Fun with generating functions

MATH 202

April 16, 2024

"The law is reason unaffected by desire." ARISTOTLE

Our problem

We would like to find an explicit formula for the sequence c that is defined recursively by

$$c_n = \begin{cases} 0 & n \in \{0, 1\} \\ c_{n-1} + c_{n-2} + 8 & n \in \mathbf{Z}_{\geq 2} \end{cases} \tag{1}$$

- The first eleven terms of the seqence c are
$$0, 0, 8, 16, 32, 56, 96, 160, 264, 432, 704 \tag{2}$$
- To complete this task, we need three new tools

Tool 1: Binomial Coefficents

For positive integers n and k with $n \geq k$, we define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{3}$$

- Spoken $\binom{n}{k}$ is "n choose k."
- For a finite set A with exactly n members, $\binom{n}{k}$ is the number of subsets of A that have exactly k members.

Tool 2: Product rule for n-th derivatives

For smooth functions f and g and a nonnegative integer n, the n-th derivative of the product of f times g is

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k}. \tag{4}$$

- This is a fun generalization of the product rule for derivatives.

Tool 3: n-th derivative of a power series

Let c be a sequence and let n be a nonnegative integer. Then

$$D_x^n \left(\sum_{k=0}^{\infty} c_k x^k \right) \Bigg|_{x=0} = n! c_n. \tag{5}$$

- The operator order is sum first, derivative second, and evaluation at zero third.
- For this to be true, it has to be the case that the radius of convergence of the series is nonzero.

Tool 4: n-th derivative of a rational function

Let $a \in \mathbf{R}_{\neq 0}$ and n a nonnegative integer. Then

$$D_x^n \left(\frac{1}{x-a} \right) \Big|_{x=0} = -\frac{n!}{a^{n+1}}.$$

And let $a, b \in \mathbf{R}_{\neq 0}$ with $a \neq b$ and let n a nonnegative integer. Then

$$D_x^n \left(\frac{1}{(x-a)(x-b)} \right) \Big|_{x=0} = -\frac{n!}{a-b} \left(\frac{1}{a^{n+1}} - \frac{1}{b^{n+1}} \right)$$

Step 0

Multiply the recursion $c_n = c_{n-1} + c_{n-2} + 8$ by z^n and sum from $n = 2$ to ∞. This gives

$$\sum_{n=2}^{\infty} c_n z^n = \sum_{n=2}^{\infty} c_{n-1} z^n + \sum_{n=2}^{\infty} c_{n-2} z^n + \sum_{n=2}^{\infty} 8 z^n \tag{6}$$

- The lowest sum index is two because the recursion $c_n = c_{n-1} + c_{n-2} + 8$ is only valid for $n \geq 2$.

Step 1

If needed, shift each sum index to make the each summand involve c_n, not c_{n-1} or c_{n-2}.

$$\sum_{n=2}^{\infty} c_n z^n = \sum_{n=1}^{\infty} c_n z^{n+1} + \sum_{n=0}^{\infty} c_n z^{n+2} + \sum_{n=2}^{\infty} 8z^n \tag{7}$$

- The lowest sum index is two because the recursion $c_n = c_{n-1} + c_{n-2} + 8$ is only valid for $n \geq 2$.

Step 2

Use the known values of c_0 and c_1 to extend the lower sum index of each sum to zero.

$$\sum_{n=0}^{\infty} c_n z^n = z \sum_{n=0}^{\infty} c_n z^n + z^2 \sum_{n=0}^{\infty} c_n z^n + \sum_{n=2}^{\infty} 8z^n \tag{8}$$

☛ If c_0 and c_1 were nonzero, we'd have a few more terms!

Step 3

Define $G(z) = \sum_{n=0}^{\infty} c_n z^n$ and $F(x) = \sum_{n=2}^{\infty} 8z^n$. We have

$$G(z) = zG(z) + z^2 G(z) + F(z) \tag{9}$$

So

$$G(z) = \frac{1}{1 - z - z^2} F(x). \tag{10}$$

Step 3

From G, determine c_n. We have

$$
\begin{aligned}
c_n &= \frac{1}{n!}\ D_z^n \frac{1}{1-z-z^2} F(x)\bigg|_{x=0}, \\
&= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} D_z^k \left(\frac{1}{1-z-z^2}\right) D_z^{n-k} F(z)\bigg|_{z=0}
\end{aligned}
$$

Step 3

Find the k-th derivative of $\left(\frac{1}{1-z-z^2}\right)$. The factors of $1-z-z^2$ are a bit messy, so let's just give them names. Say

$$
\begin{aligned}
D_z^k\left(\frac{1}{1-z-z^2}\right) &= D_z^k\left(-\frac{1}{(z-a)(z-b)}\right), \\
&= \frac{n!}{a-b}\left(\frac{1}{a^{n+1}}-\frac{1}{b^{n+1}}\right)
\end{aligned}
$$

Actually,

$$
a = -\frac{\sqrt{5}+1}{2}, \quad b = \frac{\sqrt{5}-1}{2}
$$

That makes $a-b=-\sqrt{5}$.

Step 3

Find the n-k th derivative of $F(z)$.

$$D_z^{n-k} F(z)|_{z=0} = (n-k)! \begin{cases} 0 & n-k < 2 \\ 8 & n-k \geq 2 \end{cases}. \tag{11}$$

Step 3

Collecting these results gives

$$c_n = -\frac{8}{\sqrt{5}} \sum_{k=0}^{n-2} \left(\frac{1}{a^{k+1}} - \frac{1}{b^{k+1}} \right)$$

We find an explicit sum, but the result is a bit messy. Let's settle for an approximation. The term b^{-k} grows exponentially while a^{-n} decays exponentially, so let's ignore all terms except the growing exponential. That gives

$$c_n \approx \frac{8}{\sqrt{5}} \sum_{k=0}^{n-2} \frac{1}{b^{k+1}} \approx \frac{8b}{\sqrt{5}} \frac{b^{-n}}{1-b}$$

$$c_n \approx 5.8 \times 1.6^n \tag{12}$$

The value of

$$
\begin{aligned}
c_{300} &= 287663460165266848769412532137751279241893771447 \\
&\approx 5.626 \times 10^{209}.
\end{aligned}
$$

Our aproximation gives $c_{300} \approx 5.8 \times 1.6^{300} = 5.633 \times 10^{209}$.