

## **Fun with generating functions**

MATH 420 & CYBR 304

Spring 2024

*“We have no reason to believe a priori this series will exist, but we will be optimistic.”*

DONALD KNUTH

*“If you have a good recipe and can measure accurately, you can bake it.”*

FRANCES PHILLIPS

# Our problem

We would like to find an explicit formula for the sequence  $c$  that is defined recursively by

$$c_n = \begin{cases} 0 & n = 0 \text{ or } n = 1 \\ c_{n-1} + c_{n-2} + 8 & n \geq 2 \end{cases} . \quad (1)$$

👉 The first eleven terms of the sequence  $c$  are

$$0, 0, 8, 16, 32, 56, 96, 160, 264, 432, 704. \quad (2)$$

👉 To complete this task, we need *four* new tools.

# Tool 1: Binomial Coefficients.

For positive integers  $n$  and  $k$  with  $n \geq k$ , we define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \quad (3)$$

- 👉 Spoken  $\binom{n}{k}$  is “ $n$  choose  $k$ .”
- 👉 For a set  $A$  with cardinality  $n$ , the number of subsets of  $A$  that have cardinality  $k$  is  $\binom{n}{k}$ .
- 👉 The cardinality two subsets of  $\{1, 2, 3\}$  are  $\{1, 2\}$ ,  $\{1, 3\}$ , and  $\{2, 3\}$ . So  $\binom{3}{2} = 3$ .

## Tool 2: Product rule for $n^{\text{th}}$ derivatives

For smooth functions  $f$  and  $g$  and a nonnegative integer  $n$ , the  $n^{\text{th}}$  derivative of the product of  $f$  times  $g$  is

$$D^n(fg) = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}. \quad (4)$$

- 👉  $f^{(k)}$  is the  $k^{\text{th}}$  derivative of  $f$ , not the  $k^{\text{th}}$  power.
- 👉 This is a generalization of the product rule for derivatives.
- 👉  $(fg)'' = \binom{2}{0}fg'' + \binom{2}{1}f'g' + \binom{2}{2}f''g = fg'' + 2f'g' + f''g$ .

## Tool 3: $n^{\text{th}}$ derivative of a power series

Let  $c$  be a sequence and let  $n$  be a nonnegative integer. Then

$$D_x^n \left( \sum_{k=0}^{\infty} c_k x^k \right) \Big|_{x=0} = n! c_n. \quad (5)$$

- 👉 The operator order is sum first, derivative second, and evaluation at zero third.
- 👉 For this to be true, it has to be the case that the radius of convergence of the series is nonzero.

## Tool 4: $n^{\text{th}}$ derivative of a rational function

Let  $a \in \mathbf{R}_{\neq 0}$  and  $n \in \mathbf{Z}_{\geq 0}$ . Then

$$D_x^n \left( \frac{1}{x-a} \right) \Big|_{x=0} = -\frac{n!}{a^{n+1}}.$$

And let  $a, b \in \mathbf{R}_{\neq 0}$  with  $a \neq b$  and let  $n \in \mathbf{Z}_{\geq 0}$ . Then

$$D_x^n \left( \frac{1}{(x-a)(x-b)} \right) \Big|_{x=0} = -\frac{n!}{a-b} \left( \frac{1}{a^{n+1}} - \frac{1}{b^{n+1}} \right)$$

## Step 0

Multiply the recursion  $c_n = c_{n-1} + c_{n-2} + 8$  by  $z^n$  and sum from  $n = 2$  to  $n = \infty$ . This gives

$$\sum_{n=2}^{\infty} c_n z^n = \sum_{n=2}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n + \sum_{n=2}^{\infty} 8z^n \quad (6)$$

👉 The lowest sum index is two because the recursion  $c_n = c_{n-1} + c_{n-2} + 8$  is only valid for  $n \geq 2$ .

# Step 1

If needed, shift each sum index to make each summand involve  $c_n$ , not  $c_{n-1}$  or  $c_{n-2}$ .

$$\sum_{n=2}^{\infty} c_n z^n = \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} + \sum_{n=2}^{\infty} 8z^n \quad (7)$$

👉 In  $\sum_{n=2}^{\infty} c_{n-2} z^n$ , replace every  $n$  by  $n+2$ . That gives

$$\sum_{n=2}^{\infty} c_{n-2} z^n = \sum_{n+2=2}^{n+2=\infty} c_n z^{n+2} = \sum_{n=0}^{\infty} c_n z^{n+2} \quad (8)$$



## Step 2

Use the known values of  $c_0$  and  $c_1$  to extend the lower sum index of each sum to zero.

$$\sum_{n=0}^{\infty} c_n z^n = z \sum_{n=0}^{\infty} c_n z^n + z^2 \sum_{n=0}^{\infty} c_n z^n + \sum_{n=2}^{\infty} 8z^n \quad (9)$$

👉 If  $c_0$  and  $c_1$  were nonzero, we'd have a few more terms:

$$\sum_{n=2}^{\infty} c_n z^n = -c_0 - c_1 z + \sum_{n=0}^{\infty} c_n z^n \quad (10)$$

But for our case  $c_0 = 0$  and  $c_1 = 0$ .

## Step 3

Define  $G(z) = \sum_{n=0}^{\infty} c_n z^n$  and  $F(x) = \sum_{n=2}^{\infty} 8z^n$ . We have

$$G(z) = zG(z) + z^2 G(z) + F(z) \quad (11)$$

So

$$G(z) = \frac{1}{1 - z - z^2} F(z). \quad (12)$$

👉 The function  $G$  is the *generating function for the sequence  $c$* .

## Step 4

From  $G$ , determine  $c_n$ . We have

$$\begin{aligned} c_n &= \frac{1}{n!} D_z^n \frac{1}{1-z-z^2} F(z) \Big|_{z=0}, \\ &= \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} D_z^k \left( \frac{1}{1-z-z^2} \right) \Big|_{z=0} \times D_z^{n-k} F(z) \Big|_{z=0} \end{aligned}$$

## Step 5

Find the  $k^{\text{th}}$  derivative of  $\left(\frac{1}{1-z-z^2}\right)$ . The factors of  $1 - z - z^2$  are messy, so let's just give them names. Say

$$\begin{aligned} D_z^k \left( \frac{1}{1-z-z^2} \right) \Big|_{z=0} &= D_z^k \left( -\frac{1}{(z-a)(z-b)} \right) \Big|_{z=0}, \\ &= \frac{k!}{a-b} \left( \frac{1}{a^{k+1}} - \frac{1}{b^{k+1}} \right) \end{aligned}$$

Actually,

$$a = -\frac{\sqrt{5}+1}{2}, \quad b = \frac{\sqrt{5}-1}{2}$$

That makes  $a - b = -\sqrt{5}$ .

## Step 6

Find the  $n-k$  th derivative of  $F(z)$ .

$$D_z^{n-k} F(z)|_{z=0} = (n-k)! \begin{cases} 0 & n-k < 2 \\ 8 & n-k \geq 2 \end{cases}. \quad (13)$$

## Step 7

Collecting these results gives

$$c_n = -\frac{8}{\sqrt{5}} \sum_{k=0}^{n-2} \left( \frac{1}{a^{k+1}} - \frac{1}{b^{k+1}} \right)$$

👉 What's the story with all the factorials and binomial coefficients?

👉  $\frac{1}{n!} \binom{n}{k} k!(n-k)! = 1$ , that's what!

## Step 8

We could sum these geometric sums

$$c_n = -\frac{8}{\sqrt{5}} \sum_{k=0}^{n-2} \left( \frac{1}{a^{k+1}} - \frac{1}{b^{k+1}} \right)$$

but the result is messy. So, let's settle for an approximation.

The term  $b^{-k}$  grows exponentially while  $a^{-k}$  decays exponentially, so let's ignore all terms except the growing exponential. That gives

$$c_n \approx \frac{8}{\sqrt{5}} \sum_{k=0}^{n-2} \frac{1}{b^{k+1}} \approx \frac{8b}{\sqrt{5}} \frac{b^{-n}}{1-b}$$

$$c_n \approx 5.8 \times 1.6^n \tag{14}$$

The value of

$$c_{300} = 2876634 \dots (\text{lots of digits deleted}) \dots 044718400, \\ \approx 5.626 \times 10^{209}.$$

Our approximation gives  $c_{300} \approx 5.8 \times 1.6^{300} = 5.633 \times 10^{209}$ .

- 👉 At  $10^{15}$  FLOPS, it would take about  $10^{189}$  years to compute  $P_{300}$  using the Legendre polynomial recursion without memoization.
- 👉 But with memoization, computing  $P_{300}$  takes a few microseconds.



## For further study

*CONCRETE MATHEMATICS: A Foundation for Computer Science*, 2nd Edition, Ronald Graham, Donald Knuth, and Oren Patashnik, Addison-Wesley Professional, 1994.

# What did chatGPT about our class?

Upon reviewing the logical flow of the work, I didn't find any significant logical errors. However, I'll provide some feedback on specific aspects:

- **Binomial Coefficients:** The explanation and usage of binomial coefficients seem correct.
- **Product Rule for  $n$ th Derivatives:** The expression provided for the  $n$ th derivative of the product of two functions is accurate and follows the standard mathematical notation.
- **Derivatives of Power Series:** The statement regarding the  $n$ th derivative of a power series is correct, but it's essential to ensure that the assumptions about the convergence of the series hold true.

- **Derivatives of Rational Functions:** The formulas for the  $n$ th derivative of a rational function appear to be correctly derived and presented.
- **Generating Function and Recursive Sequence:** The steps taken to derive the generating function for the given recursive sequence seem logical and well-justified. The subsequent manipulation of the generating function to find the explicit formula for the sequence also appears to be sound.
- **Approximation** The approximation made to simplify the expression seems reasonable, although it's important to note that it introduces an approximation error. The comparison with the exact value of  $c_{300}$  helps to assess the accuracy of the approximation.

Overall, the logical progression of the work seems sound, with each step building upon the previous ones in a coherent

## More from chatGPT

This lecture seems like a great fit for third-year computer science majors. It covers essential mathematical concepts that are directly relevant to their studies and future careers. By exploring generating functions, binomial coefficients, and derivatives of functions, students will gain valuable insights into algorithmic analysis and data structures.

What's particularly engaging is how the lecture applies these theoretical concepts to a practical problem: solving a recurrence relation. This hands-on approach helps students see the real-world implications of the math they're learning, making the material more relatable and exciting.

The discussion on approximation adds another layer of interest by demonstrating how to balance precision and efficiency in