Geometry of Landau orbits in the absence of rotational symmetry

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The integer quantum Hall effect (IQHE) is usually modeled using Galilean-invariant or rotationally-invariant Landau levels. However, these are not generic symmetries of electrons moving in a crystalline background, even in the low-density continuum limit. We present a treatment of the IQHE which abandons the Galilean dispersion relation while keeping inversion symmetry. We define an emergent metric g^n_{ab} for each Landau level with a reformulation of the Hall viscosity. The metric is then used to define a guiding-center coherent state and the wavefunctions are holomorphic functions of z^* times a Gaussian. By numerically studying cases with quartic dispersion, we show that the number of the zeroes of the wavefunction encircled by the semiclassical orbit, denoted by n, defines a "topological spin" s_n by $s_n = n+\frac{1}{2}$, with its original definition as an "intrinsic angular momentum" no longer valid without rotational symmetry. At the end of the paper we show our results for the density and current responses which differentiate between diagonal and Landau-level-mixing terms. In conclusion, this treatment extracts topological information without resort to Galilean or rotational symmetry, and also reveals more generic geometric structures.

I. INTRODUCTION

Most treatments of the integer and fractional quantum Hall effects (IQHE and FQHE) in the low-density two-dimensional electron gas (2DEG) subjected to a uniform magnetic flux density begin by making the simplifying assumption that the electrons obey Newtonian dynamics with 2D Galilean invariance, and that the system has a continuous rotational symmetry around any normal to the 2D surface on which the electrons move.

Among the simplifications that follow from this is the well-known result that when rotational symmetry is present, the wavefunction of an electron in the lowest Landau level is a holomorphic function $f(z) = f(x+iy)$ times a Gaussian factor, which played a key role in Laughlin's discovery[1] of a model wavefunction for the FQHE with Landau-level filling $\nu = 1/m$. The apparent importance of rotational symmetry was further reinforced when it was observed[2] that the Laughlin state wavefunctions were mathematically related to "conformal blocks" of (Euclidean) 2D conformal field theory in which the global rotation symmetry $SO(2)$ is promoted to a local Euclidean conformal symmetry $CO(2)$.

Much of the interest in the QHE relates to its robust topological properties, which by definition are not related to the presence or absence of symmetries such as rotational, Galilean or conformal symmetries, so their presence as a simplifying "toy model" feature is not problematic. However there has been recent interest in geometrical properties of the QHE[3–10], in which symmetries play an important role, and the inclusion of non-generic symmetries may in this case be misleading.

The assumption of a local Galilean structure with an inertial mass tensor given by a scalar gravitational mass times the Euclidean metric of the flat plane has been extended by some authors to a treatment of the QHE that places the electrons on a Riemann surface that can be thought of as a curved 2D surface immersed in 3D Euclidean space, from which it receives an induced metric. A related formulation of the problem has been given as "Newton-Cartan" theory[11]. However the QHE is experimentally exhibited by electrons moving on a lattice plane inside crystalline condensed matter, in which there is generically no Galilean invariance or rotational symmetry, and the non-Newtonian electron kinetic energy derives from the band structure of an underlying atomic-scale crystal structure.

Models of the QHE that include non-generic symmetries are certainly useful models which will exhibit generic topological behavior, but cannot be relied upon to exhibit generic geometrical behavior. A useful analogy is the theory of metals, for which the simple model of a Galilean-invariant Fermi gas with a spherical Fermi surface is often used as an initial "toy model" that exhibits many generic features of metals (such as a T-linear specific heat), but fails to capture the intricate geometry of true metallic Fermi surfaces.

In an attempt to clarify the nature of geometric properties of the QHE, such as the "Hall viscosity"[12], we will here present a treatment of the integer QHE which abandons the Galilean dispersion relation of non-relativistic Newtonian dynamics, where

$$\varepsilon(\boldsymbol{p}) = \frac{p_x^2 + p_y^2}{2m} \quad (1)$$

keeping only the weaker condition

$$\varepsilon(-\boldsymbol{p}) = \varepsilon(\boldsymbol{p}) \quad (2)$$

and the simplification that the classical Landau orbits $\varepsilon(\boldsymbol{p}) = E$, constant, are compact closed curves in 2D momentum space, so that classical Landau orbits are uniquely fixed by their energy and guiding center. The Galilean-invariant model is a special case of these weaker restrictions, which simplify the semiclassical quantization. The weaker restrictions we impose could themselves be dropped at the expense of introducing some additional complications, but we believe that as they permit the complete removal of the non-generic Galilean and rotational-symmetry structures from the theory of the QHE, they allow more generic geometrical structures to be revealed.

The "momentum space inversion symmetry" (2) can be justified as a condition that derives from the assumption that the 2DEG has unbroken time-reversal symmetry in the absence of an applied magnetic field. It is also compatible with spatial inversion symmetry, which is the only metric-independent point symmetry compatible with generic crystalline condensed matter, as it is the only point symmetry of a generic Bravais lattice.

In this work, the geometrical description will be restricted to that of the integer QHE of non-interacting electrons in non-Galilean Landau levels, and the extension to the richer structure of the FQHE will be given elsewhere, building on the IQHE ingredients described here.

This paper is organized as follows. In Sec.II we briefly introduce a formulation of 2D spatial geometry using consistent covariant and contravariant index placement. With the algebra of Landau orbits and guiding centers reviewed in Sec.III, we study the geometry of Landau orbits by emphasizing on a unimodular metric $\tilde{g}_{ab}$ in Sec.IV. Then in Sec.V we provide a natural choice of the metric along with a reformulation of the Hall viscosity in the absence of rotational symmetry. In Sec.VI we define a topological spin s_n by investigating the root structure of the wavefunction and a case for quartic dispersion is numerically studied in Sec.VII. After a brief discussion of some other properties of the Landau orbits in Sec.VIII, we show in Sec.IX generic results for the density and current responses in the absence of rotational symmetry. Our conclusion is in Sec.X.

II. 2D GEOMETRY WITH CONSISTENT INDEX PLACEMENT

The Euclidean plane can be parametrized by a Cartesian coordinate system

$$\boldsymbol{x} = x^a \boldsymbol{e}_a, \quad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}, \tag{3}$$

where $\{\boldsymbol{e}_a, a = 1, 2\}$ is a basis of $\boldsymbol{x}$-independent orthonormal tangent vectors, and

$$\boldsymbol{e}_a \times \boldsymbol{e}_b = \epsilon_{ab} \boldsymbol{n}, \tag{4}$$

where $\boldsymbol{n}$ is the unit normal to the plane, which is also $\boldsymbol{x}$-independent. Here δ_{ab} is the 2D Euclidean metric induced on the plane by its embedding in 3D Euclidean space, and ϵ_{ab} is the antisymmetric 2D Levi-Civita symbol, which has a sign ambiguity because the unit normal is only defined up to $\pm\boldsymbol{n}$ (the *orientation* choice).

It will prove very useful to use a formalism that distinguishes *vectors*, such as velocities $\boldsymbol{v} = v^a \boldsymbol{e}_a = \dot{x}^a \boldsymbol{e}_a$ which have components with upper indices (contravariant to those of $\boldsymbol{e}_a$), from *covectors*, such as forces, which have lower (covariant) indices $F_a \equiv \boldsymbol{e}_a \cdot \boldsymbol{F}$. Then the scalar product $\boldsymbol{F} \cdot \boldsymbol{v} \equiv F_a v^a \equiv F_1 v^1 + F_2 v^2$ is only defined as a bilinear relation between vectors and covectors, with Einstein summation convention on repeated upper/lower index pairs. In this formalism, the Euclidean metric δ_{ab}, the inverse metric δ^{ab} and the Kronecker symbol δ^a_b are distinguished, although they are numerically equal when a Cartesian coordinate system is used.

The dual Levi-Civita symbol ϵ^{ab} is also numerically-equal to (but distinguished from) ϵ_{ab}. Technically, ϵ_{ab} is a covariant pseudotensor density with weight -1, and ϵ^{ab} is a contravariant pseudotensor density with weight $+1$, If g_{ab} is a metric tensor,

$$g \equiv \det g = \epsilon^{ac} \epsilon^{bd} g_{ab} g_{cd} \tag{5}$$

is a scalar density with weight 2. A useful formal identity is

$$\epsilon^{ab} \epsilon_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c. \tag{6}$$

In any expression, covariant indices, contravariant indices, and tensor weights must balance.

The use of consistent index placement may seem pedantic in systems where the Euclidean metric plays a fundamental role, such as those governed by Newtonian dynamics, in which the inertial mass tensor of a particle m_{ab} is given by $m\delta_{ab}$, where m is its gravitational mass, However, in *generic* non-relativistic crystalline condensed matter (*i.e.*, with a trigonal as opposed to a cubic lattice, and no point-group symmetry other than spatial inversion) the Euclidean metric *plays no role whatsoever in the non-relativistic physics*, except for the definition of the non-relativistic limit $\delta_{ab} v^a v^b \ll c^2$, where $\boldsymbol{v} = v^a \boldsymbol{e}_a$ is the velocity of an electron relative to the preferred inertial frame of condensed matter, in which the underlying atomic lattice is at rest.

Consistent index placement exposes the presence of the Euclidean metric in physical equations: for example, the azimuthal angular momentum, which is conserved in Newtonian dynamics on the plane, is given by

$$L = \epsilon_{ab} \delta^{bc} x^a p_c. \tag{7}$$

The presence of the inverse Euclidean metric (required to balance the index placement) immediately identifies it as a quantity without validity in a generic crystalline environment.

III. THE ALGEBRA OF LANDAU ORBITS AND GUIDING CENTERS

The Schrödinger representation of the momentum of an electron is

$$\boldsymbol{e}_a \cdot \boldsymbol{p} = p_a = -i\hbar \frac{\partial}{\partial x^a} - eA_a(\boldsymbol{x}), \tag{8}$$

where

$$\frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} = \epsilon_{ab} B(x), \tag{9}$$

and x^a are the coordinates of the electron. With this definition, the Levi-Civita symbol is odd under time-reversal. We will concentrate on the case $B(\boldsymbol{x}) = B$, constant, leaving open the eventual extension to include a slow spatial variation $B(\boldsymbol{x})$. In the case of constant B, the commutation relations of the components of the momentum have the simple form

$$[p_a, p_b] = i\hbar eB\epsilon_{ab}. \tag{10}$$

The remaining commutation relations are unaffected by B:

$$[x^a, x^b] = 0, \quad [x^a, p_b] = i\hbar\delta^a_b. \tag{11}$$

In a system with uniform magnetic flux density, the electron coordinate may be decomposed as

$$\boldsymbol{x} = \boldsymbol{R} + (eB)^{-1}\epsilon^{ab}\boldsymbol{e}_a p_b, \tag{12}$$

where $\boldsymbol{R} = R^a\boldsymbol{e}_a$ is the *guiding center* of the Landau orbit, with commutation relations

$$[R^a, R^b] = -i\hbar(eB)^{-1}\epsilon^{ab}, \quad [R^a, p_b] = 0. \tag{13}$$

It is useful to write

$$p_a = B\epsilon_{ab}d^b, \tag{14}$$

where

$$\boldsymbol{d} = e(\boldsymbol{x} - \boldsymbol{R}) \tag{15}$$

can be interpreted as the electric dipole moment defined by the displacement of the electron relative to its guiding center. The inversion symmetry $\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$ means that in a Landau orbit eigenstate $|\psi_{n\alpha}\rangle$ of $\varepsilon(\boldsymbol{p})$, the expectation value of this electric dipole vanishes, and

$$\langle\psi_{n\alpha}|\boldsymbol{p}|\psi_{n\alpha}\rangle = 0, \tag{16}$$

which provides an unambiguous definition of the guiding center independent of the Landau level.

In a uniform magnetic field, a representation of the algebras of p_a and R^a can be given in terms of *independent harmonic oscillator degrees of freedom*

$$\begin{aligned} p_a &= \sqrt{(\tfrac{1}{2}\hbar|eB|)}\left(\tilde{e}_a a^\dagger + \tilde{e}^*_a a\right), \\ R^a &= \sqrt{(\tfrac{1}{2}\hbar|eB|^{-1})}\left(\bar{e}^a b + \bar{e}^{a*} b^\dagger\right), \end{aligned} \tag{17}$$

where

$$\epsilon^{ab}\tilde{e}^*_a\tilde{e}_b = \epsilon_{ab}\bar{e}^{a*}\bar{e}^b = i, \tag{18}$$

and $[a, a^\dagger] = [b, b^\dagger] = 1$, $[a, b] =[a, b^\dagger] = 0$. A metric is said to be *unimodular* if it has determinant 1; The condition (18) means that *two* unimodular Euclidean-signature metrics and their inverses are defined by

$$\begin{aligned} \tilde{e}^*_a\tilde{e}_a &= \tfrac{1}{2}\left(\tilde{g}_{ab} + i\epsilon_{ab}\right), \quad \tilde{g}^{ab} = \epsilon^{ac}\epsilon^{bd}\tilde{g}_{cd}, \\ \bar{e}^{a*}\bar{e}^a &= \tfrac{1}{2}\left(\bar{g}^{ab} + i\epsilon^{ab}\right), \quad \bar{g}_{ab} = \epsilon_{ac}\epsilon_{bd}\bar{g}^{cd}. \end{aligned} \tag{19}$$

The standard treatments would immediately set both $\tilde{g}_{ab}$ and $\bar{g}_{ab}$ equal to the Euclidean metric δ_{ab}, but we emphasize that they are independent free parameters of the representation (17).

It is now useful to introduce the generators of linear area-preserving $SL(2, R)$ linear deformations of momentum space

$$p_a \mapsto U(\beta)p_a U(\beta)^{-1} = \Lambda_a^{\ b}(\beta)p_b, \tag{20}$$

where

$$\det\Lambda \equiv \tfrac{1}{2}\epsilon^{ac}\epsilon_{bd}\Lambda_a^{\ b}\Lambda_c^{\ d} = 1. \tag{21}$$

This unitary transformation leaves the guiding centers invariant, and preserves the commutation relation (10). The unitary operator $U(\beta)$ is given by

$$U(\beta) = \exp(i\pi\beta^{ab}\gamma_{ab}), \tag{22}$$

where $\beta_{ab} = \beta_{ba}$ is real, and the Hermitian generators $\gamma_{ab} = \gamma_{ba}$ are given by

$$\tfrac{1}{4}\{p_a, p_b\} = \hbar eB\gamma_{ab}. \tag{23}$$

They obey the non-compact Lie algebra

$$[\gamma_{ab}, \gamma_{cd}] = \tfrac{1}{2}i\left(\epsilon_{ac}\gamma_{bd} + \epsilon_{bc}\gamma_{ad} + \epsilon_{ad}\gamma_{bc} + \epsilon_{bd}\gamma_{ac}\right) \tag{24}$$

with a quadratic Casimir

$$C_2 = \det\gamma = \tfrac{1}{2}\epsilon^{ac}\epsilon^{bd}\gamma_{ab}\gamma_{cd}. \tag{25}$$

The representation (23) of this algebra has $C_2 = -\frac{3}{16}$. Any unimodular metric $\tilde{g}_{ab}$ with $\det\tilde{g} = 1$ defines a Cartan subalgebra generated by $L_g = \tilde{g}^{ab}\gamma_{ab}$. The explicit form of this representation in terms of harmonic oscillator operators (17) is

$$\begin{aligned} \gamma_{11} &= \tfrac{1}{4}\left(a + a^\dagger\right)^2 \\ \gamma_{22} &= -\tfrac{1}{4}\left(a - a^\dagger\right)^2 \\ \gamma_{12} &= \tfrac{1}{4}i\left((a)^2 - (a^\dagger)^2\right). \end{aligned} \tag{26}$$

Then one choice of a Cartan subalgebra is

$$L = \gamma_{11} + \gamma_{22} = \tfrac{1}{2}(a^\dagger a + aa^\dagger). \tag{27}$$

This representation is in fact reducible into two irreducible representations of the Lie algebra (24) with the same value of the quadratic Casimir, distinguished by parity under $a \mapsto -a$.

The transformation (20) is a hyperbolic "squeeze mapping" of momentum space if $\det\beta < 0$, a parabolic shear if $\det\beta = 0$, and an elliptic "generalized rotation" through an angle $\theta = \pi\sqrt{(\det\beta)}$ if $\det\beta > 0$. For all β_{ab} with $\det\beta = 1$, $U(\beta)$ has the property $U(\beta)^2 = -1$, and reduces to the inversion mapping

$$U(\beta)p_a U(\beta) = -p_a, \quad \det\beta = 1. \tag{28}$$

This leaves γ_{ab} unchanged, and thus, like the quadratic Casimir, $U(\beta)$ with $\det\beta = 1$ is in the center of the Lie algebra. The full $SL(2,R)$ group is thus spanned by $U(\beta)$ with

$$\det\beta \leq 1, \tag{29}$$

where all transformations with $\det\beta = 1$ are equivalent, and equal to the inversion.

The analogous algebra that acts on the guiding centers has generators $\bar{\gamma}_{ab}$ defined by

$$\tfrac{1}{4}\{R^a, R^b\} = -\hbar(eB)^{-1}\epsilon^{ac}\epsilon^{bd}\bar{\gamma}_{cd}. \tag{30}$$

This obeys the algebra (24) with the same quadratic Casimir.

IV. LANDAU ORBITS AND THEIR GEOMETRY

In keeping with the interpretation that electronic kinetic energy $\varepsilon(\boldsymbol{p})$ derives from an expansion of a band-structure around a band minimum, it will be assumed that

$$\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p}) > \varepsilon(\boldsymbol{0}), \quad \boldsymbol{p} \neq \boldsymbol{0}. \tag{31}$$

A "*regularity condition*" on $\varepsilon(\boldsymbol{p})$ will also be assumed: this is that $\varepsilon(\lambda\boldsymbol{p})$ is analytic in λ for all $\boldsymbol{p} \neq \boldsymbol{0}$. A "*strict monotonicity condition*" will also be assumed so that, for all $\boldsymbol{p} \neq \boldsymbol{0}$, $\varepsilon(\lambda\boldsymbol{p})$ is a strictly monotonically-increasing function in the range $\varepsilon(\boldsymbol{0}) \leq \varepsilon(\lambda\boldsymbol{p}) < \infty$ for $0 \leq \lambda < \infty$, that is unbounded as $\lambda \to \infty$. This implies that all curves $\varepsilon(\boldsymbol{p}) = E > \varepsilon(\boldsymbol{0})$ are compact closed curves in the momentum plane $(p_1, p_2) \in \mathcal{R}_2$, with the topology of a circle S_1, and enclose the origin. These can be parametrized as $\boldsymbol{p}(E,\theta) = \boldsymbol{p}(E,\theta + 2\pi)$ and strict monotonicity implies that the sense of the orbit can be chosen so that, for all θ,

$$w(E,\theta) \equiv \epsilon^{ab}p_a\partial_\theta p_b(E,\theta) > 0, \tag{32}$$

where $\partial_\theta\boldsymbol{p}(E,\theta) \equiv \partial\boldsymbol{p}(E,\theta)/\partial\theta$.

The group velocity of the dispersion is

$$\boldsymbol{v}(\boldsymbol{p}) = \boldsymbol{e}_a\frac{\partial\varepsilon}{\partial p_a}, \tag{33}$$

where $\boldsymbol{v}(-\boldsymbol{p}) = -\boldsymbol{v}(\boldsymbol{p})$. The strict monotonicity and regularity conditions imply that the the only solution of the condition $\boldsymbol{v}(\boldsymbol{p}) = \boldsymbol{0}$ is $\boldsymbol{p} = 0$, and that $\boldsymbol{p}\cdot\boldsymbol{v}(\boldsymbol{p}) \equiv p_a v^a(\boldsymbol{p}) > 0$ for $\boldsymbol{p} \neq 0$. The classical Hamiltonian equation of motion is

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{p}), \quad \frac{dp_a}{dt} = eB\epsilon_{ab}v^b(\boldsymbol{p}) \tag{34}$$

and a classical particle with kinetic energy E moves on the closed momentum-space trajectory $\boldsymbol{p}(E,\theta)$ with

$$\frac{d\theta}{dt} = \frac{eBp_av^a(\boldsymbol{p})}{w(E,\theta)} > 0. \tag{35}$$

It is worth stressing that, in marked contrast to the case of Newtonian dynamics in free space, where $\varepsilon(\boldsymbol{p}) = \frac{1}{2}m^{-1}\delta^{ab}p_ap_b$, the Euclidean metric δ_{ab} has no place in the Hamiltonian equations of motion of non-relativistic Landau orbits of electrons inside condensed matter.

While the classical Landau-orbit dynamics is fundamentally Hamiltonian, a classical Lagrangian description becomes possible with an additional restriction on the function $\varepsilon(\boldsymbol{p})$, that the odd function $\boldsymbol{v}(\boldsymbol{p})$ defines an invertible map from the full momentum plane to a compact region $\mathcal{D}_v$ of the velocity plane, so that the kinetic energy function has the form $\varepsilon(\boldsymbol{p}) = \tilde{\varepsilon}(\boldsymbol{v}(\boldsymbol{p}))$, where $\tilde{\varepsilon}(\dot{\boldsymbol{x}})$, $\dot{\boldsymbol{x}} \in \mathcal{D}_v$, is the Lagrangian kinetic energy.

The momentum-space area $\mathcal{A}(E)$ enclosed by the curve $\boldsymbol{p}(E,\theta)$ is

$$\mathcal{A}(E) = \tfrac{1}{2}\int_0^{2\pi} d\theta\epsilon^{ab}p_b\partial_\theta p_b(E,\theta). \tag{36}$$

The strict monotonicity condition means that $\mathcal{A}(E)$ is a strictly monotonically increasing unbounded function of $E \geq \varepsilon(\boldsymbol{0})$, with $\mathcal{A}(E) \to 0$ as $E \to \varepsilon(\boldsymbol{0})^+$. The semiclassical quantization of Landau levels is

$$\mathcal{A}(E_n) = 2\pi\hbar eBs_n > 0, \quad (-1)^{2s_n} = -1. \tag{37}$$

The monotonicity condition ensures that for

$$s_n = (n + \tfrac{1}{2})\mathrm{sgn}(eB), \quad n = 0,1,2,\ldots, \tag{38}$$

$E_{n+1} > E_n$, and n coincides with the conventional Landau-level index.

Because the components of $\boldsymbol{p}$ no longer commute with each other when $\hbar eB \neq 0$, the one-particle quantum Hamiltonian must be given as a series expansion in p_a with a specific ordering. The form used here will be

$$H = \sum_{n=0}^{\infty}\frac{1}{2n!}(p_{a_1}\ldots p_{a_n})A_n^{\{a_i\},\{b_i\}}(p_{b_1}\ldots p_{b_n}), \tag{39}$$

where $A_n^{\{a_i\},\{b_i\}}$ is symmetric under any permutations of indices within the sets $\{a_1,\ldots a_n\}$ or $\{b_1,\ldots,b_n\}$. In the limit $\hbar eB \to 0$, H becomes the function $\varepsilon(\boldsymbol{p})$ in the momentum plane, which is then given in terms of the expansion coefficients $A_n^{\{a_i\},\{b_i\}}$. This form remains Hermitian if translational invariance is broken to make $A_n^{\{a_i\},\{b_i\}}(\boldsymbol{x})$ or $B(\boldsymbol{x})$ adiabatically-varying functions of position.

As a result of translational invariance, the Landau levels are macroscopically degenerate, with one independent state for each London quantum $\Phi_0 = \hbar/e$ of magnetic flux passing through the plane:

$$H|\Psi_{n\alpha}\rangle = E_n(\hbar eB)|\Psi_{n\alpha}\rangle. \tag{40}$$

It will be useful to generalize this Hamiltonian eigenvalue problem to a *Lagrangian* eigenvalue problem

$$\hat{L}(\boldsymbol{v}) \equiv \boldsymbol{v}\cdot\boldsymbol{p} - H,$$
$$\hat{L}(\boldsymbol{v})|\Psi_{n\alpha}(\boldsymbol{v})\rangle = L_n(\boldsymbol{v},\hbar eB)|\Psi_{n\alpha}(\boldsymbol{v})\rangle, \tag{41}$$

where the components of the velocity $\boldsymbol{v}$ are c-number parameters of the Hermitian operator $\hat{L}(\boldsymbol{v}, \hbar eB)$, and the eigenvalue $L_n(\boldsymbol{v})$ is a function of $\boldsymbol{v}$ that characterizes each Landau level, and will be the Lagrangian for the motion of the guiding center of an electron in that level. For the Galilean-invariant Newtonian problem, this Lagrangian reduces to the simple form

$$L_n(\boldsymbol{v}, \hbar eB) = \tfrac{1}{2} m\delta_{ab} v^a v^b - m^{-1}\hbar eB s_n. \tag{42}$$

The eigenstates of the Hamiltonian can be represented in a basis of harmonic oscillator states

$$|\Psi_{n\alpha}\rangle = \sum_{m=0}^{\infty} u_{n,m} \frac{(a^\dagger)^m}{\sqrt{(m!)}} |\phi_\alpha\rangle \equiv f_n(a^\dagger)|\phi_\alpha\rangle, \\ a|\phi_\alpha\rangle = 0, \tag{43}$$

where both $f_n(a^\dagger)$ and the reference state $|\phi_\alpha\rangle$ depend on the choice of unimodular metric $\tilde{g}_{ab}$. Note that while the diagonalization is carried out in a basis defined by a given choice of this metric, it will turn out to be useful to represent each Landau-level eigenstate in this form using a Landau-level-specific choice $\tilde{g}_{ab} \propto \tilde{g}^n_{ab}$, where g^n_{ab} is a (non-unimodular) metric specific to the Landau level.

The question now arises, what is the natural basis to use to resolve the degeneracy of the Landau levels? Since

$$[H, \boldsymbol{R}] = 0, \tag{44}$$

the guiding center is a classical constant of the motion. However, quantum mechanically, its components do not commute, and obey an uncertainty principle

$$\bar{g}_{ab} \left(\langle\Psi|(R^a R^b)|\Psi\rangle - \langle\Psi|R^a|\Psi\rangle\langle\Psi|R^b|\Psi\rangle \right) \geq \ell_B^2, \tag{45}$$

where $\bar{g}_{ab}$ is any Euclidean-signature unimodular metric, and

$$\ell_B = \sqrt{\left(\hbar|eB|^{-1}\right)}. \tag{46}$$

The closest quantum mechanical state to the classical state with a fixed guiding center is a *guiding-center coherent state* $|\Psi_n(\bar{\boldsymbol{x}}, \bar{g})\rangle$ that obeys the inequality (45) as an *equality*, with

$$\langle\Psi_n(\bar{\boldsymbol{x}}, \bar{g})|\boldsymbol{R}|\Psi_n(\bar{\boldsymbol{x}}, \bar{g})\rangle = \bar{\boldsymbol{x}}. \tag{47}$$

The standard treatments of Landau levels based on a Newtonian kinetic energy $\frac{1}{2}m\delta_{ab}v^a v^b$ generally use a coherent state with $\tilde{g}_{ab}$ given by the Euclidean metric δ_{ab}, but since non-Newtonian Hamiltonian dynamics has no dependence on the Euclidean metric, this is no longer the "obvious" choice. Below, we will show that there is a "natural" choice that in general is different for each Landau level, but since *any* unimodular metric is a viable choice, we leave it unspecified for now, and use it to set both $\tilde{g}_{ab}$ and $\bar{g}_{ab}$, so $\bar{g}_{ab} = \tilde{g}_{ab}$.

A unimodular metric defines a *complex structure* through the factorization

$$\tfrac{1}{2}\left(\bar{g}_{ab} + i\epsilon_{ab}\right) = e^*_a e_b \tag{48}$$

where e_a is a complex covariant vector with the property

$$\epsilon^{ab} e^*_a e_b = i. \tag{49}$$

Note that this factorization has a $U(1)$ ambiguity under $e_a \mapsto e_a \exp i\phi$. A dimensionless complex coordinate can be defined by

$$z = e_a x^a / \sqrt{2}\ell_B. \tag{50}$$

The coherent state condition is then

$$b|\Psi_n(\bar{\boldsymbol{x}}, \tilde{g})\rangle = \bar{z}|\Psi_n(\bar{\boldsymbol{x}}, \tilde{g})\rangle, \tag{51}$$

with

$$\bar{z} = e_a \bar{x}^a / \sqrt{2}\ell_B. \tag{52}$$

A common choice of electromagnetic gauge is the so-called "symmetric gauge", which derives from the factorization (48) of the Euclidean metric δ_{ab} in terms of $(e_1, e_2) = 2^{-\frac{1}{2}}(1, i)$. Then the symmetric gauge is given by

$$A_a(\boldsymbol{x}) = 2^{-\frac{1}{2}} i(z^* \boldsymbol{e}_a - z\boldsymbol{e}^*_a). \tag{53}$$

For general $\tilde{g}_{ab}$ with a factorization (48), (53) is still a valid gauge choice $A_a(\boldsymbol{x}, \tilde{g})$ that is now "symmetric" with respect to the unimodular metric $\tilde{g}_{ab} = e^*_a e_b + e_a e^*_b$. In this gauge, the Schrödinger representation of the guiding-center harmonic oscillator operators b and $b^\dagger$ is

$$b = \tfrac{1}{2}z^* + \tfrac{\partial}{\partial z}, \quad b^\dagger = \tfrac{1}{2}z - \tfrac{\partial}{\partial z^*}. \tag{54}$$

where $\partial f(z^*)/\partial z = \partial f(z)/\partial z^* = 0$. The corresponding representation of the dynamical momentum operators harmonic oscillator operators is

$$a = \tfrac{1}{2}z + \tfrac{\partial}{\partial z^*}, \quad a^\dagger = \tfrac{1}{2}z^* - \tfrac{\partial}{\partial z}. \tag{55}$$

This immediately implies that the Schrödinger wavefunction $\Psi_n(\boldsymbol{x}; \bar{\boldsymbol{x}}, \tilde{g}) \equiv \langle\boldsymbol{x}|\Psi_n(\bar{\boldsymbol{x}}, \tilde{g})\rangle$ of the coherent state has the functional form

$$\Psi_n(\boldsymbol{x}; \bar{\boldsymbol{x}}, \tilde{g}) = f_n(z^* - \bar{z}^*) e^{-\frac{1}{2}(\bar{z}^*\bar{z} - 2\bar{z}^* z + z^* z)}, \tag{56}$$

where $f_n(z^*) = (-1)^n f_n(-z^*)$ is a holomorphic function of z^*. Here $(-1)^n$ is the parity of the Landau-orbit coherent state with respect to spatial inversion around the point $\bar{x}$ at which it is centered.

A continuous rotational invariance with respect to the metric $\tilde{g}_{ab}$ exists if

$$[H, L(\tilde{g})] = 0, \quad L(\tilde{g}) = -\tfrac{1}{2}\hbar eB\tilde{g}^{ab}\gamma_{ab}. \tag{57}$$

In this case, the function $f_n(z^*)$ is the homogeneous polynomial

$$f_n(z^*) = \frac{(z^*)^n}{\sqrt{(n!)}}, \tag{58}$$

and an *arbitrary* lowest-Landau level ($n = 0$) wavefunction has the well-known form

$$\Psi_{0\alpha}(\boldsymbol{x}) = F_\alpha(z)e^{-\frac{1}{2}z^* z}, \tag{59}$$

where $F_\alpha(\boldsymbol{z})$ is a holomorphic function. This is a familiar result in the rotationally-symmetric case $\tilde{g}_{ab} = \delta_{ab}$ (when $z^* z = \frac{1}{2}\delta_{ab}x^a x^{*b}/\ell_B^2$), but remains true if (57) holds for *any* $\tilde{g}_{ab}$, when there is a "generalized rotational symmetry".

However, such a continuous symmetry is *not* a generic property of electrons moving inside condensed matter, and the holomorphic lowest-Landau-level property (59) disappears in the absence of this continuous symmetry. Starting with the Laughlin wavefunction, many models for the FQHE have incorporated this holomorphic lowest-Landau level structure, and it has often been regarded as a fundamental ingredient of a theory of the FQHE. However, (as evidenced by the fact that a Laughlin-like FQHE state also occurs in the second Landau level[2]) the holomorphic structure (59) can only have "toy model" status as a convenient simplification, and thus *can play no fundamental role in a generic theory of the FQHE.*

In contrast, the holomorphic coherent-state property (56) is quite generic, with no requirement for a continuous rotational symmetry. In the generic absence of such a symmetry, the functions $f_n(z)$ are *holomorphic, but not polynomial.* The coherent state has the structure

$$|\Psi_n(\boldsymbol{x},\tilde{g})\rangle = f_n(a^\dagger)|\bar{\boldsymbol{x}},\tilde{g}\rangle, \tag{60}$$

where

$$a|\bar{\boldsymbol{x}},\tilde{g}\rangle = 0, \quad b|\bar{\boldsymbol{x}},\tilde{g}\rangle = \bar{z}|\bar{\boldsymbol{x}},\tilde{g}\rangle. \tag{61}$$

Holomorphic functions are characterized by their zeroes (plus their asymptotic behavior for large z). However, the location of the zeroes of the functions $f_n(z^*)$ will vary continuously with the so-far arbitrary choice of the coherent state metric $\tilde{g}_{ab} = \bar{g}_{ab}$ (apart from the generic occurrence of a zero at $z^* = 0$ if n is odd, and its absence if n is even, as a consequence of momentum-space inversion symmetry (2).

To use $f_n(z^*)$ for a standardized description, a "natural choice" of metric must be uncovered. The key to this will be a property of the Landau levels that is now generally called their "Hall viscosity", first discussed in the context of 2D Landau levels by Avron, Seiler and Zograf (ASZ)[12]. This will however need a significant reformulation of the Hall viscosity in the absence of a continuous rotational symmetry.

V. HALL VISCOSITY AND THE LANDAU-ORBIT METRIC

The viscosity of a continuous fluid is the usually-dissipative relation between the stress (the current-density of momentum) in a fluid and the gradient of its flow velocity. The use of a formalism with consistent index-placement that distinguishes between covariant and contravariant indices will be important in clarifying the structure of the stress tensor when rotational invariance is not present.

Momentum, like a force, is a covariant vector, and the momentum density $T^0_{\ a}(\boldsymbol{x},t)$ is locally conserved in translationally-invariant continuum dynamics. If translational invariance is broken by coupling to background fields, the continuity relation is

$$\partial_t T^0_{\ a} + \partial_b T^b_{\ a} = F_a, \tag{62}$$

where $T^a_{\ b}(x,t)$ is the mixed-index stress tensor, and $F_a(x)$ is the body force exerted on the continuous medium by the background fields that break translational invariance.

As a mixed-index tensor, there is in general no symmetry relating the two indices. However, if the system has a rotational invariance under coordinate transformations that leaves the Euclidean metric tensor δ_{ab} invariant, the stress tensor has the Cauchy symmetry property

$$\delta_{ac}T^c_{\ b} - \delta_{bc}T^c_{\ a} = 0. \tag{63}$$

When there is such a rotational invariance, it is well-known that it is possible to obtain the expression for the stress tensor from the variation of the action with respect to the metric that defines the invariance

$$\delta S = -\tfrac{1}{2}\int d^4x\sqrt{(-g)}T_{\mu\nu}\delta g^{\mu\nu}. \tag{64}$$

The model for this is the derivation of the stress-energy tensor $T^\mu_{\ \nu}$ in General Relativity (GR), where (local) rotational invariance generalizes to (local) Lorentz invariance. and the Cauchy relation becomes

$$g_{\mu\sigma}T^\sigma_{\ \nu} - T^\sigma_{\ \mu}g_{\sigma\nu} = 0, \tag{65}$$

where $g_{\mu\nu}$ is the gravitational metric.

The relation (64) (or its dual where $g_{\mu\nu}$ is varied) is often taken as the fundamental definition of the stress-energy tensor, given in the form $T_{\mu\nu} = g_{\mu\sigma}T^\sigma_{\ \nu}$ or $T^{\mu\nu} = T^\mu_{\ \sigma}g^{\sigma\nu}$, but this is a symptom of a common practice in GR to hide dependence on the metric by using it to raise or lower indices away from their "natural" positions (*e.g.* by forming metric-dependent constructions such as $F^{\mu\nu} = g^{\mu\sigma}g^{\nu\tau}F_{\sigma\tau}$).

However, on a fundamental level, stress (or stress-energy) is the response to *strain* of the matter fields in space (or space-time), and what is arguably its true defining relation is the variation of the action under the infinitesimal *active* diffeomorphism of the matter fields $\varphi(x) \mapsto \varphi(x') = \varphi(x + \delta u(x))$, where $\delta u^\mu(x) = u^\mu(x)\delta\lambda$. This definition of the stress-energy tensor is

$$\delta S = \int d^4x\sqrt{(-g)}T^\mu_{\ \nu}\nabla_\mu\delta u^\nu. \tag{66}$$

The change in the inverse metric under the diffeomorphism (*i.e.*, after the pull-back of the coordinate system to match the diffeomorphism) is

$$g^{\mu\nu}(x) \mapsto g^{\mu\nu}(x') + \delta g^{\mu\nu}(x') \tag{67}$$

with

$$\begin{aligned}\delta g^{\mu\nu} &= \mathcal{L}_u(g^{\mu\nu})\delta\lambda \\ &= -(g^{\mu\sigma}\nabla_\sigma \delta u^\nu + g^{\nu\sigma}\nabla_\sigma \delta u^\mu),\end{aligned} \tag{68}$$

where $\mathcal{L}_u(\ldots)$ is the Lie derivative. Then the combination of the fundamental definition (66)) with the local Lorentz-invariance property (65) leads to the familiar result (64).

The QHE is exhibited by incompressible 2D electron fluids. In the case of the integer QHE, this results from the Pauli principle, where (in the zero-temperature limit) all Landau levels are either filled or empty. In this case, there are no gapless collective degrees of freedom of the electron gas that can transmit forces through the bulk, or allow dissipation of energy.

This means that the gapped quantum incompressible liquid is quite different from the notional classical incompressible liquid, which is just the limiting case of a classical liquid with a large bulk modulus. The non-relativistic classical fluid supports sound waves (pressure waves), which travel with a group velocity that diverges in the limit in which the bulk modulus becomes infinite. (This of course violates causality as the speed of sound eventually exceeds the Lorentz speed c, so the "thought experiment" of taking the incompressible limit of a classical non-relativistic fluid cannot in principle be realized.) Formally, the classical incompressible fluid supports a hydrostatic pressure which is instantaneously spatially-uniform throughout the fluid, so the trace of the stress tensor is spatially homogeneous:

$$T^a_{\ a}(\boldsymbol{x}, t) = p(\boldsymbol{t})\delta^a_a. \tag{69}$$

However the gapped quantum incompressible fluid does not support sound waves, and thus has a *traceless* stress tensor:

$$T^a_{\ a}(\boldsymbol{x}, t) = 0. \tag{70}$$

The viscosity tensor is the linear relation between the stress in a fluid and a non-uniform flow velocity field $\boldsymbol{v}(\boldsymbol{x}, t) = v^a \boldsymbol{e}_a$ that breaks translational invariance. It is also fundamentally a locally-defined mixed-index tensor

$$T^a_b(\boldsymbol{x}, t) = \eta^{a\ c}_{\ b\ d}(\boldsymbol{x}, t)\partial_c v^d(\boldsymbol{x}, t) + O(v^2). \tag{71}$$

In the zero-temperature limit, the viscosity of the gapped incompressible fluid must be dissipationless, which is the antisymmetry condition

$$\eta^{a\ c}_{\ b\ d} = -\eta^{c\ a}_{\ d\ b}. \tag{72}$$

The dissipationless viscosity must also be odd under time reversal.

In two dimensions, the tracelessness and antisymmetry of the viscosity tensor that is implied by quantum incompressibility means that it must have the structure

$$\eta^{e\ f}_{\ b\ d} = \epsilon^{ae}\epsilon^{cf}\eta^H_{abcd}, \tag{73}$$

$$\begin{aligned}\eta^H_{abcd} &= \eta^H_{bacd} = \eta^H_{abdc} = -\eta^H_{cdab} \qquad (74)\\ &= \tfrac{1}{2}\epsilon\left(\epsilon_{ac}\eta^H_{bd} + \epsilon_{ad}\eta^H_{bc} + \epsilon_{bc}\eta^H_{ad} + \epsilon_{bd}\eta^H_{ac}\right). \qquad (75)\end{aligned}$$

where $\epsilon \equiv \epsilon_{12}$ is the Pfaffian of the 2D Levi-Civita symbol. Here the rank-4 tensor η^H_{abcd} is odd under time-reversal, and $\eta^H_{ab}(\boldsymbol{x}, t)$ is a symmetric rank-2 covariant tensor that is also odd under time-reversal. This rank-2 tensor will here be called the 2D "Hall viscosity" tensor, and satisfies the stability condition

$$\det \eta^H \geq 0. \tag{76}$$

If there is rotational invariance, η^H_{ab} must be proportional to the Euclidean metric δ_{ab}. Previous work by other authors on the Hall viscosity has generally specialized to this case, with the correspondence

$$\eta^H_{ab} = \bar{\eta}^H \delta_{ab}. \tag{77}$$

In particular, Read[13] has provided an interpretation of the time-reversal-odd scalar $\bar{\eta}^H$ as

$$\bar{\eta}^H = \tfrac{1}{2}\hbar n_e \bar{s} = \tfrac{1}{2}\bar{l} \tag{78}$$

where n_e is the 2D electron density, and $\bar{s}$ is an "intrinsic orbital spin per electron", or, equivalently, that $\bar{l}$ is the (local) density of "intrinsic orbital angular momentum" of the electron fluid, a quantity that only has meaning if there is a continuous rotational symmetry. Equivalently, Read's formula can be stated as the result that, when there is such a rotational symmetry,

$$\delta^{ab}\eta^H_{ab} = \bar{l}. \tag{79}$$

To compute the stress tensor, we must consider the linear area-preserving infinitesimal active diffeomorphisms of the electronic degrees of freedom

$$x^a_i \mapsto \left(\delta^a_c + \delta\beta^{ab}\epsilon_{bc}\right) x^c_i, \tag{80}$$

$$p_{i,a} \mapsto \left(\delta^c_a - \delta\beta^{cb}\epsilon_{ba}\right) p_{i,c}, \tag{81}$$

$$R^a_i \mapsto \left(\delta^a_c + \delta\beta^{ab}\epsilon_{bc}\right) R^c_i, \tag{82}$$

where i is a particle label, and $\delta\beta^{ab} = \delta\beta^{ba}$. This preserves the commutation relations among the components of $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$, so the linear diffeomorphism produces a unitary transformation of a quantum state, and defines a family of normalized many-body states

$$|\Psi(\beta)\rangle = \prod_i \exp(i\beta^{ab}\gamma_{i,ab})\exp(i\beta^{ab}\bar{\gamma}_{i,ab})|\Psi\rangle. \tag{83}$$

where $\gamma_{i,ab}$ and $\bar{\gamma}_{i,ab}$ are the deformation operators that act on particle i.

The special property of many-body states where all Landau levels are either completely filled or completely empty is that *they are invariant (up to a phase) under diffeomorphisms of the guiding centers*, and

$$\prod_i \exp(i\beta^{ab}\bar{\gamma}_{ab,i})|\Psi\rangle = e^{i\phi}|\Psi\rangle. \tag{84}$$

This is because such a diffeomorphism does not mix Landau levels, and merely produces a unitary change of basis within each Landau level, and the Slater determinant describing a completely filled Landau level is invariant under such a change of basis, up to a phase.

Following the work of ASZ, the fundamental formula for the dissipationless Hall viscosity is obtained from the Berry curvature

$$\mathcal{F}_{ab,cd} = -i\left(\left\langle \frac{\partial\Psi}{\partial\beta^{ab}}\middle| \frac{\partial\Psi}{\partial\beta^{cd}}\right\rangle - \left\langle \frac{\partial\Psi}{\partial\beta^{cd}}\middle| \frac{\partial\Psi}{\partial\beta^{ab}}\right\rangle\right). \quad (85)$$

For a Slater determinant state of filled Landau levels, there is no contribution from $\bar{\gamma}_{ab}$ because of the guiding-center-diffeomorphism invariance, and the Berry curvature is a trace over contributions from each occupied single-particle state

$$\mathcal{F}_{ab,cd} = \sum_{n,\alpha} f^n_{ab,cd}, \quad (86)$$

where n, α labels an orthonormal basis of one-particle states present in the Slater determinant, and $f^n_{ab,cd}$ is the Berry curvature of a single-particle state $|\psi_{n\alpha}\rangle$ in the Landau level with index n:

$$f^n_{ab,cd} = -i\langle\psi_{n\alpha}|[\gamma_{ab}, \gamma_{cd}]|\psi_{n\alpha}\rangle, \quad (87)$$

where the commutator is given by (24). Then the extensive quantity $\mathcal{F}_{ab,cd}$ is given by

$$\mathcal{F}_{ab,cd} = \frac{A}{2\pi\ell_B^2}\sum_n \nu_n f^n_{ab,cd} \quad (88)$$

where A is the area of the system, and ν_n is the occupation of Landau level n. The fundamental identification with the Hall viscosity of a uniform system is

$$\hbar\mathcal{F}_{ab,cd} = A\eta^H_{abcd}, \quad (89)$$

or, since the viscosity is a local relation,

$$\hbar\mathcal{F}_{ab,cd} = \int d^2x\, \eta^H_{abcd}(\boldsymbol{x}). \quad (90)$$

In each Landau level, we can define a metric g^n_{ab} by

$$\langle\psi_{n\alpha}|\{p_a, p_b\}|\psi_{n\alpha}\rangle = -4\pi\hbar^2 s_n \mathrm{sgn(eB)} g^n_{ab} \quad (91)$$

where

$$s_n = -\mathrm{sgn}(eB)(n + \tfrac{1}{2}) \quad (92)$$

is a *topological spin* of the Landau level that appears in the semiclassical formula (37) and can also be obtained by inspection of the Landau-level coherent state calculated with $\tilde{g}_{ab} \propto g^n_{ab}$ where g^n_{ab} is the (non-unimodular) Euclidean-signature metric of a dimensionless distance measure associated with the Landau level. Note that while the "topological spin" s_n coincides with what Read has called the "intrinsic orbital angular momentum" of a rotationally-invariant Landau level, it is a topological invariant with an existence independent of the presence or absence of a rotational symmetry, described in more detail later.

The Landau-orbit contribution to the Hall viscosity is then obtained as

$$\eta^H_{ab} = \frac{\hbar}{2}\sum_n \nu_n s_n g^n_{ab}, \quad (93)$$

where in the IQHE $\nu_n = 1$ if Landau level n is filled and $\nu_n = 0$ if it is empty. For a rotationally-invariant system, all Landau levels have the same metric

$$g_{ab} = \frac{|eB|}{2\pi\hbar}\delta_{ab}. \quad (94)$$

We have now identified a "natural" unimodular metric $\tilde{g}^n_{ab} \propto g^n_{ab}$ associated with each Landau level, that allows the definition of a standard localized Landau-orbit coherent state.

VI. ZEROES OF THE LANDAU-ORBIT WAVEFUNCTION AND TOPOLOGICAL SPIN

The identification of an intrinsic unimodular metric $\tilde{g}^n_{ab}$ associated with the shape of the Landau orbit in a given Landau level now allows a systematic analysis of its wavefunction.

A basis of states in momentum space is defined by

$$\tfrac{1}{2}\tilde{g}^{ab}p_a p_b|n, \tilde{g}\rangle = \hbar|eB|(n + \tfrac{1}{2})|n, \tilde{g}\rangle. \quad (95)$$

where $\tilde{g}_{ab}$ is some unimodular metric. Assuming that the expansion (39) is truncated at some finite maximum $n = n_{\max}$, there are only non-vanishing matrix elements $\langle n_1|H|n_2\rangle$ for $|n_1 - n_2| \le 2n_{\max}$. An arbitrary choice of $\tilde{g}_{ab}$ can initially be made, and the problem can be diagonalized in a basis truncated to $n \le N$ for some suitably-large N. For each eigenstate of interest, the metric g^n_{ab} (91) is calculated, and the problem is rediagonalized in a basis (95) with $\tilde{g}_{ab} = \tilde{g}^n_{ab} \propto g^n_{ab}$, and a suitable large truncation (this process can be iterated if necessary). A Taylor-series expansion that approximates $|\psi_n\rangle$ by

$$|\psi_n\rangle = f_n(a^\dagger)|0, \tilde{g}^n\rangle \quad (96)$$

where $f_n(z^*)$ is a finite-degree polynomial is then obtained, and the zeroes of the polynomial can be obtained with a standard root finder. We studied models with quartic dispersions, $2n_{\max} = 4$, and used MPACK[14], a version of LAPACK adapted to use the GMP multiprecision library to get very accurate eigenstates, and studied how the patterns of zeroes varied with the truncation N. We observed that the polynomial roots $\{z_i\}$ converged to stable values inside some radius in the complex plane (which is also a representation of the momentum plane) which grew as N was increased. The use of MPACK allowed accurate determination of the zeroes for large N of order 1000, exposing them in a large "window" in momentum space.

The polynomial form

$$|\Psi|^2 \propto \exp V, \quad V(z,z^*) = -|z|^2 + \sum_i \ln|z-z_i|^2 \quad (97)$$

gives the amplitude of the Landau-orbit wavefunction in momentum space in terms of a potential that is a solution of Poisson's equation with a Laplacian defined by the metric $\tilde{g}^n_{ab}$, with a uniform positive charge inside a circular disk of large radius (that fixes the asymptotic form of the holomorphic function $f_n(z^*)$), and unit negative point charges at the zeroes.

For a strictly-monotonic dispersion $\varepsilon(\boldsymbol{p})$, The semiclassical quantization identifies the n'th quantized Landau orbit with a region of momentum space with total area $2\pi\hbar|eB|$ between the close contour $\mathcal{A}(E^-) = 2\pi n\hbar|eB|$ and $A(E^+) = 2\pi(n+1)\hbar|eB|$. This region has the topology of a disk (Euler characteristic $\chi = 1$) for $n = 0$, or an annulus (Euler characteristic $\chi = 0$) for $n > 0$, and can be thought of as a "fattened" semiclassical orbit. When $f_n(z^+)$ is obtained using the metric g^n_{ab}, we observed that it is possible to choose a threshold c so the region defined by $|\Psi_n(z,z^*)|^2 = |\Psi_n(\boldsymbol{p})|^2 > c$ covers an area of order $2\pi\hbar|eB|$, and defines a zero-free region with the same topology as the semiclassical prediction, that is topologically a disk for $n = 0$, and for $n > 0$ is an annulus that encircles n zeroes, hence defining a "topological spin" $s_n = n + \frac{1}{2}$ of the Landau level. It is also possible to choose c so most of the weight of the wavefunction is within the region.

This picture remains true when $f_n(z^*)$ is calculated using a different metric that is close to $\tilde{g}^n_{ab}$, but eventually breaks down for a choice of metric sufficiently different from $\tilde{g}^n_{ab}$: the pattern of zeroes changes with the metric, and for $n > 0$ the central zeroes eventually begin to "leak out" of the annulus defined by the "fattened" semiclassical orbit.

When a guiding-center coherent state of the Landau orbit is constructed using $\bar{g}_{ab} = \tilde{g}^n_{ab}$, there is also a mapping from the complex plane to real space, with the origin at the guiding center of the orbit, and the zeroes of $f(z^*)$ become the zeros of the coherent state wavefunction.

The analysis of the generic Landau orbits has now identified a topological spin s_n and a non-unimodular metric g^n_{ab} as well as its unimodular version $\tilde{g}^n_{ab}$, which in the rotationally-invariant case reduces to $\tilde{g}^n_{ab} = \delta_{ab}$.

VII. A CASE STUDY: THE QUARTIC DISPERSION

We will now present a detailed case study of the simplest model dispersion that can exhibits the generic property that there is no congruence between the shapes if different semiclassical orbits.

This is the model with quadratic and quartic terms in the dispersion:

$$H = A^{ab}_1 p_a p_b + \tfrac{1}{4} A^{abcd}_2 \{p_a, p_b\}\{p_c, p_d\}. \quad (98)$$

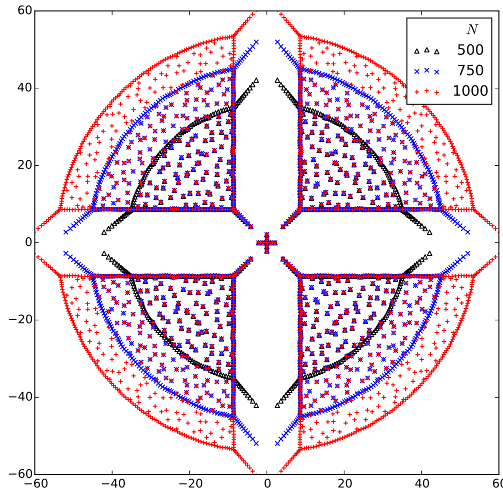

FIG. 1. Zero distribution of f_{10} for class I quartic term with $c - 1 = 2$ at different truncations N. Here N is not the degree of the polynomial, but rather the number of fixed-parity states we keep for diagonalization. For each N, we use a sufficiently high precision so that any higher precision would not do any better in terms of the locations of the zeroes. As N increases, the region covered by the zeroes also expands. However, the existing pattern of zero distribution doesn't change with higher truncation, which justifies the validity of our results.

Up to the addition of the quadratic Casimir, which is a constant, and does not affect the semiclassical orbits, there are four possible classes of quartic terms compatible with strict monotonicity. If $\tilde{g}^{ab}_i$ are the inverses of unimodular metrics:

$$\begin{aligned}
\tilde{A}^{ab,cd}_{2,\mathrm{I}} &= \tilde{g}^{ab}_1 \tilde{g}^{cd}_2 + \tilde{g}^{ab}_2 \tilde{g}^{cd}_1, \quad \tfrac{1}{2}(\tilde{g}^{ab}_1 + g^{ab}_2) = c\tilde{g}^{ab}_0, \\
\tilde{A}^{ab,cd}_{2,\mathrm{II}} &= \tilde{g}^{ab}_1 u^c_1 u^d_1 + u^a_1 u^b_1 \tilde{g}^{cd}_1, \\
\tilde{A}^{ab,cd}_{2,\mathrm{III}} &= u^a_1 u^b_1 u^c_2 u^d_2 + u^a_2 u^b_2 u^c_1 u^d_1, \ \ u^a_1 u^b_1 + u^a_2 u^b_2 = \lambda \tilde{g}^{ab}_0, \\
A^{ab,cd}_{2,\mathrm{IV}} &= u^a_1 u^b_1 u^c_1 u^d_1.
\end{aligned} \quad (99)$$

where $\lambda > 0$, and $c \geq 1$. Strict monotonicity also requires that the quadratic term A^{ab}_1 has the form $\lambda' \tilde{g}'^{ab}$, $\lambda' \geq 0$, but the weaker condition that the spectrum of H is stable (*i.e.*, bounded below) only requires that $\epsilon_{ac}\epsilon_{bd} A^{ab}_1 u^c_i u^d_i \geq 0$ in classes II, III, IV.

The class I and III quadratic terms have a C_4 symmetry with inverse metric $\tilde{g}^{ab}_0$. If $A^{ab}_1 \propto g^{ab}_0$ the full model also has this symmetry, and all Landau orbits have the same metric given by the inverse of $\tilde{g}^{ab}_0$. If, in addition, $\tilde{g}^{ab}_1 = \tilde{g}^{ab}_2 = \tilde{g}^{ab}_0$, so $c = 1$, class I models have a continuous $SO(2)$ symmetry.

We first review the purely quadratic case, where the quartic term is absent, $A^{abcd}_2 = 0$. In this case

$$H = \tfrac{1}{2} E_0 \tilde{g}^{ab} \gamma_{ab}, \quad E_0 > 0, \quad (100)$$

where $\tilde{g}^{ab}$ is the inverse of a unimodular metric $\tilde{g}_{ab}$ and there is a pseudo-Galilean invariance with δ_{ab} replaced

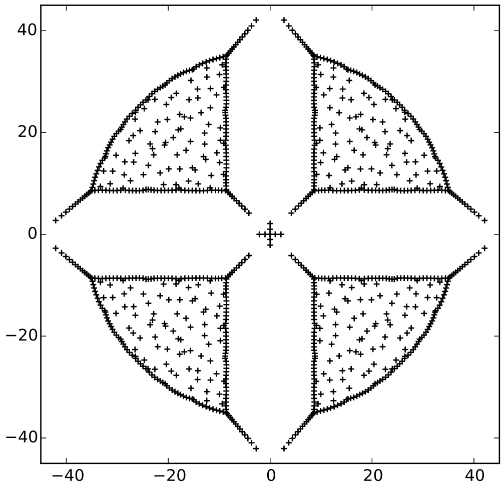

FIG. 2. Zeroes of f_{10} for class I quartic term with $c-1=2$ at a truncation of 500 fixed-parity states. Ten central zeroes, with two degenerate ones at the origin, form a cross due to the C_4 symmetry. The peripheral zeroes have such interesting features as four inward-pointing "spikes", lines with constant density that bound 2D regions of quasi-uniform distributions. The arcs on the boundary are generated due to finite truncation and should not be regarded as part of the zero pattern.

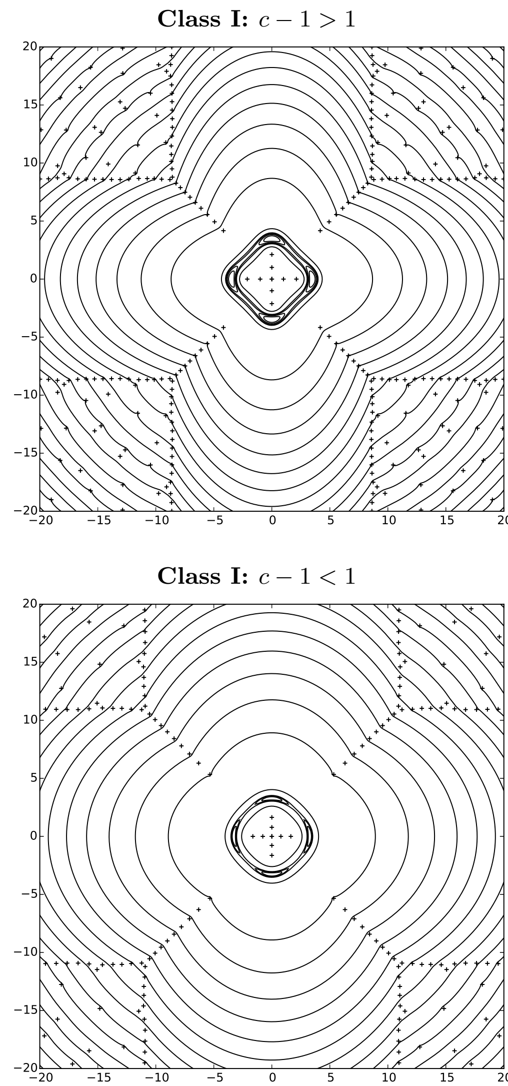

FIG. 3. Contour plots of $\ln|\Psi_{10}|^2$ for class I quartic term with $c-1>1$ and $c-1<1$ respectively. Both plots show a piece-wise feature of the contours with the four spikes and line charges as branch cuts. Another common feature is the existence of four maxima along the directions of the central cross and four saddle points along those of the spikes. Despite the similarities, the two plots also show qualitatively different shapes of the semiclassical orbits, which can be roughly pictured as those contours closest to the maxima and saddle points. We see that $c-1>1$ corresponds to concave shapes while $c-1>1$ to convex ones.

by $\tilde{g}_{ab}$, and m_e replaced by $(E_0\ell_B^2/\hbar)^{-1}$. However, the resemblance to the Newtonian model is superficial, as $(E_0\ell_B^2)^{-1}$ and $\tilde{g}_{ab}$ can be spatially-varying (and their product can have Gaussian curvature) on a completely-geometrically-flat plane with a Cartesian coordinate system. In this non-generic case, the Landau-orbit metric is $\tilde{g}_{ab}$ independent of the Landau index, with all the zeroes of the Landau-orbit coherent state at its origin.

The next-simplest case is the purely-quartic class I model, which also has the non-generic feature of a common metric $\tilde{g}_{ab}$ for all Landau orbits, because of its hidden C_4 four-fold rotation symmetry. The generic model with the strict-monotonicity property is given by

$$H = \tfrac{1}{2}E_0\{\tilde{g}_1^{ab}\gamma_{ab}, \tilde{g}_2^{cd}\gamma_{cd}\} \tag{101}$$

where $\tilde{g}_1^{ab}$ and $\tilde{g}_2^{ab}$ are the inverses of two distinct unimodular metrics $\tilde{g}^1_{ab}$ and $\tilde{g}^2_{ab}$. in this case

$$\tfrac{1}{2}\left(\tilde{g}^1_{ab} + \tilde{g}^2_{ab}\right) = c\tilde{g}_{ab}, \quad c \geq 1, \tag{102}$$

where $\tilde{g}_{ab}$ is the common unimodular metric of all the Landau orbits. After a $SL(2,R)$ transformation the Hamiltonian is put into the form

$$H = E_0\left((\gamma_{11}+\gamma_{22})^2 + (c-1)\{\gamma_{11},\gamma_{22}\}\right) \tag{103}$$

which has an explicit C_4 symmetry under $(p_1,p_2) \mapsto (-p_2,p_1)$, or $\gamma_{11} \leftrightarrow \gamma_{22}$. In the harmonic oscillator representation with $[a,a^\dagger]=1$,

$$\gamma_{11} = \tfrac{1}{2}(a+a^\dagger)^2, \quad \gamma_{22} = -\tfrac{1}{2}(a-a^\dagger)^2. \tag{104}$$

and

$$|\psi_n\rangle = f_n(a^\dagger)|0\rangle, \quad a|0\rangle = 0. \tag{105}$$

The C_4 symmetry is then the symmetry of H under $a^\dagger \mapsto ia^\dagger$.

We treated the problem numerically by projecting the Hamiltonian into the Hilbert space spanned by states where f_n are polynomials of finite degree $N \gg n$, and solving a tridiagonal Hermitian eigenproblem to obtain

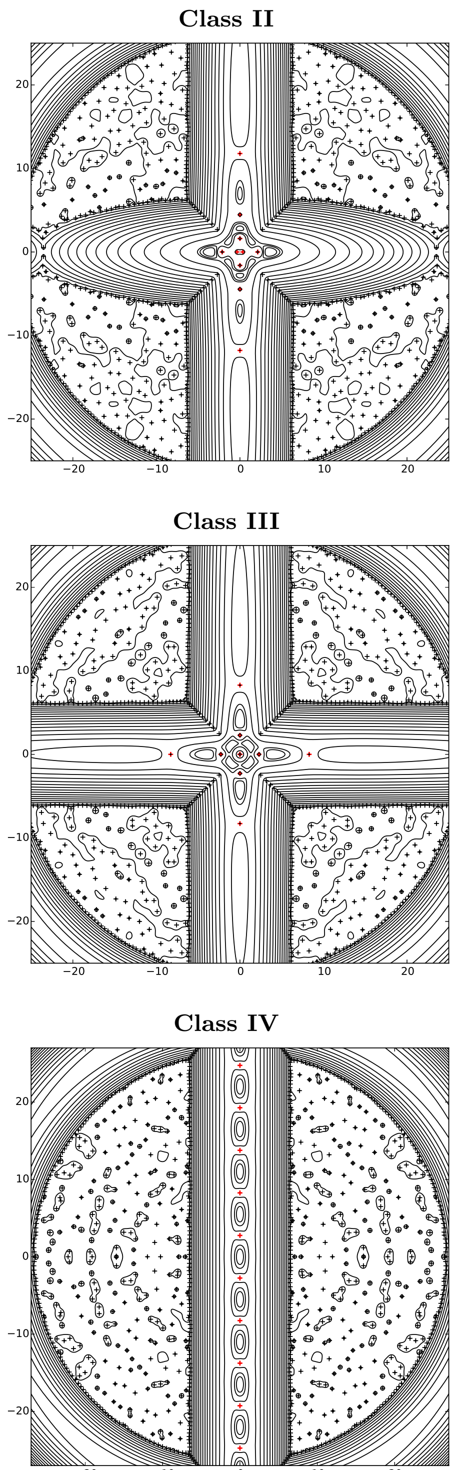

FIG. 4. Contour plots of $\ln|\Psi_{10}|^2$ for class II-IV quartic terms, with $\{p_x^2, p_x^2+p_y^2\}$, $\{p_x^2, p_y^2\}$, p_x^4 as their Hamiltonians respectively. Although classes II, III have closed contours that enclose part of the central zeroes (red pluses), none of the three possesses a clean separation between all the central zeroes and the rest ones. The contours inside the 2D-distribution regions also demonstrate qualitatively different features from those of class I: the puddle-like shapes indicate uniformly vanishing amplitudes.

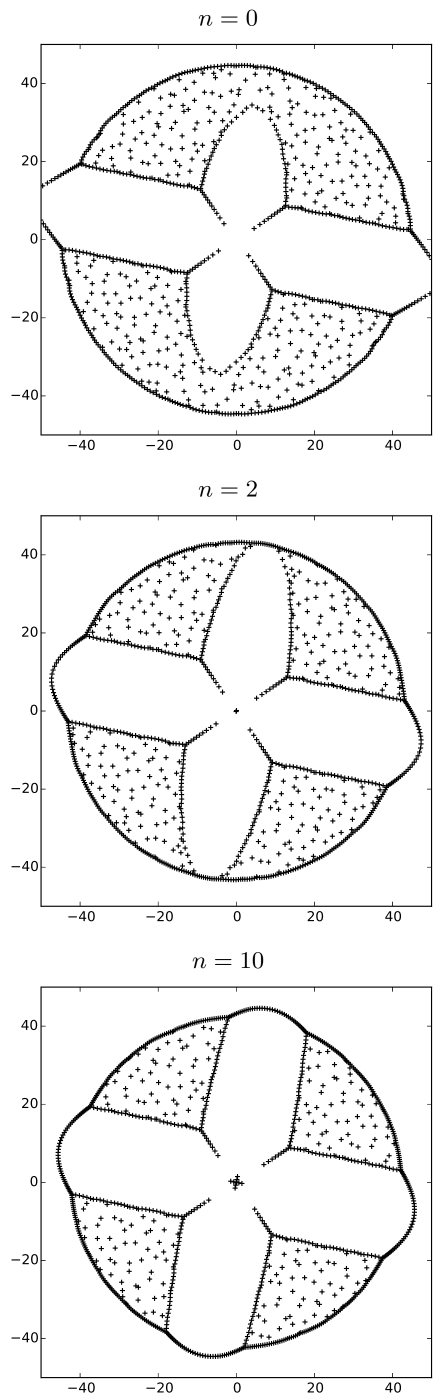

FIG. 5. Restoration of the C_4 symmetry as the Landau level index n increases. Here we adopted a different parametrization of the quartic terms. The Hamiltonian assumes the form $20p_x^2 + 20p_y^2 + 2p_x^4 + 3p_y^4 + 4\{p_x^2, p_y^2\} + \{p_x, p_y^3\}$.

the Taylor series for f_n, and then used a root finder to obtain approximate eigenstates in the form

$$|\psi_n\rangle \propto \prod_{i=1}^{N} (a^\dagger - z_i^*)|0\rangle. \tag{106}$$

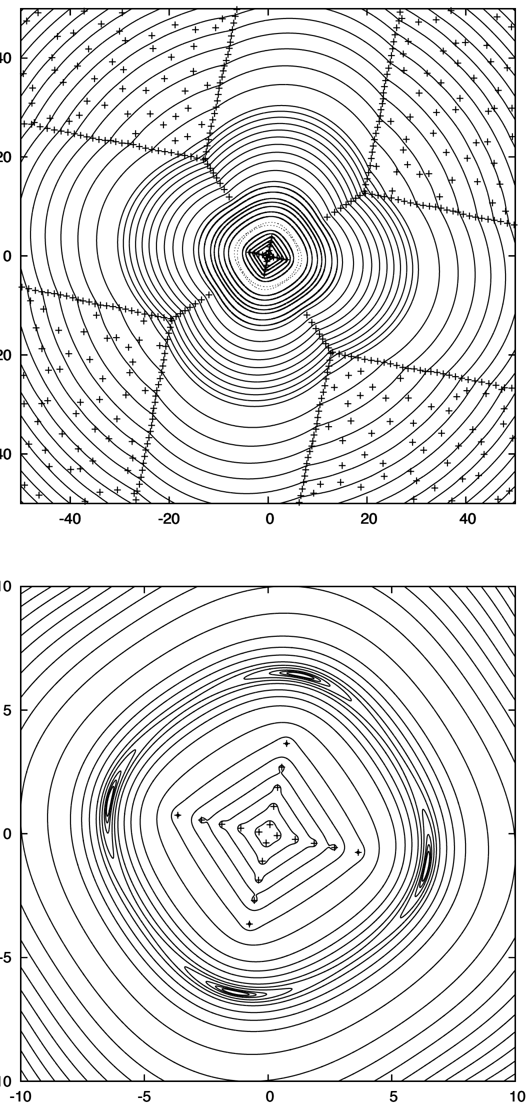

FIG. 6. Contour plots of $\ln|\Psi_{20}|^2$ for a Hamiltonian $p_x^2 + p_y^2 + 2p_x^4 + 3p_y^4 + 4\{p_x^2, p_y^2\} + \{p_x, p_y^3\}$. The contours have the same piece-wise structures as in the case of pure quartic terms. Also note that the central cross still points to the maxima, and that the spikes to the saddle points.

To accurately explore the root structure $\{z_i^*\}$, we carried out the diagonalization using MPACK[14], an adaptation of the linear algebra routines from LAPACK to use the GMP library for arbitrary-precision floating-point calculations. These calculations reveal the zeroes of $f_n(z^*)$ in some region $|z| < R_N$, with truncation-dependent features tied to the boundary $|z| \approx R_N$, but with a structure for $|z| \ll R_N$ that becomes independent of N as it, and consequently R_N, is increased, as illustrated in Fig.1. We therefore believe that this calculational method reveals the true structure of roots of the holomorphic non-polynomial function $f_n(z^*)$ in a range $|z| < R$ that can be increased at will at the expense of increasing the floating-point precision of the numerical diagonalization.

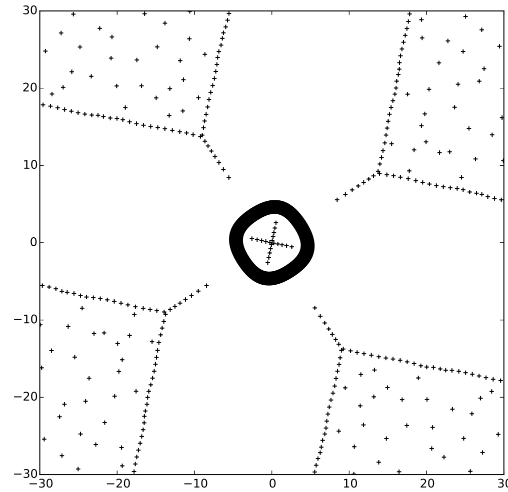

FIG. 7. Black region with 90 percent of the total probability. The annulus is bounded by contours at the same value of the Coulomb potential and cleanly separates the central zeroes from the rest of the structure.

Given that the exact eigenstate satisfies a three-term recurrence relation (or a five-term recurrence relation if quadratic terms are included) that depends on its eigenvalue, it is possible that, once the eigenvalue has been accurately determined, the Taylor series could be extended beyond the truncation point using the recursion relation, but we found that the eigenvector of the truncated tridiagonal matrix was accurate enough for finding the pattern of roots, when calculated with high precision.

A typical layout (Class I) of the roots is presented in Fig. 2. The central zeroes, the number of which determines the topological spin, are organized in a cross shape due to the C_4 symmetry. There are n mod 4 degenerate zeroes at the origin, and the degeneracy will be lifted once we add quadratic terms. Another noticeable feature is the four "spikes" that are poking inward and alternate with the four limbs of the central cross. The regions with seemingly uniform 2D distributions are bounded by line "charges" with constant density. The precise locations of the zeroes inside those 2D regions are sensitive to the perturbation of the Hamiltonian, but we believe the existence of such "darker" areas is of universal significance.

The structure of the roots strongly suggests that there is a semiclassical treatment in which $\ln f_n(z)$ is approximated by a piecewise-holomorphic function in regions separated by branch cuts along the apparent lines of zeroes, with a modification of the background 2D charge-density distribution of the Poisson equation in the regions where the roots seem to be a quasi-uniform 2D distribution. To visualize that, we show in Fig. 3 the contours of the 2D Coulomb potential $V(z, z^*)$, which is $\ln f_n(z)$ plus a quadratic potential derived from the Gaussian factor. Note that the actual topography has a "volcano crater" at the center enclosed by an annulus-like region containing the local maxima and saddle points. The fact that all the central zeroes are inside the crater and separated from the rest of the structure is critical for a semiclassical definition of the topological spin. We will see shortly that the same statement doesn't apply to classes II, III, or IV. Additionally, for class I there is a free parameter $c-1$ that controls the shape of contours of the dispersion function $\varepsilon(\boldsymbol{p})$. A simple calculation reveals that $c-1<1$ corresponds to contours in momentum space with concave shapes, while $c-1>1$ to convex ones.

Now we move on to the other three classes of quartic terms. As can be seen from the definitions, their qualitative difference from class I is that the classical dispersion $\varepsilon(\boldsymbol{p})$ is zero along one (for classes II, IV) or two (for class III) radial directions on the momentum plane. As a result, the classical orbits for those three classes are not closed, but rather extend to infinity. Though quantum mechanically the noncommutativity of p_a gives rise to effective quadratic terms for classes II and III, thus closing the quantum orbits, we expect that even in the quantum case a clean separation between the central zeroes and the outer structure by a set of contours *cannot* be achieved for any of the three classes, which is verified in Fig. 4.

To compete the discussion, we add quadratic terms to the Hamiltonian, which is parametrized as

$$\begin{aligned} H &= Ap_x^2 + Bp_y^2 + Cp_x^4 + Dp_y^4 \\ &\quad + E\{p_x^2, p_y^2\} + F\{p_x, p_y^3\} + G\{p_x^3, p_y\}. \end{aligned} \tag{107}$$

Now the C_4 symmetry is generically not there. However, as we go to high-energy states, the quartic terms will eventually dominate. Therefore we expect to see a quasi-C_4 symmetry gradually appear as the Landau index n increases, as shown in Fig. 5. Due to this asymptotic C_4 symmetry, the generic features of the root patterns as well as the contours still hold, i.e., the contours consist of segments of curves patched together with kinks, as illustrated in Fig 6.

Before concluding this section we want to revisit the semiclassical interpretation of the wave function by demonstrating that the region we pick to isolate the central zeroes can indeed encompass as much as 90 percent of the weight of the wavefunction, as shown in Fig. 7. In the rotationally invariant case, $|\Psi_n|^2$ has n zeroes at the origin, a rim of maxima at a radius of order $\sqrt{n}$ and a Gaussian tail extending to infinity. With a moderate perturbation that breaks rotational invariance, the new zeroes will only appear where the original wave function has vanishingly small amplitude, i.e., around the origin or along the Gaussian tail. That explains why we are able to extract a region that accounts for as much as 90 percent of the total probability while covering no zeroes.

VIII. OTHER PROPERTIES OF THE LANDAU ORBIT

There are some other important characteristic physical properties of the Landau orbit. The simplest is its diamagnetic magnetic moment normal to the plane, given by

$$\mu_n = \left.\frac{\partial E_n}{\partial B}\right|_{\varepsilon(\boldsymbol{p})} \tag{108}$$

which in the Newtonian model with Galilean invariance reduces to $\mu_n = \hbar e B s_n / m_e$.

The second property is the effective mass that characterizes the kinetic energy of a guiding center that flows along a line of constant $E_n(\boldsymbol{x})$, if there is slow adiabatic spatial variation of the Hamiltonian. This is given by the linear response to a perturbation that breaks the inversion symmetry of the Landau orbit around its guiding center. The perturbed Landau levels are given by

$$(H + v^a p_a)|\Psi_{n,\alpha}(\boldsymbol{v})\rangle = E_n(\boldsymbol{v})|\Psi_{n,\alpha}(\boldsymbol{v})\rangle, \tag{109}$$

where $\boldsymbol{v}$ is a parameter of the perturbed Hamiltonian. Then

$$\langle\Psi_{n\alpha}(\boldsymbol{v})|H|\Psi_{n\alpha}(\boldsymbol{v})\rangle = E_n + \tfrac{1}{2}m_{ab}^n v^a v^b + O(v^4), \tag{110}$$

and

$$E_n(\boldsymbol{v}) = E_n - \tfrac{1}{2}m_{ab}^n v^a v^b + O(v^4), \tag{111}$$

For the Newtonian model, $m_{ab}^n = m_e\delta_{ab}$. Though both m_{ab}^n and g_{ab}^n are proportional to the Euclidean metric in the Newtonian model, there is in general no proportionality between them unless there is a discrete three-, four- or six-fold crystal rotational symmetry of the lattice plane on which the electrons move (and no tangential component of the magnetic flux), in which case they both remain proportional to the Euclidean metric which is invariant under the crystal symmetry.

The generic lack of any relation between the viscosity tensor and the effective mass tensor can be traced to the fact that the viscosity derives from perturbations that mix Landau levels with the same parity, while the effective mass derives from mixing of Landau levels with opposite parity.

Since $p_a = \hbar\epsilon_{ab}(x^a - R^a)/\ell_B^2$, the perturbation can be written as

$$H(\boldsymbol{v}) = H + \epsilon_{ab}\hbar v^a (x^a - R^b)/\ell_B^2 \tag{112}$$

The effective mass tensor can therefore be interpreted as the part of the electric susceptibility associated with polarization of the Landau orbit (giving it an electric dipole moment defined as the displacement of its center of charge relative to the guiding center), with $v^a = -\epsilon^{ab}E_b/B$ (so E_a are the tangential components of the electric field in the comoving frame). The Landau-orbit polarizability contribution to the electric susceptibility per unit area is

$$\chi_E^{ab}(\boldsymbol{x}) = \epsilon^{ac}\epsilon^{bd}\frac{e^2\ell_B^2}{2\pi\hbar^2}\sum_n \nu_n m_{cd}^n(\boldsymbol{x}). \tag{113}$$

IX. DENSITY AND CURRENT RESPONSE

With the properties of the Landau orbit introduced in previous sections, we now present a generic version of the density and current responses against an electric field and a non-uniform magnetic flux in the absence of rotational symmetry. The same results have also been obtained elsewhere[15]. Detailed discussion of the derivation and related algebra can be found in Appx.A.

Denote the particle density and current in the nth Landau level by J_n^a and J_n^0. Our main results are

$$eJ_n^a = \frac{e^2}{2\pi\hbar}(1+\sigma^{cd}\partial_c\partial_d)\epsilon^{ab}E_b - \chi_m\epsilon^{ab}\partial_b B, \tag{114}$$

$$eJ_n^0 = \frac{e^2 B}{2\pi\hbar}(1+\sigma^{ab}\partial_a\partial_b \ln B) - \chi_E^{ab}\partial_a E_b, \tag{115}$$

where

$$\begin{aligned}\sigma^{ab} =&\ \frac{1}{6}s_n g^{ab} - \left[\frac{1}{6}(\tilde{R}^a, \{\tilde{R}^b, dh\})_n + a\leftrightarrow b\right]\\ &+ \frac{1}{4}(\{\tilde{R}^a,\tilde{R}^b\}, dh)_n, \end{aligned}\tag{116}$$

$$s_n g_n^{ab} = \frac{1}{2}\langle\{\tilde{R}^a,\tilde{R}^b\}\rangle_n, \tag{117}$$

$$\chi_m = \frac{e^2 l_B^2}{2\pi\hbar^2}\left(\langle dh + d^2h\rangle_n + (dh, dh)_n\right). \tag{118}$$

The notations used here are shown as follows:

$$(A,B)_n \equiv \sum_{n'\neq n}\frac{\langle n|A|n'\rangle\langle n'|B|n\rangle + \langle n|B|n'\rangle\langle n'|A|n\rangle}{E_n - E_{n'}}, \tag{119}$$

$$\langle A\rangle_n \equiv \langle n|A|n\rangle, \tag{120}$$

where $|n\rangle$ and E_n are the nth eigenstate and eigenvalue of the unperturbed single particle Hamiltonian h. The scale derivative is defines as

$$dh = B\frac{\partial h}{\partial B}, \quad d^2h = B\frac{\partial(dh)}{\partial B}. \tag{121}$$

The first term in σ^{ab} is proportional to the inverse of the Hall viscosity tensor η_{ab}^H and hence universal. The second and the third terms are due to Landau-level mixing between states with opposite and the same parity respectively, so they depend on the details of the Hamiltonian. When there is rotational symmetry, the third term vanishes because the eigenstates are invariant with respect to rescaling of B. When Galilean symmetry is present, the second term can be explicitly calculated to be

$$-[\frac{1}{6}(\tilde{R}^a, \{\tilde{R}^b, dh\})_n + a\leftrightarrow b] = \frac{4}{3}s_n g^{ab}, \tag{122}$$

which combined with the first term gives

$$\sigma^{ab} = \frac{3}{2}s_n g^{ab}. \tag{123}$$

In the Euclidean case where $g^{ab} = l_B^2\delta^{ab}$, it agrees with existing results in the literature[5,11,16].

X. CONCLUSION

We have studied the geometry of the Landau orbits in the absence of Galilean and rotational symmetry. With the help of consistent index placement, we reformulated the Hall viscosity and use it as a natural choice for the unimodular metric $\tilde{g}_{ab}^n$ that describes the shape of each Landau orbit. We thus demonstrated that the metric in the IQHE is derived from the intrinsic dynamics of the system rather than induced from the 3D space the system is set in. By investigating the root structure of the the holomorphic part, we are able to define a topological spin $s_n = n + \frac{1}{2}$ where n is the number of central zeroes enclosed by a zero-free region that can be thought of a "flattened" semiclassical orbit. Compared to the "intrinsic orbital angular momentum" interpretation, this definition is more generic and captures the topological nature of this quantity. We also introduced a mass tensor m_{ab}^n associated with the kinetic energy of guiding center, which only coincides with $\tilde{g}_{ab}^n$ when there is a continuous rotational symmetry.

With all the ingredients above, we presented the generic (without rotational symmetry) response functions for particle density and current. The results clearly show that the Hall conductivity at finite wavevector consists of a universal term that is proportional to the inverse of the Hall viscosity tensor and terms that are related to Landau-level mixing. We believe that a generic version of the effective action should also possess these features which are hidden by the simplification of applying special symmetries.

XI. ACKNOWLEDGMENTS

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Appendix A: Density and current reponses in the absence of rotational symmetry

Let the two components of the dynamical momentum $\boldsymbol{p} = \boldsymbol{p}^\dagger$ have the commutation relation

$$[p_a, p_b] = i\hbar^2 \ell_B^{-2} \epsilon_{ab}, \tag{A1}$$

where the antisymmetric Levi-Civita symbol $\epsilon_{ab} = -\epsilon_{ba}$ has $\det \epsilon = 1$. With the Schrödinger representation

$$p_a = -i\hbar\partial_a - eA_a(\boldsymbol{x}), \quad \partial_a \equiv \frac{\partial}{\partial x^a}, \tag{A2}$$

this represents the momentum of a charge-e particle moving in a uniform background magnetic flux density

$$B_0 = \partial_1 A_2(\boldsymbol{x}) - \partial_2 A_1(\boldsymbol{x}) = \mathrm{Pf}(\epsilon)(\hbar/e)\ell_B^{-2}, \tag{A3}$$

where the Pfaffian $\mathrm{Pf}(\epsilon) = \epsilon_{12}$ of the Levi-Civita symbol is odd under time-reversal. A metric-independent formalism that consistently distinguishes between covariant (lower) and contravariant (upper) 2D spatial indices will be used here. The dual symbol ϵ^{ab} is defined by

$$\epsilon^{ab}\epsilon_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c, \tag{A4}$$

where δ^a_b is the Kronecker symbol. In a metric-independent formalism, there is no symbol "δ_{ab}", which if encountered, would represent the Euclidean metric in a Cartesian coordinate system.

The algebra (A1), in which the commutator is a c-number, is the Heisenberg algebra $\mathfrak{h}_2$. Its universal enveloping algebra $\mathbb{U}(\mathfrak{h}_2)$ is spanned by the basis

$$\{1, p_a, \{p_a, p_b\}, \{p_a, p_b, p_c\}, \dots\} \tag{A5}$$

where $\{A, B, \dots, C\}$ is the symmetrized product of n operators normalized so that $\{\lambda A, \lambda A, \dots, \lambda A\} = n!(\lambda A)^n$. The kinetic energy h can then be given in the form

$$h = H(\boldsymbol{p}) \equiv \sum_n \frac{1}{n!} H_n^{a_1,a_2,\dots a_n} p_{a_1} p_{a_2} \dots p_{a_n} \tag{A6}$$

where the coefficients $H_n^{a_1\dots a_n}$ are fully symmetric in all indices.

It is useful to define

$$\tilde{R}^a = (\tilde{R}^a)^\dagger = -\hbar^{-1}\ell_B^2 \epsilon^{ab} p_b. \tag{A7}$$

Then

$$[R^a, p_b] = i\hbar\delta^a_b, \tag{A8}$$

$$[\tilde{R}^a, \tilde{R}^b] = i\ell_b^2 \epsilon^{ab}. \tag{A9}$$

The unitary boost operator, parametrized by a real c-number vector $\boldsymbol{q}$, is given by

$$U(\boldsymbol{q}) = U(-\boldsymbol{q})^\dagger = e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{R}}}, \quad [q_a, q_b] = 0. \tag{A10}$$

Note that the scalar product $\boldsymbol{q}\cdot\tilde{\boldsymbol{R}} \equiv q_a \tilde{R}^a$ can only be formed between a covariant and a contravariant vector. Then

$$U(-\boldsymbol{q})\boldsymbol{p}U(\boldsymbol{q}) = \boldsymbol{p} + \hbar\boldsymbol{q}. \tag{A11}$$

and the derivatives $h^{a_1a_2\dots a_n}$ of h are defined by

$$h(\boldsymbol{q}) \equiv U(-\boldsymbol{q})hU(\boldsymbol{q}) = \sum_{n=0}^\infty \frac{\hbar^n}{n!} h^{a_1a_2\dots a_n} q_{a_1} q_{a_2} \dots q_{a_n}, \tag{A12}$$

where the tensor-valued operators $h^{a_1a_2\dots a_n}$ are fully symmetric in their indices. In particular, the first derivative h^a is the velocity operator v^a. Note also that

$$U(-\boldsymbol{q})U(-\boldsymbol{q}')\boldsymbol{p}U(\boldsymbol{q}')U(\boldsymbol{q}) = U(-(\boldsymbol{q}+\boldsymbol{q}'))\boldsymbol{p}U(\boldsymbol{q}+\boldsymbol{q}'). \tag{A13}$$

From this

$$U(-\boldsymbol{q})h^{a_1,\dots a_m}U(\boldsymbol{q}) = \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}h^{a_1\dots a_m b_1\dots b_n}q_{b_1}\dots q_{b_n}. \tag{A14}$$

as in the differentiation of c-number functions, where $(f')' = f''$.

Now let $\boldsymbol{a} \in \mathbb{U}(\mathfrak{h})$ be a vector-valued operator and λ be a c-number. Then define the expansion

$$H(\boldsymbol{p} + \lambda\boldsymbol{a}) = h + \lambda h_1(\boldsymbol{a}) + O(\lambda^2). \tag{A15}$$

The quantity $h_1(\boldsymbol{a})$ can be expressed in two equivalent ways as the nested-commutator expansions

$$h_1(\boldsymbol{a}) = a_{a_1}h^{a_1} + \frac{1}{2!}[p_{a_1}a_{a_2}]h^{a_1a_2} + \frac{1}{3!}[p_{a_1},[p_{a_2},a_{a_3}]]h^{a_1a_2a_3} + \dots \tag{A16}$$

$$= h^{a_1}a_{a_1} + \frac{1}{2!}h^{a_1a_2}[a_{a_1},p_{a_2}] + \frac{1}{3!}h^{a_1a_2a_3}[[a_{a_1},p_{a_2}],p_{a_3}] + \dots. \tag{A17}$$

The operator h is also a function of the scale factor B_0, and more properly should be written as $h(B_0)$. The magnetization operator $m(\boldsymbol{p})$ is defined by

$$m \equiv -\left.\frac{\partial h}{\partial B_0}\right|_{\{H_n\}} = -\tfrac{1}{2}B_0^{-1}h_1(\boldsymbol{p})$$

$$= -\tfrac{1}{2}B_0^{-1}\boldsymbol{v}\cdot\boldsymbol{p} = -\tfrac{1}{2}B_0^{-1}\boldsymbol{p}\cdot\boldsymbol{v} = \tfrac{1}{2}B_0^{-1}\left(h + i\ell_B^{-2}\epsilon_{ab}\tilde{R}^a h\tilde{R}^b\right). \tag{A18}$$

If the kinetic energy is a quadratic operator, $h = \frac{1}{2}H_2^{ab}p_ap_b$, then $B_0m = -h$. For any scalar operator $h \in \mathbb{U}(\mathfrak{h}_2)$, one can define the scale derivative

$$dh \equiv B_0\frac{\partial h}{\partial B_0} = \tfrac{1}{2}\left(-i\ell_B^{-2}\epsilon_{ab}\tilde{R}^a h\tilde{R}^b - h\right), \tag{A19}$$

so

$$dh = -B_0 m. \tag{A20}$$

Higher orders of derivatives can be recursively defined as

$$d^{n+1}h \equiv B_0\frac{\partial(d^n h)}{\partial B_0}, \quad d^1 h \equiv dh. \tag{A21}$$

Note that if $Q \equiv \boldsymbol{q}\cdot\tilde{\boldsymbol{R}} \in \mathfrak{h}$, then

$$Qh\tilde{R}^a - \tilde{R}^a hQ = [Q, R^a](h + 2dh) \tag{A22}$$

where $[Q, R^a]$ is a c-number.

Now enlarge the algebra to add a second independent Heisenberg algebra, that of the *guiding centers*:

$$[R^a, R^b] = -i\epsilon^{ab}\ell_B^2, \quad [R^a, \tilde{R}^b] = 0. \tag{A23}$$

then

$$\boldsymbol{r} = \boldsymbol{R} + \tilde{\boldsymbol{R}} \tag{A24}$$

is the classical coordinate of the charged particle:

$$[r^a, r^b] = 0, \quad [r^a, p_b] = i\hbar\delta^a_b. \tag{A25}$$

Note that

$$\mathrm{Tr}(e^{i\boldsymbol{q}\cdot\boldsymbol{R}}) = 2\pi\delta^2(\boldsymbol{q}\ell_B) \tag{A26}$$

where the trace is over the Hilbert space of the guiding-center, and $\delta^2(\boldsymbol{x})$ is the 2D Dirac delta-function.

Now introduce an additional vector potential

$$A_a(\boldsymbol{x}) = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \tilde{A}_a(\boldsymbol{q}) e^{i\boldsymbol{q}\cdot\boldsymbol{x}}, \tag{A27}$$

with

$$\delta B(\boldsymbol{x}) = \partial_1 A_2(\boldsymbol{x}) - \partial_2 A_1(\boldsymbol{x}). \tag{A28}$$

Now define the operator

$$A_a \equiv A_a(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \tilde{A}_a(\boldsymbol{q}) e^{i\boldsymbol{q}\cdot\boldsymbol{R}} U(\boldsymbol{q}). \tag{A29}$$

The problem of the charged particle moving in the 2D plane $\boldsymbol{x}$ in the presence of a non-uniform magnetic flux density $B_0 + \delta B(x)$ is then described by the Hamiltonian

$$H = H(\boldsymbol{p} - e\boldsymbol{A}) \tag{A30}$$

The electric charge density operator is given by

$$J^0(\boldsymbol{x}) = e \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}(\boldsymbol{R}-\boldsymbol{x})} U(\boldsymbol{q}), \tag{A31}$$

and the electric current-density operator is given by

$$J^a(\boldsymbol{x}) = e \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}-\boldsymbol{x})} J^a(\boldsymbol{q}), \tag{A32}$$

where, to leading order in $\tilde{\boldsymbol{A}}$,

$$e_a J^a(\boldsymbol{q}) = h_1(\boldsymbol{e}U(\boldsymbol{q})) + O(\tilde{\boldsymbol{A}}), \tag{A33}$$

where $\boldsymbol{e}$ is a c-number unit vector.

The continuity equation (gauge invariance) requires that

$$(i\hbar)^{-1}[J^0(\boldsymbol{x}), H] + \partial_a J^a(\boldsymbol{x}) = 0. \tag{A34}$$

This must be satisfied at each order of an expansion in $\tilde{A}_a(\boldsymbol{q})$. In the limit of uniform magnetic flux density B_0, the continuity relation is

$$[h, U(\boldsymbol{q})] = \hbar q_a J^a(\boldsymbol{q}), \tag{A35}$$

and the functional form of the expansion of $J^a(\boldsymbol{q})$ must be

$$\hbar J^a(\boldsymbol{q}) = -i[\alpha^a(\boldsymbol{q}), h] + i\epsilon^{ba} q_b \ell_B^2 \beta(\boldsymbol{q}). \tag{A36}$$

where

$$iq_a\alpha^a(\boldsymbol{q}) = U(\boldsymbol{q}) - 1 \tag{A37}$$

or

$$\alpha^a(\boldsymbol{q}) = -i\frac{\partial}{\partial q_a}\left(\sum_{n=1}^{\infty} \frac{1}{n}\frac{(i\boldsymbol{q}\cdot\tilde{\boldsymbol{R}})^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{n!}\sum_{k=1}^{n} (iQ)^{k-1}\tilde{R}^a(iQ)^{n-k}. \tag{A38}$$

Using (A16) the explicit form of $J^a(\boldsymbol{q})$ can be written as

$$i\hbar J^a = U(\boldsymbol{q})\left([\tilde{R}^a, h] + \frac{1}{2!}[-iQ, [\tilde{R}^a, h]] + \frac{1}{3!}[-iQ, [-iQ, [\tilde{R}^a, h]]] + \cdots\right), \tag{A39}$$

where $Q = \boldsymbol{q} \cdot \tilde{\boldsymbol{R}}$ as mentioned above. Expanding $U(\boldsymbol{q})$ and the nested commutators gives

$$i\hbar J^a = \sum_{m=0}^{\infty} \frac{(iQ)^m}{m!} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (iQ)^{n-k} [\tilde{R}^a, h] (iQ)^k. \tag{A40}$$

The coefficient $C_{l,k}$ for the term

$$(iQ)^l [\tilde{R}^a, h] (iQ)^k$$

in the expansion is (using $l + k = m + n$)

$$\begin{aligned} C_{l,k} &= \sum_{m=0}^{l} \frac{1}{m!} (-1)^{l+k-m} \frac{1}{(l+k-m+1)!} (-1)^k \binom{l+k-m}{k} \\ &= \frac{1}{k!l!} \sum_{m=0}^{l} (-1)^m \frac{1}{k+m+1} \binom{l}{m} \\ &= \frac{1}{(k+l+1)!}. \end{aligned} \tag{A41}$$

Now a significantly simplified form for $J^a(\boldsymbol{q})$ can be written down

$$i\hbar J^a = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} (iQ)^{n-k} [\tilde{R}^a, h] (iQ)^k, \tag{A42}$$

which is further transformed into the desired form:

$$\begin{aligned} i\hbar J^a &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left((iQ)^n \tilde{R}^a h - h \tilde{R}^a (iQ)^n - i \sum_{k=0}^{n-1} (iQ)^{n-1-k} (Qh\tilde{R}^a - \tilde{R}^a h Q)(iQ)^k \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{1}{2} \{(iQ)^n, \tilde{R}^a\} h + \frac{1}{2} [(iQ)^n, \tilde{R}^a] h - \frac{1}{2} h \{\tilde{R}^a, (iQ)^n\} + \frac{1}{2} h [(iQ)^n, \tilde{R}^a] \right. \\ &\quad \left. - i [Q, \tilde{R}^a] \sum_{k=0}^{n-1} (iQ)^{n-1-k} (h + 2dh)(iQ)^k \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\left[\frac{1}{2} \{(iQ)^n, \tilde{R}^a\}, h \right] - i [Q, \tilde{R}^a] \sum_{k=0}^{n-1} (iQ)^{n-1-k} (2dh)(iQ)^k \right. \\ &\quad \left. + i [Q, \tilde{R}^a] \left(\frac{n}{2} (iQ)^{n-1} h + \frac{n}{2} h (iQ)^{n-1} - \sum_{k=0}^{n-1} (iQ)^{n-1-k} h (iQ)^k \right) \right) \\ &= [\alpha^a(\boldsymbol{q}), h] + i [Q, \tilde{R}^a] \beta(\boldsymbol{q}) \end{aligned} \tag{A43}$$

where

$$\alpha^a(\boldsymbol{q}) = \sum_{n=0}^{\infty} \frac{1}{2(n+1)!} \{(iQ)^n, \tilde{R}^a\}, \tag{A44}$$

$$\beta(\boldsymbol{q}) = -\sum_{n=0}^{\infty} \frac{2}{(n+1)!} \sum_{k=0}^{n-1} (iQ)^{n-1-k} (dh)(iQ)^k + \sum_{n=0}^{\infty} \frac{1}{2(n+1)!} \sum_{k=0}^{n-1} [(iQ)^{n-1-k}, [(iQ)^k, h]] \tag{A45}$$

The expansion for $\alpha^a(\boldsymbol{q})$ agrees with the universal form (A38), and the leading term in $\beta(\boldsymbol{q})$, $-dh = B_0 m$, is just the expected magnetization term in the total current density

$$J^0 = J^0_{\text{free}} - \partial_a P^a, \quad J^a = J^a_{\text{free}} + \epsilon^{ab} \partial_b M + \partial_t P^a. \tag{A46}$$

We are now in a position to give the perturbation result for a non-uniform magnetic flux density and a non-zero electric field. The perturbed Hamiltonian is

$$\begin{aligned} H = h &- e \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \tilde{A}_0(\boldsymbol{q}) U(\boldsymbol{q}) e^{i\boldsymbol{q}\cdot\boldsymbol{R}} - e \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \tilde{A}_a(\boldsymbol{q}) J^a(\boldsymbol{q}) e^{i\boldsymbol{q}\cdot\boldsymbol{R}} \\ &- e^2 \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \tilde{A}_a(-\boldsymbol{q}) \Gamma^{ab}(\boldsymbol{q}) \tilde{A}_b(\boldsymbol{q}) e^{i\boldsymbol{q}\cdot\boldsymbol{R}} + \mathcal{O}(A^3), \end{aligned} \tag{A47}$$

where $\Gamma^{ab}(\boldsymbol{q})$ is a gauge counter-term that is important for ensuring gauge invariance. Its explicit form is:

$$\Gamma^{ab}(\boldsymbol{q}) = \frac{1}{\hbar^2}\sum_{n=0}^{\infty}\frac{1}{(n+2)!}[\overbrace{iQ,\cdots[iQ}^{\text{n}},[\tilde{R}^a,[h,\tilde{R}^b]]]\cdots] + \frac{1}{\hbar^2}\sum_{n=0}^{\infty}\frac{1}{(n+2)!}[\overbrace{-iQ,\cdots[-iQ}^{\text{n}},[\tilde{R}^a,[h,\tilde{R}^b]]]\cdots] \tag{A48}$$

$$= \frac{1}{\hbar^2}\sum_{k=0}^{\infty}\frac{2}{(2k+2)!}[\overbrace{iQ,\cdots[iQ}^{\text{n}},[\tilde{R}^a,[h,\tilde{R}^b]]]\cdots]. \tag{A49}$$

The perturbed particle density and current density of a filled Landau level is

$$J_n^0(\boldsymbol{x}) = \frac{1}{2\pi\ell_B^2}\left(1 - e\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\chi_n^{00}(\boldsymbol{q})\tilde{U}(\boldsymbol{q})e^{i\boldsymbol{q}\cdot\boldsymbol{x}} - e\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\chi_n^{0a}(\boldsymbol{q})\tilde{A}_q(\boldsymbol{q})e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\right), \tag{A50}$$

$$J_n^a(\boldsymbol{x}) = -\frac{1}{2\pi\ell_B^2}\left(e\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\chi_n^{0a}(-\boldsymbol{q})\tilde{A}_0(\boldsymbol{q})e^{i\boldsymbol{q}\cdot\boldsymbol{x}} + e\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}\tilde{\chi}_n^{ab}(\boldsymbol{q})\tilde{A}_b(\boldsymbol{q})e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\right), \tag{A51}$$

where

$$\chi_n^{00}(\boldsymbol{q}) = \sum_{n'(\neq n)}\frac{\langle n|U(-\boldsymbol{q})|n'\rangle\langle n'|U^a(\boldsymbol{q})|n\rangle + \langle n|U(\boldsymbol{q})|n'\rangle\langle n'|U(-\boldsymbol{q})|n\rangle}{E_n - E_{n'}}, \tag{A52}$$

$$\chi_n^{0a}(\boldsymbol{q}) = \sum_{n'(\neq n)}\frac{\langle n|U(-\boldsymbol{q})|n'\rangle\langle n'|J^a(\boldsymbol{q})|n\rangle + \langle n|J^a(\boldsymbol{q})|n'\rangle\langle n'|U(-\boldsymbol{q})|n\rangle}{E_n - E_{n'}}, \tag{A53}$$

$$\tilde{\chi}_n^{ab}(\boldsymbol{q}) = \chi_n^{ab}(\boldsymbol{q}) + \langle n|\Gamma^{ab}(\boldsymbol{q})|n\rangle, \tag{A54}$$

$$\chi_n^{ab}(\boldsymbol{q}) = \sum_{n'(\neq n)}\frac{\langle n|J^a(-\boldsymbol{q})|n'\rangle\langle n'|J^b(\boldsymbol{q})|n\rangle + \langle n|J^b(\boldsymbol{q})|n'\rangle\langle n'|J^a(-\boldsymbol{q})|n\rangle}{E_n - E_{n'}}. \tag{A55}$$

Below we will simplify our notation by defining

$$(A,B)_n \equiv \sum_{n'\neq n}\frac{\langle n|A|n'\rangle\langle n'|B|n\rangle + \langle n|B|n'\rangle\langle n'|A|n\rangle}{E_n - E_{n'}}, \tag{A56}$$

so

$$\chi_n^{00}(\boldsymbol{q}) = (U(-\boldsymbol{q}), U(\boldsymbol{q}))_n, \tag{A57}$$

$$\chi_n^{0a}(\boldsymbol{q}) = (U(-\boldsymbol{q}), J^a(\boldsymbol{q}))_n, \tag{A58}$$

$$\chi_n^{ab}(\boldsymbol{q}) = (J^a(-\boldsymbol{q}), J^b(\boldsymbol{q}))_n. \tag{A59}$$

We will also denote the expectation value of some operator $\mathcal{O}$ in the nth Landau level by $\langle\mathcal{O}\rangle_n$.

Using the expansions of $U(\boldsymbol{q})$ and $J^a(\boldsymbol{q})$, these correlation functions are readily obtained:

$$\chi_n^{00}(\boldsymbol{q}) = -(Q,Q)_n + \mathcal{O}(|\boldsymbol{q}|^4) \tag{A60}$$

$$\chi_n^{0a}(\boldsymbol{q}) = -\frac{1}{\hbar}[Q,\tilde{R}^a]\left(1 - \frac{1}{6}\langle Q^2\rangle_n + \frac{1}{3}(Q,\{Q,dh\})_n - \frac{1}{2}(Q^2,dh)_n\right) + \mathcal{O}(|\boldsymbol{q}|^5), \tag{A61}$$

$$\tilde{\chi}_n^{ab}(\boldsymbol{q}) = -\frac{1}{\hbar^2}[Q,\tilde{R}^a][Q,\tilde{R}^b]\left(\langle dh + d^2h\rangle_n + (dh,dh)_n\right) + \mathcal{O}(|\boldsymbol{q}|^4) \tag{A62}$$

If we define

$$\chi_E^{ab} = -\frac{e^2}{2\pi l_B^2}(\tilde{R}^a,\tilde{R}^b)_n, \tag{A63}$$

$$\sigma^{ab} = \frac{1}{6}s_n g^{ab} - \left[\frac{1}{6}(\tilde{R}^a,\{\tilde{R}^b,dh\})_n + a\leftrightarrow b\right] + \frac{1}{4}(\{\tilde{R}^a,\tilde{R}^b\},dh)_n, \tag{A64}$$

$$s_n g_n^{ab} = \frac{1}{2}\langle\{\tilde{R}^a,\tilde{R}^b\}\rangle_n, \tag{A65}$$

$$\chi_m = \frac{e^2 l_B^2}{2\pi\hbar^2}\left(\langle dh + d^2h\rangle_n + (dh,dh)_n\right), \tag{A66}$$

then

$$\chi_n^{00}(\boldsymbol{q}) = -\frac{2\pi l_B^2}{e^2} q_a q_b \chi_E^{ab} + \mathcal{O}(|\boldsymbol{q}|^4), \tag{A67}$$

$$\chi_n^{0a}(\boldsymbol{q}) = -\frac{1}{\hbar}[Q, \tilde{R}^a]\left(1 - \sigma^{ab} q_a q_b\right) + \mathcal{O}(|\boldsymbol{q}|^5), \tag{A68}$$

$$\tilde{\chi}_n^{ab}(\boldsymbol{q}) = -\frac{2\pi}{e^2 l_B^2}[Q, \tilde{R}^a][Q, \tilde{R}^b]\chi_m + \mathcal{O}(|\boldsymbol{q}|^4). \tag{A69}$$

Note that χ_E^{ab} and g^{ab} are just the electric susceptibility tensor and the inverse of the Landau-orbit metric (up to normalization) defined in the main body of the paper. σ^{ab} is the tensor for the quadratic term in the $\boldsymbol{q}$ expansion of the Hall conductivity and χ_m is the magnetic Hall conductivity.

Now using

$$[Q, \tilde{R}^a] = i l_B^2 \epsilon^{ba} q_b, \tag{A70}$$

$$E(\boldsymbol{q}) = i q \tilde{A}_0(\boldsymbol{q}), \tag{A71}$$

$$\delta B(\boldsymbol{q}) = i \epsilon^{ab} q_a \tilde{A}_b(\boldsymbol{q}), \tag{A72}$$

we are able to write down the generic responses (in the absence of rotational symmetry) of charge density and current density against a non-vanishing electric field and a non-uniform magnetic field:

$$eJ_n^a = \frac{e^2}{2\pi\hbar}(1 + \sigma^{cd}\partial_c\partial_d)\epsilon^{ab}E_b - \chi_m \epsilon^{ab}\partial_b B, \tag{A73}$$

$$eJ_n^0 = \frac{e^2 B}{2\pi\hbar}(1 + \sigma^{ab}\partial_a\partial_b \ln B) - \chi_E^{ab}\partial_a E_b. \tag{A74}$$

The Hall conductivity at finite wavevector $\boldsymbol{q}$ is thus

$$\sigma_H^{ab} = \frac{e^2}{2\pi\hbar}(1 - \sigma^{cd} q_c q_d)\epsilon^{ab}, \tag{A75}$$

and we present here again the generic formula for σ^{ab}:

$$\sigma^{ab} = \frac{1}{6} s_n g^{ab} - \left[\frac{1}{6}(\tilde{R}^a, \{\tilde{R}^b, dh\})_n + a \leftrightarrow b\right] + \frac{1}{4}(\{\tilde{R}^a, \tilde{R}^b\}, dh)_n. \tag{A76}$$

The first term in σ^{ab} is proportional to the inverse of the Hall viscosity tensor η_{ab}^H and hence universal. The second and the third terms are due to inter-Landau-level mixing between states with opposite and the same parity respectively, so they depend on the details of the Hamiltonian. When there is rotational symmetry, the third term vanishes because the eigenstates are invariant with respect to rescaling of B_0. When Galilean symmetry is present, the second term can be explicitly calculated to be

$$-[\frac{1}{6}(\tilde{R}^a, \{\tilde{R}^b, dh\})_n + a \leftrightarrow b] = \frac{4}{3} s_n g^{ab}, \tag{A77}$$

which combined with the first term gives

$$\sigma^{ab} = \frac{3}{2} s_n g^{ab}. \tag{A78}$$

The case where $g^{ab} = l_B^2 \delta^{ab}$ agrees with existing results in the literature.