CALCULATION OF LLL BERRY CURVATURE IN SPACE OF METRICS

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We calculate the adiabatic curvature resulting from varying the LLL wavefunctions with respect to the torus geometry, as parametrized by flat metrics

$$g(V,\tau) = \frac{V}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$

Starting from the Landau level wavefunctions on the torus (ASZ 14), we can calculate the Berry curvature using (ASZ 6),

$$F_{ij} = \mathrm{Im}\sum_{\ell=1}^{N}\left\langle \partial_i\phi_\ell \middle| \partial_j\phi_\ell\right\rangle,$$

where ℓ indexes the single particle wavefunctions, and N is the number of particles. We will focus on obtaining this curvature for a particular index ℓ. We will find that F does not depend on ℓ, so the result for multiple particles filling N levels follows trivially.

The normalized lowest Landau level wavefunction for a given metric $g(V,\tau)$ is

$$\phi_\ell(\mathbf{x}) = \frac{(2\tau_2 B)^{1/4}}{\sqrt{V}}\sum_{n=-\infty}^{\infty} e^{i\pi\tau B(\widetilde{y}+n)^2} e^{-2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+n)} e^{2i\pi(nB+\ell)x}$$

with $\widetilde{y} = y + y_\ell = y + \ell/B + \varphi_1/B + 1/2$.

To compute components of F involving τ_1, τ_2, we note

$$\phi_\ell(\mathbf{x}) = \frac{(2\tau_2 B)^{1/4}}{V^{1/2}}\Phi_\ell(\mathbf{x})$$

with Φ holomorphic in τ. Therefore

$$\partial_{\tau_1}\phi_\ell(\mathbf{x}) = \frac{(2\tau_2 B)^{1/4}}{V^{1/2}}\partial_\tau\Phi_\ell(\mathbf{x})$$

and

$$\partial_{\tau_2}\phi_\ell(x,y) = \frac{(2\tau_2 B)^{1/4}}{V^{1/2}}\left[\frac{1}{4\tau_2}\Phi_\ell(\mathbf{x}) + i\partial_\tau\Phi_\ell(\mathbf{x})\right].$$

Explicitly,

$$\partial_\tau\Phi_\ell(\mathbf{x}) = \sum_{n=-\infty}^{\infty}\left(i\pi B(\widetilde{y}+n)^2\right) e^{i\pi\tau B(\widetilde{y}+n)^2} e^{-2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+n)} e^{2i\pi(nB+\ell)x}.$$

For F^ℓ_{V,τ_2},

$$\begin{aligned} F^\ell_{V,\tau_2} = -F^\ell_{\tau_2,V} &= \mathrm{Im}\left[\langle\partial_V\phi_\ell|\partial_{\tau_2}\phi_\ell\rangle\right] \\ &= \mathrm{Im}\left[\int_Q \sqrt{\det g}\,\mathrm{d}x\,\mathrm{d}y\,\partial_V\phi^*_\ell(\mathbf{x})\partial_{\tau_2}\phi_\ell(\mathbf{x})\right] \\ &= \mathrm{Im}\left[-\frac{1}{2V}\int_Q \mathrm{d}x\,\mathrm{d}y\,\phi^*_\ell(\mathbf{x})\left(\frac{1}{4\tau_2}\phi_\ell(\mathbf{x}) + i\sqrt{2\tau_2 B}\,\partial_\tau\Phi_\ell(\mathbf{x})\right)\right], \end{aligned}$$

where the integral is over $Q = [0,1]\times[0,1]$. The first term in the integral is clearly real, so

$$
\begin{aligned}
F^{\ell}_{V,\tau_2} &= \frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[i\int_Q \mathrm{d}x\,\mathrm{d}y\,\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x})\right]\\
&\equiv \frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[iI_1\right].
\end{aligned}
$$

The integral I_1 is

$$
\begin{aligned}
I_1 = \int_Q \mathrm{d}x\,\mathrm{d}y\,\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x}) &= \int_Q \mathrm{d}x\,\mathrm{d}y\left[\sum_{m=-\infty}^{\infty} e^{-i\pi\tau^* B(\widetilde{y}+m)^2}e^{2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+m)}e^{-2i\pi(mB+\ell)x}\right]\\
&\quad\times\left[\sum_{n=-\infty}^{\infty}\left(i\pi B(\widetilde{y}+n)^2\right)e^{i\pi\tau B(\widetilde{y}+n)^2}e^{-2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+n)}e^{2i\pi(nB+\ell)x}\right]\\
&= \int_Q \mathrm{d}x\,\mathrm{d}y\sum_{m,n} i\pi B(\tilde{y}+n)^2 e^{i\pi B\left(\tau(\tilde{y}+n)^2-\tau^*(\tilde{y}+m)^2\right)}e^{2i\pi(\phi_2+B/2)(m-n)}e^{2i\pi Bx(n-m)}
\end{aligned}
$$

Performing the x integration, we have, since B is an integer

$$
\int_0^1 \mathrm{d}x\,e^{2i\pi Bx(n-m)} = \delta_{n,m},
$$

and so

$$
\begin{aligned}
I_1 &= \int_0^1 \mathrm{d}y\,\sum_n i\pi B(\tilde{y}+n)^2 e^{i\pi B(\tau-\tau^*)(\tilde{y}+n)^2}\\
&= i\pi B\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}y\,(\tilde{y}+n)^2 e^{-2\pi B\tau_2(\tilde{y}+n)^2}\\
&= i\pi B\sum_{n=-\infty}^{\infty}\int_{y_\ell+n}^{y_\ell+n+1} \mathrm{d}u\,u^2 e^{-2\pi B\tau_2 u^2}\\
&= i\pi B\int_{-\infty}^{\infty}\mathrm{d}u\,u^2 e^{-2\pi B\tau_2 u^2}\\
&= i\pi B\left(\frac{1}{4\sqrt{2}\pi(B\tau_2)^{3/2}}\right)\\
&= \frac{i}{4\tau_2\sqrt{2\tau_2 B}}
\end{aligned}
$$

Since I_1 is pure imaginary,

$$
F^{\ell}_{V,\tau_2} = \frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[iI_1\right] = 0.
$$

For F^{ℓ}_{V,τ_1},

$$
\begin{aligned}
F^{\ell}_{V,\tau_1} &= \mathrm{Im}\left[-\frac{\sqrt{2\tau_2 B}}{2V}\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\Phi^*_\ell(\mathbf{x})\,\partial_\tau\Phi_\ell(\mathbf{x})\right] \\
&= -\frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[I_1\right] \\
&= -\frac{1}{8\tau_2 V} \neq 0\,[???]
\end{aligned}
$$

For $F^{\ell}_{\tau_1,\tau_2}$,

$$
\begin{aligned}
F^{\ell}_{\tau_1,\tau_2} = -F^{\ell}_{\tau_2,\tau_1} &= \mathrm{Im}\left[\langle\partial_{\tau_1}\phi_\ell|\partial_{\tau_2}\phi_\ell\rangle\right] \\
&= \mathrm{Im}\left[\int\limits_Q \sqrt{\det g}\,\mathrm{d}x\,\mathrm{d}y\,\partial_{\tau_1}\phi^*_\ell(\mathbf{x})\partial_{\tau_2}\phi_\ell(\mathbf{x})\right] \\
&= \mathrm{Im}\left[\sqrt{2\tau_2 B}\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\left(\frac{1}{4\tau_2}\Phi_\ell(\mathbf{x}) + i\partial_\tau\Phi_\ell(\mathbf{x})\right)\right] \\
&= \mathrm{Im}\left[\sqrt{2\tau_2 B}\left(\frac{1}{4\tau_2}\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\Phi_\ell(\mathbf{x}) + i\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x})\right)\right] \\
&= \mathrm{Im}\left[\sqrt{2\tau_2 B}\left(\frac{1}{4\tau_2}I_1^* + i\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x})\right)\right] \\
&\equiv \sqrt{2\tau_2 B}\,\mathrm{Im}\left[\frac{1}{4\tau_2}I_1^* + iI_2\right]
\end{aligned}
$$

The integral I_2 is

$$
\begin{aligned}
I_2 &= \int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x}) \\
&= \int\limits_Q \mathrm{d}x\,\mathrm{d}y\sum_{m,n}\left(-i\pi B(\widetilde{y}+m)^2\right)\left(i\pi B(\widetilde{y}+n)^2\right)e^{i\pi B\left(\tau(\widetilde{y}+n)^2-\tau^*(\widetilde{y}+m)^2\right)}e^{2i\pi(\phi_2+B/2)(m-n)}e^{2i\pi Bx(n-m)} \\
&= \pi^2 B^2\sum_{n=-\infty}^{\infty}\int\limits_0^1 \mathrm{d}y\,(\widetilde{y}+n)^4 e^{-2\pi B\tau_2(\widetilde{y}+n)^2} \\
&= \pi^2 B^2\int\limits_{-\infty}^{\infty}\mathrm{d}u\,u^4 e^{-2\pi B\tau_2(u)^2} \\
&= \pi^2 B^2\frac{3}{16\sqrt{2}\pi^2(B\tau_2)^{5/2}} \\
&= \frac{1}{\sqrt{2\tau_2 B}}\frac{3}{16\tau_2^2}.
\end{aligned}
$$

So that

$$\begin{aligned}
F_{\tau_1,\tau_2} &= \sqrt{2\tau_2 B}\,\mathrm{Im}\,\left[\frac{1}{4\tau_2}\left(\frac{-i}{4\tau_2\sqrt{2\tau_2 B}}\right) + i\frac{1}{\sqrt{2\tau_2 B}}\frac{3}{16\tau_2^2}\right] \\
&= \mathrm{Im}\,\left[i\left(-\frac{1}{16\tau_2} + \frac{3}{16\tau_2^2}\right)\right] \\
&= \frac{1}{8\tau_2^2}
\end{aligned}$$