

CYRR 304
Homework 11, Spring 2024

Name:

Homework 11 has questions 1 through 2 with a total of 50 points. The point value for each question or part of a question is in the box following each question or part of a question. This work is due **Saturday 27 April** at 11:59 PM.

For this assignment, neatly handwrite your work, digitize it, and turn in the digital copy to Canvas. If you use Julia to draw the graph for the last problem, convert it to HTML and submit that; if you use Desmos, export it as a png file and include the png file.

1. A sequence c is defined recursively by

$$c_n = \begin{cases} 0 & n = 0 \\ 2c_{n-1} + 1 & n \geq 1 \end{cases}. \quad (1)$$

- 10 (a) Find a formula for the generating function for the sequence c . That is, find a formula for $\sum_{k=0}^{\infty} c_n z^n$.

- 10 (b) Use the generating function to find a non recursive formula for the sequence c . You might like to use the fact that

$$\sum_{k=1}^{\infty} z^k = \frac{z}{1-z}. \quad (2)$$

And you might like to use the fact that

$$\frac{1}{1-2z} \times \frac{z}{1-z} = \frac{1}{z-1} - \frac{1}{2z-1}. \quad (3)$$

To earn credit for this question, you must show all of your work.

- 10 (c) Check that your formula for the sequence c matches with $c_0 = 0, c_1 = 1, c_2 = 3, c_3 = 7, c_4 = 15, c_5 = 31$.

2. Define functions $f_1(x) = \frac{1}{\sqrt{\pi}} \sin(x)$, $f_2(x) = \frac{1}{\sqrt{\pi}} \sin(2x)$, $f_3(x) = \frac{1}{\sqrt{\pi}} \sin(3x)$, $f_4(x) = \frac{1}{\sqrt{\pi}} \sin(4x)$, \dots , $f_{10}(x) = \frac{1}{\sqrt{\pi}} \sin(10x)$. Then the set of functions $\{f_0, f_1, \dots, f_{10}\}$ is orthonormal.

- 10 (a) Find numbers c_1, c_2, \dots, c_{10} that minimize the function

$$(c_1, c_2, \dots, c_{10}) \in \mathbf{R}^{10} \mapsto \int_{-\pi}^{\pi} \left(x - \sum_{k=1}^{10} \frac{1}{\sqrt{\pi}} c_k \sin(kx) \right)^2 dx.$$

You might like to use the fact that for all $k \in \mathbf{Z}_{>0}$, we have

$$\int_{-\pi}^{\pi} \frac{x \sin(kx)}{\sqrt{\pi}} dx = - \left(\frac{2\sqrt{\pi}(-1)^k}{k} \right) \quad (4)$$

- 10 (b) Using the numbers c_1 through c_{10} you found in the previous question, ask a graphing utility to graph both $y = x$ and $y = \sum_{k=1}^{10} \frac{1}{\sqrt{\pi}} c_k \sin(kx)$ on the same graph.

For a graphing utility, you can use Julia and Gadfly, or optionally, you can use Desmos; if you have never used Desmos, here is a link to a Desmos graph <https://www.desmos.com/calculator/kauijqwiri>