Localization renormalization and quantum Hall systems

Bartholomew Andrews*† Dominic Reiss* Fenner Harper Rahul Roy

University of California, Los Angeles

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Introduction

- Concepts of **renormalization** can be designed around various physical quantities for applications in condensed matter, high energy, and cosmology.

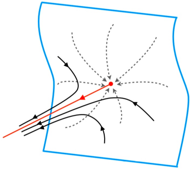
  $\longrightarrow$ position **r** [Kadanoff 1966], momentum **p** [Wilson 1971], entanglement S [Vidal 2007]

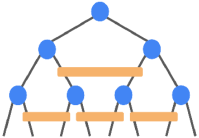
- These applications often lead to novel ideas, approaches, and algorithms.

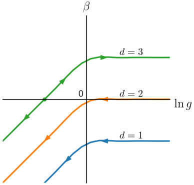
  $\longrightarrow$ MCRG [Swendsen 1979], DMRG [White 1992], FRG [Wetterich 1993], PEPS [Verstraete & Cirac 2004], TRG [Levin & Nave 2007], MERA [Vidal 2008], deep learning? [Koch & Cheng 2020]

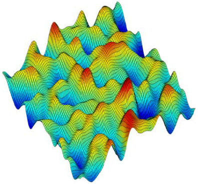
- A hallmark of several interesting condensed matter phases is the ability to construct localized degrees of freedom, which is quantified via a **localization length** ξ.

  $\longrightarrow$ Anderson insulators, MBL, plateau transitions, . . .









Images: (1) Skinner, QFT II lecture notes 2018, (2) Milstead et al. 2019, (4) Dellago & Bollhuis 2009

Localization Renormalization

We introduce a renormalization procedure based on the characteristic localization length.

1. Consider a d-dimensional quantum single-particle system with Hilbert space $\mathcal{H}$.
2. Construct a maximal set of **quasilocal operators**, corresponding to a complete basis of wavefunctions in a given band $\{|\psi\rangle\}$, and maximally localized in some metric D.
3. Define a **family of projectors** that eliminate a fraction $1-\rho$ of the degrees of freedom

$$P_\rho = P_{\text{band}} - \sum_{i \in \mathcal{L}_\rho} |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i| \,,$$

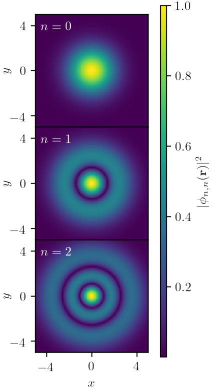
   where P_{band} is the projector to the relevant single-particle band, and $|\tilde{\psi}_i\rangle$ is the symmetrically-orthogonalized wavefunction at site i in the removal subregion $\mathcal{L}_\rho$.
4. Iteratively apply P_ρ to the system and quantify the **scaling of the characteristic localization length** ξ in the residual Hilbert space $\mathcal{H}'$ using D.
5. As $\rho \to 0$ in the thermodynamic limit, ξ diverges for delocalized systems with a **universal scaling exponent**, whereas it is constant in a localized phase.

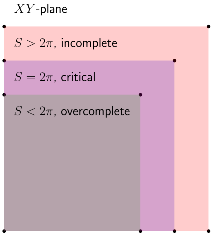
$\Longrightarrow$ Basis-independent method for classifying a wide variety of localization transitions!

Example 1: Landau Levels

Model

- Free spinless electron in 2D with a perpendicular magnetic field: $H_{\text{LL}} = (\mathbf{p} - e\mathbf{A})^2/2m = \hbar\omega(a^\dagger a + 1/2)$.
- Symmetric gauge $\Rightarrow L_z = \hbar(a^\dagger a - b^\dagger b)$, yields the eigenspectrum: $|n, m\rangle$ with $E_n = \hbar\omega(n + 1/2)$.
- We focus on **coherent states** $|\beta\rangle : b\,|\beta\rangle = \beta\,|\beta\rangle$:
  - non-dispersive minimum uncertainty states $\Delta X \Delta Y = \hbar/2$
  - magnetic translations of $|n, n\rangle$





- Coherent states form an overcomplete basis for the LL.
- Set of coherent states is **critical** when restricted to an XY-plane unit cell area $S = 2\pi$.
- Symmetric orthogonalization of a complete basis in a LL yields divergence at long distances.

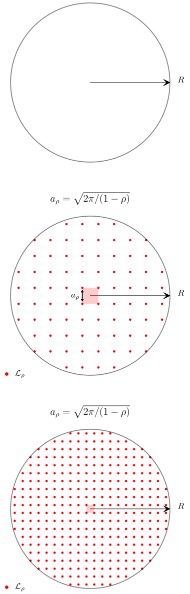
Example 1: Landau Levels

Method

1. LLs $\{|n, m\rangle\}$ defined on a **disc** of radius R, with truncated angular momentum basis $m \in \{0, 1, \ldots, \frac{3}{4}R^2\}$.
2. Introduce a removal lattice $\mathcal{L}_\rho$, with UC area $A_\rho = 2\pi/(1-\rho)$ centered at the origin, where ρ is the **fraction of states remaining** relative to $\mathcal{L}_0$: $\rho = 1 - A_0/A_\rho$. For each lattice site $\mathbf{r}_{ij} \in \mathcal{L}_\rho$, find the maximally-localized states $|\psi_{ij}\rangle$ centered at $\mathbf{r}_{ij}$.
3. Symmetrically orthogonalize these states, such that $\{|\psi_{ij}\rangle\} \to \{|\tilde{\psi}_{ij}\rangle\}$, and **project them out** of the Hilbert space:

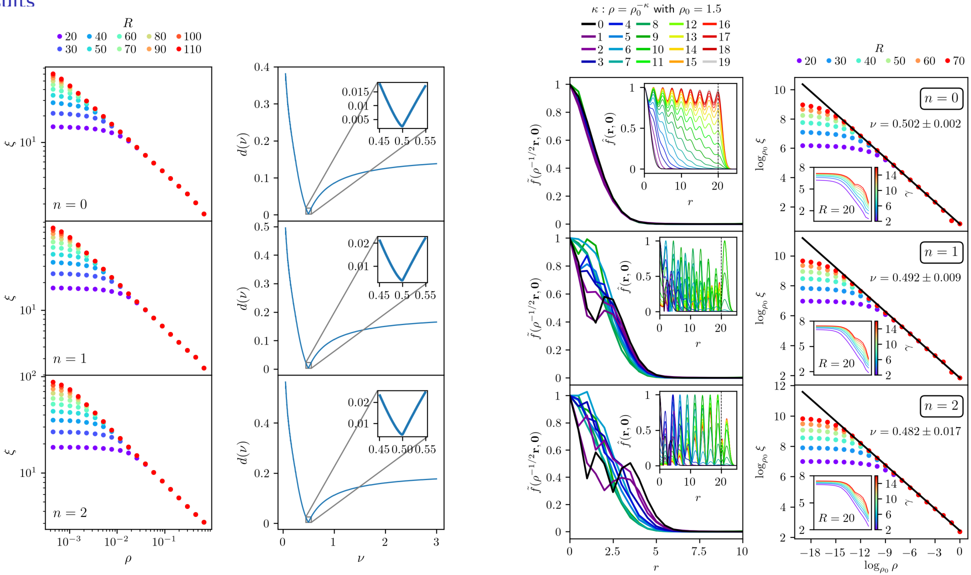
$$P_\rho^{\mathrm{LL}} = P_{n\mathrm{LL}} - \sum_{i,j \in \mathcal{L}_\rho} |\tilde{\psi}_{ij}\rangle \langle \tilde{\psi}_{ij}| \,.$$

4. Record the **localization length**, ξ, quantified by e.g. $P_\rho^{\mathrm{LL}} \mathbf{r}^2 P_\rho^{\mathrm{LL}} |\mathbf{0}_\rho\rangle = \xi^2 |\mathbf{0}_\rho\rangle$, where $|\mathbf{0}_\rho\rangle$ is the maximally-localized state at the origin, by construction.
5. Repeat for smaller ρ.



Example 1: Landau Levels

Results

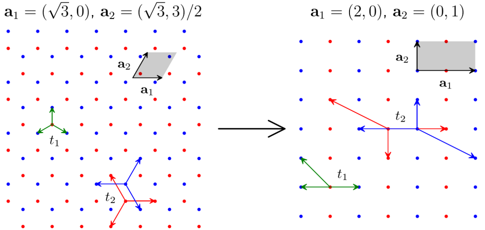


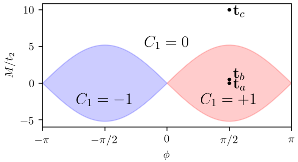
Example 2: Chern Insulators

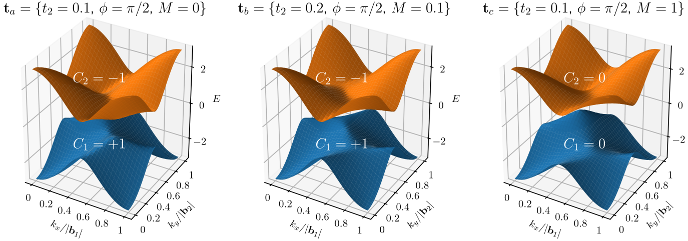
Model

Three configurations of the (square-lattice) **Haldane model**:

$$H_{\mathrm{CI}} = -t_1 \sum_{\langle ij \rangle} c_i^\dagger c_j - t_2 \sum_{\langle\langle ij \rangle\rangle} e^{\pm i\phi} c_i^\dagger c_j + M \sum_i (n_{A,i} - n_{B,i}) + \mathrm{H.c.}$$







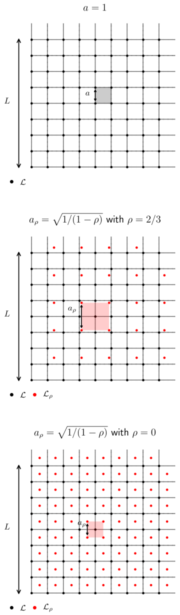
Example 2: Chern Insulators

Method

1. $L \times L$ square lattice for the Haldane model $\mathcal{L}$, with lattice constant $a = 1$, defined on a **torus**.
2. Introduce a removal lattice $\mathcal{L}_\rho$, with UC area $A_\rho = 1/(1-\rho)$ centered at an "origin" Haldane lattice site. For each lattice site $\mathbf{r}_{ij} \in \mathcal{L}_\rho$, find the maximally-localized state $|\psi_{ij}\rangle$ centered at $\mathbf{r}_{ij}$.
3. Symmetrically orthogonalize these states, such that $\{|\psi_{ij}\rangle\} \to \{|\tilde{\psi}_{ij}\rangle\}$, and **project them out** of the Hilbert space via

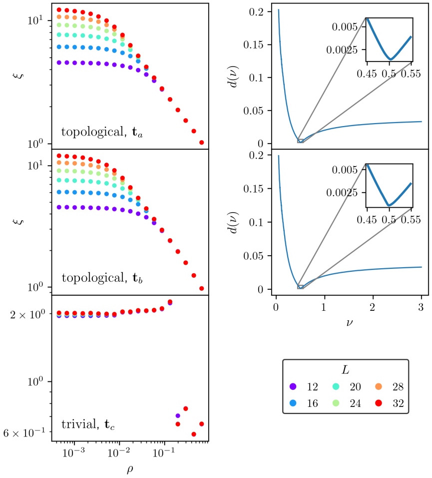
$$P_\rho^{\mathrm{CI}} = P_{\mathrm{LB}} - \sum_{i,j \in \mathcal{L}_\rho} |\tilde{\psi}_{ij}\rangle \langle \tilde{\psi}_{ij}| \,.$$

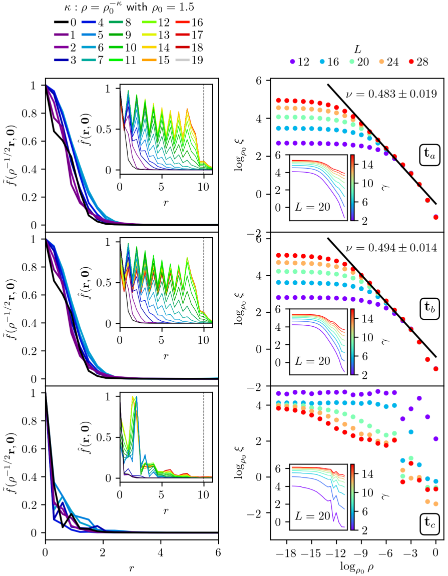
4. Record the **localization length**, ξ, quantified by, e.g. $P_\rho^{\mathrm{CI}} \mathbf{r}^2 P_\rho^{\mathrm{CI}} |\mathbf{0}_\rho\rangle = \xi^2 |\mathbf{0}_\rho\rangle$, where $|\mathbf{0}_\rho\rangle$ is the maximally-localized state at the origin, by construction.
5. Repeat for smaller ρ.



Example 2: Chern Insulators

Results





Discussion

- **Real-space RG:** Statistical self-similarity of projectors P_ρ accords with an RG picture.

Coarse-grain: $\bar{H}(\mathbf{r}) = P_\rho H(\mathbf{r}) P_\rho$ → Rescale: $\mathbf{r}' = \rho^{-1/2}\mathbf{r}$ → Renormalize: $H'(\mathbf{r}') = \rho^{-1}\bar{H}(\mathbf{r}')$

- **Entanglement RG:** *"All stable, gapped, d-dimensional phases of matter are generalized s-source RG fixed points, where s is the number of copies of an entangled ground state of linear extent L needed to describe the corresponding state of size 2L, by acting with a quasilocal unitary transformation."* [Swingle & McGreevy 2016]

Similarity	Difference
Quasilocal unitary used to scale the effective system size while limiting incremental entangling.	Degrees of freedom removed from a fraction of sites in $\mathcal{H}$ and not in increments of the system size.

- **Probing band topology:** Several advantages over existing methods...
  + versatile $\Rightarrow$ can be used in situations where traditional methods fail e.g. fractal lattices [Jha & Nielsen 2023], quasicrystals [Koshino & Oka 2021]
  + spectrum independent $\Rightarrow$ can be used to diagnose topology in disordered systems
  + generalizable $\Rightarrow$ can be extended to other symmetry classes and dimensions

Conclusion

- **Localization renormalization** is an efficient diagnostic for analyzing a diverse range of localization transitions, which we demonstrate using single-particle examples.
- For 2D class A topological insulators, we find the **universal scaling relation**

$$\lim_{\substack{\rho\to 0\\ L\to\infty}} \xi(\rho) \sim \begin{cases} \rho^{-1/2}, & \text{in a topological phase,} \\ \text{const}, & \text{in a trivial phase,} \end{cases}$$

  independent of the model, truncation algorithm, and ξ metric.
- The universal scaling exponent ν is a self-similar property of the family of projectors P_ρ, which accords with an **RG picture**.

Thank you for listening!

arXiv:2303.xxxxx [cond-mat.mes-hall] coming soon