Natural Domain

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Natural domain

Often the formula for a function is given, but not its domain. For such functions, the standard is that the domain is the largest set of real numbers such that the output of the function is a real number.

Description (Natural Domain)

The natural domain of a function is the largest set on which the output of the function is a real number. Examples of things that are not real numbers include:

- *division by zero*
- *square roots of negative numbers*
- *logarithms of non-positive numbers*

Example Find the natural domain of

$$F(x) = \frac{1}{\frac{1}{x} + \frac{1}{x-2}}.$$

- We need to require that *every* denominator is nonzero.
- But that's tricky-there are three denominators; they are x, $x - 2$, and $\frac{1}{x} + \frac{1}{x-2}$.

Solution In set builder notation, the natural domain is the set

$$\left\{ x \in \mathbf{R} \middle| (x \neq 0) \land (x - 2 \neq 0) \land \left(\frac{1}{x} + \frac{1}{x-2} \neq 0 \right) \right\}$$

✎ Arguably, this is a lovely answer. But if you were asked to express the set in interval notion, it's not such a lovely answer.

✎ To make it a truly lovely answer, we need to solve each inequation for

✎ the variable (in this case x).

✎ When we solve each inequation for the variable, we make the set builder notation *explicit*.

Explict Form

Solving $x \neq 0$ and $x - 2 \neq 0$ is easy:

$$x \neq 0, \quad x \neq 2.$$

Solving $\left(\frac{1}{x} + \frac{1}{x-2} \neq 0\right)$ takes a bit more work. Assuming that $x \neq 0$ and $x \neq 2$, we have

$$\begin{aligned}
\left[\frac{1}{x} + \frac{1}{x-2} \neq 0\right] &= \left[\frac{2x-2}{x^2-2x} \neq 0\right], && \text{(combine)} \\
&= [2x - 2 \neq 0], && \text{(algebra)} \\
&= [x \neq 1], && \text{(algebra)}
\end{aligned}$$

In *explicit* set builder notation, the natural domain Is

$$\left\{ x \in \mathbf{R} \,\middle|\, (x \neq 0) \wedge (x \neq 2) \wedge (x \neq 1) \right\}$$

Remove all Doubt

If you are a bit uncertain of why the natural domain excludes 1, try evaluating at one:

$$\frac{1}{\frac{1}{1}+\frac{1}{1-2}} = \frac{1}{1-1} = \textbf{Rubbish!}$$

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Implicit requirement to be explicit

Requirement

Unless stated otherwise, set builder notation must be explicit. The reason for this is that when expressed in explicit form, it's easy-peasy to express the set in either interval notion or pictorially.

Example

Implicit (not allowed)

$$\left\{x \in \mathbf{R} \,\middle|\, (x \neq 0) \wedge (x - 2 \neq 0) \wedge \left(\frac{1}{x} + \frac{1}{x-2} \neq 0\right)\right\}.$$

Explicit (correct)

$$\left\{x \in \mathbf{R} \,\middle|\, (x \neq 0) \wedge (x \neq 2) \wedge (x \neq 1)\right\}.$$

This is a Test

Question The natural domain of $\frac{1}{\frac{1}{x}+\frac{1}{x-2}}$ is

$$\left\{x \in \mathbf{R} \middle| (x \neq 0) \wedge (x \neq 2) \wedge (x \neq 1)\right\}$$

What's the natural domain of the similar, but different expression

$$107 + x^2 + \frac{46 + x}{\frac{1}{x} + \frac{1}{x-2}}$$

Answer Gathering the denominators, in implicit form, the natural domain is

$$\left\{x \in \mathbf{R} \middle| (x \neq 0) \wedge (x - 2 \neq 0) \wedge \left(\frac{1}{x} + \frac{1}{x-2} \neq 0\right)\right\}$$

This is no different from the condition for the domain of $\frac{1}{\frac{1}{x}+\frac{1}{x-2}}$.

Does it matter?

Question So the $107 + x^2$ and $46 + x$ don't matter?

Answer The natural domain of $\frac{1}{\frac{1}{x}+\frac{1}{x-2}}$ and $107 + x^2 + \frac{46+x}{\frac{1}{x}+\frac{1}{x-2}}$ are the same.

This is the case because both expressions have identical denominators.

Resist Simplification

Question Which calculation of the natural domain of $\frac{1}{\frac{x-3}{x-4}}$ is correct?

Answer 1

$$\left\{x \middle| (x-4 \neq 0) \wedge \left(\frac{x-3}{x-4} \neq 0\right)\right\} = \{x | (x \neq 4) \wedge (x \neq 3)\}.$$

Answer 2 First simplify $\frac{1}{\frac{x-3}{x-4}}$ to $\frac{x-4}{x-3}$. Second the domain is

$$\{x | x-3 \neq 0\} = \{x | x \neq 3\}.$$

Fact Answer 2 is wrong: the number 4 is definitely *not* in the domain of $\frac{1}{\frac{x-3}{x-4}}$. The Simplification $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ looks innocent enough, but it requires that both a and b be nonzero.