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Abstract goes here.

I. INTRODUCTION

A. Background and motivation

The quantum Hall (QH) effects have provided a rich laboratory for a wide variety of condensed matter physics concepts for decades. In particular, they serve as important prototypes for the study of topologically ordered phases. Despite a huge body of literature on the QH effects and many successful theoretical developments, a "unified theory" of QH physics has not yet emerged. Some recent attempts to expand our range of understanding have focused on removing the standard simplification of introducing non-generic symmetries to the problem, for example rotational symmetry. Put slightly differently, this amounts to studying the QH effect in settings where the single-particle bands are not Landau levels (LL), as Landau levels have symmetries unnecessary for the stability of QH phases.

An extreme example of a system without the symmetries present in the standard two-dimensional electron gas (2DEG) setup is one in which electrons live on a lattice. Even in the regime in which may treat such a system with an effective continuum theory, effects like band mass anisotropy can spoil the non-generic symmetries. While lattice theories may lack the symmetries of the 2DEG, the effective continuum theory for a generic lattice theory will, at lowest order, be one of Landau levels. Thus, if we wish to study a less-symmetric continuum model, it seems we should not start with a lattice model.

In this work, we show how one may construct a family of toy models on a lattice that do not have Landau levels as their lowest-order continuum eigenstates. In other words, we obtain single-particle bands that cannot be treated as perturbations of Landau levels.

This direction has a natural connection to the single-mode approximation (SMA) employed in studying the bulk neutral excitations of fractional quantum Hall (FQH) fluids. The lowest energy branch of these excitations is the magnetoroton or Girvin-MacDonald-Platzman (GMP) mode. This is the relevant part of the spectrum if we are interested in the stability of the gap in FQH systems. The quadropolar nature of the GMP mode suggests that this mode should be sensitive to the geometry of a FQH liquid, particularly the presence or absence of rotational symmetry.

B. Universality of Landau levels

Consider a tight-binding lattice model for an electron in two dimensions in the presence of a uniform, perpendicular magnetic field; we will refer to such a system as a Harper-Hofstadter (HH) model. We will assume that the lattice is inversion-symmetric, and that time reversal symmetry is unbroken when $B = 0$.

C. Relation to previous work

[The rest of this needs to be revised and completed.]

II. EFFECTIVE CONTINUUM HAMILTONIANS

We consider a single spinless particle moving in the periodic potential of an inversion-symmetric lattice.

$$H = \sum_{\mathbf{k}} E(\mathbf{k})\, c^\dagger_{\mathbf{k}} c_{\mathbf{k}}.$$

Since our lattice is inversion symmetric and we have not yet broken time reversal symmetry, $E(\mathbf{k}) = E(-\mathbf{k})$. We make the further assumption that E has a global minimum at $k = 0$. Taylor expanding $E(\mathbf{k})$ about this minimum, we have

$$E(\mathbf{k}) = \frac{1}{2} g_0^{ij} k_i k_j + \sum_{j=2}^{\infty} \sum_{m=0}^{2j} \Lambda_{j,m}\, k_x^m k_y^{2j-m}. \qquad (1)$$

While this expansion will produce an arbitrary symmetric g_0, we may bring g_0 to diagonal form by a $GL(2,\mathbb{R})$ transformation.

We introduce a perpendicular magnetic field $B = \partial_x A_y - \partial_y A_x$. Working with a lattice in the tight-binding approximation with hopping amplitudes t_{ij}, this field is incoporated by the Peierls substitution

$$t_{ij} \to t_{ij} \exp\left[\frac{2\pi i}{\phi_0} \int_i^j \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell}\right].$$

After expanding, the effect on the dispersion (1) is the minimal substitution $\mathbf{k} \to \mathbf{k} - \frac{e\mathbf{A}}{c}$. In the Landau gauge with $\mathbf{A} = (0, Bx)$,

$$H = \frac{p^2 + \xi^2}{2m\ell^2} + \sum_{j=2}^{\infty} \sum_{m=0}^{2j} \frac{\Lambda_{j,m}}{\ell^{2j}}\, p^m \xi^{2j-m}.$$

where $\ell = \hbar c/eB$ is the magnetic length and $\xi = \hbar(x - \ell^2 k_y)/\ell$

In the $\ell \to \infty$ limit corresponding to small magnetic field, the quadratic term, which we label H_0, dominates. Its eigenstates are the familiar Landau levels $|n, k_y\rangle$ with eigenvalues $\epsilon_n = (\hbar/m\ell^2)\left(n + \frac{1}{2}\right)$. Since we have only assumed inversion symmetry of the dispersion, we expect that generic lattice bands will approach Landau levels in the $\ell \to \infty$ limit so long as the quadratic part of the Hamiltonian does not vanish identically.

If one or both quadratic terms do vanish, the continuum limit Hamiltonian will produce non-Landau level behavior to lowest order. While an exactly vanishing H_0 is an idealization, it is in principle possible to tune the parameters of the problem so that H_0 is negligible. To demonstrate this, we consider the Hamiltonian

$$H = \frac{1}{\ell^4} H_1 + \frac{\lambda}{\ell^2} H_0 \qquad (2)$$

Where H_1 is a homogeneous, quartic polynomial in components of $\mathbf{p}$, and λ is a parameter. Each term in the Hamiltonian has been scaled to show ℓ dependence explicitly. We write the eigenstates of H_1 in a basis of Landau levels,

$$|\tilde{n}\rangle = \sum_{n=0}^{\infty} c_n\, |n\rangle\,. \qquad (3)$$

To establish quantitative bounds on observable effects of H_0, we treat the Landau level Hamiltonian $\lambda\ell^2 H_0$ as a perturbation and study the remainder terms of the zeroth order Taylor polynomials for the eigenstates and eigenvalues. With an error tolerance τ, corrections to the nth energy eigenvalue from H_0 are unobservable if the parameter λ obeys

$$\lambda < \frac{\tau \epsilon_n}{\langle\tilde{n}|H_0|\tilde{n}\rangle\, \ell^2}. \qquad (4)$$

where ϵ_n is the nth eigenvalue of H_1. Since first order corrections to the eigenstates of H vanish, (4) provides the most conservative bound on λ; observable effects arising from corrections to states will also be negligible if (4) is obeyed.

Generically, the quartic term E_1 can take three possible forms,

$$E_1^{(1)} = (g_1^{ab} k_a k_b)(g_2^{cd} k_c k_d) \qquad (5)$$

$$E_1^{(2)} = (\mathbf{a}_1 \cdot \mathbf{k})(\mathbf{a}_2 \cdot \mathbf{k})(g_1^{ab} k_a k_b) \qquad (6)$$

$$E_1^{(3)} = (\mathbf{a}_1 \cdot \mathbf{k})(\mathbf{a}_2 \cdot \mathbf{k})(\mathbf{a}_3 \cdot \mathbf{k})(\mathbf{a}_4 \cdot \mathbf{k}) \qquad (7)$$

where the g_i are symmetric 2x2 matrices with det $g_i > 0$, and the a_i are elements of $\mathbb{R}^2$. In the case that we may neglect the quadratic term, only the first of these has a unique global minimum at $\mathbf{k} = 0$, forcing the continuum limit dispersion to take this form.

[ignore this {] **Finite difference approximations**

$$\Delta^2 f = \frac{f(x-a) - 2f(x) + f(x+a)}{a^2}$$

$$\Delta^2 f - \partial^2 f = \frac{a^2}{12} \partial^4 f + O(a^4)$$

$$\Delta^4 f = \frac{f(x-2a) - 4f(x-a) + 6f(x) - 4f(x+a) + f(x+2a)}{a^4}$$

$$\Delta^4 f - \partial^4 f = \frac{a^2}{6} \partial^6 f + O(a^4)$$

[}]

As a concrete demonstration of the preceding ideas, we study lattice realizations of two models with non-Landau level behavior. We begin by considering the familiar Hofstadter model on a square lattice with nearest neighbor hopping amplitude t_1 and modify the model by including a next-nearest neighbor hopping with amplitude t_2 along the x and y directions. Setting the lattice spacing $a = 1$ and writing the eigenstates $\psi(x = n, y) = e^{ik_y y} \psi_n$, the Harper eigenvalue equation for this model is

$$\varepsilon \psi_n = -t_1 \left(\psi_{n+1} + \psi_{n-1}\right) - t_2 \left(\psi_{n+2} + \psi_{n-2}\right)$$
$$-2t_1 \cos\left(\frac{1}{\ell^2} n - k_y\right) \psi_n - 2t_2 \cos\left(2\frac{1}{\ell^2} n - 2k_y\right) \psi_n.$$

For ℓ large, we approximate ψ_n by a continuum wavefunction $\psi(x)$, Taylor expanding both the finite differences and cosine terms. This yields the differential equation

$$-(1+4t)\frac{1}{\ell^2}\frac{d^2\psi}{d\xi^2}+(1+4t)\frac{1}{\ell^2}\xi^2\,\psi(\xi)$$
$$-t\frac{1}{\ell^4}\frac{d^4\psi}{d\xi^4}-\frac{(1+16t)}{12}\frac{1}{\ell^4}\xi^4\,\psi(\xi)=\tilde{\varepsilon}\psi(\xi)$$

where $t = t_2/t_1$, $\xi = x/\ell$, and $\tilde{\varepsilon} = \varepsilon/t_1 + 4(1+t)$. With hopping amplitudes tuned so that $t = -\frac{1}{4}$, this becomes

$$\frac{1}{4\ell^4}\frac{d^4\psi}{d\xi^4}+\frac{1}{4\ell^4}\xi^4\,\psi(\xi)=\tilde{\varepsilon}\psi(\xi).$$

This is the Schrodinger eigenvalue equation for the Landau gauge Hamiltonian

$$H_1=\frac{1}{4\ell^4}\left(p^4+\xi^4\right)$$

corresponding to the dispersion $E_1(\mathbf{k}) = (k_x^4 + k_y^4)/4$. In the notation of (5), one choice of g_1, g_2 is

$$g_1=\frac{1}{2}\begin{pmatrix}1 & 1/\sqrt{2}\\ 1/\sqrt{2} & 1\end{pmatrix}, g_2=\frac{1}{2}\begin{pmatrix}1 & -1/\sqrt{2}\\ -1/\sqrt{2} & 1\end{pmatrix}.$$

A semi-classical argument applying the Bohr-Sommerfield quantization condition lets us estimate the behavior of the energy eigenvalues in the high energy limit. We write the classical energy of this system

$$E=\frac{p^4+\xi^4}{4\ell^4}$$

and impose the Bohr-Sommerfeld condition

$$\int \mathrm{d}\xi\; p(\xi)\sim n\in\mathbb{Z}.$$

This yields

$$E\sim\ell^{-4}n^2,$$

To study the eigenvalues and eigenstates of this Hamiltonian further, we work in a basis of Landau levels $\{|n\rangle\}$ and write the scaled position and momentum operators in terms of Landau level raising and lowering operators,

$$\xi=\frac{1}{\sqrt{2}}\left(a+a^\dagger\right)$$
$$p=\frac{-i}{\sqrt{2}}\left(a-a^\dagger\right)$$

with $\left[a, a^\dagger\right] = 1$ and $a|0\rangle = 0$.

[Spectrum of quartic Hamiltonian, etc....]

We now consider moving away from the fine-tuned case, letting $t = -1/4 + \delta$. The eigenvalue equation

$$-\frac{4\delta}{\ell^2}\left[\frac{d^2\psi}{d\xi^2}+\xi^2\,\psi(\xi)\right]$$
$$+\left(\frac{1}{4}-\delta\right)\frac{1}{\ell^4}\frac{d^4\psi}{\partial\xi^4}-\left(\frac{1}{4}-\frac{4}{3}\delta\right)\frac{1}{\ell^4}\xi^4\,\psi(\xi)=\tilde{\varepsilon}\psi(\xi)$$

now contains quadratic terms, and the corresponding Hamiltonian is of the form (2) with $\lambda = 4\delta$.

Using the bound (4), we can establish a range for δ such that the bands of the lattice model are distinct from Landau levels in the continuum limit; for concreteness we restrict our attention to the lowest band $|\tilde{0}\rangle$. Calculating ϵ_0 and $\langle H_0\rangle$ numerically using the finite subspace approximation described above, we find the condition

$$\delta\lesssim 0.086\frac{\tau}{(\ell/a)^2},$$

restoring the necessary factor of the lattice spacing a. We can relate this to experiment by calculating some typical experimental values of ℓ. In a recent optical lattice simulation of the Hofstadter model, researchers achieved an effective $B \approx 0.0441$ T magnetic field; the corresponding magnetic length, measured in units of the lattice spacing, is $\ell \approx 0.0143\ a$, so that

$$\delta\lesssim 423\,\tau$$

This represents the region of large lattice effects. As an example system closer to the continuum limit, we consider a GaAs heterostructure in a 30 T magnetic field. Here we have $\ell \approx 6.3\times10^{-4}\ a$, and

$$\delta\lesssim 1.36\times10^{-6}\,\tau$$

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