Exercise 4

Random Sampling

Deadline: October 19, 2016.

Note:
For calculation problem, please give details of the derivation. Answers will receive only half of the points.

1. Reading.
   (a) Lecture notes 4.
   (b) Chapter 5 of the book "Statistical Inference".
2. Let $X_1, \ldots, X_n$ be a random sample of size n from a $N(0, \sigma^2)$ population. Prove that
$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2$$
has a χ^2 distribution with $n-1$ degrees of freedom.
3. Let $X_1, \ldots, X_n$ be a random sample of size n from a $N(0, \sigma^2)$ population. Prove that $\bar{X}$ and S^2 are independent random variables.
4. Let $X_1, \ldots, X_n$ be a random sample of size n from a $N(\mu, 1)$ population. Define random variables $Y_1, \ldots, Y_n$ as
$$Y_i = \begin{cases} 1 & \text{if } X_i > \mu \\ 0 & \text{if } X_i \le \mu \end{cases}.$$

Derive the distribution of $\sum_{i=1}^{n} Y_i$

5. Two samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are obtained from two independent normal population $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$, respectively, and the condition $\sigma_X = \sigma_Y = \sigma$ holds.

   (a) Show that
   $$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}}$$
   has a standard normal distribution.

   (b) Show that
   $$\frac{(m+n-2)S_p^2}{\sigma^2}$$
   has a χ^2 distribution with $m+n-2$ degrees of freedom, where
   $$S_p^2 = \frac{(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2}$$
   is the pooled variance estimate.

   (c) With the above two conclusion, show that
   $$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}}$$
   has a student's t distribution with $m+n-2$ degrees of freedom.

6. Let X be a random variable with student t distribution with p degrees of freedom.

   (a) Derive the mean and variance of X.

   (b) Prove that when p tends to infinite, X converges in distribution to standard normal. That is,
   $$\lim_{p\to\infty} f(x|p) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ for any } -\infty < x < \infty$$

   (c) Find the distribution of X^2