Some details ASZ's calculation of Landau level Hall viscosity

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1. Berry curvature in space of flat metrics

We calculate the adiabatic curvature resulting from varying the LLL wavefunctions with respect to the torus geometry, as parametrized by flat metrics

$$g(V,\tau) = \frac{V}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$

Starting from the Landau level wavefunctions on the torus (ASZ 14), we can calculate the Berry curvature using (ASZ 6),

$$F_{ij} = \mathrm{Im}\sum_{\ell=1}^{N}\left\langle \partial_i\phi_\ell \middle| \partial_j\phi_\ell \right\rangle,$$

where ℓ indexes the single particle wavefunctions, and N is the number of particles. We will focus on obtaining this curvature for a particular index ℓ. We will find that F does not depend on ℓ, so the result for multiple particles filling N levels follows trivially.

The normalized lowest Landau level wavefunction for a given metric $g(V,\tau)$ is

$$\phi_\ell(\mathbf{x}) = \frac{(2\tau_2 B)^{1/4}}{\sqrt{V}}\sum_{n=-\infty}^{\infty} e^{i\pi\tau B(\widetilde{y}+n)^2} e^{-2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+n)} e^{2i\pi(nB+\ell)x}$$

with $\widetilde{y} = y + y_\ell = y + \ell/B + \varphi_1/B + 1/2$.

To compute components of F involving τ_1, τ_2, we note

$$\phi_\ell(\mathbf{x}) = \frac{(2\tau_2 B)^{1/4}}{V^{1/2}}\Phi_\ell(\mathbf{x})$$

with Φ holomorphic in τ. Therefore

$$\partial_{\tau_1}\phi_\ell(\mathbf{x}) = \frac{(2\tau_2 B)^{1/4}}{V^{1/2}}\partial_\tau\Phi_\ell(\mathbf{x})$$

and

$$\partial_{\tau_2}\phi_\ell(x,y) = \frac{(2\tau_2 B)^{1/4}}{V^{1/2}}\left[\frac{1}{4\tau_2}\Phi_\ell(\mathbf{x}) + i\partial_\tau\Phi_\ell(\mathbf{x})\right].$$

Explicitly,

$$\partial_\tau\Phi_\ell(\mathbf{x}) = \sum_{n=-\infty}^{\infty}\left(i\pi B(\widetilde{y}+n)^2\right) e^{i\pi\tau B(\widetilde{y}+n)^2} e^{-2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+n)} e^{2i\pi(nB+\ell)x}.$$

For F^{ℓ}_{V,τ_2},

$$\begin{aligned}
F^{\ell}_{V,\tau_2} = -F^{\ell}_{\tau_2,V} &= \mathrm{Im}\left[\langle \partial_V \phi_\ell | \partial_{\tau_2}\phi_\ell\rangle\right] \\
&= \mathrm{Im}\left[\int_Q \sqrt{\det g}\ \mathrm{d}x\ \mathrm{d}y\ \partial_V \phi^*_\ell(\mathbf{x})\partial_{\tau_2}\phi_\ell(\mathbf{x})\right] \\
&= \mathrm{Im}\left[-\frac{1}{2V}\int_Q \mathrm{d}x\ \mathrm{d}y\ \phi^*_\ell(\mathbf{x})\left(\frac{1}{4\tau_2}\phi_\ell(\mathbf{x}) + i\sqrt{2\tau_2 B}\ \partial_\tau \Phi_\ell(\mathbf{x})\right)\right],
\end{aligned}$$

where the integral is over $Q = [0,1]\times[0,1]$. The first term in the integral is clearly real, so

$$\begin{aligned}
F^{\ell}_{V,\tau_2} &= \frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[i\int_Q \mathrm{d}x\ \mathrm{d}y\ \Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x})\right] \\
&\equiv \frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[iI_1\right].
\end{aligned}$$

The integral I_1 is

$$\begin{aligned}
I_1 = \int_Q \mathrm{d}x\ \mathrm{d}y\ \Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x}) &= \int_Q \mathrm{d}x\ \mathrm{d}y\ \left[\sum_{m=-\infty}^{\infty} e^{-i\pi\tau^* B(\widetilde{y}+m)^2} e^{2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+m)} e^{-2i\pi(mB+\ell)x}\right] \\
&\qquad \times \left[\sum_{n=-\infty}^{\infty}\left(i\pi B(\widetilde{y}+n)^2\right) e^{i\pi\tau B(\widetilde{y}+n)^2} e^{-2i\pi\left(\varphi_2+B/2\right)(\widetilde{y}+n)} e^{2i\pi(nB+\ell)x}\right] \\
&= \int_Q \mathrm{d}x\ \mathrm{d}y \sum_{m,n} i\pi B(\tilde{y}+n)^2 e^{i\pi B\left(\tau(\tilde{y}+n)^2-\tau^*(\tilde{y}+m)^2\right)} e^{2i\pi(\phi_2+B/2)(m-n)} e^{2i\pi Bx(n-m)}
\end{aligned}$$

Performing the x integration, we have, since B is an integer

$$\int_0^1 \mathrm{d}x\ e^{2i\pi Bx(n-m)} = \delta_{n,m},$$

and so

$$\begin{aligned}
I_1 &= \int_0^1 \mathrm{d}y\ \sum_n i\pi B(\tilde{y}+n)^2 e^{i\pi B(\tau-\tau^*)(\tilde{y}+n)^2} \\
&= i\pi B\sum_{n=-\infty}^{\infty}\int_0^1 \mathrm{d}y\ (\tilde{y}+n)^2 e^{-2\pi B\tau_2(\tilde{y}+n)^2} \\
&= i\pi B\sum_{n=-\infty}^{\infty}\int_{y_\ell+n}^{y_\ell+n+1} \mathrm{d}u\ u^2 e^{-2\pi B\tau_2 u^2} \\
&= i\pi B\int_{-\infty}^{\infty} \mathrm{d}u\ u^2 e^{-2\pi B\tau_2 u^2} \\
&= i\pi B\left(\frac{1}{4\sqrt{2}\pi(B\tau_2)^{3/2}}\right) \\
&= \frac{i}{4\tau_2\sqrt{2\tau_2 B}}
\end{aligned}$$

Since I_1 is pure imaginary,

$$F^{\ell}_{V,\tau_2} = \frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[iI_1\right] = 0.$$

For F^{ℓ}_{V,τ_1},

$$\begin{aligned}
F^{\ell}_{V,\tau_1} &= \mathrm{Im}\left[-\frac{\sqrt{2\tau_2 B}}{2V}\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\Phi^*_\ell(\mathbf{x})\,\partial_\tau\Phi_\ell(\mathbf{x})\right] \\
&= -\frac{\sqrt{2\tau_2 B}}{2V}\mathrm{Im}\left[I_1\right] \\
&= -\frac{1}{8\tau_2 V} \neq 0\,[???]
\end{aligned}$$

For $F^{\ell}_{\tau_1,\tau_2}$,

$$\begin{aligned}
F^{\ell}_{\tau_1,\tau_2} = -F^{\ell}_{\tau_2,\tau_1} &= \mathrm{Im}\left[\langle\partial_{\tau_1}\phi_\ell|\partial_{\tau_2}\phi_\ell\rangle\right] \\
&= \mathrm{Im}\left[\int\limits_Q \sqrt{\det g}\;\mathrm{d}x\;\mathrm{d}y\;\partial_{\tau_1}\phi^*_\ell(\mathbf{x})\partial_{\tau_2}\phi_\ell(\mathbf{x})\right] \\
&= \mathrm{Im}\left[\sqrt{2\tau_2 B}\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\left(\frac{1}{4\tau_2}\Phi_\ell(\mathbf{x}) + i\partial_\tau\Phi_\ell(\mathbf{x})\right)\right] \\
&= \mathrm{Im}\left[\sqrt{2\tau_2 B}\left(\frac{1}{4\tau_2}\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\Phi_\ell(\mathbf{x}) + i\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x})\right)\right] \\
&= \mathrm{Im}\left[\sqrt{2\tau_2 B}\left(\frac{1}{4\tau_2}I_1^* + i\int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x})\right)\right] \\
&\equiv \sqrt{2\tau_2 B}\,\mathrm{Im}\left[\frac{1}{4\tau_2}I_1^* + iI_2\right]
\end{aligned}$$

The integral I_2 is

$$\begin{aligned}
I_2 &= \int\limits_Q \mathrm{d}x\,\mathrm{d}y\,\partial_\tau\Phi^*_\ell(\mathbf{x})\partial_\tau\Phi_\ell(\mathbf{x}) \\
&= \int\limits_Q \mathrm{d}x\,\mathrm{d}y\sum_{m,n}\left(-i\pi B(\widetilde{y}+m)^2\right)\left(i\pi B(\widetilde{y}+n)^2\right)e^{i\pi B\left(\tau(\widetilde{y}+n)^2-\tau^*(\widetilde{y}+m)^2\right)}e^{2i\pi(\phi_2+B/2)(m-n)}e^{2i\pi Bx(n-m)} \\
&= \pi^2B^2\sum_{n=-\infty}^{\infty}\int\limits_0^1 \mathrm{d}y\,(\widetilde{y}+n)^4e^{-2\pi B\tau_2(\widetilde{y}+n)^2} \\
&= \pi^2B^2\int\limits_{-\infty}^{\infty}\mathrm{d}u\;u^4e^{-2\pi B\tau_2(u)^2} \\
&= \pi^2B^2\frac{3}{16\sqrt{2}\pi^2(B\tau_2)^{5/2}} \\
&= \frac{1}{\sqrt{2\tau_2 B}}\frac{3}{16\tau_2^2}.
\end{aligned}$$

So that

$$\begin{aligned} F_{\tau_1,\tau_2} &= \sqrt{2\tau_2 B}\,\mathrm{Im}\left[\frac{1}{4\tau_2}\left(\frac{-i}{4\tau_2\sqrt{2\tau_2 B}}\right) + i\frac{1}{\sqrt{2\tau_2 B}}\frac{3}{16\tau_2^2}\right] \\ &= \mathrm{Im}\left[i\left(-\frac{1}{16\tau_2} + \frac{3}{16\tau_2^2}\right)\right] \\ &= \frac{1}{8\tau_2^2} \end{aligned}$$

2. Variations of strain in terms of metric variations

We want to determine how u varies under infinitesimal variations in V and τ. Importantly, we note that we'd like to produce *independent* variations in the components of u, varying one component while keeping the others fixed. To vary u_{11} independently of u_{12} and u_{22}, we perform an overall scaling of the system (vary V) and then vary τ_2 while keeping the area of the system fixed. This method can similarly be used to vary u_{22}. Consider a system initially with $\tau_1 = 0$, but V and τ_2 left general. This yields the most general *independent* variations of u_{11} and u_{22} because variation of V or τ_2 at nonzero τ_1 will simultaneously vary u_{12} The coordinates at nonzero V and τ_2 are scaled from those on $Q = [0,1] \times [0,1]$ by

$$\Delta x = \sqrt{\frac{V}{\tau_2}}$$
$$\Delta y = \sqrt{V\tau_2}$$

If we now take $V \to V' = V + dV$, $\tau_2 \to \tau_2' = \tau_2 + d\tau_2$ to obtain $\Delta x'$ and $\Delta y'$ then

$$du_{11} = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{\sqrt{\frac{V+dV}{\tau_2+d\tau_2}} - \sqrt{\frac{V}{\tau_2}}}{\sqrt{\frac{V}{\tau_2}}} = \frac{1}{2}\left(\frac{dV}{V} - \frac{d\tau_2}{\tau_2}\right)$$

and

$$du_{22} = \frac{\Delta y' - \Delta y}{\Delta y} = \frac{\sqrt{(V+dV)(\tau_2+d\tau_2)} - \sqrt{V\tau_2}}{\sqrt{V\tau_2}} = \frac{1}{2}\left(\frac{dV}{V} + \frac{d\tau_2}{\tau_2}\right)$$

[I'm sure this can be made rigorous, but this captures the general idea.]

To vary u_{12}, we vary $\tau_1 \to d\tau_1$ keeping V,τ_2 fixed. Then

$$du_{12} = \frac{1}{2}\frac{\Delta x' - \Delta x}{\Delta y} = \frac{1}{2}\frac{\sqrt{\frac{V}{\tau_2}}(1+d\tau_1) - \sqrt{\frac{V}{\tau_2}}}{\sqrt{V\tau_2}} = \frac{1}{2}\frac{d\tau_1}{\tau_2}$$

We'd like to write the Berry curvature in terms of these strain differentials rather than dV and $d\tau$ so that we can pick out the Hall viscosity component. Inverting the above,

$$\begin{aligned} dV &= V\,(du_{11} + du_{22}) \\ d\tau_1 &= 2\tau_2 du_{12} \\ d\tau_2 &= \tau_2\,(du_{22} - du_{11}) \end{aligned}$$

If we write the "viscosity part" of the Berry curvature as ASZ do,

$$\Omega = \frac{B}{4\tau_2^2} d\tau_1 \wedge d\tau_2 = \frac{B}{4\tau_2^2}\left[2\tau_2 du_{12}\right] \wedge \left[\tau_2(du_{22} - du_{11})\right] = \frac{B}{2} du_{12} \wedge (du_{22} - du_{11})$$