MATH 102, Spring 2023 **Name:** ______________________________

Final Exam Review, Part B 15 **Row and Seat:** ______________________________

"Keep cool but care." THOMAS PYNCHON, *V.*

1. Define a function W by $W(x) = \sqrt{x}$ and $\operatorname{dom}(W) = [0,4]$. Find the *range* of W.

Solution: A graph of the square root function shows that $\operatorname{range}(W) = [0,2]$.

$\boxed{5}$ 2. Find the *distance* between the points $(7,9)$ and $(-1,-2)$.

Solution: We have

$$\begin{aligned} \operatorname{dist}((7,9),(-1,-2)) &= \sqrt{(7+1)^2+(9+2)^2}, && \text{(distance formula)} \\ &= \sqrt{64+121}, && \text{(arithmetic)} \\ &= \sqrt{185}. && \text{(arithmetic)} \end{aligned}$$

- It's easy to misread a question–be careful. I often put what you need to find in italic type (in this case *distance*) in an attempt to clarify what you are looking for.
- The factors of 185 are 5 and 37. Neither of these factors are perfect squares, so $\sqrt{185}$ is properly simplified.
- Unless you are asked for a decimal approximation, you should leave your answers in an exact form. This problem *doesn't* ask for an exact solution, so 13.60147 is *not* a correct solution.
- A common error is to do the following:

$$\sqrt{8^2+11^2} = 8+11 = 19.$$

But this is **wrong**. The equation $\sqrt{x^2+y^2} = x+y$ is not an identity. You can verify this by pasting in $x=1$ and $y=1$ into the equation. The result is

$$\left[\sqrt{1^2+1^2} = 1+1\right] = \left[\sqrt{2} = 2\right] = \text{False}$$

Our Quick Reference Sheet lists other common errors.

$\boxed{5}$ 3. The *midpoint* of points P and $(5,6)$ is $(-2,3)$. Find the *coordinates* of the point P.

Solution: Let $P = (x, y)$. We have

$$\left(\frac{x+5}{2}, \frac{y+6}{2}\right) = (-2, 3).$$

So we need to solve

$$\frac{x+5}{2} = -2,$$
$$\frac{y+6}{2} = 3.$$

for x and for y. Solving the first equation for x gives

$$\left[\frac{x+5}{2} = -2\right] = [x+5 = -4], \qquad \text{(multiply by 2)}$$
$$= [x = -9]. \qquad \text{(add -5)}$$

And solving the second for y gives

$$\left[\frac{y+6}{2} = 3\right] = [y+6 = 6], \qquad \text{(multiply by 2)}$$
$$= [y = 0]. \qquad \text{(add -6)}$$

Solving these equations for x and y gives $x = -9$ and $y = 0$.

4. A line L contains the points $(x = 5, y = 2)$ and $(x = 7, y = -1)$.

$\boxed{1}$ (a) Find an *equation* of the line L.

Solution: The slope of the line L is

$$\frac{2-(-1)}{5-7} = -\frac{3}{2}$$

Using the point $(x = 5, y = 2)$ in the point-slope formula, an equation of the line is

$$y - 2 = -\frac{3}{2}(x-5).$$

Using the other point, gives an equation that looks different, but describes exactly the same line.

A good way to check our work is to paste the data in to the equation and see if it is true. Pasting $(x = 5, y = 2)$ into our equation for L, we have

$$\left[2 - 2 = -\frac{3}{2}(5-5)\right] = [0 = 0] = \text{True}.$$

And pasting in $(x=7, y=-1)$, we have

$$\left[-1-2=-\frac{3}{2}(7-5)\right]=[-3=-3]=\text{True}.$$

The question doesn't ask for a particular form for the equation of the line. We've found a point slope form for the line and we have checked our work. What should we do now? We should LIB (let it be). If the question asked for the slope-intercept form, we would need to give the answer as

$$y=-\frac{3}{2}x+\frac{19}{2}.$$

But the question gives us the *freedom* to give either form for the equation of the line. **Let's exercise our freedom and do so.**

$\boxed{1}$ (b) Find the *x-intercept* of the line L.

Solution: To find the x-intercept of the line $y-2=-\frac{3}{2}(x-5)$, set y to zero and solve for x; we have

$$\begin{aligned}\left[0-2=-\frac{3}{2}(x-5)\right]&=\left[\frac{4}{3}=x-5\right], && \text{(multiply by } -2/3)\\ &=\left[x=\frac{19}{3}\right]. && \text{(add 5)}\end{aligned}$$

$\boxed{5}$ 5. The number of lawns L a work crew can mow in a day varies jointly with the number of people N in the crew and with the time T they work in a day.

Given that $L=8$ when $N=5$ and $T=6$, find L when $N=8$ and $T=10$.

Solution: There is a constant k such that $L=kNT$. This formula should make sense–doubling the size of the work crew, we should be able to mow twice as many lawns–the formula shows that is true. The same is true for working twice as long.

Pasting in $L=8$ when $N=5$ and $T=6$ into $L=kNT$ yields $8=30k$. So $k=\frac{4}{15}$. That makes our formula

$$L=\frac{4}{15}NT.$$

Pasting in $N=8$ and $T=10$ gives

$$L=\frac{4}{15}\times 80=\frac{64}{3}\approx 21.3.$$

Teacher's embellishment A century or so ago, students were taught a short-cut method for such problems called the *double rule of three*. The Lewis Carroll poem "The Mad Gardener's Song" refers to this method.

6. Find the *vertex* of each parabola

$\boxed{2}$ (a) $y-8=x^2$

Solution: Matching $y-8=x^2$ to $y-k=a(x-h)^2$ gives $k=8, h=0$, and $a=1$. So the vertex is the point $(0,8)$.

$\boxed{2}$ (b) $y-8=\sqrt{2}(x+2)^2$

Solution: Matching $y-8=\sqrt{2}(x+2)^2$ to $y-k=a(x-h)^2$ gives $k=8, h=-2$, and $a=\sqrt{2}$. So the vertex is the point $(-2,8)$.

$\boxed{2}$ (c) $y=2x^2-28x+103$

Solution: Matching $y=2x^2-28x+103$ to $y=ax^2+bx+c$ gives $a=2, b=-28$, and $c=103$. The vertex is the point

$$\left(-\frac{b}{2a}, c-\frac{b^2}{4a}\right)=\left(-\frac{-28}{2\times 2}, 103-\frac{(-28)^2}{4\times 2}\right)=(7,5)\,. \tag{1}$$

$\boxed{2}$ (d) $y=3(x-2)(x-4)$

Solution: To match $y=3(x-2)(x-4)$ to $y=ax^2+bx+c$, we need to expand (use FOIL) $y=3(x-2)(x-4)$. We have

$$y=3(x-2)(x-4)=3(x^2-6x+8)=3x^2-18x+24.$$

So $a=3, b=-18$, and $c=24$. The vertex is the point

$$\left(-\frac{b}{2a}, c-\frac{b^2}{4a}\right)=\left(-\frac{-18}{2\times 3}, 24-\frac{(-18)^2}{4\times 3}\right)=(3,-3)\,.$$

7. Follow these steps to solve the inequality $x^2-x\geq 12$.

$\boxed{2}$ (a) *Solve* the equation $x^2-x=12$.

Solution:

$$\begin{aligned}\left[x^2 - x = 12\right] &= \left[x^2 - x - 12 = 0\right] && \text{(subtract 12)}\\ &= [(x-4)(x+3) = 0] && \text{(factor)}\\ &= [x = 4 \text{ or } x = -3] && \text{(teacher's pet fact)}\end{aligned}$$

$\boxed{2}$ (b) Check that *both of your solutions are correct* by pasting them into the equation $x^2 - x = 12$.

Solution: Pasting in $x \to 4$ into $x^2 - x = 12$ gives

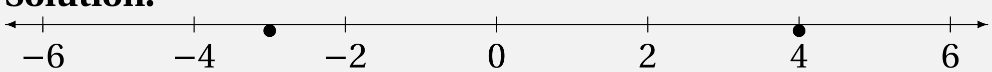
$$\left[4^2 - 4 = 12\right] = [12 = 12] = \text{True!}$$

And Pasting in $x \to -3$ into $x^2 - x = 12$ gives

$$\left[(-3)^2 + 3 = 12\right] = [12 = 12] = \text{True!}$$

$\boxed{2}$ (c) Put both of your solutions on a *number line*, correctly ordered from *least to greatest.*

Solution:



$\boxed{2}$ (d) Make a *table* of the intervals determined by the number line from the previous part, the test points, and the value of $x^2 - x \geq 12$ at each test point.

Solution:

Interval	**x**	$\mathbf{x^2 - x \geq 12}$	**true or false**
$(-\infty, -3)$	-4	$(-4)^2 + 4 \geq 12$	true
$(-3, 4)$	0	$0^2 + 0 \geq 12$	false
$(4, \infty)$	5	$5^2 + 4 \geq 12$	true

$\boxed{2}$ (e) Finish the sentence: The solution set is

Solution: $(-\infty, -3] \cup [4, \infty)$.

We include the endpoints in each interval (use a bracket, not a paren) because we're solving $x^2 - x \geq 12$. And that allows equality. Here in a bit, this rule will be modified.

8. Find the solution set to $\frac{2x+3}{4x+1} \leq 1$ by following these steps.

$\boxed{1}$ (a) Use algebra tools to find an equivalent inequality of the form $\frac{P(x)}{Q(x)} \leq 0$, where P and Q are polynomials.

Solution: We have

$$\begin{aligned}
\left[\frac{2x+3}{4x+1} \leq 1\right] &= \left[\frac{2x+3}{4x+1} - 1 \leq 0\right], && \text{(subtract 1)} \\
&= \left[\frac{2x+3}{4x+1} - \frac{4x+1}{4x+1} \leq 0\right], && \text{(make a common denominator)} \\
&= \left[\frac{-2x+2}{4x+1} \leq 0\right]. && \text{(combine numerators)}
\end{aligned}$$

$\boxed{1}$ (b) Find all x-intercepts and all VAs for $\frac{P(x)}{Q(x)}$.

Solution: To find the x-intercepts we set the numerator to zero and solve:

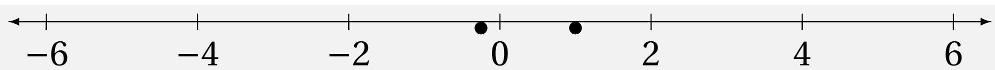
$$[-2x+2=0] = [x=1].$$

To find the VA we set the denominator to zero and solve:

$$[4x+1=0] = \left[x = -\frac{1}{4}\right]. \tag{2}$$

$\boxed{1}$ (c) Put all x-intercepts and VAs on to a number line.

Solution:



1 (d) Build the chart with columns for the interval, the test number, evaluation at the test number, and the true/false value.

Solution:

Interval	Test	$\frac{2x+3}{4x+1} \leq 1$	True or False
$(-\infty, -1/4)$	-1	$-\frac{1}{3} \leq 1$	True
$(-1/4, 1)$	0	$3 \leq 1$	False
$(1, \infty)$	2	$\frac{7}{9} \leq 1$	True

1 (e) Test each interval endpoint for inclusion or exclusion into the solution set.

Solution:

Endpoint	$\frac{2x+3}{4x+1} \leq 1$	True or False
$-1/4$	dne	False
1	$1 \leq 1$	True

1 (f) Express the solution set in either interval notation, pictorially, or set builder notation.

Solution: In interval notation the solution set is $(-\infty, -\frac{1}{4}) \cup [1, \infty)$. In set builder notation, it is $\{x | x < -\frac{1}{4} \text{ or } x \geq 1\}$.

2 9. At 6 AM, Louisa has 340 mg of caffeine circulating in her blood. After T hours, the amount of caffeine C in her blood is $C = 340 \times 0.9^T$. When Louisa goes to bed at 10 PM, how much caffeine is still in circulation?

Solution: We have

$$C = 340 \times 0.9^{16} = 63.0\,\text{mg}.$$

I rounded the value to the nearest tenth—given the context, that is reasonable.

10. Find the inverse of the function $f(x) = \frac{2x+1}{x-1}$, $x \neq 1$.

Solution: We need to solve $y = \frac{2x+1}{x-1}$ for x. When $y = 2$, the solution set is empty; otherwise, the solution is

$$x = \frac{y+1}{y-2}$$

So $f^{-1}(y) = \frac{y+1}{y-2}$ and $\text{dom}(f^{-1}) = \{y | y \neq 2\}$.