Fun with generating functions

MATH 202

April 16, 2024

"The law is reason unaffected by desire." ARISTOTLE

Our problem

We would like to find an explicit formula for the sequence c that is defined recursively by

$$c_n = \begin{cases} 0 & n \in \{0, 1\} \\ c_{n-1} + c_{n-2} + 8 & n \in \mathbf{Z}_{\geq 2} \end{cases} \tag{1}$$

- The first eleven terms of the seqence c are

$$0, 0, 8, 16, 32, 56, 96, 160, 264, 432, 704 \tag{2}$$

- To complete this task, we need three new tools

Tool 1: Binomial Coefficents

For positive integers n and k with $n \geq k$, we define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{3}$$

- Spoken $\binom{n}{k}$ is "n choose k."
- For a finite set A with exactly n members, $\binom{n}{k}$ is the number of subsets of A that have exactly k members.

Tool 2: Product rule for n-th derivatives

For smooth functions f and g and a nonnegative integer n, the n-th derivative of the product of f times g is

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k}. \tag{4}$$

- This is a fun generalization of the product rule for derivatives.

Tool 3: n-th derivative of a power series

Let c be a sequence and let n be a nonnegative integer. Then

$$D_x^n \left(\sum_{k=0}^{\infty} c_k x^k \right) \Bigg|_{x=0} = n! c_n. \tag{5}$$

- The operator order is sum first, derivative second, and evaluation at zero third.
- For this to be true, it has to be the case that the radius of convergence of the series is nonzero.

Tool 4: n-th derivative of a rational function

Let $a \in \mathbf{R}_{\neq 0}$ and n a nonnegative integer. Then

$$D_x^n \left(\frac{1}{x-a} \right) \Big|_{x=0} = -\frac{n!}{a^{n+1}}. \tag{6}$$

Step 0

Multiply the recursion $c_n = c_{n-1} + c_{n-2} + 8$ by z^n and sum from $n = 2$ to ∞. This gives

$$\sum_{n=2}^{\infty} c_n z^n = \sum_{n=2}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n + \sum_{n=2}^{\infty} 8 z^n \tag{7}$$

- The lowest sum index is two because the recursion $c_n = c_{n-1} + c_{n-2} + 8$ is only valid for $n \geq 2$.

Step 1

If needed, shift each sum index to make the each summand involve c_n, not c_{n-1} or c_{n-2}.

$$\sum_{n=2}^{\infty} c_n z^n = \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} + \sum_{n=2}^{\infty} 8z^n \tag{8}$$

- The lowest sum index is two because the recursion $c_n = c_{n-1} + c_{n-2} + 8$ is only valid for $n \geq 2$.

Step 2

Use the known values of c_0 and c_1 to extend the lower sum index of each sum to zero.

$$\sum_{n=0}^{\infty} c_n z^n = z \sum_{n=0}^{\infty} c_n z^n + z^2 \sum_{n=0}^{\infty} c_n z^n + \sum_{n=2}^{\infty} 8 z^n \tag{9}$$

☛ If c_0 and c_1 were nonzero, we'd have a few more terms!

Step 3

Define $G(z) = \sum_{n=0}^{\infty} c_n z^n$ and $F(x) = \sum_{n=2}^{\infty} 8z^n$. We have

$$G(z) = zG(z) + z^2 G(z) + F(z) \tag{10}$$

So

$$G(z) = \frac{1}{1 - z - z^2} F(x). \tag{11}$$

Step 3

From G, determine c_n. We have

$$
\begin{aligned}
c_n &= \frac{1}{n!}\, D_z^n \frac{1}{1-z-z^2} F(x)\bigg|_{x=0}, \\
&= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} D_z^k \left(\frac{1}{1-z-z^2}\right) D_z^{n-k} F(z)\bigg|_{z=0}
\end{aligned}
$$

Step 3

Find the k-th derivative of $\left(\frac{1}{1-z-z^2}\right)$

$$\begin{aligned}
D_z^k\left(\frac{1}{1-z-z^2}\right) &= D_z^k\left(-\frac{1}{z^2+z-1}\right), \\
&= D_z^k\left(-\frac{1}{(z-\alpha)(z-\beta)}\right), \\
&= \frac{1}{\beta-\alpha}D_z^k\left(\frac{1}{z-\alpha}+\frac{1}{z-\beta}\right), \\
&= -\frac{k!}{\beta-\alpha}\left(\frac{1}{\alpha^{k+1}}+\frac{1}{\beta^{k+1}}\right), \\
&= \frac{k!}{\sqrt{5}}\left(\frac{1}{\alpha^{k+1}}+\frac{1}{\beta^{k+1}}\right),
\end{aligned}$$

Tip: When things get messy, call things names

Step 3

Find the n-k th derivative of $F(z)$.

$$D_z^{n-k} F(z)|_{z=0} = (n-k)! \begin{cases} 0 & n-k < 2 \\ 8 & n-k \geq 2 \end{cases}. \tag{12}$$

Step 3

Collecting these results gives

$$
\begin{aligned}
c_n &= \frac{8}{\sqrt{5}} \sum_{k=2}^{n} \left(\frac{1}{\alpha^{k+1}} + \frac{1}{\beta^{k+1}} \right) \\
&= \frac{8}{\sqrt{5}} \left(\frac{\beta^{-n-1} - \frac{1}{\beta^2}}{\left(\frac{1}{\beta} - 1\right)\beta} + \frac{\alpha^{-n-1} - \frac{1}{\alpha^2}}{\left(\frac{1}{\alpha} - 1\right)\alpha} \right)
\end{aligned}
$$

This is a bit messy–both the term β^{-n} grows exponentially while α^{-n} decays exponentially, so let's ignore all terms except the growing exponential. That gives

$$c_n \approx 5.8 \times 1.6^n \tag{13}$$

The value of

$$\begin{aligned} c_{300} &= 287663460165266848769412532137751279241893771447 \\ &\approx 5.626 \times 10^{209}. \end{aligned}$$

Our aproximation gives $c_{300} \approx 5.8 \times 1.6^{300} = 5.633 \times 10^{209}$.