

Review for Exam I

1. Larry claims that it is true that $\lim_{x \rightarrow \sqrt{7}} \lfloor x \rfloor = \lfloor \sqrt{7} \rfloor$, but Larry can't remember the justification for this calculation. Explain to Larry what function property justifies this calculation. Write your answer in sentence form.

Solution: Larry used *direct substitution* to evaluate this limit. Direct substitution is justified provided the function is *continuous* at the limit point. Here the limit point is $\sqrt{7}$. The floor function is continuous at $\sqrt{7}$, so direct substitution is justified.

2. Find the value of $\lim_{x \rightarrow \pi} (5\lfloor x \rfloor - \lfloor 5x \rfloor)$.

Solution: The floor function is continuous at π and at 5π . Thus

$$\lim_{x \rightarrow \pi} (5\lfloor x \rfloor - \lfloor 5x \rfloor) = 5\lfloor \pi \rfloor - \lfloor 5\pi \rfloor = 0.$$

Although you might guess that $5\lfloor x \rfloor - \lfloor 5x \rfloor = 0$ for all real x , that's rubbish. For example, $5\lfloor 3/2 \rfloor - \lfloor 5 \times 3/2 \rfloor = -2$. Actually, the graph of $y = 5\lfloor x \rfloor - \lfloor 5x \rfloor$ is a curious thing—you should look at it sometime.

3. Define a function $A(x) = x^2|x|$. Use the definition of the derivative as a limit of a Newton quotient to find the value of $A'(0)$.

Solution:

$$\lim_{x \rightarrow 0} \frac{A(x) - A(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2|x| - 0}{x - 0} = \lim_{x \rightarrow 0} x|x| = 0.$$

The last step is justified in part by the fact that the absolute value function is continuous everywhere. That makes direct substitution OK.

4. Find the value of $\lim_{x \rightarrow 2^{(-)}} \lfloor x \rfloor$.

Solution: For x near, but to the left of two, $\lfloor x \rfloor$ simplifies to one. Thus

$$\lim_{x \rightarrow 2^{(-)}} \lfloor x \rfloor = \lim_{x \rightarrow 2^{(-)}} 1 = 1.$$

5. The *domain* of the natural exponential function is _____.

Solution: The *domain* of the natural exponential function is \mathbf{R} .

6. The *range* of the natural exponential function is _____.

Solution: The *range* of the natural exponential function is $(0, \infty)$.

7. The *domain* of the natural logarithm function is _____.

Solution: The *domain* of the natural logarithm function is $(0, \infty)$.

8. The *range* of the natural logarithm function is _____.

Solution: The *range* of the natural logarithm function is $(-\infty, \infty)$.

9. Find an equation of the tangent line (TL) to the curve $y = x(x - 4)$. The point of tangency is $(x = 5, y = 5)$.

Solution: We need a point on the line and its slope. The point is given—it is $(x = 5, y = 5)$. To find the slope of the TL, we need to evaluate

$$\left. \frac{dy}{dx} \right|_{x=5} = 2x - 4|_{x=5} = 6.$$

So an equation for the TL is $y - 5 = 6(x - 5)$. Since the problem asked for “an equation of the tangent line,” you are free to present your answer in either point-slope form, intercept form, or general form. Isn’t freedom of expression *wonderful*?

10. Find an equation of the tangent line (TL) to the curve $y = e^x$. The point of tangency is $(x = 0, y = 1)$.

Solution: We need a point on the line and its slope. The point is given—it is $(x = 0, y = 1)$. To find the slope of the TL, we need to evaluate $\left. \frac{dy}{dx} \right|_{x=0} = e^x|_{x=0} = 1$. So an equation for the TL is $y - 1 = x$

11. Find the *natural domain* of the function whose formula is $W(x) = \frac{5}{x} - \frac{x}{5}$.

Solution: There is only one denominator that can vanish; thus $\text{dom } W = \{x | x \neq 0\}$.

12. Find the *natural domain* of the function whose formula is $Q(x) = \frac{5}{1 - \frac{1}{x}}$.

Solution: There are two denominators—we need both of them to be nonzero. Thus

$$\left\{ x \mid 1 - \frac{1}{x} \neq 0 \text{ and } x \neq 0 \right\} = \{x | x \neq 1 \text{ and } x \neq 0\}.$$

Remember that $\{x | 1 - \frac{1}{x} \neq 0 \text{ and } x \neq 0\}$ is in *implicit form*, so it is **not** simplified and it is **unworthy** of earning full credit.

Also, the problem statement does not mandate the way to express the answer, so you have the **freedom** to present your answer in either set builder notation, interval notation, or pictorially. **It's all good!** In interval notation, the solution is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

Careful: If you “simplify” $\frac{5}{1 - \frac{1}{x}}$ To $\frac{5x}{x-1}$, you will miss the fact that zero is not in the natural domain.

13. Find each derivative

(a) $\frac{d}{dx} [\sqrt{107}]$

Solution: Ha! Don't get caught using the power rule! This is the derivative of a constant, so $\frac{d}{dx} [\sqrt{107}] = 0$.

(b) $\frac{d}{dx} [2x^2 + 31x + 107]$

Solution: We have

$$\frac{d}{dx} [2x^2 + 31x + 107] = 4x + 31.$$

(c) $\frac{d}{dx} [\sqrt{2}x - \sqrt{2x}]$

Solution: We have

$$\frac{d}{dx} [\sqrt{2}x - \sqrt{2x}] = \frac{d}{dx} [\sqrt{2}x - \sqrt{2}\sqrt{x}] = \sqrt{2} - \frac{\sqrt{2}}{2\sqrt{x}}.$$

(d) $\frac{d}{dx} [(x-5)(x-7)]$

Solution: Via the product rule

$$\frac{d}{dx} [(x-5)(x-7)] = (x-5)'(x-7) + (x-5)(x-7)' = (x-7) + (x-5) = 2x-12.$$

(e) $\frac{d}{dx} \left[\frac{x-1}{x} \right]$

Solution:

$$\frac{d}{dx} \left[\frac{x-1}{x} \right] = \frac{1}{x^2}.$$

(f) $\frac{d}{dx} [(x+6)(x+8)]$

Solution:

$$\frac{d}{dx} [(x+6)(x+8)] = 2x+14.$$

(g) $\frac{d}{dx} \left[\frac{x+6}{x+8} \right]$

Solution:

$$\frac{d}{dx} \left[\frac{x+6}{x+8} \right] = \frac{2}{(x+8)^2}.$$

(h) $\frac{d}{dx} [xe^x]$

Solution:

$$\frac{d}{dx} [xe^x] = e^x + xe^x.$$

(i) $\frac{d}{dx} \left[\frac{x^2+1}{x^2-1} \right]$

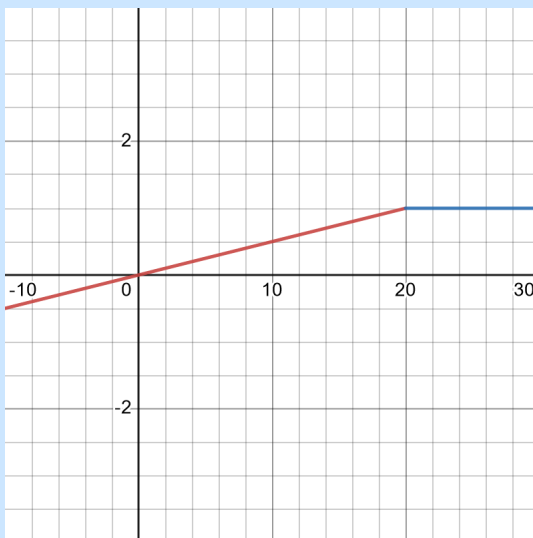
Solution:

$$\frac{d}{dx} \left[\frac{x^2+1}{x^2-1} \right] = -\frac{4x}{(x-1)^2(x+1)^2}.$$

14. Sketch a graph of $y = \begin{cases} x/20 & x < 20 \\ 1 & x \geq 20 \end{cases}$.

Find a formula for $\frac{dy}{dx}$.

Solution: A graph is



To the left of twenty, the graph is a line with slope $1/20$; and to the right of twenty, the graph is a line with slope zero. At twenty, the graph has a “corner,” (no TL), so apparently, the function is not differentiable at twenty. Accordingly, we have

$$\frac{dy}{dx} = \begin{cases} 1/20 & x < 20 \\ \text{dne} & x = 20 \\ 0 & x > 20 \end{cases}$$

15. In the year 1969 at age 11, child actress Eve Plumb purchased a Malibu beach house for \$55,000. Forty-seven years later she sold it for \$3.9 million. Her annual percent yield r on this investment is given by the solution to

$$3,900,000 = 55,000 \times (1 + r)^{47}.$$

Find Eve Plumb’s return on this investment. You will need to solve the given equation for r .

Solution: We have

$$\begin{aligned}
 [3,900,000 = 55,000 \times (1+r)^{47}] &= \left[\frac{780}{11} = (1+r)^{47} \right], && \text{(divide by 55000)} \\
 &= \left[\left(\frac{780}{11} \right)^{1/47} = (1+r) \right], && \text{(47th root)} \\
 &= \left[r = \left(\frac{780}{11} \right)^{1/47} - 1 \right], \\
 &= [r \approx 9.49\%].
 \end{aligned}$$

Since 1969, the APY for the S&P index is about 10.06%. I'd say it was a pretty good investment—of course, the home, unlike an S&P index fund requires upkeep and taxes on homes is different than taxes on stock dividends and gains.

16. After graduation, suppose your starting salary is \$64,000. Further, suppose that you expect to earn a 3.5% pay rise each year you work. What is your salary for your 40th year of work? **Hint:** Your salary for your 3rd year of work is $\$64,000 \times 1.035^2$.

Solution: In your 40th year of work, you will have earned 39 pay rises each of 4.1%. Rounded to the nearest penny, your salary for your 40th year of work is

$$46,000 \times 1.041^{39} = 220,463.29.$$

17. Define $Q(x) = x^3 + 1$ and $\text{dom}(Q) = (-\infty, \infty)$. Find the formula and the domain of Q^{-1} .

Solution: We need to solve $y = x^3 + 1$, $-\infty < x < \infty$ for x . The solution is $x = (y - 1)^{1/3}$, $-\infty < (y - 1)^{1/3} < \infty$. Solving $-\infty < (y - 1)^{1/3} < \infty$ gives $-\infty < y < \infty$. So $Q^{-1}(y) = (y - 1)^{1/3}$ and $\text{dom}(Q^{-1}) = \mathbf{R}$.

18. Find the *natural domain* of the function F whose formula is $F(x) = \frac{1}{5 + \frac{1}{x}}$

Solution: There are two denominators; we need to require that both are nonzero; thus in implicit form, the domain is

$$\text{dom}(F) = \left\{ x \mid (x \neq 0) \wedge \left(5 + \frac{1}{x} \neq 0 \right) \right\}$$

Solving each inequation for x gives an explicit form; it is

$$\text{dom}(F) = \left\{ x \mid (x \neq 0) \wedge (x \neq -\frac{1}{5}) \right\}$$

In interval notation, this is

$$\text{dom}(F) = (-\infty, -\frac{1}{5}) \cup (-\frac{1}{5}, 0) \cup (0, \infty).$$

19. Find the value of each limit:

(a) $\lim_{x \rightarrow 0} \frac{x|x|}{x}$.

Solution:

$$\lim_{x \rightarrow 0} \frac{x|x|}{x} = \lim_{x \rightarrow 0} |x| = 0.$$

(b) $\lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$

Solution: We're looking at the limit from the *left* toward 1. That makes $x < 1$, so we can simplify $\begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$ to 3. Thus

$$\begin{aligned} \lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} &= \lim_{x \rightarrow 1^{(-)}} 3 && \text{(simplification)} \\ &= 3 && \text{(limit of constant)} \end{aligned}$$

(c) $\lim_{x \rightarrow 1^{(+)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$

Solution: We're looking at the limit from the *right* toward 1. That allows us

to simplify $\begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$ to x . Thus

$$\begin{aligned} \lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} &= \lim_{x \rightarrow 1^{(-)}} x && \text{(simplification)} \\ &= 1 && \text{(limit of constant)} \end{aligned}$$

A few of you correctly simplified, but then failed to find the limit—something like

$$\lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} = x.$$

OK—there is a pending limit to evaluate. Simplifying is just the first step, but we still need to evaluate the limit.

$$(d) \lim_{x \rightarrow 1} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$$

Solution: From parts ‘a’ and ‘b’, we have $\lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} \neq \lim_{x \rightarrow 1^{(+)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$,

so $\lim_{x \rightarrow 1} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$ does not exist (aka dne).

$$(e) \lim_{x \rightarrow 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1) \sin(1/x) & 10 \leq x \end{cases}$$

Solution: The limit point is 1. For x near the limit point, we can simplify

$\lim_{x \rightarrow 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1) \sin(1/x) & 10 \leq x \end{cases}$ to 3. The ugly case of $10 \leq x$ just “simplifies away” and causes us no trouble! Thus

$$\lim_{x \rightarrow 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1) \sin(1/x) & 10 \leq x \end{cases} = \lim_{x \rightarrow 1} 3 = 3.$$

$$(f) \lim_{x \rightarrow 5} \frac{\sqrt{x+2} - \sqrt{7}}{x-5}$$

Solution: Direct substitution is not an option. To start, let's do some tricky algebra:

$$\begin{aligned}\frac{\sqrt{x+2}-\sqrt{7}}{x-5} &= \frac{\sqrt{x+2}-\sqrt{7}}{x-5} \times \frac{\sqrt{x+2}+\sqrt{7}}{\sqrt{x+2}+\sqrt{7}}, \\ &= \frac{x+2-7}{(x-5)(\sqrt{x+2}+\sqrt{7})}, \\ &= \frac{1}{\sqrt{x+2}+\sqrt{7}}.\end{aligned}$$

Now the evaluating the limit, is DS:

$$\lim_{x \rightarrow 5} \frac{\sqrt{x+2}-\sqrt{7}}{x-5} = \lim_{x \rightarrow 5} \frac{1}{\sqrt{x+2}+\sqrt{7}} = \frac{1}{2\sqrt{7}} = \frac{\sqrt{7}}{14}$$

(g) $\lim_{x \rightarrow \pi} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi}$

Solution: This one is not that much different from the previous problem:

$$\lim_{x \rightarrow \pi} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi} = \lim_{x \rightarrow \pi} \frac{1}{\sqrt{x+\pi}+\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}}$$

(h) $\lim_{x \rightarrow 3} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi}$

Solution: Ha! This one isn't similar the previous—DS is just fine!

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi} = \frac{\sqrt{3+\pi}-\sqrt{2\pi}}{3-\pi}.$$

(i) $\lim_{x \rightarrow \sqrt{107}} \frac{x}{|x|}$

Solution: Near $\sqrt{107}$, we can simplify $\frac{x}{|x|}$ to 1. So

$$\lim_{x \rightarrow \sqrt{107}} \frac{x}{|x|} = \lim_{x \rightarrow \sqrt{107}} 1 = 1.$$

(j) $\lim_{x \rightarrow -\sqrt{107}} \frac{x}{|x|}$

Solution: Near $-\sqrt{107}$, we can simplify $\frac{x}{|x|}$ to -1. So

$$\lim_{x \rightarrow -\sqrt{107}} \frac{x}{|x|} = \lim_{x \rightarrow -\sqrt{107}} (-1) = -1.$$

20. Find each of the following limits. Use the rules

Rule #0 (constant) $\lim_{x \rightarrow c} (a) = a.$

Rule #1 (linearity) $\lim_{x \rightarrow c} (aF(x) + bG(x)) = a \lim_{x \rightarrow c} (F(x)) + b \lim_{x \rightarrow c} (G(x)).$

Rule #2 (product) $\lim_{x \rightarrow c} (F(x)G(x)) = \lim_{x \rightarrow c} (F(x)) \times \lim_{x \rightarrow c} (G(x)).$

Rule #3 (quotient) Provided $\lim_{x \rightarrow c} (G(x)) \neq 0$, we have $\lim_{x \rightarrow c} \frac{F(x)}{G(x)} = \frac{\lim_{x \rightarrow c} (F(x))}{\lim_{x \rightarrow c} (G(x))}.$

Rule #4 (power) $\lim_{x \rightarrow c} F(x)^n = \left(\lim_{x \rightarrow c} F(x) \right)^n.$

Rule #5 (root) Provided $\left(\lim_{x \rightarrow c} F(x) \right)^{1/n}$ is real, $\lim_{x \rightarrow c} F(x)^{1/n} = \left(\lim_{x \rightarrow c} F(x) \right)^{1/n}.$

Rule #6 (polynomial) Provided F is a polynomial, we have $\lim_{x \rightarrow c} F(x) = F(c)$

Rule #7 (rational) Provided F is a rational function and $c \in \text{dom}(F)$, we have $\lim_{x \rightarrow c} F(x) = F(c).$

to justify each of your steps by referencing one of our rules numbered zero through seven.

(a) $\lim_{x \rightarrow \pi} (x^3 + x)$

Solution: Since $x^3 + x$ is a polynomial, we can use Rule 6; thus

$$\lim_{x \rightarrow \pi} (x^3 + x) = \pi^3 + \pi. \quad (\text{Rule 6})$$

Alternatively, we could first use Rule 1 (linearity) followed by Rule 4; thus

$$\begin{aligned} \lim_{x \rightarrow \pi} (x^3 + x) &= \lim_{x \rightarrow \pi} (x^3) + \lim_{x \rightarrow \pi} (x), & (\text{Rule 1}) \\ &= \pi^3 + \pi. & (\text{Rule 4, twice}) \end{aligned}$$

To apply Rule 4 to $\lim_{x \rightarrow \pi} (x)$, match to the algebraically equivalent $\lim_{x \rightarrow \pi} (x^1).$

(b) $\lim_{x \rightarrow \sqrt{2}} \sqrt{x+1}$

Solution: We have

$$\lim_{x \rightarrow \sqrt{2}} \sqrt{x+1} = \sqrt{\lim_{x \rightarrow \sqrt{2}} (x+1)}, \quad (\text{Rule 5})$$

$$= \sqrt{1 + \sqrt{2}}. \quad (\text{Rule 6})$$

Since $x + 1$ is a polynomial, using Rule 6 in the second step is OK.

(c) $\lim_{x \rightarrow \sqrt{2}} \frac{x+1}{x-1}$

Solution: We have

$$\lim_{x \rightarrow \sqrt{2}} \frac{x+1}{x-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1}, \quad (\text{Rule 7})$$

$$= 3 + 2\sqrt{2}. \quad (\text{simplification})$$

The simplification step is done using a multiply by one trick:

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}, \quad (\text{multiply by one})$$

$$= \frac{(\sqrt{2}+1)^2}{2 + \sqrt{2} - \sqrt{2} - 1}, \quad (\text{distribute denominator})$$

$$= \frac{(\sqrt{2}+1)^2}{1}, \quad (\text{collect like terms})$$

$$= 2 + 2\sqrt{2} + 1, \quad (\text{divisor of one})$$

$$= 3 + 2\sqrt{2}. \quad (\text{collect like terms})$$

To earn full credit with our online homework system, generally removing radicals from denominator is required.