Simulating fractional quantum Hall states using two-dimensional tensor networks

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Introduction

The **fractional quantum Hall effect (FQHE)** is the original example of a topological phase of matter with anyonic excitations. It is particularly versatile because it can host a wide selection of anyons, including the non-Abelian anyons needed for **topological quantum computing**. This year, we have demonstrated for the first time: the braiding of non-Abelian anyons to store information [1], anyonic statistics in graphene [2], and the FQHE at zero magnetic field in moiré materials [3]. The ultimate goal for many is to now combine these results and braid non-Abelian anyons in moiré mateirals at zero magnetic field.
Given this rapid experimental progress, efforts have redoubled to develop numerical tools that can efficiently simulate these strongly-correlated quantum many-body systems. Typically, this involves a combination of large-scale ED and MPS approaches. However, these methods often suffer from poor system-size scaling. Recently, **projected entangled pair states (PEPS)** algorithms have emerged as promising candidates for these 2D systems [4]. Inspired by neural quantum states [5], we evaluate the performance of **stochastic reconfiguration (SR)** VMC on a PEPS ansatz, as an alternative to common MPS approaches.

Harper-Hofstadter Model

The Harper-Hofstadter model is a popular starting point for FQHE state engineering, due to its fractal single-particle band structure with an infinite selection of Chern bands [6]. The interacting Hamiltonian is given as

$$H = -\sum_{\langle ij \rangle} \left[\exp\left\{ \frac{2\pi \mathrm{i}}{\phi_0} \int_i^j \mathbf{A} \cdot \mathrm{d}\mathbf{l} \right\} c_i^\dagger c_j + \text{H.c.} \right] + U \sum_i \rho_i \rho_i,$$

where ϕ_0 is the flux quantum, $\mathbf{A}$ is the vector potential, U is the onsite interaction strength, and $\rho = c^\dagger c$ is the density operator.

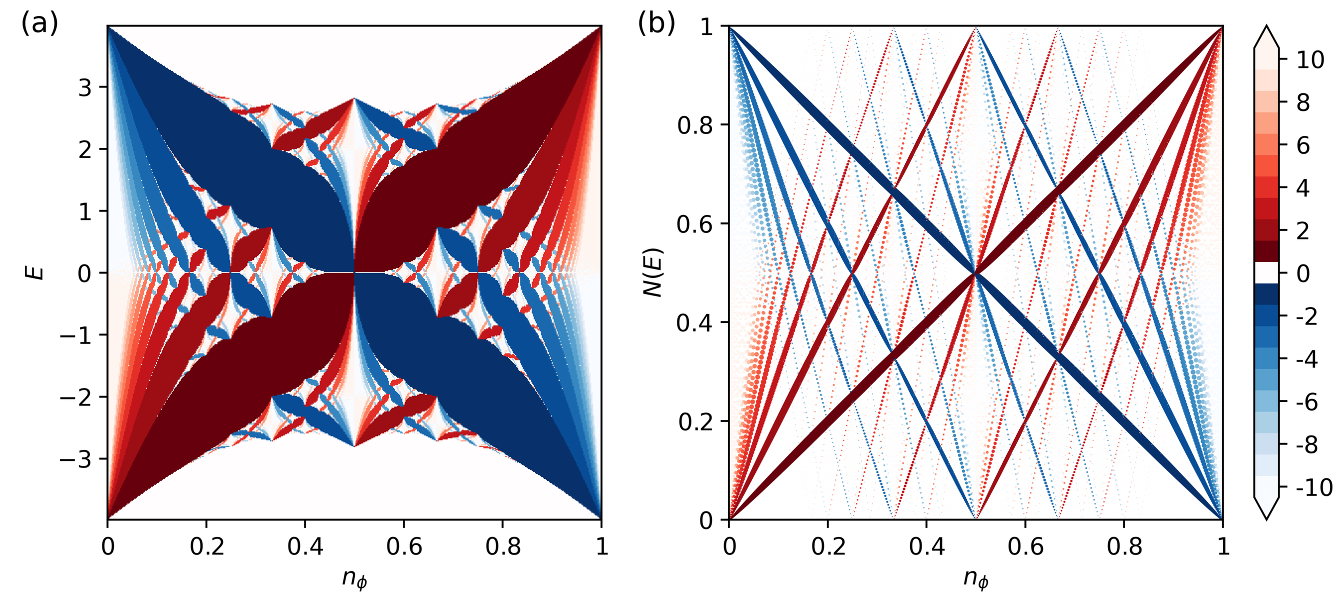

Figure 1: (a) Hofstadter butterfly and (b) Wannier diagram for the square-lattice Hofstadter model. (a) The energy E, and (b) the integrated density of states below the gap $N(E)$, are plotted as a function of flux density $n_\phi = BA_{\mathrm{min}}/\phi_0 = p/499$, where B is the perpendicular field strength, A_{min} is the area of a minimal hopping plaquette, and p is an integer. The r-th gap is colored with respect to $t = \sum_{i=0}^{r} C_i$, where C_i is the Chern number of band i. The size of the points in the Wannier diagram is proportional to the size of the gaps.[6]

Composite Fermion Theory

The complete set of fractions appearing in the FQHE is not yet fully understood. However, many of the most prominent values can be explained as the IQHE of composite fermions, which are bound states of electrons and quantized vortices. The resulting **generalized Jain series** is given as

$$\nu = \frac{r}{k|Cr|+1},$$

where $r \in \mathbb{Z} \setminus \{0\}$ is the number of filled Λ-levels, $k \in \mathbb{Z} \setminus \{0\}$ is the number of flux quanta attached to each electron (odd for bosons, even for fermions), and $C \in \mathbb{Z}$ is the Chern number of the corresponding single-particle band [7].

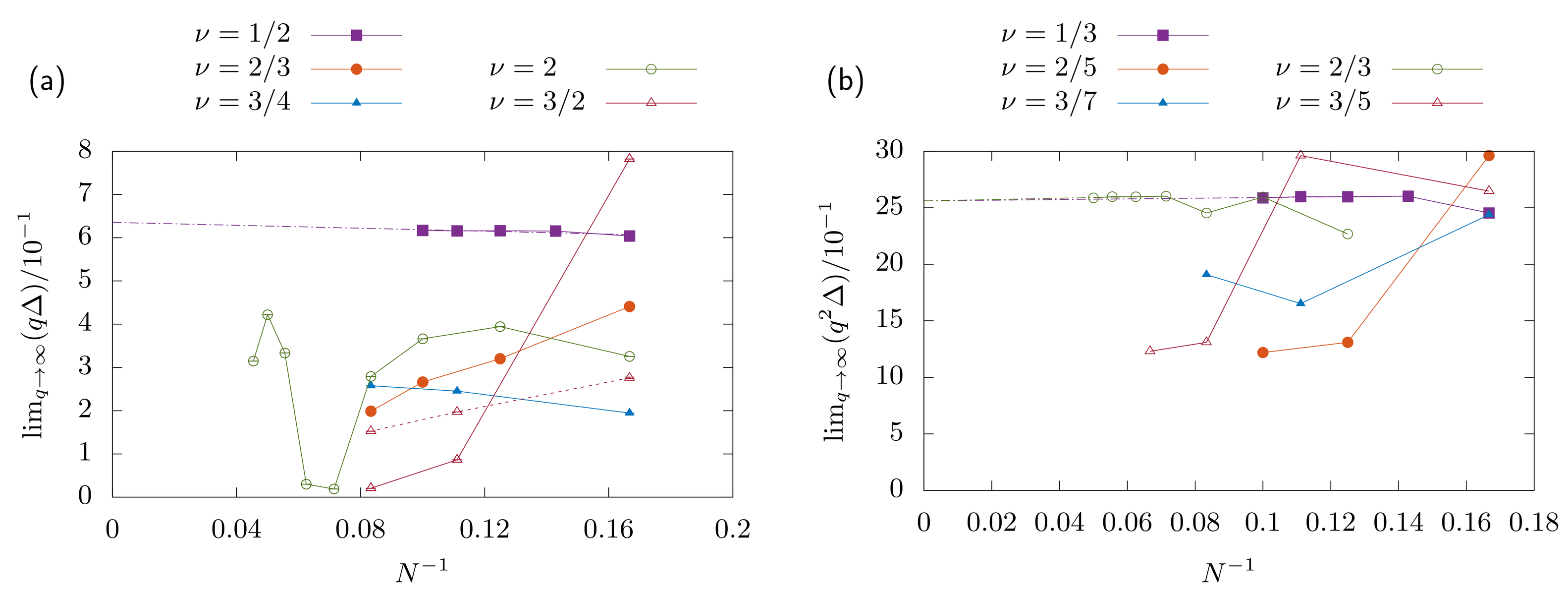

Figure 2: ED finite-size scaling of the many-body gap Δ to the thermodynamic continuum limit at fixed aspect ratio on a torus, for (a) hardcore bosons and (b) fermions with nearest-neighbor interactions, in $|C| = 1$ bands, where q is the area of the magnetic unit cell and N is the particle number. The dashed line for the bosonic $\nu = 3/2$ series corresponds to the scaling assuming a 4-fold Read-Rezayi degeneracy. [8]

Stochastic Reconfiguration

One method to find the ground state of a FQHE system is to use VMC to sample from a PEPS ansatz [9]. In SR, this involves modifying the parameters θ of our wavefunction $|\psi\rangle$, such that we approach the ground state along a path given by the projection $\mathbb{1} - \epsilon H$, where ϵ is chosen such that $\mathbb{1} - \epsilon H \geq 0$ [10, 11]. One SR update step is summarized as

$$\theta_\alpha \longrightarrow \theta_\alpha - \eta \sum_\beta (S + \epsilon \mathbb{1})^{-1}_{\alpha,\beta} R_\beta,$$

where η is the learning rate, ϵ is the regularization constant, and we define

$$S_{\alpha,\beta} = \langle O_\alpha^\dagger O_\beta \rangle - \langle O_\alpha^\dagger \rangle \langle O_\beta \rangle$$
$$R_\alpha = \langle O_\alpha^\dagger H \rangle - \langle O_\alpha^\dagger \rangle \langle H \rangle,$$

with $O_\alpha |x\rangle = \partial_{\theta_\alpha} \log(\langle x|\psi\rangle)\, |x\rangle$. SR takes advantage of the **quantum geometric tensor** for faster optimization. Moreover, the VMC framework allows for efficient GPU acceleration.

$\nu = 1/2$ Laughlin state

The most well-studied FQHE states with the clearest signatures are the **Laughlin states**. At these filling factors, all numerical algorithms agree on the scaling of the ground state energy, including ED, snake DMRG, and VMC+PEPS. The ground state can also be verified by examining edge modes and trapping anyons with fractional charge [12, 13].

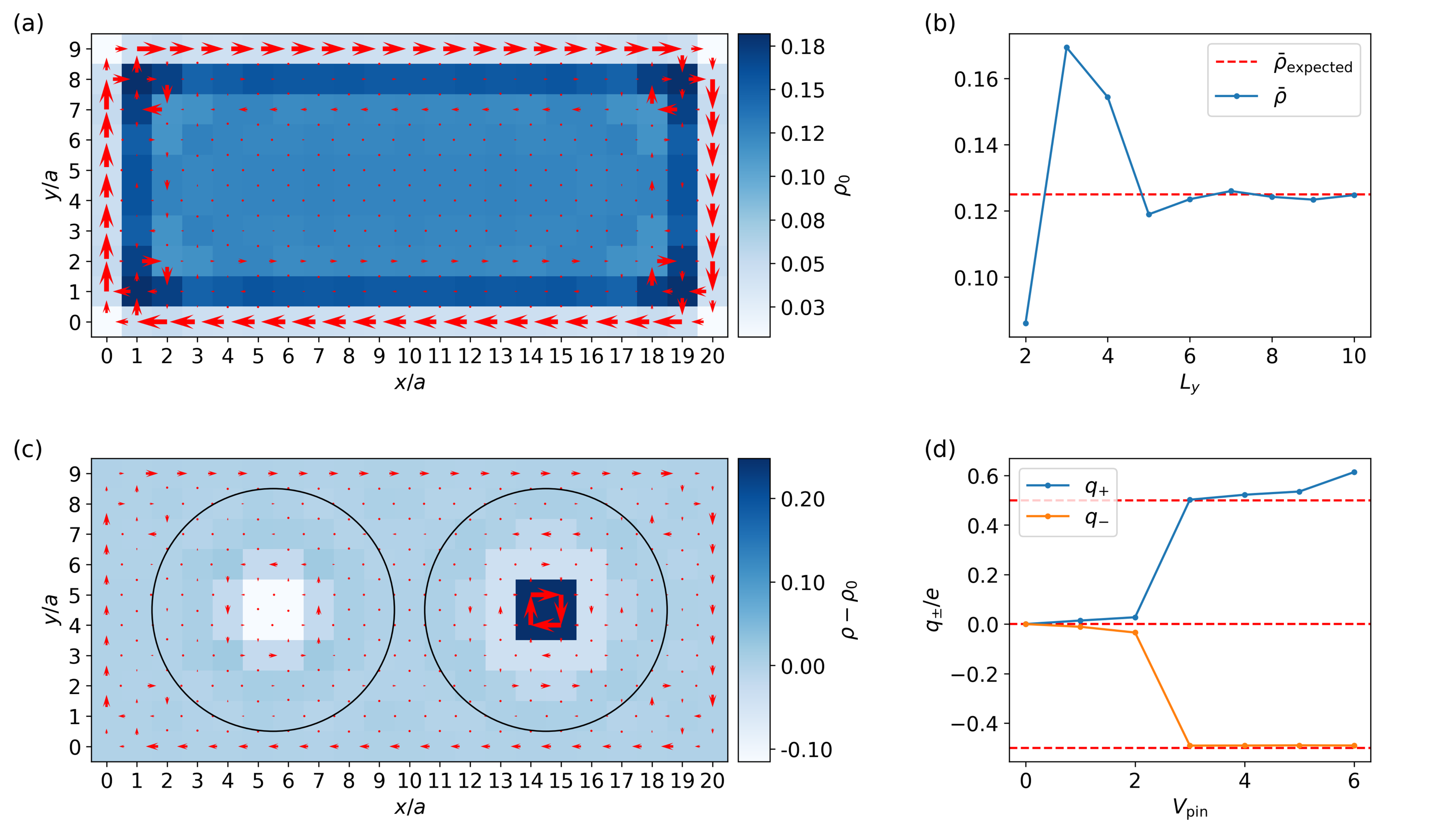

Figure 3: Ground state of the $\nu = 1/2$ bosonic Laughlin state for a 21×10 system with open boundary conditions and $n_\phi = 1/4$. (a) Density ρ_0 plotted at each position in space with the current overlaid in red. (b) Average density of the bulk $\bar{\rho}$ as a function of linear system size L_y, for a system with $\bar{\rho}_{\text{expected}} = 1/8$. This motivates the choice of $L_y \geq 10$. (c) Density difference $\rho - \rho_0$ when two positively and negatively charged anyons are trapped using point potentials $V_{\text{pin}} = \pm 5$. (d) Charge difference integrated in the black circles of (c). This shows that the anyons on the right and left have charge $q = \pm e/2$ respectively.

$\nu = 2$ bIQHE state

Intriguingly, the Jain series also predicts chiral topological states at integer fillings for bosons. The existence of these states is notoriously difficult to verify and large-scale ED and DMRG studies have so far remained not entirely conclusive [8, 14]. Using VMC+PEPS however, we can converge to a more accurate ground state and clearly uncover the chiral edge modes.

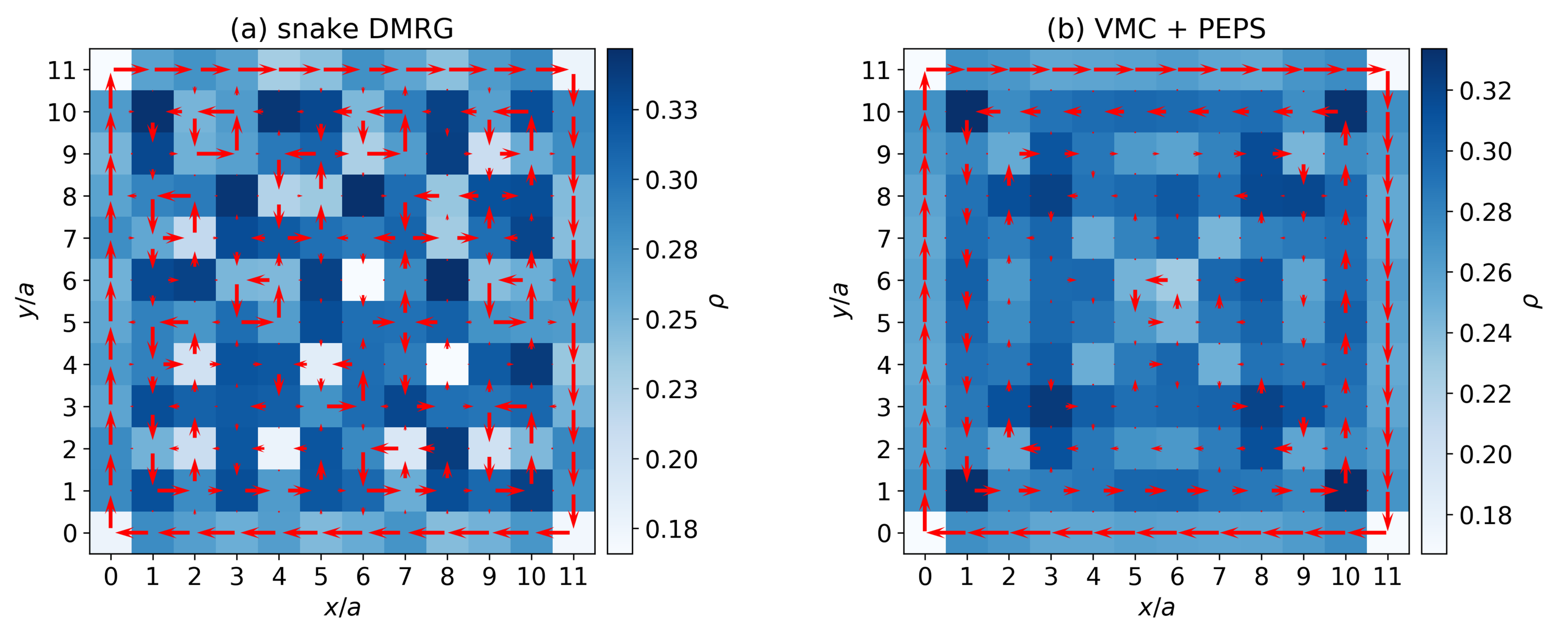

Figure 4: Two attempts to find the $\nu = 2$ bIQHE ground state for a 12×12 system with $n_\phi = 1/6$, using (a) snake DMRG and (b) VMC+PEPS. The simulation parameters were chosen to have a comparable run-time of approximately 72 hours on a cluster. (a) For snake DMRG, this equates to a bond dimension of $\chi = 1500$ with Schmidt cutoff $\lambda_{\text{min}} = 10^{-10}$. (b) For VMC+PEPS, this equates to a bulk(boundary) bond dimension of $D = 7(d = 21)$ with $\eta = 0.2$ and $\epsilon = 10^{-2}$.

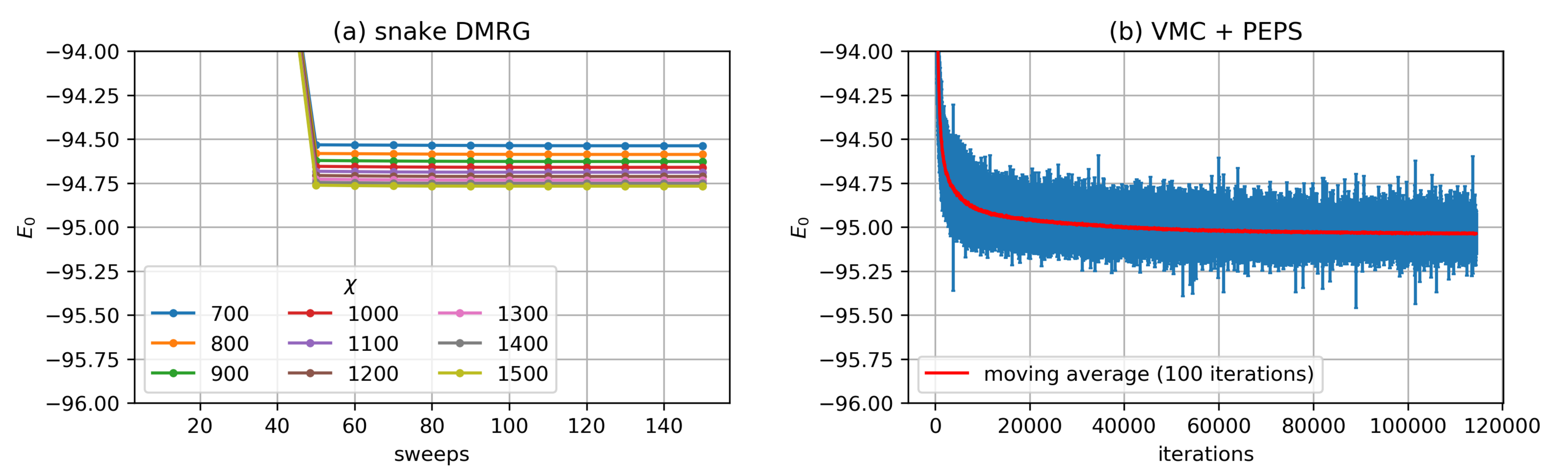

Figure 5: Convergence to the ground-state energy for the above $\nu = 2$ bIQHE state as a function of (a) DMRG sweeps and (b) VMC iterations.

Conclusion

- VMC approaches built on a PEPS ansatz have the potential to be the leading numerical method for simulating large FQHE systems.
- MPS methods, such as snake DMRG, are still the optimal choice for systems that are either small, quasi-1D, or have low entanglement.
- VMC+PEPS is more efficient for challenging systems that have high entanglement, such as the controversial $\nu = 2$ bIQHE state.

References

[1] M. Iqbal, N. Tantivasadakarn, R. Verresen *et al.*, Nature **626**, 505–511 (2024).
[2] N. L. Samuelson, L. A. Cohen, W. Wang *et al.*, arXiv:2403.19628 (2024).
[3] Z. Lu, T. Han, Y. Yao *et al.*, Nature **626**, 759–764 (2024).
[4] E. Weerda and M. Rizzi, Phys. Rev. B **109**, L241117 (2024).
[5] C.-Y. Park and M. J. Kastoryano, Phys. Rev. Res. **2**, 023232 (2020).
[6] B. Andrews, J. Open Source Softw. **9** (95), 6356 (2024).
[7] G. Möller and N. R. Cooper, Phys. Rev. Lett. **115**, 126401 (2015).
[8] B. Andrews and G. Möller, Phys. Rev. B **97**, 035159 (2018).
[9] T. Vieijra, J. Haegeman, F. Verstraete *et al.*, Phys. Rev. B **104**, 235141 (2021).
[10] S. Sorella, Phys. Rev. Lett. **80**, 4558 (1998).
[11] S. Sorella, Phys. Rev. B **64**, 024512 (2001).
[12] B. Wang, X.-Y. Dong, and A. Eckardt, SciPost Phys. **12**, 095 (2022).
[13] J. Hauschild, J. Unfried, S. Anand *et al.*, arXiv:2408.02010 (2024).
[14] Y.-C. He, F. Grusdt, A. Kaufman *et al.*, Phys. Rev. B **96**, 201103(R) (2017).

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