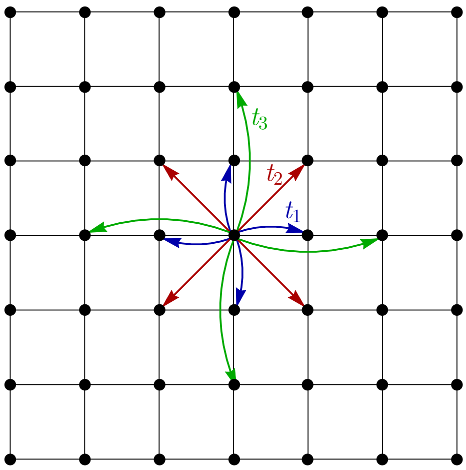
Quartic Hamiltonians in the Hofstadter Model

In this document, we obtain the Hofstadter model hopping amplitudes that lead to quartic Hamiltonians.

1 Hopping Terms on the Square Lattice

We first consider a tight-binding model on a square lattice which involves hops between each site and its three sets of nearest neighbours. We write the (assumed real) hopping amplitude to each set of neighbours as t_1, t_2 and t_3, respectively, as shown below:



In the absence of a magnetic field, the corresponding tight-binding model can be written

$$\begin{aligned}\hat{H} \quad = \quad & \sum_{\mathbf{r}} \Big[t_1 \left(c^\dagger_{\mathbf{r}+\hat{\mathbf{x}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}-\hat{\mathbf{x}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}+\hat{\mathbf{y}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}-\hat{\mathbf{y}}} c_{\mathbf{r}} \right) \\ & + t_2 \left(c^\dagger_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}} c_{\mathbf{r}} \right) \\ & + t_3 \left(c^\dagger_{\mathbf{r}+2\hat{\mathbf{x}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}-2\hat{\mathbf{x}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}+2\hat{\mathbf{y}}} c_{\mathbf{r}} + c^\dagger_{\mathbf{r}-2\hat{\mathbf{y}}} c_{\mathbf{r}} \right) \Big],\end{aligned}$$

which may be diagonalised in momentum space to give

$$\hat{H} = \sum_{\mathbf{k}} \Big[2t_1 \left[\cos(k_x) + \cos(k_y)\right] + 2t_2 \left[\cos(k_x + k_y) + \cos(k_x - k_y)\right] + 2t_3 \left[\cos(2k_x) + \cos(2k_y)\right] \Big] c^\dagger_{\mathbf{k}} c_{\mathbf{k}}. \quad (1)$$

To obtain the corresponding Hofstadter Hamiltonian in the presence of a magnetic field, we should replace $\mathbf{k}$ with its magnetic operator through

$$\mathbf{k} \quad \rightarrow \quad -i\nabla - \mathbf{A},$$

(see EM Response draft).

2 Quartic Hamiltonians

To obtain a quartic Hamiltonian, we expand Eq. (1) order by order in momentum, finding

$$\begin{aligned}\hat{H}(\mathbf{k}) &= 4(t_1+t_2+t_3) - \left(k_x^2+k_y^2\right)(t_1+2t_2+4t_3) \\ &+\frac{1}{12}\left[k_x^4(t_1+2(t_2+8t_3)) + k_y^4(t_1+2(t_2+8t_3)) + 12t_2k_x^2k_y^2\right]+\dots\end{aligned}$$

Then, we set

$$t_1 = -2t_2 - 4t_3$$

to eliminate the quadratic piece, leaving

$$\hat{H}_{O(k^4)}(\mathbf{k}) = -4(t_2+3t_3) + t_3\left[k_x^4+k_y^4\right] + t_2k_x^2k_y^2+\dots$$

In this way, we can obtain any quartic Hamiltonian we like by tuning t_2 and t_3.

Specific Parameter Choices

In particular, if we choose

$$\begin{aligned} t_2 &= 0 \\ t_1 &= -4t_3 \end{aligned}$$

we obtain the quartic Hamiltonian with no mixed terms,

$$\hat{H}_{4,4} = -12t_3 + t_3\left[k_x^4+k_y^4\right]+\dots$$

On the other hand, if we choose

$$\begin{aligned} t_3 &= 0 \\ t_1 &= -2t_2 \end{aligned}$$

we obtain the quartic Hamiltonian with only mixed terms

$$\hat{H}_{2\times 2} = -4t_2 + t_2k_x^2k_y^2+\dots$$

In all cases, we can obtain the corresponding Hofstadter Hamiltonian by replacing $\mathbf{k}$ with the magnetic momentum operator.

3 Ladder Operator Expressions

We rewrite some of the above expressions in terms of ladder operators by using the operator replacements

$$\begin{aligned}\hat{k}_x &\to -\sqrt{\frac{B}{2}}\left(a+a^\dagger\right) \\ \hat{k}_y &\to -i\sqrt{\frac{B}{2}}\left(a-a^\dagger\right)\end{aligned}$$

and being sure to replace momentum products with their fully symmetrised form.

Substituting into the expression above, we find

$$\begin{aligned}\hat{H}_{O(k^4)}(\mathbf{k}) &\to -4(t_2+3t_3) + t_3\left[\hat{k}_x^4+\hat{k}_y^4\right] + t_2\left\{\hat{k}_x^2\hat{k}_y^2\right\}+\dots \\ &= -4(t_2+3t_3) + t_3\left[6B^2a^\dagger a + 3B^2\left(a^\dagger\right)^2a^2 + \frac{1}{2}a^4B^2 + \frac{1}{2}\left(a^\dagger\right)^4B^2 + \frac{3B^2}{2}\right] \\ &\quad +t_2\left[B^2a^\dagger a + \frac{1}{2}B^2\left(a^\dagger\right)^2a^2 - \frac{1}{4}a^4B^2 - \frac{1}{4}\left(a^\dagger\right)^4B^2 + \frac{B^2}{4}\right]+\dots \\ &= -4(t_2+3t_3) + \frac{B^2}{4}\left[(-t_2+2t_3)\left(\left(a^\dagger\right)^4+a^4\right) + 2(t_2+6t_3)\left(1+2a^\dagger a+\left(a^\dagger\right)^2a^2\right)+\dots\right.\end{aligned}$$

In particular, if we set

$$t_2 + 6t_3 = 0,$$

we obtain

$$\hat{H}_{a^4,a^{\dagger 4}} = 12t_3 + 2B^2 t_3 \left(\left(a^\dagger\right)^4 + a^4\right) + \ldots,$$

which has no diagonal ladder operator terms.

Does this correspond to a realistic band structure? Substituting this new value of t_2 into the zero-field Hamiltonian, we find

$$\hat{H}_{O(k^4)}(\mathbf{k}) = 12t_3 + t_3\left(k_x^4 + k_y^4\right) - 6t_3 k_x^2 k_y^2 + \ldots$$

In this way, the $\mathbf{k} = 0$ point (which we implicitly expand about) is still a band extremum (although it will not be a global extremum). In this way, I think it still makes sense to consider $\hat{H}_{a^4,a^{\dagger 4}}$ as the approximate Hamiltonian in the weak-field limit for states at this point. The overall energy offset can be tuned to zero by applying an on-site energy, in which case the leading term in the Hamiltonian may be written

$$\tilde{H}_{a^4,a^{\dagger 4}} = 2B^2 t_3 \left(\left(a^\dagger\right)^4 + a^4\right)$$

A brief play around in Mathematica suggests that this Hamiltonian has eigenstates which are very close to the quantum Harmonic oscillator states—expect that there is now a positive energy branch and a negative energy branch (each scaling as n^2). There are four degenerate zero-energy states fairly similar ($> 90\%$) to $|0\rangle$, $|1\rangle$, $|2\rangle$ and $|3\rangle$. The positive and negative energy states with the same energy magnitude have the same density but differ by some signs (in the LL basis). Overall, I don't think the situation is very different from the case where there are $a^\dagger a$ terms (etc.) present...