Consider the hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 + C\left[\frac{p_x^4}{4m^2} + \frac{1}{4}m^2\omega^4 x^4\right] = H_2 + CH_4$$

where C is a dimensionless constant. We are thinking of this as the low-energy effective hamiltonian of the "quartic Hofstadter" model in the Landau gauge. We argue that the length scales of the wavefunctions of the two terms H_2 and H_4 are the same, regardless of the value of C. As before, we'll rescale into dimensionless variables

$$\begin{aligned} q &= x/\ell \\ p &= \ell p_x. \end{aligned}$$

Here ℓ is the magnetic length $\ell^2 = \hbar/(eB) = \hbar/(m\omega)$. Then

$$\begin{aligned} H &= \frac{p^2}{2m\ell^2} + \frac{\hbar}{2m\ell^2}q^2 + C\left[\frac{p^4}{4m^2\ell^4} + \frac{\hbar^4}{4m^2\ell^4}x^4\right] \\ &= \frac{\hbar^2}{2m\ell^2}\left(-\frac{d^2}{dq^2} + q^2\right) + C\frac{\hbar^4}{4m^2\ell^4}\left(\frac{d^4}{dq^4} + q^4\right) \end{aligned}$$

We'll now switch to units in which $\hbar^2/m = 1$ and note that in these units [energy] = [length]$^{-2}$. Now

$$H = \frac{1}{2\ell^2}\left(-\frac{d^2}{dq^2} + q^2\right) + \frac{C}{4\ell^4}\left(\frac{d^4}{dq^4} + q^4\right) = H_2 + CH_4$$

We might conclude that the second term has a length scale $\ell' = \ell/C^{1/4}$ different from that of the first term, but this is incorrent because CH_4 and H_4 have the same eigenstates. When we multiply the entire hamiltonian by a factor, we change the overall energy scale but don't affect the shape of the potential, which sets the length scale of the wavefunctions.

However, we have made a subtle mistake above. The low energy effective hamiltonain for the *ideal* quartic Hofstadter model is purely quartic, and indeed takes the form $H_4 = \frac{1}{4\ell^4}\left(\frac{d^4}{dq^4} + q^4\right)$. Rescaling by a constant does not change the length scale of the low-energy wavefunctions. However, when we perturb away from the purely quartic case – shifting the (dimensionless) hopping parameter by a constant δ – the hamiltonian becomes

$$H = \frac{4\delta}{\ell^2}\left(-\frac{d^2}{dq^2} + q^2\right) + \left(\frac{1}{4} - \delta\right)\frac{1}{\ell^4}\frac{d^4}{dq^4} + \left(\frac{1}{4} - \frac{4}{3}\delta\right)\frac{1}{\ell^4}q^4 \tag{1}$$

[NOTE: I need to double-check this algebra. In the meantime, let's consider general parameter values.]

$$H = \frac{1}{\ell^2}\left(-\frac{d^2}{dq^2} + q^2\right) + \frac{1}{\ell^4}\left(a\frac{d^4}{dq^4} + bq^4\right)$$

[...]

If we consider two different harmonic oscillators with length scales ℓ_1, ℓ_2, then

$$\langle 0_2; \ell_1 | 0_2; \ell_2 \rangle = \sqrt{\frac{2\ell_1\ell_1}{\ell_1^2 + \ell_2^2}}$$

We want to look at

$$\langle 0_2;\ell_1|0_4;\ell_2\rangle = \sum_m c_m \langle 0_2;\ell_1|m_2;\ell_2\rangle\,,$$

where

$$|0_4\rangle = \sum_m c_m\,|m_2\rangle$$

Now

$$\begin{aligned}\langle 0_2;\ell_1|n_2;\ell_2\rangle &= \frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{\pi\ell_1\ell_2}}\int_{-\infty}^{\infty} dx\,\exp\left[-\left(\frac{\ell_1^2+\ell_2^2}{2\ell_1^2\ell_2^2}\right)x^2\right] H_n\left(\frac{x}{\ell_2}\right)\\ &= \left[2\ell_1\ell_2/\sqrt{\ell_1^2+\ell_2^2}\right]^{-1}\frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{\pi\ell_1\ell_2}}\int_{-\infty}^{\infty} dx\,\exp\left(-t^2\right)H_n(\gamma t)\end{aligned}$$

$$[\text{fix this, } dx = 2\ell_1\ell_2/\sqrt{\ell_1^2+\ell_2^2}dt]$$

with $\gamma := \frac{2\ell_1}{\sqrt{\ell_1^2+\ell_2^2}}$. We can expand this as

$$H_n(\gamma t) = \sum_{j=0}^{\lfloor n/2\rfloor}\gamma^{n-2j}(\gamma^2-1)^j\binom{n}{2j}H_{n-2j}(t)$$

In the above integral, the orthogonality of the Hermite polynomials means that only the $j = n/2$ term in the sum survives. So

$$\int_{-\infty}^{\infty} dx\,\exp\left(-t^2\right)H_n(\gamma t) = (\gamma^2-1)^{n/2}\sqrt{\pi}$$

and

$$\langle 0_2;\ell_1|n_2;\ell_2\rangle = \frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{\ell_1\ell_2}}\left[\frac{3\ell_1^2-\ell_2^2}{\ell_1^2+\ell_2^2}\right]^{n/2}$$

If $\ell_2 = c\ell_1$,

$$\begin{aligned}\langle 0_2;\ell_1|n_2;\ell_2\rangle &= \frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{c}\ell_1}\left[\frac{3-c^2}{2}\right]^{n/2}\\ &= \frac{1}{2^n\sqrt{n!}}\frac{1}{\sqrt{c}\ell_1}\left(3-c^2\right)^{n/2}\end{aligned}$$

[Get something like overlap $\approx \left(1-\frac{\delta^2}{144}\right)$, i.e., very close to 1].

References