PERTURBATION THEORY FOR GENERIC HAMILTONIAN IN THE LANDAU LEVEL LIMIT

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We consider a Hamiltonian

$$H = p_x^2 + p_y^2 + \lambda \sum_{n=2}^{\infty} \lambda_{1,n} p_x^{2n} + \lambda_{2,n} p_y^{2n},$$

with p_i appropriately scaled momentum variables and λ, $\lambda_{i,n}$ are generic constants. In the Landau gauge, with $\mathbf{A} = (0, Bx)$ and $k_y = 0$, this is

$$H(B, \{\lambda\}) = p_x^2 + (x/\ell^2)^2 + \lambda \sum_{n=2}^{\infty} \lambda_{1,n} p_x^{2n} + \lambda_{2,n} (x/\ell^2)^{2n}$$

where $\ell = 1/\sqrt{eB}$ is the magnetic length. ($\hbar = c = 1$). Rescaling H in terms of $\xi = x/\ell$,

$$H(\ell, \{\lambda\}) = \frac{p_\xi^2 + \xi^2}{\ell^2} + \lambda \sum_{n=2}^{\infty} \ell^{-2n} \left(\lambda_{1,n} p_\xi^{2n} + \lambda_{2,n} \xi^{2n} \right)$$

$\hookrightarrow \ \ell^2 H(\ell, \{\}) = p_\xi^2 + \xi^2 + \lambda \sum \ell^{-(2n-2)} \lambda_{1,n} p^{2n}$

For $\lambda = 0$, the eigenvalues of $H(\ell, \lambda = 0)$ are

$$E_n(\ell) = \ell^{-2}(n + 1/2).$$

$E_\ell = (n + 1/2)$

with eigenstates $|n; \ell\rangle$ and wavefunctions $\langle x | n; \ell\rangle = \phi(\xi; \ell)$. For $\lambda \neq 0$, we can write these as Taylor series in λ,

$$E_n(\ell, \lambda) = E_n(\ell) + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \left[\frac{\partial^k}{\partial \lambda^k} E_n(\ell, \lambda) \right]_{\lambda=0} \quad (1)$$

$$|n; \ell, \lambda\rangle = |n; \ell\rangle + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \left[\frac{\partial^k}{\partial \lambda^k} |n; \ell, \lambda\rangle \right]_{\lambda=0} \quad (2)$$

As described in [1], we can resolve the derivatives of the energies and states in terms of matrix elements $\langle m; \ell | \partial H | n; \ell\rangle$ with

$$\begin{aligned} \partial H = \frac{\partial H}{\partial \lambda} &= \sum_{n=2}^{\infty} \ell^{-2n} \left(\lambda_{1,n} p_\xi^{2n} + \lambda_{2,n} \xi^{2n} \right) \\ &= \ell^{-4} \sum_{n=2}^{\infty} \ell^{-2(n-2)} \left(\lambda_{1,n} p_\xi^{2n} + \lambda_{2,n} \xi^{2n} \right) \\ &\equiv \ell^{-4}\, h(p_\xi, \xi, \{\lambda_{i,n}\}) \end{aligned}$$

We're assuming that the $\lambda_{i,n}$ are constants, but note that so long as $\lambda_{i,n}$ is at most $O(\ell^{2n-4})$, $h(p_\xi, \xi, \{\lambda_{i,n}\})$ will be at most $O(1)$ in ℓ. With this in mind, we write

$$\begin{aligned} \langle m; \ell | \partial H | n; \ell\rangle &= \ell^{-4} \langle m; \ell | h(p_\xi, \xi, \{\lambda_{i,n}\}) | n; \ell\rangle \\ &\equiv \ell^{-4} h_{nm} \end{aligned}$$

Date: August 3, 2016.

We also need to keep track of energy denominators $\Delta_{nm} = E_n(\ell) - E_m(\ell)$ for this resolution. Since $E_n \sim \ell^{-2}$, $\Delta^k_{nm} \sim O(\ell^{-2k})$. In the Taylor series for the eigenstates, each factor of h_{nm} is accompanied by a factor of Δ_{nm} in the denominator. For example,

$$\begin{aligned}\left[\frac{\partial^2}{\partial\lambda^2}\left|n;\ell,\lambda\right\rangle\right]_{\lambda=0} &= 2\sum_{m\neq n,k\neq n}\frac{\ell^{-8}h_{mk}h_{kn}}{\Delta_{nm}\Delta_{nk}}\left|m\right\rangle - 2\sum_{m\neq n}\frac{\ell^{-8}h_{mn}h_{nn}}{\Delta^2_{nm}}\left|m\right\rangle - \sum_{m\neq n}\frac{\ell^{-8}h_{nm}h_{mn}}{\Delta^2_{nm}}\left|m\right\rangle \\ &\sim O(\ell^{-4}).\end{aligned}$$

In fact, the k-th derivative term in the expansion (2) is $O(\ell^{-2k})$. [TODO: Provide more justification for this. It's "obvious" from dimensional analysis, but a proof would be nice.]

This lets us write

$$\left|n;\ell,\lambda\right\rangle = \left|n;\ell\right\rangle + \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\ell^{-2k}\left|\psi_k\right\rangle.$$

The state $\left|\psi_k\right\rangle$ appearing in each order of the expansion will be a complicated superposition of Landau levels, but to lowest order it will be independent of ℓ. From here we can see that in the limit $\ell \to \infty$ corresponding to $B \to 0$, corrections to the energy eigenstates from terms in the Hamiltonian of higher than quadratic order will vanish. More precisely, what we need is $\lambda\ell^{-2} \to 0$. Similar considerations apply to the energy eigenvalues. [i.e., I'll add that bit soon.]

References

[1] https://en.wikipedia.org/wiki/Perturbation_theory_(quantum_mechanics)#Hellman.E2.80.93Feynman_theorems

Keeping Quartic Terms

$$\mathcal{H}(\ell) = \mathcal{H}' = p_\xi^2 + \xi^2 + \lambda_4 \left(\frac{p_x^4}{\ell^2} + \frac{p_y^4}{\ell^2} \right).$$

$$\lambda_4 = \ell^2$$

$$E^{0}(\ell) = n + 1/2 \qquad \frac{\lambda_4}{\ell^2} \leq 1$$

Can safely neglect λ_4 for $n = 0$

$$\lambda_4 \left\langle 0 \right| \frac{p_x^4 + p_y^4}{\ell^2} \left| 0 \right\rangle \sim c_1$$

$$\lambda_4 \left\langle 0 \right| \frac{p_x^4 + p_y^4}{\ell^2} \left| n \right\rangle \sim c_2$$

Rough Bounds on lowest order corrections to $|\psi_n\rangle$ & E_n

→ for completeness,

Case I :- $\mathcal{H} = \underbrace{p_x^4 + p_y^4}_{} + \sum_{n=6}^{\infty} \lambda_n \left(p_x^n + \ldots \right)$

generalize to the most general quartic up to rescale

Case II :- $\mathcal{H} = \quad '' \quad + \lambda_2 \left(p_x^2 + p_y^2 + \ldots \right)$

$+ \ldots$