统计学方法及其应用

Statistical Methods with Applications

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Random Variables

统计学方法及其应用 统计学基础 随机变量

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Defining a probability function

%&'(% = "#$_!$)***)#$_"$ $(+&(, (-. / . '(0&'*(%&'($\mathcal{B}$(+&(, / 1(0.2 3 , (, 42+5 ,
6-(07+0&'0(6-(%*(%&'(&$_!$)***) &$_"$ (+&(/ 6 / / &2 , ' .8&(/ 7 3 +&50('9 , '
07 3 ('6(! *(: 65(, / 1($ $\in \mathcal{B}$)(; &-. / &(' <$=(+1

$$' <\$= = \sum_{"!\#_! \in \$\$} \&_!^*$$

>9&/ (' (.0(, (?56+ , +.4.'1(-7 / @.'6 / (6 / ($\mathcal{B}$*(>9.0(5&3 , . / 0('57&(.-
% = "#$_!$)#$_A$)***$(.0(, (@67 / ' , +4&(0&'*

Tossing coins

- 扔一枚硬币，观察到正面的概率
  - $S = \{H, T\}$
  - P(正面) $= P(\{H\}) = 1/2$
- 扔一枚硬币三次，观察到两次正面的概率
  - $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$
  - P(两次正面) $= P(\{HHT, HTH, THH\}) = 3/8$
- 扔一枚硬币一百次，观察到十次正面的概率
  - $S = \{2^{100}$ elements$\}$
  - P(十次正面) = Unable to count!
- 实际上正面出现的次数仅有101种可能

It is much easier to deal with a summary variable than with the original probability structure.

How to reduce the sample space?

- 定义计数函数
  - (<#=(B(C"D$
  - 定义域 $\mathcal{S}$ 包含 $A^{!EE}$ 个元素
  - 值域 FE)(!EEG 包含 !E! 个元素
- 观察到十次正面的次数
  - '<C"D$B!E=B '<(B!E=BH<!EE)!E=×E*I $^{!E}$×E*I ^{JE}≈!*KL×!E $^{M!L}$
- 扔任意硬币 " 次，观察到) 次正面的次数
  - '<(B)*+*")(&=(B(H<")(,=(× &, × <! —&= $^{"—,}$

Random variables

Random variable

N(!"#$%&'("!)"*+,(.0(,(-7/@'.6/(-563(,(0,3?4&(0?,@&($\mathcal{S}$(./'6
'9&(5&,4(/73+&50*

- 随机变量是定义在样本空间上的实值函数
- 随机变量用大写字母表示，例如() -) .
- 随机变量的取值用对应的小写字母表示，例如)) /) 0

Examples of random variables

- 掷一只骰子
  - (*B(观测到的点数
- 掷两只骰子
  - (*B(观测到的点数之和
  - - *B(观测到的点数之差的绝对值
- 扔一枚硬币K次
  - (*B(观测到正面的次数
- 从一副扑克牌中任意抽取五张
  - (*B(抽到K的张数

随机变量的引入简化了研究的问题，

体现了统计学中**数据简约**的思想

随机变量的取值很重要，

但随机变量以什么概率取得这些值更重要

Define a probability on the domain

%&'(% $=$ "#$_!$)***)#$_"$ $(+&(,(-./.'(0&'*(%&'(ℬ(+&(,/1(0.23 ,(,42+5,
6-(07+0&'0(6-(%*(%&'(&$_!$)***) &$_"$ (+&(/6/ /&2,'.8&(/73 +&50('9,'
073('6(!*(:65(,/1($ ∈ ℬ)(;&-./&('<$=(+1

$$'<\$= = \sum_{"!\#_! \in \$\$} \&_!*$$

>9&/(' (.0(,(?56+,+.4.'1(-7/@.'6/(6/(ℬ*(>9.0(5&3,./0('57&(.-
% $=$ "#$_!$)#$_A$)***$(.0(,(@67/',+4&(0&'*

Induce a probability on the range

O7??60&('9, '('9&(5, /2&(6-(((.0(,406(,(-./.'&(0&'($\mathcal{X}$)(P&(@, /('9&/(;&-./&

$$'_{(}<(\;=)_{!}=\;=\;'\left(\text{"}\#_{1}\;\#\#_{1}\in\mathcal{S})\;(\;<\#_{1}=\;=\;)_{!}\$\right)$$

Q6P)(4&'('9&(0.23 ,(,42&+5,($\mathcal{B}$(+&('9&(@644&@'.6/(6-(,44(07+0&'0(6-($\mathcal{X}$)

- .)%&'/ 0'-65(, /1(0&'($\$\in\mathcal{B}$)

$$\begin{aligned} '_{(}<\$= &= '\left(\bigcup_{)_{!}\in\$}\text{"}\#_{1}\;\#\#_{1}\in\mathcal{S})\;(\;<\#_{1}=\;=\;)_{!}\$\right) \\ &= \sum_{)_{!}\in\$} '\left(\text{"}\#_{1}\;\#\#_{1}\in\mathcal{S})\;(\;<\#_{1}=\;=\;)_{!}\$\right) \\ &\geq \text{E} \end{aligned}$$

- .)%&'1 0'-65('9&(&/'.5&(0, 3?4&(0?,@&($\mathcal{X}$)

$$'_{(}<\mathcal{X}= \;=\; '\left(\bigcup_{)_{!}\in\mathcal{X}}\text{"}\#_{1}\;\#\#_{1}\in\mathcal{S})\;(\;<\#_{1}=\;=\;)_{!}\$\right) = \;'<\mathcal{S}=\;=\;!$$

- .)%&'2 0'-65(?,.5P.0&(;.0R6./'(0&'0($\$_{!}$) $\$_{A}$)***)

$$\begin{aligned} '_{(}\left(\bigcup_{,=!}^{\infty}\$_{,}\right) &= '\left(\bigcup_{,=!}^{\infty}\left\{\bigcup_{)_{!}\in\$_{,}}\text{"}\#_{1}\;\#\#_{1}\in\mathcal{S})\;(\;<\#_{1}=\;=\;)_{!}\$\right\}\right) \\ &= \sum_{,=!}^{\infty} '\left(\bigcup_{)_{!}\in\$_{,}}\text{"}\#_{1}\;\#\#_{1}\in\mathcal{S})\;(\;<\#_{1}=\;=\;)_{!}\$\right) \\ &= \sum_{,=!}^{\infty} '_{(}<\$_{,}= \end{aligned}$$

Change of the sample space

- 样本空间的转换
  - 在随机变量的定义域上 $\mathcal{S}$ B("#!)(#A)(S)(#‥$
  - 在随机变量的值域上 $\mathcal{X}$ B(") !)()A)(S)() 2$
  - 随机变量建立的映射 (#($\mathcal{S} \mapsto \mathcal{X}$
- 定义在随机变量定义域上的概率函数

' <#1= B &1

' <$= B $\sum_{\#_1 \in \$}$ &1

- 定义在随机变量值域上的概率函数

' (< (*B()!= B '<"#1 $\in$ $\mathcal{S}$# (<#1= B)!$=

Distributions of random variables

- 随机变量的所有可能取值及取得每一个值的概率
- 扔一枚硬币三次，观察出现正面的次数
  - $\mathcal{S}$ B "DDD)(DD>)(D>D)(>DD)(>>D)(>D>)(D>>)(>>>$
  - $\mathcal{X}$ B "E)(!)(A)(K$
  - (#($\mathcal{S} \mapsto \mathcal{X}$
    (<DDD=BK((((<DD>=BA(((<D>D=BA((((<>DD=BA
    (<>>D=B!((((<>D>=B!(((<D>>=B!((((<>>>=BE

  - '<(*B(E=(B(!TU　　'<(*B(!=(B(KTU
    '<(*B(A=(B(KTU　　'<(*B(K=(B(!TU

Cumulative distribution function (cdf)

Distribution function

>9&(34&4+"5)(,'$)65!)*45)%#'74#35)%#(<345=(6-(,(5,/ ;63

8,5.,+4&(( )(;&/6';(+1(6 (<) =)(.0(;&-./&;(+1

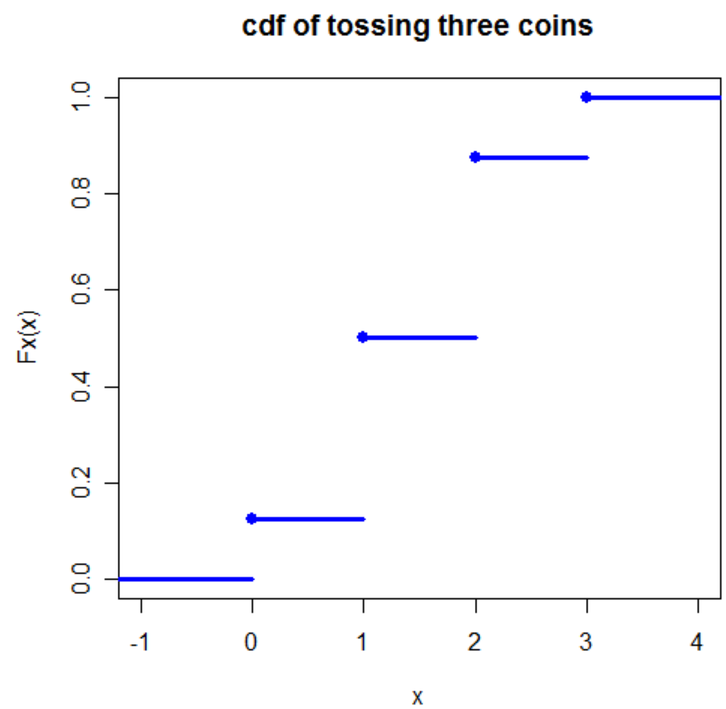
6 (<) = = ' (<(≤) =)(-65(,44()*

At most

cdf

- 扔一枚硬币三次，观察出现正面的次数
  - $\mathcal{X}$ B "E)(!)(A)(K$
  - '<(*B(E=(B(!TU　　'<(*B(!=(B(KTU
    '<(*B(A=(B(KTU　　'<(*B(K=(B(!TU
  - 分布函数

$$6_{(}\,<) = \; = \begin{cases} \mathrm{E} & .\text{-(} \quad -\infty <) < \mathrm{EV} \\ \mathrm{!}\!/\!\mathrm{U} & .\text{-(} \quad \mathrm{E} \le) < \mathrm{!V} \\ \mathrm{!}\!/\!\mathrm{A} & .\text{-(} \quad \mathrm{!} \le) < \mathrm{AV} \\ \mathrm{L}\!/\!\mathrm{U} & .\text{-(} \quad \mathrm{A} \le) < \mathrm{KV} \\ \mathrm{!} & .\text{-(} \quad \mathrm{K} \le) < \infty^{*} \end{cases}$$



Necessary and sufficient condition

Necessary and sufficient condition

>9&(-7/@'.6/(6<)=(.0(,(@;-(.-(,/;(6/41(.-('9&(-6446P./2(

58!,,(3%#$)5)%#6(964;#

!* 4.3$_{)\to -\infty}$ 6<)= $=$ E(,/;(4.3$_{)\to\infty}$ 6<)= $=$!V

A* 6<)=(.0(,(/6/;&0@5&,0./2(-7/@'.6/(6-()V

K* 6<)=(.0(5.29'M@6/'./7670V('9,'(.0)(-65(&8&51

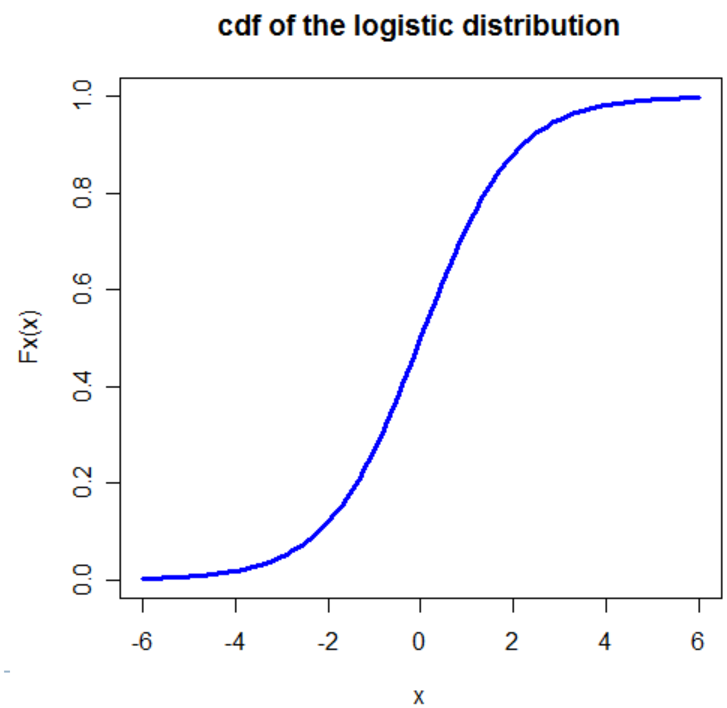
/73+&5()$_E$)(4.3$_{)\to)_E^+}$ 6<)= $=$ 6<)$_E$=*

Logistic cdf

- Logistic distribution

$$6_{(}<)= = \frac{!}{! + 7^{-)}}$$

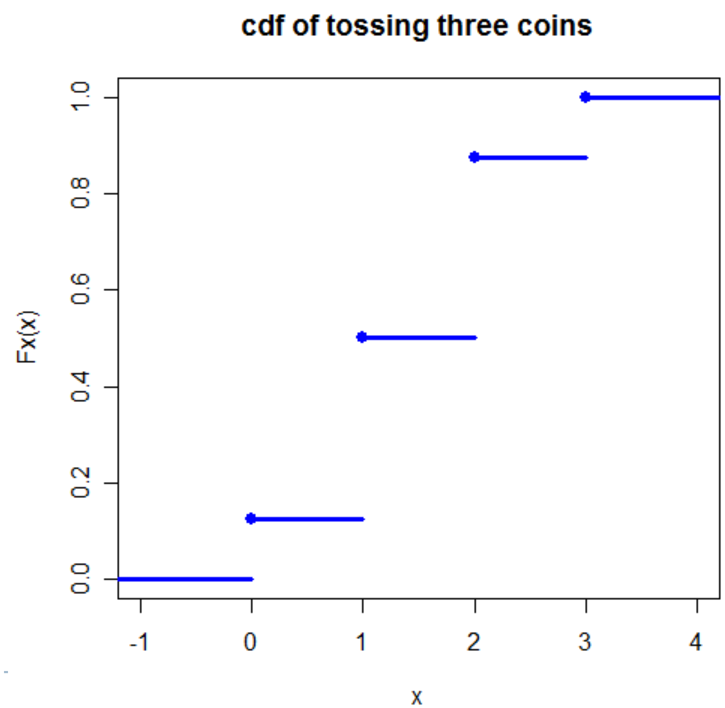
- 充要条件的满足性
  - 负无穷时为E
  - 正无穷时为!
  - 不减
  - 右连续



Discrete random variables

Discrete random variables

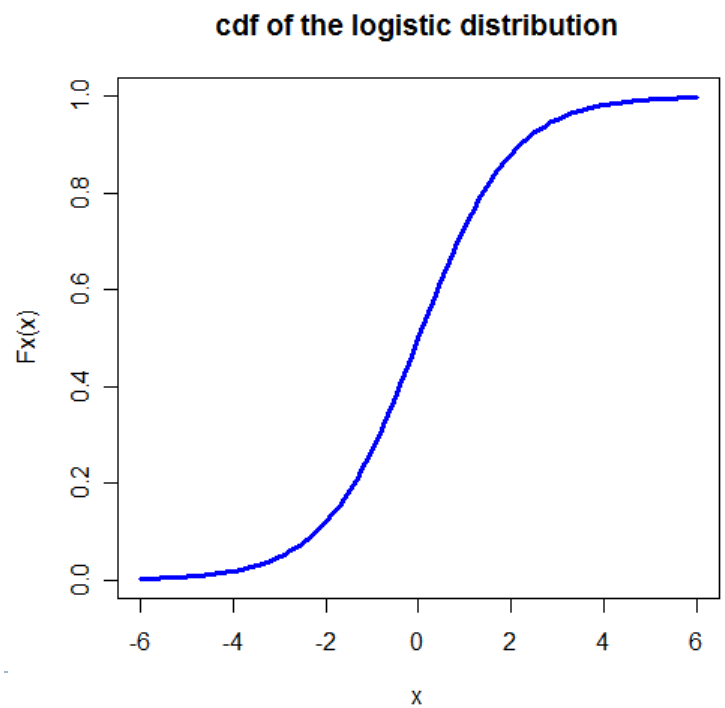
N(5 , / ; 6 3 (8 , 5. , +4&(((.0($)63! , 5 , (.-(6(<) =(.0(, (0'&?(-7 / @'.6 / (6-()*



Continuous random variables

Continuous random variables

N(5, / ;63(8,5., +4&(((.0(3%#5)#4%46(.-(6(<) =(.0(, (@6/ '. / 7670
-7 / @'.6 / (6-() *



Identically distributed

Identically distributed

>9&(5 , / ; 6 3 (8 , 5. , +4&0(((, / ; (- (, 5&()$, #5)3"++9'$)65 !)*45 , $(.-)

-65(&8&51(0&'($ $\in \mathcal{B}^{!}$)

$$'\<(\in \$= = '\<- \in \$=*$$

1. $\mathcal{B}^{!}$ is the smallest sigma algebra containing all the intervals of real numbers of the form <8)9=)(F8)(9=)(<8)9G)(and F8)9G.
2. Two identically distributed random variables are not necessarily equal.

Identically distributed

>9&(-6446P./2('P6(0','&3&/'0(,5&(&W7.8,4,/'

((!* >P6(5,/;63(8,5.,+4&0(((,/;(- (,5&(.;&/'.@,441(;.0'5.+7'&;V

((A* $6_{(}$<)= $=$ 6_{-}<)=(-65(&8&51()*

'<($\in$ $= $=$ '<- $\in$ $=(-65(,/1(0&'($ $\in \mathcal{B}$! $\Rightarrow$

'<($\in$ <$-\infty$))G= $=$ '<- $\in$ <$-\infty$))G= $\Rightarrow$

$6_{(}$<)= $=$ 6_{-}<)=

Probability mass functions

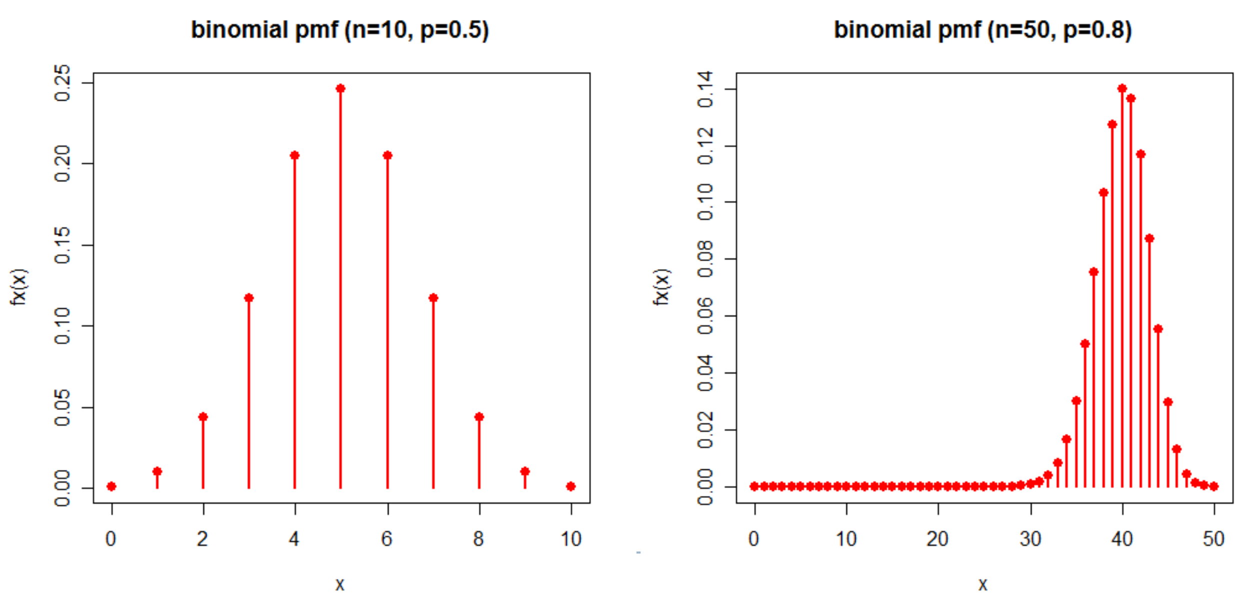
Probability mass function

>9&(: !%*"*)+)59'&"66'74#35)%#(<&25=(6-(,(;.0@5&'&(5,/;63

8,5.,+4&(()(;&/6'&;(+1(5(<)=)(.0(2.8&/(+1

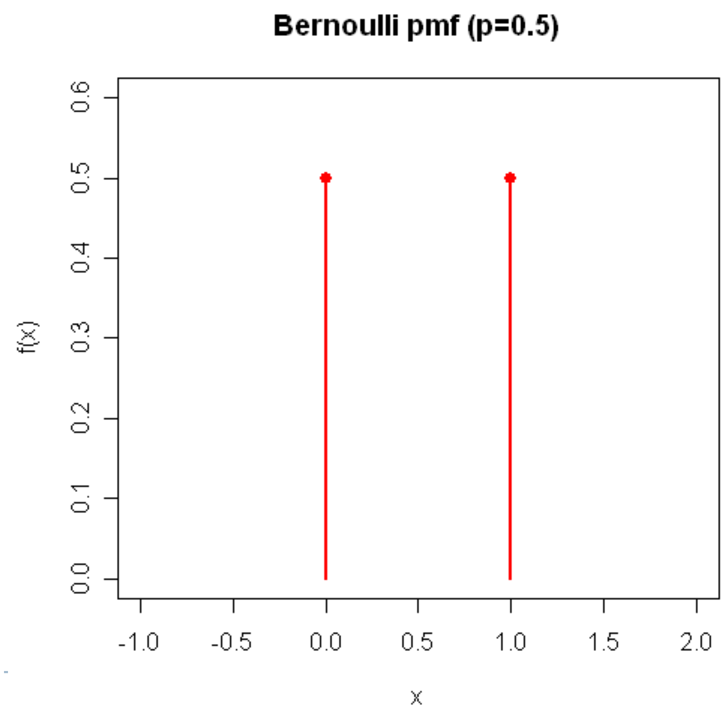
5(<)= = '(<(=)=)((-65(,44()*

Exact



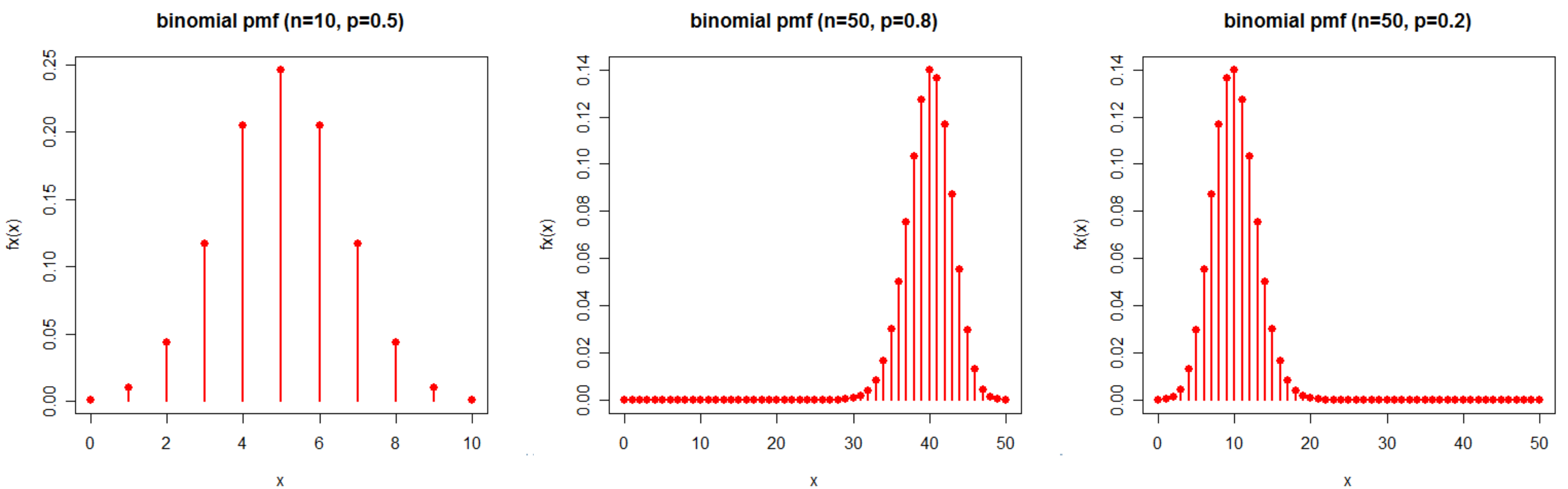
Bernoulli distribution

(= !(<07@@&00= P.'9(?56+,+.4.'1(&
E(<-,.475&= P.'9(?56+,+.4.'1(! M &



Binomial distribution

' < (=) ☓ ") &= = (" ;)) &) <! − &=" −))) = E) !) . . .) "



Relation of cdfs and pmfs

$$' \lt 8 \le (\le 9 = = \sum_{,=8}^{9} 5 \lt , =$$

$$' \lt (\le 9 = = \sum_{,=-\infty}^{9} 5 \lt , =$$

$$' \lt (\ge 9 = = \sum_{,=9}^{\infty} 5 \lt , =$$

$$6 \lt) = = ' \lt (\le) = = \sum_{,=-\infty}^{)} 5 \lt , =$$

For a continuous random variable

P; X < x=' < '>

- " (*B()$(((⊂ ") −ε Y((≤)$ -65(, / 1() , / ; (ε
- ' " (*B()$(≤ ' ") −ε Y((≤)$
  B(' " (≤)*∩ (*Z() −ε$
  B(' " (≤)*∩ < (*≤) −ε)@$
  B(' " (≤)$ − ' < (≤)*∩ (*≤) −ε$? 89>
  B(6 (<)= − 6 (<) −ε=
- E ≤ ' " (*B()$ ≤ 4. 3 $_{\varepsilon\to 0}$ F6 (<)= − 6 (<) −ε=G(B(E

P@X < xA' < 'B'7%!'"#9'x

' "8Y (Y9$B ' "8Y (≤9$B ' "8≤ (Y9$B ' "8≤ (≤9$(-65(, / 1()

Probability density functions

Probability density function

>9&(: !%*"*)+)59'$,#6)59'74#35)%#(<&45 =)(; &/6'& ; (+1($5_($ <) =) 6-(,

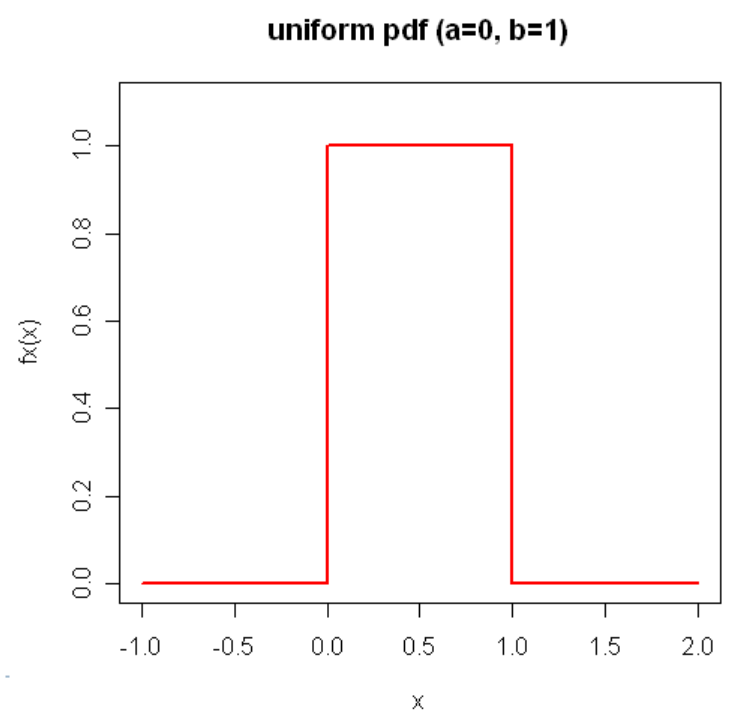
@6/'./7670(5, / ; 6 3 (8 , 5. , +4&(((.0('9&(-7/@'.6/('9,'(0,'.0-.&0

$$6_(<) = = \int_{-\infty}^{)} 5_(<:=4:)(((-65(,44()*$$

$$5_(<) = = \frac{4}{4)} 6_(<) =$$

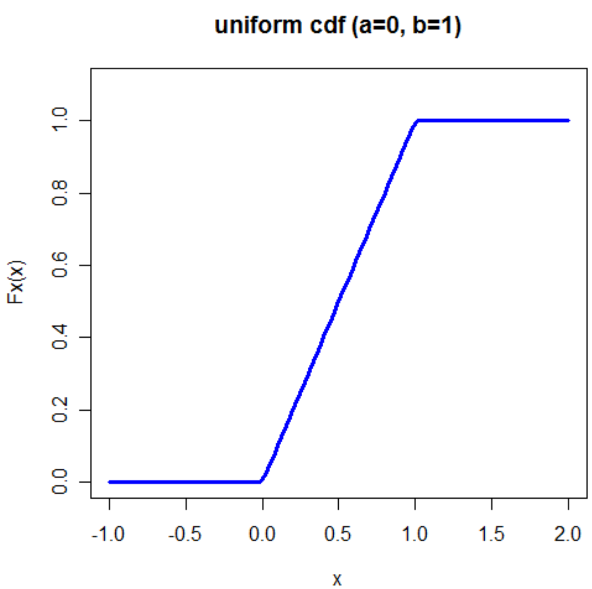
Uniform distribution

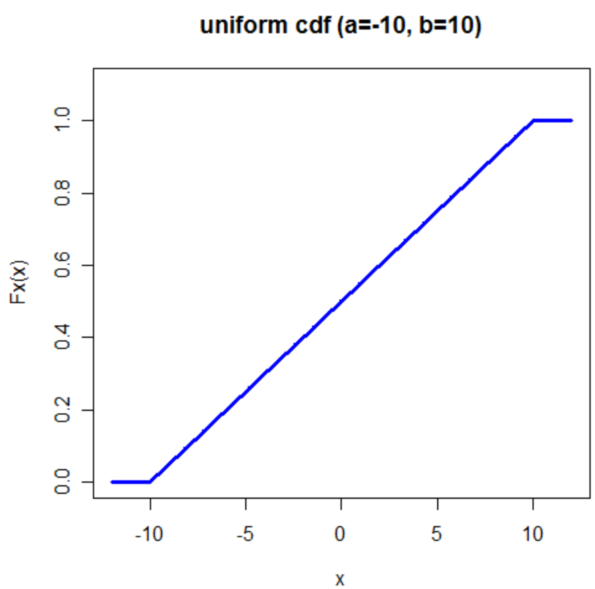
$$f_X(x;a,b) = \begin{cases} 0 & x < a \\ 1/(b-a) & a \le x \le b \\ 0 & x > b \end{cases}$$

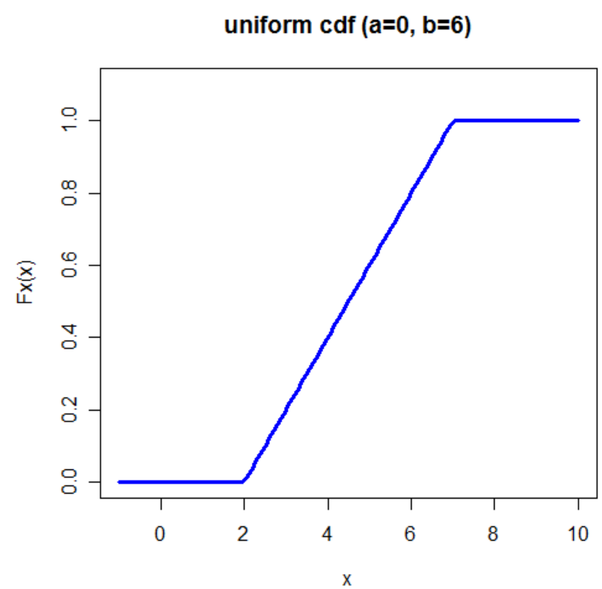


Uniform distribution

$$6_{(} <) \;\text{ᚷ}\; 8)9 = \begin{cases} \mathrm{E} &) < 8 \\ \dfrac{) - 8}{9 - 8} & 8 \le) \le 9 \\ ! &) > 9 \end{cases}$$







Relation of cdfs and pdfs

$$' <8 < (< 9= = \int_{8}^{9} 5<)=4)$$

$$' <(<)= = \int_{-\infty}^{)} 5<:=4:$$

$$' <(>)= = \int_{)}^{\infty} 5<:=4: = ! - \int_{-\infty}^{)} 5<:=4:$$

$$6<)= = ' <(\leq)= = \int_{-\infty}^{)} 5<:=4:$$

Necessary and sufficient condition

Necessary and sufficient condition

N(-7/@'.6/($5_($<)=(.0(,(?;-(65(?3-(6-(,(5,/;63(8,5.,+4&(((

.-(,/;(6/41(.-('9&(-6446P./2('P6(@6/;.'.6/0(964;#

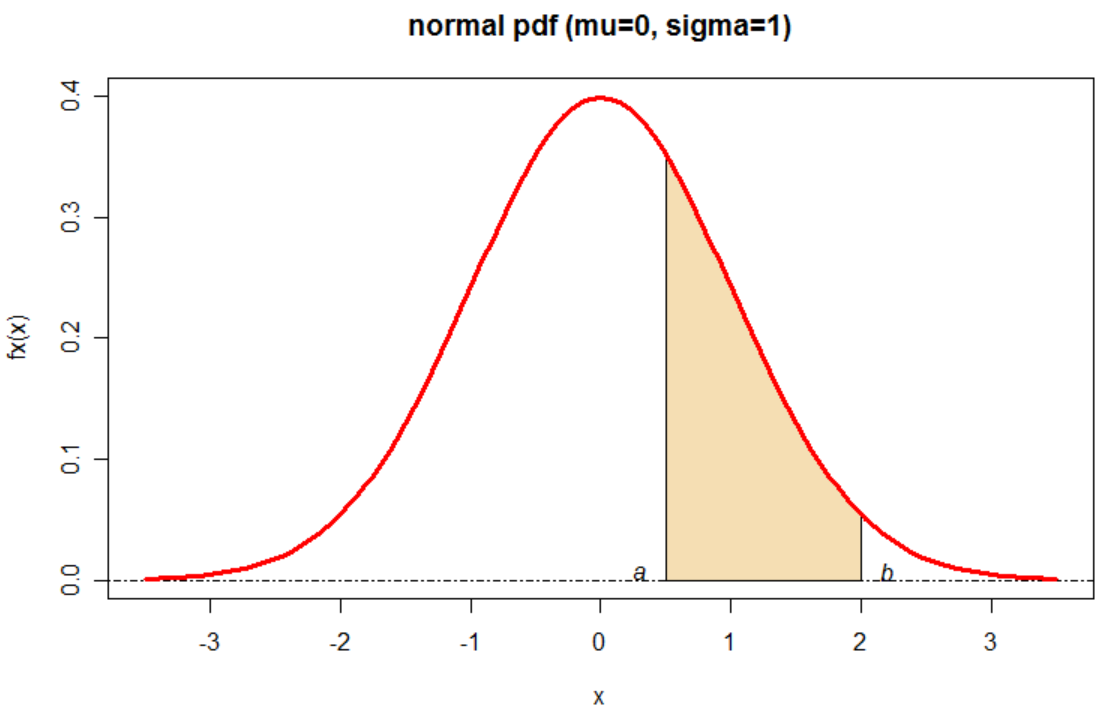
!* $5_($<)= $\geq$ E(-65(,44()V

A* $\sum_)$ $5_($<)=B!(<?3-=(65(

$\int_{-\infty}^{\infty}$ $5_($<)=B!(<?;-=*

Standard normal distribution

$$5<)= = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^A}{A}}$$



$$' <8 \le) \le 9=$$
$$= ' <8 <) \le 9=$$
$$= ' <8 \le) < 9=$$
$$= ' <8 <) < 9=$$
$$= 6_(<9= - 6_(<8=$$
$$= \int_8^9 5_(<)=4)$$

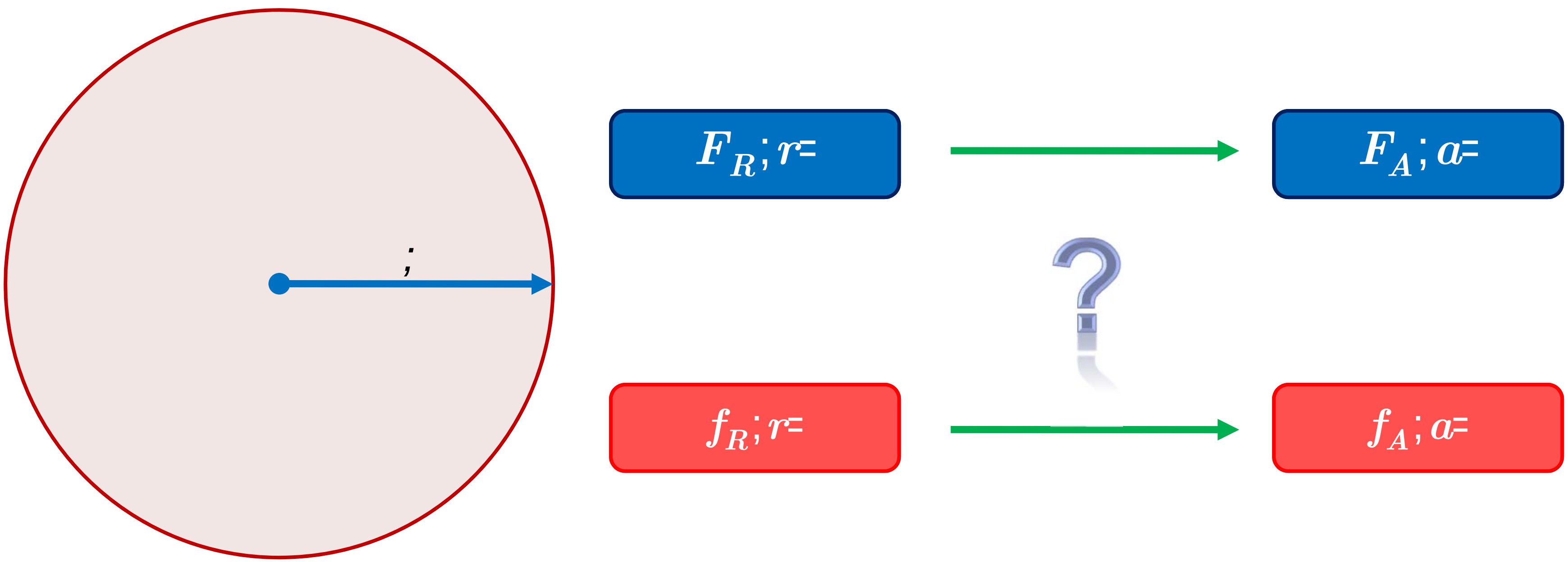
Transformations

统计学方法及其应用 > 统计学基础 > 随机变量的函数

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Why need functions of random variables

- 已知一些量的分布，而关心的是另一些量的分布
  - 半径 ; ~ [/ .-65 3 <E)(! =
  - 面积 $ ~ \

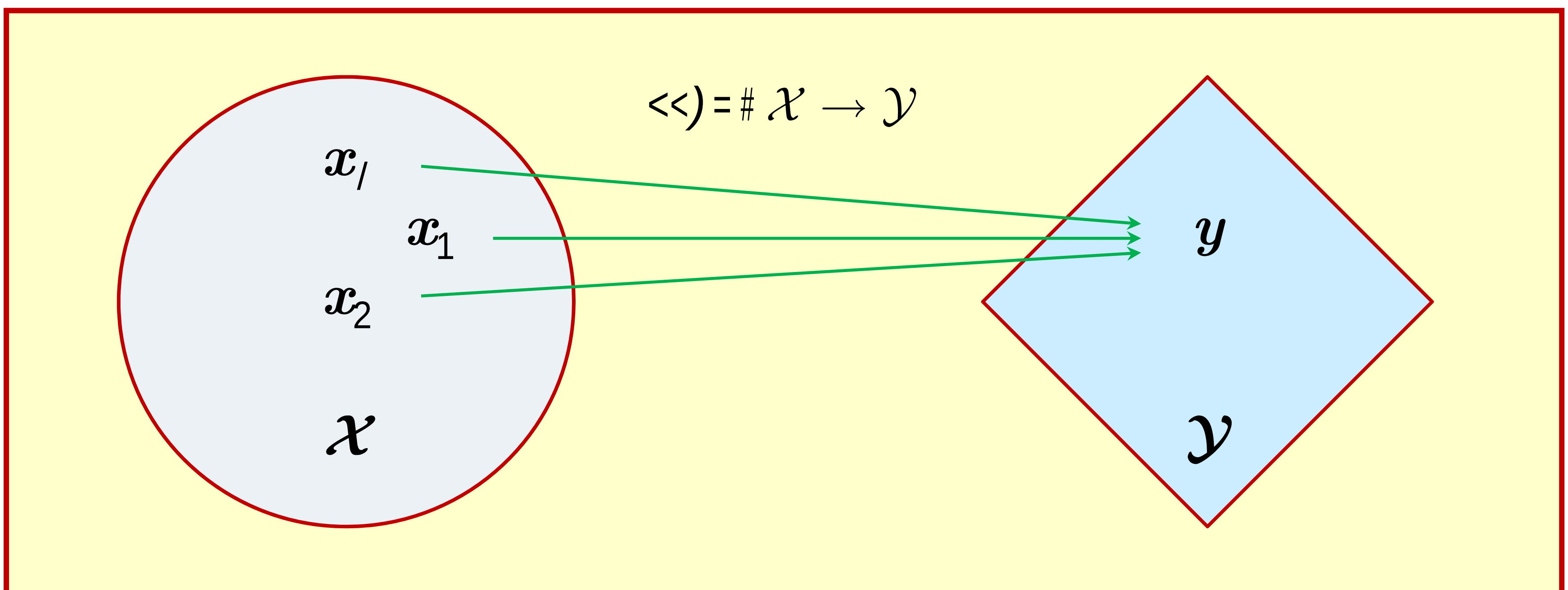


Function of a random variable

- If (is a random variable with cdf 6 (<)=, then any function of (, say, - *B(<< (=, is also a random variable
- The probability behavior of - can be described using (

$$' <- \in \$= = ' <<< (= \in \$=$$

depending on the distribution of (and the function <



Transformation of a pmf

$$
\begin{aligned}
' <- \ \in \$= &= ' <<< (= \in \$= \\
&= ' <) \in \mathcal{X} \# <<) = \in \$= \\
&= ' < (\ \in <^{-!} < \$ ==
\end{aligned}
$$

]-(((.0(; .0@5&'&)('9&/(- (.0(, 406(; .0@5&'&

<+&@ , 70&(+6'9($\mathcal{X}$ (, / ; ($\mathcal{Y}$ (, 5&(@67' , +4&=

]-(/ $\in \mathcal{Y}$)('9&/

$$
\begin{aligned}
5_{-} </= &= ' <- \ = /= \\
&= \sum_{) \in <^{-!} </=} ' < (\ =) = \\
&= \sum_{) \in <^{-!} </=} 5_{(} <) =*
\end{aligned}
$$

]-(/ $\notin \mathcal{Y}$)('9&/

$$5_{-} </= \ = E*$$

$$<^{-!} < \$ = = ") \in \mathcal{X} \# <<) = \in \$\$$$

$$<^{-!} </= = ") \in \mathcal{X} \# <<) = = /\$$$

Binomial transformation

O7??60&(5(<)= = ' <(=) Ⅹ ")&= = (" ;)) &) <! − &=" −))) = E) !)…) ")

'9, '(.0)((((,(+./63.,4(;.0'5.+7'.6/(P.'9(?,5, 3&'&50("(, / ;(&)(, / ;(

- = <<(= = " − ((.0(,('5, /0-653 , '.6/*(>9&/)(/ = <<)= = " −))(

, / ;

5- </ Ⅹ ")&= = ∑) ∈<−! </= 5(<) Ⅹ ")&=

= 5(<" − / Ⅹ ")&=

= (" ; " − /) &" −/ <! − &=" −<" −/=

= (" ; /) <! − &=/ &" −/

.0(,406(,(+./63.,4(;.0'5.+7'.6/(P.'9(?,5, 3&'&50("(, / ;(! − &*

Transformation of a cdf

>9&(@;-(6-(- = <<(=(.0

6- </= = '<- ≤ /=

= '<<<(= ≤ /=

= '<") ∈ 𝒳 # <<) = ≤ /$=

= ∫ ") ∈𝒳#<<) =≤/$ 5(<) =4)

D6P('6(6+' ,./(") ∈ 𝒳 # <<) = ≤ /$\

1. The function <<)=(is monotone increasing
2. The function <<)=(is monotone decreasing
3. The function <<)=(is piecewise monotone

Monotone increasing

%&'(((9,8&(@;-(6_(<)=*(%&'(- = <<)=*(%&'(

$\mathcal{X}$ = ") # 5_(<)= > E$)(,/;(

$\mathcal{Y}$ = "/ # / = <<)=(-65(06 3&() $\in$ $\mathcal{X}$$*

]-(<<)=(.0(3 6/6'6/&(./@5&,0./2)('9&/('9&(3 ,??./2() $\longrightarrow$ <<)=(.0

^6/&M'6M6/&^(,/;(^6/'6^)(,/;(<$^{-!}$</=(.0(0./24&M8,47&;(,/;

,406(3 6/6'6/&(./@5&,0./2*(>9&5&-65&)

") $\in \mathcal{X}$ # <<)= $\leq$ /$ = ") $\in \mathcal{X}$ #) $\leq$ <$^{-!}$</=$

6_- </= = $\int$_{") ∈ 𝒳#<<)=≤/$} 5_(<)=4)

= $\int$_{") ∈ 𝒳#) ≤<^{-!}</=$} 5_(<)=4)

= 6_(<<$^{-!}$</==

Monotone decreasing

%&'(((9 , 8&(@ ; -(6 (<) =*(%&'(- = <<) =*(%&'(

𝒳 = ") # 5 (<) = > E$)(, / ; (

𝒴 = "/ # / = <<) =(-65(06 3 &() ∈ 𝒳$*

]-(<<) =(.0(3 6 / 6 ' 6 / &(; &@5& , 0. / 2)(' 9& / (' 9&(3 , ??. / 2() ⟶ <<) =(.0

^6 / &M ' 6M6 / &^(, / ; (^6 / ' 6^)(, / ; (< −! </=(.0(0. / 24&M8 , 47& ; (, / ;

, 406(3 6 / 6 ' 6 / &(. / @5& , 0. / 2*(>9&5&-65&)

") ∈ 𝒳 # <<) = ≤ /$ = ") ∈ 𝒳 #) ≥ < −! </=$

6 - </= = ∫ ") ∈𝒳#<<) =≤/$ 5 (<) =4)

= ∫ ") ∈𝒳#) ≥< −! </=$ 5 (<) =4)

= ! − 6 (<< −! </==

Transformation of a cdf

%&'(((9,8&(@;-(6(<)=)(4&'(- = <<)=)(,/;(4&'(

𝒳 = ") # 5(<)=> E$)(,/;(

𝒴 = "/ # / = <<)=(-65(06 3&() ∈ 𝒳$*

>9&/

]-(<(.0(,/(./@5&,0./2(-7/@'.6/(6/(𝒳)

6- </= = 6(<<−!</==(-65(/ ∈ 𝒴*

]-(<(.0(,(;&@5&,0./2(-7/@'.6/(6/(𝒳)

6- </= = ! − 6(<<−!</==(-65(/ ∈ 𝒴*

Exponential distribution

O7??60&('9 , '(5(<)= = !(-65() ∈ <E) !=(, / ; (E(6'9&5P.0&)('9 , '(.0)(((

9 , 0(, (7 / .-65 3 (; .0'5.+7'.6 /)(, / ; (- = << (= = −λ 462 ((<λ > E=(.0

, ('5 , / 0-65 3 , '.6 / *(>9& /)(

6(<) = =) (-65() ∈ <E) !=*(

<<) = = −λ 462) (.0(; &0@5& , 0. / 2(6 / (.'0(07??65')(, / ; (

<−!</= = &_?<−/ T λ=(.0(, 406(; &@5& , 0. / 2(6 / (.'0(; 6 3 , . / (E < / < ∞*

>9&5&-65&)

6- </= = ! − 6(<<−!</== = ! − 6(<&_?<−/ T λ==

= ! − &_?<−/ T λ=) E < / < ∞*(

Probability integral transformation

%&'(((9,8&(@6/'./7670(@;-(6(<) =(,/;(;&-./&(,(5,/;63(8,5.,+4&
- (,0(- = 6(<(=*(>9&/(- (.0(7/.-653 41(;.0'5.+7'&;(6/(<E) !=)(
'9,'(.0)

'<- ≤ /= = /) (EY1Y!*

'<- ≤ /= = '<6(<(= ≤ /=

= '<6(−! F6(<(=G ≤ 6(−! </==

= '<(≤ 6(−! </==

= 6(<6(−! </==

= /

cdfs → pdfs

]-(<(.0(,/(./@5&,0./2(-7/@'.6/(6/($\mathcal{X}$)

$$6_{\text{-}}\,\text{</=} = 6_{(}\,\text{<<}^{-!}\text{</==(-65(/} \in \mathcal{Y}\text{*}$$

$$5_{\text{-}}\,\text{</=} = \frac{4}{4/}\,6_{\text{-}}\,\text{</=} = \frac{4}{4/}\,6_{(}\,\text{<<}^{-!}\text{</==} = 5_{(}\,\text{<<}^{-!}\text{</==}\frac{4}{4/}\text{<}^{-!}\text{</=*}$$

` /('9&(6'9&5(9,/ ;)(.-(<(.0(,(;&@5&,0./2(-7/@'.6/(6/($\mathcal{X}$)

$$6_{\text{-}}\,\text{</=} = ! - 6_{(}\,\text{<<}^{-!}\text{</==(-65(/} \in \mathcal{Y}\text{*}$$

$$5_{\text{-}}\,\text{</=} = \frac{4}{4/}\,6_{\text{-}}\,\text{</=} = \frac{4}{4/}\left[! - 6_{(}\,\text{<<}^{-!}\text{</==}\right] = -5_{(}\,\text{<<}^{-!}\text{</==}\frac{4}{4/}\text{<}^{-!}\text{</=*}$$

Transformation of a pdf

%&'(((9 , 8&(? ; -(5(<) =)(4&'(- = <<) =)(P9&5&(< (.0(, (&%#%5%# , (
-7 / @'.6 / *(O7??60&(5(<) =(.0(@6 / '. / 7670(6 / ($\mathcal{X} =$ ") # 5(<) = > E$(
, / ; (<$^{-!}$</=(9 , 0(, (@6 / '. / 7670(; &5.8 , '.8&(6 /

$\mathcal{Y}=$ "/ # / $=$ <<) =(-65(06 3 &() $\in \mathcal{X}$$*

>9&/('9&(? ; -(6-(- (.0(2.8&/(+1

$$5_{-} </= = \begin{cases} 5_{(} << ^{-!} </== \left| \frac{4}{4/} < ^{-!} </= \right| & / \in \mathcal{Y} \\ E & 6'9\&5P.0\&* \end{cases}$$

Cauchy distribution

O7??60&('9 , '(5(<) = = ! T π(-65() ∈ <−π T A) π T A=(, / ; (E(6'9&5P.0&)(
'9 , '(.0)((((9 , 0(, (7 / .-65 3 (; .0'5.+7'.6 /)(, / ; (- = << (= = ' , / ((.0(, (

'5 , / 0-65 3 , '.6 / *(>9& /)(. / ('9&(. / '&58 , 4(/ ∈ <−∞) ∞=

) = <−! </= = , 5@' , / /) (

4/4/ <−! <) = = 4/4/ , 5@' , / / = !/(! + /A))

, / ; 5- </= = 5((<−! </=) |4/4/ <−! <)=| = !/π !/(! + /A)) −∞ < / < ∞*

Non-monotone transformation

O7??60&(5$_($ <) = = <$\sqrt{A\pi}$=$^{-!}$ &_?< −)^A T A=(-65() ∈ <−∞) ∞=)('9 , '(.0)

((9 , 0(, (0' , / ; , 5 ; (/ 65 3 , 4(; .0'5.+7 ' .6 /)(, / ; (- = << (= = (^{A} (.0

, ('5 , / 0-65 3 , ' .6 / *(>9& /)(. / ('9&(. / '&58 , 4(/ ∈ <E)(∞=

$$
\begin{aligned}
6_- </= &= \ ' <- \ \leq /= = \ ' < (^A \leq /= \\
&= \ ' <-\sqrt{/} \leq (\leq \sqrt{/}= \\
&= \ ' < (\leq \sqrt{/}= - \ ' < (\leq -\sqrt{/}= \\
&= 6_(<\sqrt{/}= - 6_(<-\sqrt{/}=(\\
5_- </= &= \frac{4}{4/} 6_- </= \\
&= \frac{!}{A\sqrt{/}}\left[5_(<\sqrt{/}= + 5_(<-\sqrt{/}= \right] \\
&= \frac{!}{\sqrt{A\pi}} \frac{!}{\sqrt{/}} 7^{-\frac{/}{A}})E < / < \infty *
\end{aligned}
$$

Piecewise monotone

%&'(((9,8&(?;-(5(<) =)(4&'(- = <<) =)(,/;(;&-./&('9&(0,3?4&(0?,@&(𝒳 ,0 '9&(07??65'(0&'(6-(a*(O7??60&('9&5&(&_.0'0(,(?,5'.'.6/)($E) $!)...) $,)(6- 𝒳 07@9('9,'('<(∈ $E= = E(,/;(5(<) =(.0(@6/'./7670(6/(&,@9($!* : 75'9&5)(07??60&('9&5&(&_.0'(-7/@'.6/0(<! <) =)...)<, <) =)(;&-./&;(6/ $!)...) $,)(5&0?&@'.8&41)(0,'.0-1./2

!*(<<) = = <! <) =)(-65() ∈ $!)

A*(<! <) =(.0(36/6'6/&(6/($!)

K*('9&(0&'(𝒴= "/ # / = <! <) =(-65(063&() ∈ $! $(.0('9&(0,3& (((-65(&,@9(! = !)...),)

>9&/

5- </= = { ∑,!=! 5(<<!−! </== |4 / 4/ <!−! </=| / ∈ 𝒴 ; E 6'9&5P.0&* }

Chi-squared distribution

Q6'.@&('9,'(- $= <<(= = (^{A}$

$) \in <-\infty)E=)/ = <_{!}<) = =)^{A})(=_{!}</= = -\sqrt{/})$;&@5&,0./2\

$) \in <E)+\infty=)/ = <_{A}<) = =)^{A})(=_{A}</= = \sqrt{/})$ ./@5&,0./2\

$) =$ E(<P.'9(?56+,+.4.'1(E=*

b&-./& $\$_{E} =$ "E$\ $\$_{!} = <-\infty)E=\ \$_{A} = <E)\infty=*$

>9&/ $\$_{E} \cap \$_{!} \cap \$_{A} = \not{E}(,/;(\$_{E} \cup \$_{!} \cup \$_{A} = <-\infty)\infty=*$

$$]/(\$_{!})\quad 5_{!}</= = \frac{!}{\sqrt{A\pi}}7^{-\frac{<-\sqrt{/}=^{A}}{A}}\left|-\frac{!}{A}\frac{!}{\sqrt{/}}\right| = \frac{!}{A}\frac{!}{\sqrt{A\pi}}\frac{!}{\sqrt{/}}7^{-\frac{/}{A}}$$

$$]/(\$_{A})\quad 5_{A}</= = \frac{!}{\sqrt{A\pi}}7^{-\frac{<\sqrt{/}=^{A}}{A}}\left|\frac{!}{A}\frac{!}{\sqrt{/}}\right| = \frac{!}{A}\frac{!}{\sqrt{A\pi}}\frac{!}{\sqrt{/}}7^{-\frac{/}{A}}$$

$$>9\&/)\quad 5</= = 5_{!}</= + 5_{A}</= = \frac{!}{\sqrt{A\pi}}\frac{!}{\sqrt{/}}7^{-\frac{/}{A}} \sim \chi^{A}_{!}$$

Summary

F_X; x=

6_ </== 6_(<<^{−!}</==) (<<) =(单调增时

6_ </== ! − 6_(<<^{−!}</==) (<<) =(单调减时

F_Y; y=

6<) = = ∫_{−∞}^{)} 5<: =4:

5<: = = 4/4) 6<) =

6<) = = ∫_{−∞}^{)} 5<: =4:

5<: = = 4/4) 6<) =

f_X; x=

5_ </== 5_(<<^{−!}</== 4/4) <^{−!}</=) (<<) =(单调增时

5_ </== −5_(<<^{−!}</== 4/4) <^{−!}</=) (<<) =(单调减时

f_Y; y=

Location family

O7??60&(((.0(, (5 , / ; 6 3 (8 , 5. , +4&(9 , 8. / 2(? ; -(5<) =*

H6 / 0. ; &5('9&('5 , / 0-65 3

$$- = (+ \mu*$$

O. / @&

($$(= - - \mu)$$

$$\frac{4)}{4/} = !*$$

>9&5&-65&

<</== 5</ $- \mu$=

>9&(- , 3 .41(6-(? ; -0(5<) $- \mu$=) . / ; &_& ; (+1('9&(? , 5 , 3 &'&5(μ)$-\infty < \mu < \infty$)

.0(@ , 44& ; ('9&(+%3"5)%#'7" &)+9(P.'9(65"#$"!$' : $7 (5<) =) (, / ; ($\mu$(.0(@ , 44& ;

'9&(+%3"5)%#' : "!" & , 5 , ! (-65('9&(- , 3 .41*

Normal location family

O7??60&(((.0(,(0' , / ; , 5 ; (/ 65 3 , 4(5 , / ; 6 3 (8 , 5. , +4&*(>9&/)

$$5<)= = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^{A}}{A}}$$

H6/0.; &5('9&('5, /0-65 3

$$- = (+ \mu*$$

O./@&

(

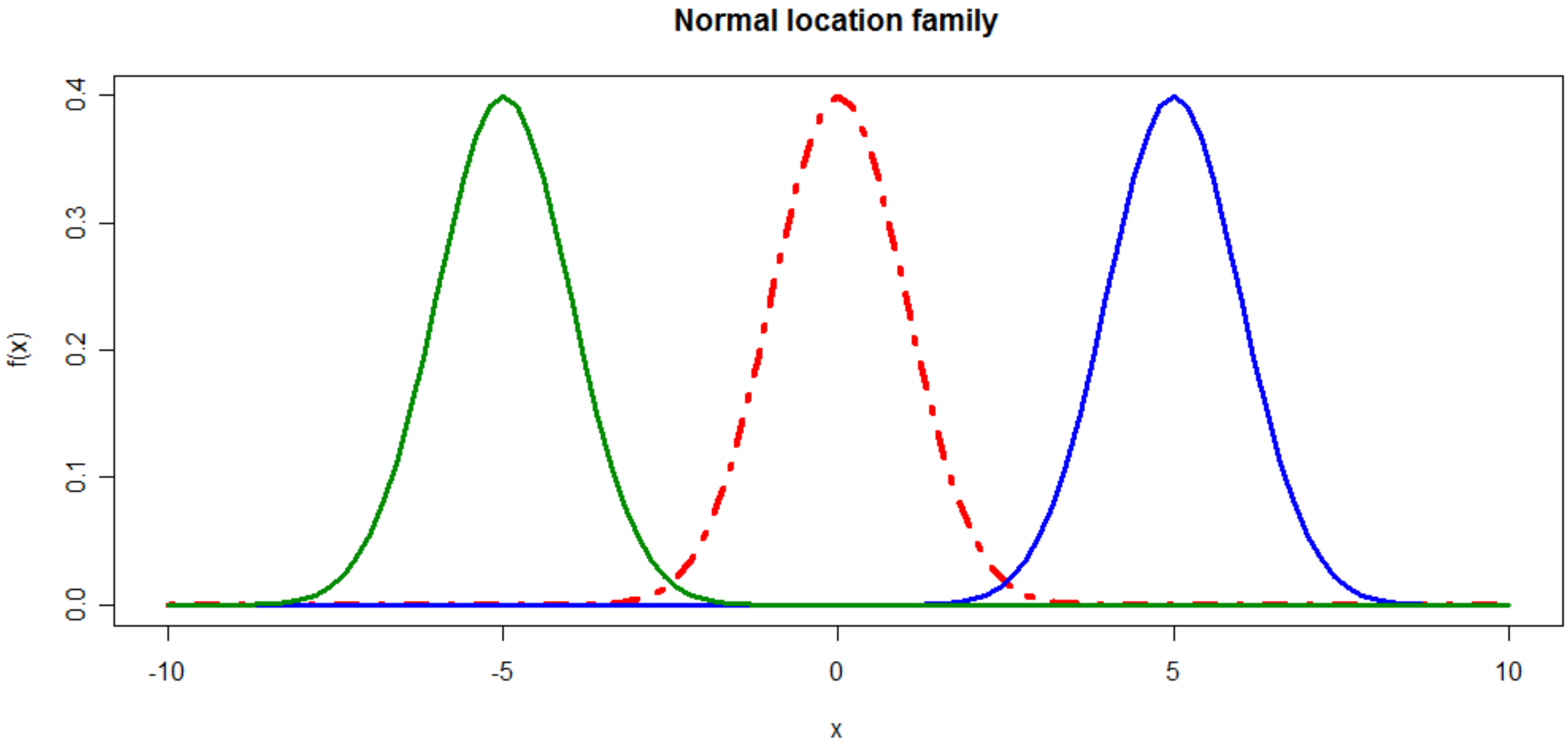
$$(= - - \mu)$$

$$\frac{4)}{4/} = !*$$

>9&5&-65&

$$5</= = \frac{!}{\sqrt{A\pi}} 7^{-\frac{</-\mu=^{A}}{A}}$$

Normal location family



Scale family

O7??60&(((.0(, (5 , / ; 6 3 (8 , 5. , +4&(9 , 8. / 2(? ; -(5 <) =*

H6 / 0. ; &5('9&('5 , / 0-65 3

$$- = \sigma (*$$

O. / @&(

$$(= - \ T \sigma)$$

$$\frac{4)}{4/} = \frac{!}{\sigma} *$$

>9&5&-65&

$$<</= = \frac{!}{\sigma} 5 \left(\frac{/}{\sigma} \right)$$

>9&(- , 3 .41(6-(? ; -0($\frac{!}{\sigma} 5 \left(\frac{)}{\sigma} \right)$) . / ; &_& ; (+1('9&(? , 5 , 3 &'&5(σ) $\sigma >$ E) (.0(@ , 44& ; '9&(63"+ , '7" &)+9 (P. '9(65"#$" !$' : $7 (5 <) =V ($\sigma$ (.0(@ , 44& ; ('9&(63"+ , ' : " ! " & , 5 , ! -65('9&(- , 3 .41*

Normal scale family

O7??60&(((.0(,(0' , / ; ,5;(/65 3 ,4(5, / ; 6 3 (8,5.,+4&*(>9&/)

$$5<)= = \frac{!}{\sqrt{A\pi}}7^{-\frac{)^{A}}{A}}$$

H6/0.;&5('9&('5,/0-65 3

$$- = \sigma (*$$

O./@&

(

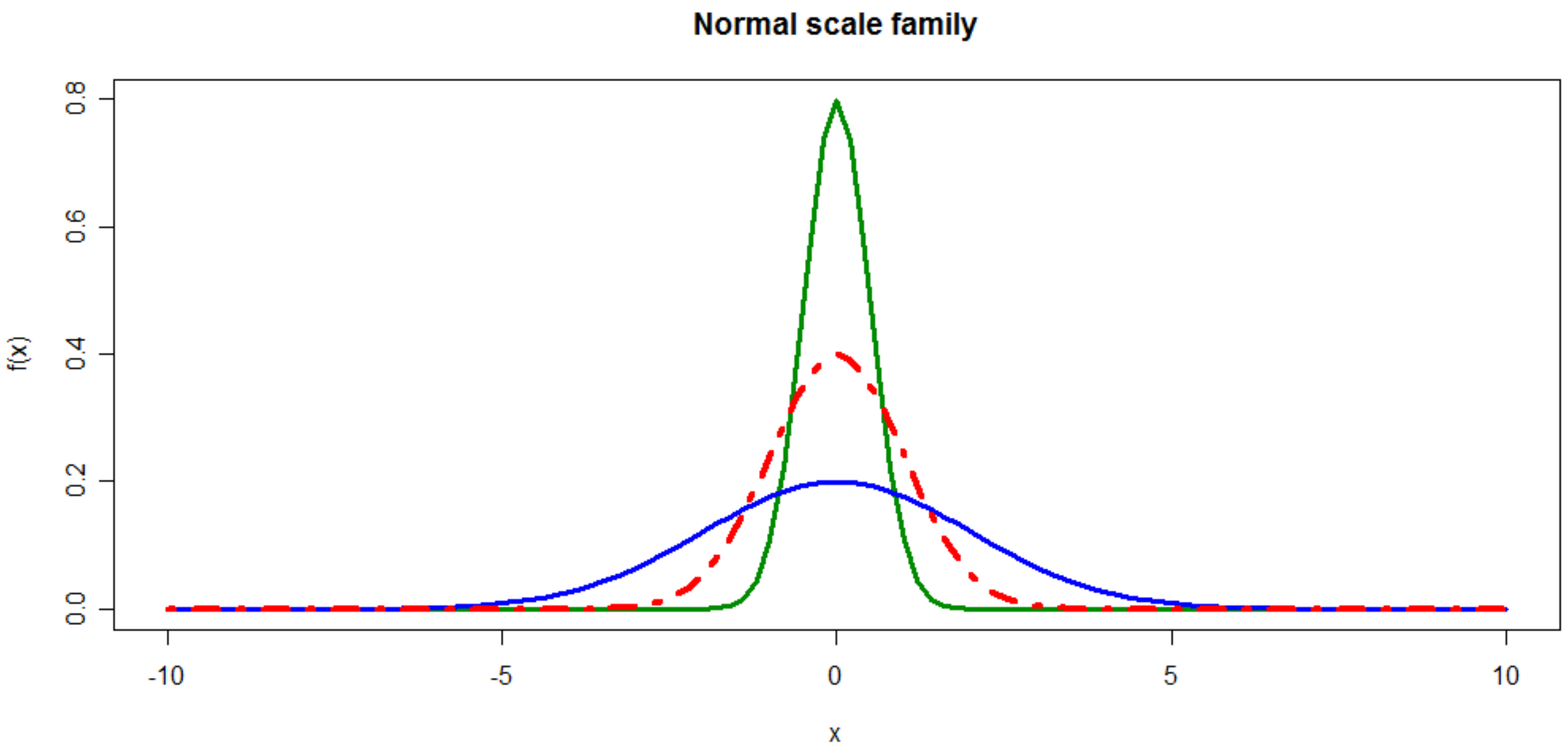
$$(= - \mathrm{T}\,\sigma)$$

$$\frac{4)}{4/} = !\,\mathrm{T}\,\sigma*$$

>9&5&-65&

$$5</= = \frac{!}{\sqrt{A\pi\sigma}}7^{-\frac{/^{A}}{A\sigma^{A}}}$$

Normal scale family



Location-scale family

O7??60&(((.0(, (5 , / ; 6 3 (8 , 5. , +4&(9 , 8. / 2(? ; -(5 <) =*

H6 / 0. ; &5('9&('5 , / 0-65 3

$$- = \sigma (+ \mu *$$

O. / @&(

$$(= <- - \mu = T \sigma)$$

$$\frac{4)}{4/} = \frac{!}{\sigma} *$$

>9&5&-65&

$$<</= = \frac{!}{\sigma} 5 \left(\frac{/ - \mu}{\sigma} \right)$$

>9&(- , 3 .41(6-(? ; -0($\frac{!}{\sigma} 5 \left(\frac{) - \mu}{\sigma} \right)$) . / ; &_& ; (+1('9&(? , 5 , 3 &'&50(μ(, / ; (σ

<σ > E=) (.0(@ , 44& ; ('9&(+%3"5)%#M63"+ , '7" &)+9(P. '9(65"#$"!$' : $7 (5 <) =V

μ(.0(@ , 44& ; ('9&(46@ , '.6 / ' : "!" & , 5 , ! (-65('9&(- , 3 .41) (, / ; (σ(.0(@ , 44& ; ('9&

63"+ , ' : "!" & , 5 , ! (-65('9&(- , 3 .41*

Normal location-scale family

O7??60&(((.0(, (0' , / ; , 5 ; (/ 65 3 , 4(5 , / ; 6 3 (8 , 5. , +4&*(>9& /)

$$5<) = = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^A}{A}}$$

H6 / 0. ; &5('9&('5 , / 0-65 3

$$- \;=\sigma (\;+\mu *$$

O. / @&

($$(\;= <- \;-\mu = \mathrm{T}\, \sigma)$$

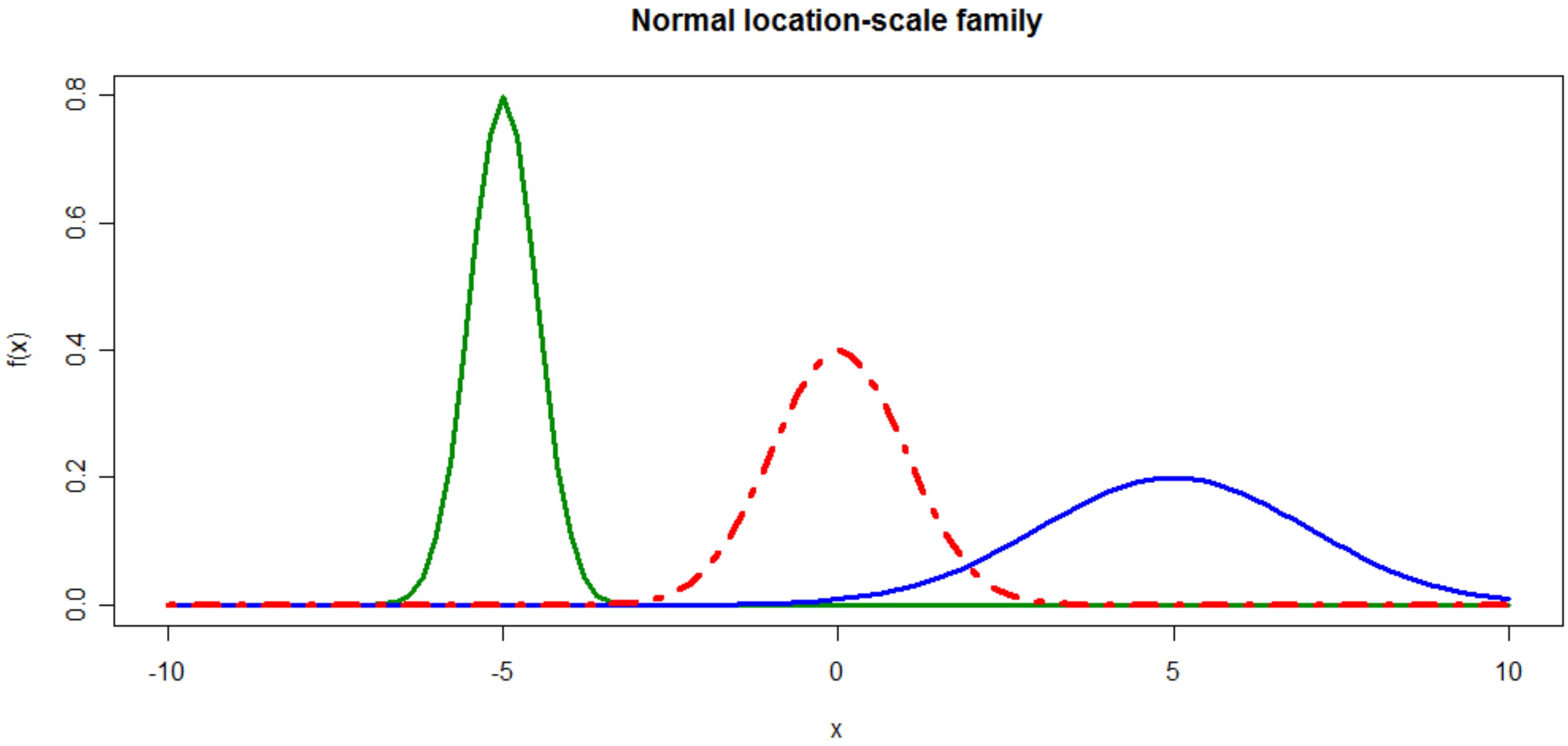
$$\frac{4)}{4/} = !\ \mathrm{T}\ \sigma *$$

>9&5&-65&

$$5</= = \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{</-\mu=^A}{A\sigma^A}}$$

>9.0(.0('9&(? ; -(6-(, (#%! & "+'$)65 !) * 45)%#*(] / (6'9&5(P65 ; 0)(- $\sim$ ><μ) σ^A=*

Normal location-scale family



Exponential families

N(-, 3 .41(6-(? ; -0(65(? 3 -0(.0(@, 44& ; (, / (7)&?"7":!8@(58 2 !@/ (
.-(.'(@, / (+&(&_?5&00& ; (, 0

$$5<) \vee \theta = \; = \; =<) = 3<\theta = \&\_? \left(\sum_{!=!}^{,} A_! <\theta = :_! <) = \right))$$

P9&5&(=<) = $\geq$ E)3<$\boldsymbol{\theta}$= $\geq$ E)$A_!$<θ=(, 5&(5& , 4(8 , 47& ; (-7 / @'.6 / 0
6-('9&(? , 5 , 3 &'&5($\boldsymbol{\theta}$)(, / ; ($:_!$<) =(, 5&(5& , 4(8 , 47& ; (-7 / @'.6 / 0
6-('9&(6+0&58 , '.6 / ()*

Binomial exponential family

- Binomial pmf

$$
\begin{aligned}
p(x \mid \mu) &= \binom{N}{x} \mu^{x} (1-\mu)^{N-x} \\
&= \binom{N}{x} (1-\mu)^{N} \left(\frac{\mu}{1-\mu} \right)^{x} \\
&= \binom{N}{x} (1-\mu)^{N} \exp\left[\ln\left(\frac{\mu}{1-\mu} \right) x \right]
\end{aligned}
$$

- Exponential family pmf

$$
p(x \mid \boldsymbol{\theta}) = h(x)\, g(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^{k} \eta_i(\theta)\, u_i(x) \right)
$$

$$
h(x) = \binom{N}{x} \qquad g(\mu) = (1-\mu)^{N}
$$

$$
\eta_1(\mu) = \ln\left(\frac{\mu}{1-\mu} \right) \qquad u_1(x) = x
$$

Normal exponential family

- Normal pdf

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$= \underbrace{1}_{h(x)} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right)}_{c(\mu,\sigma^2)} \exp\left(\underbrace{\frac{1}{2\sigma^2}}_{w_1(\mu,\sigma^2)} \underbrace{(-x^2)}_{t_1(x)} + \underbrace{\frac{\mu}{\sigma^2}}_{w_2(\mu,\sigma^2)} \underbrace{x}_{t_1(x)}\right)$$

- $d = 2$ parameters, $k = 2$ items in the sum in the exponent
  - $d < k \mapsto$ ***curved*** exponential family, e.g., $N(\mu, \mu^2)$
  - $d = k \mapsto$ ***full*** exponential family, e.g., $N(\mu, \sigma^2)$

Expectations of Random Variables

统计学方法及其应用 | 统计学基础 | 随机变量的期望

“A random variable is a quantity whose values are random and to which a probability distribution is assigned.”

Mode

Mode

>9&(&%$,(6-(,(5,/;63(8,5.,+4&(((.0('9&(8,47&('9,'
6@@750('9&(360'(-5&W7&/'41(./('9&(?56+,+.4.'1(;.0'5.+7'.6/)
@655&0?6/ ;./2('6('9&(3 ,_.373(8,47&(./('9&(?3-(65(?;-*

Median

Median

>9&(&,$)"#(6-(,(5,/;63(8,5.,+4&((((.0(,(8,47&(*2* (07@9('9,'

$$'<(\leq 2 = \geq \frac{!}{A} (,/;('<(\geq 2 = \geq \frac{!}{A}$$

: 65(,(@6/'./7670(5,/;63(8,5.,+4&((( )('9&(3&;..,/(*2* (0,'.0-.&0

$$\int_{-\infty}^{2} 5<)=4) = \int_{2}^{\infty} 5<)=4) = \frac{!}{A}$$

Expectations

Expected value

>9&(, . : ,35,$'("+4, (65(&, "#(6-(, (5, / ; 63(8, 5., +4&(<<(=)
; & / 6 ' & ; (+1(c<<) =)(.0

$$c<<) = = \begin{cases} \int_{-\infty}^{\infty} <<) =5_{(} <) =4) & .-((\ (.0(@6/'./7670 \\ \sum_{) \in \mathcal{X}} <<) =5_{(} <) = & .-((\ (.0(;\&0@5\&'\& \end{cases}$$

?568. ; & ; ('9, '('9&(./ '&25, 4(65(07 3 (&_.0'0*(]-(c $\left| <<(= \right| = \infty$)

P&(0,1('9, '(c<<(=(; 6&0(/6'(&_.0'*

Normal mode

O7??60&(((.0(,(/653 ,4(<μ) σ^{A}=(5, / ;63(8,5.,+4&*(>9&/)

$$5<)= \quad = \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{<)-\mu=^{A}}{A\sigma^{A}}}$$

$$462\ 5<)= \quad = -\frac{!}{A} 462<A\pi= - \frac{!}{A} 462<\sigma^{A}= - \frac{!}{A} <\sigma^{A}=<) - \mu=^{A}$$

` +8.67041)('9&(3 ,_.373(8,47&(.0(6+',./&;(,'() $= \mu$*

>9&5&-65&)

>9&(36;&(6-(,(/653 ,4(;.0'5.+7'.6/(.0(.'0(46@,'.6/(?,5, 3&'&5*

Normal median

O7??60&(((.0(,(/65 3 ,4(<μ) σ^A=(5, / ; 6 3 (8,5.,+4&*(>9&/)

$$5<) = \qquad = \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{<)-\mu=^A}{A\sigma^A}}$$

$$\int_{-\infty}^{\mu} 5<)=4) \quad = \int_{-\infty}^{\mu} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{<)-\mu=^A}{A\sigma^A}} 4) \xrightleftharpoons{/=)-\mu} \int_{-\infty}^{E} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{/^A}{A\sigma^A}} 4/$$

$$\int_{\mu}^{\infty} 5<)=4) = \int_{\mu}^{\infty} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{<)-\mu=^A}{A\sigma^A}} 4) \xrightleftharpoons{/=-<)-\mu=} \int_{-\infty}^{E} \frac{!}{\sqrt{A\pi}\sigma} 7^{-\frac{/^A}{A\sigma^A}} 4/$$

` +8.67041)('9&0&('P6(./'&25,40(,5&(&W7,4

>9&5&-65&)

>9&(3 &;.,/(6-(,(/65 3 ,4(;.0'5.+7'.6/(.0(.'0(46@,'.6/(?,5, 3 &'&5*

Standard normal expectation

Suppose

$$5<)= = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^{A}}{A}}) -\infty \le) < \infty$$

that is, X has an **standard normal distribution** $N(0,1)$.

Then,

$$\begin{aligned} \mathsf{c}\,(&= \int_{-\infty}^{\infty}) 7^{-\frac{)^{A}}{A}} 4) \\ &= -\int_{-\infty}^{\infty} 7^{-\frac{)^{A}}{A}} 4\left(-\frac{)^{A}}{A}\right) \\ &= 7^{-\frac{)^{A}}{A}}\Big|_{-\infty}^{\infty} \\ &= \mathsf{E} \end{aligned}$$

Cauchy expectation

- Suppose

$$f(x;\lambda) = \frac{1}{\pi}\frac{1}{1+x^2},\ -\infty < x < \infty,$$

that is, X has a Cauchy distribution, denoted as $X \sim$ Cauchy. Then,

$$\begin{aligned}
E|X| &= \int_{-\infty}^{\infty} \frac{|x|}{\pi}\frac{1}{1+x^2}dx \\
&= \frac{2}{\pi}\int_0^{\infty}\frac{x}{1+x^2}dx \\
&= \lim_{b\to\infty}\frac{2}{\pi}\int_0^{b}\frac{x}{1+x^2}dx \\
&= \frac{1}{\pi}\lim_{b\to\infty}\log(1+b^2) \\
&= \infty
\end{aligned}$$

Properties of expectation

Properties of expectation

%&'(((+&(,(5, / ;63(8,5.,+4&(, / ; (4&'(8)9) (, / ; (3(+&(@6/0' , / '0*
>9&/(-65(, /1(-7/@'.6/0(<!<(=(, / ; (<A<(=(P960&(&_?&@' , '.6/0
&_.0'0)

!* cF8<!<(= + 9<A<(= + 3G = 8c<!<(= + 9c<A<(= + 3V

A*]-(<!<(= ≥ E(-65(,44())('9&/(c<!<(= ≥ EV

K*]-(<!<(= ≥ <A<(=(-65(,44())('9&/(c<!<(= ≥ c<A<(=V

d*]-(8 ≤ <!<(= ≤ 9(-65(,44())('9&/(8 ≤ c<!<(= ≤ 9*

Moments of random variables

Moment

: 65(&W@9(. / '&2&5(")('9&(n58'&%&,#5(6-(,(5,/;63(8,5.,+4&(( )
$\mu'_{"}$)(.0(

$$\mu'_{"} = \mathrm{c}\,(\ ^{"}*$$

>9&(n58'3,#5!"+'&%&,#5(6-(( )($\mu_{"}$)(.0

$$\mu_{"} = \mathrm{c}<(\ - \mu=^{"})$$

P9&5&($\mu = \mu'_{"} = \mathrm{c}\,($*

Mean

Mean

>9&(& , "#(6-(, (5 , / ; 6 3 (8 , 5. , +4&(((.0(. '0(-.50'(3 6 3 &/ '

$\mu =$ c (*

Variance

Variance

>9&(("!)"#3,(6-(,(5, / ;63(8,5.,+4&(((.0(.'0(0&@6/ ;
@&/'5,4(363&/'

e,5(= c<(− c(=A*

>9&(?60.'.8&(0&W75&(566'(6-(e,5((.0('9&(65"#$"!$'
$,()"5)%#(6-((*

Properties of variance

Properties of variances

]-(((.0(,(5,/;63(8,5.,+4&(P.'9(-./.'&(8,5.,/@&)('9&/(-65(,/1 @6/0',/'0(8(,/;(9

e,5<8(+9= = 8^A e,5(*

0./@&(e,5<8(+9= = c<<8(+9= – C<8(+9==^A
= c<8(– 8C(=^A
= 8^A c<(– c(=^A
= 8^A e,5<(=*

e,5(= c(^A – <c(=^A

0./@&(e,5(= c<(– c(=^A
= c<(^A – A(c(+ <c(=^A=
= c(^A – A<c(=^A + <c(=^A
= c(^A – <c(=^A*

Bernoulli variance

- O7??60&

$$(= \begin{cases} ! & \text{P.'9(?56+,+.4.'1(\&} \\ E & \text{P.'9(?56+,+.4.'1(!} - \& \end{cases})$$

'9,'(.0) (**9,0(,(f &5/6744.(;.0'5.+7'.6/)(;&/6'&;(,0

($\sim$ f &5/6744.<&=*(>9&/)

$$\begin{aligned} \text{c(} &= \& \times ! + \text{<!} - \&\text{=} \times E = \& \\ \text{c(}^{A} &= \& \times ! + \text{<!} - \&\text{=} \times E = \& \\ \text{e,5(} &= \text{c(}^{A} - \text{<c(=}^{A} \\ &= \& - \&^{A} \\ &= \&\text{<!} - \&\text{=*} \end{aligned}$$

Standard normal variance

O7??60&(((.0(, (0' , / ; , 5 ; (/ 65 3 , 4(5 , / ; 6 3 (8 , 5. , +4&*(>9& /)

$$5<) = \quad = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^{A}}{A}}$$

$$c\,(\quad = \int_{-\infty}^{\infty}) <A\pi=^{-!TA} \&\_?<-)^{A}\, T\, A=4)$$

$$= -<A\pi=^{-!TA} \int_{-\infty}^{\infty} \&\_?<-)^{A}\, T\, A=4<-)^{A}\, T\, A=$$

$$= -<A\pi=^{-!TA} \&\_?<-)^{A}\, T\, A=\Big|_{-\infty}^{\infty}$$

$$= E$$

$$c\,(^{A} \quad = \int_{-\infty}^{\infty})^{A}<A\pi=^{-!TA} \&\_?<-)^{A}\, T\, A=4)$$

$$= -<A\pi=^{-!TA} \int_{-\infty}^{\infty})4\; \&\_?<-)^{A}\, T\, A=$$

$$= -<A\pi=^{-!TA})\; \&\_?<-)^{A}\, T\, A=\Big|_{-\infty}^{\infty} + <A\pi=^{-!TA} \int_{-\infty}^{\infty} \&\_?<-)^{A}\, T\, A=4)$$

$$= !$$

>9&5&-65&

$$e\,,5\,(\; = c\,(^{A} - <c\,(=^{A} = !$$

$$\int_{-\infty}^{\infty} \&\_?<-\&^{A})^{A} + D)=4) = \&\_?\left(\frac{D^{A}}{d\&^{A}}\right)\frac{\sqrt{\pi}}{\&}\;(<\& > E=$$

Skewness

Skewness

>9&(6C , D# , 66(6-(, (5 , / ; 6 3 (8 , 5. , +4&(((.0(.'0('9.5 ; (@&/ '5 , 4

3 6 3 &/ '(68&5('9&(@7+&(6-('9&(0' , / ; , 5 ; (; &8. , '.6/

$$\beta_{\#} = \mathrm{c}\left[\left(\frac{\text{(} \ - \mu}{\sigma}\right)^{K}\right] = \frac{\mu_{K}}{\sigma^{K}} *$$

Standard normal skewness

O7??60&(((.0(,(0' , / ; , 5 ; (/653 ,4(5, / ; 6 3 (8,5., +4&*(>9&/)

$$5<) = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^A}{A}}$$

$$c\,(^K = \int_{-\infty}^{\infty})^K <A\pi=^{-!TA}\ \&\_?<-)^A\ T\ A=4)$$

$$= -<A\pi=^{-!TA} \int_{-\infty}^{\infty})^A 4\ \&\_?<-)^A\ T\ A=$$

$$= -<A\pi=^{-!TA})^A\ \&\_?<-)^A\ T\ A=\Big|_{-\infty}^{\infty} + <A\pi=^{-!TA}\int_{-\infty}^{\infty} \&\_?<-)^A\ T\ A=4)^A$$

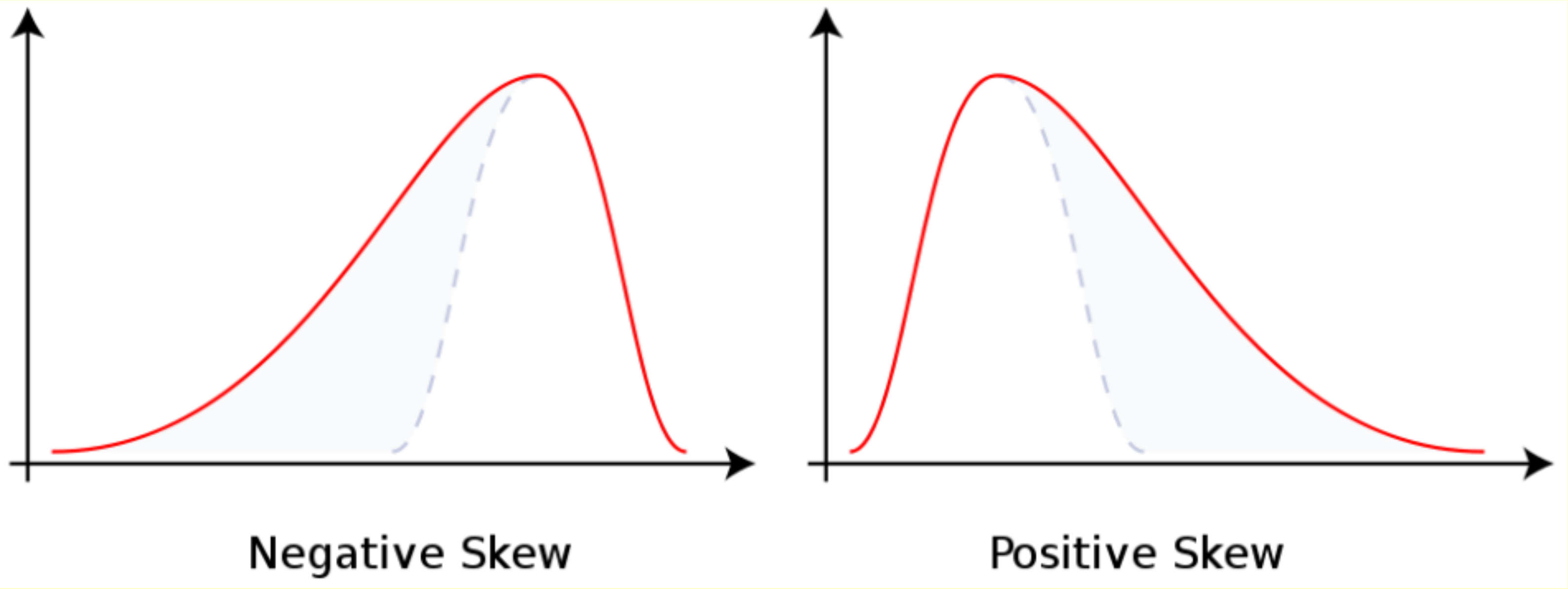
$$= -A<A\pi=^{-!TA}\int_{-\infty}^{\infty}\&\_?<-)^A\ T\ A=4<-)^A\ T\ A=$$

$$= E$$

>9&5&-65&

$$\beta_{\#} = E$$

Skewness



Kurtosis

Kurtosis

>9&(E4!5%6)6(6-(,(5,/;63(8,5.,+4&(((.0(.'0(-675'9(@&/'5,4

363&/'(68&5('9&(-675'9(?6P&5(6-('9&(0',/;,5;(;&8.,'.6/

$$\beta_{,} = c\left[\left(\frac{(- \mu}{\sigma}\right)^{d}\right] = \frac{\mu_{d}}{\sigma^{d}}*$$

Standard normal kurtosis

O7??60&(((.0(,(0' , / ; ,5;(/653 ,4(5, / ;63(8,5.,+4&*(>9&/)

$$5<)= \quad = \frac{!}{\sqrt{A\pi}} 7^{-\frac{)^{A}}{A}}$$

$$c\,(^{d} \quad = \int_{-\infty}^{\infty})^{d} <A\pi=^{-!TA} \&\_?<-)^{A}\ T\ A=4)$$

$$= -<A\pi=^{-!TA} \int_{-\infty}^{\infty})^{K} 4\ \&\_?<-)^{A}\ T\ A=$$

$$= -<A\pi=^{-!TA})^{K}\ \&\_?<-)^{A}\ T\ A=\Big|_{-\infty}^{\infty} + <A\pi=^{-!TA} \int_{-\infty}^{\infty} \&\_?<-)^{A}\ T\ A=4)^{K}$$

$$= K\left[<A\pi=^{-!TA} \int_{-\infty}^{\infty})^{A}\ \&\_?<-)^{A}\ T\ A=4)\right]$$

$$= K$$

>9&5&-65&

$$\beta_{,} = K$$

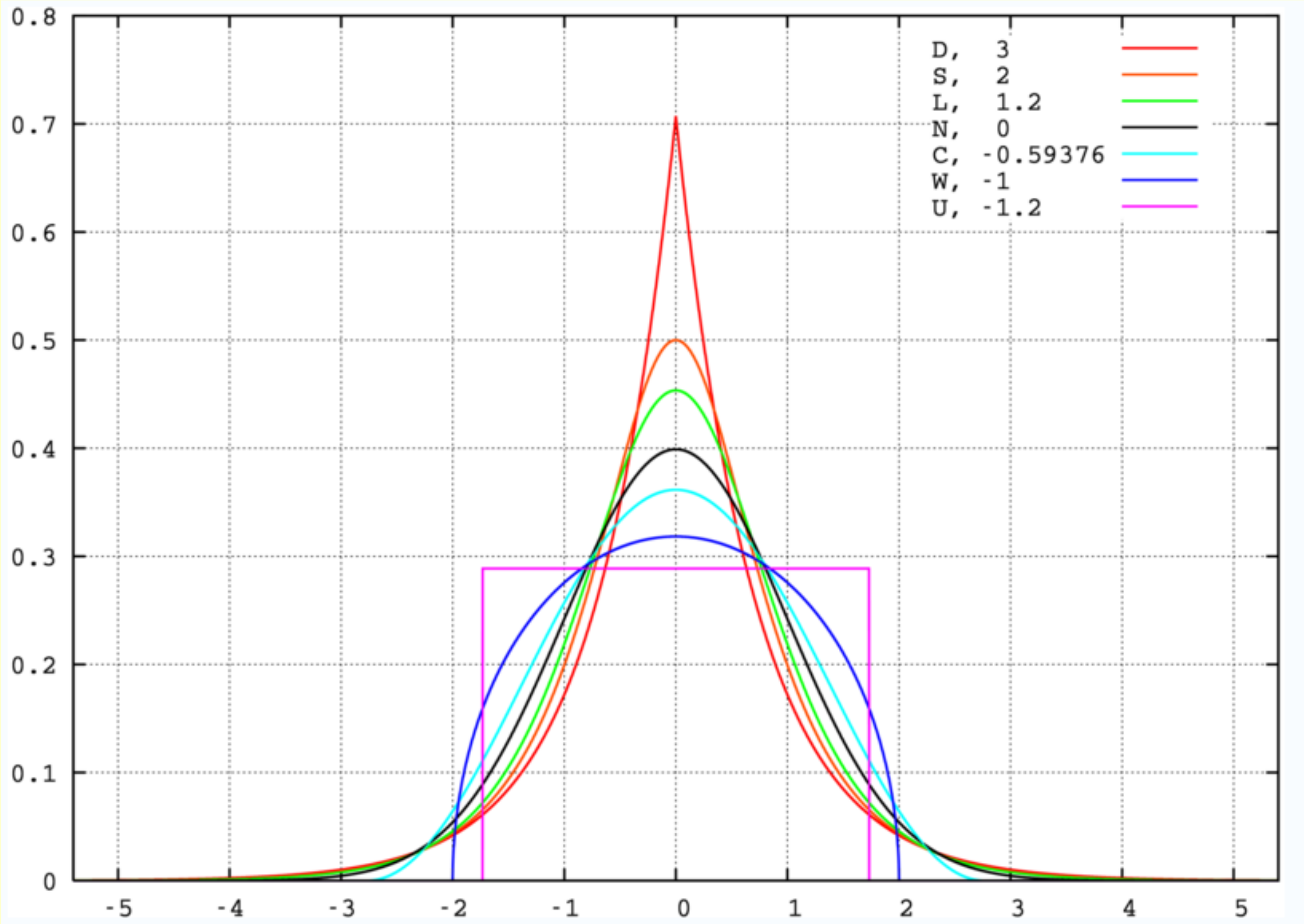
Excess kurtosis

Excess kurtosis

>9&(F.3,66'E4!5%6)6(6-(,(5,/;63(8,5.,+4&(((.0(.'0(-675'9 @&/'5,4(363&/'(68&5('9&(-675'9(?6P&5(6-('9&(0',/;,5; ;&8.,'.6/(3./70(K

$$\beta_{,} = \mathrm{c}\left[\left(\frac{(- \mu}{\sigma}\right)^{\mathrm{d}}\right] - \mathrm{K} = \frac{\mu_{\mathrm{d}}}{\sigma^{\mathrm{d}}} - \mathrm{K}^{*}$$

Excess kurtosis



Moment generating function

Moment generating function

%&'(((+&(,(5,/;63(8,5.,+4&(P.'9(@;-(6(<)=*(>9&

&%&,#5'G,#,!"5)#G'74#35)%#';&G7='6-(()(B(<:=)(.0

$$B_{(}<:= = c7^{:(})$$

?568.;&;('9,'('9&(&_?&@',',.6/(&_.0'0(-65(:(./(063&

/&.29+65966;(6-(E*

$$B_{(}<:= = \int_{-\infty}^{\infty} 7^{:)} 5_{(}<)=4)$$

$$B_{(}<:= = \sum_{)=-\infty}^{\infty} 7^{:)}\ '<(\ =\ )=$$

Normal moment generation function

$$
\begin{aligned}
B<:= &= \int_{-\infty}^{\infty} 7^{:)} \frac{!}{\sqrt{A\pi\sigma}} 7^{-\frac{<) - \mu=^A}{A\sigma^A}} 4) \\
&= \int_{-\infty}^{\infty} \frac{!}{\sqrt{A\pi\sigma}} \&\_? \left(-\frac{<) - \mu=^A}{A\sigma^A} + :) \right) 4) \\
&= \int_{-\infty}^{\infty} \frac{!}{\sqrt{A\pi\sigma}} \&\_? \left(-\frac{)^A - A\mu) + \mu^A + A\sigma^A:)}{A\sigma^A} \right) 4) \\
&= \int_{-\infty}^{\infty} \frac{!}{\sqrt{A\pi\sigma}} \&\_? \left(-\frac{)^A - A<\mu + \sigma^A:=) + <\mu + \sigma^A:=^A + \mu^A - <\mu + \sigma^A:=^A}{A\sigma^A} \right) 4) \\
&= \int_{-\infty}^{\infty} \frac{!}{\sqrt{A\pi\sigma}} \&\_? \left[-\frac{!}{A\sigma^A}\left() - <\mu + \sigma^A:=\right)^A + \left(\mu: g \frac{!}{A}\sigma^A:^A \right) \right] 4) \\
&= {\color{red}\&\_? \left(\mu: g \frac{!}{A}\sigma^A:^A \right)} \underbrace{\int_{-\infty}^{\infty} \frac{!}{\sqrt{A\pi\sigma}} \&\_? \left[-\frac{!}{A\sigma^A}\left() - <\mu + \sigma^A:=\right)^A \right] 4)}_{=!}
\end{aligned}
$$

Deriving moments from mgf

Deriving moments

]-(((9,0(32-($B_{(}$<:=)('9&/

$$c\,(^{''} = B_{(}^{<''=}<E= = \frac{4^{''}}{4:^{''}} B_{(}<:=\Big|_{:=E} \quad *$$

>9,'(.0)('9&("M'9(363&/'(.0(&W7,4('6('9&("M'9

;&5.8,'.8&(6-($B_{(}$<:=)(&8,47,'&;(,'(: = E*

Standard normal moments

O' , / ; , 5 ; (/ 6 5 3 , 4 (3 2-(. 0

$$B <:= \;=\; \&\_?\left(\frac{:^{A}}{A}\right)$$

$$\frac{4}{4)}B <:= \;=\; :\,\&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu'_{!} = E \Rightarrow \mu = E$$

$$\frac{4^{A}}{4)^{A}}B <:= \;=\; <:^{A} + !=\&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu'_{A} = ! \Rightarrow \sigma^{A} = !$$

$$\frac{4^{K}}{4)^{K}}B <:= \;=\; <:^{K} + K:=\&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu'_{K} = E \Rightarrow \beta_{\#} = E$$

$$\frac{4^{d}}{4)^{d}}B <:= \;=\; <:^{d} + h:^{A} + K=\&\_?\left(\frac{:^{A}}{A}\right) \Rightarrow \mu'_{d} = K \Rightarrow \beta_{,} = K$$

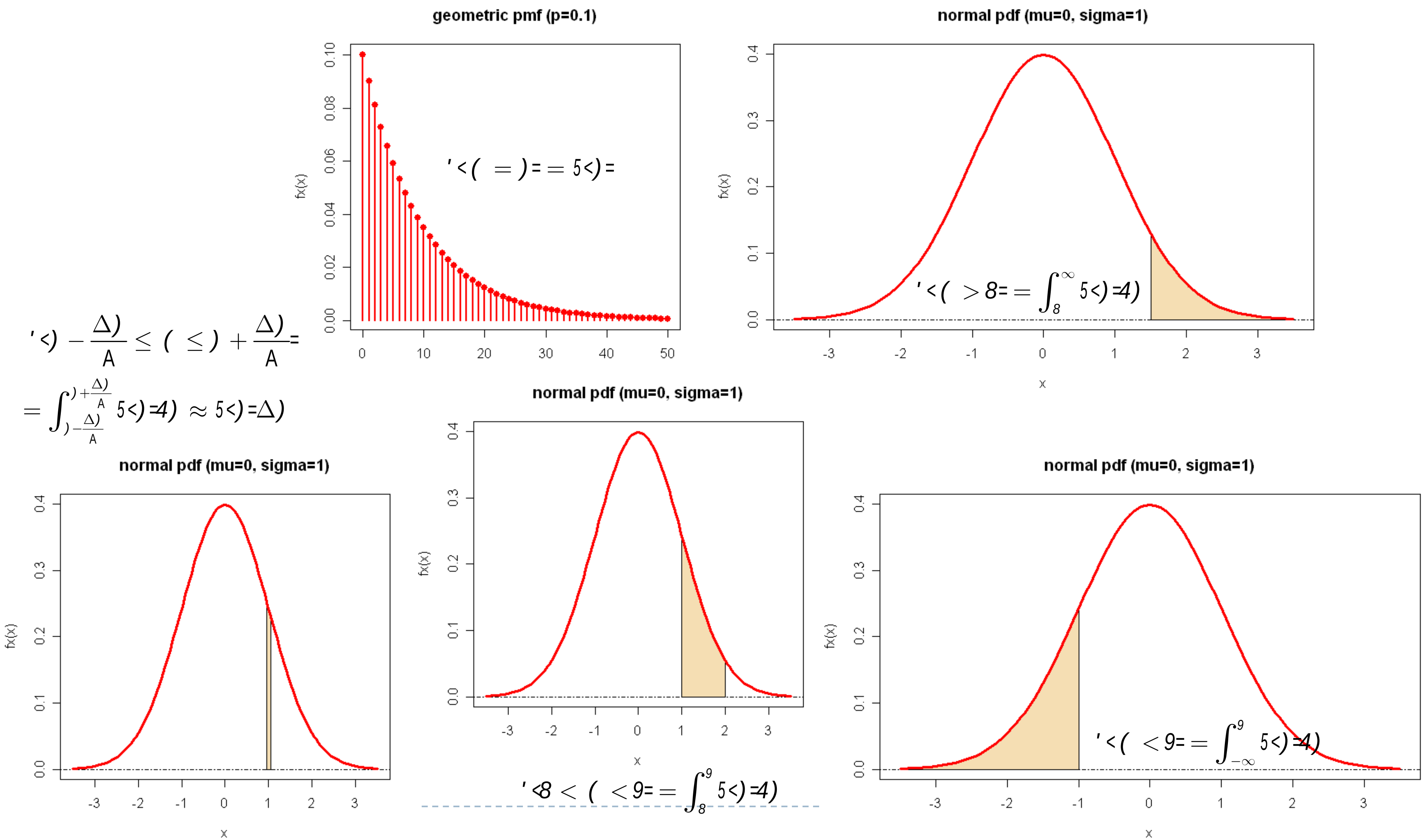
Distribution Functions of Random Variables

"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

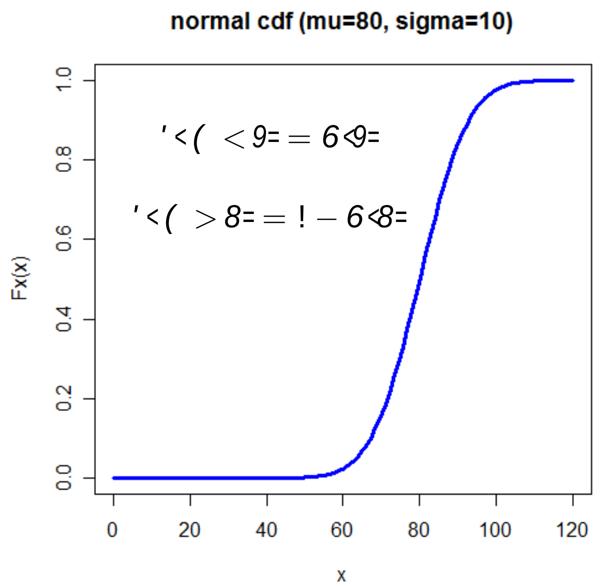
Distribution functions

- Probability mass (density) function (pmf, pdf)
  - Probability at or near a particular value
- Cumulative distribution function (cdf)
  - Probability less than or equal to a particular value
- Quantile function
  - The particular value corresponding to a probability, on the basis of the cdf
- Random numbers
  - Points distributed as the given distribution

Probability mass/density functions

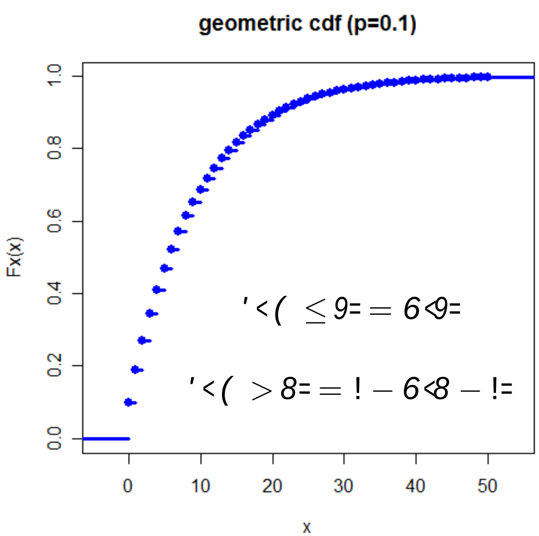


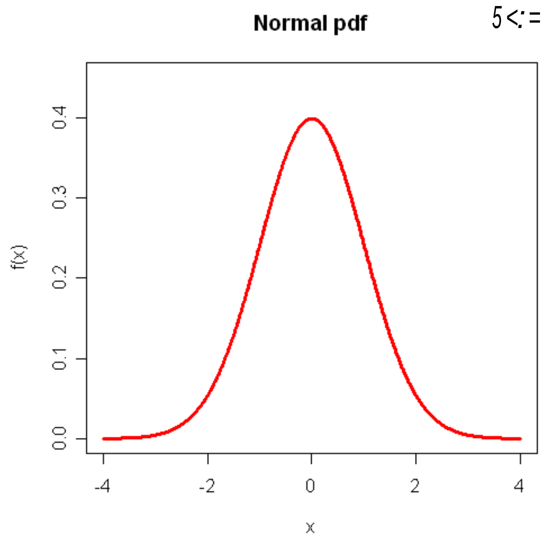
Cumulative distribution functions

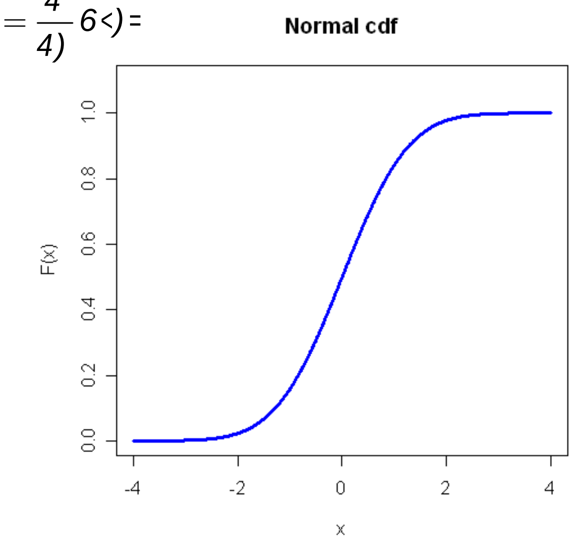


$$6<)= = \int_{-\infty}^{)} 5<:=4:$$

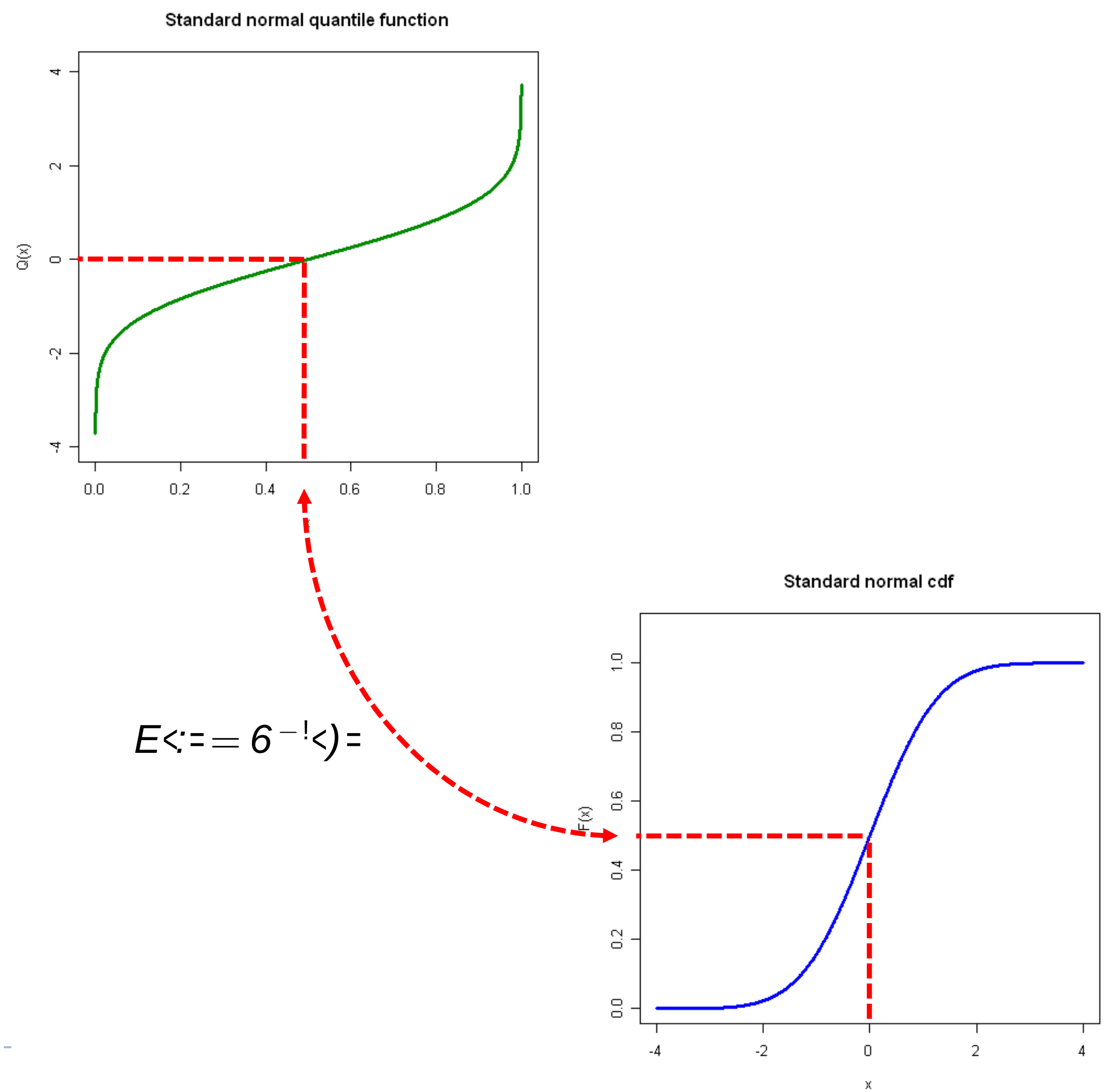
$$5<:= = \frac{4}{4)} 6<)=$$



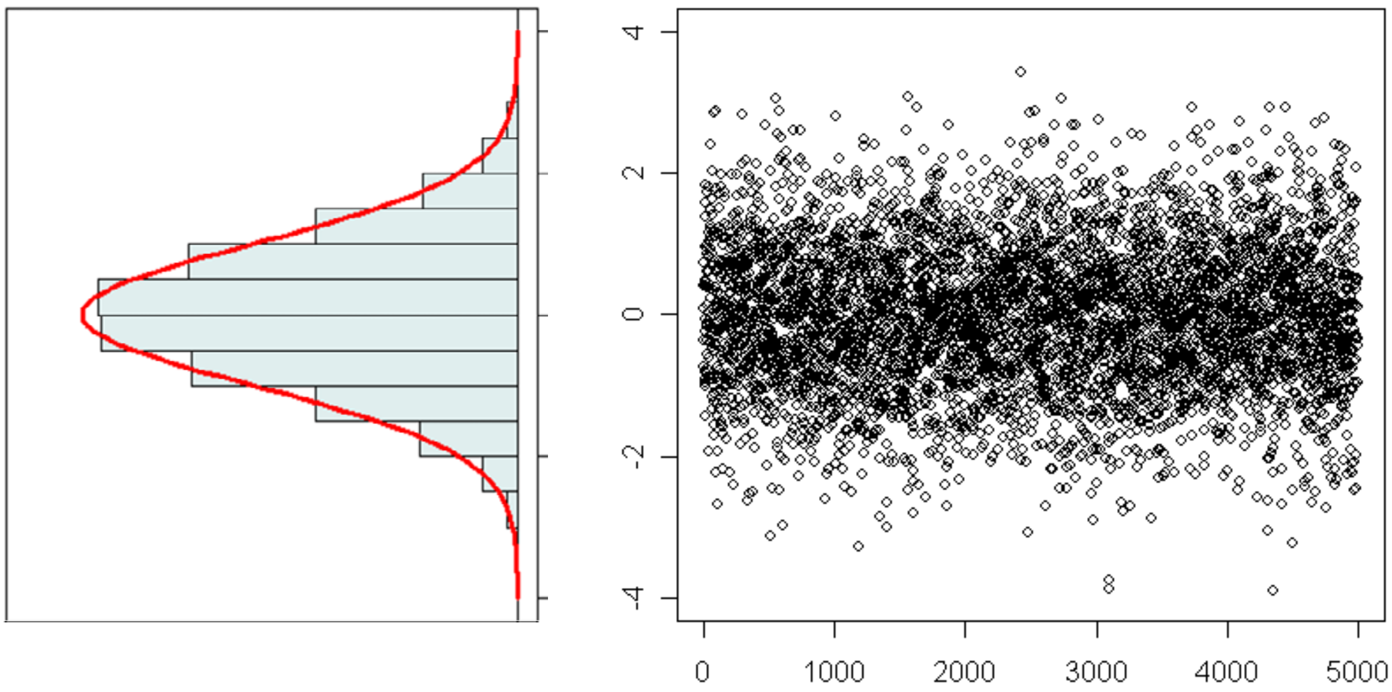




Quantile functions



Random number generators



Distribution functions in R

- ?3- 65(?;-
  5 (<)=(**d**`xxxx(x, parameters)`

- @;-
  6 (<)=   **p**`xxxx(q, parameters)`

- i 7 , / ' .4& -7 / @ ' .6 /
  6 (−!<&=(**q**`xxxx(p, parameters)`

- j , / ; 6 3 (/ 7 3 +&50
  r`xxxx(n, parameters)`

Thank you very much