

2016年秋《统计方法与应用》作业-2（随机变量）

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1 Reading.

- (a) Lecture notes 2.
- (b) Chpaters 2 and 3 of the book " Statistical Inference" .

2 In each of the following show that the given function is a cdf and find $F_X^{-1}(y)$

根据课本定理1.5.3, 验证下列函数是否满足累积分布函数的三个性质即可。

(a) 对于

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$$

有:

- $\lim_{x \rightarrow -\infty} F(x) = 0$, 因为 $x < 0$ 时, $F_X(x) = 0$; 又因为 $\lim_{x \rightarrow +\infty} e^{-x} = 0$, 所以 $\lim_{x \rightarrow +\infty} F(x) = 1$;
- 因为 e^x 是单调增函数, 因此 $1 - e^{-x}$ 还是单调增函数; 或者可以证明导数大于0; 直接求导有, $\frac{d}{dx} F_X(x) = (1 - e^{-x})' = -(e^{-x})'(-x)' = e^{-x} > 0$
- 由于 $F_X(x)$ 是连续函数, 因此 $F_X(x)$ 一定是右连续函数。

即得证。

求 $F_X^{-1}(y)$:

$$\text{有 } y = 1 - e^{-x} \Leftrightarrow$$

$$e^{-x} = 1 - y \Leftrightarrow$$

$$\ln e^{-x} = \ln 1 - y \Leftrightarrow$$

$$x = -\ln 1 - y$$

$$\text{即, } F_X^{-1}(y) = -\ln(1 - y)$$

(b) 对于

$$F_X(x) = \begin{cases} e^x/2 & \text{if } x < 0 \\ 1/2 & \text{if } 0 \leq x < 1 \\ 1 - (e^{1-x}/2) & \text{if } 1 \leq x \end{cases}$$

有:

- $\lim_{x \rightarrow -\infty} F(x) = 0$, 因为 $x < 0$ 时, $\lim_{x \rightarrow -\infty} e^x = 0$; 又因为 $\lim_{x \rightarrow +\infty} e^{1-x} = 0$, 所以 $\lim_{x \rightarrow +\infty} F(x) = 1$;
- 因为 e^x 是单调增函数, 因此 $1 - e^{-x}$ 还是单调增函数, 所以 $1 - (e^{1-x}/2)$ 还是单调增函数, 而 $F_X(x)$ 在区间 $[0, 1)$ 是常数, 常数是非单调递减, 因此 $F_X(x)$ 在整个定义域上是单调增函数;
- 由于 $F_X(x)$ 是连续函数, 因为 $\lim_{x \rightarrow 0} F(x) = 1/2$ 且 $\lim_{x \rightarrow 1} F(x) = 1/2$, 所以 $F_X(x)$ 一定是右连续函数。

即得证。

求 $F_X^{-1}(y)$:

$$\text{有当 } x \in (-\infty, 0) \text{ 时, 有 } y \in [0, 1/2], \quad y = e^x/2 \text{ (疑问2)} \Leftrightarrow$$

$$e^x = 2y \Leftrightarrow$$

$$\ln e^x = \ln(2y) \Leftrightarrow$$

$$x = \ln(2y)$$

即, 当 $y \in [0, 1/2]$ 时, $F_X^{-1}(y) = \ln 2y$

当 $x \in [1, +\infty)$ 时, 有 $y \in [1/2, 1)$, $y = 1 - (e^{1-x}/2) \Leftrightarrow$

$$e^{1-x} = 2(1-y) \Leftrightarrow$$

$$\ln e^{1-x} = \ln(2(1-y)) \Leftrightarrow$$

$$1-x = \ln(2(1-y))$$

\Leftrightarrow

$$x = 1 - \ln(2(1-y))$$

即, 当 $y \in [1/2, 1)$ 时, $F_X^{-1}(y) = 1 - \ln(2(1-y))$

故,

$$F_X^{-1}(y) = \begin{cases} \ln(2y) & \text{if } y \in [0, 1/2] \\ 1 - \ln(2(1-y)) & \text{if } y \in [1/2, 1) \end{cases}$$

(c) 对于

$$F_X(x) = \begin{cases} e^x/4 & \text{if } x < 0 \\ 1 - (e^{-x}/4) & \text{if } x \geq 0 \end{cases}$$

有:

- $\lim_{x \rightarrow -\infty} F(x) = 0$, 因为 $x < 0$ 时, $\lim_{x \rightarrow -\infty} e^x = 0$; 又因为 $\lim_{x \rightarrow +\infty} e^{-x} = 0$, 所以 $\lim_{x \rightarrow +\infty} F(x) = 1$;

- 因为 e^x 是单调增函数, 因此 $1 - e^{-x}$ 还是单调增函数, 所以 $1 - (e^{-x}/4)$ 还是单调增函数, 而 $F_X(x)$ 在区间 $[0, 1)$ 是常数, 常数是非单调递减, 因此 $F_X(x)$ 在整个定义域上是单调增函数; (略微有点疑问)

- 因为 $\lim_{x \rightarrow 0^+} F(x) = 3/4$ 且 $F(0) = 3/4$, 所以 $F_X(x)$ 是右连续函数。

即得证。

求 $F_X^{-1}(y)$:

有当 $x \in (-\infty, 0)$ 时, 有 $y \in [0, 1/4)$, $y = e^x/4 \Leftrightarrow$

$$e^x = 4y \Leftrightarrow$$

$$\ln e^x = \ln(4y) \Leftrightarrow$$

$$x = \ln(4y)$$

即, 当 $y \in [0, 1/4)$ 时, $F_X^{-1}(y) = \ln(4y)$

当 $x \in [0, +\infty)$ 时, 有 $y \in [3/4, 1)$, $y = 1 - (e^{-x}/4) \Leftrightarrow$

$$e^{-x} = 4(1 - y) \Leftrightarrow$$

$$\ln e^{-x} = \ln 4(1 - y) \Leftrightarrow$$

$$x = -\ln(4(1 - y))$$

即, 当 $y \in [1/4, 1)$ 时, $F_X^{-1}(y) = 1 - \ln(4(1 - y))$

故,

$$F_X^{-1}(y) = \begin{cases} \ln(4y) & \text{if } y \in [0, 1/4) \\ 1 - \ln(4(1 - y)) & \text{if } y \in [3/4, 1) \end{cases}$$

3 Let X have the pdf,

$$f(x) = \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-x^2/\beta^2}, x \in [0, \infty), \beta > 0$$

(a) Verify $f(x)$ is a valid pdf.

证明: 两个性质不难证明性质a, 即 $f_X(x) \geq 0$.

不难推导,

$$\int_0^\infty f(x) dx = \int_0^\infty \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-x^2/\beta^2} dx \approx 1$$

即可证 $f(x)$ 是概率密度函数。

(b) Find $\mathbb{E}(X)$ and $\text{Var} X$.

解: 首先, 因为,

$$\int_0^\infty x e^{-x} dx = \Gamma(2) = 1;$$

先求期望,

$$\mathbb{E}X = \int_0^\infty \frac{4}{\beta^3 \sqrt{\pi}} x^3 e^{-x^2/\beta^2} dx$$

令 $t = x/\beta$, 有

$$\mathbb{E}X = \frac{4\beta}{\sqrt{\pi}} \int_0^\infty t^3 e^{-t^2} dt$$

再令 $m = t^2$, 有

$$\mathbb{E}X = \frac{4\beta}{\sqrt{\pi}} \frac{1}{2} \int_0^\infty m e^{-m} dm$$

进而,

$$\mathbb{E}X = \frac{2\beta}{\sqrt{\pi}} \int_0^\infty me^{-m} dm = \frac{2\beta}{\sqrt{\pi}}$$

再求平方的期望, 因为

$$\int_{-\infty}^\infty x^2 e^{-x^2} dx = \int_0^\infty xe^{-x^2} 2x dx = \int_0^\infty u^{\frac{1}{2}} e^{-u} du = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

, 且这个函数是关于0对称, 因此

$$\int_{-\infty}^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\mathbb{E}X = \int_0^\infty \frac{4}{\beta^3 \sqrt{\pi}} x^4 e^{-x^2/\beta^2} dx$$

令 $t = x/\beta$, 有

$$\mathbb{E}X = \frac{4\beta^2}{\sqrt{\pi}} \int_0^\infty t^4 e^{-t^2} dt$$

$$\mathbb{E}X = \frac{4\beta^2}{\sqrt{\pi}} \left(-\frac{1}{2}\right) \int_0^\infty t^3 e^{-t^2} d(-t^2)$$

进而,

$$\begin{aligned} \mathbb{E}X &= -\frac{2\beta^2}{\sqrt{\pi}} \int_0^\infty t^3 d(e^{-t^2}) = -\frac{2\beta^2}{\sqrt{\pi}} \left(t^3 e^{-t^2} \Big|_0^{+\infty} - \int_0^\infty e^{-t^2} d(t^3) \right) \\ &= -\frac{2\beta^2}{\sqrt{\pi}} \left(t^3 e^{-t^2} \Big|_0^{+\infty} - \int_0^\infty 3t^2 e^{-t^2} dt \right) \\ &= -\frac{2\beta^2}{\sqrt{\pi}} \left(-3 \frac{\sqrt{\pi}}{4} \right) = \frac{3\beta^2}{2} \end{aligned}$$

因此方差 $VarX = \mathbb{E}(x^2) - (\mathbb{E}(x))^2 = \frac{3\beta^2}{2} - \left(\frac{2\beta}{\sqrt{\pi}}\right)^2$

4 证明

(a) 设X是连续且非负的随机变量, 证明 $\mathbb{E}X = \int_0^\infty [1 - F_X(x)] dx$

证明: 由于 $F_X(x) = P(X \leq x)$, 且, $1 - F_X(x) = P(X > x)$

那么, 有

$$\int_0^{\infty} (1 - F_X(x))dx = \int_0^{\infty} P(X > x)dx$$

而根据定义

$$\begin{aligned} EX &= \int_0^{\infty} x f_X(x)dx \\ &= \int_0^{\infty} \int_x^{\infty} f_X(x)dydx \\ &= \int_0^{\infty} \int_0^y dx f_X(y)dy \\ &= \int_0^{\infty} y f_X(y)dy \\ &= \int_0^{\infty} x f_X(x)dx \end{aligned}$$

即

$$= EX$$

故得证。

(b) 设 X 是取值为非负整数的离散随机变量, 证明: $EX = \sum_{k=0}^{\infty} (1 - F_X(k))$

证明:

5 设 $f(x)$ 为一概率密度函数, 如果存在数 a 使得: 对于任意 $\varepsilon > 0$ 都有 $f(a + \varepsilon) = f(a - \varepsilon)$, 则称 $f(x)$ 关于 a 对称。

(a) 三个对称的概率密度函数:

- 正态分布:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- 柯西分布:

$$f(x) = \frac{1}{\pi(1+x^2)}$$

- 罗吉斯蒂克概率函数:

$$f(x) = \frac{1}{1+e^{-x}}$$

(b) 因为概率分布的中位数满足 $P(X \leq m) \geq \frac{1}{2}$ 且 $P(X \geq m) \geq \frac{1}{2}$,

$$\int_{-\infty}^a f(x)dx = \int_a^{\infty} f(2a - x)dx = 1/2$$

.

令 $\varepsilon = x - a$ 那么有,

$$\int_a^{\infty} f(x)dx = \int_0^{\infty} f(a + \varepsilon)d\varepsilon = \int_0^{\infty} f(a - \varepsilon)d\varepsilon$$

令 $x = a - \varepsilon$ 那么有上式等于

$$= \int_{-\infty}^a f(x)dx$$

即

$$\int_a^{\infty} f(x)dx = \int_{-\infty}^a f(x)dx$$

而对于一个概率密度函数有

$$\int_a^{\infty} f(x)dx + \int_{-\infty}^a f(x)dx = 1$$

, 因此有

$$\int_a^{\infty} f(x)dx = \int_{-\infty}^a f(x)dx = \frac{1}{2}$$

,

那么根据中位数的性质, 可得该函数的中位数就是 a 。

(c) 根据期望的定义可得,

$$EX = \int_{-\infty}^{\infty} xf(x)dx$$

且有,

$$EX - a = E(X - a)$$

因此

$$EX - a = E(X - a) = \int_{-\infty}^{\infty} (x - a)f(x)dx$$

此时令 $\varepsilon = x - a$, 有上式等于

$$= \int_0^\infty -\varepsilon f(a - \varepsilon) d\varepsilon + \int_0^\infty \varepsilon f(a + \varepsilon) d\varepsilon$$

又有对称函数的性质, $f(a + \varepsilon) = f(a - \varepsilon)$ 可得, 上式为0

即

$$EX - a = 0$$

, 因此有

$$EX = a$$

(d) 对于 $f(x) = e^{-x}$ 有, $f(a + \varepsilon) = e^{-a-\varepsilon}$, $f(a - \varepsilon) = e^{-a+\varepsilon}$,

$$\text{可得 } \frac{f(a+\varepsilon)}{f(a-\varepsilon)} = \frac{e^{-\varepsilon}}{e^{\varepsilon}} = \frac{1}{e^{2\varepsilon}}$$

因为 $\varepsilon \geq 0$, 因此 $\frac{1}{e^{2\varepsilon}} \neq 1$, 因此 $f(x) = e^{-x}$ 不是对称的概率密度函数。

(e) 对于 $f(x) = e^{-x}$, 可求得中值为 $\log(2)$, 而期望 $EX = \int_{-\infty}^\infty xf(x)dx = 1$

即中位数小于期望。

6 求下列分布的矩母函数

(a) $f(x) = \frac{1}{c}, 0 < x < c$;

解: 根据矩母函数的定义, 有:

$$Ee^{tX} = \int_0^c e^{tx} f(x) dx$$

$$= \int_0^c e^{tx} \frac{1}{c} dx = \frac{1}{ct} e^{tx} \Big|_0^c = \frac{1}{ct} (e^{tc} - 1)$$

(b) $f(x) = \frac{2x}{c^2}, 0 < x < c$;

解: 根据矩母函数的定义, 有:

$$Ee^{tX} = \int_0^c e^{tx} f(x) dx$$

$$= \int_0^c e^{tx} \frac{2x}{c^2} dx = \frac{2x}{c^2} e^{tx} \Big|_0^c = \frac{2}{c^2 t^2} (ct e^{tc} - e^{tc} + 1)$$

(c) $f(x) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta}$, $-\infty < x < \infty$, $-\infty < \alpha < \infty$, $\beta > 0$;

解: 根据矩母函数的定义, 有:

$$\begin{aligned} Ee^{tX} &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\beta} e^{-|x-\alpha|/\beta} dx \\ &= \int_{-\infty}^{\alpha} e^{tX} \frac{1}{2\beta} e^{(x-\alpha)/\beta} dx + \int_{\alpha}^{\infty} e^{tX} \frac{1}{2\beta} e^{-(x-\alpha)/\beta} dx \\ &= \frac{e^{-\alpha/\beta}}{2\beta} \frac{1}{1/\beta + t} e^{-x(1/\beta+t)} \Big|_{-\infty}^{\alpha} - \frac{e^{\alpha/\beta}}{2\beta} \frac{1}{1/\beta - t} e^{-x(1/\beta-t)} \Big|_{\alpha}^{\infty} \\ &= \frac{4}{4 - \beta^2 t^2} e^{\alpha t} \end{aligned}$$

7 求出下列Y的概率密度函数

(a) $Y = X^2$ and $f_X(x) = 1, 0 < x < 1$

解: 令 $Y = g(x)$, 则 $g^{-1}(y) = y^{1/2}$ 且 $\frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}$

对于, $0 < x < 1$, 有 $Y = g(x)$ 是单调增函数, 因此由课本定理2.1.5可得, 概率密度函数 $f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$, $0 < y < 1$

即

$$f_Y(y) = 1 * \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}$$

, 且 $0 < y < 1$.

(b) $Y = -\log(X)$ and X has pdf, $f_X(x) = \frac{(m+n+1)!}{n!m!} x^n (1-x)^m$, $0 < x < 1$, m, n 为正整数。

解: 令 $Y = g(x)$, 则 $g^{-1}(y) = e^{-y}$ 且 $\frac{d}{dy} g^{-1}(y) = -e^{-y}$

对于, $0 < x < 1$, 有 $Y = g(x)$ 是单调减函数, 因此由课本定理2.1.5可得, 概率密度函数 $f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$, $0 < y < 1$

即

$$f_Y(y) = -\frac{(m+n+1)!}{n!m!} (e^{-y})^n (1-e^{-y})^m - e^{-y} = \frac{(m+n+1)!}{n!m!} e^{-y(n+1)} (1-e^{-y})^m$$

, $0 < y < \infty$.

(c) $Y = e^X$ and X has pdf, $f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2}$, $0 < x < \infty$, σ^2 为正数。

解: 令 $Y = g(x)$, 则 $g^{-1}(y) = \log y$ 且 $\frac{d}{dy} g^{-1}(y) = 1/y$

对于, $0 < x < 1$, 有 $Y = g(x)$ 是单调增函数, 因此由课本定理2.1.5可得, 概率密度函数 $f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$, $0 < y < 1$

即

$$f_Y(y) = \frac{1}{\sigma^2} (\log y) e^{-(\log y/\sigma)^2/2} * (1/y) = \frac{\log y}{y\sigma^2} e^{-(\log y/\sigma)^2/2}$$

, $0 < y < \infty$.

8 A random variable X is said to have a Gamma distribution if its pdf is:

$$f(x|shape = \alpha, scale = \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, x \in [0, \infty), \alpha > 0, \theta > 0$$

(a) Verify $f(x|\alpha, \theta)$ is a valid pdf.

证明: 两个性质不难证明性质a, 即 $f_X(x) \geq 0$.

不难推导,

$$\begin{aligned} f_X(x|\alpha, \theta) &= \int_{-\infty}^{\infty} f_X(x|\alpha, \theta) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta} dx \approx 1 \end{aligned}$$

即可证 $f(x|\alpha, \theta)$ 是概率密度函数。

(b) Find the mode of a Gamma random variable (for $\alpha > 1$);

解: 当 $\alpha > 1$ 时, $f(x)$ 先递增, 后递减, mode 为 $(\alpha - 1)\theta$

(c) Find the moment generating function $M(t)$ of a Gamma random variable;

解：根据 $\Gamma(\alpha)$ 函数的性质，其对应的矩母函数为：

$$\begin{aligned} M_X(t) &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^{+\infty} e^{tx} x^{\alpha-1} e^{-x/\theta} dx \\ &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^{+\infty} x^{\alpha-1} e^{-(1/\theta-t)x} dx \end{aligned}$$

根据伽玛函数的性质,对于任意大于0的常数 α, β :

$$f(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}$$

,都是某随机变量的概率密度函数, 于是

$$\int_0^{+\infty} \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} dx = 1$$

也就是

$$\int_0^{+\infty} x^{a-1} e^{-x/b} dx = \Gamma(a)b^a$$

即得：当 $t < 1/\theta$ 有

$$M_X(t) = \frac{1}{\Gamma(\alpha)\theta^\alpha} \Gamma(\alpha) \left(\frac{\theta}{1-\theta t} \right)^\alpha = \left(\frac{1}{1-\theta t} \right)^\alpha$$

而当 $t \geq 1/\theta$ 时, 没有矩母函数, 因为 $M_X(x)$ 。

(d) Find mean , variance of X , the skewness and the kurtosis of a Gamma random variable;

解：先求期望,

$$EX = \frac{d}{dx} M_X(t)|_{t=0} = \frac{\alpha\theta}{(1-\theta t)^{\alpha+1}}|_{t=0} = \alpha\theta$$

再求平方的期望,

$$EX^2 = \frac{d^{(2)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{d}{dx} \frac{\alpha\theta}{(1-\theta t)^{\alpha+1}}|_{t=0} = \frac{(\alpha+1)\alpha\theta^2}{(1-\theta t)^{\alpha+2}}|_{t=0} = (\alpha+1)\alpha\theta^2$$

, 因此方差 $Var X = EX^2 - (EX)^2 = \alpha\theta^2$

The skewness of a random variable X is its third central moment ,因此

$$EX^3 = \frac{d^{(3)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{2}{\sqrt{\alpha}}$$

The Kurtosis of a random variable X is its fourth central moment, 因此

$$EX^4 = \frac{d^{(4)}}{dx} M_X^{(2)}(t)|_{t=0} = 3 + \frac{6}{\alpha}$$

(e) Let $Y = 1/X$. What is the pdf of Y ? (Y is said to have an inverse gamma distribution)

解: 令 $Y = 1/X$, 则有, $\frac{dx}{dy} = \frac{1}{y^2}$, 且 Y 的概率密度函数为:

$$\begin{aligned} f_Y(y|\alpha, \theta) &= f_X(y|\alpha, \theta) \frac{dx}{dy} \\ f_Y(y|\alpha, \theta) &= \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{1}{y}\right)^{\alpha-1} e^{-1/(\theta y)} \left| \frac{1}{y^2} \right| \\ f_Y(y|\alpha, \theta) &= \frac{1}{\Gamma(\alpha)\theta^\alpha} y^{-\alpha-1} e^{-1/(\theta y)} \end{aligned}$$

用 β 替换 θ^{-1} 得:

$$f_Y(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\beta/y}$$

因此, Y 的概率密度函数为逆Gamma分布。

(f) Find the mean, the variance, the skewness and the kurtosis of a inverse Gamma random variable; 解: 计算公式, 当 $\alpha > n$ 时,

$$EX^n = \frac{d^{(n)}}{dx} M_X(t)|_{t=0} = \frac{\beta^n}{(\alpha-1)(\alpha-2)\dots(\alpha-n)}$$

那么对于 $\alpha > 1$ 时, 期望

$$EX = \frac{d}{dx} M_X(t)|_{t=0} = \frac{\beta^n}{(\alpha-1)(\alpha-2)\dots(\alpha-n)} \Big|_{n=1} = \frac{\beta}{\alpha-1}$$

, 再求平方的期望,

$$EX^2 = \frac{d^{(2)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{\beta^n}{(\alpha-1)(\alpha-2)\dots(\alpha-n)} \Big|_{n=2} = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$$

, 因此方差 $VarX = EX^2 - (EX)^2 = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$

那么对于 $\alpha > 3$ 时, skewness:

$$EX^3 = \frac{d^{(3)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{4\sqrt{(\alpha-2)}}{\alpha-3}$$

那么对于 $\alpha > 4$ 时, Kurtosis :

$$EX^4 = \frac{d^{(4)}}{dx} M_X^{(2)}(t)|_{t=0} = \frac{(30 * \alpha - 66)}{((\alpha - 3) * (\alpha - 4))}$$

9 A random variable X is said to have a Poisson distribution if its pdf is:

$$f(x|shape = \alpha, scale = \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, x \in [0, \infty), \alpha > 0, \theta > 0$$

(a) Verify $f(X = k)$ is a valid pdf.

证明: 两个性质不难证明性质a, 即 $f(X = k) \geq 0$ 。

不难推导,

$$\begin{aligned} f(X = k) &= \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, \dots \\ &= \sum_{x=0}^{\infty} P(X = x|\lambda) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1 \end{aligned}$$

即可证 $f(X = k)$ 是概率密度函数。

(b) Find the moment generating function M(t) of a Gamma random variable;

解: 矩母函数为:

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!} = e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)} \end{aligned}$$

(c) Find the mean, the variance, the skewness and the kurtosis of a Poisson random variable;

解: 先求期望,

$$EX = \frac{d}{dx} M_X(t)|_{t=0} = e^{\lambda(e^t-1)} \lambda e^t|_{t=0} = \lambda$$

再求平方的期望,

$$EX^2 = \frac{d^{(2)}}{dx} M_X^{(2)}(t)|_{t=0} = \lambda^2 + \lambda$$

, 因此方差 $Var X = EX^2 - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

The skewness of a random variable X is its third central moment ,因此

$$EX^3 = \frac{d^{(3)}}{dx} M_X^{(2)}(t)|_{t=0} = \lambda^{-1/2}$$

The Kurtosis of a random variable X is its fourth central moment,因此

$$EX^4 = \frac{d^{(4)}}{dx} M_X^{(2)}(t)|_{t=0} = \lambda^{-1}$$

10 Show that

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}, \alpha = 1, 2, 3, \dots$$

(Hint: Use integration by parts.) Express this formula as a probabilistic relationship between Poisson and gamma random variables.

解: 因为 $\alpha = 1, 2, 3, \dots$, 即 α 是正整数, 故有 $\Gamma(\alpha) = (\alpha - 1)!$

因此有

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz$$

$$= \frac{1}{(\alpha-1)!} \int_x^\infty z^{\alpha-1} e^{-z} dz$$

令 $u = z^{\alpha-1}$, $dv = e^{-z}$ 进行分部积分, 有上式

$$\begin{aligned} &= \frac{1}{(\alpha-1)!} \left(-z^{\alpha-1} e^{-z} \Big|_x^\infty + \int_x^\infty (\alpha-1) z^{\alpha-2} e^{-z} dz \right) \\ &= \frac{z^{\alpha-1} e^{-z}}{(\alpha-1)!} + (\alpha-1) \int_x^\infty z^{\alpha-2} e^{-z} dz \end{aligned}$$

重复上述分部积分过程, 可得:

$$\begin{aligned} &= \frac{z^{\alpha-1} e^{-z}}{(\alpha-1)!} + \frac{z^{\alpha-2} e^{-z}}{(\alpha-2)!} + (\alpha-2) \int_x^\infty z^{\alpha-3} e^{-z} dz \\ &= \frac{z^{\alpha-1} e^{-z}}{(\alpha-1)!} + \frac{z^{\alpha-2} e^{-z}}{(\alpha-2)!} + \dots + \frac{z e^{-z}}{1!} + (1) \int_x^\infty z^0 e^{-z} dz \end{aligned}$$

令 $y = \alpha - 1$, 代入到上式可得

$$= \sum_{y=0}^{\alpha-1} \frac{z^y e^{-z}}{y!}$$

因此, 可得题中左右两式相等, 而等式左边为 $P(X \geq x)$, 等式右边为 $P(Y \leq y)$, 即可得出结论, 如果 $X \sim \Gamma(\alpha, 1)$ 且 $Y \sim \text{Poisson}(x)$, 那么它们就有一种关系:

$$P(X \geq x) = P(Y \leq \alpha - 1)$$