CYBR 304
Spring 2024
Review for Second Examination

1. Know how the Google page rank is determined (our class disscussion doesn't include the so-called damping factor–know the method as described in class, and *not* as you might find it described elsewhere.)

2. Given the links between webpages, be able to write the linear equations that determine the Google page rank (actually, the method we learned in class without the damping factor).

3. Consider three webpages, A, B, and C. The links between these pages are $A \to B$, $A \to A$, $A \to B$, $B \to C$, $C \to C$, and $C \to A$. Find the linear equations for the page ranks. Include the condition that the sum of the ranks is one.

4. Know the three properties of a *vector norm* (either classnotes or `https://en.wikipedia.org/wiki/Norm_(mathematics)`

5. Know the definitions of the one, two, and infinity vector norms. (Either classnotes or `https://en.wikipedia.org/wiki/Norm_(mathematics)#p-norm`.

6. Given a vector (say with three components), compute its one, two, and infinity norms.

7. Find the one, two, and infinity norms of the vector $\langle -6, -9, 23 \rangle$.

8. Know the process of row reduction for matrices.

9. Use row reduction to solve the linear system $\begin{bmatrix} 1 & 2 \\ -7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$.

10. Know the definition of the one and infinity matrix norms. Again, I'd say the best source for this is our classnotes: For any square matrix the one and infinity norms are defined by

$$\|A\|_1 = \text{maximum absolute column sum}, \quad \|A\|_\infty = \text{maximum absolute row sum}.$$

11. Know the definition of the matrix condition number.

12. Understand the meaning of the bound

$$\|\mathbf{x} - \widehat{\mathbf{x}}\|_p \le 2\varepsilon_m \frac{\text{cond}_A}{1 - \varepsilon_m \text{cond}_A} \|\mathbf{x}\|_p,$$

provided $1 - \varepsilon_m \text{cond}_A > 0$.

13. Given that

$$\begin{bmatrix} 1 & 2 \\ -7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{15} & -\frac{2}{15} \\ \frac{7}{15} & \frac{1}{15} \end{bmatrix}$$

find the infinity norm condition number of the matrix $\begin{bmatrix} 1 & 2 \\ -7 & 1 \end{bmatrix}$.

14. The condition number of a matrix M is 10^{12}. You need to solve a linear system $Mx = b$ and you need to ensure that the true solution x and the numerical solution $\widehat{x}$ satisfy

$$\frac{\|x-\widehat{x}\|_\infty}{\|x\|_\infty} < 10^{-14}.$$

What this the greatest value of the machine epsilon that will allow for this?

15. Know how to solve an interpolation problem using a Vandermonde matrix; specifically know how to find the coefficient matrix and write the linear equations in matrix form.

16. Know how to solve an interpolation problem using the Lagrange polynomials; know the properties of the Lagrange polynomials; and given the knots, know how to find the formula for the Lagrange polynomials.

17. Know what it means to say that a polynomial interpolates a function.

18. Find the matrix product $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

19. Using integration by parts, show that $\int_a^b F(x)\,\mathrm{d}x = (b-a)F(a) - \int_a^b (x-b)F'(x)\,\mathrm{d}x$.

20. In terms of a *Vandermonde matrix*, find linear equations that determine c_0, c_1, and c_2 such that the polynomial $P(x) = c_0 + c_1 x + c_2 x^2$ interpolates the points $(1,8), (2,9), (10,12)$. Write the equations in matrix form, but *do not solve them.*

21. Find the *three* Lagrange polynomials for the three knots $1,2,3$.

22. Let L_0, L_1, and L_2 be the Lagrange polynomials for the three knots $1,2,3$. Find numbers c_0, c_1, and c_2, such that for $P(x) = c_0 L_0(x) + c_1 L_1 + c_2 L_2$, we have $P(1) = 0, P(2) = 9$, and $P(10) = 12$.

$\boxed{5}$ 23. In terms of a *Vandermonde matrix*, find linear equations that determine c_0, c_1, and c_2 such that the polynomial $P(x) = c_0 + c_1 x + c_2 x^2$ interpolates the points $(1,1), (2,4), (3,8)$. Write the equations in matrix form, but *do not solve them.*

$\boxed{5}$ 24. Find the *three* Lagrange polynomials for the three knots $-1,0,1$.

$\boxed{5}$ 25. Let L_0, L_1, and L_2 be the Lagrange polynomials for the three knots $-1,0,1$. Find numbers c_0, c_1, and c_2, such that for $P(x) = c_0 L_0(x) + c_1 L_1 + c_2 L_2$, we have $P(-1) = 10, P(0) = 2$, and $P(1) = 5$.

26. Given that

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

find the infinity norm condition number of the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

27. Let T_n be the n-panel trapezodial rule estimation for $\int_0^1 F$. Given that

$$\max_{[0,1]}|F| = \pi^5, \quad \max_{[0,1]}|F'| = \sqrt{107\pi}, \text{ and } \max_{[0,1]}|F''| = 120,$$

Find the least integer n such that you can be certain that $|T_n(F) - \int_0^1 F| < 10^{-8}$.