Boolean Logic

Lesson 1

Statements

Quasi-definition

A *statement*, also known as a *proposition* or an *assertion*, is a sentence that has a truth value of either true or false. A *theorem* is a statement that has a truth value of true.

1. Boolean logic is named in honor of **George Boole** (1815 – 1864).
2. In boolean logic, the truth values are either **true** or **false**.
3. A statement is a concept that we can describe, but don't define.
4. An *axiom* is a statement that is *assumed* to have a truth value of true. Generally, the truth value of an axiom cannot be determined by the truth value of other theorems.

Example

Examples of statements:

1. $1 = 1$.
2. Every square is a rectangle.
3. Some integers are divisible by 42.

Examples of non-statements:

1. Square houses are boring.
2. Please make your bed, brush your teeth, and take out the garbage.

Logical notation

We'll use the ISO standard names for logical functions. These names are

negation	$\neg$
and	$\wedge$
or	$\vee$
implies	$\implies$
equivalent	$\equiv$
for all	$\forall$
there exists	$\exists$

1. For a quick review of these functions, see `https://en.wikipedia.org/wiki/Boolean_algebra`.
2. For additional ISO math symbols, see `https://en.wikipedia.org/wiki/ISO_31-11`.
3. In mathematics, for statements P and Q, the statement $P \vee Q$ is true when both P and Q are true; that is, we use the disjunction inclusive.

Negation

Definition

For a statement P, we define its *logical negation*, denoted by $\neg P$, with the *truth table*

P	$\neg P$
T	F
F	T

.

Equality

Definition

Let P and Q be statements. We define *equivalence* $P \equiv Q$ by the truth table

P	Q	$P \equiv Q$
T	T	T
T	F	F
F	T	F
F	F	T

.

1. Statements P and Q are equivalent provided the statements have the same truth value.
2. Since both P and Q have two possible values, the truth table has $4(= 2 \times 2)$ rows.
3. $P \equiv Q$ is an example of a *compound statement*. Its constituent parts are the statements P and Q.

Disjunctions

Definition

Let P and Q be statements. The *disjunction* of P with Q, denoted by $P \vee Q$, is a statement whose truth value is given by

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

.

1. That is $P \vee Q$ is false when both P and Q are false; otherwise $P \vee Q$ is true.
2. $P \vee Q$ is another example of a *compound statement*.
3. In mathematical logic, notice that True $\vee$ True has a truth value of true.

Conjunctions

Definition

Let P and Q be a statements. The *conjunction* of P with Q, denoted by $P \wedge Q$, is a statement whose truth value is given by

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

.

1. That is $P \wedge Q$ is true provided both P and Q are true; otherwise $P \wedge Q$ is false.

Tautologies

Definition

A compound statement that has a truth value of true for all possible truth values of its constituent parts is a *tautology*.

Example

Each of the following are tautologies:

1. $P \vee \neg P$,
2. $P \equiv P$,
3. $P \equiv \neg\neg P$,
4. $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$.

Example

Example

Let's show that $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is a tautology. There are two constituent parts, so we need a truth table with four rows. How many columns it has depends on how many steps we are willing to skip.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	T

.

The last column shows that regardless of the truth values for P and Q, the statement $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is true; therefore $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is a tautology.

1. Possibly the truth table should have columns for $\neg P$ and $\neg Q$.
2. The tautology $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is due to De Morgan, and is known as *De Morgan's law* (Augustus De Morgan (1806 – 1871).

Conditionals

The conditional is a logical connective that allows us to form a compound statement with the meaning "if P, then Q." Specifically:

Definition

Let P and Q be a statements. We define $P \implies Q$ with the truth table

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

.

.

1. In the conditional $P \implies Q$, we say that P is the *hypothesis* and Q is the *conclusion*.
2. A conditional is false when the hypothesis is true, but the conclusion is false; otherwise, a conditional is true.

Converse

Definition

The *converse* of the conditional $P \implies Q$ is the conditional $Q \implies P$.

Fact

A truth table shows that $(P \implies Q) \equiv (Q \implies P)$ is not a tautology. Specifically, $T \implies F$ is false, but $F \implies T$ is true.

Example

Consider the statement

If $x < 5$, then $x < 7$

and its converse

If $x < 7$, then $x < 5$.

The first statement is true, but its converse is false (because, for example, x could be six, making $x < 7$ true, but $x < 5$ false.

Contrapositive

Definition

The *contrapositive* of the conditional $P \implies Q$ is the conditional $\neg Q \implies \neg P$.

Fact

A truth table shows that $(P \implies Q) \equiv (\neg Q \implies \neg P)$ is a tautology.

Example

Consider the statements:

If $x < 5$, *then* $x < 7$

and its contrapositive

If $x \geq 7$, *then* $x \geq 5$

These statements are logically equivalent.

Extra conditional

Fact

A truth table shows that $(P \implies Q) \equiv \neg P \vee Q$ is a tautology. This makes $P \not\Rightarrow Q$ equivalent to $P \wedge \neg Q$.