In class work 12 has questions 1 through 2 with a total of 10 points. Turn in your work at the end of class *on paper*. This assignment is due *Wednesday 9 November 13:15* P M.

5 1. Show that among all rectangles with a given perimeter, a square has the greatest area.

   To do this, let the lengths of the two perpendicular sides of the rectangle be x and y and let L be the perimeter of the rectangle. That makes $L = 2x + 2y$ a constraint. The other constraints are $0 \le x$ and $0 \le y$. We have $A = xy$. Your task is to maximize A subject to the constraints $L = 2x + 2y, 0 \le x$, and $0 \le y$ with L given.

   Solution: To start, we need to solve the constraint $L = 2x + 2y$ for either x or y. Since interchanging x and y doesn't change the constraint, it hardly matters if we solve for x or y. So let's solve for y. The solution is $y = \frac{L-2x}{2}$. Pasting this solution into $0 \le x$ and $0 \le y$ yields

   $$\left[0 \le x \text{ and } 0 \le \frac{L-2x}{2} \right] = \left[0 \le x \text{ and } x \le \frac{L}{2} \right].$$

   The constraint $x \le L/2$ isn't mysterious–if $x = L/2$, then $y = 0$. And making x bigger than $L/2$ makes y negative.

   Pasting $y = \frac{L-2x}{2}$ into $A = xy$ gives $A = x(\frac{L-2x}{2})$. The graph of A as a function of x is a downward facing parabola that intersects the x-axis at 0 and at $L/2$. The x-coordinate of the vertex of the parabola is halfway between the x-intercepts; thus the x-coordinate of the vertex is $x = L/4$. Using $y = \frac{L-2x}{2}$, the y-coordinate of the vertex is $y = L/4$.

   So to maximize the area of the rectangle, we need $x = L/4$ and $y = L/4$. And that's a square.

5 2. Show that among all isosceles triangles with a given perimeter, the equilateral triangle has the greatest area.

   To do this, let the lengths of the sides of the triangle be x, x, and y and let L be the perimeter. That makes $L = 2x + y$ a constraint. The other constraints are $0 \le x$ and $0 \le y$. The area A of the triangle is (this is a specialization of the wonderful formula for the area of a triangle that is due to Hero of Alexandria 10 AD – c. 70 AD)

   $$16A^2 = 4x^2 y^2 - y^4.$$

   Solve the constraint $L = 2x + y$ for y and paste that result in the formula for $16A^2$. Now do some calculus. **Hint:** Maximizing $16A^2$ also maximizes A. Thus alternatively, maximize

the value of Q where
$$Q = 4x^2 y^2 - y^4.$$

You don't have to use this hint, but it's the easy way, I think.   If the algebra seems daunting to you, set $L = 3$ and work that specialization.

Solution: Let's begin by solving the constraint $L = 2x + y$ for y; thus $y = L - 2x$. But we also have $0 \le x$ and $0 \le y$. But $y = L - 2x$, so $0 \le L - 2x$. Put together, these inequalities tell us that $0 \le x \le L/2$.

But there is another inequality constraint that's subtle. Given any two sides of a triangle, the sum of these lengths must be greater than the length of the other side. Thus in addition to $0 \le x$ and $0 \le y$, we need

$$[y \le 2x \text{ and } x \le x + y] = [L - 2x \le 2x \text{ and } 0 \le L - 2x],$$
$$= [L/4 \le x \text{ and } x \le L/2].$$

So we're looking at an optimization on the closed interval $[L/4, L/2]$.

Now paste $y = L - 2x$ into Q. We have

$$Q = 4x^2 y^2 - y^4,$$
$$= 4(L - 2x)^2 x^2 - (L - 2x)^4,$$

We could either expand, factor, or LIB. We need to find the derivative of Q, so I think that LIB is a bad option. Expanding is tempting, but I see some opportunity to factor–let's factor.

$$= L(2x - L)^2 (4x - L).$$

This shows that Q vanishes (and thus A vanishes as well) when either $x = L/2$ or when $x = L/4$. Thus, at each endpoint, the area is zero.

Now find the derivative of Q. We have

$$\frac{dQ}{dx} = 4L(2x - L)^2 + 4L(2x - L)(4x - L),$$

Again, we have a choice between LIB, factor, or expand. Let's try the road less traveled and factor.

$$= 8L(2x - L)(3x - L).$$

Now solve $\dfrac{dQ}{dx} = 0$. We have

$$[8L(2x - L)(3x - L) = 0] = \left[x = \frac{L}{2}, x = \frac{L}{3}\right]. \tag{1}$$

So there are two CNs. Wait! We need to check: are the CNs in the interval $[L/4, L/2]$? Yes, they are–the CN $x = \frac{L}{2}$ is also an endpoint.

To find the maximum of Q, we need the chart:

CN	Q
$L/4$	0
$L/3$	$L^4/27$
$L/2$	0

So the maximum area happens when $x = L/3$. And that makes $y = L/3$. So the three side lengths are the same.

There is a wonderful algorithm for solving systems of linear inequalities–it is called Fourier elimination. The Maxima Computer Algebra system has implementation of this method:

(% i1)   load(fourier_elim)\$

(% i2)   fourier_elim([0 < x, 0 < y, y < 2*x, x < x + y, 2*x+y = L],[y,x]);

$$[y = L - 2x, \frac{L}{4} < x, x < \frac{L}{2}, 4L > 0] \qquad \text{(\% o2)}$$

The time complexity of Fourier elimination is high. Although the Simplex Method is far faster, it doesn't fully solve linear inequations.

For information about the Maxima CAS, see `https://maxima.sourceforge.io/`. I'm one of about two dozen developers for Maxima–for a map of the developers, see `https://maxima.sourceforge.io/project.html`