**MATH 115** **Name:** ____________________

**In class work 12, Fall 2022** **Row and Seat**:____________________

In class work 12 has questions 1 through 2 with a total of 10 points. Turn in your work at the end of class *on paper*. This assignment is due *Wednesday 9 November 13:15* PM.

$\boxed{5}$ 1. Show that among all rectangles with a given perimeter, a square has the greatest area.

To do this, let the lengths of the two perpendicular sides of the rectangle be $x$ and $y$ and let $L$ be the perimeter of the rectangle. That makes $L = 2x + 2y$ a constraint. The other constraints are $0 \le x$ and $0 \le y$. We have $A = xy$. Your task is to maximize $A$ subject to the constraints $L = 2x + 2y, 0 \le x$, and $0 \le y$ with $L$ given.

**Solution:** Let's solve the constraint $L = 2x + 2y$ for $y$; the solution is $y = \frac{L-2x}{2}$. Pasting this into $A = xy$ gives $A = x(\frac{L-2x}{2})$. The graph of $A$ as a function of $x$ is a downward facing parabola that intersects the x-axis at 0 and at $L/2$. The x-coordinate of the vertex of the parabola is halfway between the x-intercepts; thus the x-coordinate of the vertex is $x = L/4$. Using $y = \frac{L-2x}{2}$, the y-coordinate of the vertex is $y = L/4$.

So to maximize the area of the rectangle, we need $x = L/4$ and $y = L/4$. And that's a square.

$\boxed{5}$ 2. Show that among all isosceles triangles with a given perimeter, the equilateral triangle has the greatest area.

To do this, let the lengths of the sides of the triangle be $x, x$, and $y$ and let $L$ be the perimeter. That makes $L = 2x + y$ a constraint. The other constraints are $0 \le x$ and $0 \le y$. The area $A$ of the triangle is (this is a specialization of the wonderful formula for the area of a triangle that is due to Hero of Alexandria 10 AD – c. 70 AD)

$$16A^2 = 4x^2 y^2 - y^4.$$

Solve the constraint $L = 2x + y$ for $y$ and paste that result in the formula for $16A^2$. Now do some calculus. **Hint:** Maximizing $16A^2$ also maximizes $A$. Thus alternatively, maximize the value of $Q$ where

$$Q = 4x^2 y^2 - y^4.$$

You don't have to use this hint, but it's the easy way, I think. If the algebra seems daunting to you, set $L = 3$ and work that specialization.

**Solution:** Let's begin by solving the constraint $L = 2x + y$ for $y$; thus $y = L - 2x$. But we also have $0 \le x$ and $0 \le y$. But $y = L - 2x$, so $0 \le L - 2x$. Put together, these inequalities tell us that $0 \le x \le L/2$. So we're looking at an optimization on a closed interval.

Paste $y = L - 2x$ into $Q$. We have

$$\begin{aligned} Q &= 4x^2 y^2 - y^4, \\ &= 4(L - 2x)^2 x^2 - (L - 2x)^4, \end{aligned}$$

We could either expand, factor, or LIB. We need to find the derivative of $Q$, so I think that LIB is a bad option. Expanding is tempting, but I see some opportunity to factor–let's factor.

$$= L\,(2x - L)^2\,(4x - L)\,.$$

This shows that $Q$ vanishes (and thus $A$ vanishes as well) when either $x = L/2$ or when $x = L/4$. Why does the area vanish when $x = L/2$. Oh, when $x = L/2$, we have $y = 0$. That gives a side length of zero, so the area is zero. A bit more mysterious is why the area is zero when $x = L/4$. That makes $y = L/2$. Try drawing a triangle with side lengths of $L/4, L/4$, and $L/2$. What's the height? It's zero.

Now find the derivative of $Q$. We have

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = 4L\,(2x - L)^2 + 4L\,(2x - L)\,(4x - L)\,,$$

Again, we have a choice between LIB, factor, or expand. Let's try the road less traveled and factor.

$$= 8L\,(2x - L)\,(3x - L)\,.$$

Now solve $\frac{\mathrm{d}Q}{\mathrm{d}x} = 0$. We have

$$[8L\,(2x - L)\,(3x - L) = 0] = \left[x = \frac{L}{2}, x = \frac{L}{3}\right]. \tag{1}$$

So there are two CNs. But if $x = \frac{L}{2}$, then $y = L - L = 0$. Now that minimizes, not maximizes the area. Actually, we have $0 \le x$ and $0 \le y$. But $y = L - 2x$, so $0 \le L - 2x$. So $0 \le x \le L/2$.

To find the maximum of $Q$, we need the chart:

| CN | Q |
|---|---|
| $0$ | $0$ |
| $L/3$ | $L^4/27$ |
| $L/2$ | $0$ |

So the maximum area happens when $x = L/3$. And that makes $y = L/3$. So the three side lengths are the same.

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