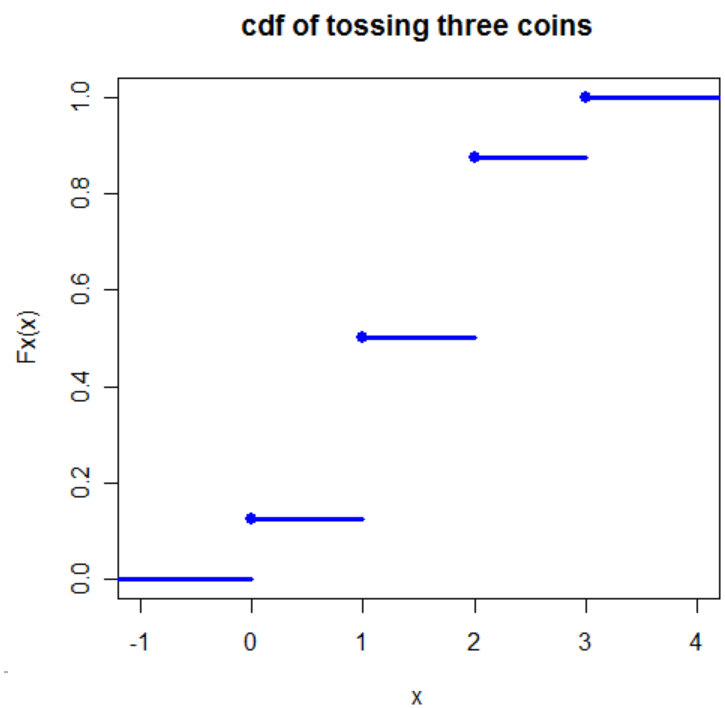
- 充要条件的满足性
  - 负无穷时为E
  - 正无穷时为!
  - 不减
  - 右连续



Discrete random variables

Discrete random variables

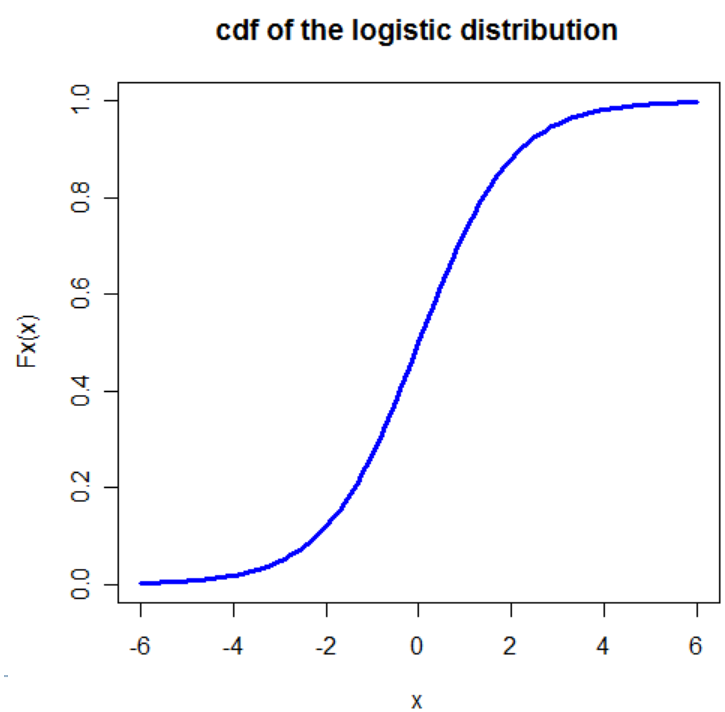
N(5 , / ; 6 3 (8 , 5. , +4&(((.0($)63! , 5 , (.-(6 (<) =(.0(, (0'&?(-7 / @'.6 / (6-() *



Continuous random variables

Continuous random variables

N(5, / ;63(8,5., +4&(((.0(3%#5)#4%46(.-(6(<) =(.0(, (@6/ '. / 7670
-7/@'.6/(6-()*



Identically distributed

Identically distributed

>9&(5 , / ; 6 3 (8 , 5. , +4&0(((, / ; (- (, 5&()$, #5)3"++9'$)65 !)*45 , $(.-)

-65(&8&51(0&'($ $\in \mathcal{B}^{!}$)

'<($\in$ $= $=$ '<- $\in$ $=*

1. $\mathcal{B}^{!}$ is the smallest sigma algebra containing all the intervals of real numbers of the form <8)9=)(F8)(9=)(<8)9G)(and F8)9G.
2. Two identically distributed random variables are not necessarily equal.

Identically distributed

>9&(-6446P./2('P6(0','&3&/'0(,5&(&W7.8,4,/'

((!* >P6(5,/;63(8,5.,+4&0(((,/;(- (,5&(.;&/'.@,441(;.0'5.+7'&;V

((A* $6_{(}$ <)= $=$ 6_{-} <)=(-65(&8&51()*

'<($\in$ \$= $=$ '<- $\in$ \$=(-65(,/1(0&'(\$ $\in \mathcal{B}^{!}$ $\Rightarrow$

'<($\in$ <$-\infty$)) G= $=$ '<- $\in$ <$-\infty$)) G= $\Rightarrow$

$6_{(}$ <)= $=$ 6_{-} <)=

Probability mass functions

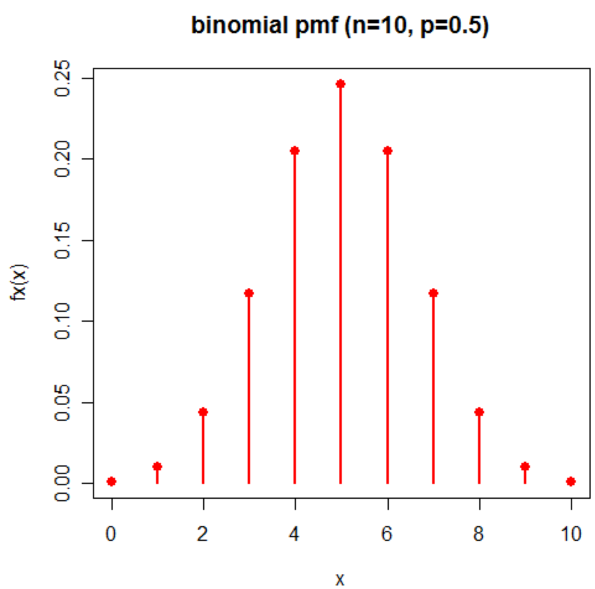
Probability mass function

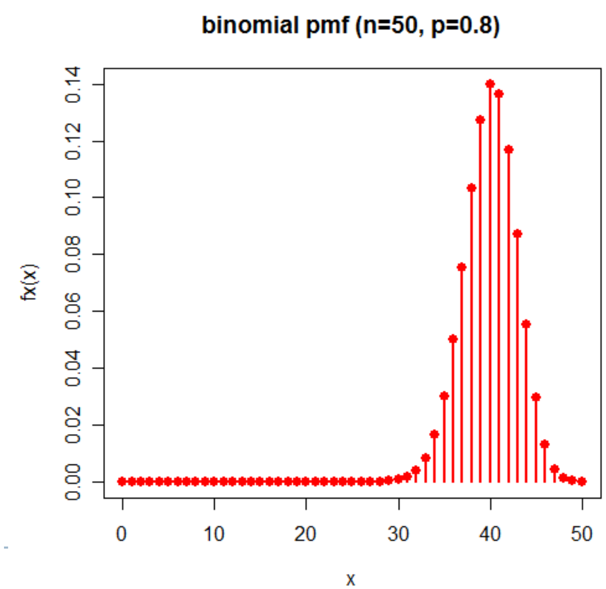
>9&(: !%*"*)+)59'&"66'74#35)%#(<&25=(6-(,(;.0@5&'&(5,/;63

8,5.,+4&(()(;&/6'&;(+1(5(<)=)(.0(2.8&/(+1

5(<)= = '(<(=))=)((-65(,44()*

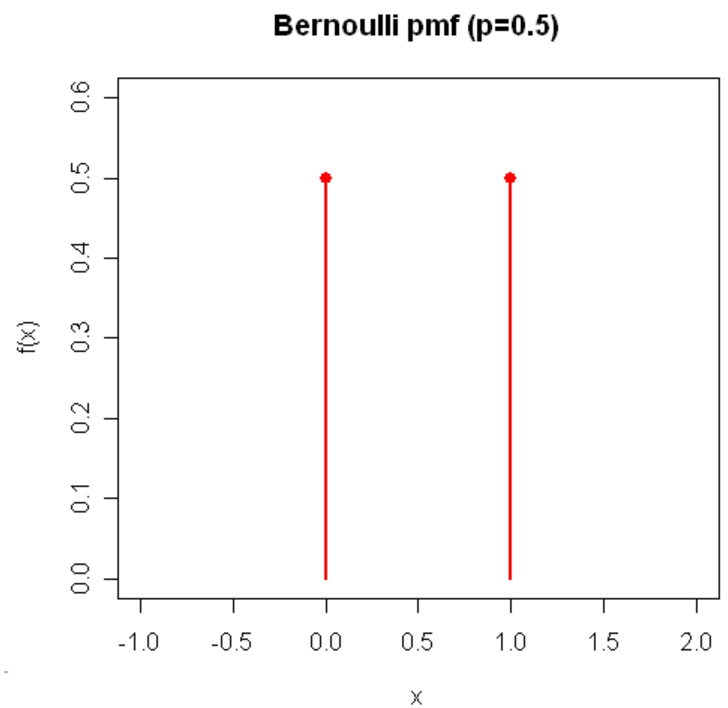
Exact





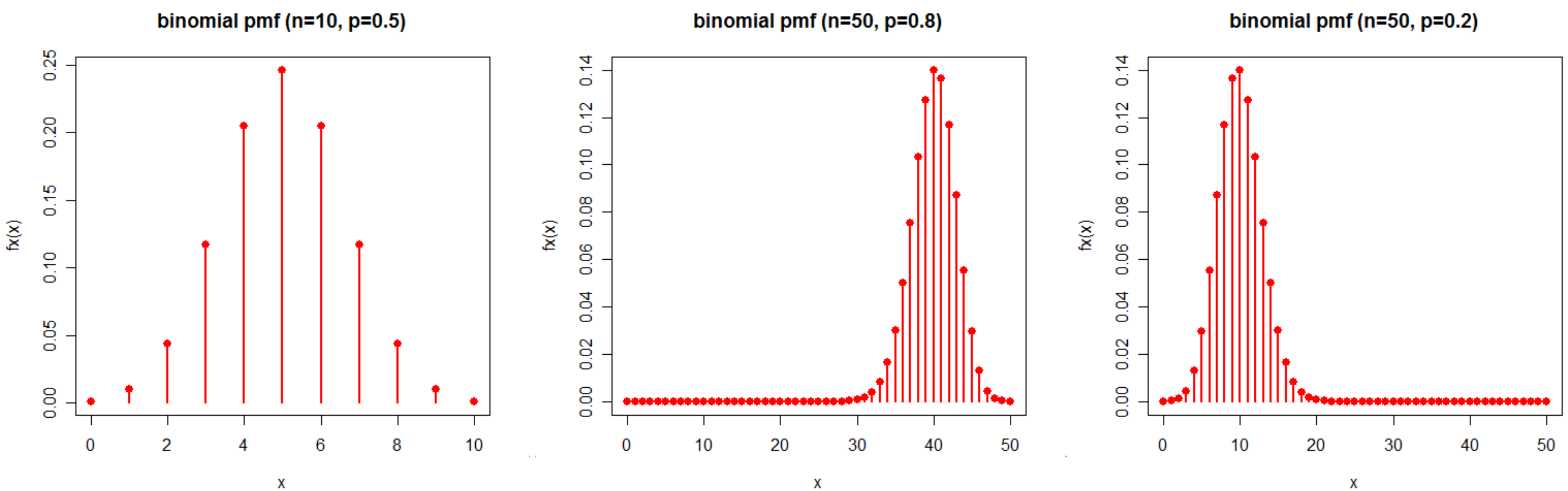
Bernoulli distribution

(= { !(<07@@&00= P.'9(?56+,+.4.'1(&
      E(<-,.475&= P.'9(?56+,+.4.'1(! M &



Binomial distribution

$$' < (\ =) \ \chi \ '') \, \&= \ = \begin{pmatrix} '' \\ ) \end{pmatrix} \&^{)} <! - \&=^{''-)})) \ = \text{E}) \ !) \ldots) \ ''$$



Relation of cdfs and pmfs

RELATION?

$$' < 8 \leq (\;\leq 9 = \;=\; \sum_{, =8}^{9} 5<, =$$

$$' < (\;\leq 9 = \;=\; \sum_{, =-\infty}^{9} 5<, =$$

$$' < (\;\geq 9 = \;=\; \sum_{, =9}^{\infty} 5<, =$$

$$6<) = \;=\; ' < (\;\leq) = \;=\; \sum_{, =-\infty}^{)} 5<, =$$

For a continuous random variable

P; X < x=' < '>

- " (*B()$(((⊂ ") −ε Y((≤)$ -65(, / 1() , / ; (ε
- ' " (*B()$(≤ ' ") −ε Y((≤)$
  B(' " (≤)*∩ (*Z() −ε$
  B(' " (≤)*∩ < (*≤) −ε)@$
  B(' " (≤)$ − ' < (≤)*∩ (*≤) −ε$? 89>
  B(6 (<)= − 6 (<) −ε=
- E ≤ ' " (*B()$ ≤ 4. 3 $_{\varepsilon \to 0}$ F6 (<)= − 6 (<) −ε=G(B(E

P@X < xA' < 'B'7%!'"#9'x

' "8Y (Y9$B ' "8Y (≤9$B ' "8≤ (Y9$B ' "8≤ (≤9$(-65(, / 1()

Probability density functions

Probability density function

>9&(: !%*"*)+)59'$,#6)59'74#35)%#(<&45 =)(; &/6'&; (+1($5_($ <) =) 6-(,

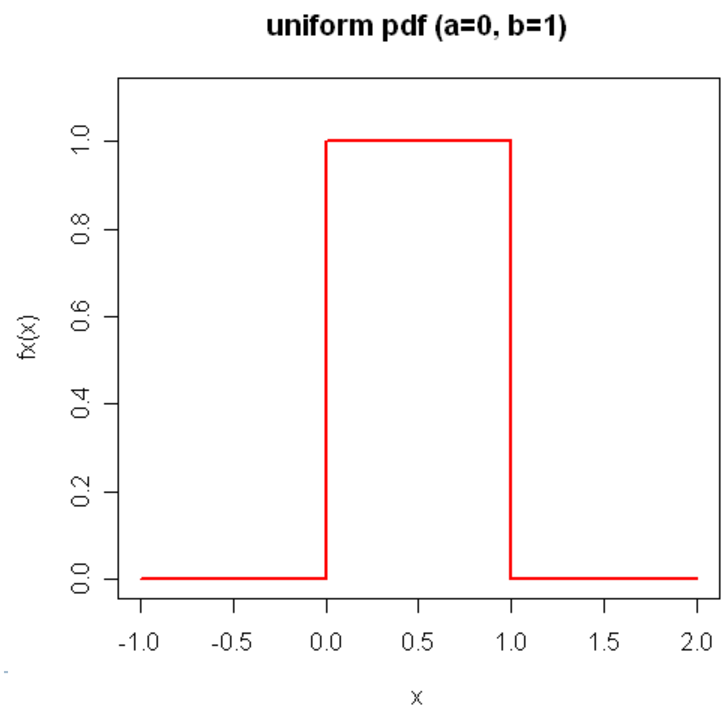
@6/'./7670(5, / ; 63 (8,5.,+4&(((.0('9&(-7/@'.6/('9,'(0,'.0-.&0

$$6_(<) = = \int_{-\infty}^{)} 5_(<:=4:)(((-65(,44()*$$

$$5_(<) = = \frac{4}{4)} 6_(<) =$$

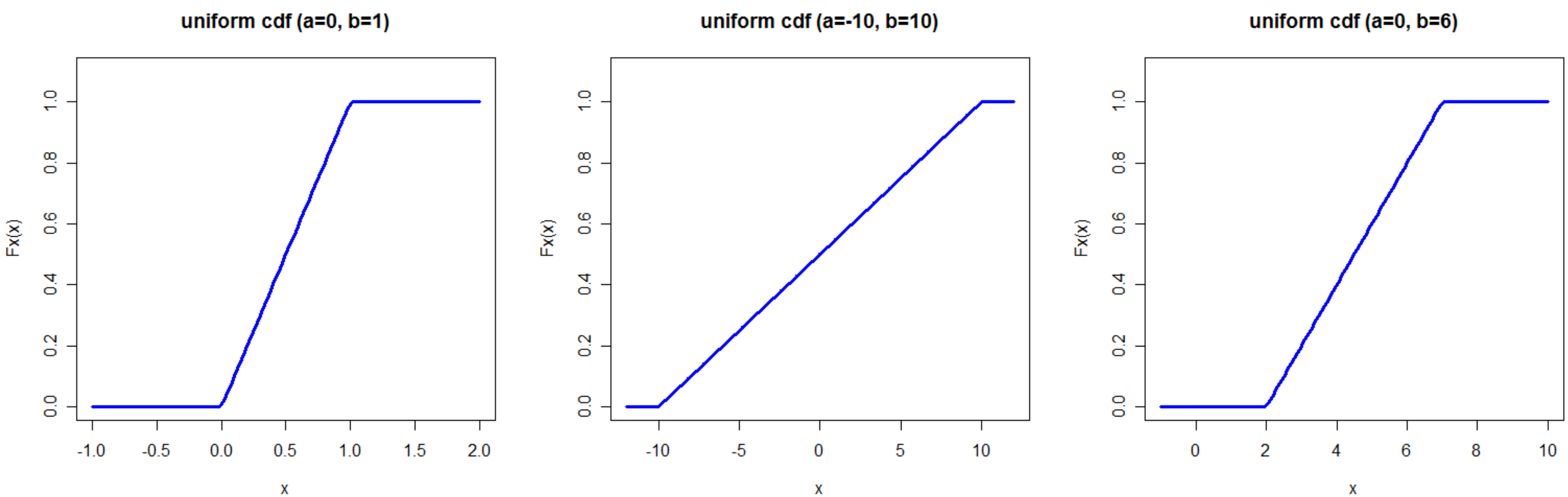
Uniform distribution

$$f_X(x;a,b) = \begin{cases} 0 & x < a \\ 1/(b-a) & a \le x \le b \\ 0 & x > b \end{cases}$$



Uniform distribution

$$6_{(}<)\,\wr\,8)9= = \begin{cases} \mathrm{E} &) < 8 \\ \dfrac{) - 8}{9 - 8} & 8 \le) \le 9 \\ ! &) > 9 \end{cases}$$



Relation of cdfs and pdfs

$$' <8 < (< 9= = \int_{8}^{9} 5<)=4)$$

$$' <(<)= = \int_{-\infty}^{)} 5<:=4:$$

$$' <(>)= = \int_{)}^{\infty} 5<:=4: = ! - \int_{-\infty}^{)} 5<:=4:$$

$$6<)= = ' <(\leq)= = \int_{-\infty}^{)} 5<:=4:$$

Necessary and sufficient condition

Necessary and sufficient condition

N(-7/@'.6/(5(<)=(.0(,(?;-(65(?3-(6-(,(5,/;63(8,5.,+4&(((

.-(,/;(6/41(.-('9&(-6446P./2('P6(@6/;.'.6/0(964;#

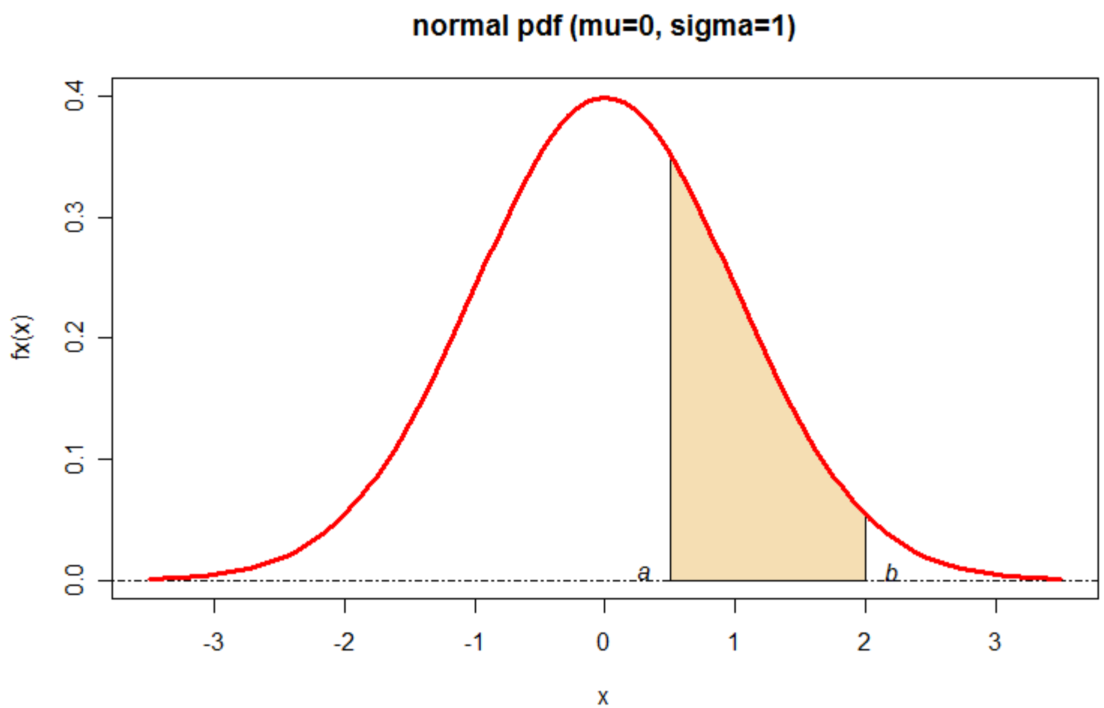
!* 5(<)= $\geq$ E(-65(,44()V

A* $\sum_{)}$ 5(<)=B!(<?3-=(65(

$\int_{-\infty}^{\infty}$ 5(<)=B!(<?;-=*

Standard normal distribution

$$5<)= = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^{A}}{A}}$$



$$' <8 \leq) \leq 9=$$

$$= ' <8 <) \leq 9=$$

$$= ' <8 \leq) < 9=$$

$$= ' <8 <) < 9=$$

$$= 6_{(} <9= - 6_{(} <8=$$

$$= \int_{8}^{9} 5_{(} <) =4)$$

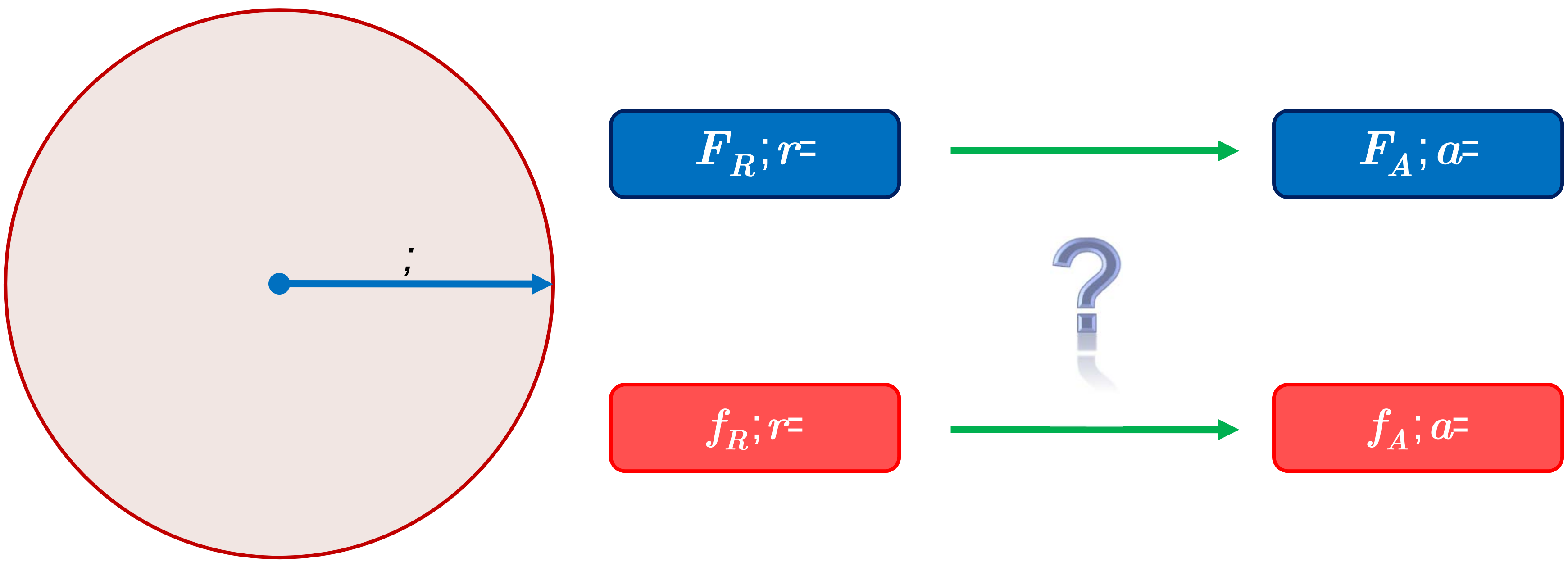
Transformations

统计学方法及其应用 统计学基础 随机变量的函数

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Why need functions of random variables

- 已知一些量的分布，而关心的是另一些量的分布
  - 半径 ; ~ [/ .-65 3 <E)(! =
  - 面积 $ ~ \

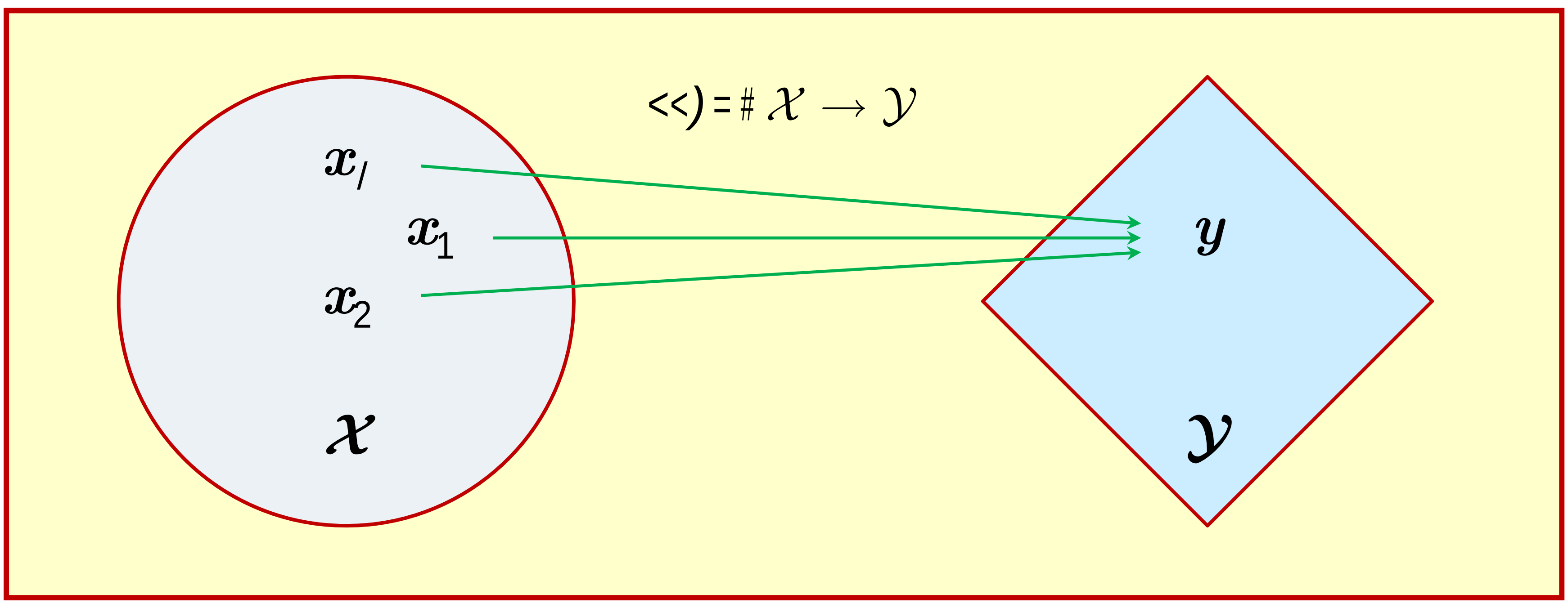


Function of a random variable

- If X is a random variable with cdf $F_X(x)$, then any function of X, say, $Y = g(X)$, is also a random variable
- The probability behavior of Y can be described using X

$$P(Y \in A) = P(g(X) \in A)$$

depending on the distribution of X and the function g



Transformation of a pmf

' <- ∈ $= = ' <<< (= ∈ $=

= ' <) ∈ 𝒳 # <<) = ∈ $=

= ' < (∈ <⁻ ! <$==

]-(((.0(; .0@5&'&)('9&/(- (.0(, 406(; .0@5&'&

<+&@ , 70&(+6'9(𝒳(, / ; (𝒴(, 5&(@67' , +4&=

]-(/ ∈ 𝒴)('9&/

5 <"/= = ' <- = /=

= Σ ' < (=)=

= Σ 5 <) =*

]-(/ ∉ 𝒴)('9&/

5 </= = E*

<⁻ ! <$= = ") ∈ 𝒳 # <<) = ∈ $$

<⁻ ! </= = ") ∈ 𝒳 # <<) = = /$

Binomial transformation

O7??60&(5(<) = = ' < (=) X ") &= = (" ;)) &) <! − &=" −))) = E) !) . . .) ")

'9 , '(.0)(((, (+. / 6 3 . , 4(; .0'5.+7'.6 / (P.'9(? , 5 , 3 & '&50(" (, / ; (&)(, / ; (

- = << (= = " − ((.0(, ('5 , / 0-65 3 , '.6 / *(>9& /)(/ = <<) = = " −)) (

, / ;

5- </ X ") &= = ∑) ∈<−! </ = 5(<) X ") &=

= 5(<" − / X ") &=

= (" ; " − /) & " −/ <! − &= " −<" −/=

= (" ; /) <! − &=/ & " −/

.0(, 406(, (+. / 6 3 . , 4(; .0'5.+7'.6 / (P.'9(? , 5 , 3 & '&50(" (, / ; (! − &*

Transformation of a cdf

>9&(@;-(6-(- $=$ <<(=(.0

6- </= $=$ '<- $\leq$ /=

$=$ '<<<(= $\leq$ /=

$=$ '<") $\in \mathcal{X}$ # <<)= $\leq$ /$=

$=$ $\int$ ") $\in\mathcal{X}$#<<)=$\leq$/$ 5(<)=4)

D6P('6(6+' , . / (") $\in \mathcal{X}$ # <<)= $\leq$ /$ \

1. The function <<)=(is monotone increasing
2. The function <<)=(is monotone decreasing
3. The function <<)=(is piecewise monotone

Monotone increasing

%&'(((9,8&(@;-(6(<)=*(%&'(- = <<)=*(%&'(

𝒳 = ") # 5(<)= > E$)(,/;(

𝒴 = "/ # / = <<)=(-65(06 3&() ∈ 𝒳$*

]-(<<)=(.0(3 6/6'6/&(./@5&,0./2)('9&/('9&(3 ,??./2() → <<)=(.0

^6/&M'6M6/&^(,/;(^6/'6^)(,/;(<⁻!</=(.0(0./24&M8,47&;(,/;

,406(3 6/6'6/&(./@5&,0./2*(>9&5&-65&)

") ∈ 𝒳 #<<)= ≤ /$ = ") ∈ 𝒳 #) ≤ <⁻!</=$

6- </= = ∫ ")∈𝒳#<<)=≤/$ 5(<)=4)

= ∫ ")∈𝒳#)≤<⁻!</=$ 5(<)=4)

= 6(<<⁻!</==

Monotone decreasing

%&'(((9,8&(@;-($6_($ <)=*(%&'(- $=$ <<)=*(%&'(

$\mathcal{X} =$ ") # 5$_($ <)= $>$ E$)(,/;(

$\mathcal{Y} =$ "/ # / $=$ <<)=(-65(06 3&() $\in \mathcal{X}$$*

]-(<<)=(.0(3 6/6'6/&(;&@5&,0./2)('9&/('9&(3 ,??./2() $\longrightarrow$ <<)=(.0

^6/&M'6M6/&^(,/;(^6/'6^)(,/;(<$^{-!}$</=(.0(0./24&M8,47&;(,/;

,406(3 6/6'6/&(./@5&,0./2*(>9&5&-65&)

$$") \in \mathcal{X}\ \# <<)= \leq /\$ = ") \in \mathcal{X}\ \#)\ \geq <^{-!}</=\$$$

$$\begin{aligned} 6_{-}</= &= \int_{")\in\mathcal{X}\#<<)=\leq/\$} 5_{(}<)=4) \\ &= \int_{")\in\mathcal{X}\#)\geq<^{-!}</=\$} 5_{(}<)=4) \\ &= ! - 6_{(}<<^{-!}</== \end{aligned}$$

Transformation of a cdf

%&'(((9,8&(@;-(6(<) =)(4&'(- = <<) =)(, / ; (4&'(

𝒳 = ") # 5(<) = > E$)(, / ; (

𝒴 = "/ # / = <<) =(-65(06 3 &() ∈ 𝒳$*

>9&/

]-(<(.0(, / (. / @5&, 0. / 2(-7 / @'.6 / (6 / (𝒳)

6- </= = 6(<<−! </==(-65(/ ∈ 𝒴*

]-(<(.0(, (; &@5&, 0. / 2(-7 / @'.6 / (6 / (𝒳)

6- </= = ! − 6(<<−! </==(-65(/ ∈ 𝒴*

easy

Exponential distribution

O7??60&('9 , '(5(<) = = !(-65() ∈ <E) !=(, / ; (E(6'9&5P.0&)('9 , '(.0)(((

9 , 0(, (7 / .-65 3 (; .0'5.+7'.6 /)(, / ; (- = << (= = −λ 462 ((<λ > E=(.0

, ('5 , / 0-65 3 , '.6 / *(>9& /)(

6(<) = =) (-65() ∈ <E) !=*(

<<) = = −λ 462) (.0(; &0@5& , 0. / 2(6 / (.'0(07??65')(, / ; (

<−! </= = &_?<−/ T λ=(.0(, 406(; &@5& , 0. / 2(6 / (.'0(; 6 3 , . / (E < / < ∞*

>9&5&-65&)

6- </= = ! − 6(<<−! </== = ! − 6(<&_?<−/ T λ==

= ! − &_?<−/ T λ=)E < / < ∞*(

Probability integral transformation

%&'(((9,8&(@6/'./7670(@;-(6(<) =(, / ;(;&-./&(,(5, / ;6 3 (8,5.,+4&
- (,0(- = 6(<(=*(>9&/(- (.0(7/.-65 3 41(;.0'5.+7'&;(6/(<E) !=)(
'9,'(.0)

$$'<- \leq / = = /)(EY1Y!*$$

p-value

$$
\begin{aligned}
'<- \leq / = &= '<6_(<(= \leq / = \\
&= '<6_(^{-!} F6_(<(= G \leq 6_(^{-!} </== \\
&= '<(\leq 6_(^{-!} </== \\
&= 6_(<6_(^{-!} </== \\
&= /
\end{aligned}
$$

cdfs → pdfs

]-(<(.0(,/(./@5&,0./2(-7/@'.6/(6/($\mathcal{X}$)

$$6_{-}</= = 6_{(}<<^{-!}</==(-65(/ \in \mathcal{Y}*$$

$$5_{-}</= = \frac{4}{4/}6_{-}</= = \frac{4}{4/}6_{(}<<^{-!}</== = 5_{(}<<^{-!}</==\frac{4}{4/}<^{-!}</=*$$

` /('9&(6'9&5(9,/;)(.-(<(.0(,(;&@5&,0./2(-7/@'.6/(6/($\mathcal{X}$)

$$6_{-}</= = ! - 6_{(}<<^{-!}</==(-65(/ \in \mathcal{Y}*$$

$$5_{-}</= = \frac{4}{4/}6_{-}</= = \frac{4}{4/}\left[! - 6_{(}<<^{-!}</==\right] = -5_{(}<<^{-!}</==\frac{4}{4/}<^{-!}</=*$$

Transformation of a pdf

%&'(((9,8&(?;-($5_($ <)=)(4&'(- $=$ <<)=)(P9&5&(<(.0(,(&%#%5%#,(
-7/@'.6/*(O7??60&($5_($ <)=(.0(@6/'./7670(6/($\mathcal{X} =$ ") # $5_($ <)= $>$ E$(
,/;(<$^{-!}$</=(9,0(,(@6/'./7670(;&5.8,'.8&(6/

$$\mathcal{Y} = \text{"/ \# /} = \text{<<)=(-65(063\&()} \in \mathcal{X}\text{\$*}$$

>9&/('9&(?;-(6-(- (.0(2.8&/(+1

$$5_{-}\text{</=} = \begin{cases} 5_{(}\text{<<}^{-!}\text{</==}\left|\dfrac{4}{4/}\text{<}^{-!}\text{</=}\right| & / \in \mathcal{Y} \\ \text{E} & \text{6'9\&5P.0\&*} \end{cases}$$

Cauchy distribution

O7??60&('9, '(5(<) = = ! T π(-65() ∈ <−π T A) π T A=(, / ; (E(6'9&5P.0&)(
'9, '(.0)(((9, 0(, (7/.-65 3 (; .0'5.+7'.6/)(, / ; (- = <<(= = ', / ((.0(, (

'5, /0-65 3 , '.6/*(>9&/)(./('9&(./'&58, 4(/ ∈ <−∞) ∞=

) = <^{−!}</= = , 5@', / /)(

重尾巴，均值不存在?

4/4/ <^{−!}<) = = 4/4/ , 5@', / / = !/(! + /^A))

, / ; 5_- </= = 5((<^{−!}</=) |4/4/ <^{−!}<) =| = !/π · !/(! + /^A)) −∞ < / < ∞*

Non-monotone transformation

O7??60&(5$_($ <) = = <$\sqrt{A\pi}$=$^{-!}$ &_?< $-$)^A T A=(-65() $\in$ <$-\infty$) ∞=)('9 , '(.0)

((9 , 0(, (0' , / ; , 5 ; (/ 65 3 , 4(; .0'5.+7'.6 /)(, / ; (- = << (= = (^{A} (.0

, ('5 , / 0-65 3 , '.6 / *(>9&/)(./('9&(./'&58 , 4(/ $\in$ <E)(∞=

$$
\begin{aligned}
6_{\text{-}} </= &= \text{'}<\text{-} \le /== \text{'}<(^{A} \le /= \\
&= \text{'}<-\sqrt{/} \le (\le \sqrt{/}= \\
&= \text{'}<(\le \sqrt{/}= - \text{'}<(\le -\sqrt{/}= \\
&= 6_{(} <\sqrt{/}= - 6_{(} <-\sqrt{/}=(\\
5_{\text{-}} </= &= \frac{4}{4/} 6_{\text{-}} </= \\
&= \frac{!}{A\sqrt{/}} \left[5_{(} <\sqrt{/}= + 5_{(} <-\sqrt{/}= \right] \\
&= \frac{!}{\sqrt{A\pi}} \frac{!}{\sqrt{/}} 7^{-\frac{/}{A}})E < / < \infty *
\end{aligned}
$$

Piecewise monotone

%&'(((9 , 8&(? ; -(5$_($ <) =)(4&'(- $=$ <<) =)(, / ; (; &-. / &('9&(0 , 3 ?4&(0? , @&($\mathcal{X}$, 0 '9&(07??65'(0&'(6-(a*(O7??60&('9&5&(&_.0'0(, (? , 5' . ' .6 /)($\$_E$) $\$_!$)…) $\$_,$)(6- $\mathcal{X}$ 07@9('9 , '(' < ($\in$ $\$_E$= $=$ E(, / ; (5$_($ <) =(.0(@6 / ' . / 7670(6 / (& , @9($\$_!$* : 75'9&5)(07??60&('9&5&(&_.0'(-7 / @' .6 / 0(<$_!$<) =)…)<$_,$ <) =)(; &-. / & ; (6 / $\$_!$)…) $\$_,$)(5&0?&@' .8&41)(0 , ' .0-1. / 2

!*(<<) = $=$ <$_!$<) =)(-65() $\in$ $\$_!$)

A*(<$_!$<) =(.0(3 6 / 6'6 / &(6 / ($\$_!$)

K*('9&(0&'($\mathcal{Y}$= "/ # / $=$ <$_!$<) =(-65(06 3 &() $\in$ $\$_!$$(.0('9&(0 , 3 &

(((-65(& , @9(! $=$!)…) ,)

>9& /

$$5_- </= = \begin{cases} \sum_{!=!}^{,} 5_(<<_!^{-!}</== \left| \frac{4}{4/} <_!^{-!}</= \right| & / \in \mathcal{Y} \\ E & 6'9\&5P.0\&* \end{cases}$$

Chi-squared distribution

$$\text{Q6'.@\&('9,'(-} = \text{<<(=} = (^{A}$$

$$) \in \text{<}-\infty\text{)E=)/} = \text{<}_{!}\text{<)=} =)^{A}\text{)(=}_{!}\text{</=} = -\sqrt{/}\text{) ;\&@5\&,0./2V}$$

$$) \in \text{<E)} + \infty\text{=)/} = \text{<}_{A}\text{<)=} =)^{A}\text{)(=}_{A}\text{</=} = \sqrt{/}\text{) ./@5\&,0./2V}$$

$$) = \text{E(<P.'9(?56+,+.4.'1(E=*}$$

$$\text{b\&-./\&}\quad \$_{E} = \text{"E\$V}\ \$_{!} = \text{<}-\infty\text{)E=V}\ \$_{A} = \text{<E)}\infty\text{=*}$$

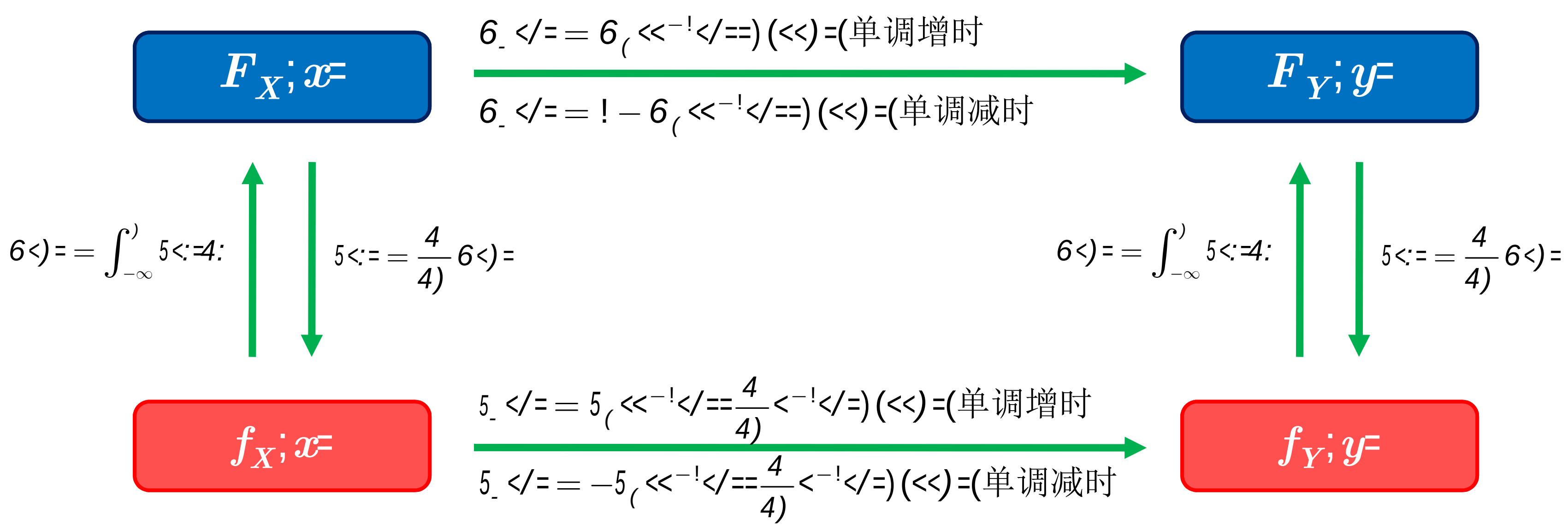
$$\text{>9\&/}\quad \$_E \cap \$_! \cap \$_A = \not{E}\text{(,/;(}\$_E \cup \$_! \cup \$_A = \text{<}-\infty\text{)}\infty\text{=*}$$

$$\text{]/(}\$_!\text{)}\quad 5_!\text{</=} = \frac{!}{\sqrt{A\pi}} 7^{-\frac{\text{<}-\sqrt{/}\text{=}^A}{A}} \left| -\frac{!}{A}\frac{!}{\sqrt{/}} \right| = \frac{!}{A}\frac{!}{\sqrt{A\pi}}\frac{!}{\sqrt{/}} 7^{-\frac{/}{A}}$$

$$\text{]/(}\$_A\text{)}\quad 5_A\text{</=} = \frac{!}{\sqrt{A\pi}} 7^{-\frac{\text{<}\sqrt{/}\text{=}^A}{A}} \left| \frac{!}{A}\frac{!}{\sqrt{/}} \right| = \frac{!}{A}\frac{!}{\sqrt{A\pi}}\frac{!}{\sqrt{/}} 7^{-\frac{/}{A}}$$

$$\text{>9\&/)}\quad 5\text{</=} = 5_!\text{</=} + 5_A\text{</=} = \frac{!}{\sqrt{A\pi}}\frac{!}{\sqrt{/}} 7^{-\frac{/}{A}} \sim \chi^A_!$$

Summary



Location family

O7??60&(((.0(, (5 , / ; 6 3 (8 , 5. , +4&(9 , 8. / 2(? ; -(5 <) =*

H6 / 0. ; &5('9&('5 , / 0-65 3

- = (+ μ*

O. / @&

(

(= - − μ)

4) / 4/ = !*

>9&5&-65&

<</== 5</ − μ=

>9&(- , 3 .41(6-(? ; -0(5 <) − μ=) . / ; &_& ; (+1('9&(? , 5 , 3 &'&5(μ) −∞ < μ < ∞)

.0(@ , 44& ; ('9&(+%3"5)%#'7" &)+9(P. '9(65"#$" !$' : $7 (5 <) =) (, / ; (μ (.0(@ , 44& ;

'9&(+%3"5)%#' : " ! " & , 5 , ! (-65('9&(- , 3 .41*

Normal location family

O7??60&(((.0(,(0' , / ; ,5;(/653 ,4(5, / ;63(8,5., +4&*(>9&/)

$$5<)= = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^A}{A}}$$

H6/0.;&5('9&('5, /0-653

$$- = (+ \mu*$$

O./@&

(

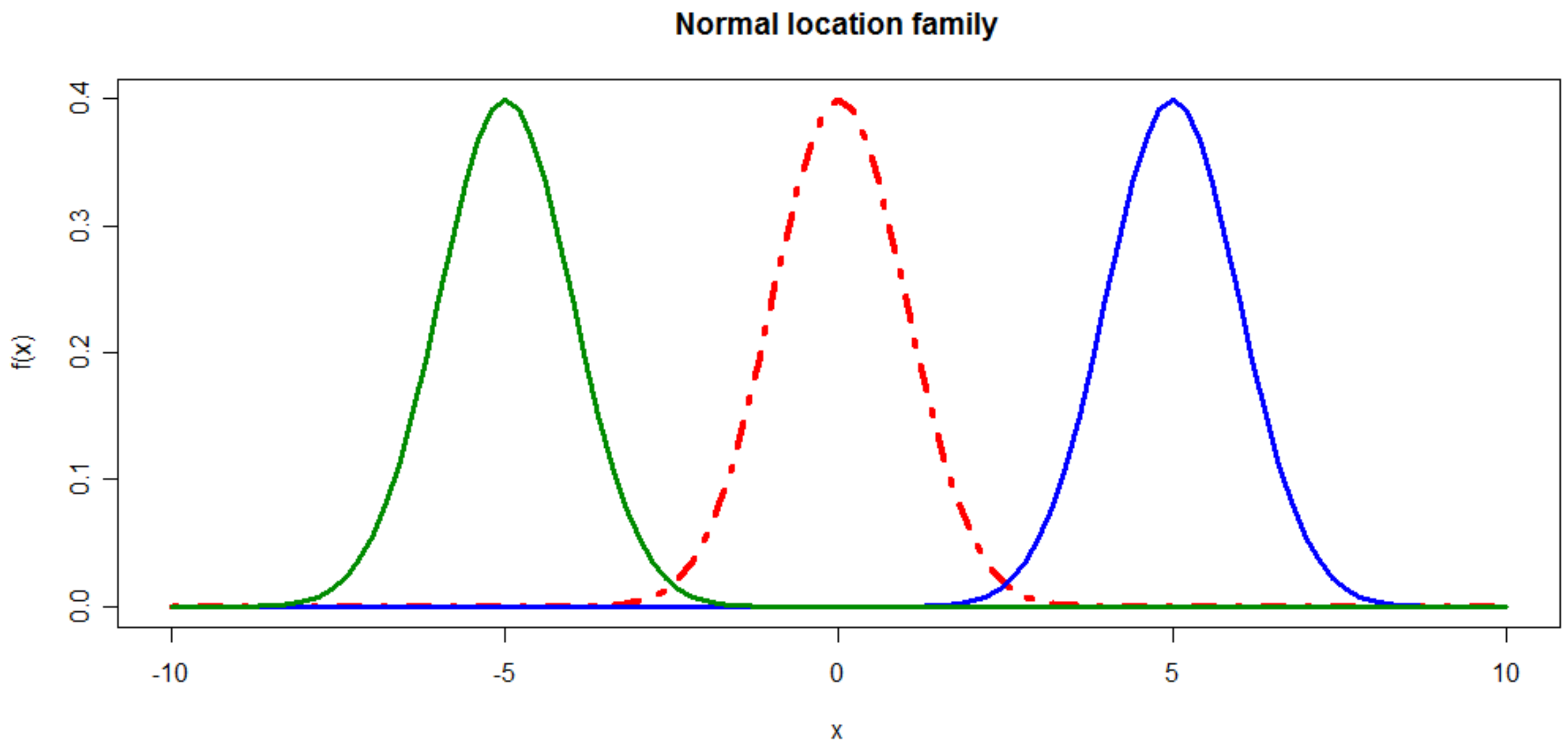
$$(= - - \mu)$$

$$\frac{4)}{4/} = !*$$

>9&5&-65&

$$5</= = \frac{!}{\sqrt{A\pi}} 7^{-\frac{</-\mu=^A}{A}}$$

Normal location family



Scale family

O7??60&(((.0(, (5 , / ; 6 3 (8 , 5. , +4&(9 , 8. / 2(? ; -(5<) =*

H6 / 0. ; &5('9&('5 , / 0-65 3

$$- = \sigma (*$$

O. / @&(

$$(= - \; \mathrm{T} \sigma)$$

$$\frac{4)}{4/} = \frac{!}{\sigma} *$$

>9&5&-65&

$$<</= = \frac{!}{\sigma} 5 \left(\frac{/}{\sigma} \right)$$

>9&(- , 3 .41(6-(? ; -0($\frac{!}{\sigma} 5 \left(\frac{)}{\sigma} \right)$) . / ; &_& ; (+1('9&(? , 5 , 3 &'&5(σ) $\sigma >$ E) (.0(@ , 44& ;

'9&(63"+ , '7" &)+9 (P. '9(65"#$" !$' : $7 (5<) =V ($\sigma$ (.0(@ , 44& ; ('9&(63"+ , ' : " ! " & , 5 , !

-65('9&(- , 3 .41*

Normal scale family

O7??60&(((.0(,(0' , / ; ,5;(/653 ,4(5, / ;63(8,5.,+4&*(>9&/)

$$5<)= = \frac{!}{\sqrt{A\pi}}7^{-\frac{)^A}{A}}$$

H6/0.;&5('9&('5,/0-653

$$- = \sigma(*$$

O./@&

(

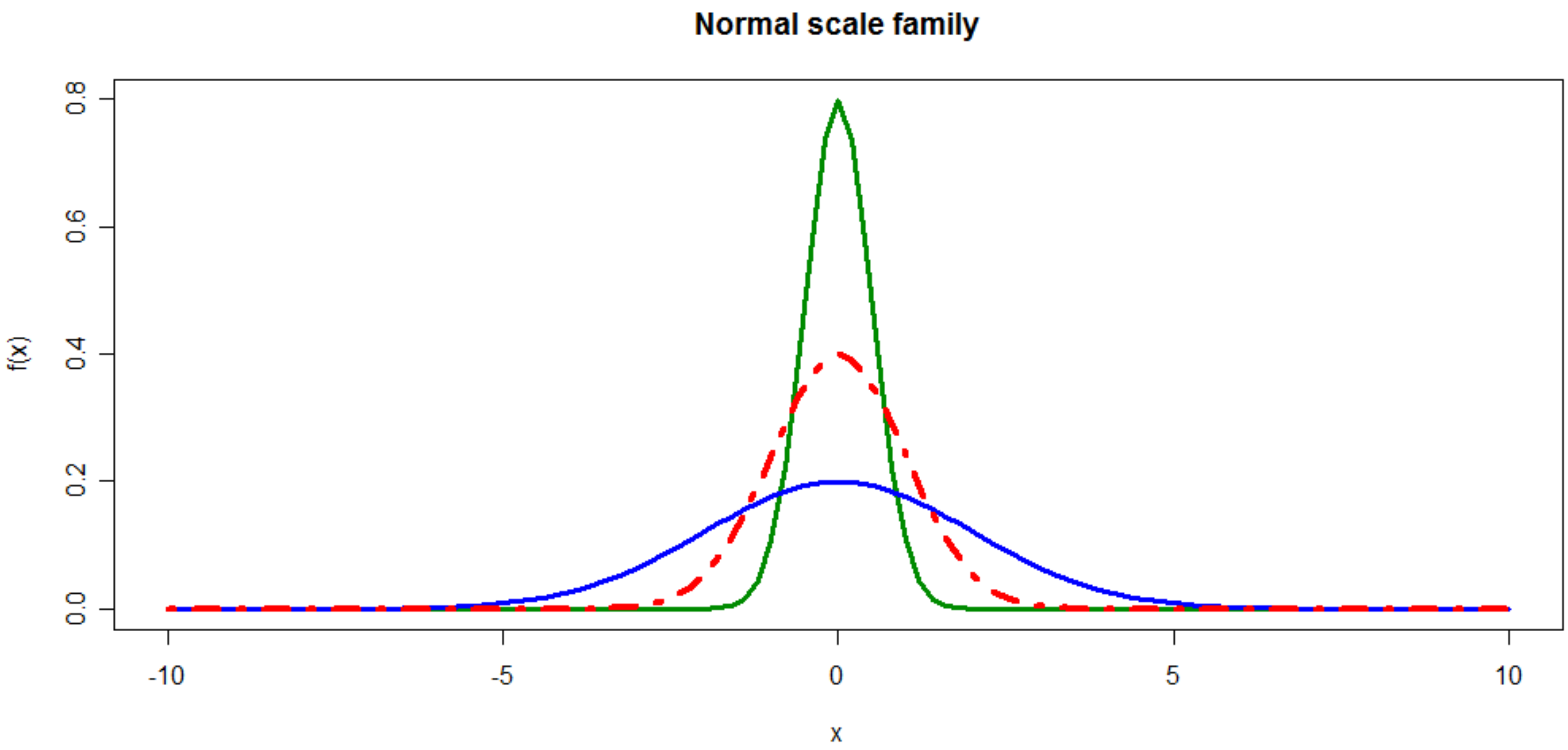
$$(= - T\sigma)$$

$$\frac{4)}{4/} = ! T\sigma*$$

>9&5&-65&

$$5</= = \frac{!}{\sqrt{A\pi\sigma}}7^{-\frac{/^A}{A\sigma^A}}$$

Normal scale family



Location-scale family

O7??60&(((.0(, (5 , / ; 6 3 (8 , 5. , +4&(9 , 8. / 2(? ; -(5 <) =*

H6 / 0. ; &5('9&('5 , / 0-65 3

$$- \;=\sigma (\; + \mu *$$

O. / @&($$(\; = <- \; - \mu = T\,\sigma)$$

$$\frac{4)}{4/} = \frac{!}{\sigma} *$$

>9&5&-65&

$$<</= = \frac{!}{\sigma} 5 \left(\frac{/ - \mu}{\sigma} \right)$$

>9&(- , 3 .41(6-(? ; -0($\frac{!}{\sigma} 5 \left(\frac{) - \mu}{\sigma} \right)$) . / ; &_& ; (+1('9&(? , 5 , 3 &'&50(μ (, / ; (σ

< σ > E=) (.0(@ , 44& ; ('9&(+%3"5)%#M63"+ , '7" &)+9(P.'9(65"#$"!$' : $7 (5 <) =V

μ (.0(@ , 44& ; ('9&(46@ , '.6 / ' : "!" & , 5 , ! (-65('9&(- , 3 .41) (, / ; (σ (.0(@ , 44& ; ('9&

63"+ , ' : "!" & , 5 , ! (-65('9&(- , 3 .41*

Normal location-scale family

O7??60&(((.0(, (0' , / ; , 5 ; (/ 65 3 , 4(5 , / ; 6 3 (8 , 5. , +4&*(>9& /)

$$5<) = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^{A}}{A}}$$

H6 / 0. ; &5('9&('5 , / 0-65 3

$$- = \sigma (+ \mu*$$

O. / @&

(

$$(= <- - \mu = T \sigma)$$

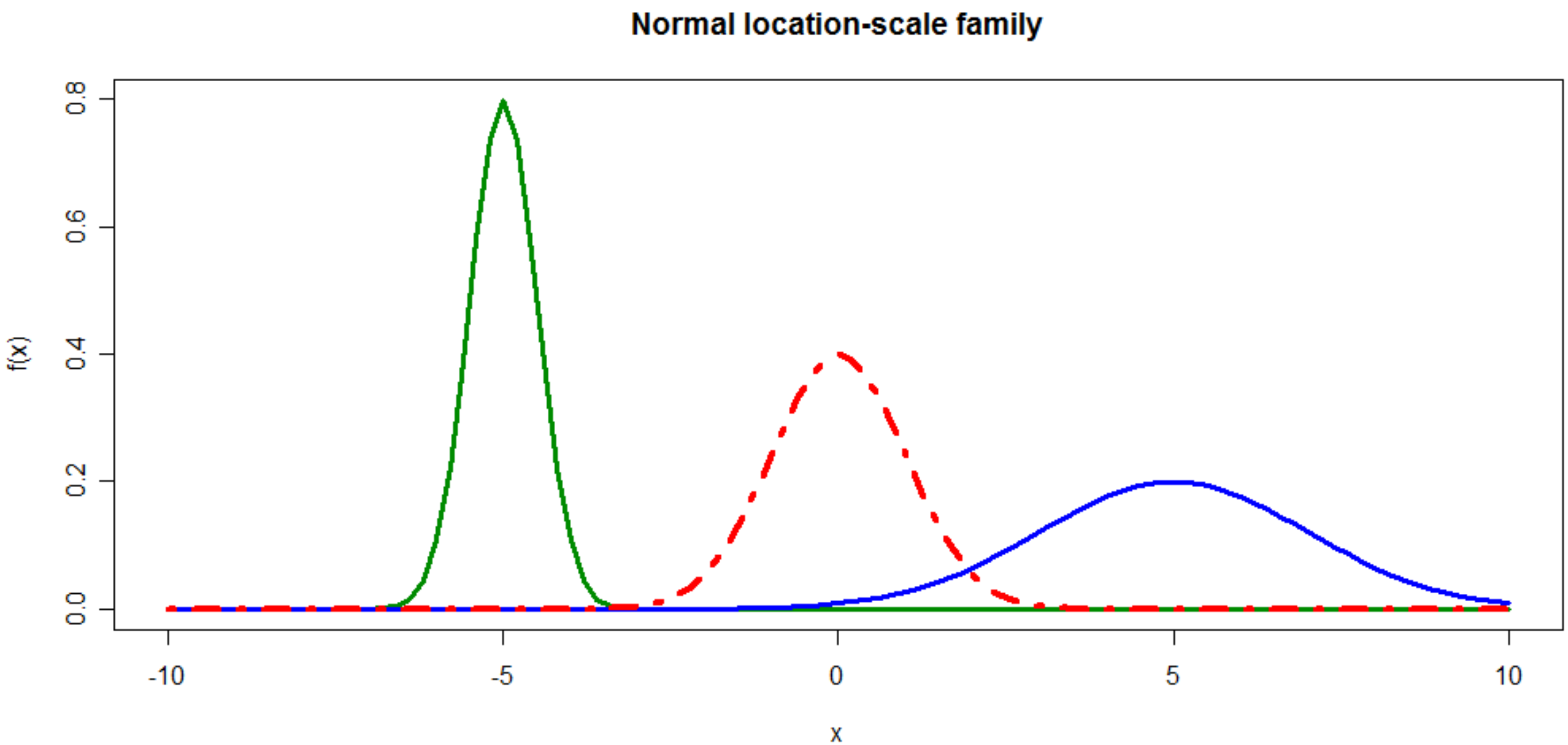
$$\frac{4)}{4/} = ! T \sigma*$$

>9&5&-65&

$$5</= = \frac{!}{\sqrt{A\pi\sigma}} 7^{-\frac{</ - \mu=^{A}}{A\sigma^{A}}}$$

>9.0(.0('9&(? ; -(6-(, (#%! & "+'$)65!)*45)%#*(] / (6'9&5(P65 ; 0)(- $\sim$ ><μ) σ^{A}=*

Normal location-scale family



Exponential families

N(-, 3 .41(6-(? ; -0(65(? 3 -0(.0(@, 44& ; (, / (7) &?"7" :!8@ (58 2 !@/ (
.-(.'(@, / (+&(&_?5&00& ; (, 0

$$5<) \vee \theta= = =<) =3<\theta= \&\_? \left(\sum_{!=!}^{,} A_! <\theta=:_! <) = \right))$$

P9&5&(=<) = ≥ E)3<θ= ≥ E)$A_!$<θ=(, 5&(5& , 4(8 , 47& ; (-7 / @'.6 / 0
6-('9&(? , 5 , 3 &'&5(θ)(, / ; (:$_!$<) =(, 5&(5& , 4(8 , 47& ; (-7 / @'.6 / 0
6-('9&(6+0&58 , '.6 / ()*

Binomial exponential family

- Binomial pmf

$$p(x\mid\pi) = \binom{n}{x}\pi^{x}(1-\pi)^{n-x}$$

$$= \binom{n}{x}(1-\pi)^{n}\left(\frac{\pi}{1-\pi}\right)^{x}$$

$$= \binom{n}{x}(1-\pi)^{n}\exp\left[\log\left(\frac{\pi}{1-\pi}\right)x\right]$$

- Exponential family pmf

$$p(x\mid\boldsymbol{\theta}) = h(x)\,g(\boldsymbol{\theta})\exp\left(\sum_{i=1}^{k}\eta_i(\theta)T_i(x)\right)$$

$$h(x) = \binom{n}{x} \qquad g(\pi) = (1-\pi)^{n}$$

$$\eta_1(\pi) = \log\left(\frac{\pi}{1-\pi}\right) \qquad T_1(x) = x$$

Normal exponential family

- Normal pdf

$$f(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$=\underbrace{1}_{h(x)}\cdot\underbrace{\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\mu^2}{2\sigma^2}\right)}_{a(\mu,\sigma^2)}\exp\left(\underbrace{\frac{1}{2\sigma^2}}_{\eta_1(\mu,\sigma^2)}\underbrace{(-x^2)}_{T_1(x)}+\underbrace{\frac{\mu}{\sigma^2}}_{\eta_2(\mu,\sigma^2)}\underbrace{x}_{T_2(x)}\right)$$

- $d = 2$ parameters, $k = 2$ items in the sum in the exponent
  - $d < k \mapsto$ ***curved*** exponential family, e.g., $N(\mu, \mu^2)$
  - $d = k \mapsto$ ***full*** exponential family, e.g., $N(\mu, \sigma^2)$

Expectations of Random Variables

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Mode

Mode

>9&(&%$,(6-(,(5,/;63(8,5.,+4&(((.0('9&(8,47&('9,'
6@@750('9&(360'(-5&W7&/'41(./('9&(?56+,+.4.'1(;.0'5.+7'.6/)
@655&0?6/;./2('6('9&(3,_.373(8,47&(./('9&(?3-(65(?;-*

Median

Median

>9&(&,$)"#(6-(,(5, / ;6 3 (8,5.,+4&((((.0(,(8,47&(2 (07@9('9,'

$$' < (\leq 2 = \geq \frac{!}{A} \; (, / ; (' < (\geq 2 = \geq \frac{!}{A}$$

: 65(,(@6 / './7670(5, / ;6 3 (8,5.,+4&((( )('9&(3 &;.., / (2 (0, '.0-.&0

$$\int_{-\infty}^{2} 5<)=4) = \int_{2}^{\infty} 5<)=4) = \frac{!}{A}$$

Expectations

Expected value

>9&(, . : ,35,$'("+4, (65(&,"#(6-(,(5, / ;63(8,5. ,+4&(<<(=)
;&/6'&;(+1(c<<)=)(.0

$$
c<<) = = \begin{cases} \int_{-\infty}^{\infty} <<)=5_{(} <)=4) & \text{.-(((.0(@6/'./7670} \\ \sum_{) \in \mathcal{X}} <<)=5_{(} <)= & \text{.-(((.0(;&0@5&'&} \end{cases}
$$

?568.;&;('9,'('9&(./'&25,4(65(07 3(&_.0'0*(]-(c $\left|<<(=\right| = \infty$)

P&(0,1('9,'(c<<(=(;6&0(/6'(&_.0'*

Normal mode

O7??60&(((.0(,(/653 ,4(<μ) σ^A=(5, / ;63(8,5.,+4&*(>9&/)

$$5<)= \quad = \frac{!}{\sqrt{A\pi\sigma}} 7^{-\frac{<)-\mu=^A}{A\sigma^A}}$$

$$462\ 5<)= \quad = -\frac{!}{A} 462<A\pi= -\frac{!}{A} 462<\sigma^A= -\frac{!}{A} <\sigma^A=<) - \mu=^A$$

` +8.67041)('9&(3 ,_.373(8,47&(.0(6+' ,./&;(,'() $= \mu$*

>9&5&-65&)

>9&(36;&(6-(,(/653 ,4(;.0'5.+7'.6/(.0(.'0(46@, '.6/(?,5, 3&'&5*

Normal median

O7??60&(((.0(, (/ 65 3 , 4(<μ) σ^{A}=(5 , / ; 6 3 (8 , 5. , +4&*(>9& /)

$$5<) = \qquad = \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{<)-\mu=^{A}}{A\sigma^{A}}}$$

$$\int_{-\infty}^{\mu} 5<)=4) \quad = \int_{-\infty}^{\mu} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{<)-\mu=^{A}}{A\sigma^{A}}} 4) \xrightarrow{/=)-\mu} \int_{-\infty}^{E} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{/^{A}}{A\sigma^{A}}} 4/$$

$$\int_{\mu}^{\infty} 5<)=4) = \int_{\mu}^{\infty} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{<)-\mu=^{A}}{A\sigma^{A}}} 4) \xrightarrow{/=-<)-\mu=} \int_{-\infty}^{E} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{/^{A}}{A\sigma^{A}}} 4/$$

` +8.67041)('9&0&('P6(. / '&25 , 40(, 5&(&W7 , 4

>9&5&-65&)

>9&(3 & ; . , / (6-(, (/ 65 3 , 4(; .0'5.+7'.6 / (.0(.'0(46@ , '.6 / (? , 5 , 3 &'&5*

Standard normal expectation

Suppose

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty \leq x < \infty$$

that is, X has an **standard normal distribution** $N(0,1)$.

Then,

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \\ &= -\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} d\left(-\frac{x^2}{2}\right) \\ &= e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty} \\ &= 0 \end{aligned}$$

Cauchy expectation

- Suppose

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2},\quad -\infty < x < \infty$$

that is, X has a Cauchy distribution, denoted as $X \sim$ Cauchy. Then,

$$\begin{aligned}
E|X| &= \int_{-\infty}^{\infty} \frac{|x|}{\pi}\frac{1}{1+x^2}\,dx \\
&= \frac{2}{\pi}\int_{0}^{\infty} \frac{x}{1+x^2}\,dx \\
&= \lim_{b\to\infty} \frac{2}{\pi}\int_{0}^{b} \frac{x}{1+x^2}\,dx \\
&= \frac{1}{\pi}\lim_{b\to\infty} \log(1+b^2) \\
&= \infty
\end{aligned}$$

Properties of expectation

Properties of expectation

%&'(((+&(,(5,/;63(8,5.,+4&(,/;(4&'(8)9)(,/;(3(+&(@6/0',/'0*
>9&/(-65(,/1(-7/@'.6/0(<!<(=(,/;(<A<(=(P960&(&_?&@','.6/0
&_.0'0)

!* cF8<!<(= + 9<A<(= + 3G = 8c<!<(= + 9c<A<(= + 3V

A*]-(<!<(= ≥ E(-65(,44())('9&/(c<!<(= ≥ EV

K*]-(<!<(= ≥ <A<(=(-65(,44())('9&/(c<!<(= ≥ c<A<(=V

d*]-(8 ≤ <!<(= ≤ 9(-65(,44())('9&/(8 ≤ c<!<(= ≤ 9*

Moments of random variables

Moment

: 65(&W@9(./'&2&5(")('9&(n58'&%&,#5(6-(,(5,/;63(8,5.,+4&(( ) $\mu'_{"}$)(.0(

$$\mu'_{"} = c(\ ^{"}*$$

>9&(n58'3,#5!"+'&%&,#5(6-(( )($\mu_{"}$)(.0

$$\mu_{"} = c<(- \mu=^{"})$$

P9&5&($\mu = \mu'_{"} = c(*$

Mean

Mean

>9&(& , "#(6-(, (5 , / ; 6 3 (8 , 5. , +4&(((.0(.'0(-.50'(3 6 3 &/ '

$\mu =$ c (*

Variance

Variance

>9&(("!)"#3,(6-(,(5, / ;63(8,5.,+4&(((.0(.'0(0&@6/ ;
@&/'5,4(363&/'

e ,5(= c<(− c (=A*

>9&(?60.'.8&(0&W75&(566'(6-(e ,5((.0('9&(65"#$"!$'
$, ()"5)%#(6-((*

Properties of variance

Properties of variances

]-(((.0(, (5 , / ; 6 3 (8 , 5. , +4&(P. '9(-. / . '&(8 , 5. , / @&)('9& / (-65(, / 1
@6 / 0' , / '0(8 (, / ; (9

e , 5<8 (+ 9= = 8 A e , 5 (*

0. / @&(e , 5<8 (+ 9= = c<<8 (+ 9= – C<8 (+ 9== A
= c<8 (– 8C (= A
= 8 A c< (– c (= A
= 8 A e , 5< (=*

e , 5 (= c (A – <c (= A

0. / @&(e , 5 (= c< (– c (= A
= c< (A – A (c (+ <c (= A =
= c (A – A<c (= A + <c (= A
= c (A – <c (= A *

Bernoulli variance

- O7??60&

(= { ! P.'9(?56+,+.4.'1(&
E P.'9(?56+,+.4.'1(! − &)

'9,'(.0) (**9,0(,(f &5/6744.(;.0'5.+7'.6/)(;&/6'&;(,0
(∼ f &5/6744.<&=*(>9&/)

c(= & × ! + <! − &= × E = &

c(A = & × ! + <! − &= × E = &

e,5(= c(A − <c(=A

= & − &A

= &<! − &=*

Standard normal variance

O7??60&(((.0(, (0' , / ; , 5 ; (/ 65 3 , 4(5 , / ; 6 3 (8 , 5. , +4&*(>9& /)

$$5<)= \quad = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^{A}}{A}}$$

$$c\,(\quad = \int_{-\infty}^{\infty})<A\pi=^{-!TA} \&\_?<-)^{A}\ T\ A=4)$$

$$= -<A\pi=^{-!TA} \int_{-\infty}^{\infty} \&\_?<-)^{A}\ T\ A=4<-)^{A}\ T\ A=$$

$$= -<A\pi=^{-!TA} \&\_?<-)^{A}\ T\ A= \Big|_{-\infty}^{\infty}$$

$$= E$$

$$c\,(^{A} \quad = \int_{-\infty}^{\infty})^{A}<A\pi=^{-!TA} \&\_?<-)^{A}\ T\ A=4)$$

$$= -<A\pi=^{-!TA} \int_{-\infty}^{\infty})4\ \&\_?<-)^{A}\ T\ A=$$

$$= -<A\pi=^{-!TA}) \ \&\_?<-)^{A}\ T\ A= \Big|_{-\infty}^{\infty} + <A\pi=^{-!TA} \int_{-\infty}^{\infty} \&\_?<-)^{A}\ T\ A=4)$$

$$= !$$

>9&5&-65&

$$e\,,5\,(\ = c\,(^{A} - <c\,(=^{A} = !$$

$$\int_{-\infty}^{\infty} \&\_?<-\&^{A})^{A} + D)=4) = \&\_?\left(\frac{D^{A}}{d\&^{A}}\right)\frac{\sqrt{\pi}}{\&} \ (<\& > E=$$

Skewness

Skewness

>9&(6C,D#,66(6-(,(5,/;63(8,5.,+4&(((.0(.'0('9.5;(@&/'5,4

363&/'(68&5('9&(@7+&(6-('9&(0',/;,5;(;&8.,'.6/

$$\beta_{\#} = \mathsf{c}\left[\left(\frac{(\;-\mu}{\sigma}\right)^{K}\right] = \frac{\mu_{K}}{\sigma^{K}}*$$

Standard normal skewness

O7??60&(((.0(,(0' , / ; ,5;(/653 ,4(5, / ;63(8,5.,+4&*(>9&/)

$$5<)= \quad = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^A}{A}}$$

$$c\,(^{K} \quad = \int_{-\infty}^{\infty})^{K} <A\pi=^{-!TA} \&\_?<-)^{A}\ T\ A=4)$$

$$= -<A\pi=^{-!TA} \int_{-\infty}^{\infty})^{A} 4\ \&\_?<-)^{A}\ T\ A=$$

$$= -<A\pi=^{-!TA})^{A}\ \&\_?<-)^{A}\ T\ A=\Big|_{-\infty}^{\infty} + <A\pi=^{-!TA} \int_{-\infty}^{\infty} \&\_?<-)^{A}\ T\ A=4)^{A}$$

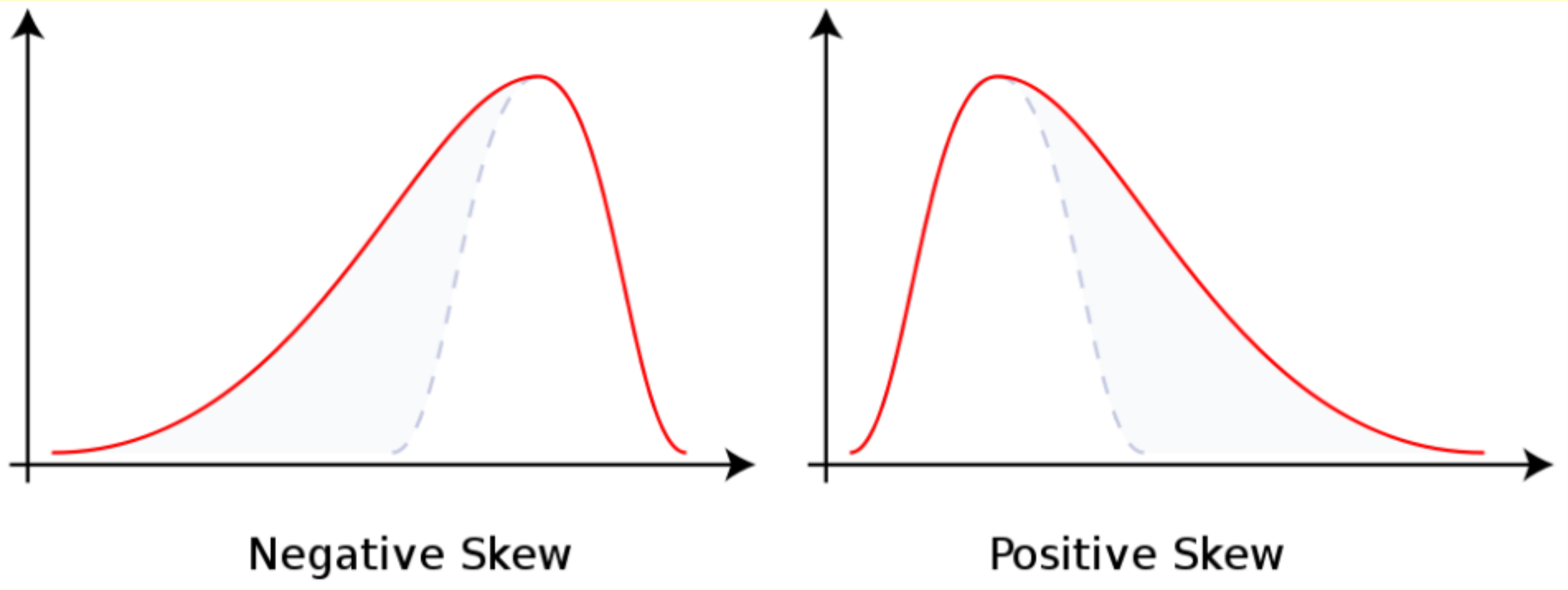
$$= -A<A\pi=^{-!TA} \int_{-\infty}^{\infty} \&\_?<-)^{A}\ T\ A=4<-)^{A}\ T\ A=$$

$$= E$$

>9&5&-65&

$$\beta_{\#} = E$$

Skewness



Kurtosis

Kurtosis

>9&(E4!5%6)6(6-(,(5,/;63(8,5.,+4&(((.0(.'0(-675'9(@&/'5,4

363&/'(68&5('9&(-675'9(?6P&5(6-('9&(0',/;,5;(;&8.,'.6/

$$\beta_{,} = \mathrm{c}\left[\left(\frac{(- \mu}{\sigma}\right)^{\mathrm{d}}\right] = \frac{\mu_{\mathrm{d}}}{\sigma^{\mathrm{d}}}*$$

Standard normal kurtosis

O7??60&(((.0(,(0' , / ; ,5;(/653 ,4(5, / ;63(8,5., +4&*(>9&/)

$$
\begin{aligned}
5<)= \quad &= \frac{!}{\sqrt{A\pi}}7^{-\frac{)^{A}}{A}} \\
c\,(^{d} \quad &= \int_{-\infty}^{\infty})^{d} <A\pi=^{-!TA} \&\_?<-)^{A}\ T\ A=4) \\
&= -<A\pi=^{-!TA}\int_{-\infty}^{\infty})^{K}4\ \&\_?<-)^{A}\ T\ A= \\
&= -<A\pi=^{-!TA})^{K}\ \&\_?<-)^{A}\ T\ A=\Big|_{-\infty}^{\infty} + <A\pi=^{-!TA}\int_{-\infty}^{\infty} \&\_?<-)^{A}\ T\ A=4)^{K} \\
&= K\left[<A\pi=^{-!TA}\int_{-\infty}^{\infty})^{A}\ \&\_?<-)^{A}\ T\ A=4)\right] \\
&= K
\end{aligned}
$$

>9&5&-65&

$$\beta_{,} = K$$

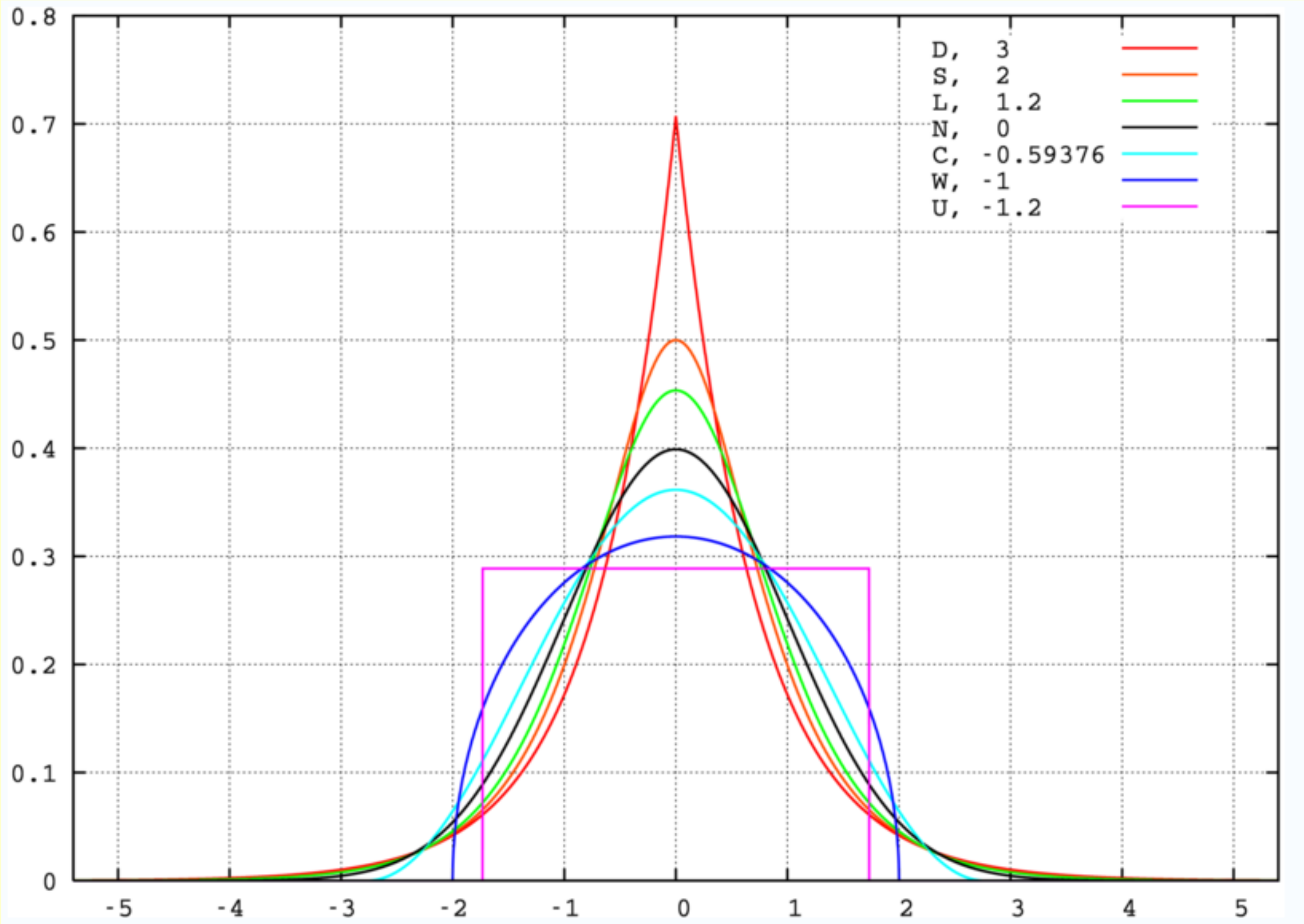
Excess kurtosis

Excess kurtosis

>9&(F.3,66'E4!5%6)6(6-(,(5,/;63(8,5.,+4&(((.0(.'0(-675'9 @&/'5,4(363&/'(68&5('9&(-675'9(?6P&5(6-('9&(0',/;,5; ;&8.,'.6/(3./70(K

$$\beta_{,} = \mathrm{c}\left[\left(\frac{(- \mu}{\sigma}\right)^{\mathrm{d}}\right] - \mathrm{K} = \frac{\mu_{\mathrm{d}}}{\sigma^{\mathrm{d}}} - \mathrm{K*}$$

Excess kurtosis



Moment generating function

Moment generating function

%&'(((+&(,(5, / ; 6 3 (8 , 5. , +4&(P.'9(@ ; -(6(<) =*(>9&
&%& ,#5'G,#, !"5)#G'74#35)%#'; &G7='6-(( )(B (<:=)(.0

B (<:= = c7:( )

?568. ; & ; ('9 , '('9&(&_?&@' , '.6 / (&_.0'0(-65(: (. / (06 3 &
/ &.29+65966 ; (6-(E*

B (<:= = ∫ from −∞ to ∞ 7:) 5(<) =4)

B (<:= = Σ from) = −∞ to ∞ 7:) '<(=)=

Normal moment generation function

$$B<:= \;=\int_{-\infty}^{\infty} 7^{:)}\,\frac{!}{\sqrt{A\pi\sigma}}\,7^{-\frac{<)-\mu=^{A}}{A\sigma^{A}}}\,4)$$

$$=\int_{-\infty}^{\infty}\frac{!}{\sqrt{A\pi\sigma}}\,\&\_?\left(-\frac{<)-\mu=^{A}}{A\sigma^{A}}+:)\right)4)$$

$$=\int_{-\infty}^{\infty}\frac{!}{\sqrt{A\pi\sigma}}\,\&\_?\left(-\frac{)^{A}-A\mu)+\mu^{A}+A\sigma^{A}:)}{A\sigma^{A}}\right)4)$$

$$=\int_{-\infty}^{\infty}\frac{!}{\sqrt{A\pi\sigma}}\,\&\_?\left(-\frac{)^{A}-A<\mu+\sigma^{A}:=)+<\mu+\sigma^{A}:=^{A}+\mu^{A}-<\mu+\sigma^{A}:=^{A}}{A\sigma^{A}}\right)4)$$

$$=\int_{-\infty}^{\infty}\frac{!}{\sqrt{A\pi\sigma}}\,\&\_?\left[-\frac{!}{A\sigma^{A}}\left()-<\mu+\sigma^{A}:=\right)^{A}+\left(\mu: g\,\frac{!}{A}\sigma^{A}:^{A}\right)\right]4)$$

$$=\&\_?\left(\mu: g\,\frac{!}{A}\sigma^{A}:^{A}\right)\underbrace{\int_{-\infty}^{\infty}\frac{!}{\sqrt{A\pi\sigma}}\,\&\_?\left[-\frac{!}{A\sigma^{A}}\left()-<\mu+\sigma^{A}:=\right)^{A}\right]4)}_{=!}$$

Deriving moments from mgf

Deriving moments

]-(((9,0(32-(B (<:=)('9&/

c (" = B (<"= <E= = 4" / 4:" B (<:= | :=E *

>9,'(.0)('9&("M'9(363&/'(.0(&W7,4('6('9&("M'9

;&5.8,'.8&(6-(B (<:=)(&8,47,'&;(,'(: = E*

Standard normal moments

O' , / ; ,5;(/653 ,4(32-(.0

$$B<:= = \&\_?\left(\frac{:^{A}}{A}\right)$$

$$\frac{4}{4)} B<:= = :\&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu_{!}' = E \Rightarrow \mu = E$$

$$\frac{4^{A}}{4)^{A}} B<:= = <:^{A} + != \&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu_{A}' = ! \Rightarrow \sigma^{A} = !$$

$$\frac{4^{K}}{4)^{K}} B<:= = <:^{K} + K:= \&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu_{K}' = E \Rightarrow \beta_{\#} = E$$

$$\frac{4^{d}}{4)^{d}} B<:= = <:^{d} + h:^{A} + K= \&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu_{d}' = K \Rightarrow \beta_{,} = K$$

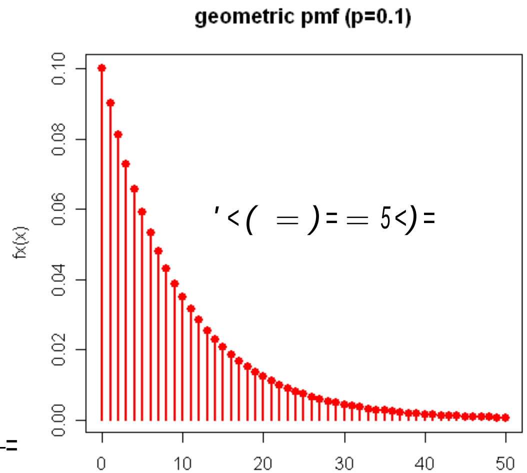
Distribution Functions of Random Variables

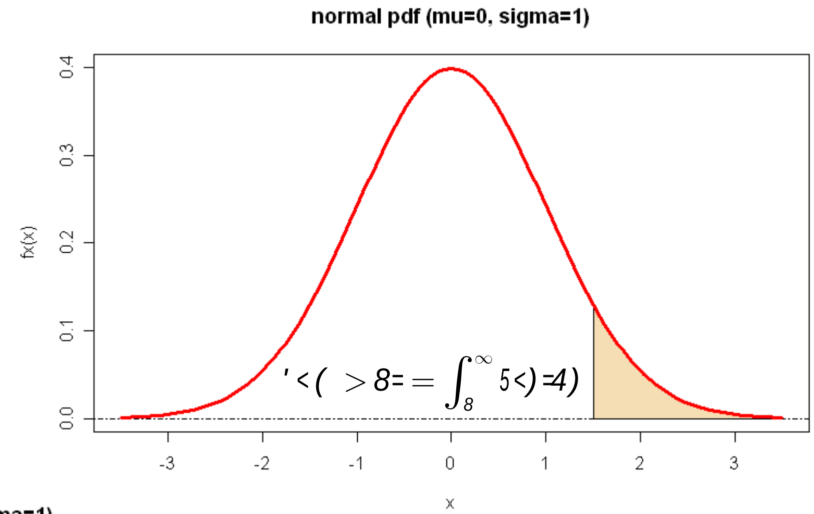
"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Distribution functions

- Probability mass (density) function (pmf, pdf)
  - Probability at or near a particular value
- Cumulative distribution function (cdf)
  - Probability less than or equal to a particular value
- Quantile function
  - The particular value corresponding to a probability, on the basis of the cdf
- Random numbers
  - Points distributed as the given distribution

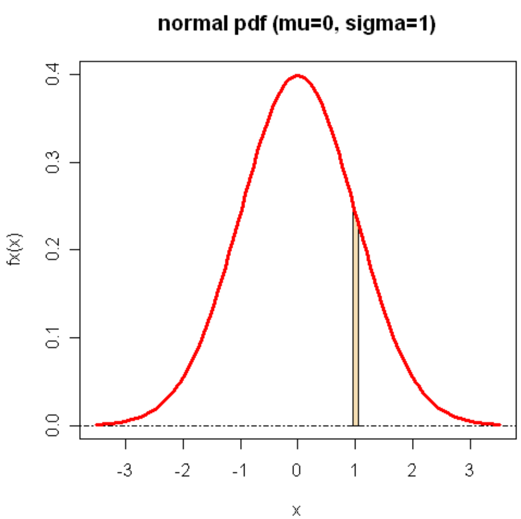
Probability mass/density functions

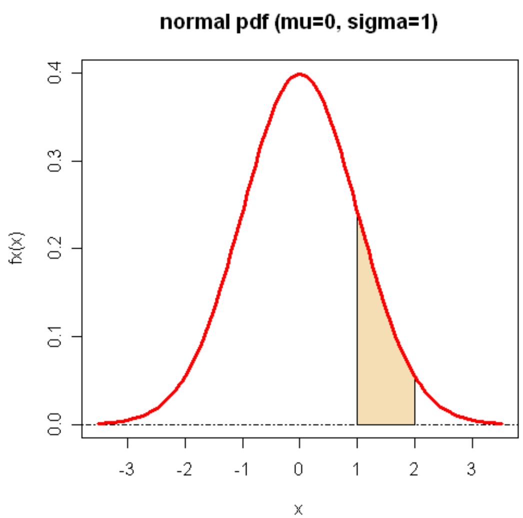




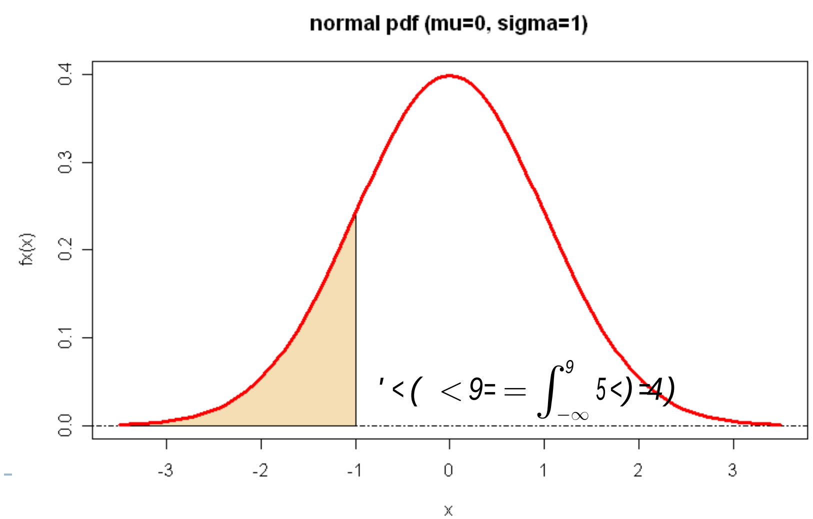
$$' <) - \frac{\Delta)}{A} \leq (\leq) + \frac{\Delta)}{A} =$$

$$= \int_{) - \frac{\Delta)}{A}}^{) + \frac{\Delta)}{A}} 5<) = 4) \approx 5<) = \Delta)$$

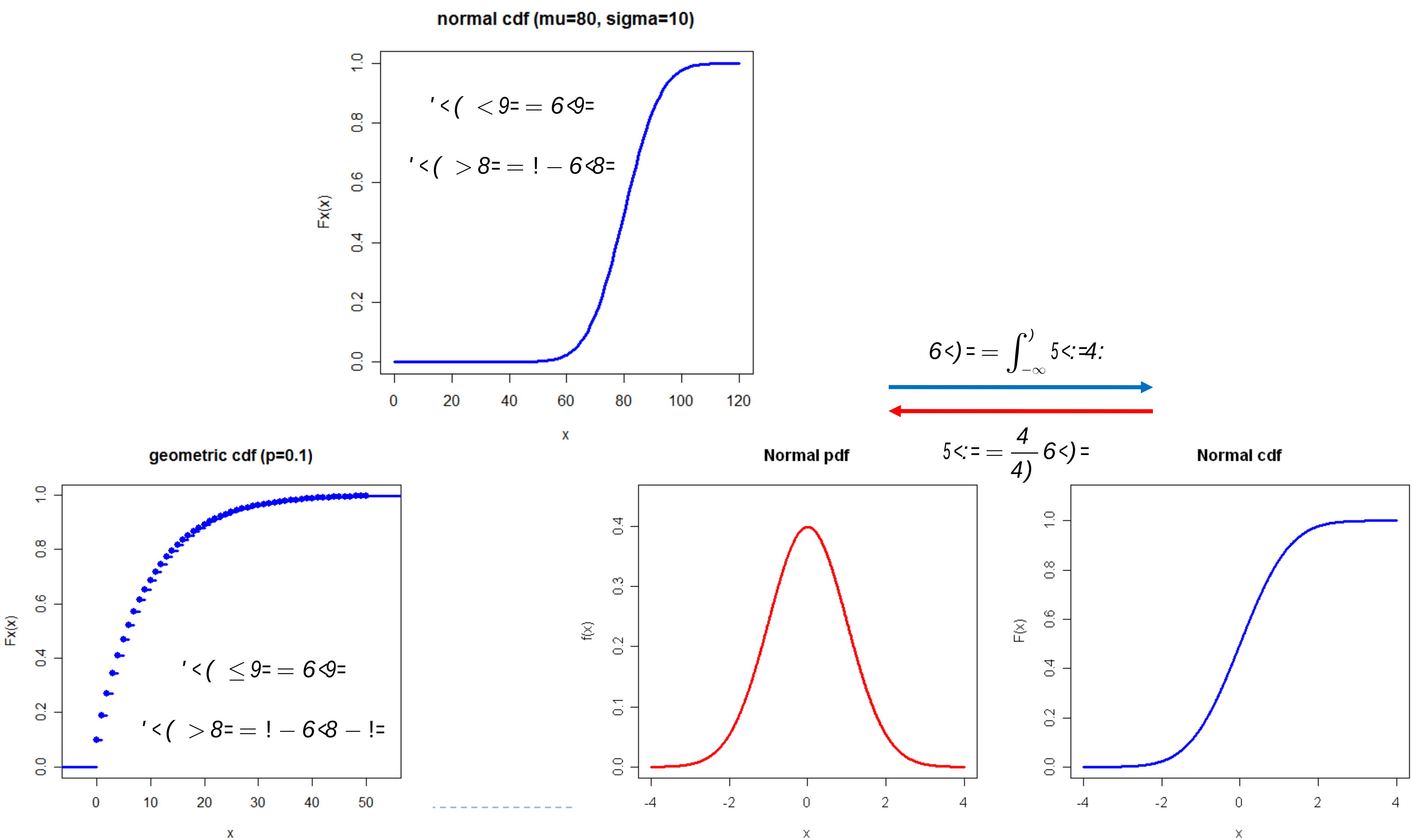




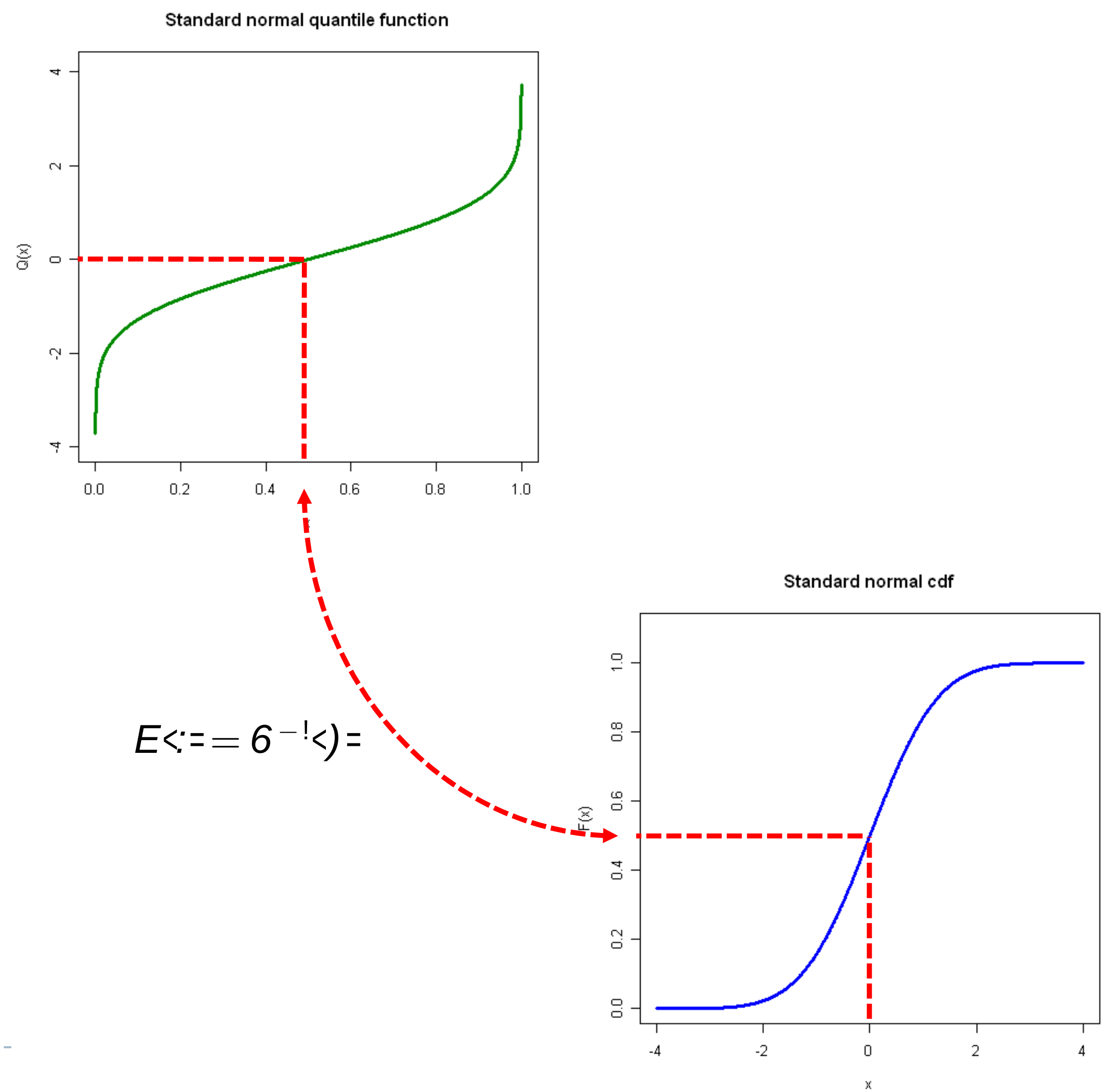
$$' < 8 < (< 9 = = \int_8^9 5<) = 4)$$



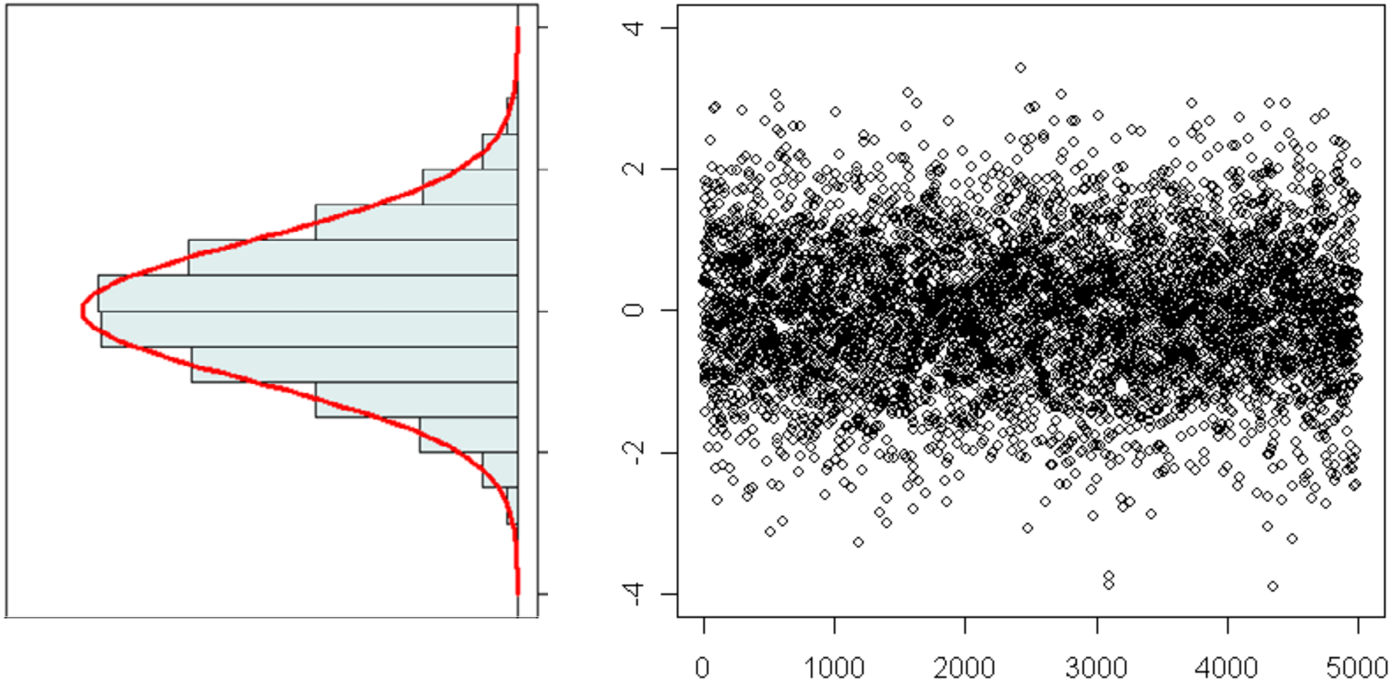
Cumulative distribution functions



Quantile functions



Random number generators



Distribution functions in R

- ?3- 65(?;-
  5(<)=(　　`dxxxx(x, parameters)`

- @;-
  6(<)=　　`pxxxx(q, parameters)`

- i7,/'.4& -7/@'.6/
  6(−!<&=(　　`qxxxx(p, parameters)`

- j ,/;63(/73+&50
  `rxxxx(n, parameters)`

Thank you very much