CYBR 304
Spring 2024
Review for Third Examination

1. Know the properties of an inner product (page 205 of our textbook).
2. Know the definition of the Kronecker delta function (classnotes Friday).
3. Find a simple representation of $\sum_{k=-10}^{10} (\pi^2 + 42)\,\delta_{k,1}$.
4. Find a simple representation of $\sum_{k=-10}^{10} (\pi^2 + 42)\,\delta_{k,28}$.
5. Given an innerproduct on a set of function, know what it means for functions to be perpendicular.
6. Let f, g, and h be functions. For some innner product, suppose

$$\begin{bmatrix} \langle f,f\rangle & \langle f,g\rangle & \langle f,h\rangle \\ \langle g,f\rangle & \langle g,g\rangle & \langle g,h\rangle \\ \langle h,f\rangle & \langle h,g\rangle & \langle h,h\rangle \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 8 \\ 4 & 8 & 1 \end{bmatrix}. \tag{1}$$

   (a) Find the numerical value of $\langle f, g-h\rangle$

   Solution: $\langle f, g-h\rangle = \langle f,g\rangle - \langle f,h\rangle = 3-4 = -1$

   (b) Find the numerical value of $\langle f, 2f+5g\rangle$

   Solution: $\langle f, 2f+5g\rangle = \langle f, 2f\rangle\langle f, 5g\rangle = 2\langle f,f\rangle + 5\langle f,g\rangle = 2\times 2 + 5\times 3 = 4+15 = 19.$

   (c) Find a real number α such that $\langle \alpha f, \alpha f\rangle = 1$.

   Solution: We need to solve $\langle \alpha f, \alpha f\rangle = 1$ for α. We have

$$\begin{aligned} \left[\langle \alpha f, \alpha f\rangle = 1\right] &= \left[\alpha^2\langle f,f\rangle = 1\right] \\ &= \left[2\alpha^2 = 1\right] \\ &= \left[\alpha = -\frac{1}{\sqrt{2}} \text{ or } \alpha = \frac{1}{\sqrt{2}}\right] \end{aligned}$$

   (d) Find a real number β such that $\langle \beta g, \beta g\rangle = 1$.

   Solution: We need to solve $\langle \beta g, \beta g\rangle = 1$ for α. We have

$$\begin{aligned} \left[\langle \beta g, \beta g\rangle = 1\right] &= \left[\beta^2\langle g,g\rangle = 1\right] \\ &= \left[2\beta^2 = 1\right] \\ &= \left[\beta = -\frac{1}{\sqrt{2}} \text{ or } \beta = \frac{1}{\sqrt{2}}\right] \end{aligned}$$

(e) Find a real number α such that f and $f + \alpha g$ are perpendicular.

Solution: $\left[\langle f, f + \alpha g\rangle\right.$

(f) Find a real number β such that g and $g + \alpha h$ are perpendicular.

7. Know what it means to say that two functions are orthgonal with respect to an inner product (Definition 5.3.2, page 5.3.2).

8. Be able to show that the functions $x \in [-1, 1] \mapsto x$ and $x \in [-1, 1] \mapsto 3x^2 - 1$ are orthogonal with respect to the innerproduct $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$.

9. Be able to show that the functions $x \in [-1, 1] \mapsto \mathrm{cis}(x)$ and $x \in [-1, 1] \mapsto \mathrm{cis}(2x)$ are orthogonal with respect to the innerproduct $\langle f, g\rangle = \int_{0}^{2\pi} \overline{f(x)}g(x)\,\mathrm{d}x$.

10. Know how to solve a continuous least squares problem using an orthogonal basis (see Examples 5.3.3 and 5.3.4, page 209. The actual computation for this example (top of page 210) is too involved for an exam problem, but you should understand the logic behind the calculation).

11. Given that the set of functions $\{f, g\}$ are orthonormal with respect to the innerproduct $\langle f, g\rangle = \int_{0}^{1} f(x)g(x)\,\mathrm{d}x$, find integral representations for the numbers that minimize the function

$$(a, b) \mapsto \int_0^1 \left(x - af(x) - bg(x)\right)^2 \mathrm{d}x.$$

You are not given the formula for the functions f and g, so you will not be able to find the explicit values of the numbers that minimize the given function.

12. Know the Gram-Schmidt process (page 206 of our textbook). Be able to work through a few steps of the process for a given set of functions and an innner product (Example 5.3.1, page 206).

13. For the innerproduct $\langle f, g\rangle = \int_{0}^{1} \sqrt{x} f(x)g(x)\,\mathrm{d}x$, do the Gram-Schmidt process for the functions $f_0 = x \in [0, 1] \mapsto 1$, $f_1 = x \in [0, 1] \mapsto x$, and $f_2 = x \in [0, 1] \mapsto x^2$.

14. Know the properties of the function cis. (Class on Monday)

15. Know the definition of the Fourier coefficients (Handout and classes on Frirday and Monday)

16. Understand the process of approximating the value of the Fourier coefficients by the right point rule to "discover" the DFT.