**CYRR 304** **Name:**
**Homework 10, Spring 2024**

Homework 10 has questions 1 through 5 with a total of of 50 points. The point value for each question or part of a question is in the box following each question or part of a question. This work is due **Saturday 20 April** at 11:59 PM.

In class (and in our textbook), we applied the Gram-Schmidt process to the functions $x \mapsto 1$, $x \mapsto x, x \mapsto x^2, \ldots$ using the interval $[-1,1]$ and an inner product defined as

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x.$$

For this homework, we'll do the same, except that we will change the inner product to

$$\langle f, g\rangle = \int_{-1}^{1} \sqrt{1-x^2} f(x)g(x)\,\mathrm{d}x.$$

This inner product has all the required properties and the weight function satisfies the requirements on page 205 of our textbook.

Doing the Gram-Schimidt process by hand for these functions will develop strength and character, but it's tedious and challenging to get correct. The result of doing so is

$$\begin{aligned}
\mathrm{C}_0(x) &= \frac{\sqrt{2}}{\sqrt{\pi}},\\
\mathrm{C}_1(x) &= \frac{\sqrt{8}}{\sqrt{\pi}}x,\\
\mathrm{C}_2(x) &= \frac{\sqrt{2}}{\sqrt{\pi}}(2x-1)(2x+1),\\
\mathrm{C}_3(x) &= \frac{\sqrt{32}}{\sqrt{\pi}}x(2x^2-1),\\
\mathrm{C}_4(x) &= \frac{\sqrt{2}}{\sqrt{\pi}}(4x^2-2x-1)(4x^4+2x+1).
\end{aligned}$$

These functions have been normalized so that $\langle \mathrm{C}_k, \mathrm{C}_\ell\rangle = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$. These functions are a special case of the well-known *ultraspherical* polynomials that are known as the *Chebyshev polynomials of the second kind*. The zeros of $\mathrm{C}_4$ are

$$x_0 = -\left(\frac{\sqrt{5}+1}{4}\right), x_1 = -\left(\frac{\sqrt{5}-1}{4}\right), x_2 = \frac{\sqrt{5}-1}{4}, x_3 = \frac{\sqrt{5}+1}{4}$$

$\boxed{10}$ 1. These polynomials can be defined recursively by

$$\mathrm{C}_{n+1}(x) = 2x\,\mathrm{C}_n(x) - \mathrm{C}_{n-1}(x) = 0,$$

and $\mathrm{C}_0(x) = \frac{\sqrt{2}}{\sqrt{\pi}}, \mathrm{C}_1(x) = \frac{\sqrt{8}}{\sqrt{\pi}}x$. Write a Julia function `ultraspherical(n::Integer, x::Number)` that evaluates these polynomials.

$\boxed{10}$ 2. Check that $C_{10}(0.23) \approx 0.6818339026993401$ and $C_{15}(-0.19) \approx 0.06737434860812452$

$\boxed{10}$ 3. Use Gadfly to graph $C_0, C_1, C_2, C_3$, and $C_4$ on the interval $[-1, 1]$.

$\boxed{10}$ 4. As we did for the Legendre polynomials, let's build a quadrature rule whose knots are the zeros of $C_4$. The corresponding weights are

$$\mathrm{w0} = -\left(\frac{\left(\sqrt{5}-5\right)\pi}{40}\right), \mathrm{w1} = \frac{\left(\sqrt{5}+5\right)\pi}{40}, \mathrm{w2} = \frac{\left(\sqrt{5}+5\right)\pi}{40}, \mathrm{w3} = -\left(\frac{\left(\sqrt{5}-5\right)\pi}{40}\right)$$

The resulting quadrature rule is

$$\int_{-1}^{1} \sqrt{1-x^2}\,F(x)\,dx \approx \sum_{k=0}^{3} w_k F(x_k). \tag{1}$$

Write a Julia function that implements this quadrature rule.

$\boxed{10}$ 5. Show that this quadrature rule is pretty close to exact when F is a polynomial of degree seven or less. To do this, test the rule on each of the following:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2},$$
$$\int_{-1}^{1} x\sqrt{1-x^2}\,dx = 0,$$
$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = \frac{\pi}{8},$$
$$\int_{-1}^{1} x^3\sqrt{1-x^2}\,dx = 0,$$
$$\int_{-1}^{1} x^4\sqrt{1-x^2}\,dx = \frac{\pi}{16},$$
$$\int_{-1}^{1} x^5\sqrt{1-x^2}\,dx = 0,$$
$$\int_{-1}^{1} x^6\sqrt{1-x^2}\,dx = \frac{5\pi}{128},$$
$$\int_{-1}^{1} x^7\sqrt{1-x^2}\,dx = 0.$$