

Homework 9, Spring 2024

Homework 9 has questions 1 through 5 with a total of 50 points. The point value for each question or part of a question is in the box following each question or part of a question. This work is due **Saturday 20 April** at 11:59 PM.

In class (and in our textbook), we applied the Gram-Schmidt process to the functions $x \mapsto 1$, $x \mapsto x$, $x \mapsto x^2, \dots$ using the interval $[-1, 1]$ and an inner product defined as

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx.$$

For this homework, we'll do the same, except that we will change the inner product to

$$\langle f, g \rangle = \int_{-1}^1 \sqrt{1-x^2} f(x)g(x) dx.$$

This inner product has all the required properties and the weight function satisfies the requirements on page 205 of our textbook.

Doing the Gram-Schmidt process by hand for these functions might develop your character, but it's tedious and challenging to get correct. The result of doing so is

$$C_0(x) = \frac{\sqrt{2}}{\sqrt{\pi}},$$

$$C_1(x) = \frac{\sqrt{8}}{\sqrt{\pi}}x,$$

$$C_2(x) = \frac{\sqrt{2}}{\sqrt{\pi}}(2x-1)(2x+1),$$

$$C_3(x) = \frac{\sqrt{32}}{\sqrt{\pi}}x(2x^2-1),$$

$$C_4(x) = \frac{\sqrt{2}}{\sqrt{\pi}}(4x^2-2x-1)(4x^4+2x+1).$$

These functions have been normalized so that $\langle C_k, C_\ell \rangle = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$. These functions are a special case of the well-known *ultraspherical* polynomials that are known as the *Chebyshev polynomials of the second kind*. The zeros of C_4 are

$$x_0 = -\left(\frac{\sqrt{5}+1}{4}\right), x_1 = \frac{\sqrt{5}-1}{4}, x_2 = -\left(\frac{\sqrt{5}-1}{4}\right), x_3 = \frac{\sqrt{5}+1}{4}.$$

- 10 1. These polynomials can be defined recursively by

$$(n+1)C_{n+1}(x) = 2\left(n + \frac{3}{2}\right)x C_n(x) - (n+2)C_{n-1}(x) = 0,$$

and $C_0(x) = \frac{\sqrt{2}}{\sqrt{\pi}}$, $C_1(x) = \frac{\sqrt{8}}{\sqrt{\pi}}x$. Write a Julia function `ultraspherical(n::Integer, x::Number)` that evaluates these polynomials.

- 10 2. Check that $C_{10}(0.23) \approx 0.6818339026993401$ and $C_{15}(-0.19) \approx 0.06737434860812452$
- 10 3. Use Gadfly to graph C_0, C_1, C_2, C_3 , and C_4 on the interval $[-1, 1]$.
- 10 4. As we did for the Legendre polynomials, let's build a quadrature rule whose knots are the zeros of C_4 . The corresponding weights are

$$w_0 = -\left(\frac{(\sqrt{5}-5)\pi}{40}\right), w_1 = \frac{(\sqrt{5}+5)\pi}{40}, w_2 = \frac{(\sqrt{5}+5)\pi}{40}, w_3 = -\left(\frac{(\sqrt{5}-5)\pi}{40}\right)$$

The resulting quadrature rule is

$$\int_{-1}^1 \sqrt{1-x^2} F(x) dx \approx \sum_{k=0}^3 w_k F(x_k). \quad (1)$$

Write a Julia function that implements this quadrature rule.

- 10 5. Show that this quadrature rule is pretty close to exact when F is a polynomial of degree eight or less. To do this, test the rule on

$$\begin{aligned} \int_{-1}^1 \sqrt{1-x^2} dx &= \frac{\pi}{2}, \\ \int_{-1}^1 x \sqrt{1-x^2} dx &= 0, \\ \int_{-1}^1 x^2 \sqrt{1-x^2} dx &= \frac{\pi}{8}, \\ \int_{-1}^1 x^3 \sqrt{1-x^2} dx &= 0, \\ \int_{-1}^1 x^4 \sqrt{1-x^2} dx &= \frac{\pi}{16}, \\ \int_{-1}^1 x^5 \sqrt{1-x^2} dx &= 0, \\ \int_{-1}^1 x^6 \sqrt{1-x^2} dx &= \frac{5\pi}{128}, \\ \int_{-1}^1 x^7 \sqrt{1-x^2} dx &= 0. \end{aligned}$$