3B1B Optimization – Solutions

A. Zisserman, Michaelmas Term 2016

1. The Rosenbrock function is

$$f(x,y) = 100(y - x^2)^2 + (1-x)^2$$

(a) Compute the gradient and Hessian of $f(x,y)$.

(b) Show that that $f(x,y)$ has zero gradient at the point $(1,1)$.

(c) By considering the Hessian matrix at $(x,y) = (1,1)$, show that this point is a minimum.

(a) Gradient and Hessian

$$\nabla f = \begin{pmatrix} 400x^3 - 400xy + 2x - 2 \\ 200(y - x^2) \end{pmatrix} \qquad \mathtt{H} = \begin{bmatrix} 1200x^2 - 400y + 2 & -400x \\ -400x & 200 \end{bmatrix}$$

(b) gradient at the point $(1,1)$

$$\nabla f = \begin{pmatrix} 400x^3 - 400xy + 2x - 2 \\ 200(y - x^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b) Hessian at the point $(1,1)$

$$\mathtt{H} = \begin{bmatrix} 1200x^2 - 400y + 2 & -400x \\ -400x & 200 \end{bmatrix} = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$$

Examine eigenvalues:

- det is positive, so eigenvalues have same sign (thus not saddle point)
- trace is positive, so eigenvalues are positive
- Thus a minimum
- $\lambda_1 = 1001.6006, \lambda_2 = 0.39936077$

2. In Newton type minimization schemes the update step is of the form

$$\delta \mathbf{x} = -\mathtt{H}^{-1}\mathbf{g}$$

where $\mathbf{g} = \nabla f$. By considering $\mathbf{g}.\delta\mathbf{x}$ compare convergence of:

(a) Newton, to

(b) Gauss Newton

for a general function $f(\mathbf{x})$ (i.e. where $\mathtt{H}$ may not be positive definite).

A note on positive definite matrices

An $n \times n$ symmetric matrix $\mathtt{M}$ is **positive definite** if

- $\mathbf{x}^\top \mathtt{M}\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x}$
- All the eigen-values of $\mathtt{M}$ are positive

In each case consider $df = \mathbf{g}.\boldsymbol{\delta}\mathbf{x}$. This should be negative for convergence.

(a) Newton

$$\mathbf{g}.\boldsymbol{\delta}\mathbf{x} = -\mathbf{g}^\top \mathtt{H}^{-1}\mathbf{g}$$

Can be positive if $\mathtt{H}$ **not** positive definite.

(b) Gauss Newton

$$\mathbf{g}.\boldsymbol{\delta}\mathbf{x} = -\mathbf{g}^\top (2\mathtt{J}^\top \mathtt{J})^{-1}\mathbf{g}$$

Non positive, since $\mathtt{J}^\top \mathtt{J}$ is positive definite.

3. Explain how you could use the Gauss Newton method to solve a set of simultaneous non-linear equations.

Square the non-linear equations and add them – the resulting cost is then a sum of squared residuals, and so has a structure suitable for the Gauss Newton method.

For example, the set of equations:

$$\begin{aligned} g_1(x, y) &= 0 \\ g_2(x, y) &= 0 \end{aligned}$$

can be solved for $\mathbf{x} = (x, y)$ by the following optimization problem which has the required sum of squares form

$$\min_{\mathbf{x}} f(\mathbf{x}) = g_1(\mathbf{x})^2 + g_2(\mathbf{x})^2$$

4. Sketch the feasible regions defined by the the following inequalities and comment on the possible optimal values.

(a)

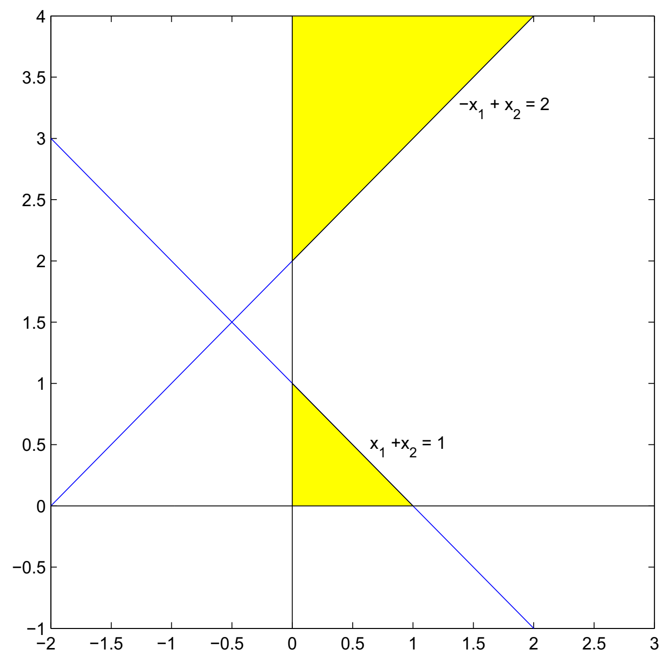
$$\begin{aligned} -x_1 + x_2 &\geq 2 \\ x_1 + x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - x_2 &\geq 2 \\ x_1 &\leq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

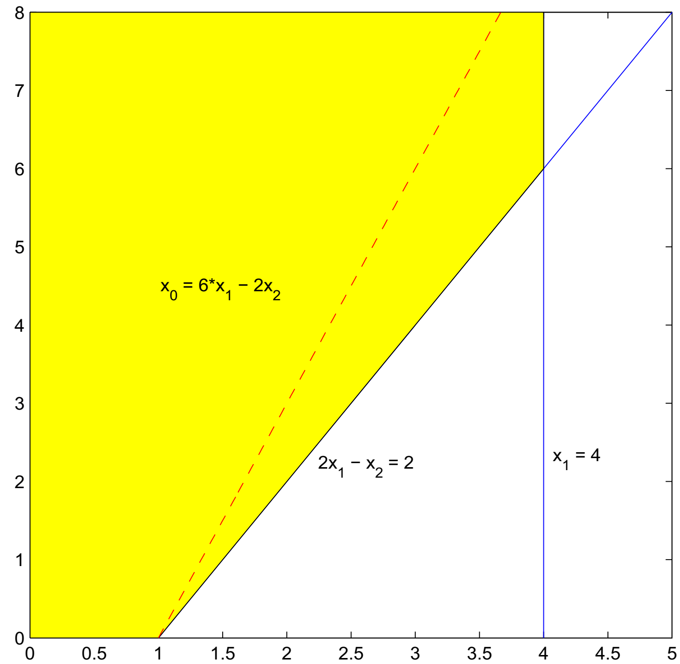
(a)

$$\begin{aligned} -x_1 + x_2 &\geq 2 \\ x_1 + x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



There is no feasible region, and therefore no possible solutions.

(b)
$$\begin{aligned} 2x_1 - x_2 &\leq 2 \\ x_1 &\leq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



The feasible region is unbounded, but the optimum can still be bounded for some cost functions.

5. More on linear programming.

(a) Show that the optimization

$$\min_{\mathbf{x}} \sum_i |\mathbf{a}_i{}^\top \mathbf{x} - b_i|$$

where the vectors $\mathbf{a}_i$ and scalars b_i are given, can be formulated as a linear programming problem.

(b) Solve the following linear programming problem using Matlab:

$$\begin{aligned}
\max_{x_1,x_2} \quad & 40x_1 + 88x_2 \\
\text{subject to} \quad & \\
2x_1 + 8x_2 &\leq 60 \\
5x_1 + 2x_2 &\leq 60 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}$$

(a) The absolute value operator $|.|$ is not linear, so at first sight this does not look like a linear programming problem. However, it can be transformed into one by adding extra variables and constraints

Introduce additional variables α_i with the constraints for each i that $|\mathbf{a}_i{}^\top x - b_i| \leq \alpha_i$. This can be written as the two linear constraints:

$$\begin{aligned}
\mathbf{a}_i{}^\top x - b_i &\leq \alpha_i \\
\mathbf{a}_i{}^\top x - b_i &\geq -\alpha_i
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathbf{a}_i{}^\top x - b_i &\leq \alpha_i \\
b_i - \mathbf{a}_i{}^\top x &\leq \alpha_i
\end{aligned}$$

Then the linear programming problem

$$\min_{\mathbf{x},\alpha_i} \sum_i \alpha_i$$

subject to these constraints, optimizes the original problem.

(b)

$$\begin{aligned} \max_{x_1,x_2} \quad & 40x_1 + 88x_2 \\ \text{subject to} \quad & \\ 2x_1 + 8x_2 &\leq 60 \\ 5x_1 + 2x_2 &\leq 60 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

```
Matlab code

f = [-40; -88];

A = [ 2 8
      5 2 ];

b = [ 60; 60 ];

lb = zeros(2,1);

x = linprog(f,A,b,[],[],lb);

solution is x_1 = 10.0000, x_2 =  5.0000.
```

6. Interior point method using a barrier function. Show that the following 1D problem

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = x^2, x \in \mathbb{R} \\ \text{subject to} & x - 1 \geq 0 \end{array}$$

can be reformulated using a logarithmic barrier method as

$$\text{minimize } x^2 - r \log(x-1)$$

Determine the solution (as a function of r), and show that the global optimum is obtained as $r \to 0$.

We need to find the optimum of

$$\min_x B(x,r) = x^2 - r \log(x-1)$$

Differentiating wrt x gives

$$2x - \frac{r}{x-1} = 0$$

and rearranging gives

$$2x^2 - 2x - r = 0$$

Use the standard formula for a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to obtain

$$x = \frac{2 \pm \sqrt{4 + 8r}}{4} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+2r}$$

and since only $x > 1$ is admissible (due to the log)

$$x^*(r) = \frac{1}{2} + \frac{1}{2}\sqrt{1+2r}$$

and as $r \to 0$, $x \to 1$, which is the global optimum.

7. Mean and median estimates. For a set of measurements $\{a_i\}$, show that

(a)

$$\min_x \sum_i (x - a_i)^2$$

is the mean of $\{a_i\}$.

(b)

$$\min_x \sum_i |x - a_i|$$

is the median of $\{a_i\}$.

(a)

$$\min_x \sum_i^N (x - a_i)^2$$

To find the minimum, differentiate $f(x)$ wrt x, and set to zero:

$$\frac{df(x)}{dx} = \sum_i^N 2(x - a_i) = 0$$

and rearranging

$$\sum_i^N x = \sum_i^N a_i$$

and so

$$x = \frac{1}{N} \sum_i^N a_i$$

i.e. the mean of $\{a_i\}$.

(b)

$$\min_x f(x) = \sum_i |x - a_i|$$

- The derivative of $|x - a_i|$ wrt x is $+1$ when $x > a_i$ and -1 when $x < a_i$.
- The derivative of $f(x)$ is zero when there are as many values of a_i less than x as there are greater than x.
- Thus $f(x)$ mininized at the median of values of $\{a_i\}$

Note, the median is immune to changes in a_i that lie far from the median – the value of the cost function changes, but not the position of the median.

8. Determine in each case if the following functions are convex:

(a) The sum of quadratic functions $f(x) = a_1(x - b_1)^2 + a_2(x - b_2)^2$, for $a_i > 0$

(b) The piecewise linear function $f(x) = \max_{i=1,\dots,m}(\mathbf{a_i}^\top \mathbf{x} + b_i)$

(c) $f(x) = \max\{x, 1/x\}$

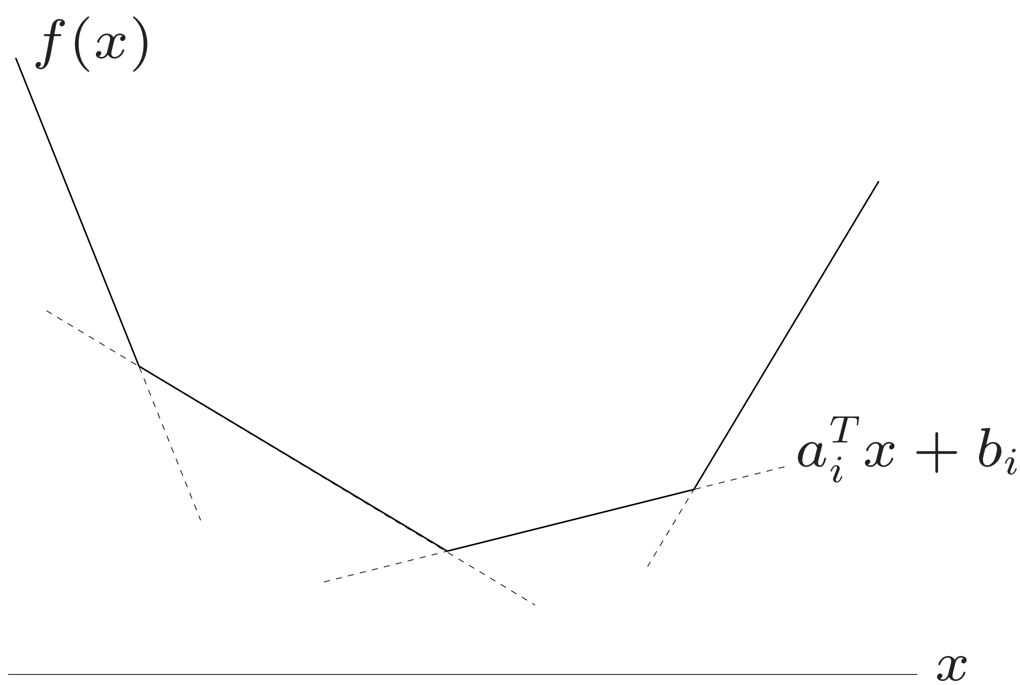
(d) $f(\mathbf{x}) = ||\mathtt{A}\mathbf{x} - \mathbf{b}||^2$

(a) The sum of quadratic functions $f(x) = a_1(x - b_1)^2 + a_2(x - b_2)^2$, for $a_i > 0$
Consider expanding the two quadratics, then the coefficient of x^2 is $a_1 + a_2$. Using the second derivative test for convexity:

$$\frac{d^2 f}{d x^2} \geq 0$$

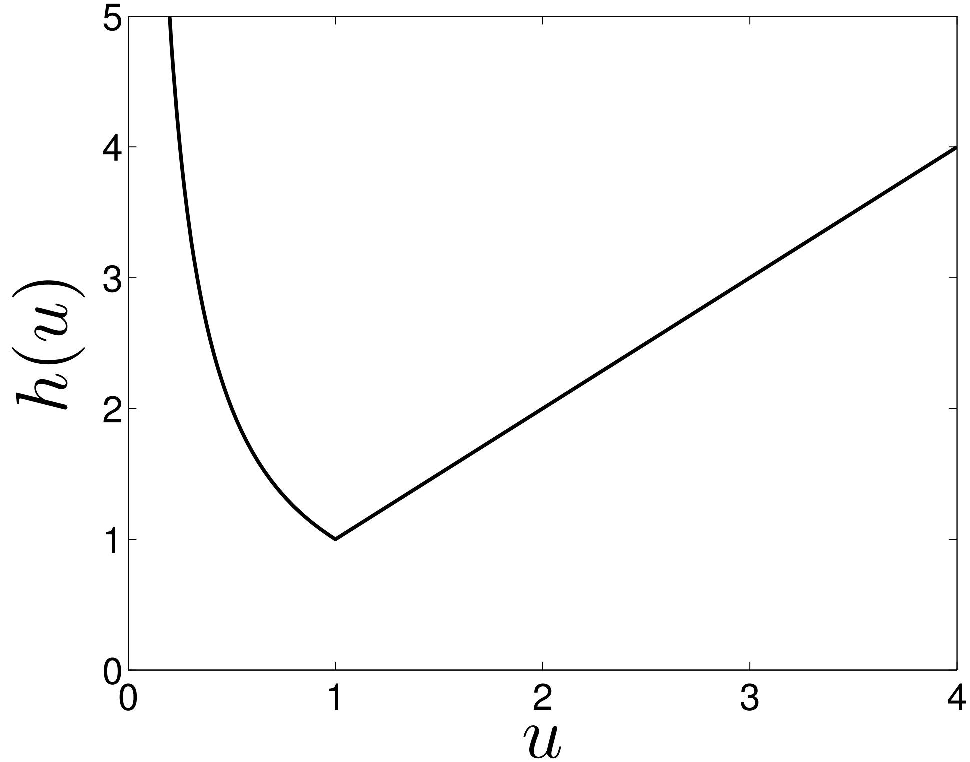
then the sum is convex provided that $a_1 + a_2 \geq 0$. So the function *is convex* since $a_i > 0$.

(b) The piecewise linear function $f(x) = \max_{i=1,\dots,m}(\mathbf{a_i}^\top \mathbf{x} + b_i)$
This *is convex*. Most easily seen by drawing and applying the geometric chord test for convexity.



(c) $f(x) = \max\{x, 1/x\}$ for $x > 0$

This *is convex*. Most easily seen by drawing and applying the geometric chord test for convexity.



(d) $f(\mathbf{x}) = ||\mathtt{A}\mathbf{x} - \mathbf{b}||^2$

Compute the Hessian

$$\nabla f(\mathbf{x}) = 2\mathtt{A}^\top(\mathtt{A}\mathbf{x} - \mathbf{b}) = 2\mathtt{A}^\top\mathtt{A}\mathbf{x} - 2\mathtt{A}^\top\mathbf{b}$$

$$\mathtt{H} = 2\mathtt{A}^\top\mathtt{A}$$

Hence $\mathtt{H}$ is positive definite (because $\mathtt{A}^\top\mathtt{A}$ is positive definite), and so the function is convex.