Lattice models for generic Landau orbits

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Abstract goes here.

I. INTRODUCTION

A. Background and motivation

The incompressible liquid phases of the fractional quantum Hall (FQH) effect serve as prototypical examples of topologically ordered or gapped quantum liquid phases.

Two recent developments in FQH physics have each inspired a large body of literature. The first is the numerical discovery of fractional Chern insulators (FCI), which are fractionalized phases of interacting particles in the presence of a strong lattice potential. This is in contrast to the traditional picture of the quantum Hall effect, which is formulated in the continuum. Since FQH phases are topologically stable against local perturbations,

FCI phases have been discovered experimentally in bilayer graphene heterostructures[1] The single-particle Chern bands that are the FCI analogues of Landau levels have also been observed experimentally in cold atoms [cite].

A second development in the FQH literature is based on observations that, in spite of their topological nature, FQH liquids are also characterized by non-trivial *geometrical* properties.

As pointed out by Haldane,[2] the geometry inherent in the FQH problem has been obscured by the introduction of symmetries – for example rotational invariance – that are unnecessary for the stability of the FQH liquid.

These two threads of inquiry are not disconnected. The addition of a non-negligible lattice potential introduces additional geometric data that may break some non-generic symmetries. For example, in the case of a square lattice, $SO(2)$ rotational invariance in the coordinate plane is explicitly broken to C_4 lattice symmetry. From this point of view, a complete theory of the FQHE formulated as generically as possible should furnish a theory of FCI phases, or at least those FCI phases that have FQH analogoues. [It is not clear to me whether there are topologically-ordered phases in FCIs that require going beyond this generic FQH framework.]

Although the FQHE is essentially a many-particle phenomenon, we expect that the nature of the host band should play a crucial role.

In the following section, we show that under some fairly generic assumptions, the effective hamiltonian will be quadratic in the momenta. In other words, Landau levels are a generic.

B. Universality of Landau orbits

Let us begin by briefly reviewing the Landau level problem. Here we have a hamiltonian $H_0 = \frac{1}{2m}\left(\pi_x^2 + \pi_y^2\right)$. The operators H_0, π_x and π_y form a Heisenberg Lie algebra, with commutators $[\pi_x, \pi_y] = i\hbar eB$, $[H_0, \pi_x] = 0$ and $[H_0, \pi_y] = 0$.

Now consider a tight-binding model of an electron living on a Bravais lattice in two spatial dimensions. We will assume that the hopping amplitude between any two sites on the lattice is generically non-zero. We index the sites of the lattice by a vector $\mathbf{m} = (m_1, m_2)$ with integer components. Let $c_\mathbf{m}^\dagger$ ($c_\mathbf{m}$) create (annihilate) an electron at site $\mathbf{m}$ of the lattice. We have the usual fermion anticommutation relations $\left\{c_\mathbf{m}, c_\mathbf{n}^\dagger\right\} = 2\delta_{\mathbf{mn}}$. The single-particle Hilbert space is spanned by the space of states $|\mathbf{m}\rangle$ with the electron occupying site $\mathbf{m}$. The hamiltonian in this representation is

$$\begin{aligned} H &= -\sum_{\mathbf{m}\neq\mathbf{n}} t_{\mathbf{mn}} c_\mathbf{n}^\dagger c_\mathbf{m} + t_{\mathbf{nm}} c_\mathbf{m}^\dagger c_\mathbf{n} \\ &= \sum_{\mathbf{m}\neq\mathbf{n}} t_{\mathbf{mn}} |\mathbf{n}\rangle\langle\mathbf{m}| + t_{\mathbf{nm}} |\mathbf{m}\rangle\langle\mathbf{n}| \end{aligned}$$

Note that we are excluding onsite/mass terms from the above hamiltonian; this is because a translation-invariant mass term simply shifts H by a constant. We define lattice translation operators $T_a = \sum_\mathbf{m} |\mathbf{m} + \mathbf{e}_a\rangle\langle\mathbf{m}|$. We can write the hamiltonian in terms of these as

$$H = -\sum_{j,k} t_{jk}\left(T_1^j T_2^k + (T_2^\dagger)^k (T_1^\dagger)^j\right) \tag{1}$$

We introduce a uniform background magnetic field B perpendicular to the spatial extent of the lattice. We choose the value of B to be such that the flux per lattice plaquette is $Ba^2 = \frac{P}{Q}\Phi_0$, where $\Phi_0 = 2\pi\hbar/e$ is the magnetic flux quantum and P and Q are relatively prime integers. In terms of the magnetic length $\ell = \sqrt{\frac{\hbar}{eB}}$, we have $a^2/\ell^2 = 2\pi P/Q$. We now define $\epsilon^2 := a^2/\ell^2$. In the presence of the magnetic field, the above translation operators are no longer gauge invariant. Instead, the correct lattice translation operators in this case are

$$\widetilde{T}_a = \sum_\mathbf{m} e^{i\theta_a(\mathbf{m})} |\mathbf{m} + \mathbf{e}_a\rangle\langle\mathbf{m}|.$$

where the phases $e^{i\theta_a(\mathbf{m})}$ satisfy

$$\theta_1(\mathbf{m}) + \theta_2(\mathbf{m} + \mathbf{e}_1) - \theta_1(\mathbf{m} + \mathbf{e}_2) - \theta_2(\mathbf{m}) = \epsilon^2.$$

Note that we regard the phases θ as residing on the links of the lattice, but we canonically identify $\theta_a(\mathbf{m}) = \theta(\mathbf{m}, \mathbf{m} +$

$\mathbf{e}_a$). The lattice translation operators $\widetilde{T}_a$ are unitary, so we can write them in terms of hermitian generators $\widetilde{T}_a = \exp\left[-i\widetilde{\Pi}_a\right]$. The components of $\widetilde{\mathbf{T}}$ do not commute, but satisfy $\widetilde{T}_1\widetilde{T}_2 = \exp(i\epsilon^2)\widetilde{T}_2\widetilde{T}_1$. This implies the commutator $\left[\widetilde{\Pi}_1, \widetilde{\Pi}_2\right] = i\epsilon^2$.

[Actually, it seems like it implies $[\Pi_1, \Pi_2] = i\epsilon + 2\pi i M$ for $M \in \mathbf{Z}$. I think it's probably worthwhile to keep track of the M, although I'm not sure of its significance. (Also I guess for the same reason the phases should really satisfy $\sum_\square \theta = \epsilon + 2\pi M$). Anyway, I think I'm ok to ignore this here.]

Because the $\widetilde{T}_a$ do not commute with each other, there is an ambiguity in passing from the $B = 0$ hamiltonian (1) to one for $B \neq 0$, which appears as a choice of phase of the (now-complex) $t_{\mathbf{mn}}$. This is analogous to the ordering ambiguity in quantizing polynomials on classical phase space. For now we fix this ambiguity by making the arbitrary choice that $T_1^j T_2^k \to \widetilde{T}_1^j\widetilde{T}_2^k + \widetilde{T}_2^k\widetilde{T}_1^j$ in the presence of nonzero B. (We could also have fixed this ambiguity by a choice of gauge for the $\theta(\mathbf{m})$.) Then

$$H = -\sum_{j,k} t_{jk}\left(\widetilde{T}_1^j\widetilde{T}_2^k + \widetilde{T}_2^k\widetilde{T}_1^j\right) + \text{h.c.}$$

Let's look at just the local part of this hamiltonian, where by "local" we mean containing only first powers of the translation operators: $H_{\text{loc}} = -t_{10}\left(T_1 + T_1^\dagger\right) - t_{01}\left(T_2 + T_2^\dagger\right)$. This part of the hamiltonian is special, because no ordering prescription (or gauge choice) is necessary. If we set $t_{10} = t_{01}$, H_{loc} is just the hamiltonian of the Hofstadter model. Since $\widetilde{T}_a = e^{-i\widetilde{\Pi}_a}$, we can at least formally write $H_{\text{loc}} = -2t_{10}\cos\left(\widetilde{\Pi}_x\right) - 2t_{01}\cos\left(\widetilde{\Pi}_y\right)$. Now we rescale the $\widetilde{\Pi}_a$ operators, defining $\epsilon\Pi_a = \widetilde{\Pi}_a$, and we have the commutation relation $[\Pi_x, \Pi_y] = i$. Although the commutator of the Π_a is $O(1)$ with respect to ϵ, we can not necessarily infer that the Π_a are individually $O(1)$. [If there is an argument for this, I would be happy to know.] However, in what follows we will assume this is the case. Then

$$\begin{aligned} H_{\text{loc}} &= -2\sum_{n=0}^{\infty}\frac{(-1)^n\epsilon^{2n}}{(2n)!}\left(t_{10}\Pi_x^{2n} + t_{01}\Pi_y^{2n}\right) \\ &= -2 + 2\epsilon^2\frac{\left(t_{10}\Pi_x^2 + t_{01}\Pi_y^2\right)}{2} + O(\epsilon^4). \end{aligned}$$

Now let $t_{10} = t$, and $t_{01} = \alpha^2 t$, i.e., t is a common hopping energy scale and α parameterizes anisotropy in the hopping amplitudes. Then to lowest order in $\epsilon = a/\ell$, we have a small-ϵ effective hamiltonian $H_{\text{eff}} = t\epsilon^2\left(\Pi_x^2 + \alpha^2\Pi_y^2\right)$. We can rewrite this in terms of momentum operators that satisfy $[\pi_x, \pi_y] = i\hbar^2/\ell^2$ as

$$H_{\text{eff}} = \frac{1}{2m_*}\left(\pi_x^2 + \pi_y^2\right),$$

showing that our problem is isomorphic to the Landau level problem with effective mass $m_* = \hbar^2/(2ta^2\alpha)$. [I am thinking about how this connects with the massless/LLL projection limit in the Landau level problem. In our case, we effectively have $a \to 0$, so the mass should be large unless ta^2 is large.]

[Note: in the above, I did not consider the possibility that the t depends on ϵ. Within the tight-binding approximation as we're employing it here, this is kosher. However, it could be interesting to see whether this approximation breaks down at all when $\epsilon > 0$.]

C. Chern bands and geometry

Given a tight-binding hamiltonian as described above, we can define magnetic translation operators that commute with the hamiltonian and an associated magnetic unit cell in the usual way.[3] The magnetic unit cell translations in each lattice direction commute, and we can define their simulatenous eigenstates $|\mathbf{k}\rangle$, where $\mathbf{k}$ takes values in the magnetic Brillouin zone.

D. Relation to previous work

[lattice FQH, FCIs] [Haldane]

E. Outline and summary

In the following section, we will consider particular lattice models for which the continuum-limit hamiltonian contains both quadratic and quartic terms. We will

Then we will specialize to the case in which the quadratic term vanishes identically.

Finally, we will comment on anisotropic modifications to these models when the lattice symmetry is broken further by varying the relative hopping strengths in each lattice direction.

II. GENERIC QUARTIC MODEL

Following the above discussion, the most general effective hamiltonian we can write down, keeping terms quartic in the momenta, is

$$H = \frac{1}{2}h^{(1)}_{ab}\pi_a\pi_b + \frac{1}{4}h^{(2)}_{abcd}\pi_a\pi_b\pi_c\pi_d$$

As discussed, there is ambiguity in ordering the components of the momentum. We will for now elide this ambiguity by a particular choice of ordering. Namely, we choose the coefficients $h^{(n)}$ to be totally symmetric in their $2n$ indices.

Under an $SL(2,R)$ transformation of the momenta, the commutator $[\pi, \pi]$ is preserved.

In general, there are no restrictions on the form of $h^{(2)}_{abcd}$ beyond total symmetry of its indices. However, in the next section we will consider the case in which $h^{(1)}$ vanishes. In this case, H will not have a stable minimum unless $h^{(2)}_{abcd} = \gamma^1_{ab}\gamma^2_{cd}$ with $\gamma^j_{ab} = \gamma^j_{ba}$ symmetric.

III. PURELY QUARTIC MODEL

Now we consider the continuum limit of the following lattice model.

IV. ANISTROPIC MODELS

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