Frame-dragging and bending of Light in Kerr and Kerr-(anti) de Sitter spacetimes.

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Abstract

The equations of general relativity in the form of timelike and null geodesics that describe motion of test particles and photons in Kerr spacetime are solved exactly including the contribution from the cosmological constant. We then perform a systematic application of the exact solutions obtained to the following cases. The exact solutions derived for null, spherical, polar and non-polar orbits are applied for the calculation of frame dragging (Lense-Thirring effect) for the orbit of a photon around the galactic centre, assuming that the latter is a Kerr black hole for various values of the Kerr parameter including those supported by recent observations. Unbound null polar orbits are investigated, and an analytical expression for the deviation angle of a polar photon orbit from the gravitational Kerr field is derived. In addition, we present the exact solution for timelike and null equatorial orbits. In the former case, we derive an analytical expression for the precession of the point of closest approach (perihelion, periastron) for the orbit of a test particle around a rotating mass whose surrounding curved spacetime geometry is described by the Kerr field. In the latter case, we calculate an exact expression for the deflection angle for a light ray in the gravitational field of a rotating mass (the Kerr field). We apply this calculation for the bending of light from the gravitational field of the galactic centre, for various values of the Kerr parameter, and the impact factor.

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1 Introduction

1.1 Motivation

Most of the celestial bodies deviate very little from spherical symmetry, and the Schwarzschild spacetime is an appropriate approximation for their gravitational field [2]. However, for some astrophysical bodies the rotation of the mass distribution cannot be neglected. A more general spacetime solution of the gravitational field equations should take this property into account. In this respect, the Kerr solution [3] represents, the curved spacetime geometry surrounding a rotating mass [4]. Moreover, the above solution is also important for probing the strong field regime of general relativity [5]. This is significant, since general relativity has triumphed in large-scale cosmology [8, 7, 9, 10], and in predicting solar system effects on planetary orbits like the perihelion precession of Mercury with a very high precision [1, 6] ¹.

As was discussed in [11], the investigation of spacetime structures near strong gravitational sources, like neutron stars or candidate black hole (BH) systems is of paramount importance for testing the predictions of the theory in the strong field regime. The study of geodesics are crucial in this respect, in providing information of the structure of spacetime in the strong field limit.

The study of the geodesics from the Kerr metric are additionally motivated by recent observational evidence of stellar orbits around the galactic centre, which indicates that the spacetime surrounding the Sgr A* radio source, which is believed to be a supermassive black hole of 3.6 million solar masses, is described by the Kerr solution rather than the Schwarzschild solution, with the Kerr parameter [12]

$$\frac{J}{GM_{\rm BH}/c} = 0.52 \; (\pm 0.1, \pm 0.08, \pm 0.08) \tag{1}$$

where the reported high-resolution infrared observations of Sgr A* revealed 'quiescent' emission and several flares. This is half the maximum value for a Kerr black hole [13]. In the above equation J^2 denotes the angular momentum of the black hole (The error estimates here the uncertainties in the period, black hole mass ($M_{\rm BH}$) and distance to the galactic centre, respectively; G is the gravitational constant and c the velocity of light.)

Taking into account the cosmological constant Λ contribution, the generalization of the Kerr solution is described by the Kerr -de Sitter metric element which in Boyer-Lindquist (BL) coordinates ³ is given by [18, 19]:

 $^{^1{\}rm See}$ also about the Bepi Colombo science mission on Mercury http://sci.esa.int/home/bepi colombo/ $^2J=ca$ where a is the Kerr parameter. The interpretation of ca as the angular momentum

 $^{^2}J=ca$ where a is the Kerr parameter. The interpretation of ca as the angular momentum per unit mass was first given by Boyer and Price [15]. In fact, by comparing with the Lense-Thirring calculations [16] they determined the Kerr parameter to be: $a=-\frac{2\Omega l^2}{5c}$, where Ω and l denote the angular velocity and radius of the rotating sphere.

³These coordinates have the advantage that reduce to the Schwarzschild solution with a cosmological constant in the limit $a \to 0$, see [14].

$$ds^{2} = \frac{\Delta_{r}}{\Xi^{2}\rho^{2}} \left(cdt - a\sin^{2}\theta d\phi\right)^{2} - \frac{\rho^{2}}{\Delta_{r}}dr^{2} - \frac{\rho^{2}}{\Delta_{\theta}}d\theta^{2}$$
$$- \frac{\Delta_{\theta}\sin^{2}\theta}{\Xi^{2}\rho^{2}} \left(acdt - (r^{2} + a^{2})d\phi\right)^{2}$$
(2)

where

$$\Delta_{r} := (1 - \frac{\Lambda}{3}r^{2})(r^{2} + a^{2}) - \frac{2GMr}{c^{2}}$$

$$\Delta_{\theta} := 1 + \frac{a^{2}\Lambda}{3}\cos^{2}\theta$$

$$\Xi := 1 + \frac{a^{2}\Lambda}{3}, \quad \rho^{2} := r^{2} + a^{2}\cos^{2}\theta$$
(3)

In a recent paper [11], we derived the timelike geodesic equations in Kerr spacetime with a cosmological constant by solving the Hamilton-Jacobi partial differential equation by separation of variables. Subsequently, we solved exactly the corresponding differential equations for an interesting class of possible types of motion for a test particle in Kerr and Kerr-(anti) de Sitter spacetimes. The exact solution of non-spherical geodesics was obtained by using the transformation theory of elliptic functions.

The exact solutions of the timelike geodesic equations obtained in [11] were applied to the following situations:

Frame dragging from rotating gravitational mass. An essential property of the geodesics in Schwarzschild spacetime is that although the orbit precesses relativistically it remains in the same plane; the Kerr rotation adds longitudinal dragging to this precession. For instance, in the spherical polar orbits we discussed in [11], (where the particle traverses all latitudes, passes through the symmetry axis z, infinitely many times) the angle of longitude increases after a complete oscillation in latitude. This phenomenon, is in accordance with Mach's principle.

More specifically, in [11] we calculated the dragging of inertial frames in the following situations. (a) Dragging of a satellite's spherical polar orbit in the gravitational field of Earth assuming Kerr geometry, using as radii, the semi-major axis of the polar orbit of the GP-B mission ⁴ launched in April 2004. (b) Dragging of a stellar, spherical polar orbit, in the gravitational field of a rotating galactic black hole.

It is the purpose of this paper to extend the analysis and applications of the exact solutions obtained in [11] to an interesting class of possible types of motion for a test particle in Kerr and Kerr-(anti) de Sitter spacetimes, as well as to derive the exact solutions of null geodesics in the same spacetimes and explore their physical implications. In the latter case, we apply the exact solutions obtained, to the following situations:

 $^{^4\}mathrm{http://einstein.stanford.}$ See also [17].

- (a) Dragging of a photon's spherical polar and non-polar orbit in the gravitational field of a rotating galactic centre black hole.
- (b) The deflection angle of a light ray from the gravitational field of a rotating black hole, for various values of the Kerr parameter and the impact factor.

The material of this paper is organized as follows. In section 2 we review the derivation of the relevant geodesic equations. In section 2.1 we discuss the definition of the Lauricella's hypergeometric function of many variables, as well as the integral representations that it admits, which are important in our exact treatment of geodesic equations that describe motion of a test particle and photon in Kerr-(anti) de Sitter spacetime. In sections 3.2,3.1 and 6, we solve exactly spherical polar or non-polar null geodesics with and without the cosmological constant. In the case of spherical polar photonic geodesics and for a vanishing cosmological constant the exact solution for the orbit is given by the Weierstraß elliptic function. The exact expression for frame dragging is proportional to the real half-period of the Weierstraß modular Jacobi form. In the case of non-polar spherical photonic orbits, the exact expression for the Lense-Thirring precession of the photon is given in terms of a hypergeometric function of one variable and Appell's first generalized hypergeometric function of two variables F_1 [20]. Assuming a vanishing cosmological constant and that the galactic centre is a supermassive rotating black hole, we apply the exact solutions obtained for the determination of Lense-Thirring precession for a photon in spherical polar and non-polar orbits around the galactic centre. The corresponding exact expressions in the presence of the cosmological constant are also derived and discussed in sections 3.2 and 7.

In section 4 we perform a precise calculation for the deflection angle of a photon's $non - spherical\ polar$ orbit from the gravitational Kerr field. In this novel case, the exact expressions obtained were written in terms of Lauricella's hypergeometric function F_D .

Timelike spherical polar (with $\Lambda \neq 0$) and non-polar (with $\Lambda = 0$) orbits are treated in sections 5, 8 respectively.

In sections 8.4 and 8.5 we study the exact solution of non-spherical timelike and null equatorial orbits respectively. The amount of relativistic precession for a test particle in a timelike orbit, confined to the equatorial plane, in the presence of rotation of the central mass is given in terms of Appell's first hypergeometric function of two variables F_1 . On the other hand, the exact expression for the deflection angle of a photonic orbit from the Kerr gravitational field surrounding a rotating central mass is given in terms of Appell's F_1 hypergeometric function and Lauricella's fourth hypergeometric function F_D of three variables [21]. We use section 9 for our conclusions. In the appendices, we collect some of our formal calculations, as well as some useful properties of Appell's hypergeometric function and definitions of genus-2 theta functions.

2 Separability of Hamilton-Jacobi's differential equation in Kerr-(anti) de Sitter metric and derivation of geodesics.

In the presence of the cosmological constant it was proved in [11] the important result that the Hamilton-Jacobi differential equation can be solved by separation of variables. Thus in this case, the characteristic function separates and takes the form [11]

$$W = -Ect + L\phi + \int \frac{\sqrt{[Q + (L - aE)^2 \Xi^2 - \mu^2 a^2 \cos^2 \theta] \Delta_\theta - \frac{\Xi^2 (aE \sin^2 \theta - L)^2}{\sin^2 \theta}}}{\Delta_\theta} d\theta + \int \frac{\sqrt{\Xi^2 [(r^2 + a^2)E - aL]^2 - \Delta_r (\mu^2 r^2 + Q + \Xi^2 (L - aE)^2)}}{\Delta_r} dr$$

By differentiating now with respect to constants of integration, Q, L, E, μ , we obtain the following set of geodesic differential equations

$$\int \frac{dr}{\sqrt{R'}} = \int \frac{d\theta}{\sqrt{\Theta'}}$$

$$\rho^2 \frac{d\phi}{d\lambda} = -\frac{\Xi^2}{\Delta_\theta \sin^2 \theta} \left(aE \sin^2 \theta - L \right) + \frac{a\Xi^2}{\Delta_r} \left[(r^2 + a^2)E - aL \right]$$

$$c\rho^2 \frac{dt}{d\lambda} = \frac{\Xi^2 (r^2 + a^2) \left[(r^2 + a^2)E - aL \right]}{\Delta_r} - \frac{a\Xi^2 (aE \sin^2 \theta - L)}{\Delta_\theta}$$

$$\rho^2 \frac{dr}{d\lambda} = \pm \sqrt{R'}$$

$$\rho^2 \frac{d\theta}{d\lambda} = \pm \sqrt{\Theta'}$$
(4)

where

$$R' := \Xi^{2} \left[(r^{2} + a^{2})E - aL \right]^{2} - \Delta_{r} \left(\mu^{2}r^{2} + Q + \Xi^{2}(L - aE)^{2} \right)$$

$$\Theta' := \left[Q + (L - aE)^{2}\Xi^{2} - \mu^{2}a^{2}\cos^{2}\theta \right] \Delta_{\theta} - \Xi^{2} \frac{(aE\sin^{2}\theta - L)^{2}}{\sin^{2}\theta}$$
 (5)

The first line of Eq.(4) is a differential equation that relates a *hyperelliptic* Abelian integral to an elliptic integral which is the generalisation of the theory of transformation of elliptic functions discussed in [11], in the case of non-zero cosmological constant. The mathematical treatment of such a relationship was first discussed by Abel in [34].

Assuming a zero cosmological constant, as was shown by Carter, one gets

$$W = -Ect + \int \frac{\sqrt{R}}{\Delta} dr + \int \sqrt{\Theta} d\theta + L\phi \tag{6}$$

where

$$\Theta := Q - \left[a^2 (\mu^2 - E^2) + \frac{L^2}{\sin^2 \theta} \right] \cos^2 \theta \tag{7}$$

and

$$R := \left[(r^2 + a^2)E - aL \right]^2 - \Delta \left[\mu^2 r^2 + (L - aE)^2 + Q \right]$$
 (8)

with $\Delta := r^2 + a^2 - \frac{2GMr}{c^2}$. Also E, L are constants of integration associated with the isometries of the Kerr metric. Carter's constant of integration is denoted by Q. Differentiation of (6), with respect to the integration constants E, L, Q, μ leads to the following set of first-order equations of motion [37]:

$$\rho^{2} \frac{cdt}{d\lambda} = \frac{r^{2} + a^{2}}{\Delta} P - a \left(a E \sin^{2} \theta - L \right)$$

$$\rho^{2} \frac{dr}{d\lambda} = \pm \sqrt{R}$$

$$\rho^{2} \frac{d\theta}{d\lambda} = \pm \sqrt{\Theta}$$

$$\rho^{2} \frac{d\phi}{d\lambda} = \frac{a}{\Delta} P - aE + \frac{L}{\sin^{2} \theta}$$
(9)

where

$$P := E(r^2 + a^2) - aL \tag{10}$$

Null-geodesics are derived by setting $\mu = 0$.

2.1 Lauricella's multivariable hypergeometric functions

Giuseppe Lauricella, building on the work of Appell who had developed hypergeometric functions of two variables, investigated in a systematic way, multiple hypergeometric functions at the end of the nineteenth century [21]. He defined, four functions which are named after him and have both multiple series and integral representations. In particular, the fourth of these functions, denoted by F_D , admits integral representations of importance in our exact treatment of geodesic equations, which describe motion of a test particle in Kerr-(anti) de Sitter spacetime.

The fourth Lauricella function of m-variables is given by

$$F_{D}(\alpha, \boldsymbol{\beta}, \gamma; \mathbf{z}) = \sum_{n_{1}, n_{2}, \dots, n_{m} = 0}^{\infty} \frac{(\alpha)_{n_{1} + \dots + n_{m}} (\beta_{1})_{n_{1}} \dots (\beta_{m})_{n_{m}}}{(\gamma)_{n_{1} + \dots + n_{m}} (1)_{n_{1}} \dots (1)_{n_{m}}} z_{1}^{n_{1}} \dots z_{m}^{n_{m}}$$

$$= \sum_{n_{1}, n_{2}, \dots, n_{m} = 0}^{\infty} \frac{\sum (\alpha, n_{1} + \dots + n_{m})(\beta_{1}, n_{1}) \dots (\beta_{m}, n_{m}) z_{1}^{n_{1}} \dots z_{m}^{n_{m}}}{(\gamma, n_{1} + \dots + n_{m})n_{1}! \dots n_{m}!}$$

where

$$\mathbf{z} = (z_1, \dots, z_m)$$

$$\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$$
(11)

The Pochhammer symbol $(\alpha)_m = (\alpha, m)$ is defined by

$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } m=0, \\ \alpha(\alpha+1)\cdots(\alpha+m-1), & \text{if } m=1,2,3,\cdots \end{cases}$$

The series admits the following integral representation

$$F_D(\alpha, \boldsymbol{\beta}, \gamma; \mathbf{z}) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha - 1} (1 - t)^{\gamma - \alpha - 1} (1 - z_1 t)^{-\beta_1} \cdots (1 - z_m t)^{-\beta_m} dt$$
(12)

which is valid for $\text{Re}(\alpha) > 0$, $\text{Re}(\gamma - \alpha) > 0$. It converges absolutely inside the m-dimensional cuboid:

$$|z_j| < 1, \qquad (j = 1, \cdots, m) \tag{13}$$

3 Spherical polar null geodesics

Depending on whether or not the coordinate radius r is constant along a given geodesic, the corresponding particle orbit is characterized as spherical or non-spherical respectively. In this section, we will concentrate on spherical polar photon orbits with a vanishing cosmological constant. We should mention at this point the extreme black hole solutions a=1 of spherical non-polar photon geodesics obtained in [22] in terms of formal integrals.

The exact solution of the corresponding timelike orbits and their physical applications have been derived and investigated in [11].

Assuming a zero cosmological constant, $r = r_f$, where r_f is a constant value setting $\mu = 0$ and using the last two equations of (9) we obtain:

$$\frac{d\phi}{d\theta} = \frac{\frac{aP}{\Delta} - aE + L/\sin^2\theta}{\sqrt{\Theta}} \tag{14}$$

where Θ now is given by

$$\Theta = Q - \left[-a^2 E^2 + \frac{L^2}{\sin^2 \theta} \right] \cos^2 \theta \tag{15}$$

It is convenient to introduce the parameters

$$\Phi := L/E, \quad \mathcal{Q} := Q/E^2 \tag{16}$$

Now by defining $z := \cos^2 \theta$, the previous equation can be written as follows,

$$d\phi = -\frac{1}{2} \frac{dz}{\sqrt{z^3 \alpha - z^2 (\alpha + \beta) + Qz}} \times \left\{ \frac{aP}{\Delta} - a + \frac{\Phi}{1 - z} \right\}$$
 (17)

where

$$\alpha := -a^2, \quad \beta := \mathcal{Q} + \Phi^2 \tag{18}$$

It has been shown [23] that a necessary condition for an orbit to be *polar* (meaning to intersect the symmetry axis of the Kerr gravitational field) is the vanishing of the parameter L, i.e. L=0. Assuming $\Phi=0$, in equation (17), we can transform it into the Weierstraß form of an elliptic curve by the following substitution

$$z := -\frac{\xi + \frac{\alpha + \beta}{12}}{-\alpha/4} \tag{19}$$

Thus, we obtain the integral equation

$$\int d\phi = \int -\frac{1}{2} \frac{d\xi}{\sqrt{4\xi^3 - q_2\xi - q_3}} \times \left\{ \frac{aP'}{\Delta} - a \right\}$$
 (20)

and this orbit integral can be inverted by the Weierstraß modular Jacobi form 5

$$\xi = \wp \left(\phi / A \right) \tag{21}$$

where $A:=-\frac{1}{2}\left(\frac{aP'}{\Delta}-a\right), P'=(r^2+a^2)$ and the Weierstraß invariants take the form

$$g_{2} = \frac{1}{12}(\alpha + \beta)^{2} - \frac{Q\alpha}{4}$$

$$g_{3} = \frac{1}{216}(\alpha + \beta)^{3} - \frac{Q\alpha^{2}}{48} - \frac{Q\alpha\beta}{48}$$
(22)

3.1 Exact solution for spherical polar null geodesics with a vanishing cosmological constant

In terms of the original variables, the exact solution for the polar orbit of the photon ($\Phi = 0$) takes the form

$$\wp(\phi + \epsilon) = \frac{\alpha''}{4}\cos^2\theta - \frac{1}{12}(\alpha'' + \beta'') \tag{23}$$

where $\alpha'':=\alpha'/A'^2=\frac{\alpha}{a^2A'^2}=-\frac{1}{A'^2},\ \beta'':=\beta'/A'^2=\frac{\mathcal{Q}}{a^2A'^2},\ \mathcal{Q}''=\mathcal{Q}'/A'^2=\frac{\mathcal{Q}}{a^2A'^2}.$ Also A' is given by the expression

$$A' := \frac{-\frac{GMr}{c^2a^2}}{\frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2a^2}} \tag{24}$$

Equation (23) represents the first exact solution of a spherical polar photonic orbit assuming a zero cosmological constant, in closed analytic form, in terms of the Weierstraß Jacobi modular form of weight 2.

⁵For more information on the properties of the Weierstraß function, the reader is referred to the monographs [24, 25], and the appendix of [10].

The Weierstraß invariants are given by

$$\begin{split} g_2'' &= \frac{(\alpha'' + \beta'')^2}{12} - \frac{\mathcal{Q}''\alpha''}{4} \\ &= \frac{1}{12} \frac{(-a^2 + \mathcal{Q})^2}{a^4 A'^4} + \frac{\mathcal{Q}}{4a^2 A'^4} \\ g_3'' &= \frac{\left(\alpha'' + \beta''\right)^3}{216} - \frac{\mathcal{Q}''\alpha''^2}{48} - \frac{\mathcal{Q}''\alpha''\beta''}{48} \\ &= \frac{1}{432a^6 A'^6} \left[-2a^6 - 3a^4 \mathcal{Q} + 3a^2 \mathcal{Q}^2 + 2\mathcal{Q}^3 \right] \end{split}$$

The sign of the discriminant Δ^c ($\Delta^c = g_2^3 - 27g_3^2$) determines the roots of the elliptic curve: $\Delta^c > 0$, corresponds to three real roots while for $\Delta^c < 0$ two roots are complex conjugates and the third is real. In the degenerate case $\Delta^c = 0$, (where at least two roots coincide) the elliptic curve becomes singular and the solution is not given by modular functions. The analytic expressions for the three roots of the cubic, which can be obtained by applying the algorithm of Tartaglia and Cardano [26], are given by

$$e_{1} = \frac{(a^{2} + 2Q) \Delta^{2}}{12a^{2}r^{2}(GM/c^{2})^{2}}$$

$$e_{2} = \frac{(a^{2} - Q) \Delta^{2}}{12a^{2}r^{2}(GM/c^{2})^{2}}$$

$$e_{3} = -\frac{(2a^{2} + Q) \Delta^{2}}{12a^{2}r^{2}(GM/c^{2})^{2}}$$
(25)

Since we are assuming spherical orbits, there are two conditions from the vanishing of the polynomial R(r) and its first derivative ⁶. Implementing these two conditions, expressions for the parameter Φ and Carter's constant \mathcal{Q} are obtained

$$\Phi = \frac{a^2 + r^2}{a}, \mathcal{Q} = -\frac{r^4}{a^2}$$

$$\Phi = \frac{a^2 \frac{GM}{c^2} + a^2 r - 3 \frac{GM}{c^2} r^2 + r^3}{a(\frac{GM}{c^2} - r)}, \mathcal{Q} = -\frac{r^3 (-4a^2 \frac{GM}{c^2} + r(\frac{-3GM}{c^2} + r)^2)}{a^2 (\frac{GM}{c^2} - r)^2}$$
(26)

However, only the second solution is physical [22].

⁶These orbits are unstable since $\frac{d^2R}{dr^2}\Big|_{r=r_f} > 0$. However, they represent interesting new possible types of motion in the Kerr spacetime. They represent a non-trivial generalisation of the unstable circular closed orbit (photonsphere) in the Schwarzschild black hole.

parameters	half-period	predicted dragging
$a_{\rm Galactic} = 0.52$	$\omega = 0.331117$	$\Delta \phi^{\rm GTR} = 1.32447$
$\Phi = 0, r = 2.87313, \mathcal{Q} = 25.8829$		$=75.88^{\circ}$ per revolution $=273192\frac{\text{arcs}}{\text{revolution}}$
$a_{\text{Galactic}} = 0.9939$	$\omega = 0.784737$	CAMID
$\Phi = 0, r = 2.42451, \mathcal{Q} = 22.3842$		= 179.8° per revolution = $647455 \frac{\text{arcs}}{\text{revolution}}$

Table 1: Predictions for frame dragging from galactic black hole for a photonic spherical polar orbit, for Kerr parameter $a_{\rm Galactic} = 0.52 \frac{GM_{\rm BH}}{c^2}$, $a_{\rm Galactic} = 0.9939 \frac{GM_{\rm BH}}{c^2}$, respectively. The values of the radii are in units of $GM_{\rm BH}/c^2$, while those of Carter's constant $\mathcal Q$ in units of $(GM_{\rm BH}/c^2)^2$. The period ratios, τ , are 2.33615i, 1.88276i respectively.

The two half-periods ω and ω' are given by the following Abelian integrals (for $\Delta^c > 0$) [27]:

$$\omega = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \quad \omega' = i \int_{-\infty}^{e_3} \frac{dt}{\sqrt{-4t^3 + g_2 t + g_3}}$$
 (27)

The values of the Weierstraß function at the half-periods are the three roots of the cubic. For positive discriminant Δ^c one half-period is real while the second is imaginary ⁷. The period ratio is defined as $\tau = \frac{\omega'}{\omega}$.

An alternative expression for the real half-period ω of the Weierstraß function is: $\omega = \frac{1}{\sqrt{e_1 - e_3}} \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}, 1, \frac{e_2 - e_3}{e_1 - e_3})$, where $F(\alpha, \beta, \gamma, x)$ is the hypergeometric function $1 + \frac{\alpha.\beta}{1.\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \cdots$.

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$$\omega = \frac{2}{\sqrt{\frac{(a^2 + \mathcal{Q})(a^2 + (-2+r)r)^2}{a^2r^2}}} \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2}{a^2 + \mathcal{Q}})$$
(28)

After a complete oscillation in latitude, the angle of longitude, which determines the amount of dragging for the spherical photon polar orbit in the general theory of relativity (GTR), increases by

$$\Delta \phi^{\rm GTR} = 4\omega \tag{29}$$

We can also integrate the first and the third equation in (9). Then we get

$$\int c \, dt = -4 \left[\int_0^1 -\frac{1}{2} \frac{a^2 \left[\frac{r^2}{a^2} + 1 \right]^2}{\frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2 a^2}} \frac{dz}{\sqrt{\mathcal{Q}} \sqrt{z} \sqrt{(1-z)} \sqrt{1 - \left(\frac{-a^2}{\mathcal{Q}} \right) z}} \right] + \int_0^1 \frac{a^2 (1-z) dz}{2\sqrt{\mathcal{Q}} \sqrt{z} \sqrt{(1-z)} \sqrt{1 - \left(\frac{-a^2}{\mathcal{Q}} \right) z}} \right]$$

⁷We organize the roots as: $e_1 > e_2 > e_3$.

⁸Yet another equivalent representation for ω is $\omega = \frac{a\pi}{\sqrt{\mathcal{Q}}} \frac{\frac{GMr}{c^2a^2}}{\frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2a^2}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{-a^2}{\mathcal{Q}}\right)$.

Thus we obtain the following exact expression for t

$$ct = -4 \left[-\frac{1}{2} \frac{a^2}{\sqrt{\mathcal{Q}}} \frac{\left[\frac{r^2}{a^2} + 1\right]^2}{\frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2 a^2}} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{-a^2}{\mathcal{Q}}\right) \right]$$

$$+ \frac{a^2}{2\sqrt{\mathcal{Q}}} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} F\left(\frac{1}{2}, \frac{1}{2}, 2, \frac{-a^2}{\mathcal{Q}}\right) \right]$$

$$(30)$$

Similarly, if we integrate the differential equations for t and ϕ we obtain

$$\frac{cdt}{d\phi} = \frac{\frac{r^2 + a^2}{\Delta} P' - a^2 \sin^2 \theta}{\frac{a}{\Delta} P' - a}$$

$$= a + \frac{\frac{r^2 P'}{\Delta}}{\frac{aP'}{\Delta} - a} + \frac{a^2 \cos^2 \theta}{\frac{aP'}{\Delta} - a} \tag{31}$$

or

$$ct + \mathcal{E} = a\phi + \frac{r^2 P'/\Delta}{(-2aA')}\phi - \frac{4a^2/\alpha''}{(-2aA')}\left(\zeta(\phi) - \frac{1}{12}(\alpha'' + \beta'')\phi\right)$$
 (32)

where we used Eq.(23) and the fact that, $\int \wp(\phi)d\phi = -\zeta(\phi)$, where $\zeta(z)$ denotes the Weierstraß zeta function. Also \mathcal{E} denotes a constant of integration.

Assuming, that the centre of the Milky Way is a black hole and that the structure of spacetime near the region Sgr A*, is described by the Kerr geometry as is indicated by observations Eq.(1), we determined the precise frame dragging (Lense-Thirring effect) of a null orbit with a spherical polar geometry. We repeated the analysis for a value of the Kerr parameter as high as $a_{\rm Galactic} = 0.9939$. Such high values for the angular momentum of the black hole, have been recently reported from x-ray flare analysis of the galactic centre [28]. The results are displayed in table 1. Let us also mention that in [29] it has been argued that an upper bound of a is given by a = 0.99616.

3.2 Null spherical polar geodesics with the cosmological constant, Lense-Thirring effect and Appell hypergeometric functions

We now derive an exact expression for the amount of dragging for a photonic spherical polar orbit in the presence of the cosmological constant, thus generalizing the results of the previous section. After a complete oscillation in latitude, the angle of longitude $\Delta \phi$, which determines the amount of dragging for the spherical polar orbit, is given by

$$\Delta \phi^{\text{GTR}} = -4 \left[\alpha_1 \int_0^1 \frac{dz}{\sqrt{f(z)}} + \beta_1 \int_0^1 \frac{zdz}{\sqrt{f(z)}} \right]
= -4 \left[\alpha_1 \frac{1}{\sqrt{Q}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} \times F_1(\frac{1}{2}, \frac{1}{2}, 1, 1, -\frac{Qa^2 \frac{\Lambda}{3} + \Xi^3 a^2}{Q}, \frac{-a^2 \Lambda}{3}) \right]
+ \beta_1 \frac{1}{\sqrt{Q}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} \times F_1(\frac{3}{2}, \frac{1}{2}, 1, 2, -\frac{Qa^2 \frac{\Lambda}{3} + \Xi^3 a^2}{Q}, \frac{-a^2 \Lambda}{3}) \right]$$
(33)

where $f(z) = z(1-z)(Q+z(Qa^2\frac{\Lambda}{3}+\Xi^3a^2))(1+\frac{a^2\Lambda}{3}z)^2$ and $\alpha_1 = \frac{\Xi^2a}{2}-\frac{1}{2}\frac{a\Xi^2(r^2+a^2)}{\Delta_r}$, $\beta_1 = -\frac{1}{2}\frac{a\Xi^2(r^2+a^2)}{\Delta_r}\frac{a^2\Lambda}{3}$. The function $F_1(\alpha,\beta,\beta',\gamma,x,y)$ is the first of the four Appell's hypergeometric functions of two variables x,y [20] 9 ,

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n$$
(34)

which admits the following integral representation

$$\int_0^1 u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} (1-uy)^{-\beta'} du = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_1(\alpha,\beta,\beta',\gamma,x,y)$$
(35)

The double series converges when |x|<1 and |y|<1. The above Euler integral representation is valid for $\operatorname{Re}(\alpha)>0$ and $\operatorname{Re}(\gamma-\alpha)>0$. Also $\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx$ denotes the gamma function.

For a zero cosmological constant $(\Lambda = 0, \beta_1 = 0)$ and we obtain the correct limit.

$$\Delta \phi^{\text{GTR}} = -4 \frac{\alpha_1 \pi}{\sqrt{\mathcal{Q}}} F(\frac{1}{2}, \frac{1}{2}, 1, \frac{-a^2}{\mathcal{Q}})$$
(36)

where $\alpha_1 = -\frac{aGMr}{c^2\Delta}$, in the limit of a vanishing cosmological constant. We can also obtain an exact expression for time. After a quarter of an

We can also obtain an exact expression for time. After a quarter of an oscillation in latitude the time elapses as

$$c t = \int_{0}^{1} \frac{(\gamma_{1} + \delta_{1}z)dz}{\sqrt{f(z)}}$$

$$= \frac{\gamma_{1}}{\sqrt{Q}} \pi F_{1} \left(\frac{1}{2}, \frac{1}{2}, 1, 1, -\frac{Qa^{2}\frac{\Lambda}{3} + \Xi^{3}a^{2}}{Q}, \frac{-a^{2}\Lambda}{3}\right)$$

$$+ \frac{\delta_{1}}{\sqrt{Q}} \frac{\pi}{2} F_{1} \left(\frac{3}{2}, \frac{1}{2}, 1, 2, -\frac{Qa^{2}\frac{\Lambda}{3} + \Xi^{3}a^{2}}{Q}, \frac{-a^{2}\Lambda}{3}\right)$$
(37)

⁹The expression $(\lambda, \kappa) = \lambda(\lambda + 1) \cdots (\lambda + \kappa - 1)$, and the symbol $(\lambda, 0)$ represents 1.

In the limit $\Lambda = 0$, $\gamma_1 = -\frac{1}{2} \frac{(r^2 + a^2)^2}{\Delta} + \frac{a^2}{2}$, $\delta_1 = -\frac{a^2}{2}$, and Eq.(30) is recovered. The conditions from the vanishing of the polynomial R and its first derivative result in equations which generalize (26) and are provided in Appendix C.

4 Non-spherical polar null geodesics

In this case, the relevant differential equation for the calculation of deviation angle $\Delta \phi$ of light from the rotating black hole (or rotating central mass) is the following

$$\frac{d\phi}{dr} = \frac{2aGMr}{c^2\Delta} \frac{1}{\sqrt{R}} \tag{38}$$

where the quartic polynomial R(r) is given by the expression

$$R = r^4 + r^2(a^2 - Q) + \frac{2GMr}{c^2}(a^2 + Q) - a^2Q$$
 (39)

Expressing the roots of Δ as r_+, r_- which are the locations of the event horizons of the black hole, and using partial fractions we derive the expression

$$\frac{d\phi}{dr} = \frac{A_{+}^{P}}{(r - r_{+})\sqrt{R}} + \frac{A_{-}^{P}}{(r - r_{-})\sqrt{R}}$$

$$= \frac{A_{+}^{P}}{(r - r_{+})\sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}}$$

$$+ \frac{A_{-}^{P}}{(r - r_{-})\sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}}$$

where A_{\pm}^{P} are given by the equations

$$A_{\pm}^{P} = \pm \frac{2aGMr_{\pm}}{c^{2}(r_{+} - r_{-})} \tag{40}$$

Also the radii of the event horizons are located at

$$r_{\pm} = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2} \tag{41}$$

In order to calculate the angle of deflection we need to integrate the above equation from the distance of closest approach (e.g. from the maximum positive root of the quartic) to infinity. We denote the roots of the quartic by $\alpha, \beta, \gamma, \delta, \quad \alpha > \beta > \gamma > \delta$. Thus $\Delta \phi = 2 \int_{\alpha}^{\infty}$. We organize all roots in ascending order of magnitude as follows, ¹⁰

$$\alpha_{\mu} > \alpha_{\nu} > \alpha_{i} > \alpha_{\rho} \tag{42}$$

 $^{^{10}}$ We have the correspondence $\alpha_{\mu+1}=\alpha, \alpha_{\mu+2}=\beta, \alpha_{\mu-1}=r_+=\alpha_{\mu-2}, \alpha_{\mu-3}=\gamma, \alpha_{\mu}=\delta.$

where $\alpha_{\mu} = \alpha_{\mu+1}, \alpha_{\nu} = \alpha_{\mu+2}, \alpha_{\rho} = \alpha_{\mu}$ and $\alpha_i = \alpha_{\mu-i}, i = 1, 2, 3$ and we have that $\alpha_{\mu-1} \ge \alpha_{\mu-2} > \alpha_{\mu-3}$. By applying the transformation

$$r = \frac{\omega z \alpha_{\mu+2} - \alpha_{\mu+1}}{\omega z - 1} \tag{43}$$

or equivalently

$$z = \left(\frac{\alpha_{\mu} - \alpha_{\mu+2}}{\alpha_{\mu} - \alpha_{\mu+1}}\right) \left(\frac{r - \alpha_{\mu+1}}{r - \alpha_{\mu+2}}\right) \tag{44}$$

where

$$\omega := \frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}} \tag{45}$$

we can bring our integrals into the familiar integral representation of Lauricella's F_D and Appell's hypergeometric function F_1 of three and two variables respectively. Indeed, we derive

$$\Delta\phi = 2 \left[\int_{0}^{1/\omega} \frac{-A_{+}^{P}\omega(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{+}} \frac{dz}{\sqrt{z(1-z)}(1-\kappa_{+}^{2}z)\sqrt{1-\mu^{2}z}} \right]$$

$$+ \int_{0}^{1/\omega} \frac{A_{+}^{P}\omega^{2}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{+}} \frac{zdz}{\sqrt{z(1-z)}(1-\kappa_{+}^{2}z)\sqrt{1-\mu^{2}z}}$$

$$+ \int_{0}^{1/\omega} \frac{-A_{-}^{P}\omega(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{-}} \frac{dz}{\sqrt{z(1-z)}(1-\kappa_{-}^{2}z)\sqrt{1-\mu^{2}z}}$$

$$+ \int_{0}^{1/\omega} \frac{A_{-}^{P}\omega^{2}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{-}} \frac{zdz}{\sqrt{z(1-z)}(1-\kappa_{-}^{2}z)\sqrt{1-\mu^{2}z}} \right]$$

$$(46)$$

where the moduli κ_{\pm}^2, μ^2 are

$$\kappa_{\pm}^{2} = \left(\frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}}\right) \left(\frac{\alpha_{\mu+2} - \alpha_{\mu-1}^{\pm}}{\alpha_{\mu+1} - \alpha_{\mu-1}^{\pm}}\right)$$

$$\mu^{2} = \left(\frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}}\right) \left(\frac{\alpha_{\mu+2} - \alpha_{\mu-3}}{\alpha_{\mu+1} - \alpha_{\mu-3}}\right)$$
(47)

Also

$$H^{\pm} = \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})(\alpha_{\mu+1} - \alpha_{\mu-1}^{\pm})\sqrt{\alpha_{\mu+1} - \alpha_{\mu}}\sqrt{\alpha_{\mu+1} - \alpha_{\mu-3}}$$
 (48)

and $\alpha_{\mu-1}^{\pm} = r_{\pm}$. By defining a new variable $z' := \omega z$ we can express the angle $\Delta \phi$ in terms of Lauricella's hypergeometric function F_D ,

$$\Delta\phi^{\text{GTR}} = 2 \left[\frac{-2A_{+}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{+}} F_{D} \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_{+}^{\prime 2}, \mu^{\prime 2} \right) \right. \\
+ \frac{A_{+}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{+}} F_{D} \left(\frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{5}{2}, \frac{1}{\omega}, \kappa_{+}^{\prime 2}, \mu^{\prime 2} \right) \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \\
+ \frac{-2A_{-}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{-}} F_{D} \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_{-}^{\prime 2}, \mu^{\prime 2} \right) \\
+ \frac{A_{-}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{-}} F_{D} \left(\frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{5}{2}, \frac{1}{\omega}, \kappa_{-}^{\prime 2}, \mu^{\prime 2} \right) \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right]$$

$$(49)$$

where the variables of the function F_D are given in terms of the roots of the quartic and the radii of the event horizons by the expressions

$$\frac{1}{\omega} = \frac{\alpha_{\mu} - \alpha_{\mu+2}}{\alpha_{\mu} - \alpha_{\mu+1}} = \frac{\delta - \beta}{\delta - \alpha}$$

$$\kappa_{\pm}^{\prime 2} = \frac{\alpha_{\mu+2} - \alpha_{\mu-1}^{\pm}}{\alpha_{\mu+1} - \alpha_{\mu-1}^{\pm}} = \frac{\beta - r_{\pm}}{\alpha - r_{\pm}}$$

$$\mu^{\prime 2} = \frac{\alpha_{\mu+2} - \alpha_{\mu-3}}{\alpha_{\mu+1} - \alpha_{\mu-3}} = \frac{\beta - \gamma}{\alpha - \gamma}$$
(50)

An equivalent expression is as follows

$$\Delta\phi^{\text{GTR}} = 2 \left[\frac{-2A_{+}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{+}} F_{D} \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_{+}^{\prime 2}, \mu^{\prime 2} \right) \right. \\
+ \frac{A_{+}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{+}} \left(-\frac{1}{\kappa_{+}^{\prime 2}} F_{1} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu^{\prime 2} \right) 2 \right. \\
+ \frac{1}{\kappa_{+}^{\prime 2}} F_{D} \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_{+}^{\prime 2}, \mu^{\prime 2} \right) 2 \right) \\
+ \frac{-2A_{-}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{-}} F_{D} \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_{-}^{\prime 2}, \mu^{\prime 2} \right) 2 \\
+ \frac{A_{-}^{P}\sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^{-}} \left(-\frac{1}{\kappa_{-}^{\prime 2}} F_{1} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu^{\prime 2} \right) 2 \right. \\
+ \frac{1}{\kappa_{-}^{\prime 2}} F_{D} \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_{-}^{\prime 2}, \mu^{\prime 2} \right) 2 \right] \right] \tag{51}$$

In going from Eq.(49) to Eq.(51) we used the identity which is proven in Appendix C

$$F_{D}\left(\frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{5}{2}, \frac{1}{\omega}, \kappa_{+}^{\prime 2}, \mu^{\prime 2}\right) \frac{\Gamma(3/2)}{\Gamma(5/2)}$$

$$= -\frac{1}{\kappa_{+}^{\prime 2}} F_{1}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu^{\prime 2}\right) \frac{\Gamma(1/2)}{\Gamma(3/2)}$$

$$+ \frac{1}{\kappa_{+}^{\prime 2}} F_{D}\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa_{+}^{\prime 2}, \mu^{\prime 2}\right) \frac{\Gamma(1/2)}{\Gamma(3/2)}$$
(52)

The phenomenological applications of Eq.(49) for gravitational bending and lensing studies from a galactic black hole, as well as its generalization in the presence of the cosmological constant will be the subject of detailed investigation in a future publication.

Exact solution of timelike spherical polar orbits with 4.1 a cosmological constant

Using the second and the fifth line of Eq.(4), for L=0 and assuming a constant value for r, we obtain

$$\frac{d\phi}{d\theta} = \frac{-\frac{\Xi^2 aE}{\Delta_{\theta}} + \frac{a\Xi^2 (r^2 + a^2)E}{\Delta_r}}{\sqrt{\Theta'}}$$

$$= \frac{-\Xi^2 E a + B\Delta_{\theta}}{\Delta_{\theta}\sqrt{\Theta'}} \tag{53}$$

where $B:=\frac{a\Xi^2(r^2+a^2)E}{\Delta_r}$. Similarly, using the third and fifth line we obtain

$$\frac{cdt}{d\theta} = \frac{\Xi^2 (r^2 + a^2)^2 E}{\Delta_r \sqrt{\Theta'}} - \frac{a^2 \Xi^2 E \sin^2 \theta}{\Delta_\theta \sqrt{\Theta'}}$$

$$= \frac{\Gamma \Delta_\theta - a^2 \Xi^2 E \sin^2 \theta}{\Delta_\theta \sqrt{\Theta'}} \tag{54}$$

and $\Gamma := \frac{\Xi^2 (r^2 + a^2)^2 E}{\Delta_r}$.

Now using the variable $z = \cos^2 \theta$, we obtain the following system of integral equations:

$$\phi = \int \frac{\Xi^2 a E/2}{\sqrt{f(z)}} dz + \int \frac{B(1 + \frac{a^2 \Lambda}{3} z)/(-2)}{\sqrt{f(z)}} dz$$

$$ct = \int \frac{\frac{-\Gamma}{2} (1 + \frac{a^2 \Lambda}{3} z)}{\sqrt{f(z)}} dz - \int \frac{\frac{a^2}{-2} \Xi^2 E(1 - z) dz}{\sqrt{f(z)}}$$
(55)

or

$$\phi = \int^{z} \frac{(\alpha_{1} + \beta_{1}z)dz}{\sqrt{f(z)}}$$

$$ct = \int^{z} \frac{(\gamma_{1} + \delta_{1}z)dz}{\sqrt{f(z)}}$$
(56)

where $f(z)=z(1-z)(Q+z(Qa^2\frac{\Lambda}{3}+\Xi^3a^2E^2-\mu^2a^2)+z^2(-\mu^2a^4\frac{\Lambda}{3}))(1+a^2\frac{\Lambda}{3}z)^2$. Also we have defined

$$\alpha_{1} = \Xi^{2} \frac{aE}{2} - \frac{1}{2} \frac{a\Xi^{2}(r^{2} + a^{2})E}{\Delta_{r}}$$

$$\beta_{1} = -\frac{1}{2} \frac{a\Xi^{2}(r^{2} + a^{2})E}{\Delta_{r}} \frac{a^{2}\Lambda}{3}$$

$$\gamma_{1} = -\frac{1}{2} \frac{\Xi^{2}(r^{2} + a^{2})^{2}E}{\Delta_{r}} + \frac{a^{2}\Xi^{2}E}{2}$$

$$\delta_{1} = -\frac{a^{2}\Lambda}{6} \frac{\Xi^{2}(r^{2} + a^{2})^{2}E}{\Delta_{r}} - \frac{a^{2}\Xi^{2}E}{2}$$
(57)

Equation (55) is a system of equations of Abelian integrals, whose inversion in principle, involves genus-2 Abelian-Siegelsche modular functions. Indeed, this system is a particular case of Jacobi's inversion problem of hyperelliptic Abelian integrals of genus 2 [43]-[49] (see Appendix A for details). Then, one can express z as a single valued genus two Abelian theta function with argumenents t, ϕ . However, since the polynonial f(z) of sixth degree posses a double root it may well be that the Abelian genus-2 theta function degenerates and the final result can be expressed in terms of genus-1 modular functions.

5 Frame dragging in spherical polar timelike geodesics with a cosmological constant

After a quarter of oscillation in latitude the change of longitude is

$$\Delta \phi = \int_{0}^{1} \frac{(\alpha_{1} + \beta_{1}z)dz}{\sqrt{f(z)}}$$

$$= \frac{\alpha_{1}}{\sqrt{Q}} \frac{\Gamma(\frac{1}{2})(\frac{1}{2})}{\Gamma(1)} F_{D}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1; z'_{1}, z'_{2}, \frac{-a^{2}\Lambda}{3}) + \frac{\beta_{1}}{\sqrt{Q}} \frac{\Gamma(\frac{3}{2})(\frac{1}{2})}{\Gamma(2)} F_{D}(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 1, 2; z'_{1}, z'_{2}, \frac{-a^{2}\Lambda}{3})$$

where $z_i' = 1/z_i$ and

$$z_{1} = -\frac{-3a^{2}E^{2}\Xi^{3} + 3a^{2} - a^{2}Q\Lambda - \sqrt{12Q\Lambda a^{4} + (3a^{2}E^{2}\Xi^{3} - 3a^{2} + a^{2}Q\Lambda)^{2}}}{2a^{4}\Lambda},$$

$$z_{2} = \frac{-3a^{2}E^{2}\Xi^{3} + 3a^{2} - a^{2}Q\Lambda + \sqrt{12Q\Lambda a^{4} + (3a^{2}E^{2}\Xi^{3} - 3a^{2} + a^{2}Q\Lambda)^{2}}}{2a^{4}\Lambda}$$

6 Spherical non-polar null geodesics

In this section, we investigate spherical null geodesics with a nonzero value of the parameter Φ . Now one has to calculate the integral

$$\Delta \phi = -\frac{1}{2} \int \frac{dz \left(\frac{a}{\Delta} P - a\right)}{\sqrt{\alpha z^3 - z^2(\alpha + \beta) + \mathcal{Q}z}} - \frac{1}{2} \int \frac{dz \,\Phi}{(1 - z)\sqrt{\alpha z^3 - z^2(\alpha + \beta) + \mathcal{Q}z}}$$
(58)

where

$$\alpha := -a^2, \beta := Q + \Phi^2, \quad P = r^2 + a^2 - a\Phi$$
 (59)

The first integral can be brought into the Weierstraß form with the invariants g_2,g_3

$$g_2'' = \frac{1}{12} (\alpha'' + \beta'')^2 - \frac{\mathcal{Q}''\alpha''}{4}$$

$$= \frac{1}{12} \frac{1}{a^4 A'^4} \left(-a^2 + \mathcal{Q} + \Phi^2 \right)^2 + \frac{\mathcal{Q}}{4a^2 A'^4}$$

$$g_3'' = \frac{1}{432a^6 A'^6} \left[2(-a^2 + \mathcal{Q} + \Phi^2)^3 - 9\mathcal{Q}a^4 + 9\mathcal{Q}a^2(\mathcal{Q} + \Phi^2) \right]$$

and

$$A' = \frac{\frac{\Phi}{2a} - \frac{GMr}{c^2a^2}}{\frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2a^2}}$$
 (60)

Now let us discuss the second integral in Eq.(58). We define a new variable

$$u_1'z \equiv v \tag{61}$$

then

$$- \frac{1}{2} \int_{0}^{1} \frac{dz \, \Phi}{(1-z)\sqrt{\alpha z^{3} - z^{2}(\alpha + \beta) + \mathcal{Q}z}}$$

$$= -\int_{0}^{u'_{1}} \frac{\Phi}{2\sqrt{\mathcal{Q}}\sqrt{u'_{1}}} \frac{dv}{\sqrt{v}\sqrt{1-v}(1-\chi_{1}v)\sqrt{1-\chi_{2}v}}$$

$$= -\frac{\Phi}{2\sqrt{\mathcal{Q}}\sqrt{u'_{1}}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} F_{1}\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \chi_{1}, \chi_{2}\right)$$

$$= -\frac{\Phi}{2\sqrt{\mathcal{Q}}\sqrt{u'_{1}}} \pi F_{1}\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \chi_{1}, \chi_{2}\right)$$

where $u_i' = \frac{1}{u_i}, i = 1, 2$ and

$$\begin{array}{rcl} u_1 & = & -\frac{-a^2 + \mathcal{Q} + \Phi^2 - \sqrt{4a^2\mathcal{Q} + (a^2 - \mathcal{Q} - \Phi^2)^2}}{2a^2} \\ \\ u_2 & = & -\frac{-a^2 + \mathcal{Q} + \Phi^2 + \sqrt{4a^2\mathcal{Q} + (a^2 - \mathcal{Q} - \Phi^2)^2}}{2a^2} \end{array}$$

parameters	predicted dragging
$\Phi = 1, \mathcal{Q} = 22.693, r = 2.7452$	$\Delta \phi^{\rm GTR} = 7.72736 = 442.7^{\circ} \text{ per revolution} = 1.59 \times 10^{6} \frac{\rm arcs}{\rm revolution}$
$\Phi = -1, \mathcal{Q} = 26.9984, r = 2.99523$	$\Delta \phi^{\rm GTR} = -5.0551 = -289.6^{\circ} \text{ per revolution} = -1.04 \times 10^{6} \frac{\text{arcs}}{\text{revolution}}$
$\Phi = -3, \mathcal{Q} = 23.0508, r = 3.2239$	$\Delta \phi^{\rm GTR} = -5.2014 = -298^{\circ} \text{ per revolution} = -1.07 \times 10^6 \frac{\text{arcs}}{\text{revolution}}$

Table 2: Predictions for frame dragging from a galactic black hole, with Kerr parameter $a_{\rm Galactic} = 0.52 \frac{GM_{\rm BH}}{c^2}$, for different values of photon angular momentum and Carter's Constant. The values of the radii and Φ are in units of $GM_{\rm BH}/c^2$, while Carter's constant $\mathcal Q$ in units of $(GM_{\rm BH}/c^2)^2$.

parameters	predicted dragging
$\Phi = 1, \mathcal{Q} = 16.1443, r = 2.02083$	$\Delta \phi^{\rm GTR} = 10.7355 = 615^{\circ} \text{ per revolution} = 2.2 \times 10^{6} \frac{\rm arcs}{\rm revolution}$
$\Phi = -1, \mathcal{Q} = 25.8865, r = 2.73783$	
$\Phi = -3, \mathcal{Q} = 25.8628, r = 3.23713$	$\Delta \phi^{\text{GTR}} = -4.32779 = -247.9^{\circ} \text{ per revolution} = -892671 \frac{\text{arcs}}{\text{revolution}}$

Table 3: Predictions for frame dragging from a galactic black hole, with Kerr parameter $a_{\rm Galactic} = 0.9939 \frac{GM_{\rm BH}}{c^2}$, for different values of photon angular momentum and Carter's Constant. The values of the radii and Φ are in units of $GM_{\rm BH}/c^2$, while Carter's constant Q in units of $(GM_{\rm BH}/c^2)^2$.

Also

$$\chi_1 := \frac{1}{u_1'}, \quad \chi_2 := \frac{u_2'}{u_1'}$$
(62)

Thus, we expressed the above integral in terms of Appell's first hypergeometric function of two variables, $F_1(\alpha, \beta, \beta', \gamma, x, y)$.

After a complete oscillation in latitude, the angle of longitude, which determines the amount of dragging for the spherical non-polar photonic orbit in the General Theory of Relativity (GTR), is given by

$$\Delta \phi^{\text{GTR}} = 4 \frac{\pi}{2\sqrt{Q}} \frac{\left[\frac{a}{\Delta}P - a\right]}{\sqrt{u_1'}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u_2'}{u_1'}\right) + 4 \frac{\Phi}{2\sqrt{Q}\sqrt{u_1'}} \pi F_1\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \chi_1, \chi_2\right)$$

$$= 4 \frac{\pi}{2\sqrt{Q}} \frac{1}{\sqrt{u_1'}} \frac{a\left(\frac{-\Phi}{a} + \frac{2GMr}{c^2a^2}\right)}{\frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2a^2}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u_2'}{u_1'}\right) + 4 \frac{\Phi}{2\sqrt{Q}\sqrt{u_1'}} \pi F_1\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \chi_1, \chi_2\right)$$
(63)

Orbits with $\Delta \phi^{\rm GTR} > 0$ are called *prograde* and those with $\Delta \phi^{\rm GTR} < 0$ are called retrogade.

As before we can obtain an exact expression for time. After a quarter of

oscillation in latitude

$$\int c \, dt = -\left[-\int_{0}^{1} \frac{dz}{2\sqrt{z}} \frac{\frac{(r^{2}+a^{2})}{\Delta} \left[(r^{2}+a^{2}) - a\Phi \right]}{\sqrt{\alpha z^{2} - z(\alpha + \beta) + Q}} + \frac{dz}{2\sqrt{z}} \frac{a[a(1-z) - \Phi]}{\sqrt{\alpha z^{2} - z(\alpha + \beta) + Q}} \right] \\
= -\left[\left[-\frac{1}{2} \frac{r^{2} + a^{2}}{\Delta} \left[(r^{2} + a^{2}) - a\Phi \right] - a\frac{\Phi}{2} \right] \frac{1}{\sqrt{Q}} \frac{1}{\sqrt{u'_{1}}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u'_{2}}{u'_{1}}\right) \pi + \frac{a^{2}}{2\sqrt{Q}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} F_{1}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, u'_{1}, u'_{2}\right) \right]$$
(64)

The above equation has the correct limit for $\Phi = 0$ and reproduces the corresponding exact expression, Eqn.(30) for spherical null polar geodesics. Indeed, for $\Phi = 0$, $\frac{u_2'}{u_1'} = \frac{-a^2}{Q}$, $u_1' = 1$, and the Appell function has the following limit:

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2)}{\Gamma\left(\frac{1}{2}\right)}F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{2},1,\frac{-a^2}{\mathcal{Q}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}F\left(\frac{1}{2},\frac{1}{2},2,\frac{-a^2}{\mathcal{Q}}\right) \tag{65}$$

Assuming that the centre of the Milky Way is a rotating black hole and that the structure of spacetime near the region Sgr A* is described by the Kerr geometry, we determined the precise frame dragging of a null orbit with a spherical non-polar geometry. The results are displayed in tables 2 and 3.

7 Frame dragging in spherical non-polar null geodesics with a cosmological constant

Including the contribution from the cosmological constant, the relevant differential equation is

$$\frac{d\phi}{d\theta} = \frac{\frac{-a\Xi^2}{\Delta_{\theta}} + \frac{\Phi\Xi^2}{\Delta_{\theta}\sin^2\theta}}{\sqrt{\Theta'}} + \frac{B'}{\Delta_{r}\sqrt{\Theta'}}$$
(66)

where $B' := \Xi^2 a P$.

Introducing the variable z, we find that the angle of longitude that measures the frame-dragging of a spherical non-polar photon orbit, after a complete oscillation in latitude is given by the exact expression

$$\Delta\Phi^{GTR} = -4 \left[-\frac{1}{2} \frac{\Phi\Xi^{2}}{\sqrt{Q}} \frac{1}{\sqrt{u'_{1}}} \frac{\pi}{1} F_{D} \left(\frac{1}{2}, 1, 1, \frac{1}{2}, 1, \frac{-a^{2}\Lambda}{3u'_{1}}, \frac{1}{u'_{1}}, \frac{u'_{2}}{u'_{1}} \right) + \frac{1}{2} \frac{a\Xi^{2}}{\sqrt{Qu'_{1}}} \pi F_{1} \left(\frac{1}{2}, 1, \frac{1}{2}, 1, -\frac{a^{2}\Lambda}{3u'_{1}}, \frac{u'_{2}}{u'_{1}} \right) - \frac{B'}{\Delta_{r}} \frac{1}{\sqrt{Qu'_{1}}} \frac{\pi}{2} F \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u'_{2}}{u'_{1}} \right) \right]$$
(67)

where $u_i' = 1/u_i, i = 1, 2$ and the roots u_i are

$$u_{1} = \frac{\Lambda \Xi^{2} a^{4} - 2\Lambda \Xi^{2} \Phi a^{3} + 3\Xi^{2} a^{2} + \Lambda \Xi^{2} \Phi^{2} a^{2} + Q\Lambda a^{2} - 3\Xi^{2} \Phi^{2} - 3Q + \sqrt{H_{\Lambda}}}{2a^{2} \left(\left(\Lambda a^{2} - 2\Lambda \Phi a + \Lambda \Phi^{2} + 3 \right) \Xi^{2} + Q\Lambda \right)}$$

$$u_{2} = \frac{\Lambda \Xi^{2} a^{4} - 2\Lambda \Xi^{2} \Phi a^{3} + 3\Xi^{2} a^{2} + \Lambda \Xi^{2} \Phi^{2} a^{2} + Q\Lambda a^{2} - 3\Xi^{2} \Phi^{2} - 3Q - \sqrt{H_{\Lambda}}}{2a^{2} \left(\left(\Lambda a^{2} - 2\Lambda \Phi a + \Lambda \Phi^{2} + 3 \right) \Xi^{2} + Q\Lambda \right)}$$
(68)

and

$$H_{\Lambda} := 12\mathcal{Q}\left(\left(\Lambda a^2 - 2\Lambda\Phi a + \Lambda\Phi^2 + 3\right)\Xi^2 + \mathcal{Q}\Lambda\right)a^2 + \left(\left(a - \Phi\right)\left(\Lambda a^3 - \Lambda\Phi a^2 + 3a + 3\Phi\right)\Xi^2 + \mathcal{Q}\left(a^2\Lambda - 3\right)\right)^2$$

$$(69)$$

Equation (67) involves three hypergeometric functions, Gauß's F, Appell's F_1 and Lauricella's F_D . For vanishing cosmological constant the above expression reduces exactly to (63).

Similarly the time elapses after a complete oscillation in latitude is

$$ct = -4 \left[-\frac{1}{2} \frac{\Xi^{2}(r^{2} + a^{2})[r^{2} + a^{2} - a\Phi]}{\Delta_{r}} \frac{1}{\sqrt{Qu'_{1}}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u'_{2}}{u'_{1}}\right) \pi \right]$$

$$- \frac{1}{2} \frac{a\Phi\Xi^{2}}{\sqrt{Qu'_{1}}} F_{1}\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{a^{2}\Lambda}{3u'_{1}}, \frac{u'_{2}}{u'_{1}}\right) \pi$$

$$+ \frac{a^{2}\Xi^{2}}{2\sqrt{Qu'_{1}}} F_{D}\left(\frac{1}{2}, -1, 1, \frac{1}{2}, 1, \frac{1}{u'_{1}}, \frac{a^{2}\Lambda}{3u'_{1}}, \frac{u'_{2}}{u'_{1}}\right) \pi$$

$$(70)$$

8 Spherical timelike geodesics with $L \neq 0$

The relevant equation for integration is [11]

$$d\phi = -\frac{1}{2} \frac{dz}{\sqrt{z^3 \alpha - z^2 (\alpha + \beta) + Qz}} \times \left\{ \frac{aP}{\Delta} - aE + \frac{L}{1 - z} \right\}$$
 (71)

where

$$\alpha = a^2(1 - E^2), \quad \beta = Q + L^2$$
 (72)

and P is provided from Eq.(10) ¹¹.

Let us define:

$$\Pi := \int \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} \tag{73}$$

thus $\xi = \wp(\Pi + \epsilon)$, and $A'' \int \frac{dz}{\sqrt{z^3 \alpha - z^2 (\alpha + \beta) + Qz}} = A'' \int \frac{d\xi}{\sqrt{4\xi^3 - g_2 \xi - g_3}} = A'' \Pi$, $A'' := -\frac{1}{2} \left(\frac{aP}{\Delta} - aE \right)$ and ϵ is a constant of integration. Now

$$-\frac{1}{2}L\int \frac{dz}{(1-z)\sqrt{z^3\alpha - z^2(\alpha+\beta) + Qz}}$$
 (74)

under the substitution (19) becomes

$$-\frac{L\alpha}{8} \int \frac{d\xi}{\left(\frac{\alpha}{4}(1 - \frac{\alpha + \beta}{3\alpha}) - \xi\right) \sqrt{4\xi^3 - g_2\xi - g_3}}$$

$$= -\frac{L\alpha}{8} \int \frac{d\xi}{(w - \xi) \sqrt{4\xi^3 - g_2\xi - g_3}}$$

$$= -\frac{L\alpha}{8} \int \frac{\wp'(\Pi)d\Pi}{(w - \wp(\Pi + \epsilon)) \sqrt{4\wp^3(\Pi) - g_2\wp(\Pi) - g_3}}$$

$$= -\frac{L\alpha}{8} \int \frac{d\Pi}{(w - \wp(\Pi + \epsilon))}$$

$$= -\frac{L\alpha}{8} \left[\text{Log} \frac{\sigma(\Pi + \epsilon - v_0)}{\sigma(\Pi + \epsilon + v_0)} + 2\Pi \zeta(v_0) \right] \times \frac{1}{\wp'(v_0)}$$
(75)

where $w:=\frac{\alpha}{4}\left(1-\frac{\alpha+\beta}{3\alpha}\right)=\wp(v_0)$. Also $\sigma(z)$ denotes the Weierstraß sigma function. Thus the equation for the orbit is given by

$$\phi = \int d\phi = A''\Pi - \frac{L\alpha}{8} \left[\text{Log} \frac{\sigma(\Pi + \epsilon - v_0)}{\sigma(\Pi + \epsilon + v_0)} + 2\Pi\zeta(v_0) \right] \times \frac{1}{\wp'(v_0)}$$
 (76)

and $\wp^{2\prime}(v_0)=4\wp^3(v_0)-g_2\wp(v_0)-g_3=4w^3-g_2w-g_3$. In terms of the integration constants, w is given by the expression:

$$w = \frac{a^2(1 - E^2)}{4} - \frac{a^2(1 - E^2) + Q + L^2}{12}$$
 (77)

Similarly using the first and third line of Eq. (9), we obtain for t the expression

$$c t = \frac{r^2 + a^2}{\Delta} P \frac{\Pi}{-2} + \frac{a\Pi}{2} (aE - L) + \frac{a^2 E}{2} \Pi \left(-\frac{1}{3} \frac{\alpha + \beta}{\alpha} \right) + \frac{a^2 E}{2} \frac{4}{\alpha} \zeta(\Pi) \quad (78)$$

¹¹The extreme black hole a=1 solutions for $L\neq 0$ have been investigated in [38]

An alternative exact expression for time t in terms of ordinary hypergeometric of one-variable F and Appell's first hypergeometric function of two variables is as follows:

$$c t = -\frac{\pi}{2\sqrt{Q}} \frac{\left[\frac{r^2 + a^2}{\Delta}P + aL\right]}{\sqrt{u_1'}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u_2'}{u_1'}\right) + \frac{a^2 E}{2\sqrt{Q}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} F_1\left(\frac{1}{2}, -1, \frac{1}{2}, 1, \frac{1}{u_1'}, \frac{u_2'}{u_1'}\right)$$
(79)

The variables u_i are given in terms of the constants of integration and the Kerr parameter a by the expressions

$$u_{1} = \frac{a^{2}(1-E^{2}) + L^{2} + Q - \sqrt{(-a^{2}(1-E^{2}) - L^{2} - Q)^{2} - 4a^{2}(1-E^{2})Q}}{2a^{2}(1-E^{2})}$$

$$u_{2} = \frac{a^{2}(1-E^{2}) + L^{2} + Q + \sqrt{(-a^{2}(1-E^{2}) - L^{2} - Q)^{2} - 4a^{2}(1-E^{2})Q}}{2a^{2}(1-E^{2})}$$
(80)

and $u'_i = 1/u_i, i = 1, 2.$

Similarly an alternative expression for the amount of dragging for timelike non-polar spherical orbits is as follows,

$$\Delta \phi^{\text{GTR}} = 4 \frac{1}{2\sqrt{Q}} \frac{a \left(\frac{-L}{a} + \frac{2GMEr}{c^2 a^2}\right)}{\frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2 a^2}} \frac{\pi}{\sqrt{u'_1}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u'_2}{u'_1}\right) + 4 \frac{L}{2\sqrt{Q}\sqrt{u'_1}} \pi F_1\left(\frac{1}{2}, 1, \frac{1}{2}, 1, \chi_1, \chi_2\right)$$
(81)

where $\chi_1 = \frac{1}{u_1'}$, $\chi_2 = \frac{u_2'}{u_1'}$. We have calculated the amount of frame dragging for galactic black holes for the values of Kerr parameter given in equation (1) and for fixed values for radii and Carter's constant Q. The invariant parameters L, E are determined by the two conditions for spherical orbits. We repeated the analysis for different values of the Kerr parameter [28]. The results are displayed in table 4.

8.1 General solution for non-spherical geodesics in Kerr

In the general case with non-zero cosmological constant, of non-spherical orbits one has to solve the differential equations

$$\int^{\theta} \frac{d\theta}{\sqrt{\Theta'}} = \int^{r} \frac{dr}{\sqrt{R'}}$$
 (82)

parameters	predicted dragging
a = 0.52, L = -2.03566, E = 0.957665, Q = 11	$\Delta \phi^{\text{GTR}} = -6.06933 = -1.2519 \times 10^6 \frac{\text{arcs}}{\text{revolution}}$
a = 0.9939, L = -2.25773, E = 0.959284, Q = 11	$\Delta \phi^{\text{GTR}} = -5.86166 = -1.2090 \times 10^6 \frac{\text{arcs}}{\text{revolution}}$
a = 0.99616, L = -2.25883, E = 0.959292, Q = 11	$\Delta \phi^{\rm GTR} = -5.86161 = -1.20904 \times 10^6 \frac{\rm arcs}{\rm revolution}$

Table 4: Predictions for frame dragging from a galactic black hole, with the indicated values of Kerr parameter $a_{\rm Galactic}$, for different values of test particle's angular momentum, a particular value for Carter's constant and for a fixed radius r=10. The values of the radii a,L are in units of $GM_{\rm BH}/c^2$, while Carter's constant $\mathcal Q$ in units of $(GM_{\rm BH}/c^2)^2$. The period ratios, τ , are 3.058i,2.69743i, 2.69625i respectively.

where R'(r) is a quartic polynomial for null geodesics ($\mu = 0$ in Eq.(5)) which is given by

$$R' = E^{2} \left\{ \Xi^{2} \left[(r^{2} + a^{2}) - a\Phi \right]^{2} - \Delta_{r} \left[\Xi^{2} (\Phi - a)^{2} + \mathcal{Q} \right] \right\}$$
(83)

and $\Theta'(\theta)$ is given by

$$\Theta' = E^2 \left\{ \left[\mathcal{Q} + (\Phi - a)^2 \Xi^2 \right] \Delta_\theta - \Xi^2 \frac{(a \sin^2 \theta - \Phi)^2}{\sin^2 \theta} \right\}$$
(84)

Note that this equation is an equation between two elliptic integrals, in the general case when all roots are distinct. The left hand side can be transformed into an elliptic integral with variable z or ξ in Weierstraß normal form, see Eq.(19). In order to solve this differential equation and determine r as a function of θ , one can employ the theory of Abel for the transformation of elliptic functions [32], which was first applied in [11] for the case of timelike orbits. A detailed exposition of the theory of transformation of elliptic functions based on [32] can also be found in [11]. The interesting connection with modular equations [11] is outlined in Appendix B.

We note at this point that the corresponding relationship for non-spherical timelike orbits in the presence of the cosmological constant relates a hyperelliptic integral (the polynomial R' is a sectic in this case) to an elliptic integral, and therefore involves the transformation theory of hyperelliptic functions. This fact was first observed in [11] and will be a subject of another publication.

8.2 Transforming the geodesic elliptic integrals into Abel's form

The transformation

$$x \to e_3 + \frac{(e_1 - e_3)}{x^2}$$
 (85)

transforms the elliptic integral in Weierstraß form into Abel's and Jacobi's form:

$$\int \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}} \to -\frac{1}{\sqrt{e_1-e_3}} \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$
(86)

with $k^2 = \frac{e_2 - e_3}{e_1 - e_3}$. Similarly, the quartic can be brought to Jacobi's form $y^2 = (1 - x^2)(1 - k_1^2 x^2)$. Indeed, as we saw in section 3 we have

$$\frac{d\theta}{\sqrt{\Theta}} = -\frac{1}{2} \frac{dz}{\sqrt{z^3 \alpha - z^2 (\alpha + \beta) + \mathcal{Q}z}} = -\frac{1}{2} \frac{d\xi}{\sqrt{4\xi^3 - q_2 \xi - q_3}} \tag{87}$$

where

$$g_{2} = \frac{1}{12} \left(-a^{2} + Q + \Phi^{2} \right)^{2} - \frac{Q}{4} (-a^{2})$$

$$g_{3} = \frac{1}{216} \left(-a^{2} + Q + \Phi^{2} \right)^{3} - \frac{Q}{48} a^{4} - \frac{Q}{48} (-a^{2} (Q + \Phi^{2}))$$
(88)

In this case the three roots of the cubic $4\xi^3 - g_2\xi - g_3$ are given by the expressions

$$e_{1} = \frac{1}{24} \left(-a^{2} + \Phi^{2} + Q + 3\sqrt{a^{4} + 2a^{2}Q + Q^{2} - 2a^{2}\Phi^{2} + 2Q\Phi^{2} + \Phi^{4}} \right)$$

$$e_{2} = \frac{1}{12} \left(a^{2} - Q - \Phi^{2} \right)$$

$$e_{3} = \frac{1}{24} \left(-a^{2} + \Phi^{2} + Q - 3\sqrt{a^{4} + 2a^{2}Q + Q^{2} - 2a^{2}\Phi^{2} + 2Q\Phi^{2} + \Phi^{4}} \right)$$
(89)

Thus the Jacobi modulus $k^2 = \frac{e_2 - e_3}{e_1 - e_3}$ is given by

$$k^{2} = \frac{a^{2} - \mathcal{Q} - \Phi^{2} + \sqrt{a^{4} + 2a^{2}(\mathcal{Q} - \Phi^{2}) + (\mathcal{Q} + \Phi^{2})^{2}}}{2\sqrt{a^{4} + 2a^{2}(\mathcal{Q} - \Phi^{2}) + (\mathcal{Q} + \Phi^{2})^{2}}}$$
(90)

It has the correct limit for photon spherical geodesics with $\Phi = 0$

$$k^{2}(\Phi = 0) = \frac{a^{2}}{a^{2} + \mathcal{Q}} \tag{91}$$

Also

$$\frac{1}{\sqrt{e_1 - e_3}} = \frac{2}{(a^4 + 2a^2(Q - \Phi^2) + (Q + \Phi^2)^2)^{1/4}}$$
(92)

and it also has the correct limit for spherical photon orbits with zero Φ

$$\frac{1}{\sqrt{e_1 - e_3}} = \frac{2}{\sqrt{a^2 + \mathcal{O}}} \tag{93}$$

Thus, we get

$$\int \frac{d\theta}{\sqrt{\Theta}} = -\frac{1}{2} \frac{1}{\sqrt{e_1 - e_3}} \int \frac{d\xi'}{\sqrt{(1 - \xi'^2)(1 - k^2 \xi'^2)}}$$
(94)

Applying Luchterhand's transformation formula [36] on the radial integral the Jacobi's form can be recovered:

$$\frac{\partial x}{M\sqrt{(1-x^2)(1-k_1^2x^2)}} = \frac{\partial y}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$$
(95)

where the Jacobi modulus k_1 and the coefficient M, are given in terms of the roots $\alpha, \beta, \gamma, \delta$ of the quartic, by the following expressions

$$k_1 = \frac{\sqrt{(\alpha - \delta)(\beta - \gamma)}}{\sqrt{(\alpha - \gamma)(\beta - \delta)}}, \quad M = \sqrt{(\alpha - \gamma)(\beta - \delta)}$$
 (96)

and the integration variables are related by

$$\frac{1-x}{1+x} = \frac{(\gamma - \delta)(y - \alpha)(y - \beta)}{(\alpha - \beta)(y - \gamma)(y - \delta)}$$
(97)

Also we assume that the roots $\alpha, \beta, \gamma, \delta$ of the quartic are real and are organized in the following ascending order of magnitude: $\alpha > \beta > \gamma > \delta$.

We can also provide a nice formula for r in terms of the Jacobi's sinus amplitudinus function ¹²

$$\frac{(\gamma - \delta)(r - \alpha)(r - \beta)}{(\alpha - \beta)(r - \gamma)(r - \delta)} = \frac{1 - \operatorname{sn}\left(M \int \frac{d\theta}{\sqrt{\Theta}}, k_1\right)}{1 + \operatorname{sn}\left(M \int \frac{d\theta}{\sqrt{\Theta}}, k_1\right)}$$
(98)

8.3 Equatorial geodesics including the contribution of the cosmological constant

The equatorial geodesics (i.e. $\theta = \pi/2, Q = 0$), with a nonzero cosmological constant, may be obtained by Eq.(4) for the particular values of Q, θ . The characteristic function in this case, has the form [11]

$$W = -Ect + \int \frac{\sqrt{R'}}{\Delta_r} dr + L\phi \tag{99}$$

and the geodesics are given by the expressions:
$$\frac{1}{1} \text{We wright } \int \frac{dr}{\sqrt{R}} = \int \frac{dr}{\sqrt{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}} = \int \frac{\partial x}{M\sqrt{(1-x^2)(1-k_1^2x^2)}} = \int \frac{d\theta}{\sqrt{\Theta}}.$$

$$\frac{dr}{\sqrt{R'}} = \frac{d\lambda}{r^2}$$

$$r^2 \frac{d\phi}{d\lambda} = \frac{a(1 + \frac{1}{3}a^2\Lambda)^2 (E(r^2 + a^2) - La)}{(1 - \frac{\Lambda}{3}r^2)(r^2 + a^2) - \frac{2GMr}{c^2}} + (L - aE)(1 + \frac{1}{3}a^2\Lambda)^2$$

$$cr^2 \frac{dt}{d\lambda} = \frac{(1 + \frac{1}{3}a^2\Lambda)^2 (r^2 + a^2) \left[(r^2 + a^2)E - aL \right]}{\Delta_r} + (1 + \frac{1}{3}a^2\Lambda)^2 a(L - aE)$$
(100)

where

$$R' = (1 + \frac{1}{3}a^2\Lambda)^2 \left[((r^2 + a^2)E - aL)^2 - \Delta_r((L - aE)^2) \right]$$
 (101)

for null geodesics and

$$R' = (1 + \frac{1}{3}a^2\Lambda)^2 \left[((r^2 + a^2)E - aL)^2 - \Delta_r((L - aE)^2) \right] - \Delta_r(\mu^2 r^2)$$
 (102)

for timelike geodesics.

Equatorial orbits are of particular interest for various astrophysics applications. The exact solution of circular equatorial orbits with a cosmological constant was presented in [11]. In what follows, we shall concentrate on the cases of non-circular equatorial timelike and null orbits that describe motion of test particles and photons under the assumption of a vanishing cosmological constant. We shall derive new exact expressions for the precession of perihelion or periapsis for the orbit of test particle in the gravitational field of Kerr, as well as for the deflection angle of a light ray from the Kerr gravitational field, in terms of multivariable hypergeometric functions. In the latter case, we apply the exact calculation obtained for determining the bending angle of a light ray from the gravitational field of the galactic centre of Milky-Way assuming that the Sgr A* region is a supermassive rotating black hole for various values of the Kerr parameter which are supported by recent observations and of the impact factor. The more general case, in the presence of the cosmological constant is a task for a future publication.

8.4 Exact solution of timelike equatorial geodesics

We now proceed to determine the exact expression for the precession of equatorial timelike orbits in Kerr spacetime. We have $r^2(\dot{r}) = \sqrt{R}$. This can be rewritten as

$$(\dot{r})^2 = E^2 + \frac{a^2 E^2}{r^2} - \frac{L^2}{r^2} + \frac{2GM}{c^2 r^3} (L - aE)^2 - \frac{\Delta}{r^2}$$
 (103)

By defining a new variable u = 1/r we get the following expression

$$u^{-4}\dot{u}^2 = E^2 + a^2E^2u^2 - L^2u^2 + \frac{2GM}{c^2}(L - aE)^2u^3 - (1 + a^2u^2 - \frac{2GM}{c^2}u) \equiv B^t(u) \tag{104}$$

Similarly $\dot{\phi}^2 = u^4 \frac{A^2(u)}{D^2(u)}$ where

$$A(u) = L + u\alpha_S(aE - L), \ D(u) = 1 + a^2u^2 - \alpha_S u, \ \alpha_S := \frac{2GM}{c^2}$$
 (105)

Thus, we obtain the differential equation

$$\frac{d\phi}{du} = \frac{A(u)}{D(u)} \frac{1}{\sqrt{B^t(u)}} \tag{106}$$

We now write

$$\frac{A(u)}{D(u)} = \frac{A_{+}}{u_{+} - u} + \frac{A_{-}}{u_{-} - u} \tag{107}$$

where $u_{+} = \frac{r_{+}}{a^{2}}, u_{-} = \frac{r_{-}}{a^{2}}$ and

$$r_{\pm} = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}$$
 (108)

where the quantities A_+, A_- are given by

$$A_{+} = \frac{\frac{L}{a^{2}} + \frac{\alpha_{S}}{a^{2}} (aE - L) u_{+}}{u_{-} - u_{+}}$$

$$A_{-} = \frac{-\frac{L}{a^{2}} - \frac{\alpha_{S}}{a^{2}} (aE - L) u_{-}}{u_{-} - u_{+}}$$
(109)

For the calculation of the perihelion (periapsis) precession of a test particle in orbit around a rotating mass we need to calculate the integral $\Delta\phi^{\rm GTR}=2\int_{u_3'}^{u_2'}d\phi$. Then

$$\int d\phi = \int \frac{du'A_{+}}{\left(\frac{Mr_{+}}{a^{2}} - u'\right)} \frac{1}{\sqrt{\frac{\alpha_{S}(L - aE)^{2}}{\left(\frac{GM}{c^{2}}\right)^{3}}}} \frac{1}{\sqrt{(u' - u'_{3})(u'_{1} - u')(u'_{2} - u')}} + \int \frac{du'A_{-}}{\left(\frac{Mr_{-}}{a^{2}} - u'\right)} \frac{1}{\sqrt{\frac{\alpha_{S}(L - aE)^{2}}{\left(\frac{GM}{c^{2}}\right)^{3}}}} \frac{1}{\sqrt{(u' - u'_{3})(u'_{1} - u')(u'_{2} - u')}}$$
(110)

and we have defined $u' := u \frac{GM}{c^2}$. Using new variables

$$u' \to u'_3 + \xi^2 (u'_2 - u'_3)$$

we can bring our expression to the integral representation of Appell's first hypergeometric function

$$\Delta \phi^{\text{GTR}} = \frac{2}{\sqrt{u'_1 - u'_3}} \frac{1}{\sqrt{\frac{\alpha_S (L - aE)^2}{\left(\frac{GM}{c^2}\right)^3}}} \left\{ \frac{A_+}{\frac{GMr_+}{c^2 a^2} - u'_3} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_+}{c^2 a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3}\right) \frac{\Gamma(1/2)^2}{\Gamma(1)} + \frac{A_-}{\frac{GMr_-}{c^2 a^2} - u'_3} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_-}{c^2 a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3}\right) \frac{\Gamma(1/2)^2}{\Gamma(1)} \right\}$$
(111)

We can now provide an exact expression for time t. The first differential equation in Eq.(9) for equatorial orbits can be written in terms of the variable u as follows

$$c\dot{t} = \frac{E(1+a^2u^2) + a\alpha_S u^3(aE-L)}{Du}$$
 (112)

Subsequently by dividing with \dot{u} we derive the equation

$$\frac{cdt}{du} = \frac{E(1 + a^{2}u^{2}) + a\alpha_{S}u^{3}(aE - L)}{u^{2}D(u)\sqrt{B^{t}(u)}}
= \frac{E}{u^{2}D(u)\sqrt{B^{t}(u)}} + \frac{Ea^{2}}{D(u)\sqrt{B^{t}(u)}} + \frac{au\alpha_{S}(aE - L)}{D(u)\sqrt{B^{t}(u)}}$$
(113)

By integrating we can express t in terms of Appell's and Lauricella generalized hypergeometric functions

$$ct = \frac{2}{\sqrt{u'_1 - u'_3}} \frac{1}{\sqrt{\frac{\alpha_S(L - aE)^2}{\left(\frac{GM}{c^2}\right)^3}}} \left\{ \frac{A_+^t}{\frac{GMr_+}{c^2a^2} - u'_3} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_+}{c^2a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \right.$$

$$+ \frac{A_-^t}{\frac{GMr_-}{c^2a^2} - u'_3} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_-}{c^2a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \right\}$$

$$+ 2 \frac{A'_+}{\sqrt{u'_1 - u'_2}} \frac{1}{\frac{u'_2^2}{G^2M^2c^{-4}}} \frac{1}{\sqrt{\frac{\alpha_S(L - aE)^2}{\left(\frac{GM}{c^2}\right)^3}}} \left\{ \frac{1}{\frac{GMr_+}{c^2a^2} - u'_2} F_D \left(\frac{1}{2}, 2, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{u'_2}, \frac{u'_3 - u'_2}{\frac{GMr_+}{c^2a^2} - u'_2}, \frac{u'_3 - u'_2}{u'_1 - u'_2} \right) \pi$$

$$- \frac{1}{\frac{GMr_-}{c^2a^2} - u'_2} F_D \left(\frac{1}{2}, 2, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{u'_2}, \frac{u'_3 - u'_2}{\frac{GMr_-}{c^2a^2} - u'_2}, \frac{u'_3 - u'_2}{u'_1 - u'_2} \right) \pi \right\}$$

$$(114)$$

8.5 Exact solution of null equatorial geodesics

In this case, we arrive at the differential equation first derived in [14]

$$\frac{d\phi}{du} = \frac{\Phi + u\alpha_S(a - \Phi)}{D(u)} \frac{1}{\sqrt{B^N(u)}}$$
(115)

where the cubic polynomial $B^{N}(u)$ is given by the expression

$$B^{N}(u) = \alpha_{S}(\Phi - a)^{2}u^{3} + u^{2}(a^{2} - \Phi^{2}) + 1$$
(116)

In order to calculate the angle of deflection is necessary to calculate the integral: $\Delta\phi^{\rm GTR}=2\int_0^{u_2'}d\phi$.

Using partial fractions as in the timelike case we obtain an elegant exact expression for $\Delta \phi^{\text{GTR}}$ in terms of Lauricella's fourth, hypergeometric function of three variables F_D , and Appell's first hypergeometric function of two variables F_1 ,

$$\Delta\phi^{\text{GTR}} = \frac{2}{\sqrt{u'_1 - u'_3}} \frac{1}{\sqrt{\frac{\alpha_S(\Phi - a)^2}{G^3 M^3}}} \left(\frac{A_+}{\frac{GMr_+}{c^2 a^2} - u'_3} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_+}{c^2 a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \right) \\
+ \frac{A_-}{\frac{GMr_-}{c^2 a^2} - u'_3} F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_-}{c^2 a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \right) \\
+ \frac{2}{\sqrt{u'_1 - u'_3}} \frac{1}{\sqrt{\frac{\alpha_S(\Phi - a)^2}{G^3 M^3}}}} \\
\left(-A_+ \sqrt{\frac{-u'_3}{u'_2 - u'_3}} \frac{1}{\left(\frac{GMr_+}{c^2 a^2} - u'_3\right)} 2F_D\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{-u'_3}{\frac{GMr_+}{c^2 a^2} - u'_3}, \frac{-u'_3}{u'_1 - u'_3}, \frac{-u'_3}{u'_2 - u'_3} \right) \\
- A_- \sqrt{\frac{-u'_3}{u'_2 - u'_3}} \frac{1}{\left(\frac{GMr_-}{c^2 a^2} - u'_3\right)} 2F_D\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{-u'_3}{\frac{GMr_-}{c^2 a^2} - u'_3}, \frac{-u'_3}{u'_1 - u'_3}, \frac{-u'_3}{u'_2 - u'_3} \right) \right)$$
(117)

or

$$\Delta\phi^{\text{GTR}} = \frac{2}{\sqrt{u'_1 - u'_3}} \frac{1}{\sqrt{\frac{\alpha_S(\Phi - a)^2}{c^3 M^3}}} \left\{ \frac{A_+}{\frac{GMr_+}{c^2 a^2} - u'_3} \left(F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_+}{c^2 a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \right. \\
- 2\sqrt{\frac{-u'_3}{u'_2 - u'_3}} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{-u'_3}{\frac{GMr_+}{c^2 a^2} - u'_3}, \frac{-u'_3}{u'_1 - u'_3}, \frac{-u'_3}{u'_2 - u'_3} \right) \right) \\
+ \frac{A_-}{\frac{GMr_-}{c^2 a^2} - u'_3} \left(F_1 \left(\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u'_2 - u'_3}{\frac{GMr_-}{c^2 a^2} - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \right. \\
- 2\sqrt{\frac{-u'_3}{u'_2 - u'_3}} F_D \left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{-u'_3}{\frac{GMr_-}{c^2 a^2} - u'_3}, \frac{-u'_3}{u'_1 - u'_3}, \frac{-u'_3}{u'_2 - u'_3} \right) \right) \right\}$$
(118)

parameters	predicted deflection
$a_{\mathrm{Galactic}} = 0.52, \Phi = 5$	$\Delta \phi^{\text{GTR}} - \pi = 1.84869 = 105.9^{\circ} = 381319 \text{arcs}$
$a_{\text{Galactic}} = 0.52, \Phi = 10$	$\Delta \phi^{\text{GTR}} - \pi = 0.537 = 30.77^{\circ} = 110792 \text{arcs}$
$a_{\text{Galactic}} = 0.52, \Phi = 15$	$\Delta \phi^{\text{GTR}} - \pi = 0.320 = 18.34^{\circ} = 66015.6 \text{arcs}$
$a_{\text{Galactic}} = 0.52, \Phi = 20$	$\Delta \phi^{\text{GTR}} - \pi = 0.228 = 13.08^{\circ} = 47093 \text{arcs}$
$a_{\text{Galactic}} = 0.52, \Phi = 40$	$\Delta \phi^{\text{GTR}} - \pi = 0.1065 = 6.10^{\circ} = 21973 \text{arcs}$

Table 5: Predictions for light deflection from a galactic rotating black hole with Kerr parameter $a_{\rm Galactic} = 0.52 \frac{GM_{\rm BH}}{c^2}$. The values of the impact parameter Φ are in units of $\frac{GM_{\rm BH}}{c^2}$.

During the derivation of Eq.(118) we have used at an intermediate step of the calculation the identity

$$\frac{1}{\frac{Mr_{\pm}}{a^{2}}\sqrt{u'_{1}u'_{2}}}F_{D}\left(1,1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{u'_{3}}{\frac{Mr_{\pm}}{a^{2}}},\frac{u'_{3}}{u'_{1}},\frac{u'_{3}}{u'_{2}}\right)$$

$$= \frac{1}{\frac{Mr_{\pm}}{a^{2}}-u'_{3}\sqrt{u'_{1}-u'_{3}}\sqrt{u'_{2}-u'_{3}}}F_{D}\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{-u'_{3}}{\frac{Mr_{\pm}}{a^{2}}-u'_{3}},\frac{-u'_{3}}{u'_{1}-u'_{3}},\frac{-u'_{3}}{u'_{2}-u'_{3}}\right)$$
(119)

The quantities A_{\pm} are now given in terms of the impact parameter Φ and the Kerr parameter a by the expressions

$$A_{\pm} = \frac{\pm \Phi \pm \alpha_S (a - \Phi) \frac{r_{\pm}}{a^2}}{-2\sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}}$$
(120)

The angle of deflection δ of light-rays from the gravitational field of a Galactic rotating black hole or a massive star is defined to be the deviation of $\Delta\phi^{\rm GTR}$ from the transcendental number π

$$\delta = \Delta \phi^{\text{GTR}} - \pi \tag{121}$$

We have calculated the deflection angle δ of light rays from the gravitational field of a galactic rotating black hole for different values of the Kerr parameter a and the impact parameter Φ . The results are displayed in figure (1) and tables 5,6. It is clear from figure (1) that especially for smaller values of the impact parameter Φ , there is a strong dependence of the deflection angle on the spin of the black hole. This has implications for gravitational lensing studies and can lead in principle to an independent measurement of the Kerr parameter at the strong field regime.

The roots of the cubic are organized as $u'_1 > u'_2 > 0 > u'_3$.

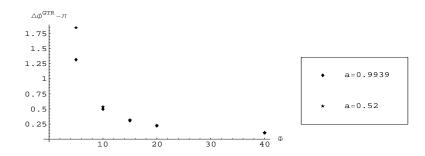


Figure 1: Deflection angle δ for various values of impact parameter Φ and for two values of the Kerr parameter a for the rotating black hole.

parameters	predicted deflection
$a_{\text{Galactic}} = 0.9939, \Phi = 5$	$\Delta \phi^{\text{GTR}} - \pi = 1.317 = 75.47^{\circ} = 271710 \text{arcs}$
$a_{\text{Galactic}} = 0.9939, \Phi = 10$	$\Delta \phi^{\text{GTR}} - \pi = 0.497 = 28.49^{\circ} = 102581 \text{arcs}$
$a_{\text{Galactic}} = 0.9939, \Phi = 15$	$\Delta \phi^{\text{GTR}} - \pi = 0.306 = 17.57^{\circ} = 63254 \text{arcs}$
$a_{\text{Galactic}} = 0.9939, \Phi = 20$	$\Delta \phi^{\text{GTR}} - \pi = 0.2217 = 12.70^{\circ} = 45728 \text{arcs}$
$a_{\text{Galactic}} = 0.9939, \Phi = 40$	$\Delta \phi^{\text{GTR}} - \pi = 0.1052 = 6.026^{\circ} = 21694.3 \text{arcs}$

Table 6: Predictions for light deflection from a galactic rotating black hole with Kerr parameter $a_{\rm Galactic} = 0.9939 \frac{GM_{\rm BH}}{c^2}$. The values of the impact parameter Φ are in units of $\frac{GM_{\rm BH}}{c^2}$.

8.5.1 Null equatorial geodesics with a cosmological constant

In this case, the generalization of equation (115) is given by

$$\frac{d\phi}{du} = \frac{\Phi u^2 - \alpha_S u^3 (\Phi - a) + (\Phi - a)(-\frac{\Lambda}{3})(1 + a^2 u^2)}{\left[\left(u^2 - \frac{\Lambda}{3}\right)(1 + a^2 u^2) - \alpha_S u^3\right] \sqrt{B^{\Lambda}(u)}}$$
(122)

where $B^{\Lambda}(u)$ is still a cubic polynomial

$$B^{\Lambda}(u) = 1 + a^2 u^2 - \Phi^2 u^2 + \alpha_S u^3 (\Phi - a)^2 + \frac{\Lambda}{3} (1 + a^2 u^2) (\Phi - a)^2$$
 (123)

For a vanishing cosmological constant the above cubic polynomial reduces to Eq. (116).

9 Conclusions

In this work, we have investigated the motion of a test particle and light in the gravitational field of Kerr spacetime with and without the cosmological constant. We have derived a number of useful analytical expressions for measurable physical quantities.

In the case of null orbits, we solved exactly the geodesic equations for spherical polar and non-polar photon orbits. The exact solution for the orbit of a photon with zero angular momentum Φ and vanishing cosmological constant was provided by the Weierstraß elliptic function $\wp(z)$.

The exact expressions that determine the amount of frame-dragging (Lense-Thirring effect) for the corresponding photon orbits, assuming vanishing cosmological constant, were written in terms of Weierstraß function real half-period or equivalently in terms of Gauß hypergeometric function F for a photon spherical orbit with $\Phi = 0$ and Gauß hypergeometric function F and Appell's generalized hypergeometric function of two variables F_1 for photon orbits with $\Phi \neq 0$ and constant radius. The corresponding expressions in the presence of the cosmological constant were given in terms of Appell's hypergeometric function F_1 for the

case of spherical null polar orbits, and in terms of Lauricella's hypergeometric function of three variables F_D , Appell's F_1 and Gauß ordinary hypergeometric function F, in the case of spherical photonic orbits with nonvanishing value for the invariant parameter Φ .

We subsequently applied our exact solutions for the determination of the Lense-Thirring effect that a photon experiences in a spherical polar and non-polar orbit around and close to our galactic centre, assuming the latter is a rotating black hole whose surrounding spacetime structure is described by the Kerr geometry as supported by recent observations. We repeated the analysis for various values of the Kerr parameter.

We also solved exactly non-spherical polar null unbound orbits. We derived analytical results for the deflection angle of a light ray from the gravitational field of a rotating black hole's pole. The resulting expression for the deflection angle was written elegantly in terms of Lauricella's hypergeometric function F_D .

We then investigated, non-circular orbits confined to the equatorial plane (timelike and null-like equatorial geodesics) for which the value of Carter's constant invariant parameter vanishes. For the case of a vanishing cosmological constant, we derived an exact expression for the amount of relativistic precession for a test particle in a timelike orbit, around a rotating central mass. The corresponding novel expression was given in terms of Appell's first hypergeometric function of two variables F_1 . The application of this exact solution as well as of those that describe non-spherical orbits not necessarily confined to the equatorial plane, for the determination of the effect of rotation of central mass on the perihelion precession of a test particle (Mercury around Sun) or periapsis precession for a star such as S2 in a high eccentricity orbit around the galactic centre [42] is beyond the scope of this work and will be the subject of a future publication [51].

We have also derived an exact expression for the deflection angle of a light ray from the gravitational field of a rotating mass (the Kerr field). The corresponding expression was given in terms of Lauricella's F_D and Appell's F_1 generalized hypergeometric functions. We applied this calculation for the bending of light from the gravitational field of the galactic centre of Milky Way, assuming the latter is a supermassive Kerr black hole for various values of the Kerr parameter and the impact factor. We emphasized in the main text the strong dependence of the bending angle on the Kerr parameter for small values of the impact factor. These results should be useful for gravitational lensing studies, where one treats the black hole as a gravitational "lens" [50], especially in the strong field region, when the bending angle can be very large.

The synergy between theory and experiment for probing and measuring relativistic effects is going to be one of the most exciting and fruitful scientific endeavours.

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A Definitions of genus-2 theta functions that solve Jacobi's inversion problem

Riemann's theta function [48] for genus g is defined as follows:

$$\Theta(u) := \sum_{n_1, \dots, n_g} e^{2\pi i u n + i\pi \Omega n^2}$$
(124)

where $\Omega n^2 := \Omega_{11} n_1^2 + \cdots 2\Omega_{12} n_1 n_2 + \cdots$ and $un := u_1 n_1 + \cdots u_g n_g$. The symmetric $g \times g$ complex matrix Ω whose imaginary part is positive definite is a member of the set called Siegel upper-half-space denoted as \mathcal{L}_{S_g} . It is clearly the generalization of the ratio of half-periods τ in the genus g = 1 case. For genus g = 2 the Riemann theta function can be written in matrix form:

$$\Theta(u,\Omega) = \sum_{\mathbf{n}\in\mathbb{Z}^2} e^{\pi i^{\mathbf{t}} \mathbf{n}\Omega \mathbf{n} + 2\pi i^{\mathbf{t}} \mathbf{n}\mathbf{u}}$$

$$= \sum_{n_1,n_2} e^{\pi i \begin{pmatrix} n_1 & n_2 \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + 2\pi i \begin{pmatrix} n_1 & n_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}$$
(125)

Riemann's theta function with characteristics is defined by:

$$\Theta(u; q, q') := \sum_{n_1, \dots, n_g} e^{2\pi i u(n+q') + i\pi\Omega(n+q')^2 + 2\pi i q(n+q')}$$
(126)

herein q denotes the set of g quantities q_1, \dots, q_g and q' denotes the set of g quantities q'_1, \dots, q'_g . Eq.(126) can be rewritten in a suggestive matrix form:

$$\Theta \begin{bmatrix} q' \\ q \end{bmatrix} (u, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i^t (n+q')\Omega(n+q') + 2\pi i^t (n+q')(u+q)}, \quad q, q' \in \mathbb{Q}^g \quad (127)$$

The theta functions whose quotients provide a solution to Abel-Jacobi's inversion problem are defined as follows [49]:

$$\theta(u;q,q') := \sum e^{au^2 + 2hu(n+q') + b(n+q')^2 + 2i\pi q(n+q')}$$
(128)

where the summation extends to all positive and negative integer values of the g integers n_1, \dots, n_g , a is any symmetrical matrix whatever of g rows and columns, h is any matrix whatever of g rows and columns, in general not symmetrical, b is any symmetrical matrix whatever of g rows and columns, such that the real part of the quadratic form bm^2 is necessarily negative for all real values of the quantities m_1, \dots, m_g , other than zero, and q, q' constitute the characteristics of the function. The matrix b depends on $\frac{1}{2}g(g+1)$ independent

constants; if we put $i\pi\Omega = b$ and denote the g-quantities hu by $i\pi U$, we obtain the relation with Riemann's theta function:

$$\theta(u;q,q') = e^{au^2}\Theta(U;q,q') \tag{129}$$

The dependence of genus-2 theta functions on two complex variables is denoted by: $\theta(u;q,q') = \theta(u_1,u_2;q,q')$, the dependence on the Siegel moduli matrix Ω by: $\theta(u_1,u_2,\Omega;q,q')$. To every half-period one can associate a set of characteristics. For instance, the period $u^{a,a_1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ while $\theta(u)$ is a theta function of two variables with zero characteristics, i.e. $\theta(u) = \theta(u;0,0) = \theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(u,\Omega)$. Also, Weierstraß had associated a symbol for each of the six odd theta functions with characteristics and the ten even theta functions of genus two. For example, $\theta(u)$ is associated with the Weierstraß symbol 5 or occasionaly the number appears as a subscript, i.e. $\theta(u)_5$.

Let the genus g Riemann hyperelliptic surface be described by the equation:

$$y^{2} = 4(x - a_{1}) \cdots (x - a_{g})(x - c)(x - c_{1}) \cdots (x - c_{g})$$
(130)

For g=2 the above hyperelliptic Riemann algebraic equation reduces to:

$$y^{2} = 4(x - a_{1})(x - a_{2})(x - c)(x - c_{1})(x - c_{2})$$
(131)

where a_1, a_2, c, c_1, c_2 denote the finite branch points of the surface.

The Jacobi's inversion problem involves finding the solutions, for x_i in terms of u_i , for the following system of equations of Abelian integrals [49]:

$$u_1^{x_1,a_1} + \dots + u_1^{x_g,a_g} \equiv u_1$$

$$\vdots + \dots + \vdots \qquad \vdots$$

$$u_g^{x_1,a_1} + \dots + u_g^{x_g,a_g} \equiv u_g$$
(132)

where $u_1^{x,\mu} = \int_{\mu}^{x} \frac{dx}{y}, u_2^{x,\mu} = \int_{\mu}^{x} \frac{xdx}{y}, \cdots, u_g^{x,\mu} = \int_{\mu}^{x} \frac{x^{g-1}dx}{y}.$ For g=2 the above system of equations takes the form:

$$\int_{a_1}^{x_1} \frac{dx}{y} + \int_{a_2}^{x_2} \frac{dx}{y} \equiv u_1$$

$$\int_{a_1}^{x_1} \frac{x \, dx}{y} + \int_{a_2}^{x_2} \frac{x \, dx}{y} \equiv u_2$$
(133)

where u_1, u_2 are arbitrary. The solution is given by the five equations [49]

$$\frac{\theta^{2}(u|u^{b,a})}{\theta^{2}(u)} = A(b-x_{1})(b-x_{2})\cdots(b-x_{g})$$

$$= A(b-x_{1})(b-x_{2})$$

$$= \pm \frac{(b-x_{1})(b-x_{2})}{\sqrt{e^{\pi i PP'} f'(b)}};$$
(134)

where $f(x) = (x-a_1)(x-a_2)(x-c)(x-c_1)(x-c_2)$, and $e^{\pi i P P'} = \pm 1$ accordingly as $u^{b,a}$ is an odd or even half-period. Also b denotes a finite branch point and the branch place a being at infinity [49]. The symbol $\theta(u|u^{b,a})$ denotes a genus 2 theta function with characteristics: $\theta(u;q,q')$ [49], where $u,=(u_1,u_2)$, denotes two independent variables. From any 2 of these equations, eq.(134), the upper integration bounds x_1, x_2 of the system of differential equations eq.(133) can be expressed as single valued functions of the arbitrary arguments u_1, u_2 . For instance,

$$x_1 = a_1 + \frac{1}{A_1(x_2 - a_1)} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)}$$
(135)

and

$$x_{2} = -\frac{\left[(a_{2} - a_{1})(a_{2} + a_{1}) + \frac{1}{A_{1}} \frac{\theta^{2}(u|u^{a_{1},a})}{\theta^{2}(u)} - \frac{1}{A_{2}} \frac{\theta^{2}(u|u^{a_{2},a})}{\theta^{2}(u)}\right]}{2(a_{1} - a_{2})}$$

$$\pm \frac{\sqrt{\left[(a_{2} - a_{1})(a_{2} + a_{1}) + \frac{1}{A_{1}} \frac{\theta^{2}(u|u^{a_{1},a})}{\theta^{2}(u)} - \frac{1}{A_{2}} \frac{\theta^{2}(u|u^{a_{2},a})}{\theta^{2}(u)}\right]^{2} - 4(a_{1} - a_{2})\eta}}{2(a_{1} - a_{2})}$$

$$(136)$$

where

$$\eta := a_2 \ a_1(a_1 - a_2) - \frac{a_2}{A_1} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)} + \frac{a_1}{A_2} \frac{\theta^2(u|u^{a_2,a})}{\theta^2(u)}$$
(137)

Also,
$$A_i = \pm \frac{1}{\sqrt{e^{\pi i PP'} f'(a_i)}}$$

The solution can be reexpressed in terms of generalized Weierstraß functions:

$$x_k^{(1,2)} = \frac{\wp_{2,2}(u) \pm \sqrt{\wp_{2,2}^2(u) + 4\wp_{2,1}(u)}}{2}, \quad k = 1, 2$$
 (138)

where

$$\wp_{2,2}(u) = \frac{(a_1 - a_2)(a_2 + a_1) - \frac{1}{A_1} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)} + \frac{1}{A_2} \frac{\theta^2(u|u^{a_2,a})}{\theta^2(u)}}{a_1 - a_2}$$
(139)

and

$$\wp_{2,1}(u) = \frac{-a_1 a_2 (a_1 - a_2) - \frac{a_1}{A_2} \frac{\theta^2 (u | u^{a_2, a})}{\theta^2 (u)} + \frac{a_2}{A_1} \frac{\theta^2 (u | u^{a_1, a})}{\theta^2 (u)}}{a_1 - a_2}$$
(140)

Thus, x_1, x_2 , that solve Jacobi's inversion problem Eq.(133), are solutions of a quadratic equation [44, 49]

$$Ux^2 - U'x + U'' = 0 (141)$$

where U, U', U'' are functions of u_1, u_2 . In the particular case that the coefficient of x^5 in the quintic polynomial is equal to 4, $U = 1, U' = \wp_{2,2}(u), U'' = \wp_{2,1}(u)$.

The matrix elements h_{ij} , Ω_{ij} can be explicitly written in terms of the half-periods $U_r^{x,a}$. For clarity, $U_2^{e_4,e_3} = \int_{e_3}^{e_4} x dx/y$, $U_1^{e_4,e_3} = \int_{e_3}^{e_4} dx/y$ etc. The roots have been arranged in ascending order of magnitude and are denoted by $e_{2g}, e_{2g-1}, \cdots, e_0, g=2$, so that e_{2i}, e_{2i-1} are respectively, c_{g-i+1}, a_{g-i+1} and e_0 is c. For instance, the matrix element $h_{11} = \frac{U_2^{e_4,e_3}}{2(U_1^{e_4,e_3}U_2^{e_2,e_1}U_1^{e_2,e_1}U_2^{e_4,e_3})} \times \pi i$, while $\Omega_{11} = \frac{U_1^{e_1,e_0}U_2^{e_4,e_3}-U_2^{e_1,e_0}U_1^{e_4,e_3}}{U_2^{e_2,e_1}U_1^{e_4,e_3}-U_1^{e_2,e_1}U_2^{e_4,e_3}}$.

A.1 A particular inversion problem of Jacobi

An indefinite elliptic integral of the third kind can be regarded as a special form of a hyperelliptic integral

$$\int_0^x \frac{(\alpha + \beta x)dx}{\sqrt{x(1-x)(1-\kappa^2 x)(1-\lambda^2 x)(1-\mu^2 x)}}$$
(142)

in the special case when two of the moduli are equal, e.g. $\lambda = \mu$, and therefore one can consider the following Jacobi's inversion problem

$$u = \int_{0}^{x_{1}} \frac{(\alpha + \beta x)dx}{(1 - \lambda^{2}x)\sqrt{x(1 - x)(1 - \kappa^{2}x)}} + \int_{0}^{x_{2}} \frac{(\alpha + \beta x)dx}{(1 - \lambda^{2}x)\sqrt{x(1 - x)(1 - \kappa^{2}x)}}$$

$$v = \int_{0}^{x_{1}} \frac{(\alpha' + \beta'x)dx}{(1 - \lambda^{2}x)\sqrt{x(1 - x)(1 - \kappa^{2}x)}} + \int_{0}^{x_{2}} \frac{(\alpha' + \beta'x)dx}{(1 - \lambda^{2}x)\sqrt{x(1 - x)(1 - \kappa^{2}x)}}$$
(143)

For a convenient choice of the constants $\alpha, \beta, \alpha', \beta'$ the solution of the above Jacobi's inversion problem can be expressed in terms of genus 1, Jacobi theta functions

$$\pm \kappa \sqrt{x_1 x_2} = \frac{\theta(a) e^{-v} \theta(u-a) - e^{v} \theta(u+a)}{\theta_1(a) e^{-v} \theta_1(u-a) + e^{v} \theta_1(u+a)}$$

$$\frac{\kappa}{\kappa'} \sqrt{(1-x_1)(1-x_2)} = \frac{\theta_3(a) e^{-v} \theta_2(u-a) - e^{v} \theta_2(u+a)}{\theta_1(a) e^{-v} \theta_1(u-a) + e^{v} \theta_1(u+a)}$$

$$\frac{1}{\kappa'} \sqrt{(1-\kappa^2 x_1)(1-\kappa^2 x_2)} = \frac{\theta_2(a) e^{-v} \theta_3(u-a) - e^{v} \theta_3(u+a)}{\theta_1(a) e^{-v} \theta_1(u-a) + e^{v} \theta_1(u+a)}$$
(144)

B Transformation theory of elliptic functions and modular equations

One of the applications supplied by the transformation theory of Elliptic functions, which is of great importance in Number theory, are the modular equations described below [32, 33, 35].

For a rational solution of the differential equation

$$\frac{dy}{\sqrt{(1-y^2)(1-e_1^2y^2)}} = C\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$$
(145)

the necessary conditions among the periods

$$K(e_1) = a_0 CK(e) + a_1 CiK'(e)$$

 $iK'(e_1) = b_0 CK(e) + b_1 CiK'(e)$
(146)

with the period ratios (moduli) of the associated modular theta functions being given by

$$\tau = \frac{b_0 + b_1 \tau'}{a_0 + a_1 \tau'} \tag{147}$$

are also sufficient, when

$$a_0b_1 - a_1b_0 = n (148)$$

is a positive integer number. The integer n is called the degree of transformation. Equation (148) for $a_0, b_1, a_1, b_0 \in \mathbb{Z}$ when viewed as the determinant of a matrix $\in GL(2, \mathbb{Z})$, sometimes is called a modular correspondence of level n.

It can be shown that the $inequivalent\ reduced\ forms\ of\ modular\ correspondences$

$$\left(\begin{array}{cc} a_0 & a_1 \\ b_0 & b_1 \end{array}\right),\,$$

are of the form

$$\left(\begin{array}{cc} q & 0 \\ 16\xi & q' \end{array}\right)$$

where q a positive part of n represents, $q':=\frac{n}{q}$, and $0\leq \xi\leq q'-1$. For instance for n=p a prime number, there are p+1 inequivalent reduced forms of the form 13

$$\left(\begin{array}{cc}1&0\\0&p\end{array}\right),\left(\begin{array}{cc}1&0\\16&p\end{array}\right),\left(\begin{array}{cc}1&0\\16.2&p\end{array}\right),\cdots\left(\begin{array}{cc}1&0\\16(p-1)&p\end{array}\right),\left(\begin{array}{cc}p&0\\0&1\end{array}\right)$$

Also the multiplication factor C in Eq.(145) is given by

$$C = \frac{1}{q} \frac{K(e_1)}{K(e)} \tag{149}$$

which for a degree of transformation that is a prime number (n = p) is equal to $\frac{K(e_1)}{K(e)}$ or $(1/n)\frac{K(e_1)}{K(e)}$. Equation (149) can be re-expressed in terms of Jacobi theta functions as follows:

$$C = \frac{1}{q} \frac{\vartheta_3^2(0,\tau)}{\vartheta_3^2(0,\frac{q\tau - 16\xi}{q'})}$$
 (150)

$$\alpha_i = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$
, $ad = n$, $(a, b, d) = 1$, $0 \le b < d$ and $a, b, d \in Z$

 $^{^{13}\}mathrm{In}$ a more familiar notation these classes of inequivalent reduced forms are

The modular equations are equations relating the Jacobi modulus $e(p\tau)$ to $e(\tau)$ which are of the form

$$F_p\left[\left(\frac{2}{p}\right)\sqrt[4]{e(\tau)}, \sqrt[4]{e\left(\frac{\tau - 16\xi}{p}\right)}\right] = 0$$
 (151)

where $\left(\frac{2}{n}\right)$ denotes the Legendre symbol ¹⁴. Equivalently, modular equations can be written in terms of the absolute modular invariant function $j(\tau)$, and relate the reduced absolute modular invariant j^* to j by polynomial equations of the form

$$\Phi_p(j^*, j) = 0 (152)$$

where $j^* := j.\alpha_i = j\left(\frac{a\tau+b}{d}\right)$ The explicit form of $\Phi_2(j^*,j) = 0$, has been given

Conditions for radii for spherical null geodesics with a cosmological constant and differential equations of Appell's function

The conditions from the vanishing of the polynomial R and its first derivative result in the equations which generalize (26)

$$\Phi = \frac{r^3 \Lambda^2 a^5 + Y a^3 + X a \pm \sqrt{3} \sqrt{f_1}}{r^3 \Lambda^2 a^4 + (2\Lambda^2 r^5 + 6\frac{GM}{c^2} \Lambda r^2 - 9r + 9\frac{GM}{c^2}) a^2 + r^4 \Lambda (\Lambda r^3 - 3r + 6\frac{GM}{c^2})}$$
(153)

where

$$f_1 := r^2 (Za^6 + r(3\frac{GM}{c^2}K + r(2\Lambda^3r^6 - 15\Lambda^2r^4 + 9\Lambda r^2 + 54))$$

$$+ r^2 (54(\Lambda r^2 + 2)(\frac{GM}{c^2})^2 + 12rK_2\frac{GM}{c^2} + r^2K_1)a^2$$

$$+ 3(3\frac{GM}{c^2} - r)r^6\Lambda(\Lambda r^3 - 3r + 6\frac{GM}{c^2}))$$

Indeed, for a vanishing cosmological constant it has the correct limit $\Phi \to$ $\frac{a^2+r^2}{a}$ derived in the previous subsection. Then the parameter $\mathcal Q$ is given by the expression

$$Q = \frac{\Xi^2 r^4 + \Xi^2 r^2 (a^2 - \Phi^2) + \frac{2GM}{c^2} r \Xi^2 (\Phi - a)^2 + \frac{\Lambda}{3} r^2 (r^2 + a^2) \Xi^2 (\Phi - a)^2}{\Delta_r}$$
(154)

$$14\left(\frac{2}{a_0}\right) = e^{\frac{a_0^2 - 1}{8}i\pi}.$$

where

$$X := \Lambda^{2}r^{7} + 6\frac{GM}{c^{2}}\Lambda r^{4} - 9\frac{GM}{c^{2}}r^{2}$$

$$Y := 2\Lambda^{2}r^{5} + \frac{GM}{c^{2}}(6\Lambda r^{2} + 9)$$

$$Z := \Lambda^{3}r^{6} - 6\Lambda^{2}r^{4} + 27$$

$$K := 5\Lambda^{2}r^{4} - 9\Lambda r^{2} - 36$$

$$K_{1} := \Lambda^{3}r^{6} - 12\Lambda^{2}r^{4} + 18\Lambda r^{2} + 27$$

$$K_{2} = 2\Lambda^{2}r^{4} - 6\Lambda r^{2} - 9$$
(155)

C.0.1 Proof of Identity Eq.(52)

$$\int_{0}^{1} \frac{z'dz'}{\sqrt{z'(1-\frac{z'}{\omega})}} \frac{1}{(1-\kappa_{+}^{\prime 2}z')} \frac{1}{\sqrt{1-\mu'^{2}z'}}$$

$$= \int_{0}^{1} \frac{dz'}{\sqrt{z'(1-\frac{z'}{\omega})}} \left[\frac{z'}{1-\kappa_{+}^{\prime 2}z'} \right] \frac{1}{\sqrt{1-\mu'^{2}z'}}$$

$$= \int_{0}^{1} \frac{dz'}{\sqrt{z'(1-\frac{z'}{\omega})}} \frac{-1}{\kappa_{+}^{\prime 2}} \left[1 - \frac{1}{1-\kappa_{+}^{\prime 2}z'} \right] \frac{1}{\sqrt{1-\mu'^{2}z'}}$$

$$= -\frac{1}{\kappa_{+}^{\prime 2}} \int_{0}^{1} \frac{dz'}{\sqrt{z'(1-\frac{z'}{\omega})}} \frac{1}{\sqrt{1-\mu'^{2}z'}} + \frac{1}{\kappa_{+}^{\prime 2}} \int_{0}^{1} \frac{dz'}{\sqrt{z'(1-\frac{z'}{\omega})}} \frac{1}{1-\kappa_{+}^{\prime 2}z'} \frac{1}{\sqrt{1-\mu'^{2}z'}}$$
(156)

Picard, had developed a theory for finding solutions of the system of differential equations that the Appell hypergeometric function obeys. More precisely he showed, by direct substitution, that solutions are provided by definite integrals of the form [30]

$$\int_{q}^{h} u^{b_1 - 1} (u - 1)^{b_2 - 1} (u - x)^{\mu - 1} (u - y)^{\lambda - 1} du \tag{157}$$

where g,h denotes two of the quantities $0,1,x,y,\infty$ and we have the correspondence

$$b_1 = 1 + \beta + \beta' - \gamma, b_2 = \gamma - \alpha, \mu = 1 - \beta, \lambda = 1 - \beta'$$
 (158)

This is the generalisation of Kummer's work who found that the standard hypergeometric equation has 24 solutions [31]. The system of linear differential

equations of Appell's function F_1 is

$$x(1-x)(x-y)\frac{\partial^{2}F_{1}}{\partial x^{2}} + \left[\gamma(x-y) - (\alpha+\beta+1)x^{2} + (\alpha+\beta-\beta'+1)xy + \beta'y\right]\frac{\partial F_{1}}{\partial x}$$

$$-\beta y(1-y)\frac{\partial F_{1}}{\partial y} - \alpha\beta(x-y)F_{1} = 0,$$

$$y(1-y)(y-x)\frac{\partial^{2}F_{1}}{\partial y^{2}} + \left[\gamma(y-x) - (\alpha+\beta'+1)y^{2} + (\alpha+\beta'-\beta+1)xy + \beta x\right]\frac{\partial F_{1}}{\partial y}$$

$$-\beta'x(1-x)\frac{\partial F_{1}}{\partial x} - \alpha\beta'(y-x)F_{1} = 0,$$

$$(x-y)\frac{\partial^{2}F_{1}}{\partial x\partial y} = \beta'\frac{\partial F_{1}}{\partial x} - \beta\frac{\partial F_{1}}{\partial y}$$

$$(159)$$

For instance the following integral is represented as follows

$$\int_{1}^{\infty} u^{\beta+\beta'-\gamma} (u-1)^{\gamma-\alpha-1} (u-x)^{-\beta} (u-y)^{-\beta'} du = B(1+\beta+\beta'-\gamma,\gamma-\alpha) x^{-\beta} y^{-\beta'} F_1 \left(1+\beta+\beta'-\gamma,\beta,\beta',1+\beta+\beta'-\alpha,\frac{1}{x},\frac{1}{y}\right)$$
(160)

and $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ denotes the beta function.

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