3B1B Optimization Examples

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1. The Rosenbrock function is
$$f(x, y) = 100(y - x^2)^2 + (1 - x)^2$$
   (a) Compute the gradient and Hessian of $f(x, y)$.
   (b) Show that that $f(x, y)$ has zero gradient at the point $(1, 1)$.
   (c) By considering the Hessian matrix at $(x, y) = (1, 1)$, show that this point is a minimum.

2. In Newton type minimization schemes the update step is of the form
$$\delta \mathbf{x} = -\mathtt{H}^{-1}\mathbf{g}$$
   where $\mathbf{g} = \nabla f$. By considering $\mathbf{g}.\delta \mathbf{x}$ compare the convergence of:
   (a) Newton, to
   (b) Gauss Newton
   for a general function $f(\mathbf{x})$ (i.e. where $\mathtt{H}$ may not be positive definite).

3. Explain how you could use the Gauss Newton method to solve a set of simultaneous non-linear equations.

4. Sketch the feasible regions defined by the the following inequalities and comment on the possible optimal values.
   (a)
$$\begin{aligned} -x_1 + x_2 &\geq 2 \\ x_1 + x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$
   (b)
$$\begin{aligned} 2x_1 - x_2 &\leq 2 \\ x_1 &\leq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

5. More on linear programming.
   (a) Show that the optimization
$$\min_{\mathbf{x}} \sum_i |\mathbf{a}_i{}^\top \mathbf{x} - b_i|$$
   where the vectors $\mathbf{a}_i$ and scalars b_i are given, can be formulated as a linear programming problem.

(b) Solve the following linear programming problem using Matlab:

$$\begin{array}{rcl} \max\limits_{x_1,x_2} & & 40x_1 + 88x_2 \\ \text{subject to} & & \\ 2x_1 + 8x_2 & \leq & 60 \\ 5x_1 + 2x_2 & \leq & 60 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array}$$

6. Interior point method using a barrier function. Show that the following 1D problem

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = x^2, x \in \mathbb{R} \\ \text{subject to} & x - 1 \geq 0 \end{array}$$

can be reformulated using a logarithmic barrier method as

$$\text{minimize } x^2 - r \log(x-1)$$

Determine the solution (as a function of r), and show that the global optimum is obtained as $r \to 0$.

7. Mean and median estimates. For a set of measurements $\{a_i\}$, show that

(a)

$$\min_x \sum_i (x - a_i)^2$$

is the mean of $\{a_i\}$.

(b)

$$\min_x \sum_i |x - a_i|$$

is the median of $\{a_i\}$.

8. Determine in each case if the following functions are convex:

(a) The sum of quadratic functions $f(x) = a_1(x - b_1)^2 + a_2(x - b_2)^2$, for $a_i > 0$

(b) The piecewise linear function $f(x) = \max_{i=1,\ldots,m}(\mathbf{a}_i{}^\top \mathbf{x} + b_i)$

(c) $f(x) = \max\{x, 1/x\}$ for $x > 0$

(d) $f(\mathbf{x}) = ||\mathtt{A}\mathbf{x} - \mathbf{b}||^2$