Hall of fame orthogonal set

MATH 420 & CYBR 304

Spring 2024

Long ago, we defined

Definition Let $\{f_1, f_2, \ldots, f_n\} \subset C_{[a,b]}$ and let $\langle \cdot, \cdot \rangle$ be an inner product on $C_{[a,b]}$. The set of functions $\{f_1, f_2, \ldots, f_n\}$ is *orthogonal* provided for all $k, \ell \in 1 \ldots n$ with $k \neq \ell$, we have

$$\langle f_k, f_\ell \rangle = 0. \tag{1}$$

The set $\{f_1, f_2, \ldots, f_n\}$ is *orthonormal* provided for all $k, \ell \in 1 \ldots n$, we have

$$\langle f_k, f_\ell \rangle = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}. \tag{2}$$

Facts

- Every orthonormal set is orthogonal.
- When $\{f_1, f_2, \ldots, f_n\}$ is orthogonal, but not orthonormal, we can define
$$\widehat{f_k} = \frac{f_k}{\sqrt{\langle f_k, f_k \rangle}}. \tag{3}$$
Then the set $\{\widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_n}\}$ is orthonormal.
- We've assumed that for all $k \in 1 \ldots n$, we have $\langle f_k, f_k \rangle \neq 0$.
- We assumed that our functions are continuous, but piecewise continuous or just Riemann integrable would be OK too.

More facts from the past

Let $\{f_1, f_2, \dots, f_n\} \subset C_{[a,b]}$, and suppose that this set is orthonormal. Given $F \in C_{[a,b]}$, we would like to find $c_1, c_2, \dots, c_n \in \mathrm{R}$ that minimize the function

$$(c_1, c_2, \dots, c_n) \in \mathrm{R}^n \mapsto \|F - \sum_{k=0}^{n} c_k f_k, F\|_2^2 \tag{4}$$

The solution is

$$c_k = \langle F, f_k \rangle, \text{ for } k \in 1 \dots n. \tag{5}$$

☛ Recall $\|f\|_2^2 = \langle f, f \rangle$

A hall of fame orthogonal set

Fact For $k, \ell \in \mathbb{Z}$, we have

$$\int_{-\pi}^{\pi} \cos(kx)\cos(\ell x)\,\mathrm{d}x = \begin{cases} 2\pi & k = 0 \text{ and } \ell = 0 \\ \pi & k = \ell \text{ and } k \neq 0\,, \\ 0 & k \neq \ell \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(kx)\sin(\ell x)\,\mathrm{d}x = 0,$$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(\ell x)\,\mathrm{d}x = \begin{cases} \pi & k = \ell \\ 0 & k \neq \ell \end{cases}.$$

For a proof, use

$$
\begin{aligned}
\cos(x)\cos(y) &= \frac{\cos(y+x)}{2} + \frac{\cos(y-x)}{2}, \\
\cos(x)\sin(y) &= \frac{\sin(y+x)}{2} + \frac{\sin(y-x)}{2}, \\
\sin(x)\sin(y) &= \frac{\cos(y-x)}{2} - \frac{\cos(y+x)}{2}.
\end{aligned}
$$

Trig made orthonormal

Fact For any $n \in \mathbb{Z}_{>0}$, the union of these sets of functions

$$\{x \mapsto \frac{1}{\sqrt{2\pi}}\}$$
$$\{x \mapsto \frac{1}{\sqrt{\pi}} \cos(kx) \mid k \in 1 \dots n\}$$
$$\{x \mapsto \frac{1}{\sqrt{\pi}} \sin(kx) | k \in 1 \dots n\}$$

is orthonormal on the closed interval $[-\pi, \pi]$ with respect to the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x.$$

The basis of our digital life

This set of orthonormal functions is involved with:

1. Image compression, including digital TV
2. Digitizing sound, including noise reduction
3. Medical imaging, including MRI and CAT scans

We'll learn a little about the algorithm that makes all this possible.

Every class must have an example

Let's find the continuous least squares approximation to the absolute value function using our trigonometric functions.

It's not too fun to prove, but for all nonzero integers k, we have

$$\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}|x|\cos(kx)\,\mathrm{d}x = \frac{2(-1)^k-2}{\sqrt{\pi}k^2}$$

$$\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}|x|\sin(kx)\,\mathrm{d}x = 0$$

And

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}|x|\,\mathrm{d}x = \frac{\pi^{\frac{3}{2}}}{\sqrt{2}}.$$

The verdict

For any $n \in \mathbb{Z}_{\geq 0}$, we

$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{n} \frac{\cos((2k+1)x)}{(2k+1)^2} \tag{6}$$

Does it really work?

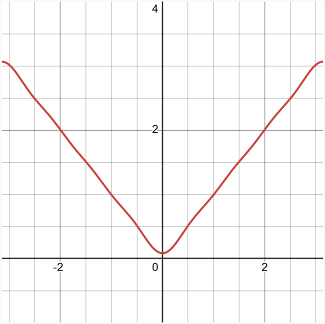

Figure 1: Graph of $x \mapsto \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{3} \frac{\cos((2k+1)x)}{(2k+1)^2}$ on the interval $[-\pi, \pi]$.

The bigger picture

Our basis functions are periodic with period 2π. Outside the interval $[-\pi, \pi]$, our approximation is bad:

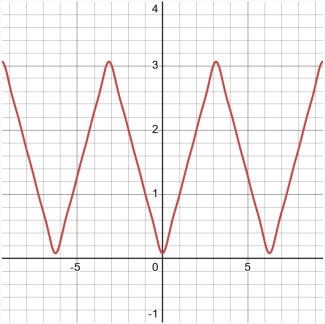

Figure 2: Graph of $x \mapsto \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{3} \frac{\cos((2k+1)x)}{(2k+1)^2}$ on the interval $[-3\pi, 3\pi]$.