CYRR 304 **Name:**

Homework 10, Spring 2024

Homework 11 has questions 1 through 2 with a total of of 50 points. The point value for each question or part of a question is in the box following each question or part of a question. This work is due **Saturday 20 April** at 11:59 PM.

1. A sequence c is defined recursively by

$$c_n = \begin{cases} 0 & n = 0 \\ 2c_{n-1} + 1 & n \geq 1 \end{cases}. \tag{1}$$

$\boxed{10}$ (a) Find a formula for the generating function for the sequence c. That is, find a formula for $\sum_{k=0}^{\infty} c_n z^n$.

$\boxed{10}$ (b) Use the generating function to find a non recursive formula for the sequence c. You might like to use the fact that

$$\sum_{k=1}^{\infty} z^k = \frac{z}{1-z}. \tag{2}$$

And you might like to use the fact that

$$\frac{1}{1-2z} \times \frac{z}{1-z} = \frac{1}{z-1} - \frac{1}{2z-1}. \tag{3}$$

To earn credit for this question, you must show all of your work.

$\boxed{10}$ (c) Check that your formula for the sequence c matches with $c_0 = 0, c_1 = 1, c_2 = 3, c_3 = 7, c_4 = 15, c_5 = 31$.

2. Define functions $f_1(x) = \frac{1}{\sqrt{\pi}} \sin(x)$, $f_2(x) = \frac{1}{\sqrt{\pi}} \sin(2x)$, $f_3(x) = \frac{1}{\sqrt{\pi}} \sin(3x)$, $f_4(x) = \frac{1}{\sqrt{\pi}} \sin(4x)$, $\cdots$, $f_{10}(x) = \frac{1}{\sqrt{\pi}} \sin(10x)$. Then the set of functions $\{f_0, f_1, \ldots, f_{10}\}$ is orthonormal.

$\boxed{10}$ (a) Find numbers $c_1, c_2, \ldots, c_{10}$ that minimize the function

$$(c_1, c_2, \ldots, c_{10}) \in \mathbf{R}^{10} \mapsto \int (x - \sum_{k=1}^{10} \frac{1}{\sqrt{\pi}} c_k \sin(kx))^2 \, \mathrm{d}x.$$

You might like to use the fact that for all $k \in \mathbf{Z}_{>0}$, we have

$$\int_{-\pi}^{\pi} \frac{x \sin(kx)}{\sqrt{\pi}} dx = -\left(\frac{2\sqrt{\pi}(-1)^k}{k} \right) \tag{4}$$

$\boxed{10}$ (b) Using the numbers c_1 through c_{10} you found in the previous question, ask a graphing utility to graph both $y = x$ and $y = \sum_{k=1}^{10} \frac{1}{\sqrt{\pi}} c_k \sin(kx))$ on the same graph.