

## Review for Exam I

1. Larry claims that it is true that  $\lim_{x \rightarrow \sqrt{7}} \lfloor x \rfloor = \lfloor \sqrt{7} \rfloor$ , but Larry can't remember the justification for this calculation. Explain to Larry what function property justifies this calculation. Write your answer in sentence form.

**Solution:** Larry used *direct substitution* to evaluate this limit. Direct substitution is justified provided the function is *continuous* at the limit point. Here the limit point is  $\sqrt{7}$ . The floor function is continuous at  $\sqrt{7}$ , so direct substitution is justified.

2. Find the value of  $\lim_{x \rightarrow \pi} (5\lfloor x \rfloor - \lfloor 5x \rfloor)$ .

**Solution:** The floor function is continuous at  $\pi$  and at  $5\pi$ . Thus

$$\lim_{x \rightarrow \pi} (5\lfloor x \rfloor - \lfloor 5x \rfloor) = 5\lfloor \pi \rfloor - \lfloor 5\pi \rfloor = 0.$$

Although you might guess that  $5\lfloor x \rfloor - \lfloor 5x \rfloor = 0$  for all real  $x$ , that's rubbish. For example,  $5\lfloor 3/2 \rfloor - \lfloor 5 \times 3/2 \rfloor = -2$ . Actually, the graph of  $y = 5\lfloor x \rfloor - \lfloor 5x \rfloor$  is a curious thing—you should look at it sometime.

3. Define a function  $A(x) = x^2|x|$ . Use the definition of the derivative as a limit of a Newton quotient to find the value of  $A'(0)$ .

**Solution:**

$$\lim_{x \rightarrow 0} \frac{A(x) - A(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2|x| - 0}{x - 0} = \lim_{x \rightarrow 0} x|x| = 0.$$

The last step is justified in part by the fact that the absolute value function is continuous everywhere. That makes direct substitution OK.

4. Find the value of  $\lim_{x \rightarrow 2^{(-)}} \lfloor x \rfloor$ .

**Solution:** For  $x$  near, but to the left of two,  $\lfloor x \rfloor$  simplifies to one. Thus

$$\lim_{x \rightarrow 2^{(-)}} \lfloor x \rfloor = \lim_{x \rightarrow 2^{(-)}} 1 = 1.$$

5. The *domain* of the natural exponential function is \_\_\_\_\_.

**Solution:** The *domain* of the natural exponential function is  $\mathbf{R}$ .

6. The *range* of the natural exponential function is \_\_\_\_\_.

**Solution:** The *range* of the natural exponential function is  $(0, \infty)$ .

7. The *domain* of the natural logarithm function is \_\_\_\_\_.

**Solution:** The *domain* of the natural logarithm function is  $(0, \infty)$ .

8. The *range* of the natural logarithm function is \_\_\_\_\_.

**Solution:** The *range* of the natural logarithm function is  $(-\infty, \infty)$ .

9. Find an equation of the tangent line (TL) to the curve  $y = x(x - 4)$ . The point of tangency is  $(x = 5, y = 5)$ .

**Solution:** We need a point on the line and its slope. The point is given—it is  $(x = 5, y = 5)$ . To find the slope of the TL, we need to evaluate

$$\left. \frac{dy}{dx} \right|_{x=5} = 2x - 4|_{x=5} = 6.$$

So an equation for the TL is  $y - 5 = 6(x - 5)$ . Since the problem asked for “an equation of the tangent line,” you are free to present your answer in either point-slope form, intercept form, or general form. Isn’t freedom of expression *wonderful*?

10. Find an equation of the tangent line (TL) to the curve  $y = e^x$ . The point of tangency is  $(x = 0, y = 1)$ .

**Solution:** We need a point on the line and its slope. The point is given—it is  $(x = 0, y = 1)$ . To find the slope of the TL, we need to evaluate  $\left. \frac{dy}{dx} \right|_{x=0} = e^x|_{x=0} = 1$ . So an equation for the TL is  $y - 1 = x$

11. Find the *natural domain* of the function whose formula is  $W(x) = \frac{5}{x} - \frac{x}{5}$ .

**Solution:** There is only one denominator that can vanish; thus  $\text{dom } W = \{x | x \neq 0\}$ .

12. Find the *natural domain* of the function whose formula is  $Q(x) = \frac{5}{1 - \frac{1}{x}}$ .

**Solution:** There are two denominators—we need both of them to be nonzero. Thus

$$\left\{ x \mid 1 - \frac{1}{x} \neq 0 \text{ and } x \neq 0 \right\} = \{x | x \neq 1 \text{ and } x \neq 0\}.$$

Remember that  $\{x | 1 - \frac{1}{x} \neq 0 \text{ and } x \neq 0\}$  is in *implicit form*, so it is **not** simplified and it is **unworthy** of earning full credit.

Also, the problem statement does not mandate the way to express the answer, so you have the **freedom** to present your answer in either set builder notation, interval notation, or pictorially. **It's all good!** In interval notation, the solution is  $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ .

**Careful:** If you “simplify”  $\frac{5}{1 - \frac{1}{x}}$  To  $\frac{5x}{x-1}$ , you will miss the fact that zero is not in the natural domain.

13. Find each derivative

(a)  $\frac{d}{dx} [\sqrt{107}]$

**Solution:** Ha! Don't get caught using the power rule! This is the derivative of a constant, so  $\frac{d}{dx} [\sqrt{107}] = 0$ .

(b)  $\frac{d}{dx} [2x^2 + 31x + 107]$

**Solution:** We have

$$\frac{d}{dx} [2x^2 + 31x + 107] = 4x + 31.$$

(c)  $\frac{d}{dx} [\sqrt{2}x - \sqrt{2x}]$

**Solution:** We have

$$\frac{d}{dx} [\sqrt{2}x - \sqrt{2x}] = \frac{d}{dx} [\sqrt{2}x - \sqrt{2}\sqrt{x}] = \sqrt{2} - \frac{\sqrt{2}}{2\sqrt{x}}.$$

(d)  $\frac{d}{dx} [(x-5)(x-7)]$

**Solution:** Via the product rule

$$\frac{d}{dx} [(x-5)(x-7)] = (x-5)'(x-7) + (x-5)(x-7)' = (x-7) + (x-5) = 2x-12.$$

(e)  $\frac{d}{dx} \left[ \frac{x-1}{x} \right]$

**Solution:**

$$\frac{d}{dx} \left[ \frac{x-1}{x} \right] = \frac{1}{x^2}.$$

(f)  $\frac{d}{dx} [(x+6)(x+8)]$

**Solution:**

$$\frac{d}{dx} [(x+6)(x+8)] = 2x+14.$$

(g)  $\frac{d}{dx} \left[ \frac{x+6}{x+8} \right]$

**Solution:**

$$\frac{d}{dx} \left[ \frac{x+6}{x+8} \right] = \frac{2}{(x+8)^2}.$$

(h)  $\frac{d}{dx} [xe^x]$

**Solution:**

$$\frac{d}{dx} [xe^x] = e^x + xe^x.$$

(i)  $\frac{d}{dx} \left[ \frac{x^2+1}{x^2-1} \right]$

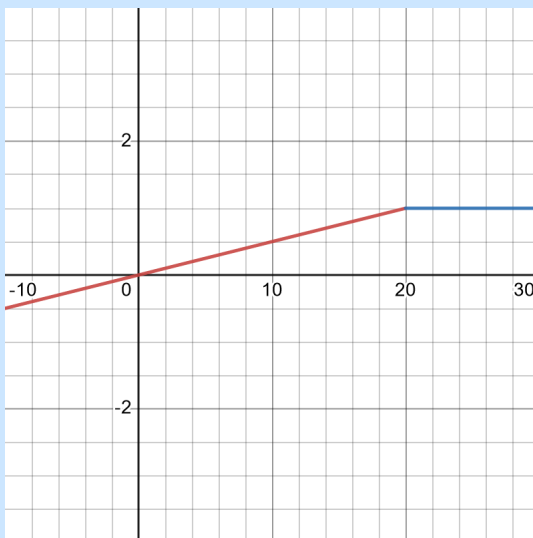
**Solution:**

$$\frac{d}{dx} \left[ \frac{x^2+1}{x^2-1} \right] = -\frac{4x}{(x-1)^2(x+1)^2}.$$

14. Sketch a graph of  $y = \begin{cases} x/20 & x < 20 \\ 1 & x \geq 20 \end{cases}$ .

Find a formula for  $\frac{dy}{dx}$ .

**Solution:** A graph is



To the left of twenty, the graph is a line with slope  $1/20$ ; and to the right of twenty, the graph is a line with slope zero. At twenty, the graph has a “corner,” (no TL), so apparently, the function is not differentiable at twenty. Accordingly, we have

$$\frac{dy}{dx} = \begin{cases} 1/20 & x < 20 \\ \text{dne} & x = 20 \\ 0 & x > 20 \end{cases}$$

15. In the year 1969 at age 11, child actress Eve Plumb purchased a Malibu beach house for \$55,000. Forty-seven years later she sold it for \$3.9 million. Her annual percent yield  $r$  on this investment is given by the solution to

$$3,900,000 = 55,000 \times (1 + r)^{47}.$$

Find Eve Plumb’s return on this investment. You will need to solve the given equation for  $r$ .

**Solution:** We have

$$\begin{aligned} [3,900,000 = 55,000 \times (1+r)^{47}] &= \left[ \frac{780}{11} = (1+r)^{47} \right], && \text{(divide by 55000)} \\ &= \left[ \left( \frac{780}{11} \right)^{1/47} = (1+r) \right], && \text{(47th root)} \\ &= \left[ r = \left( \frac{780}{11} \right)^{1/47} - 1 \right], \\ &= [r \approx 9.49\%]. \end{aligned}$$

Since 1969, the APY for the S&P index is about 10.06%. I'd say it was a pretty good investment—of course, the home, unlike an S&P index fund requires upkeep and taxes on homes is different than taxes on stock dividends and gains.

16. After graduation, suppose your starting salary is \$64,000. Further, suppose that you expect to earn a 3.5% pay rise each year you work. What is your salary for your 40<sup>th</sup> year of work? **Hint:** Your salary for your 3<sup>rd</sup> year of work is  $\$64,000 \times 1.035^2$ .

**Solution:** In your 40<sup>th</sup> year of work, you will have earned 39 pay rises each of 3.5%. Rounded to the nearest penny, your salary for your 40<sup>th</sup> year of work is

$$64,000 \times 1.035^{39} \approx 244,823.79.$$

17. Define  $Q(x) = x^3 + 1$  and  $\text{dom}(Q) = (-\infty, \infty)$ . Find the formula and the domain of  $Q^{-1}$ .

**Solution:** We need to solve  $y = x^3 + 1, -\infty < x < \infty$  for  $x$ . The solution is  $x = (y-1)^{1/3}, -\infty < (y-1)^{1/3} < \infty$ . Solving  $-\infty < (y-1)^{1/3} < \infty$  gives  $-\infty < y < \infty$ . So  $Q^{-1}(y) = (y-1)^{1/3}$  and  $\text{dom}(Q^{-1}) = \mathbf{R}$ .

If you loath naming the independent variable  $y$ , I suggest that you get over it. Until then, the formulae  $Q^{-1}(y) = (y-1)^{1/3}$  and  $Q^{-1}(x) = (x-1)^{1/3}$  are semantically (but not syntactically) equivalent.

18. Find the *natural domain* of the function  $F$  whose formula is  $F(x) = \frac{1}{5+\frac{1}{x}}$

**Solution:** There are two denominators; we need to require that both are nonzero; thus in implicit form, the domain is

$$\text{dom}(F) = \left\{ x \mid (x \neq 0) \wedge \left(5 + \frac{1}{x} \neq 0\right) \right\}$$

Solving each inequation for  $x$  gives an explicit form; it is

$$\text{dom}(F) = \left\{ x \mid (x \neq 0) \wedge \left(x \neq -\frac{1}{5}\right) \right\}.$$

In interval notation, this is

$$\text{dom}(F) = (-\infty, -\frac{1}{5}) \cup (-\frac{1}{5}, 0) \cup (0, \infty).$$

19. Find the value of each limit:

(a)  $\lim_{x \rightarrow 0} \frac{x|x|}{x}.$

**Solution:**

$$\lim_{x \rightarrow 0} \frac{x|x|}{x} = \lim_{x \rightarrow 0} |x| = 0.$$

(b)  $\lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$

**Solution:** We're looking at the limit from the *left* toward 1. That makes  $x < 1$ ,

so we can simplify  $\begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$  to 3. Thus

$$\begin{aligned} \lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} &= \lim_{x \rightarrow 1^{(-)}} 3 && \text{(simplification)} \\ &= 3 && \text{(limit of constant)} \end{aligned}$$

(c)  $\lim_{x \rightarrow 1^{(+)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$

**Solution:** We're looking at the limit from the *right* toward 1. That allows us to simplify  $\begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$  to  $x$ . Thus

$$\begin{aligned} \lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} &= \lim_{x \rightarrow 1^{(-)}} x && \text{(simplification)} \\ &= 1 && \text{(limit of constant)} \end{aligned}$$

A few of you correctly simplified, but then failed to find the limit—something like

$$\lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} = x.$$

OK—there is a pending limit to evaluate. Simplifying is just the first step, but we still need to evaluate the limit.

$$(d) \lim_{x \rightarrow 1} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$$

**Solution:** From parts 'a' and 'b', we have  $\lim_{x \rightarrow 1^{(-)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases} \neq \lim_{x \rightarrow 1^{(+)}} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$ ,  
so  $\lim_{x \rightarrow 1} \begin{cases} 3 & x < 1 \\ x & 1 \leq x \end{cases}$  does not exist (aka dne).

$$(e) \lim_{x \rightarrow 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1) \sin(1/x) & 10 \leq x \end{cases}$$

**Solution:** The limit point is 1. For  $x$  near the limit point, we can simplify  $\begin{cases} 3 & x < 10 \\ \ln(x^x + 1) \sin(1/x) & 10 \leq x \end{cases}$  to 3. The ugly case of  $10 \leq x$  just “simplifies away” and causes us no trouble! Thus

$$\lim_{x \rightarrow 1} \begin{cases} 3 & x < 10 \\ \ln(x^x + 1) \sin(1/x) & 10 \leq x \end{cases} = \lim_{x \rightarrow 1} 3 = 3.$$

$$(f) \lim_{x \rightarrow 5} \frac{\sqrt{x+2} - \sqrt{7}}{x-5}$$



**Solution:** Direct substitution is not an option. To start, let's do some tricky algebra:

$$\begin{aligned}\frac{\sqrt{x+2}-\sqrt{7}}{x-5} &= \frac{\sqrt{x+2}-\sqrt{7}}{x-5} \times \frac{\sqrt{x+2}+\sqrt{7}}{\sqrt{x+2}+\sqrt{7}}, \\ &= \frac{x+2-7}{(x-5)(\sqrt{x+2}+\sqrt{7})}, \\ &= \frac{1}{\sqrt{x+2}+\sqrt{7}}.\end{aligned}$$

Now the evaluating the limit, is DS:

$$\lim_{x \rightarrow 5} \frac{\sqrt{x+2}-\sqrt{7}}{x-5} = \lim_{x \rightarrow 5} \frac{1}{\sqrt{x+2}+\sqrt{7}} = \frac{1}{2\sqrt{7}} = \frac{\sqrt{7}}{14}$$

(g)  $\lim_{x \rightarrow \pi} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi}$

**Solution:** This one is not that much different from the previous problem:

$$\lim_{x \rightarrow \pi} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi} = \lim_{x \rightarrow \pi} \frac{1}{\sqrt{x+\pi}+\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}}$$

(h)  $\lim_{x \rightarrow 3} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi}$

**Solution:** Ha! This one isn't similar the previous—DS is just fine!

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+\pi}-\sqrt{2\pi}}{x-\pi} = \frac{\sqrt{3+\pi}-\sqrt{2\pi}}{3-\pi}.$$

(i)  $\lim_{x \rightarrow \sqrt{107}} \frac{x}{|x|}$

**Solution:** Near  $\sqrt{107}$ , we can simplify  $\frac{x}{|x|}$  to 1. So

$$\lim_{x \rightarrow \sqrt{107}} \frac{x}{|x|} = \lim_{x \rightarrow \sqrt{107}} 1 = 1.$$

(j)  $\lim_{x \rightarrow -\sqrt{107}} \frac{x}{|x|}$

**Solution:** Near  $-\sqrt{107}$ , we can simplify  $\frac{x}{|x|}$  to -1. So

$$\lim_{x \rightarrow -\sqrt{107}} \frac{x}{|x|} = \lim_{x \rightarrow -\sqrt{107}} (-1) = -1.$$

20. Find each of the following limits. Use the rules

**Rule #0 (constant)**  $\lim_{x \rightarrow c} (a) = a.$

**Rule #1 (linearity)**  $\lim_{x \rightarrow c} (aF(x) + bG(x)) = a \lim_{x \rightarrow c} (F(x)) + b \lim_{x \rightarrow c} (G(x)).$

**Rule #2 (product)**  $\lim_{x \rightarrow c} (F(x)G(x)) = \lim_{x \rightarrow c} (F(x)) \times \lim_{x \rightarrow c} (G(x)).$

**Rule #3 (quotient)** Provided  $\lim_{x \rightarrow c} (G(x)) \neq 0$ , we have  $\lim_{x \rightarrow c} \frac{F(x)}{G(x)} = \frac{\lim_{x \rightarrow c} (F(x))}{\lim_{x \rightarrow c} (G(x))}.$

**Rule #4 (power)**  $\lim_{x \rightarrow c} F(x)^n = \left( \lim_{x \rightarrow c} F(x) \right)^n.$

**Rule #5 (root)** Provided  $\left( \lim_{x \rightarrow c} F(x) \right)^{1/n}$  is real,  $\lim_{x \rightarrow c} F(x)^{1/n} = \left( \lim_{x \rightarrow c} F(x) \right)^{1/n}.$

**Rule #6 (polynomial)** Provided  $F$  is a polynomial, we have  $\lim_{x \rightarrow c} F(x) = F(c)$

**Rule #7 (rational)** Provided  $F$  is a rational function and  $c \in \text{dom}(F)$ , we have  $\lim_{x \rightarrow c} F(x) = F(c).$

to justify each of your steps by referencing one of our rules numbered zero through seven.

(a)  $\lim_{x \rightarrow \pi} (x^3 + x)$

**Solution:** Since  $x^3 + x$  is a polynomial, we can use Rule 6; thus

$$\lim_{x \rightarrow \pi} (x^3 + x) = \pi^3 + \pi. \quad (\text{Rule 6})$$

Alternatively, we could first use Rule 1 (linearity) followed by Rule 4; thus

$$\begin{aligned} \lim_{x \rightarrow \pi} (x^3 + x) &= \lim_{x \rightarrow \pi} (x^3) + \lim_{x \rightarrow \pi} (x), & (\text{Rule 1}) \\ &= \pi^3 + \pi. & (\text{Rule 4, twice}) \end{aligned}$$

To apply Rule 4 to  $\lim_{x \rightarrow \pi} (x)$ , match to the algebraically equivalent  $\lim_{x \rightarrow \pi} (x^1).$

(b)  $\lim_{x \rightarrow \sqrt{2}} \sqrt{x+1}$

**Solution:** We have

$$\lim_{x \rightarrow \sqrt{2}} \sqrt{x+1} = \sqrt{\lim_{x \rightarrow \sqrt{2}} (x+1)}, \quad (\text{Rule 5})$$

$$= \sqrt{1 + \sqrt{2}}. \quad (\text{Rule 6})$$

Since  $x + 1$  is a polynomial, using Rule 6 in the second step is OK.

(c)  $\lim_{x \rightarrow \sqrt{2}} \frac{x+1}{x-1}$

**Solution:** We have

$$\lim_{x \rightarrow \sqrt{2}} \frac{x+1}{x-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1}, \quad (\text{Rule 7})$$

$$= 3 + 2\sqrt{2}. \quad (\text{simplification})$$

The simplification step is done using a multiply by one trick:

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}, \quad (\text{multiply by one})$$

$$= \frac{(\sqrt{2}+1)^2}{2 + \sqrt{2} - \sqrt{2} - 1}, \quad (\text{distribute denominator})$$

$$= \frac{(\sqrt{2}+1)^2}{1}, \quad (\text{collect like terms})$$

$$= 2 + 2\sqrt{2} + 1, \quad (\text{divisor of one})$$

$$= 3 + 2\sqrt{2}. \quad (\text{collect like terms})$$

To earn full credit with our online homework system, generally removing radicals from denominator is required.