Review for Exam I

1. Larry claims that it is true that $\lim_{x \to \sqrt{7}} \lfloor x \rfloor = \lfloor \sqrt{7} \rfloor$, but Larry can't remember the justification for this calculation. Explain to Larry what function property justifies this calculation. Write your answer in sentence form.

Solution: Larry used *direct substitution* to evaluate this limit. Direct substitution is justified provided the function is *continuous* at the limit point. Here the limit point is $\sqrt{7}$. The floor function is continuous at $\sqrt{7}$, so direct substitution is justified.

2. Find the value of $\lim_{x \to \pi} (5\lfloor x \rfloor - \lfloor 5x \rfloor)$.

Solution: The floor function is continuous at π and at 5π. Thus

$$\lim_{x \to \pi} (5\lfloor x \rfloor - \lfloor 5x \rfloor) = 5\lfloor \pi \rfloor - \lfloor 5\pi \rfloor = 0.$$

Although you might guess that $5\lfloor x \rfloor - \lfloor 5x \rfloor = 0$ for all real x, that's rubbish. For example, $5\lfloor 3/2 \rfloor - \lfloor 5 \times 3/2 \rfloor = -2$. Actually, the graph of $y = 5\lfloor x \rfloor - \lfloor 5x \rfloor$ is a curious thing–you should look at it sometime.

3. Define a function $A(x) = x^2 |x|$. Use the definition of the derivative as a limit of a Newton quotient to find the value of $A'(0)$.

Solution:

$$\lim_{x \to 0} \frac{A(x) - A(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 |x| - 0}{x - 0} = \lim_{x \to 0} x|x| = 0.$$

The last step is justified in part by the fact that the absolute value function is continuous everywhere. That makes direct substitution OK.

4. Find the value of $\lim_{x \to 2^{(-)}} \lfloor x \rfloor$.

Solution: For x near, but to the left of two, $\lfloor x \rfloor$ simplifies to one. Thus

$$\lim_{x \to 2^{(-)}} \lfloor x \rfloor = \lim_{x \to 2^{(-)}} 1 = 1.$$

5. The *domain* of the natural exponential function is ________.

Solution: The *domain* of the natural exponential function is $\mathbf{R}$.

6. The *range* of the natural exponential function is ________.

Solution: The *range* of the natural exponential function is $(0, \infty)$.

7. The *domain* of the natural logarithm function is ________.

Solution: The *domain* of the natural logarithm function is $(0, \infty)$.

8. The *range* of the natural logarithm function is ________.

Solution: The *range* of the natural logarithm function is $(-\infty, \infty)$.

9. Find an equation of the tangent line (TL) to the curve $y = x(x-4)$. The point of tangency is $(x = 5, y = 5)$.

Solution: We need a point on the line and its slope. The point is given–it is $(x = 5, y = 5)$. To find the slope of the TL, we need to evaluate

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=5} = 2x - 4|_{x=5} = 6.$$

So an equation for the TL is $y - 5 = 6(x - 5)$. Since the problem asked for "an equation of the tangent line," you are free to present your answer in either point-slope form, intercept form, or general form. Isn't freedom of expression *wonderful?*

10. Find an equation of the tangent line (TL) to the curve $y = \mathrm{e}^x$. The point of tangency is $(x = 0, y = 1)$.

Solution: We need a point on the line and its slope. The point is given–it is $(x = 0, y = 1)$. To find the slope of the TL, we need to evaluate $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = \mathrm{e}^x\big|_{x=0} = 1$. So an equation for the TL is $y - 1 = x$

11. Find the *natural domain* of the function whose formula is $W(x) = \frac{5}{x} - \frac{x}{5}$.

Solution: There is only one denominator that can vanish; thus $\operatorname{dom}(W) = \{x | x \neq 0\}$.

12. Find the *natural domain* of the function whose formula is $Q(x) = \frac{5}{1-\frac{1}{x}}$.

Solution: There are two denominators–we need both of them to be nonzero. Thus

$$\left\{x \middle| 1 - \frac{1}{x} \neq 0 \text{ and } x \neq 0\right\} = \{x | x \neq 1 \text{ and } x \neq 0\}.$$

Remember that $\left\{x \middle| 1 - \frac{1}{x} \neq 0 \text{ and } x \neq 0\right\}$ is in *implicit form*, so it is **not** simplified and it is **unworthy** of earning full credit.

Also, the problem statement does not mandate the way to express the answer, so you have the **freedom** to present your answer in either set builder notation, interval notation, or pictorially. **It's all good!** In interval notation, the solution is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

Careful: If you "simplify" $\frac{5}{1-\frac{1}{x}}$ To $\frac{5x}{x-1}$, you will miss the fact that zero is not in the natural domain.

13. Find each derivative

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{107}\right]$

Solution: Ha! Don't get caught using the power rule! This is the derivative of a constant, so $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{107}\right] = 0$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[2x^2 + 31x + 107\right]$

Solution: We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2x^2 + 31x + 107\right] = 4x + 31.$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{2}x - \sqrt{2x}\right]$

Solution: We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{2}x - \sqrt{2x}\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{2}x - \sqrt{2}\sqrt{x}\right] = \sqrt{2} - \frac{\sqrt{2}}{2\sqrt{x}}.$$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left[(x-5)(x-7)\right]$

Solution: Via the product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x-5)(x-7)\right] = (x-5)'(x-7)+(x-5)(x-7)' = (x-7)+(x-5) = 2x-12.$$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x-1}{x}\right]$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x-1}{x}\right] = \frac{1}{x^2}.$$

(f) $\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+6)(x+8)\right]$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+6)(x+8)\right] = 2x+14.$$

(g) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x+6}{x+8}\right]$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x+6}{x+8}\right] = \frac{2}{(x+8)^2}.$$

(h) $\frac{\mathrm{d}}{\mathrm{d}x}\left[x\mathrm{e}^x\right]$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x\mathrm{e}^x\right] = \mathrm{e}^x + x\mathrm{e}^x.$$

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2+1}{x^2-1}\right]$

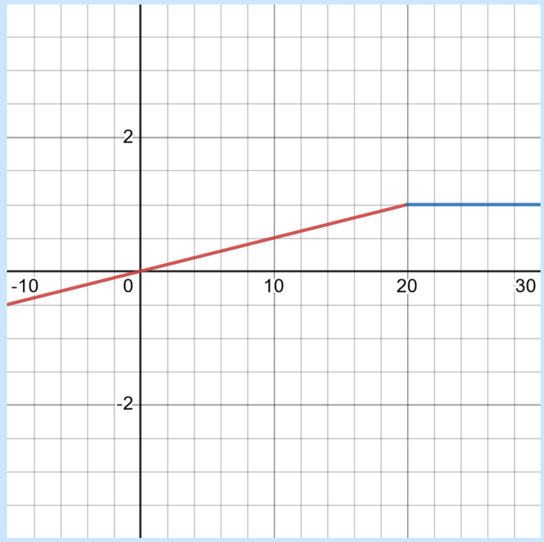
Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2+1}{x^2-1}\right] = -\frac{4x}{(x^2-1)^2}.$$

14. Sketch a graph of $y = \begin{cases} x/20 & x < 20 \\ 1 & x \geq 20 \end{cases}$.

Find a formula for $\frac{dy}{dx}$.

Solution: A graph is



To the left of twenty, the graph is a line with slope $1/20$; and to the right of twenty, the graph is a line with slope zero. At twenty, the graph has a "corner," (no TL), so apparently, the function is not differentiable at twenty. Accordingly, we have

$$\frac{dy}{dx} = \begin{cases} 1/20 & x < 20 \\ \text{dne} & x = 20 \\ 0 & x > 20 \end{cases}.$$

15. In the year 1969 at age 11, child actress Eve Plumb purchased a Malibu beach house for \$55,000. Forty-seven years later she sold it for \$3.9 million. Her annual percent yield r on this investment is given by the solution to

$$3,900,000 = 55,000 \times (1 + r)^{47}.$$

Find Eve Plumb's return on this investment. You will need to solve the given equation for r.

Solution: We have

$$\begin{aligned}
\left[3,900,000 = 55,000 \times (1+r)^{47}\right] &= \left[\frac{780}{11} = (1+r)^{47}\right], && \text{(divide by 55000)}\\
&= \left[\left(\frac{780}{11}\right)^{1/47} = (1+r)\right], && \text{(47th root)}\\
&= \left[r = \left(\frac{780}{11}\right)^{1/47} - 1\right],\\
&= [r \approx 9.49\%].
\end{aligned}$$

Since 1969, the APY for the S&P index is about 10.06%. I'd say it was a pretty good investment–of course a home, unlike an S&P index fund, requires upkeep; and taxes on homes is different from taxes on stock dividends and gains.

16. After graduation, suppose your starting salary is \$64,000. Further, suppose that you expect to earn a 3.5% pay rise each year you work. What is your salary for your 40^{th} year of work? **Hint:** Your salary for your 3^{rd} year of work is $\$64,000 \times 1.035^2$.

Solution: In your 40^{th} year of work, you will have earned 39 pay rises each of 3.5%. Rounded to the nearest penny, your salary for your 40^{th} year of work is

$$64,000 \times 1.035^{39} \approx 244,823.79.$$

17. Define $Q(x) = x^3 + 1$ and $\operatorname{dom}(Q) = (-\infty, \infty)$. Find the formula and the domain of Q^{-1}.

Solution: We need to solve $y = x^3 + 1, -\infty < x < \infty$ for x. The solution is $x = (y-1)^{1/3}$ and $-\infty < (y-1)^{1/3} < \infty$. Solving $-\infty < (y-1)^{1/3} < \infty$ gives $-\infty < y < \infty$. So $Q^{-1}(y) = (y-1)^{1/3}$ and $\operatorname{dom}(Q^{-1}) = \mathbf{R}$.

If you loath naming the independent variable y, I suggest that you get over it. Until then, the formulae $Q^{-1}(y) = (y-1)^{1/3}$ and $Q^{-1}(x) = (x-1)^{1/3}$ are semantically (but not syntactically) equivalent.

18. Find the *natural domain* of the function F whose formula is $F(x) = \frac{1}{5+\frac{1}{x}}$

Solution: There are two denominators; we need to require that both are nonzero; thus in implicit form, the domain is

$$\operatorname{dom}(F) = \left\{ x | (x \neq 0) \wedge (5 + \frac{1}{x} \neq 0) \right\}$$

Solving each inequation for x gives an explicit form; it is

$$\operatorname{dom}(F) = \left\{ x | (x \neq 0) \wedge (x \neq -\frac{1}{5}) \right\}.$$

In interval notation, this is

$$\operatorname{dom}(F) = (-\infty, -\frac{1}{5}) \cup (-\frac{1}{5}, 0) \cup (0, \infty).$$

19. Find the value of each limit:

(a) $\lim_{x \to 0} \frac{x|x|}{x}$.

Solution:

$$\lim_{x \to 0} \frac{x|x|}{x} = \lim_{x \to 0} |x| = 0.$$

(b) $\lim_{x \to 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$

Solution: We're looking at the limit from the *left* toward 1. That makes $x < 1$, so we can simplify $\begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$ to 3. Thus

$$\begin{aligned} \lim_{x \to 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} &= \lim_{x \to 1^{(-)}} 3, && \text{(simplification)} \\ &= 3. && \text{(limit of constant)} \end{aligned}$$

(c) $\lim_{x \to 1^{(+)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$

Solution: We're looking at the limit from the *right* toward 1. That allows us to simplify $\begin{cases} 3 & x < 1 \\ x & 1 \le x \end{cases}$ to x. Thus

$$\begin{aligned} \lim_{x \to 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \le x \end{cases} &= \lim_{x \to 1^{(-)}} x && \text{(simplification)} \\ &= 1 && \text{(limit of constant))} \end{aligned}$$

A few of you correctly simplified, but then failed to find the limit–something like

$$\lim_{x \to 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \le x \end{cases} = x.$$

OK–there is a pending limit to evaluate. Simplifying is just the first step, but we still need to evaluate the limit.

(d) $\lim_{x \to 1} \begin{cases} 3 & x < 1 \\ x & 1 \le x \end{cases}$

Solution: From parts 'a' and 'b', we have $\lim_{x \to 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \le x \end{cases} \ne \lim_{x \to 1^{(+)}} \begin{cases} 3 & x < 1 \\ x & 1 \le x \end{cases}$, so $\lim_{x \to 1} \begin{cases} 3 & x < 1 \\ x & 1 \le x \end{cases}$ does not exist (aka dne).

(e) $\lim_{x \to 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1)\sin(1/x) & 10 \le x \end{cases}$

Solution: The limit point is 1. For x near the limit point, we can simplify $\lim_{x \to 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1)\sin(1/x) & 10 \le x \end{cases}$ to 3. The ugly case of $10 \le x$ just "simplifies away" and causes us no trouble! Thus

$$\lim_{x \to 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1)\sin(1/x) & 10 \le x \end{cases} = \lim_{x \to 1} 3 = 3.$$

(f) $\lim_{x \to 5} \dfrac{\sqrt{x+2} - \sqrt{7}}{x - 5}$

Solution: Direct substitution is not an option. To start, let's do some tricky algebra:

$$\begin{aligned}\frac{\sqrt{x+2}-\sqrt{7}}{x-5} &= \frac{\sqrt{x+2}-\sqrt{7}}{x-5} \times \frac{\sqrt{x+2}+\sqrt{7}}{\sqrt{x+2}+\sqrt{7}}, \\ &= \frac{x+2-7}{(x-5)(\sqrt{x+2}+\sqrt{7})}, \\ &= \frac{1}{\sqrt{x+2}+\sqrt{7}}.\end{aligned}$$

Now the evaluating the limit, is DS:

$$\lim_{x\to 5} \frac{\sqrt{x+2}-\sqrt{7}}{x-5} = \lim_{x\to 5} \frac{1}{\sqrt{x+2}+\sqrt{7}} = \frac{1}{2\sqrt{7}} = \frac{\sqrt{7}}{14}$$

(g) $\displaystyle \lim_{x\to\pi} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi}$

Solution: This one is not that much different from the previous problem:

$$\lim_{x\to\pi} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi} = \lim_{x\to\pi} \frac{1}{\sqrt{x+\pi}+\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}}$$

(h) $\displaystyle \lim_{x\to 3} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi}$

Solution: Ha! This one isn't similar the previous–DS is just fine!

$$\lim_{x\to 3} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi} = \frac{\sqrt{3+\pi}-\sqrt{2\pi}}{3-\pi}.$$

(i) $\displaystyle \lim_{x\to\sqrt{107}} \frac{x}{|x|}$

Solution: Near $\sqrt{107}$, we can simplify $\frac{x}{|x|}$ to 1. So

$$\lim_{x\to\sqrt{107}} \frac{x}{|x|} = \lim_{x\to\sqrt{107}} 1 = 1.$$

(j) $\displaystyle \lim_{x\to-\sqrt{107}} \frac{x}{|x|}$

Solution: Near $-\sqrt{107}$, we can simplify $\frac{x}{|x|}$ to -1. So

$$\lim_{x\to\sqrt{107}} \frac{x}{|x|} = \lim_{x\to\sqrt{107}} (-1) = -1.$$

20. Find each of the following limits. Use the rules

Rule #0 (constant) $\lim_{x\to c}(a) = a$.

Rule #1 (linearity) $\lim_{x\to c}(aF(x) + bG(x)) = a\lim_{x\to c}(F(x)) + b\lim_{x\to c}(G(x))$.

Rule #2 (product) $\lim_{x\to c}(F(x)G(x)) = \lim_{x\to c}(F(x)) \times \lim_{x\to c}(G(x))$.

Rule #3 (quotient) Provided $\lim_{x\to c}(G(x)) \neq 0$, we have $\lim_{x\to c} \frac{F(x)}{G(x)} = \frac{\lim_{x\to c}(F(x))}{\lim_{x\to c}(G(x))}$.

Rule #4 (power) $\lim_{x\to c} F(x)^n = \left(\lim_{x\to c} F(x)\right)^n$.

Rule #5 (root) Provided $\left(\lim_{x\to c} F(x)\right)^{1/n}$ is real, $\lim_{x\to c} F(x)^{1/n} = \left(\lim_{x\to c} F(x)\right)^{1/n}$.

Rule #6 (polynomial) Provided F is a polynomial, we have $\lim_{x\to c} F(x) = F(c)$

Rule #7 (rational) Provided F is a rational function and $c \in \operatorname{dom}(F)$, we have $\lim_{x\to c} F(x) = F(c)$.

to justify each of your steps by referencing one of our rules numbered zero through seven.

(a) $\lim_{x\to\pi} \left(x^3 + x\right)$

Solution: Since $x^3 + x$ is a polynomial, we can use Rule 6; thus

$$\lim_{x\to\pi} \left(x^3 + x\right) = \pi^3 + \pi. \qquad \text{(Rule 6)}$$

Alternatively, we could first use Rule 1 (linearity) followed by Rule 4; thus

$$\begin{aligned} \lim_{x\to\pi} \left(x^3 + x\right) &= \lim_{x\to\pi}(x^3) + \lim_{x\to\pi}(x), && \text{(Rule 1)} \\ &= \pi^3 + \pi. && \text{(Rule 4, twice)} \end{aligned}$$

To apply Rule 4 to $\lim_{x\to\pi}(x)$, match to the algebraically equivalent $\lim_{x\to\pi}(x^1)$.

(b) $\lim_{x\to\sqrt{2}} \sqrt{x+1}$

Solution: We have

$$\begin{aligned}\lim_{x\to\sqrt{2}} \sqrt{x+1} &= \sqrt{\lim_{x\to\sqrt{2}} (x+1)}, && \text{(Rule 5)}\\ &= \sqrt{1+\sqrt{2}}. && \text{(Rule 6)}\end{aligned}$$

Since $x+1$ is a polynomial, using Rule 6 in the second step is OK.

(c) $\displaystyle\lim_{x\to\sqrt{2}} \frac{x+1}{x-1}$

Solution: We have

$$\begin{aligned}\lim_{x\to\sqrt{2}} \frac{x+1}{x-1} &= \frac{\sqrt{2}+1}{\sqrt{2}-1}, && \text{(Rule 7)}\\ &= 3+2\sqrt{2}. && \text{(simplification)}\end{aligned}$$

The simplification step is done using a multiply by one trick:

$$\begin{aligned}\frac{\sqrt{2}+1}{\sqrt{2}-1} &= \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}, && \text{(multiply by one)}\\ &= \frac{(\sqrt{2}+1)^2}{2+\sqrt{2}-\sqrt{2}-1}, && \text{(distribute denominator)}\\ &= \frac{(\sqrt{2}+1)^2}{1}, && \text{(collect like terms)}\\ &= 2+2\sqrt{2}+1, && \text{(divisor of one)}\\ &= 3+2\sqrt{2}. && \text{(collect like terms)}\end{aligned}$$

To earn full credit with our online homework system, generally removing radicals from denominator is required.