MATH 115 **Name:** ______________________

In class work 12, Fall 2022 **Row and Seat**:______________________

In class work 12 has questions 1 through 2 with a total of 10 points. Turn in your work at the end of class *on paper*. This assignment is due *Wednesday 9 November 13:15* PM.

$\boxed{5}$ 1. Show that among all rectangles with a given perimeter, a square has the greatest area.

To do this, let the lengths of the two perpendicular sides of the rectangle be x and y and let L be the perimeter of the rectangle. That makes $L = 2x + 2y$ a constraint. The other constraints are $0 \le x$ and $0 \le y$. We have $A = xy$. Your task is to maximize A subject to the constraints $L = 2x + 2y, 0 \le x$, and $0 \le y$ with L given.

Solution: To start, we need to solve the constraint $L = 2x + 2y$ for either x or y. Since interchanging x and y doesn't change the constraint, it hardly matters if we solve for x or y. So let's solve for y. The solution is $y = \frac{L-2x}{2}$. Pasting this solution into $0 \le x$ and $0 \le y$ yields

$$\left[0 \le x \text{ and } 0 \le \frac{L-2x}{2}\right] = \left[0 \le x \text{ and } x \le \frac{L}{2}\right].$$

The constraint $x \le L/2$ isn't mysterious–if $x = L/2$, then $y = 0$. And making x bigger than $L/2$ makes y negative.

Pasting $y = \frac{L-2x}{2}$ into $A = xy$ gives $A = x(\frac{L-2x}{2})$. The graph of A as a function of x is a downward facing parabola that intersects the x-axis at 0 and at $L/2$. The x-coordinate of the vertex of the parabola is halfway between the x-intercepts; thus the x-coordinate of the vertex is $x = L/4$. Using $y = \frac{L-2x}{2}$, the y-coordinate of the vertex is $y = L/4$.

So to maximize the area of the rectangle, we need $x = L/4$ and $y = L/4$. And that's a square.

$\boxed{5}$ 2. Show that among all isosceles triangles with a given perimeter, the equilateral triangle has the greatest area.

To do this, let the lengths of the sides of the triangle be x, x, and y and let L be the perimeter. That makes $L = 2x + y$ a constraint. The other constraints are $0 \le x$ and $0 \le y$. The area A of the triangle is (this is a specialization of the wonderful formula for the area of a triangle that is due to Hero of Alexandria 10 AD – c. 70 AD)

$$16A^2 = 4x^2y^2 - y^4.$$

Solve the constraint $L = 2x + y$ for y and paste that result in the formula for $16A^2$. Now do some calculus. **Hint:** Maximizing $16A^2$ also maximizes A. Thus alternatively, maximize the value of Q where

$$Q = 4x^2y^2 - y^4.$$

You don't have to use this hint, but it's the easy way, I think. If the algebra seems daunting to you, set $L = 3$ and work that specialization.

Solution: Let's begin by solving the constraint $L = 2x + y$ for y; thus $y = L - 2x$. But we also have $0 \le x$ and $0 \le y$. Using $y = L - 2x$, gives $0 \le L - 2x$. Put together, these inequalities tell us that $0 \le x \le L/2$.

But there is another inequality constraint that's subtle. Given any two sides of a triangle, the sum of these lengths must be greater than the length of the other side. Thus in addition to $0 \le x$ and $0 \le y$, we need

$$\begin{aligned}\left[y \le 2x \text{ and } x \le x + y\right] &= [L - 2x \le 2x \text{ and } 0 \le L - 2x],\\ &= [L/4 \le x \text{ and } x \le L/2].\end{aligned}$$

So we're looking at an optimization on the closed interval $[L/4, L/2]$.

Now paste $y = L - 2x$ into Q. We have

$$\begin{aligned}Q &= 4x^2y^2 - y^4,\\ &= 4(L-2x)^2x^2 - (L-2x)^4,\end{aligned}$$

We could either expand, factor, or LIB. We need to find the derivative of Q, so I think that LIB is a bad option. Expanding is tempting, but I see some opportunity to factor–let's factor.

$$= L\,(2x - L)^2\,(4x - L)$$

Had we expanded, we would get (after lots of work)

$$= 16Lx^3 - 20L^2x^2 + 8L^3x - L^4.$$

This shows that Q vanishes (and thus A vanishes as well) when either $x = L/2$ or when $x = L/4$. Thus, at each endpoint, the area is zero.

Now find the derivative of Q. We have

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = 4L\,(2x - L)^2 + 4L\,(2x - L)\,(4x - L),$$

Again, we have a choice between LIB, factor, or expand. Let's try the road less traveled and factor.

$$= 8L\,(2x - L)\,(3x - L).$$

Now solve $\frac{\mathrm{d}Q}{\mathrm{d}x} = 0$. We have

$$[8L\,(2x-L)\,(3x-L)=0] = \left[x=\frac{L}{2}, x=\frac{L}{3}\right]. \tag{1}$$

So there are two CNs. Wait! We need to check: are the CNs in the interval $[L/4, L/2]$? Yes, they are–the CN $x = \frac{L}{2}$ is also an endpoint.

To find the maximum of Q, we need the chart:

CN	Q
$L/4$	0
$L/3$	$L^4/27$
$L/2$	0

So the maximum area happens when $x = L/3$. And that makes $y = L/3$. So the three side lengths are the same.

Is the problem easier if we are given a specific value for L? Yes and no. If we're given a numeric value for L, we'll do less algebra, but we'd have to re-do the problem from the start if we needed to a different value for L. But there is another advantage to not using a numeric value for L. Since x and L are both lengths, it follows that the answer must look like $x = \text{number} \times L$. If we goof and get $x = 2 + L$, we know immediately that we have flubed.

There is a wonderful algorithm for solving systems of linear inequalities–it is called Fourier elimination. The Maxima Computer Algebra system has implementation of this method:

(% i1) load(fourier_elim)$

(% i2) fourier_elim([0 < x, 0 < y, y < 2*x, x < x + y, 2*x+y = L],[y,x]);

$$\left[y = L-2x, \frac{L}{4}< x, x< \frac{L}{2}, 4L> 0\right] \tag{\% o2}$$

The time complexity of Fourier elimination is high. Although the Simplex Method is far faster, it doesn't fully solve linear inequations.

For information about the Maxima CAS, see `https://maxima.sourceforge.io/`. I'm one of about two dozen developers for Maxima–for a map of the developers, see `https://maxima.sourceforge.io/project.html`