

The Discrete Fourier Transform

MATH 420 & CYBR 304

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*“The profound study of nature is the most fertile source of
mathematical discovery.”* JOSEPH FOURIER

A trigonometric basis

Let $n \in \mathbf{Z}_{\geq 0}$ and let a_0 through a_n and b_1 through b_n be real numbers. Define a function F by

$$F(x) = a_0 + \sum_{k=1}^n a_k \cos(kx) + b_k \sin(kx).$$

Expressing \cos and \sin in exponential form, the formula is

$$\begin{aligned} F(x) &= a_0 + \sum_{k=1}^n a_k \frac{\operatorname{cis}(kx) + \overline{\operatorname{cis}(kx)}}{2} + b_k \frac{\operatorname{cis}(kx) - \overline{\operatorname{cis}(kx)}}{2i}, \\ &= a_0 + \sum_{k=1}^n \frac{1}{2} (a_k - ib_k) \operatorname{cis}(kx) + \frac{1}{2} (a_k + ib_k) \overline{\operatorname{cis}(kx)}. \end{aligned}$$

An exponential basis

So

$$\begin{aligned} F(x) &= a_0 + \sum_{k=1}^n \frac{1}{2}(a_k - ib_k)\text{cis}(kx) + \frac{1}{2}(a_k + ib_k)\overline{\text{cis}(kx)}, \\ &= a_0 + \text{Re} \left(\sum_{k=1}^n (a_k - ib_k)\text{cis}(kx) \right). \end{aligned}$$

Defining complex numbers c_0, c_1, \dots, c_n as

$$c_k = \begin{cases} a_0 & k = 0 \\ a_k - ib_k & k \neq 0 \end{cases},$$

we have

$$F(x) = \text{Re} \left(\sum_{k=0}^n c_k \text{cis}(kx) \right).$$

Unreal functions

If we drop the condition that F is real-valued and use the fact that $\overline{\text{cis}(kx)} = \text{cis}(-kx)$ we have

$$\begin{aligned} F(x) &= a_0 + \sum_{k=1}^n \frac{1}{2}(a_k - \mathrm{i}b_k)\text{cis}(kx) + \frac{1}{2}(a_k + \mathrm{i}b_k)\text{cis}(-kx), \\ &= \sum_{k=-n}^n c_k \text{cis}(kx). \end{aligned}$$

where

$$c_k = \begin{cases} \frac{1}{2}(a_k + \mathrm{i}b_k) & k < 0 \\ a_0 & k = 0 \\ \frac{1}{2}(a_k - \mathrm{i}b_k) & k > 0 \end{cases}.$$

Unreal but orthogonal

For any $n \in \mathbf{Z}_{\geq 0}$, the set of functions

$$\{x \in [0, 2\pi] \mapsto \text{cis}(kx) \mid k \in -n \dots n\}$$

is orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(x)} g(x) \, dx.$$

In particular for $k, \ell \in \mathbf{Z}$, we have

$$\int_0^{2\pi} \overline{\text{cis}(kx)} \text{cis}(\ell x) \, dx = \begin{cases} 0 & k \neq \ell \\ 2\pi & k = \ell \end{cases} = 2\pi \delta_{k,\ell}.$$

The Kronecker delta function

We define the Kronecker delta function δ as

$$\delta_{k,\ell} = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}.$$

👉 The arguments to the Kronecker delta function are almost expressed as subscripts.

👉 Example $\delta_{\pi,3} = 0$ and $\delta_{\pi,\pi} = 1$.

👉 We have $\sum_{k=0}^n f_k \delta_{k\ell} = \begin{cases} f_\ell & \ell \in 0 \dots n \\ 0 & \ell \notin 0 \dots n \end{cases}.$

A trigonometric puzzle

Problem Let $n \in \mathbf{Z}_{\geq 0}$ and let c_{-n} through c_n be complex numbers. Define a function F whose formula is

$$F(x) = \sum_{k=-n}^n c_k e^{ikx}$$

For a given function F , we would like to find a way to find the complex numbers c_{-n} through c_n .

Step #1

Step #1 Multiply $F(x) = \sum_{k=-n}^n c_k e^{ikx}$ by $e^{-i\ell x}$. This gives

$$e^{-i\ell x} F(x) = \sum_{k=-n}^n c_k e^{-i\ell x} e^{ikx} \quad (1)$$

Using a rule of exponents, we have

$$e^{-i\ell x} F(x) = \sum_{k=-n}^n c_k e^{i(k-\ell)x}. \quad (2)$$

Step #2

Step #2 Integrate with respect to x over the interval $[0, 2\pi]$

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = \int_0^{2\pi} \sum_{k=-n}^n c_k e^{i(k-\ell)x} dx. \quad (3)$$

Swap the integration and the finite sum:

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = \sum_{k=-n}^n c_k \int_0^{2\pi} e^{i(k-\ell)x} dx. \quad (4)$$

Step #3

Step #3 Integrate the orthogonal functions:

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = \sum_{k=-n}^n c_k 2\pi \delta_{k,\ell}. \quad (5)$$

Simplify the sum

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = 2\pi c_\ell. \quad (6)$$

What did we just do?

We've shown that if

$$F(x) = \sum_{k=-n}^n c_k e^{ikx}, \quad (7)$$

then for all $\ell \in -n \dots n$, we have

$$c_\ell = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\ell x} F(x) dx \quad (8)$$

- 👉 Given the function F , the complex numbers c_{-n} through c_n are uniquely determined.

The Fourier coefficients

For any function F that is integrable on the interval $[0, 2\pi]$ the numbers c_k defined as

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \text{cis}(kx) F(x) \, dx$$

are the *Fourier coefficients* of the function F . Generally, $k \in \mathbf{Z}$.

Continuous least squares

Fact For any integrable function F and any nonnegative integer n , the numbers c_{-n} through c_n that minimize the function

$$(c_{-n}, c_{-n+1}, \dots, c_n) \mapsto \int_0^{2\pi} \left| F(x) - \sum_{k=-n}^n c_k \operatorname{cis}(kx) \right|^2 dx$$

are the Fourier coefficients.

I know what you are thinking

Unless F is fairly simple, we have no chance at finding a formula for the numbers c_{-n} through c_n .

Have no fear: we have a tool for that. Let's use the right-point rule integration to find approximate values for the Fourier coefficients. Using n equal length subintervals of $[0, 2\pi]$, we have

$$\begin{aligned}c_\ell &\approx \frac{1}{2\pi} \frac{2\pi}{n} \sum_{k=0}^{n-1} e^{-i\frac{2\pi}{n}\ell k} F(x_k), \\&= \frac{1}{n} \sum_{k=0}^{n-1} e^{-i\frac{2\pi}{n}\ell k} F\left(\frac{2\pi}{n}k\right).\end{aligned}$$

Caveat

For large values of the integer ℓ , the integrand of

$$c_\ell = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\ell x} F(x) dx$$

becomes more and more “wiggly.” So we should expect that our right point rule quadrature rule will give less and less accurate results for larger and larger ℓ .

Nice coincidence

You might be thinking “Why the low accuracy right point rule?” Why not something like the trapezoidal rule? Ha!

For a function F with period 2π , the right-point, left-point, and trapezoidal rule all give the same value for

$$\int_0^{2\pi} \text{cis}(kx) F(x) \, dx \tag{9}$$

where $k \in \mathbf{Z}$.

DFT defined

Definition Let a_0, a_1, \dots, a_{n-1} be numbers (either real or complex). For every $\ell \in 0 \dots, n-1$, define

$$\hat{a}_\ell = \sum_{k=0}^{n-1} e^{-i\frac{2\pi}{n}\ell k} a_k$$

The list of numbers $\hat{a}_0, \hat{a}_1, \dots, \hat{a}_{n-1}$ is the *discrete Fourier transform* (DFT) of a_0, a_1, \dots, a_{n-1} .

Doubly indexed things are matrices

For any $n \in \mathbf{Z}_{>0}$, define $\mathcal{F}_{\ell,k} = e^{-i\frac{2\pi}{n}\ell k}$. We have

$$\hat{a}_\ell = \sum_{k=0}^{n-1} \mathcal{F}_{\ell,k} a_k$$

we see that the DFT is matrix multiplication. Arranging $\hat{a}_0, \dots, \hat{a}_{n-1}$ and a_0, \dots, a_{n-1} as column vectors \mathbf{a} and $\hat{\mathbf{a}}$, respectively, we have

$$\hat{\mathbf{a}} = \mathcal{F} \mathbf{a}$$

Magic properties

- 👉 When the size of a matrix \mathcal{F} is $n \times n$, ordinarily the effort required to do the matrix product $\mathcal{F} \mathbf{a}$ is proportional to n^2 .
- 👉 But the matrix \mathcal{F} has some special (almost magic) properties that makes the effort only proportional to $n \log_{10}(n)$.
- 👉 The algorithm that utilizes these properties to perform the multiplication quickly is the *Fast Fourier Transform* (FFT).
- 👉 The FFT isn't really a transform, but an algorithm that computes a transform.
- 👉 When n is large, $n^2 \gg n \log_{10}(n)$.

Inverses

For a Fourier matrix \mathcal{F} of size $n \times n$, we have

$$\overline{\mathcal{F}}\mathcal{F} = n\mathbf{I},$$

where \mathbf{I} is the $n \times n$ identity matrix. Thus

$$n\mathcal{F}^{-1} = \overline{\mathcal{F}}.$$

What does chatGPT have to say?

Question Would this document make for a good lecture for 3rd year computer science students?

Answer Yes, this document could serve as a good lecture for 3rd year computer science students, particularly those interested in topics related to mathematics, signal processing, or algorithms. (Specifics deleted).