

## Functions

We'll use some notation that isn't in our book. This notation is standard and it allows us to express facts with fewer words. For additional standard notation, see [https://en.wikipedia.org/wiki/ISO\\_31-11](https://en.wikipedia.org/wiki/ISO_31-11).

Notation	Meaning
$\text{dom}(F)$	The <i>domain</i> of a function $F$ ; thus $\text{dom}(F)$ is the set of all <i>inputs</i> to the function $F$ .
$\text{range}(F)$	The <i>range</i> of a function $F$ is the set $\text{range}(F)$ ; thus $\text{range}(F)$ is the set of all <i>outputs</i> to the function $F$ .
$F : A \rightarrow B$	This means that $F$ is a function whose domain is the set $A$ and whose range is a <i>subset</i> of the set $B$ .
$x \in A \mapsto F(x)$	Defines a function whose domain is the set $A$ and whose formula is $F(x)$ .
$C_A$	The set of functions that are continuous on the set $A$ .
$C_A^n$	The set of functions $F$ such that that are continuous whose first through $n^{\text{th}}$ derivatives are continuous on the set $A$ .

## Examples

- (a) Every real number is a valid input to the natural exponential function  $\exp$ ; thus  $\text{dom}(\exp) = \mathbf{R}$ .
- (b) The set of outputs to the natural exponential function  $\ln$  is  $(0, \infty)$ ; thus  $\text{range}(\exp) = (0, \infty)$ .
- (c) The domain of the sine function is  $\mathbf{R}$  and every output of sine is in  $\mathbf{R}$ ; thus  $\sin : \mathbf{R} \rightarrow \mathbf{R}$ .
- (d) The domain of the natural logarithm  $\ln$  is the interval  $(0, \infty)$  and every output of  $\ln$  is a real number; thus  $\ln : (0, \infty) \rightarrow \mathbf{R}$ .
- (e) The domain of the sine function is  $\mathbf{R}$  and every output of sine is in  $[-1, 1]$ ; thus  $\sin : \mathbf{R} \rightarrow [-1, 1]$ . It's somewhat confusing that both  $\sin : \mathbf{R} \rightarrow \mathbf{R}$  and  $\sin : \mathbf{R} \rightarrow [-1, 1]$ , but remember that the notation  $F : A \rightarrow B$  means that  $B \subset \text{range}(F)$ . Specifically, the notation  $F : A \rightarrow B$  tells us the domain of  $F$ , but it doesn't tell us the range of  $F$  or its formula.
- (f)  $x \in [-1, 1] \mapsto x^2$  defines a function whose domain is the set  $[-1, 1]$  and whose output is the square of the input. This notation allows us to define a function without giving it a name. It also gives a way to combine defining the domain of the function with its formula.
- (g) Since the sine function is continuous on  $\mathbf{R}$ , we have  $\sin \in C_{\mathbf{R}}$ .
- (h) Since the sine function is continuous on  $[0, 2\pi]$ , we have  $\sin \in C_{[0, 2\pi]}$ .
  - (i)  $F \in C_{[-1, 1]}^2$  is equivalent to  $F \in C_{[-1, 1]}$  and  $F' \in C_{[-1, 1]}$ .
  - (j) The square root function  $\sqrt{\phantom{x}}$  is continuous on  $[0, \infty)$ , but its derivative is not; thus we have  $\sqrt{\phantom{x}} \in C_{[0, \infty)}$  and  $\sqrt{\phantom{x}} \notin C_{[0, \infty)}^1$ .

## Theorems

- (a)  $F, G \in C_{[a, b]} \implies F + G \in C_{[a, b]}$ .

In words, this says that the sum of functions that are continuous on an interval  $[a, b]$  is continuous on the interval  $[a, b]$ .

- (b)  $F, G \in C_{[a, b]} \implies FG \in C_{[a, b]}$ .

In words, this says that the product of functions that are continuous on an interval  $[a, b]$  is continuous on the interval  $[a, b]$ .