Analytic Wavefunctions for Quartic Hamiltonian

1 Zero-Energy Wavefunction for $a^4 + \left(a^\dagger\right)^4$ Hamiltonian

We try to obtain an analytic expression for the lowest energy wavefunction of a particular quartic model. We take the Hamiltonian

$$\hat{H} \quad = \quad \omega \left[a^4 + \left(a^\dagger\right)^4\right]$$

and write a general wavefunction as

$$|\psi\rangle \quad = \quad \sum_k C_k \, |k\rangle$$

as a sum over Landau level states. The eigenvalue equation becomes

$$\begin{aligned}
\hat{H}\,|\psi\rangle &= \omega \sum_k C_k \left[\sqrt{k(k-1)(k-2)(k-3)}\,|k-4\rangle + \sqrt{(k+1)(k+2)(k+3)(k+4)}\,|k+4\rangle\right] \\
&= E \sum_k C_k \,|k\rangle\,.
\end{aligned}$$

Taking inner products with different Landau level states yields

$$\begin{array}{lcrl}
\langle 0| : & \omega C_4 \alpha_{-4}(4) & = & EC_0 \\
\langle 4| : & \omega\left[C_8 \alpha_{-4}(8) + C_0 \alpha_{+4}(0)\right] & = & EC_4 \\
\langle 8| : & \omega\left[C_{12} \alpha_{-4}(12) + C_4 \alpha_{+4}(4)\right] & = & EC_8 \\
\langle 12| : & \omega\left[C_{16} \alpha_{-4}(16) + C_8 \alpha_{+4}(8)\right] & = & EC_{12} \\
 & \vdots & & \\
\langle 4p| : & \omega\left[C_{4(p+1)} \alpha_{-4}(4(p+1)) + C_{4(p-1)} \alpha_{+4}(4(p-1))\right] & = & EC_{4p},
\end{array}$$

where we have used the shorthand

$$\begin{aligned}
\alpha_-(k) &= \sqrt{k(k-1)(k-2)(k-3)} \\
\alpha_+(k) &= \sqrt{(k+1)(k+2)(k+3)(k+4)}.
\end{aligned}$$

We first try to obtain a state with energy $E = 0$. Substituting this into some of the coefficient equations, we find

$$\begin{aligned}
\omega C_4 \alpha_{-4}(4) &= 0 \\
\Rightarrow C_4 &= 0 \\
\omega \left[C_8 \alpha_{-4}(8) + C_0 \alpha_{+4}(0)\right] &= 0 \\
\Rightarrow C_8 &= -C_0 \frac{\alpha_{+4}(0)}{\alpha_{-4}(8)} \\
\omega \left[C_{12} \alpha_{-4}(12) + 0\right] &= 0 \\
\Rightarrow C_{12} &= 0 \\
\omega \left[C_{16} \alpha_{-4}(16) + C_8 \alpha_{+4}(8)\right] &= 0 \\
\Rightarrow C_{16} &= -C_8 \frac{\alpha_{+4}(8)}{\alpha_{-4}(16)} \\
&= C_0 \frac{\alpha_{+4}(8)\alpha_{+4}(0)}{\alpha_{-4}(16)\alpha_{-4}(8)}.
\end{aligned}$$

The pattern of coefficients continues in the same manner. We define

$$\beta(k) = \frac{\alpha_{+4}(8(k-1))}{\alpha_{-4}(8k)}$$

so that

$$\begin{aligned}
C_8 &= -\beta(1) C_0 \\
C_{16} &= +\beta(2)\beta(1) C_0 \\
&\vdots \\
C_{8p} &= (-1)^p C_0 \prod_{k=1}^{p} \beta(k) \\
&\equiv \gamma(p) C_0,
\end{aligned}$$

where we have defined $\gamma(p)$ in the final line. Mathematica 'simplifies' the function γ to

$$\gamma(n) = \frac{\sqrt[4]{\frac{2}{\pi}}\,\Gamma\left(\frac{7}{8}\right)\sqrt{\Gamma\left(n+\frac{9}{8}\right)\Gamma\left(n+\frac{5}{4}\right)\Gamma\left(n+\frac{11}{8}\right)\Gamma\left(n+\frac{3}{2}\right)}}{\Gamma\left(\frac{1}{8}\right)\sqrt{\Gamma\left(n+\frac{13}{8}\right)\Gamma\left(n+\frac{7}{4}\right)\Gamma\left(n+\frac{15}{8}\right)\Gamma(n+2)}},$$

with

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x.$$

We now have both a recursive and a non-recursive relation for the coefficient C_{8p} (with all other $C_k = 0$). The complete wavefunction is

$$\begin{aligned}
|\psi\rangle &= \sum_k C_k |k\rangle \\
&= C_0 \sum_k \gamma(k) |k\rangle,
\end{aligned}$$

which leads to the normalisation condition

$$1 = |C_0|^2 \left[1 + |\gamma(1)|^2 + |\gamma(2)|^2 + \ldots\right].$$

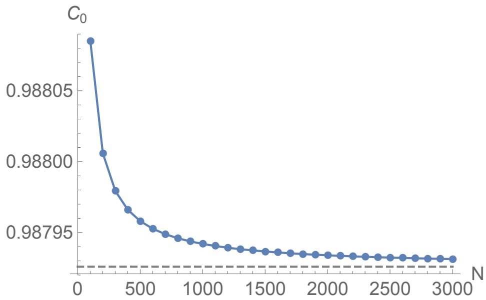
Using Mathematica, we find

$$\frac{1}{|C_0|^2} = {}_4F_3\left(\left[\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}\right], \left[\frac{5}{8}, \frac{3}{4}, \frac{7}{8}\right], 1\right),$$

where ${}_4F_3(a;b;z)$ is a generalised hypergeometric function. Solving for C_0, we find

$$C_0 \quad = \quad 0.987926,$$

which agrees very well with numerics:



2 Lowest-Energy Wavefunction for $p^4 + x^4$ Hamiltonian