CYBR 304 & MATH 420
Exam III

Name: ______________________________
Row and Seat: ______________________________

Exam III has questions 1 through 7 with a total of 50 points.

1. **Show all of your work.** Do not expect to earn full credit for a correct answer without the needed work.
2. Divine intervention is *not* a substitute for showing your work.
3. Respect equality. Do not write equal (=) between expressions that are not equal.
4. If your answer is wrong, but your work shows me that you know the major steps in solving a problem, you will likely earn some partial credit.
5. Your work should convince me that not only could you correctly solve the given problem, but you could also solve any related problem.
6. If a question asks for an explanation, write your answer as an English sentence.
7. No talking, no sharing calculators, and no scratch paper.
8. Turn your phone off and put it out of sight.
9. Clear your desk of everything, except a pencil, an eraser, and a calculator.
10. If you never make a mistake, you may use ink; otherwise, use a pencil.
11. Do not unstaple the pages of your exam.
12. We'll all start at the same time; it is the polite thing to do.
13. Write your answers in the space provided.
14. If you do not want something graded, erase it or clearly cross it out.
15. You may stare at your feet, your paper, or the ceiling, but nowhere else.
16. If you wear a baseball cap, wear it backwards, so I can see your eyes.
17. Work each problem correctly.
18. When you are finished, collect your things, place your exam paper in the folder on the front desk, and quietly leave the room.
19. After you turn in your paper, I will not answer questions about the test until after it is graded.
20. Not knowing the rules is not a valid excuse for not following them.
21. Read all directions and problems carefully.

1. Find a simple representation of each sum:

5 (a) $\sum_{k=-10}^{10} \delta_{k,5}$.

Solution: We have

$$\sum_{k=-10}^{10} \delta_{k,5} = \delta_{-10,5}+\delta_{-9,5}+\cdots+\delta_{5,5}+\delta_{6,5}+\cdots+\delta_{10,5} = 0+0+\cdots+1+0+0+\cdots+0 = 1.$$

Since the Kronecker delta $\delta_{k,5}$ is 1 if $k = 5$ and 0 otherwise, the sum is simply $\delta_{5,5} = 1$.

5 (b) $\sum_{k=-10}^{10} \delta_{k,15}$.

Solution: We have

$$\sum_{k=-10}^{10} \delta_{k,15} = \delta_{-10,15} + \delta_{-9,15} + \cdots + \delta_{10,15} = 0+0+\cdots+0+0 = 0.$$

Since the Kronecker delta $\delta_{k,15}$ is 1 if $k = 15$ and 0 otherwise, the summand simplifies to 0, making the sum zero.

2. Let f, g, and h be functions. For some inner product, suppose

$$\begin{bmatrix} \langle f,f\rangle & \langle f,g\rangle & \langle f,h\rangle \\ \langle g,f\rangle & \langle g,g\rangle & \langle g,h\rangle \\ \langle h,f\rangle & \langle h,g\rangle & \langle h,h\rangle \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 8 \\ 4 & 8 & 1 \end{bmatrix}.$$

5 (a) Find the numerical value of $\langle f, g+h\rangle$.

Solution: Using linearity of the inner product,

$$\langle f, g+h\rangle = \langle f,g\rangle + \langle f,h\rangle = 3+4 = 7.$$

5 (b) Find a positive real number α such that $\langle \alpha f, \alpha f\rangle = 1$.

Solution: We need $\langle \alpha f, \alpha f\rangle = \alpha^2\langle f,f\rangle = 2\alpha^2 = 1$. Solving for α and keeping in mind that we need a positive solution, we get $\alpha = \frac{1}{\sqrt{2}}$.

5 (c) Find a real number α such that f and $f+\alpha g$ are perpendicular.

Solution: We need $\langle f, f+\alpha g\rangle = 0$. Using linearity of the inner product,

$$0 = \langle f, f+\alpha g\rangle = \langle f,f\rangle + \alpha\langle f,g\rangle = 2+3\alpha.$$

Thus, $\alpha = -\frac{2}{3}$.

$\boxed{5}$ 3. Define $P_1(x) = x$ and $P_2(x) = 3x^2 - 1$. Show that P_1 and P_2 are perpendicular with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$.

Solution: To show that P_1 and P_2 are perpendicular, we need to show that $\langle P_1, P_2\rangle = 0$. Using the given inner product,

$$\langle P_1, P_2\rangle = \int_{-1}^{1} x(3x^2 - 1)\,\mathrm{d}x = \left[\frac{3}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{1} = 0.$$

Thus, P_1 and P_2 are perpendicular.

$\boxed{5}$ 4. Let $F \in C_{[0,2\pi]}$. For $k \in \mathbf{Z}$, define the numbers c_k as

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} F(x)\,\mathrm{d}x.$$

Using an n-panel right point rule, find approximate values for the numbers c_k in terms of summation notation.

Solution: The right point rule for an interval $[a, b]$ with n panels and function $F(x)$ is given by:

$$\int_a^b F(x)\,\mathrm{d}x \approx \sum_{k=1}^{n} F(x_k)\Delta x,$$

where $x_k = a + k\Delta x$ for $k = 0, 1, \ldots, n$, and $\Delta x = \frac{b-a}{n}$. Applying this to our integral, we get:

$$c_k \approx \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{-\mathrm{i}kx_j} F(x_j)$$

where $x_j = \frac{2\pi j}{n}$ for $j = 0, 1, \ldots, n-1$.

$\boxed{5}$ 5. Apply the Gram-Schmidt process to the functions $f_0(x) = 1$ and $f_1(x) = x$ with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$. You should choose $g_0(x) = 1$. Your only task is to find a formula for g_1.

Solution: We start with $g_0(x) = f_0(x) = 1$. Then,

$$g_1(x) = f_1(x) - \frac{\langle f_1, g_0\rangle}{\langle g_0, g_0\rangle} g_0(x),$$

where

$$\langle f_1, g_0\rangle = \int_0^1 x \cdot 1\,\mathrm{d}x = \frac{1}{2}, \quad \text{and} \quad \langle g_0, g_0\rangle = \int_0^1 1 \cdot 1\,\mathrm{d}x = 1.$$

Therefore,

$$g_1(x) = x - \frac{1/2}{1} \cdot 1 = x - \frac{1}{2}.$$

6. Consider the family polynomials defined by the recurrence relation:

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x),$$

where $P_0(x) = 1$ and $P_1(x) = x$.

$\boxed{5}$ (a) Using the recurrence relation, find a formula for P_2.

Solution: Using the recurrence relation, we have:

$$P_2(x) = 2xP_1(x) - P_0(x) = 2x^2 - 1.$$

$\boxed{5}$ 7. Consider the Julia code for evaluation these polynomials

```
function P(n::Integer,x::Number)
    if n==0
        1
    elseif n==1
        x
    else
        2*x*P(n-1,x) - P(n-2,x)
    end
```

Solution: The cost to evaluate `2*x*P(n-1,x)-P(n-2,x)` is the cost to evaluate `P(n-1,x)`, plus the cost to evaluate `P(n-2,x)`, plus three additional floating point operations; these three additional floating point operations come from the two explicit multiplications in `2*x*P(n-1,x)` and from the subtraction in `2*x*P(n-1,x)-P(n-2,x)`.

The cost to evaluate `P(n-1,x)` is C_{n-1}; the cost to evaluate `P(n-2,x)` is C_{n-2}. Thus to evaluate $P(n,x)$ using the given Julia code, we can express the number of floating point operations C_n recursively as follows:

$$C_0 = 0, \quad C_1 = 0, \quad \text{and} \quad C_n = C_{n-1} + C_{n-2} + 3 \quad \text{for } n \geq 2.$$