How do you want me to simplify this?

September 14, 2021

> *My (admittedly perverse) answer is that "to simplify" means to write an equivalent expression that the instructor/marker likely wants or expects as an answer. It is an exercise in mind-reading.*
>
> B. S. THOMSON[1]

Undoubtedly, a math teacher has told you that you *must* simply your answers. And maybe you have been frustrated by not earning full credit for an answer that was correct, but not expressed in *exactly* the form required by the teacher, making math exams a perverse game without clear rules.

If you asked your teacher *exactly* what it means to be simplified that means, it's unlikely that you got a clear answer. You needn't be especially cynical to speculate that the only reason for the requirement of simplified answers is to reduce the teacher's burden of grading papers–it is much easier to simply mark any answer that isn't visually identical to the answer key as wrong.

Compounding the frustration is that some simplification rules, such as eliminating radicals in the denominator seem arbitrary, turning the process of simplification into a silly game. Why is $\sqrt{2}/2$ a simplification of of $1/\sqrt{2}$? Certainly if you need a decimal approximation, entering $1/\sqrt{2}$ into a calculator takes no more effort than does $\sqrt{2}/2$. Although that's true today, it wasn't true before the era of electronic calculators. Long ago doing the long division problem

$$1.414213562373095\cdots \,\overline{)\,1.000000000000000000}$$

by hand was odious, but doing the algebraically identical calculation $(\frac{\sqrt{2}}{2})$

$$2.0000000000000\cdots \,\overline{)\,1.414213562373095\cdots}$$

was a snap. That fact tipped the scale, so it's standard that promoting radicals from the denominator to the numerator results in a simplified expression.

Promoting a single factor of a radical in a denominator to the numerator is arguably a nice skill to learn, but what about a sum of two or more radicals? Possibly you were taught the process for a sum of two radicals; for example,

$$\frac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{3}-\sqrt{2}. \tag{1}$$

But what about a sum of three radicals? What is the simplification of

$$\frac{1}{\sqrt{7}+\sqrt{5}+\sqrt{3}+\sqrt{2}}? \tag{2}$$

$$\frac{22\sqrt{105}-34\sqrt{70}-50\sqrt{42}+62\sqrt{30}+135\sqrt{7}-133\sqrt{5}-145\sqrt{3}+185\sqrt{2}}{215}. \tag{3}$$

[1] Professor Emeritus at Simon Fraser University; see `https://www.quora.com/What-does-it-mean-to-simplify-an-expression?share=1`

Even for the question if a polynomial should simplified by expanding or factoring is unclear. If the goal of simplification is reduce the number of terms in an expression, it seems that we should favor factoring over expanding. Indeed, factoring $x^3-8x^2+21x-18$ yields $(x-3)^2(x-2)$, which has fewer terms and is arguably simpler.

But if the goal of of simplification is to make all algebraically identical expressions (also known as semanatically equal) to be syntactically the same, it's a fool's errand. To illustrate, although

$$(a-b)\sin(a-b)=(b-a)\sin(b-a) \tag{4}$$

is an identity, it's hard to argue that one expression is more simple than the other. Sure, we could so devise an rule (similar to alphabetizing) that would determine that $(a-b)\sin(a-b)$ is more simple than $(b-a)\sin(b-a)$, but it hardly seems worth the effort. Similarly, we would need to decide if $1+x+x^2$ is more simple than x^2+x+1. Again, rules for such things hardly seems worth the effort.

Actually, its more than a fool's errand. Richardson's theorem tells us that once we allow expressions to involve trigonometric and exponential functions, the absolute value function, rational numbers, and constants form by apply the logarithm to a number, there is no algorithm that can prove always

`https://en.wikipedia.org/wiki/Richardson%27s_theorem`

What it means to be simplified can be context dependent. But there are some simplifications that nearly everybody agrees should generally be done. These are:

(a) Reduce all rational numbers to lowest terms.

(b) All arithmetic in sums, products, and exponents should be done.

(c) All common additive and multiplicative terms should be combined.

(d) For any real valued expression, use the identities $1\times x=x, 0x=0, 1^x=1$ and $x^1=x$ to replace the left side by the right side.

(e) Provided x is a nonzero and real valued expression, use the identities $\frac{x}{x}=1, x^0=1$ to replace the left side by the right side.

(f) Provided x is a nonnegative and real valued expression, use the identity $(x^a)^b=x^{ab}$ to replace the left side by the right side.

(g) Use the well known values of the trigonometric functions at the integer multiples of $\pi/6$ and $\pi/4$ to simplify these values.

(h) For any odd function O, replace $O(x)+O(-x)$ by zero. For any odd function E, replace $E(x)-E(-x)$ by zero.

(i) Use the well known values of the logarithms to simplify these values.

(j) For a positive integer n, replace $\frac{1}{\sqrt{n}}$ by $\frac{\sqrt{n}}{n}$.

(k) For a positive integers m and n, replace $\sqrt{mn^2}$ by $n\sqrt{m}$.

In all of these rules, x can match any expression or any subexpression, not just an explicit match to the variable x. And sometimes, we may need to use factoring some other identities to find a match. Thus our guideline is a guideline, not an algorithm.

- The subexpressions of the quotient $\dfrac{x(x^2+1)}{x^2+1}$ are $x(x^2+1)$ and x^2+1, but $\frac{x^2+1}{x^2+1}$ isn't a subexpression. But rearranging the quotient $\dfrac{x(x^2+1)}{x^2+1}$ to the product $x\dfrac{x^2+1}{x^2+1}$, we now have $\frac{x^2+1}{x^2+1}$ as a subexpression of the product, and this subexpression explicitly matches rule 'e.' So we simplify $\dfrac{x(x^2+1)}{x^2+1}$ to x.

- Although there are no common additive terms in $6|x| - |28x|$, there are if we use the identity $|xy| = |x||y|$ Applying this identity first yields

$$6|x| - |28x| = 6|x| - |28||x| = 6|x| - 28|x| = -22|x|.$$

- The expression $\sqrt{50}$ doesn't match any of these rules. But it does match the last rule if we use the factorization $50 = 2 \times 5^2$.

Examples

(a) Using rules 'a' through 'd,' we would simplify

$$\frac{2}{3} + 6^3 + (x+1)^1 + 0 \times x^2 + 1^{10^9} + z - 107z = \frac{656}{3} + x - 106z.$$

The ordering of the terms in the sum $\frac{656}{3} + x - 106z$ is a detail that few teachers would require.

(b) Using rule 'e,' we would simplify

$$\frac{46(x^2+1)}{x^2+1} + (|x|+1)^0 = 46 + 1 = 47.$$

Here both x^2+1 and $|x|+1$ are nonzero and real valued. When it's not certain that x matches with a nonzero expression and we use these simplications, we should make a note of the assumptions; for example:

$$28\frac{x+1}{x+1} = 28, \text{ provided } x+1 \neq 0.$$

And $0^{x-216} = 1,$ provided $x - 216 \neq 0$. Actually, the question of whether or not 0^0 is undefined or if it is equal to 1 is controversial.

(c) Using rule 'f,' we would simplify

$$\sqrt{x^2+1}^{\,2} = \left((x^2+1)^{1/2}\right)^2 = x^2 + 1.$$

Again, if it's not certain that x matches with a nonnegative real valued expression, our work should note the assumption; for example: $\sqrt{x}^{\,2} = x,$ provided $x \geq 0$.

(d) Using rules 'g' and 'h,' we have

$$\cos(5\pi/3) + \log_{10}(100) = \frac{5}{2}.$$

(e) Using rule 'i,' we have

$$\frac{107}{\sqrt{5}} = \frac{107\sqrt{5}}{5}.$$

Please be careful with rule 'f.' Using this rule without adhering to the condition yields rubbish:

$$\left((-1)^2\right)^{1/2} = 1^{1/2} = 1.$$

But

$$(-1)^{2 \times \frac{1}{2}} = (-1)^1 = -1.$$

So the proviso isn't optional.

Non Examples

(a) Our simplification guidelines say nothing about expanding or factoring a polynomial. So by our standard, both $(x-1)(x+1)$ and x^2-1 are simplified.

(b) Our simplification guide also says that both $|xy|$ and $|x||y|$ are simplified. But maybe we should append a rule for this case.

(c) Although $\sin(-x) = -\sin(x)$, our simplification guide says that both $\sin(x)$ and $\sin(-x)$ are simplified. Similarly both $\sin(x-x^2)$ and $-\sin(x^2-x)$ are simplified, but algebraically equivalent. But we do have a rule that requires that we simplify $\sin(x)+\sin(-x)$ to zero.

(d)

Likely you have been taught that to simplify a quotient with a radical in the demonontator such as $\frac{1}{\sqrt{2}}$, you need to multiply by a well chosen representation of one; for example

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

And you were told that $\frac{\sqrt{2}}{2}$ is a simplification of $\frac{1}{\sqrt{2}}$. Long ago, before when calculators were exotic, doing the long division of $1 \div 1.414213562373095\ldots$ was tedious, but the equivalent calcuatoin of $1.414213562373095 \cdots \div 2$ is easy.

$$1.414213562373095\overline{)1.000000000000000000}$$

but the long division

$$2\overline{)1.414213562373095\ldots}$$

is easy.

$$56\overline{)3678}$$

$$\frac{1}{\sqrt{3}+\sqrt{5}+\sqrt{7}} = \frac{\sqrt{7}+5\sqrt{3}+\sqrt{243}-2\sqrt{105}}{59}$$

Ideally

From a teacher's perspective, we'd like to make paper grading as fast as possible. And one way to sped that task is for all correct answer to look exactly alike, and for all wrong answers to not look exactly like the correct answer. If we stick to answers that are polynomials with coefficients that are explicit rational numbers, we can achieve this by requiring that all answers be fully expanded with coefficients expressed as improver rational numbers in reduced form and arranged from low to high power. And example of a simplified expression would be

$$3+5x-\frac{107}{46}x^2.$$

Under these rules this answer is correct and no other answer, including the algebraically equivalent $-\frac{107}{46}x^2 + 5x+3$ is wrong. Imposing this

Efficiency

(Horner's method)

Accuracy

And if you have studied a bit more physics and learned about Einstien's special theory of relativity, you might remember that the kenetic energy is given by

$$T = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right),$$

where c is the speed of light. Let's find T given $m = 1, c = 299792458$ and $v = 10^{-8}c$. Pasting in these values into a calculator that uses about 15 decimal digits, we get

$$T = 299792458^2 \left(\frac{1}{1 - \sqrt{1 - 10^{-16}}} - 1 \right) = 299792458^2 \times 2.22044604925031310^{-16} = 19.95637385869426.$$