Fun with generating functions

MATH 420 & CYBR 304

Spring 2024

Long ago, we defined

Definition Let $\{f_1, f_2, \ldots, f_n\} \subset C_{[a,b]}$ and let $\langle \cdot, \cdot \rangle$ be an inner product on $C_{[a,b]}$. The set of functions $\{f_1, f_2, \ldots, f_n\}$ is *orthogonal* provided distinct pairs of these functions are orthogonal. The set is *orthonormal* provided

$$\langle f_k, f_\ell \rangle = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases} \tag{1}$$

- Every orthonormal set is orthogonal.
- When $\{f_1, f_2, \ldots, f_n\}$ is orthogonal, but not orthonormal, we can define
$$\widehat{f_k} = \frac{f_k}{\sqrt{\langle f_k, f_k \rangle}}. \tag{2}$$
Then the set $\{\widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_n}\}$ is orthonormal.

More facts from the past

Let $\{f_1, f_2, \ldots, f_n\} \subset C_{[a,b]}$, and suppose that this set is orthnormal. Given $F \in C_{[a,b]}$, we would like to find $c_1, c_2, \ldots c_n$ that minimize the function

$$(c_1, c_1, \ldots, c_n) \in \mathbf{R}^n \mapsto \|F - \sum_{k=0}^{n} c_k f_k, F\|_2^2 \tag{3}$$

The solution is

$$c_k = \langle F, f_k \rangle, \text{ for } k \in 1 \ldots n. \tag{4}$$

☛ Recall $\|f\|_2^2 = \langle f, f \rangle$

A Hall of Fame Orthogonal set

Fact For $k, \ell \in \mathbf{Z}$, we have

$$\int_{-\pi}^{\pi} \cos(kx)\cos(\ell x)\,\mathrm{d}x = \begin{cases} \pi & k = \ell \\ 0 & k \neq \ell \end{cases},$$

$$\int_{-\pi}^{\pi} \cos(kx)\sin(\ell x)\,\mathrm{d}x = 0,$$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(\ell x)\,\mathrm{d}x = \begin{cases} \pi & k = \ell \\ 0 & k \neq \ell \end{cases}.$$