Viscosity of quartic systems from Kubo forumla

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1. Quadratic hamiltonians without rotational symmetry

For a rotationally invariant Hamiltonian written in terms of a metric $g_{\mu\nu}$ we can define stress tensor as the derivative of the Hamiltonian with respect to the inverse metric,

$$T_{\mu\nu} = -2\frac{\partial H}{\partial g^{\mu\nu}} \tag{1}$$

In the absence of rotational symmetry, we need a more general prescription. Consider a uniform strain Λ under which the coordinates and momenta transform as

$$\begin{aligned} x_\nu &= \Lambda_{\nu\mu} x_\mu \\ p_\nu &= \Lambda^{-1}_{\nu\mu} p_\mu \end{aligned}$$

We first consider the case of zero magnetic field and a quadratic Hamiltonian

$$H = g^{\mu\nu} p_\mu p_\nu.$$

Under the strain Λ the Hamiltonian transforms as

$$H_\Lambda = g^{\alpha\beta}\Lambda_{\alpha\mu}\Lambda_{\beta\nu} p_\mu p_\nu := g^{\alpha\beta}\exp\left(\lambda_{\alpha\mu}\right)\exp\left(\lambda_{\beta\nu}\right) p_\mu p_\nu.$$

To implement this transformation with a unitary operator

$$U(t) = \exp\left[-i\lambda_{\mu\nu}(t) J_{\mu\nu}\right],$$

we require strain generators $J_{\mu\nu}$ such that

$$H_\Lambda = U(t) H_0 U^{-1}(t)$$

We can write the stress tensor at zero strain as

$$T_{\mu\nu} = -\left.\frac{\partial H_\Lambda}{\partial \lambda_{\mu\nu}}\right|_{\Lambda=\mathbb{1}}$$

We use the notation $T_{\mu\nu}$ for the $t = 0$ stress tensor $T_{\mu\nu} = T_{\mu\nu}(0)$ [...]

Taking the derivative of (...), we have

$$\begin{aligned} \frac{\partial H_\Lambda}{\partial \lambda_{\mu\nu}} &= \frac{\partial}{\partial \lambda_{\mu\nu}}\left(g^{\alpha\beta} e^{\lambda_{\alpha\rho}} e^{\lambda_{\beta\sigma}} p_\rho p_\sigma\right) \\ &= g^{\mu\beta} e^{\lambda_{\mu\nu}} e^{\lambda_{\beta\sigma}} p_\nu p_\sigma + g^{\alpha\mu} e^{\lambda_{\alpha\rho}} e^{\lambda_{\mu\nu}} p_\rho p_\nu \\ &= 2e^{\lambda_{\mu\nu}} p_\nu g^{\mu\alpha} e^{\lambda_{\alpha\beta}} p_\beta \end{aligned}$$

Evaluating this derivative at zero strain gives

$$T_{\mu\nu} = 2g^{\mu\alpha} p_\alpha p_\nu \tag{2}$$

In the case of the Euclidean metric, $g_{\mu\nu} = \delta_{\mu\nu}/2m$, this is $T_{\mu\nu} = p_\mu p_\nu/m$.

Date: March 9, 2017.

Having obtained the stress tensor, we can use the (stress-stress form of the) linear response formula from Ref. [1] for the extensive viscosity tensor

$$X_{\mu\nu\alpha\beta} = \frac{1}{\omega_+}\left\{\left\langle\left[T_{\mu\nu}, J_{\alpha\beta}\right]\right\rangle + \int_0^\infty dt\, e^{i\omega_+ t}\left\langle\left[T_{\mu\nu}(t), T_{\alpha\beta}\right]\right\rangle\right\} \tag{3}$$

where $\omega_+ = \omega + i\epsilon$.

The commutator of the stress tensor with the strain generator is

$$\begin{aligned}\left[T_{\mu\nu}, J_{\alpha\beta}\right] &= -g^{\mu\sigma}\left[p_\nu p_\sigma, x_\alpha p_\beta + p_\beta x_\alpha\right]\\ &= i g^{\mu\sigma} p_\beta\left(p_\sigma \delta_{\nu\alpha} + p_\nu \delta_{\sigma\alpha}\right)\\ &= i g^{\mu\sigma}\delta_{\nu\alpha} p_\sigma p_\beta + i g^{\mu\alpha} p_\nu p_\beta.\end{aligned}$$

The expectation of the commutator inside the integral vanishes, and so

$$X_{\mu\nu\alpha\beta} = \frac{i}{\omega_+}\left[g^{\mu\sigma}\delta_{\nu\alpha}\left\langle p_\sigma p_\beta\right\rangle + g^{\mu\alpha}\left\langle p_\nu p_\beta\right\rangle\right].$$

We evaluate the expectation values in an eigenstate of the hamiltonian in a box of size L^2. (We work in two dimensions throughout.) For the quadratic Hamiltonian we can diagonalize $g_{\mu\nu}$ by a linear transformation, so we now take $g_{\mu\nu} \to g'_{\mu\nu} = \delta_{\mu\nu}$. Then

$$X_{\mu\nu\alpha\beta} = \frac{iE_0}{\omega_+}\left[\delta_{\nu\alpha}\delta_{\mu\beta} + \delta_{\mu\alpha}\delta_{\nu\beta}\right]$$

We now turn on a nonzero magnetic field and repeat the above calculation. We start with the rotationally symmetric case. The unstrained Hamiltonian is

$$H_0 = \frac{1}{2m}\pi_\mu \pi_\mu$$

$$\boldsymbol{\pi} = \mathbf{p} - \boldsymbol{A}$$

$$\left[x_\mu, \pi_\nu\right] = \delta_{\mu\nu}$$

$$\left[\pi_\mu, \pi_\nu\right] = iB\epsilon_{\mu\nu}$$

The stress tensor at zero strain is

$$T_{\mu\nu} = \frac{1}{2}\left\{\pi_\mu, \pi_\nu\right\}$$

and the strain generators are

$$J_{\mu\nu} = J^{\mathrm{sh}}_{\mu\nu} + \frac{K}{2}\delta_{\mu\nu}$$

with

$$J^{\mathrm{sh}}_{\mu\nu} = \frac{1}{2}\left(-\left\{x_\mu, \pi_\nu\right\} + \mathcal{B}\epsilon_{\nu\alpha}x_\mu x_\alpha\right)$$

and

$$K = -\frac{1}{2}\left\{x_\mu, \pi_\mu\right\} + \{\mathcal{B}, \Xi\}.$$

The x_μ and π_μ operators may be written in terms of two sets of boson operators a, $a^\dagger$ and b, $b^\dagger$, where

$$b =$$

$$a =$$

For convenience, we introduce the operators

$$\begin{aligned} Q_\pm &= b^{\dagger 2} \pm b^2 \\ n &= b^\dagger b \\ A_{\mu\nu} &= Q_+\sigma^3_{\mu\nu} + Q_-\sigma^1_{\mu\nu} \\ B_{\mu\nu} &= Q_-\sigma^3_{\mu\nu} + Q_+\sigma^1_{\mu\nu} \end{aligned}$$

with commutators

$$\begin{aligned} [Q_+, Q_-] &= 4(2n+1) \\ [n, Q_\pm] &= 2Q_\mp \\ \left[A_{\mu\nu}, A_{\alpha\beta}\right] &= [Q_+, Q_-]\sigma^3_{\mu\nu}\sigma^1_{\alpha\beta} + [Q_-, Q_+]\sigma^1_{\mu\nu}\sigma^3_{\alpha\beta} \\ &= 4(2n+1)\left(\sigma^3_{\mu\nu}\sigma^1_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^3_{\alpha\beta}\right) \\ \left[B_{\mu\nu}, B_{\alpha\beta}\right] &= -\left[A_{\mu\nu}, A_{\alpha\beta}\right] \\ \left[A_{\mu\nu}, B_{\alpha\beta}\right] &= 4(2n+1)\left(\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\right). \end{aligned}$$

In terms of these,

$$\begin{aligned} H_0 &= \omega_c(n + \frac{1}{2}) \\ T_{\mu\nu} &= \omega_c(n+\frac{1}{2})\delta_{\mu\nu} + \frac{\omega_c}{2}A_{\mu\nu} \end{aligned}$$

and

$$\left[H_0, T_{\mu\nu}\right] = \frac{\omega_c^2}{2}\left[n, A_{\mu\nu}\right] = \omega_c^2 B_{\mu\nu}.$$

We now compute

$$\begin{aligned} T_{\mu\nu}(t) &= e^{iH_0 t}T_{\mu\nu}E^{-iH_0 t} \\ &= T_{\mu\nu} + (-it)\left[H_0, T_{\mu\nu}\right] + \frac{(-it)^2}{2!}\left[H_0, \left[H_0, T_{\mu\nu}\right]\right] + \ldots \\ &= \left(H_0\delta_{\mu\nu} + \frac{\omega_c}{2}A_{\mu\nu}\right) + (-it)\omega_c^2 B_{\mu\nu} + \frac{(-it)^2}{2!}2\omega_c^3 A_{\mu\nu} + \ldots \\ &= H_0\delta_{\mu\nu} + \frac{\omega_c}{2}A_{\mu\nu}\left[1 + \frac{(-2it\omega_c)^2}{2!} + \frac{(-2it\omega_c)^4}{4!} + \ldots\right] + \frac{\omega_c}{2}B_{\mu\nu}\left[-2it\omega_c^2 + \frac{(-2it\omega_c)^3}{3!} + \ldots\right] \\ &= H_0\delta_{\mu\nu} + \frac{\omega_c}{2}A_{\mu\nu}\cos(2\omega_c t) + \frac{\omega_c}{2}B_{\mu\nu}(-i\sin(2\omega_c t)), \end{aligned}$$

giving

$$\left[T_{\mu\nu}(t), T_{\alpha\beta}\right] = \left[H_0\delta_{\mu\nu} + \frac{\omega_c}{2}\cos(2\omega_c t)A_{\mu\nu} - \frac{i\omega_c}{2}\sin(2\omega_c t)B_{\mu\nu}, H_0\delta_{\alpha\beta} + \frac{\omega_c}{2}A_{\alpha\beta}\right].$$

Then we have

$$\begin{aligned} \left\langle\left[T_{\mu\nu}(t), T_{\alpha\beta}\right]\right\rangle &= \frac{\omega_c^2}{4}\cos(2\omega_c t)\left\langle\left[A_{\mu\nu}, A_{\alpha\beta}\right]\right\rangle - \frac{i\omega_c}{2}\sin(2\omega_c t)\left\langle\left[B_{\mu\nu}, A_{\alpha\beta}\right]\right\rangle \\ &= \omega_c^2\cos(2\omega_c t)\left\langle(2n+1)\right\rangle\left(\sigma^3_{\mu\nu}\sigma^1_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^3_{\alpha\beta}\right) + i\omega_c^2\sin(2\omega_c t)\left\langle(2n+1)\right\rangle\left(\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\right) \\ &= 2\omega_c\left\langle H_0\right\rangle\left[\cos(2\omega_c t)\left(\sigma^3_{\mu\nu}\sigma^1_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^3_{\alpha\beta}\right) + i\sin(2\omega_c t)\left(\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\right)\right] \end{aligned}$$

With

$$\int_0^\infty dt\, e^{i\omega_+ t}\cos(2\omega_c t) = \frac{i\omega_+}{\omega_+^2 - 4\omega_c^2}$$
$$\int_0^\infty dt\, e^{i\omega_+ t}\sin(2\omega_c t) = \frac{-2\omega_c}{\omega_+^2 - 4\omega_c^2},$$

we find

$$\begin{aligned}\int_0^\infty dt\, e^{i\omega_+ t}\left\langle\left[T_{\mu\nu}(t), T_{\alpha\beta}\right]\right\rangle &= 2\omega_c\,\langle H_0\rangle\left[\frac{i\omega_+}{\omega_+^2-4\omega_c^2}\left(\sigma^3_{\mu\nu}\sigma^1_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^3_{\alpha\beta}\right) + \frac{-2\omega_c}{\omega_+^2-4\omega_c^2}\left(\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\right)\right]\\ &= \frac{2\omega_c\,\langle H_0\rangle}{\omega_+^2-4\omega_c^2}\left[i\omega_+\left(\sigma^3_{\mu\nu}\sigma^1_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^3_{\alpha\beta}\right) - 2i\omega_c\left(\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\right)\right]\end{aligned}$$

We note the identities

$$\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} + \sigma^1_{\mu\nu}\sigma^1_{\alpha\beta} = \delta_{\nu\alpha}\delta_{\mu\beta} - \epsilon_{\nu\alpha}\epsilon_{\mu\beta}$$
$$\sigma^3_{\mu\nu}\sigma^1_{\alpha\beta} - \sigma^1_{\mu\nu}\sigma^3_{\alpha\beta} = \delta_{\nu\alpha}\epsilon_{\mu\beta} - \delta_{\mu\beta}\epsilon_{\nu\alpha}$$

[...]

$$\begin{aligned}\left[T_{\mu\nu}, J^{\mathrm{sh}}_{\alpha\beta}\right] &= \left[H_0\delta_{\mu\nu} + \frac{\mathcal{B}}{2m}\left((b^{\dagger 2}+b^2)\sigma^3_{\mu\nu} + (b^{\dagger 2}-b^2)\sigma^1_{\mu\nu}\right), \frac{i}{4}(b^{\dagger 2}-b^2)\sigma^3_{\alpha\beta}\right]\\ &\quad - \left[H_0\delta_{\mu\nu} + \frac{\mathcal{B}}{2m}\left((b^{\dagger 2}+b^2)\sigma^3_{\mu\nu} + (b^{\dagger 2}-b^2)\sigma^1_{\mu\nu}\right), \frac{1}{4}(b^{\dagger 2}+b^2)\sigma^1_{\alpha\beta}\right]\\ &\quad + \left[H_0\delta_{\mu\nu} + \frac{\mathcal{B}}{2m}\left((b^{\dagger 2}+b^2)\sigma^3_{\mu\nu} + (b^{\dagger 2}-b^2)\sigma^1_{\mu\nu}\right), \frac{1}{2}b^\dagger b\epsilon_{\alpha\beta}\right]\\ &= \frac{i}{4}\left[H_0, b^{\dagger 2}-b^2\right]\delta_{\mu\nu}\sigma^3_{\alpha\beta} - \frac{1}{4}\left[H_0, b^{\dagger 2}+b^2\right]\delta_{\mu\nu}\sigma^3_{\alpha\beta}\\ &\quad + \frac{i\mathcal{B}}{8m}\left[b^{\dagger 2}+b^2, b^{\dagger 2}-b^2\right]\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} - \frac{\mathcal{B}}{8m}\left[b^{\dagger 2}-b^2, b^{\dagger 2}+b^2\right]\sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\\ &\quad + \frac{\mathcal{B}}{4m}\left[\left((b^{\dagger 2}+b^2)\sigma^3_{\mu\nu} + (b^{\dagger 2}-b^2)\sigma^1_{\mu\nu}\right), b^\dagger b\right]\epsilon_{\alpha\beta}\\ &= \omega_c(\frac{i}{2}\left(b^{\dagger 2}+b^2\right)\delta_{\mu\nu}\sigma^3_{\alpha\beta} - \frac{1}{2}\left(b^{\dagger 2}-b^2\right)\delta_{\mu\nu}\sigma^3_{\alpha\beta}\\ &\quad + \frac{i}{2}(bb^\dagger + b^\dagger b)\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} + \frac{1}{2}(bb^\dagger + b^\dagger b)\sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\\ &\quad + \frac{1}{4}\left[b^{\dagger 2}+b^2, b^\dagger b\right]\sigma^3_{\mu\nu}\epsilon_{\alpha\beta} + \frac{1}{4}\left[b^{\dagger 2}-b^2, b^\dagger b\right]\sigma^1_{\mu\nu}\epsilon_{\alpha\beta})\end{aligned}$$

$$\left\langle\left[T_{\mu\nu}, J^{\mathrm{sh}}_{\alpha\beta}\right]\right\rangle = E_0\left(i\sigma^3_{\mu\nu}\sigma^3_{\alpha\beta} + \sigma^1_{\mu\nu}\sigma^1_{\alpha\beta}\right).$$

We also note that

$$\left[T_{\mu\nu}, K\right] = 0.$$

[...]

$$X_{\mu\nu\alpha\beta}(\omega) = \frac{E_0}{\omega_+^2 - 4\omega_c^2}\left(i\omega_+\left(\delta_{\mu\beta}\delta_{\nu\alpha} - \epsilon_{\mu\beta}\epsilon_{\nu\alpha}\right) - 2\omega_c\left(\delta_{\nu\alpha}\epsilon_{\mu\beta} - \delta_{\mu\beta}\epsilon_{\alpha\nu}\right)\right) + \ldots$$

2. Quartic hamiltonians

We repeat the above calcuation for an unstrained hamiltonian

$$H_0 = \frac{\pi_x^4 + \pi_y^4}{4}$$

which is not rotationally symmetric.

To do so we will need to compute the strain generators and stress tensor for this system. As before, the strain generators are found by imposing the condition

$$H_\Lambda = e^{-i\lambda_{\mu\nu}J_{\mu\nu}} H_0 e^{i\lambda_{\mu\nu}J_{\mu\nu}}$$

For the moment, we assume that the strain generators take the same form as for the quadratic Hamiltonian,

$$J_{\mu\nu} = \frac{1}{2}\left(-\left\{x_\mu, \pi_\nu\right\} + \mathcal{B}\epsilon_{\nu\alpha}x_\mu x_\alpha\right) + \frac{\delta_{\mu\nu}}{2}\left(-\frac{1}{2}\left\{x_\mu, \pi_\mu\right\} + \{\mathcal{B}, \Xi\})\right)$$

The stress tensor is then

$$T_{\mu\nu} = -i[H_0, J_{\mu\nu}]$$

As before, we can write the coordinates and kinetic momentum in terms of raising and lowering operators. We see

$$H_0 = \ldots$$
$$J_{\mu\nu}^{\mathrm{sh}} = \ldots$$

[...] If we consider a more general quartic Hamiltonian in *zero* magnetic field,

$$H' = g_1^{\mu\nu} g_2^{\rho\sigma} p_\mu p_\nu p_\rho p_\sigma,$$

again applying the transformation $p_\mu \to \Lambda^{\mu\nu} p_\nu$ and taking the derivative with respect to $\lambda_{\mu\nu}$, we find

$$\frac{\partial H'}{\partial \lambda_{ij}} = 2g_1^{j\mu} g_2^{\rho\sigma} e^{\lambda_{\beta\mu}} e^{\lambda_{\gamma\rho}} e^{\lambda_{\delta\sigma}} p_i p_\beta p_\gamma p_\delta + 2g_1^{\mu\nu} g_2^{j\sigma} e^{\lambda_{\beta\mu}} e^{\lambda_{\gamma\rho}} e^{\lambda_{\delta\sigma}} p_\alpha p_\beta p_i p_\delta$$

so that

$$T_{\mu\nu} = 2\left(g_1^{\nu\alpha} g_2^{\beta\gamma} + g_1^{\alpha\beta} g_2^{\nu\gamma}\right) p_\mu p_\alpha p_\beta p_\gamma$$

References

[1] B. Bradlyn, M. Goldstein, and N. Read, Phys. Rev. B 86, 1 (2012).