2016年秋《统计方法与应用》作业-2（随机变量）

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1 Reading.

(a) Lecture notes 2.

(b) Chpaters 2 and 3 of the book ” Statistical Inference” .

2 In each of the following show that the given function is a cdf and find $F_X^{-1}(y)$

根据课本定理1.5.3，验证下列函数是否满足累积分布函数的三个性质即可。

(a) 对于

$$F_X(x) = \begin{cases} 0 & \text{if x < 0} \\ 1 - e^{-x} & \text{if x} \geq 0 \end{cases}$$

有:

- $\lim_{x \to -\infty} F(x) = 0$，因为$x < 0$时，$F_X(x) = 0$;又因为$\lim_{x \to +\infty} e^{-x} = 0$, 所以$\lim_{x \to +\infty} F(x) = 1$;
- 因为e^x是单调增函数，因此$1 - e^{-x}$还是单调增函数；或者可以证明导数大于0;
  直接求导有，$\frac{d}{dx} F_X(x) = (1 - e^{-x})' = -(e^{-x})'(-x)' = e^{-x} > 0$
- 由于$F_X(x)$是连续函数，因此$F_X(x)$一定是右连续函数。

即得证。

求$F_X^{-1}(y)$:

有$y = 1 - e^{-x} \Leftrightarrow$

$e^{-x} = 1 - y \Leftrightarrow$

$\ln e^{-x} = \ln 1 - y \Leftrightarrow$

$x = -\ln 1 - y$

即，$F_X^{-1}(y) = -\ln(1-y)$

(b) 对于

$$F_X(x) = \begin{cases} e^x/2 & \text{if x < 0} \\ 1/2 & \text{if } 0 \leq \text{x < 1} \\ 1-(e^{1-x}/2) & \text{if } 1 \leq \text{x} \end{cases}$$

有:

- $\lim\limits_{x\to-\infty} F(x) = 0$，因为$x < 0$时，$\lim\limits_{x\to-\infty} e^x = 0$;又因为$\lim\limits_{x\to+\infty} e^{1-x} = 0$, 所以$\lim\limits_{x\to+\infty} F(x) = 1$;
- 因为e^x是单调增函数，因此$1-e^{-x}$还是单调增函数, 所以$1-(e^{1-x}/2)$还是单调增函数，而$F_X(x)$在区间[0,1)是常数，常数是非单调递减, 因此$F_X(x)$在整个定义域上是单调增函数;
- 由于$F_X(x)$是连续函数，因为$\lim\limits_{x\to0} F(x) = 1/2$且$\lim\limits_{x\to1} F(x) = 1/2$，所以$F_X(x)$一定是右连续函数。

即得证。

求$F_X^{-1}(y)$:

有当$x \in (-\infty, 0)$时, 有$y \in [0, 1/2]$，$y = e^x/2$（疑问2） $\Leftrightarrow$

$e^x = 2y \Leftrightarrow$

$\ln e^x = \ln(2y) \Leftrightarrow$

$x = \ln(2y)$

即，当$y \in [0,1/2]$时，$F_X^{-1}(y) = \ln 2y$

当$x \in [1,+\infty)$时，有$y \in [1/2,1)$，$y = 1-(e^{1-x}/2) \Leftrightarrow$

$e^{1-x} = 2(1-y) \Leftrightarrow$

$\ln e^{1-x} = \ln(2(1-y)) \Leftrightarrow$

$1-x = \ln(2(1-y))$

$\Leftrightarrow$

$x = 1-\ln(2(1-y))$

即，当$y \in [1/2,1)$时，$F_X^{-1}(y) = 1-\ln(2(1-y))$

故，

$$F_X^{-1}(y) = \begin{cases} ln(2y) & \text{if y} \in \text{[0,1/2]} \\ 1-ln(2(1-y)) & \text{if y} \in \text{[1/2,1)} \end{cases}$$

(c) 对于

$$F_X(x) = \begin{cases} e^x/4 & \text{if x} < 0 \\ 1-(e^{-x}/4) & \text{if x} \geq 0 \end{cases}$$

有:

- $\lim\limits_{x\to-\infty} F(x) = 0$，因为$x<0$时，$\lim\limits_{x\to-\infty} e^x = 0$;又因为$\lim\limits_{x\to+\infty} e^{-x} = 0$, 所以$\lim\limits_{x\to+\infty} F(x) = 1$;
- 因为e^x是单调增函数，因此$1-e^{-x}$还是单调增函数, 所以$1-(e^{-x}/4)$还是单调增函数，而$F_X(x)$在区间[0,1)是常数，常数是非单调递减, 因此$F_X(x)$在整个定义域上是单调增函数;（略微有点疑问）
- 因为$\lim\limits_{x\to 0^+} F(x) = 3/4$且$F(0) = 3/4$，所以$F_X(x)$是右连续函数。

即得证。

求$F_X^{-1}(y)$:

有当$x \in (-\infty,0)$时, 有$y \in [0,1/4)$，$y = e^x/4 \Leftrightarrow$

$e^x = 4y \Leftrightarrow$

$\ln e^x = \ln(4y) \Leftrightarrow$

$x = \ln(4y)$

即，当$y \in [0,1/4)$时，$F_X^{-1}(y) = \ln(4y)$

当$x \in [0, +\infty)$时, 有$y \in [3/4, 1)$，$y = 1 - (e^{-x}/4) \Leftrightarrow$

$e^{-x} = 4(1-y) \Leftrightarrow$

$\ln e^{-x} = \ln 4(1-y) \Leftrightarrow$

$x = -\ln(4(1-y))$

即，当$y \in [1/4, 1)$时，$F_X^{-1}(y) = 1 - \ln(4(1-y))$

故,

$$F_X^{-1}(y) = \begin{cases} ln(4y) & \text{if y} \in \text{[0,1/4)} \\ 1 - ln(4(1-y)) & \text{if y} \in \text{[3/4,1)} \end{cases}$$

3 Let X have the pdf,

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, x \in [0, \infty), \beta > 0$$

(a) Verify $f(x)$ is a valid pdf.

证明：两个性质不难证明性质a, 即$f_X(x) \geq 0$。

不难推导,

$$f(x) = \int_0^\infty f(x)dx = \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2} dx \approx 1$$

即可证$f(x)$ 是概率密度函数。

(b) Find $\mathbb{E}(X)$ and $VarX$.

解: 首先，因为,

$$\int_0^\infty xe^{-x}dx = \Gamma(2) = 1;$$

先求期望,

$$\mathbb{E}X = \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} x^3 e^{-x^2/\beta^2} dx$$

令$t = x/\beta$, 有

$$\mathbb{E}X = \frac{4\beta}{\sqrt{\pi}} \int_0^\infty t^3 e^{-t^2} dt$$

再令$m = t^2$,有

$$\mathbb{E}X = \frac{4\beta}{\sqrt{\pi}} \frac{1}{2} \int_0^\infty me^{-m}dm$$

进而,

$$\mathbb{E}X = \frac{2\beta}{\sqrt{\pi}} \int_0^{\infty} me^{-m} dm = \frac{2\beta}{\sqrt{\pi}}$$

再求平方的期望，因为

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \int_0^{\infty} xe^{-x^2} 2x dx = \int_0^{\infty} u^{\frac{1}{2}} e^{-u} du = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

, 且这个函数是关于0对称，因此

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$$

$$\mathbb{E}X = \int_0^{\infty} \frac{4}{\beta^3 \sqrt{\pi}} x^4 e^{-x^2/\beta^2} dx$$

令$t = x/\beta$, 有

$$\mathbb{E}X = \frac{4\beta^2}{\sqrt{\pi}} \int_0^{\infty} t^4 e^{-t^2} dt$$

$$\mathbb{E}X = \frac{4\beta^2}{\sqrt{\pi}} (-\frac{1}{2}) \int_0^{\infty} t^3 e^{-t^2} d(-t^2)$$

进而,

$$\begin{aligned}
\mathbb{E}X = -\frac{2\beta^2}{\sqrt{\pi}} \int_0^{\infty} t^3 d(e^{-t^2}) &= -\frac{2\beta^2}{\sqrt{\pi}} \left(t^3 e^{-t^2} |_0^{+\infty} - \int_0^{\infty} e^{-t^2} d(t^3) \right) \\
&= -\frac{2\beta^2}{\sqrt{\pi}} \left(t^3 e^{-t^2} |_0^{+\infty} - \int_0^{\infty} 3t^2 e^{-t^2} dt \right) \\
&= -\frac{2\beta^2}{\sqrt{\pi}} \left(-3 \frac{\sqrt{\pi}}{4} \right) = \frac{3\beta^2}{2}
\end{aligned}$$

因此方差$VarX = \mathbb{E}(x^2) - (\mathbb{E}(x))^2 = \frac{3\beta^2}{2} - (\frac{2\beta}{\sqrt{\pi}})^2$

4 证明

(a) 设X是连续且非负的随机变量，证明$EX = \int_0^{\infty} [1 - F_X(x)] dx$

证明：由于$F_X(x) = P(X \leq x)$, 且，$1 - F_X(x) = P(X > x)$

那么，有

$$\int_0^\infty (1-F_X(x))dx = \int_0^\infty P(X>x)dx$$

而根据定义

$$\begin{aligned}EX &= \int_0^\infty xf_X(x)dx \\ &= \int_0^\infty \int_x^\infty f_X(x)dydx \\ &= \int_0^\infty \int_0^y dxf_X(y)dy \\ &= \int_0^\infty yf_X(y)dy \\ &= \int_0^\infty xf_X(x)dx\end{aligned}$$

即

$$= EX$$

故得证。

(b) 设X是取值为非负整数的离散随机变量，证明：$EX = \sum_{k=0}^\infty (1-F_X(k))$

证明:

5 设$f(x)$为一概率密度函数，如果存在数a使得：对于任意$\varepsilon > 0$都有$f(a+\varepsilon) = f(a-\varepsilon)$, 则称$f(x)$关于a对称。

(a) 三个对称的概率密度函数:

- 正态分布:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

- 柯西分布:

$$f(x) = \frac{1}{\pi(1+x^2)}$$

- 罗吉斯蒂克概率函数:

$$f(x) = \frac{1}{1+e^{-x}}$$

(b) 因为概率分布的中位数满足$P(X \le m) \ge \frac{1}{2}$ 且$P(X \ge m) \ge \frac{1}{2}$，

$$\int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(2a-x)dx = 1/2$$

.

令$\varepsilon = x - a$那么有,

$$\int_{a}^{\infty} f(x)dx = \int_{0}^{\infty} f(a+\varepsilon)d\varepsilon = \int_{0}^{\infty} f(a-\varepsilon)d\varepsilon$$

令$x = a - \varepsilon$那么有上式等于

$$= \int_{-\infty}^{a} f(x)dx$$

即

$$\int_{a}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx$$

而对于一个概率密度函数有

$$\int_{a}^{\infty} f(x)dx + \int_{-\infty}^{a} f(x)dx = 1$$

， 因此有

$$\int_{a}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx = \frac{1}{2}$$

,

那么根据中位数的性质，可得该函数的中位数就是a。

(c) 根据期望的定义可得,

$$EX = \int_{-\infty}^{\infty} xf(x)dx$$

且有,

$$EX - a = E(X - a)$$

因此

$$EX - a = E(X - a) = \int_{-\infty}^{\infty} (x-a)f(x)dx$$

此时令$\varepsilon = x - a$,有上式等于

$$= \int_0^\infty -\varepsilon f(a-\varepsilon)d\varepsilon + \int_0^\infty \varepsilon f(a+\varepsilon)d\varepsilon$$

又有对称函数的性质，$f(a+\varepsilon) = f(a-\varepsilon)$可得，上式为0

即

$$EX - a = 0$$

, 因此有

$$EX = a$$

(d) 对于$f(x) = e^{-x}$有，$f(a+\varepsilon) = e^{-a-\varepsilon}$，$f(a-\varepsilon) = e^{-a+\varepsilon}$,
可得$\frac{f(a+\varepsilon)}{f(a-\varepsilon)} = \frac{e^{-}\varepsilon}{e^{\varepsilon}} = \frac{1}{e^{2\varepsilon}}$
因为$\varepsilon \geq 0$, 因此$\frac{1}{e^{2\varepsilon}} \neq 1$， 因此$f(x) = e^{-x}$不是对称的概率密度函数。

(e) 对于$f(x) = e^{-x}$，可求得中值为$log(2)$, 而期望$EX = \int_{-\infty}^{\infty} xf(x)dx = 1$
即中位数小于期望。

6 求下列分布的矩母函数

(a) $f(x) = \frac{1}{c}, 0 < x < c$;
解：根据矩母函数的定义，有:

$$Ee^{tX} = \int_0^c e^{tx} f(x)dx$$

$$= \int_0^c e^{tx} \frac{1}{c} dx = \frac{1}{ct} e^{tx}|_0^c = \frac{1}{ct}(e^{tc} - 1)$$

(b) $f(x) = \frac{2x}{c^2}, 0 < x < c$;
解：根据矩母函数的定义，有:

$$Ee^{tX} = \int_0^c e^{tx} f(x)dx$$

$$= \int_0^c e^{tx}\frac{2x}{c^2}dx = \frac{2x}{c^2}e^{tx}|_0^c = \frac{2}{c^2t^2}(cte^{tc} - e^{tc} + 1)$$

(c) $f(x) = \frac{1}{2\beta}e^{-|x-\alpha|/\beta}, -\infty < x < \infty, -\infty < \alpha < \infty, \beta > 0$;

解：根据矩母函数的定义，有:

$$Ee^{tX} = \int_{-\infty}^{\infty} e^{tx}f(x)dx$$

$$= \int_{-\infty}^{\infty} e^{tx}\frac{1}{2\beta}e^{-|x-\alpha|/\beta}dx$$

$$= \int_{-\infty}^{\alpha} e^{tX}\frac{1}{2\beta}e^{(x-\alpha)/\beta}dx + \int_{\alpha}^{\alpha} e^{tX}\frac{1}{2\beta}e^{-(x-\alpha)/\beta}dx$$

$$= \frac{e^{-\alpha/\beta}}{2\beta}\frac{1}{1/\beta + t}e^{-x(1/\beta+t)}|_{-\infty}^{\alpha} - \frac{e^{\alpha/\beta}}{2\beta}\frac{1}{1/\beta - t}e^{-x(1/\beta-t)}|_{\alpha}^{\infty}$$

$$= \frac{4}{4 - \beta^2t^2}e^{\alpha t}$$

7 求出下列Y的概率密度函数

(a) $Y = X^2$ and $f_X(x) = 1, 0 < x < 1$

解：令$Y = g(x)$,则$g^{-1}(y) = y^{1/2}$且$\frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y}}$

对于，$0 < x < 1$,有$Y = g(x)$是单调增函数，因此由课本定理2.1.5可得，概率密度函数$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$, $0 < y < 1$

即

$$f_Y(y) = 1 * \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}$$

,且 $0 < y < 1$。

(b) $Y = -log(X)$ and X has pdf, $f_X(x) = \frac{(m+n+1)!}{n!m!}x^n(1-x)^m, 0 < x < 1, m, n$ 为正整数。

解：令$Y = g(x)$,则$g^{-1}(y) = e^{-y}$且$\frac{d}{dy}g^{-1}(y) = -e^{-y}$

对于，$0 < x < 1$,有$Y = g(x)$是单调减函数，因此由课本定理2.1.5可得，概率密度函数$f_Y(y) = -f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$, $0 < y < 1$

即

$$f_Y(y) = -\frac{(m+n+1)!}{n!m!}(e^{-y})^n(1-e^{-y})^m - e^{-y} = \frac{(m+n+1)!}{n!m!}e^{-y(n+1)}(1-e^{-y})^m$$

, $0 < y < \infty$。

(c) $Y = e^X$ and X has pdf, $f_X(x) = \frac{1}{\sigma^2}xe^{-(x/\sigma)^2/2}$, $0 < x < \infty$, σ^2为正数。

解：令$Y = g(x)$,则$g^{-1}(y) = \log y$且$\frac{d}{dy}g^{-1}(y) = 1/y$

对于，$0 < x < 1$,有$Y = g(x)$是单调增函数，因此由课本定理2.1.5可得，概率密度函数$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$, $0 < y < 1$

即

$$f_Y(y) = \frac{1}{\sigma^2}(\log y)e^{-(\log y/\sigma)^2/2} * (1/y) = \frac{\log y}{y\sigma^2}e^{-(\log y/\sigma)^2/2}$$

, $0 < y < \infty$。

8 A random variable X is said to have a Gamma distribution if its pdf is:

$$f(x|shape = \alpha, scale = \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-x/\theta}, x \in [0, \infty), \alpha > 0, \theta > 0$$

(a) Verify $f(x|\alpha, \theta)$ is a valid pdf.

证明：两个性质不难证明性质a, 即$f_X(x) \geq 0$。

不难推导,

$$f_X(x|\alpha, \theta) = \int_{-\infty}^{\infty} f_X(x|\alpha, \theta)dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-x/\theta}dx \approx 1$$

即可证$f(x|\alpha, \theta)$ 是概率密度函数。

(b) Find the mode of a Gamma random variable (for $\alpha > 1$);

解：当 $\alpha > 1$时，$f(x)$先递增，后递减，mode为$(\alpha - 1)\theta$

(c) Find the moment generating function M(t) of a Gamma random variable;

解: 根据$\Gamma(\alpha)$函数的性质，其对应的矩母函数为:

$$M_X(t) = \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^{+\infty} e^{tx}x^{\alpha-1}e^{-x/\theta}dx$$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^{+\infty} x^{\alpha-1}e^{-(1/\theta-t)x}dx$$

根据伽玛函数的性质,对于任意大于0的常数α, β:

$$f(x) = \frac{1}{\Gamma(a)b^a}x^{a-1}e^{-x/b}$$

,都是某随机变量的概率密度函数，于是

$$\int_0^{+\infty} \frac{1}{\Gamma(a)b^a}x^{a-1}e^{-x/b}dx = 1$$

也就是

$$\int_0^{+\infty} x^{a-1}e^{-x/b}dx = \Gamma(a)b^a$$

即得: 当$t < 1/\theta$有

$$M_X(t) = \frac{1}{\Gamma(\alpha)\theta^\alpha}\Gamma(\alpha)\left(\frac{\theta}{1-\theta t}\right)^\alpha = \left(\frac{1}{1-\theta t}\right)^\alpha$$

而当$t \geq 1/\theta$时，没有矩母函数，因为$M_X(x)$。

(d) Find mean , variance of X, the skewness and the kurtosis of a Gamma random variable;

解: 先求期望,

$$EX = \frac{d}{dx}M_X(t)|_{t=0} = \frac{\alpha\theta}{(1-\theta t)^{\alpha+1}}|_{t=0} = \alpha\theta$$

再求平方的期望,

$$EX^2 = \frac{d^{(2)}}{dx}M_X^{(2)}(t)|_{t=0} = \frac{d}{dx}\frac{\alpha\theta}{(1-\theta t)^{\alpha+1}}|_{t=0} = \frac{(\alpha+1)\alpha\theta^2}{(1-\theta t)^{\alpha+2}}|_{t=0} = (\alpha+1)\alpha\theta^2$$

, 因此方差$VarX = EX^2 - (EX)^2 = \alpha\theta^2$

The skewness of a random variable X is its third central moment ,因此

$$EX^3 = \frac{d^{(3)}}{dx}M_X^{(2)}(t)|_{t=0} = \frac{2}{\sqrt{\alpha}}$$

The Kurtosis of a random variable X is its fourth central moment,因此

$$EX^4 = \frac{d^{(4)}}{dx} M_X^{(2)}(t)|_{t=0} = 3 + \frac{6}{\alpha}$$

(e) Let $Y = 1/X$. What is the pdf of Y?(Y is said to have an inverse gamma distribution)

解: 令$Y = 1/X$, 则有, $\frac{dx}{dy} = \frac{1}{y^2}$, 且$Y$的概率密度函数为:

$$f_Y(y|\alpha,\theta) = f_X(y|\alpha,\theta)\frac{dx}{dy}$$

$$f_Y(y|\alpha,\theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha}(\frac{1}{y})^{\alpha-1}e^{-1/(\theta y)} \mid \frac{1}{y^2} \mid$$

$$f_Y(y|\alpha,\theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha}y^{-\alpha-1}e^{-1/(\theta y)}$$

用β替换θ^{-1}得:

$$f_Y(y|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}y^{-\alpha-1}e^{-\beta/y}$$

因此, Y的概率密度函数为逆Gamma分布。

(f) Find the mean, the variance, the skewness and the kurtosis of a inverse Gamma random variable; 解: 计算公式, 当$\alpha > n$时,

$$EX^n = \frac{d^{(n)}}{dx} M_X(t)|_{t=0} = \frac{\beta^n}{(\alpha-1)(\alpha-2)...()\alpha-n}$$

那么对于$\alpha > 1$时, 期望

$$EX = \frac{d}{dx} M_X(t)|_{t=0} = \frac{\beta^n}{(\alpha-1)(\alpha-2)...()\alpha-n}|_{n=1} = \frac{\beta}{\alpha-1}$$

, 再求平方的期望,

$$EX^2 = \frac{d^{(2)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{\beta^n}{(\alpha-1)(\alpha-2)...()\alpha-n}|_{n=2} = \frac{\beta}{(\alpha-1)(\alpha-2)}$$

, 因此方差$VarX = EX^2 - (EX)^2 = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$

那么对于$\alpha > 3$时, skewness:

$$EX^3 = \frac{d^{(3)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{4\sqrt{(\alpha-2)}}{\alpha-3}$$

那么对于$\alpha > 4$时，Kurtosis :

$$EX^4 = \frac{d^{(4)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{(30 * \alpha - 66)}{((\alpha - 3) * (\alpha - 4))}$$

9 A random variable X is said to have a Possion distribution if its pdf is:

$$f(x|shape = \alpha, scale = \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, x \in [0, \infty), \alpha > 0, \theta > 0$$

(a) Verify $f(X = k)$ is a valid pdf.

证明：两个性质不难证明性质a, 即$f(X = k) \geq 0$。

不难推导,

$$f(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, ...$$

$$= \sum_{x=0}^{\infty} P(X = x|\lambda) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$$

即可证$f(X = k)$ 是概率密度函数。

(b) Find the moment generating function M(t) of a Gamma random variable;

解：矩母函数为:

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{(e^t\lambda)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

(c) Find the mean, the variance, the skewness and the kurtosis of a Poisson random variable;

解: 先求期望,

$$EX = \frac{d}{dx} M_X(t)|_{t=0} = e^{\lambda(e^t-1)}\lambda e^t|_{t=0} = \lambda$$

再求平方的期望,

$$EX^2 = \frac{d^{(2)}}{dx} M_X^{(2)}(t)|_{t=0} = \lambda^2 + \lambda$$

, 因此方差$VarX = EX^2 - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

The skewness of a random variable X is its third central moment ,因此

$$EX^3 = \frac{d^{(3)}}{dx} M_X^{(2)}(t)|_{t=0} = \lambda^{-1/2}$$

The Kurtosis of a random variable X is its fourth central moment,因此

$$EX^4 = \frac{d^{(4)}}{dx} M_X^{(2)}(t)|_{t=0} = \lambda^{-1}$$

10 Show that

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}, \alpha = 1, 2, 3, ...$$

(Hint:Use integration by parts.) Express this formula as a probabilistic relationship between Possion and gamma random variables.

解: 因为$\alpha = 1$, 2, $3, ...$, 即α是正整数, 故有$\Gamma(\alpha) = (a-1)!$

因此有

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz$$

$$= \frac{1}{(\alpha-1)!} \int_x^{\infty} z^{\alpha-1} e^{-z} dz$$

令$u = z^{\alpha-1}$, $dv = e^{-z}$进行分部积分， 有上式

$$= \frac{1}{(\alpha-1)!} \left(-z^{\alpha-1} e^{-z} |_x^{\infty} + \int_x^{\infty} (\alpha-1) z^{\alpha-2} e^{-z} dz \right)$$

$$= \frac{z^{\alpha-1} e^{-z}}{(\alpha-1)!} + (\alpha-1) \int_x^{\infty} z^{\alpha-2} e^{-z} dz$$

重复上述分部积分过程，可得:

$$= \frac{z^{\alpha-1} e^{-z}}{(\alpha-1)!} + \frac{z^{\alpha-2} e^{-z}}{(\alpha-2)!} + (\alpha-2) \int_x^{\infty} z^{\alpha-3} e^{-z} dz$$

$$= \frac{z^{\alpha-1} e^{-z}}{(\alpha-1)!} + \frac{z^{\alpha-2} e^{-z}}{(\alpha-2)!} + ... + \frac{z e^{-z}}{1!} + (1) \int_x^{\infty} z^0 e^{-z} dz$$

令$y = \alpha - 1$,代入到上式可得

$$= \sum_{y=0} \alpha - 1 \frac{z^y e^{-z}}{y!}$$

因此，可得题中左右两式相等，而等式左边为$P(X \geqslant x)$，等式左边为$P(Y \leqslant y)$，即可得出结论，如果$X \backsim \Gamma(\alpha, 1)$且$Y \backsim Poisson(x)$,那么它们就有一种关系:

$$P(X \geq x) = P(Y \leq \alpha - 1)$$