

# The Discrete Fourier Transform

MATH 420 & CYBR 304

Spring 2024

*“The profound study of nature is the most fertile source of mathematical discovery.”* Joseph Fourier

# A trigonometric basis

Let  $n \in \mathbb{Z}_{\geq 0}$  and let  $a_0$  through  $a_n$  and  $b_1$  through  $b_n$  be real numbers. Define a function  $F$  by

$$F(x) = a_0 + \sum_{k=1}^n a_k \cos(kx) + b_k \sin(kx).$$

Expressing  $\cos$  and  $\sin$  in exponential form, the formula is

$$\begin{aligned} F(x) &= a_0 + \sum_{k=1}^n a_k \frac{\operatorname{cis}(kx) + \overline{\operatorname{cis}(kx)}}{2} + b_k \frac{\operatorname{cis}(kx) - \overline{\operatorname{cis}(kx)}}{2i}, \\ &= a_0 + \sum_{k=1}^n \frac{1}{2} (a_k - ib_k) \operatorname{cis}(kx) + \frac{1}{2} (a_k + ib_k) \overline{\operatorname{cis}(kx)}. \end{aligned}$$

# An exponential basis

So

$$\begin{aligned} F(x) &= a_0 + \sum_{k=1}^n \frac{1}{2} (a_k - ib_k) \operatorname{cis}(kx) + \frac{1}{2} (a_k + ib_k) \overline{\operatorname{cis}(kx)}, \\ &= a_0 + \operatorname{Re} \left( \sum_{k=1}^n (a_k - ib_k) \operatorname{cis}(kx) \right). \end{aligned}$$

Defining complex numbers  $c_0, c_1, \dots, c_n$  as

$$c_k = \begin{cases} a_0 & k = 0 \\ a_k - ib_k & k \neq 0 \end{cases},$$

we have

$$F(x) = \operatorname{Re} \left( \sum_{k=0}^n c_k \operatorname{cis}(kx) \right).$$

# Unreal functions

If we drop the condition that  $F$  is real-valued and use the fact that  $\overline{\text{cis}(kx)} = \text{cis}(-kx)$  we have

$$\begin{aligned} F(x) &= a_0 + \sum_{k=1}^n \frac{1}{2}(a_k - ib_k)\text{cis}(kx) + \frac{1}{2}(a_k + ib_k)\text{cis}(-kx), \\ &= \sum_{k=-n}^n c_k \text{cis}(kx). \end{aligned}$$

where

$$c_k = \begin{cases} \frac{1}{2}(a_k + ib_k) & k < 0 \\ a_0 & k = 0 \\ \frac{1}{2}(a_k - ib_k) & k > 0 \end{cases}.$$

# Unreal but orthogonal

For any  $n \in \mathbb{Z}_{\geq 0}$ , the set of functions

$$\{x \in [0, 2\pi] \mapsto \text{cis}(kx) \mid k \in -n \dots n\}$$

is orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(x)} g(x) \, dx.$$

In particular for  $k, \ell \in \mathbb{Z}$ , we have

$$\int_0^{2\pi} \overline{\text{cis}(kx)} \text{cis}(\ell x) \, dx = \begin{cases} 0 & k \neq \ell \\ 2\pi & k = \ell \end{cases} = 2\pi \delta_{k,\ell}.$$

# The Kronecker delta function

We define the Kronecker delta function  $\delta$  as

$$\delta_{k,\ell} = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}.$$

👉 The arguments to the Kronecker delta function are almost expressed as subscripts.

👉 Example  $\delta_{\pi,3} = 0$  and  $\delta_{\pi,\pi} = 1$ .

👉 We have  $\sum_{k=0}^n f_k \delta_{k\ell} = \begin{cases} f_\ell & \ell \in 0 \dots n \\ 0 & \ell \notin 0 \dots n \end{cases}.$

# A trigonometric puzzle

**Problem** Let  $n \in \mathbb{Z}_{\geq 0}$  and let  $c_{-n}$  through  $c_n$  be complex numbers. Define a function  $F$  whose formula is

$$F(x) = \sum_{k=-n}^n c_k e^{ikx}$$

For a given function  $F$ , we would like to find a way to find the complex numbers  $c_{-n}$  through  $c_n$ .

# Step #1

**Step #1** Multiply  $F(x) = \sum_{k=-n}^n c_k e^{ikx}$  by  $e^{-i\ell x}$ . This gives

$$e^{-i\ell x} F(x) = \sum_{k=-n}^n c_k e^{-i\ell x} e^{ikx} \quad (1)$$

Using a rule of exponents, we have

$$e^{-i\ell x} F(x) = \sum_{k=-n}^n c_k e^{-i(k-\ell)x}. \quad (2)$$



## Step #2

**Step #2** Integrate with respect to  $x$  over the interval  $[0, 2\pi]$

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = \int_0^{2\pi} \sum_{k=-n}^n c_k e^{-i(k-\ell)x} dx. \quad (3)$$

Swap the integration and the finite sum:

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = \sum_{k=-n}^n c_k \int_0^{2\pi} e^{-i(k-\ell)x} dx. \quad (4)$$

## Step #3

**Step #3** Integrate the orthogonal functions:

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = \sum_{k=-n}^n c_k 2\pi \delta_{k,\ell}. \quad (5)$$

Simplify the sum

$$\int_0^{2\pi} e^{-i\ell x} F(x) dx = 2\pi c_\ell. \quad (6)$$

# What did we just do?

We've shown that if

$$F(x) = \sum_{k=-n}^n c_k e^{ikx}, \quad (7)$$

then for all  $\ell \in -n \dots n$ , we have

$$c_\ell = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\ell x} F(x) dx \quad (8)$$

👉 Given the function  $F$ , the complex numbers  $c_{-n}$  through  $c_n$  are uniquely determined.

# The Fourier coefficients

For any function  $F$  that is integrable on the interval  $[0, 2\pi]$  the numbers  $c_k$  defined as

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \text{cis}(kx) F(x) dx$$

are the *Fourier coefficients* of the function  $F$ . Generally,  $k \in \mathbb{Z}$ .

# I know what you are thinking

Unless  $F$  is fairly simple, we have no chance at finding a formula for the numbers  $c_{-n}$  through  $c_n$ .

Have no fear: we have a tool for that. Let's use the right-point rule integration to find approximate values for the Fourier coefficients. Using  $n$  equal length subintervals of  $[0, 2\pi]$ , we have

$$\begin{aligned} c_\ell &\approx \frac{1}{2\pi} \frac{2\pi}{n} \sum_{k=0}^{n-1} e^{-i\frac{2\pi}{n}\ell k} F(x_k), \\ &= \frac{1}{n} \sum_{k=0}^{n-1} e^{-i\frac{2\pi}{n}\ell k} F\left(\frac{2\pi}{n}k\right). \end{aligned}$$

# Caveat

For large values of the integer  $\ell$ , the integrand of

$$c_\ell = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\ell x} F(x) dx$$

becomes more and more “wiggly.” So we should expect that our right point rule quadrature rule will give less and less accurate results for larger and larger  $\ell$ .

# Nice coincidence

You might be thinking “Why the low accuracy right point rule?” Why not something like the trapezoidal rule? Ha!

For a function  $F$  with period  $2\pi$ , the right-point, left-point, and trapezoidal rule all give the same value for

$$\int_0^{2\pi} \text{cis}(kx)F(x)dx \tag{9}$$

where  $k \in \mathbb{Z}$ .

# DFT defined

**Definition** Let  $a_0, a_1, \dots, a_{n-1}$  be numbers (either real or complex). For every  $\ell \in 0 \dots, n-1$ , define

$$\hat{a}_\ell = \sum_{k=0}^{n-1} e^{-i\frac{2\pi}{n}\ell k} a_k$$

The list of numbers  $\hat{a}_0, \hat{a}_1, \dots, \hat{a}_{n-1}$  is the *discrete Fourier transform* (DFT) of  $a_0, a_1, \dots, a_{n-1}$ .



# Doubly indexed things are matrices

For any  $n \in \mathbb{Z}_{>0}$ , define  $\mathcal{F}_{\ell,k} = e^{-i\frac{2\pi}{n}\ell k}$ . We have

$$\hat{a}_\ell = \sum_{k=0}^{n-1} \mathcal{F}_{\ell,k} a_k$$

we see that the DFT is matrix multiplication. Arranging  $\hat{a}_0, \dots, \hat{a}_{n-1}$  and  $a_0, \dots, a_{n-1}$  as column vectors  $\mathbf{a}$  and  $\hat{\mathbf{a}}$ , respectively, we have

$$\hat{\mathbf{a}} = \mathcal{F} \mathbf{a}$$

# Magic properties

- 👉 When the size of a matrix  $\mathcal{F}$  is  $n \times n$ , ordinarily the effort required to do the matrix product  $\mathcal{F}a$  is proportional to  $n^2$ .
- 👉 But the matrix  $\mathcal{F}$  has some special (almost magic) properties that makes the effort only proportional to  $n \log_{10}(n)$ .
- 👉 The algorithm that utilizes these properties to perform the multiplication quickly is the *Fast Fourier Transform* (FFT).
- 👉 The FFT isn't really a transform, but an algorithm that computes a transform.
- 👉 When  $n$  is large,  $n^2 \gg n \log_{10}(n)$ .

# Inverses

For a Fourier matrix  $\mathcal{F}$  of size  $n \times n$ , we have

$$\overline{\mathcal{F}} \mathcal{F} = nI,$$

where  $I$  is the  $n \times n$  identity matrix. Thus

$$n\mathcal{F}^{-1} = \overline{\mathcal{F}}.$$

# What does chatGPT have to say?

**Question** Would this document make for a good lecture for 3rd year computer science students?

**Answer** Yes, this document could serve as a good lecture for 3rd year computer science students, particularly those interested in topics related to mathematics, signal processing, or algorithms. (Specifics deleted).