

How do you want me to simplify this?

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My (admittedly perverse) answer is that “to simplify” means to write an equivalent expression that the instructor/marker likely wants or expects as an answer. It is an exercise in mind-reading.

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Undoubtedly, you have had a math teacher tell you that you must simply your answers. And if you asked your teacher exactly what it means to be simplified that means, it's unlikely that you got a clear and complete answer. You needn't be overly cynical to speculate that your teacher wants all answers to be simplified so that all correct answers are exactly the same to make the papers easier to grade.

Actually, there is no clear and complete answer to the question. What it means to be simplified can be context dependent. But there are some simplifications that nearly everybody agrees should generally be done. These are:

- (a) Reduce all rational numbers to lowest terms.
- (b) All arithmetic in sums, products, and exponents should be done.
- (c) All common additive and multiplicative terms should be combined.
- (d) For any real valued expression, use the identities $1 \times x = x$, $0x = 0$, $1^x = 1$ and $x^1 = x$ to replace the left side by the right side.
- (e) Provided x is a nonzero and real valued expression, use the identities $\frac{x}{x} = 1$, $x^0 = 1$ to replace the left side by the right side.
- (f) Provided x is a nonnegative and real valued expression, use the identity $(x^a)^b = x^{ab}$ to replace the left side by the right side.
- (g) Use the well known values of the trigonometric functions at the integer multiplies of $\pi/6$ and $\pi/4$ to simplify these values.
- (h) For any odd function O , replace $O(x) + O(-x)$ by zero. For any odd function E , replace $E(x) - E(-x)$ by zero.
- (i) Use the well known values of the logarithms to simplify these values.
- (j) For a positive integer n , replace $\frac{1}{\sqrt{n}}$ by $\frac{\sqrt{n}}{n}$.
- (k) For a positive integers m and n , replace $\sqrt{mn^2}$ by $n\sqrt{m}$.

¹Professor Emeritus at Simon Fraser University; see <https://www.quora.com/What-does-it-mean-to-simplify-an-expression?share=1>

In all of these rules, x can match any expression or any subexpression, not just an explicit match to the variable x . And sometimes, we may need to use factoring some other identities to find a match. Thus our guideline is a guideline, not an algorithm.

- The subexpressions of the quotient $\frac{x(x^2 + 1)}{x^2 + 1}$ are $x(x^2 + 1)$ and $x^2 + 1$, but $\frac{x^2 + 1}{x^2 + 1}$ isn't a subexpression.

But rearranging the quotient $\frac{x(x^2 + 1)}{x^2 + 1}$ to the product $x \frac{x^2 + 1}{x^2 + 1}$, we now have $\frac{x^2 + 1}{x^2 + 1}$ as a subexpression of the product, and this subexpression explicitly matches rule 'e.' So we simplify $\frac{x(x^2 + 1)}{x^2 + 1}$ to x .

- Although there are no common additive terms in $6|x| - |28x|$, there are if we use the identity $|xy| = |x||y|$. Applying this identity first yields

$$6|x| - |28x| = 6|x| - |28||x| = 6|x| - 28|x| = -22|x|.$$

- The expression $\sqrt{50}$ doesn't match any of these rules. But it does match the last rule if we use the factorization $50 = 2 \times 5^2$.

Examples

(a) Using rules 'a' through 'd,' we would simplify

$$\frac{2}{3} + 6^3 + (x + 1)^1 + 0 \times x^2 + 1^{10^9} + z - 107z = \frac{656}{3} + x - 106z.$$

The ordering of the terms in the sum $\frac{656}{3} + x - 106z$ is a detail that few teachers would require.

(b) Using rule 'e,' we would simplify

$$\frac{46(x^2 + 1)}{x^2 + 1} + (|x| + 1)^0 = 46 + 1 = 47.$$

Here both $x^2 + 1$ and $|x| + 1$ are nonzero and real valued. When it's not certain that x matches with a nonzero expression and we use these simplifications, we should make a note of the assumptions; for example:

$$28 \frac{x + 1}{x + 1} = 28, \text{ provided } x + 1 \neq 0.$$

And $0^{x-216} = 1$, provided $x - 216 \neq 0$. Actually, the question of whether or not 0^0 is undefined or if it is equal to 1 is controversial.

(c) Using rule 'f,' we would simplify

$$\sqrt{x^2 + 1}^2 = ((x^2 + 1)^{1/2})^2 = x^2 + 1.$$

Again, if it's not certain that x matches with a nonnegative real valued expression, our work should note the assumption; for example: $\sqrt{x^2} = x$, provided $x \geq 0$.

(d) Using rules 'g' and 'h,' we have

$$\cos(5\pi/3) + \log_{10}(100) = \frac{5}{2}.$$

(e) Using rule 'i,' we have

$$\frac{107}{\sqrt{5}} = \frac{107\sqrt{5}}{5}.$$

Please be careful with rule 'f.' Using this rule without adhering to the condition yields rubbish:

$$((-1)^2)^{1/2} = 1^{1/2} = 1.$$

But

$$(-1)^{2 \times \frac{1}{2}} = (-1)^1 = -1.$$

So the proviso isn't optional.

Non Examples

- (a) Our simplification guidelines say nothing about expanding or factoring a polynomial. So by our standard, both $(x-1)(x+1)$ and x^2-1 are simplified.
- (b) Our simplification guide also says that both $|xy|$ and $|x||y|$ are simplified. But maybe we should append a rule for this case.
- (c) Although $\sin(-x) = -\sin(x)$, our simplification guide says that both $\sin(x)$ and $\sin(-x)$ are simplified. Similarly both $\sin(x-x^2)$ and $-\sin(x^2-x)$ are simplified, but algebraically equivalent. But we do have a rule that requires that we simplify $\sin(x) + \sin(-x)$ to zero.
- (d)

Likely you have been taught that to simplify a quotient with a radical in the demonontator such as $\frac{1}{\sqrt{2}}$, you need to multiply by a well chosen representation of one; for example

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

And you were told that $\frac{\sqrt{2}}{2}$ is a simplification of $\frac{1}{\sqrt{2}}$. Long ago, before when calculators were exotic, doing the long division of $1 \div 1.414213562373095\dots$ was tedious, but the equivalent calcuatoin of $1.414213562373095\dots \div 2$ is easy.

$$1.414213562373095)1.000000000000000000$$

but the long division

$$2)1.414213562373095\dots$$

is easy.

$$56)3678$$

$$\frac{1}{\sqrt{3} + \sqrt{5} + \sqrt{7}} = \frac{\sqrt{7} + 5\sqrt{3} + \sqrt{243} - 2\sqrt{105}}{59}$$

Ideally

From a teacher's perspective, we'd like to make paper grading as fast as possible. And one way to speed that task is for all correct answers to look exactly alike, and for all wrong answers to not look exactly like the correct answer. If we stick to answers that are polynomials with coefficients that are explicit rational numbers, we can achieve this by requiring that all answers be fully expanded with coefficients expressed as improper rational numbers in reduced form and arranged from low to high power. An example of a simplified expression would be

$$3 + 5x - \frac{107}{46}x^2.$$

Under these rules this answer is correct and no other answer, including the algebraically equivalent $-\frac{107}{46}x^2 + 5x + 3$ is wrong. Imposing this

Efficiency

(Horner's method)

Accuracy

And if you have studied a bit more physics and learned about Einstein's special theory of relativity, you might remember that the kinetic energy is given by

$$T = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right),$$

where c is the speed of light. Let's find T given $m = 1$, $c = 299792458$ and $v = 10^{-8}c$. Pasting in these values into a calculator that uses about 15 decimal digits, we get

$$T = 299792458^2 \left(\frac{1}{1 - \sqrt{1 - 10^{-16}}} - 1 \right) = 299792458^2 \times 2.22044604925031310^{-16} = 19.95637385869426.$$