**MATH 102, Spring 2023** **Name:** ______________________________

**In class work 5** **Row and Seat**: ______________________________

In class work 5 has questions 1 through 8 with a total of 85 points. This assignment is due at the end of the class period (9:55 AM).

$\boxed{5}$ 1. Find the *distance* between the points $(7,2)$ and $(-1,-2)$.

**Solution:** We have

$$\begin{aligned}
\operatorname{dist}((7,2),(-1,-2)) &= \sqrt{(7+1)^2+(2+2)^2}, && \text{(distance formula)}\\
&= \sqrt{64+16}, && \text{(arithmetic)}\\
&= \sqrt{80}, && \text{(arithmetic)}\\
&= 4\sqrt{5}. && \text{(factor)}
\end{aligned}$$

$\boxed{5}$ 2. Find the *midpoint* of the points $(2,4)$ and $(5,7)$.

**Solution:**

$$\text{mid point} = \left(\frac{2+5}{2}, \frac{4+7}{2}\right) = \left(\frac{7}{2}, \frac{11}{2}\right).$$

$\boxed{5}$ 3. A line $L$ contains the points $(x=5, y=7)$ and $(x=7, y=-1)$.

$\boxed{5}$ (a) Find an *equation* of the line $L$.

**Solution:** The slope is $\frac{7+1}{5-7} = -4$. An equation of the line is $y-7 = -4(x-5)$.

$\boxed{5}$ (b) Find the *x-intercept* of the line $L$.

**Solution:** Replace $y$ by zero and solve for $x$; we have

$$[0-7=-4(x-5)] = \left[\frac{7}{4} = x-5\right] = \left[x = \frac{27}{4}\right].$$

$\boxed{5}$ 4. Find an equation of the line that is parallel to the line $3y + 6x = 1$ and that contains the point $(x = 1, y = 1)$.

**Solution:** The slope of the line $3y + 6x = 1$ is $-2$. So an equation of the line we are looking for is $y - 1 = -2(x - 1)$.

$\boxed{5}$ 5. Find the *center* and *radius* of the circle $x^2 + 2x + y^2 - 6y = -6$.

**Solution:**

$$\left[x^2 + 2x + y^2 - 6y = -6\right] = \left[x^2 + 2x + 1 + y^2 - 6y + 9 = -6 + 1 + 9\right].$$

So the center is $(x = 1, y = -3)$ and the radius is $\sqrt{4} = 2$.

$\boxed{5}$ 6. The number of doghouses $L$ a work crew can build in a day varies jointly with the number of people $N$ in the crew and with the time $T$ they work in a day. Given that $L = 12$ when $N = 5$ and $T = 6$, find $L$ when $N = 20$ and $T = 10$.

**Solution:** There is a constant $k$ such that

$$L = kNT.$$

Pasting in the data $L = 12$ when $N = 5$ and $T = 6$ gives $k = \frac{12}{30}$. When $N = 20$ and $T = 10$, we have

$$L = \frac{12}{30} \times 20 \times 10 = 80. \tag{1}$$

7. Shown below is a graph of the equation $y = U(x)$. Some points on the graph are labeled. The domain of $U$ is the closed interval $[-2, 2]$

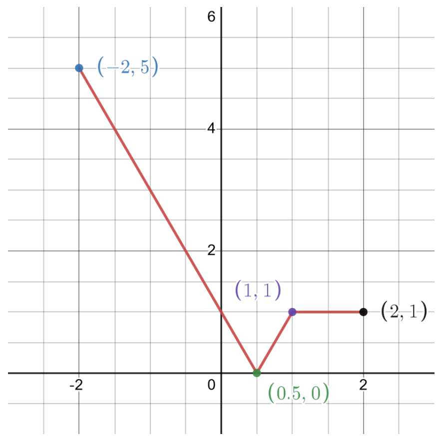



Figure 1: Graph of the equation $y = U(x)$.

$\boxed{5}$ (a) Find the numerical value of $U(-2)$.

**Solution:** $U(-2) = 5$.

$\boxed{5}$ (b) Find the *range* of $U$.

**Solution:** $\text{range}(U) = [0, 5]$

$\boxed{5}$ (c) Find the interval(s) on which $U$ is *decreasing*.

**Solution:** $[-2, \frac{1}{2}]$.

$\boxed{5}$ (d) Find the interval(s) on which $U$ is *increasing*.

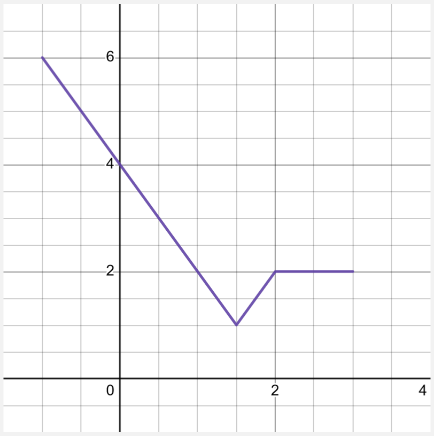
**Solution:** $[\frac{1}{2}, 1]$.

$\boxed{5}$ (e) Find the interval(s) on which $U$ is *constant*.

**Solution:** $[1, 2]$.

$\boxed{5}$ (f) Sketch a graph of the equation $y - 1 = U(x - 1)$.

**Solution:** We need to translate the graph of $y = U(x)$ one unit to the right and one unit up. Here it is!




8. Define a function $Q$ by $Q(x) = \begin{cases} x & x < 1 \\ 5 & 1 \le x \end{cases}$.

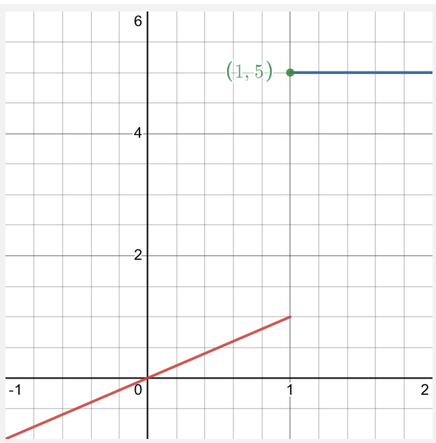
$\boxed{5}$ (a) Find the numerical value of $Q(-2)$.

**Solution:** $Q(-2) = -2$

$\boxed{5}$ (b) Find the numerical value of $Q(2)$.

**Solution:** $Q(2) = 5$

$\boxed{5}$ (c) Sketch a graph of $Q$.

**Solution:** 


(d) Find the *average rate of change* of $Q$ on the interval $[-2, 2]$.

**Solution:**

$$\underset{[-2,2]}{\mathrm{ARC}}(Q) = \frac{Q(2) - Q(-2)}{2 - (-2)} = \frac{5+2}{2+2} = \frac{7}{4}.$$

## Greek characters

| Name | Symbol | Typical use(s) |
|---|---|---|
| alpha | $\alpha$ | angle, constant |
| beta | $\beta$ | angle, constant |
| gamma | $\gamma$ | angle, constant |
| delta | $\delta$ | limit definition |
| epsilon | $\epsilon$ or $\varepsilon$ | limit definition |
| theta | $\theta$ or $\vartheta$ | angle |
| pi | $\pi$ or $\varpi$ | circular constant |
| phi | $\phi$ or $\varphi$ | angle, constant |

## Named sets

| | |
|---|---|
| empty set | $\varnothing$ |
| real numbers | $\mathbf{R}$ |
| ordered pairs | $\mathbf{R}^2$ |

| | |
|---|---|
| integers | $\mathbf{Z}$ |
| positive integers | $\mathbf{Z}_{>0}$ |
| positive reals | $\mathbf{R}_{>0}$ |

## Set symbols

| Meaning | Symbol |
|---|---|
| is a member | $\in$ |
| subset | $\subset$ |
| intersection | $\cap$ |

| Meaning | Symbol |
|---|---|
| union | $\cup$ |
| complement | superscript$^{C}$ |
| set minus | $\setminus$ |

## Logic symbols

| Meaning | Symbol |
|---|---|
| negation | $\neg$ |
| and | $\wedge$ |
| or | $\vee$ |
| implies | $\implies$ |

| Meaning | Symbol |
|---|---|
| equivalent | $\equiv$ |
| iff | $\iff$ |
| for all | $\forall$ |
| there exists | $\exists$ |

## Arithmetic properties of R

$(\forall a, b \in \mathbf{R})(a + b = b + a)$ commutivity

$(\forall a, b, c \in \mathbf{R})(a + (b + c) = (a + b) + c)$ associative

$(\forall a, b \in \mathbf{R})(ab = ba)$ commutivity

$(\forall a, b, c \in \mathbf{R})(a(bc) = (ab)c)$ associative

$(\forall a, b, c \in \mathbf{R})(a(b + c) = ab + ac)$ distributive

## Intervals

For numbers $a$ and $b$, we define the intervals

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$$
$$[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$$
$$(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$$
$$[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$

## Distance & Midpoint

The distance between the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The midpoint is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

## Exponents

For $a, b > 0$ and $m, n$ real:

$$a^0 = 1, \qquad 0^a = 0$$
$$1^a = 1, \qquad a^n a^m = a^{n+m}$$
$$a^n/a^m = a^{n-m}, \qquad (a^n)^m = a^{n \cdot m}$$
$$a^{-m} = 1/a^m, \qquad (a/b)^m = a^m/b^m$$

## Radicals

$$\sqrt[n]{a} = a^{1/n}$$
$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \quad \text{(provided } a, b \geq 0)$$
$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$
$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$
$$\sqrt[n]{a^n} = \begin{cases} a & n \text{ odd} \\ |a| & n \text{ even} \end{cases}$$

## Identities

$$a(b + c) = ab + ac$$
$$((a + b)(c + d)) = ac + ad + bc + bd$$
$$\frac{ab + ac}{a} = b + c \quad \text{(provided } a \neq 0)$$
$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc} \quad \text{(provided } b, d \neq 0)$$
$$\sqrt{ab} = \sqrt{a}\sqrt{b} \quad \text{(provided } a \geq 0, b \geq 0)$$
$$\ln(ab) = \ln(a) + \ln(b) \quad \text{(provided } a \geq 0, b \geq 0)$$

## Solution of Equations

**Algebraic**

$$[ab = 0] \equiv [a = 0 \textbf{ or } b = 0]$$
$$[a^2 = b^2] \equiv [a = b \textbf{ or } a = -b]$$
$$\left[\frac{a}{b} = 0\right] \equiv [a = 0 \textbf{ and } b \neq 0]$$
$$\left[\frac{a}{b} = \frac{c}{d}\right] \equiv [ad = bc \textbf{ and } b \neq 0 \textbf{ and } d \neq 0]$$
$$[|a| = |b|] \equiv [a = b \textbf{ or } a = -b]$$
$$[\sqrt{a} = b] \equiv [a = b^2 \textbf{ and } b \geq 0]$$

For $a \neq 0$,

$$[ax^2 + bx + c = 0] \equiv \left[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

**Exponential**

$$[\ln(a) = 0] \equiv [a = 1]$$
$$[\mathrm{e}^a = 1] \equiv [a = 0]$$
$$[\ln(a) = b] \equiv [a = \mathrm{e}^b]$$

## Logarithms

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

## Graph Translations

For the graph of $F(x, y) = 0$

- The graph of $F(x - h, y) = 0$ is the graph of $F(x, y) = 0$ translated $h$ units to the right.
- The graph of $F(x, y - k) = 0$ is the graph of $F(x, y) = 0$ translated $k$ units up.
- The graph of $F(x/c, y) = 0$ is the graph of $F(x, y) = 0$ stretched a factor of $c$ horizontally.
- The graph of $F(x, y/c) = 0$ is the graph of $F(x, y) = 0$ stretched a factor of $c$ vertically.

## Circles

Equation of circle centered at $(h, k)$ with radius $r$ is

$$(x - h)^2 + (y - k)^2 = r^2.$$

Expanded the equation is

$$x^2 - 2hx + y^2 - 2ky = r^2 - h^2 - k^2.$$

## Parabolas & Lines

The vertex of the parabola $ax^2 + bx + c = y$ is

$$\left(x = -\frac{b}{2a}, y = c - \frac{b^2}{4a}\right).$$

An equation of the line that contains the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

The number $\frac{y_2 - y_1}{x_2 - x_1}$ is the slope.

## Function notation

| | |
|---|---|
| $\mathrm{dom}(F)$ | domain of function $F$ |
| $\mathrm{range}(F)$ | range of function $F$ |

## Domains, Ranges, and Zeros

| Function | Domain | Range | Zeros |
|---|---|---|---|
| $\ln, \log$ | $(0, \infty)$ | $(-\infty, \infty)$ | 1 |
| exp | $(-\infty, \infty)$ | $(0, \infty)$ | $\varnothing$ |
| abs | $(-\infty, \infty)$ | $(0, \infty)$ | 0 |
| $\sqrt{\ }$ | $(0, \infty)$ | $(0, \infty)$ | 0 |
| $\sqrt[3]{\ }$ | $(-\infty, \infty)$ | $(-\infty, \infty)$ | 0 |
| floor | $(-\infty, \infty)$ | $\mathbf{Z}$ | $[0, 1)$ |
| ceiling | $(-\infty, \infty)$ | $\mathbf{Z}$ | $(-1, 0]$ |

## Compound Interest

Interest rate $r$ compounded $n$ times per year

$$A = P(1 + r/n)^{nt}$$

Continuous compounding:

$$A = Pe^{rt}$$

[c]

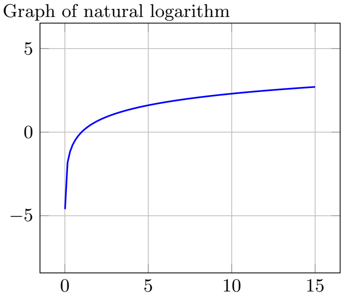
## Exponential Growth

The exponential function that contains the points $(t = t_o, y = y_o)$ and $(t = t_1, y = y_1)$ is

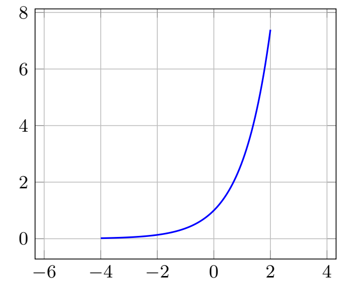
$$y = y_o \left(\frac{y_1}{y_o}\right)^{\frac{t - t_o}{t_1 - t_o}}.$$

## Graphs

Graph of natural logarithm




Graph of natural exponential




## Common Errors

| **Error** | **Correct or Example** |
|---|---|
| ${}^{x}/_{0} = 0$ or $x$ | ${}^{x}/_{0}$ is undefined |
| $-x^2 = x^2$ | $-x^2 = -(x^2)$ |
| ${}^{a}/_{(b+c)} = {}^{a}/_{b} + {}^{a}/_{c}$ | ${}^{1}/_{(1+1)} \neq {}^{1}/_{1} + {}^{1}/_{1}$ |
| ${}^{a+bx}/_{a} = 1 + bx$ | ${}^{a+bx}/_{a} = 1 + {}^{bx}/_{a}$ |
| $(a + b)^2 = a^2 + b^2$ | $(a + b)^2 = a^2 + 2ab + b^2$ |
| $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ | $\sqrt{1 + 1} \neq \sqrt{1} + \sqrt{1}$ |