Functions

We'll use some standard mathematical notation that allows us to express facts compactly and precisely. Much of this notation isn't in our book, but we'll use it because it's not hard to remember and it is convenient. For additional standard notation, see `https://en.wikipedia.org/wiki/ISO_31-11`.

Notation	**Meaning**
$\mathrm{dom}(F)$	The *domain* of a function F; thus $\mathrm{dom}(F)$ is the set of all *inputs* to the function F.
$\mathrm{range}(F)$	The *range* of a function F is the set $\mathrm{range}(F)$; thus thus $\mathrm{range}(F)$ is the set of all *outputs* to the function F.
$F : A \to B$	This means that F is a function whose domain is the set A and whose range is a *subset* of the set B.
$F(S)$	When F is a function and S is a subset of its domain, we define $F(S) = \{F(x) \mid x \in \mathrm{dom}(F)\}$.
$x \in A \mapsto F(x)$	Defines a function whose domain is the set A and whose formula is $F(x)$.
C_A	The set of functions that are continuous on the set A.
C^1_A	The set of functions that are continuous and whose first derivatives are continuous on the set A.
C^n_A	The set of functions that are continuous and whose first through n^{th} derivatives are continuous on the set A.
C^∞_A	The set of functions that are continuous and whose derivative of all orders are continuous on the set A.

Examples

(a) Every real number is a valid input to the natural exponential function exp; thus $\mathrm{dom}(\exp) = \mathbf{R}$.

(b) The set of outputs to the natural exponential function is $(0,\infty)$; thus $\mathrm{range}(\exp) = (0,\infty)$.

(c) The domain of the sine function is $\mathbf{R}$ and every output of sine is in $\mathbf{R}$; thus $\sin : \mathbf{R} \to \mathbf{R}$.

(d) The domain of the sine function is $\mathbf{R}$ and every output of sine is in $[-1,1]$; thus $\sin : \mathbf{R} \to [-1,1]$. It's somewhat confusing that both $\sin : \mathbf{R} \to \mathbf{R}$ and $\sin : \mathbf{R} \to [-1,1]$, remember that the notation $F : A \to B$ means that $B \subset \mathrm{range}(F)$ and *not* $B = \mathrm{range} F$.

(e) We have $\sqrt{[0,4]} = [0,2]$. And $\sin([0,\pi]) = [0,1]$

(f) The domain of the natural logarithm ln is the interval $(0,\infty)$ and every output of ln is a real number; thus $\ln : (0,\infty) \to \mathbf{R}$.

(g) $x \in [-1,1] \mapsto x^2$ defines a function whose domain is the set $[-1,1]$ and whose output is the square of the input.

(h) Since the sine function is continuous on $\mathbf{R}$, we have $\sin \in C_{\mathbf{R}}$.

(i) Since the sine function is infinityly differentiable on $\mathbf{R}$, we have $\sin \in C^\infty_{\mathbf{R}}$.

(j) Since the sine function is continuous on $[0,2\pi]$, we have $\sin \in C_{[0,2\pi]}$.

(k) The square root function $\sqrt{}$ is continuous on $[0,\infty)$, but its derivative is not; thus we have $\sqrt{} \in C_{[0,\infty)}$ and $\sqrt{} \notin C^1_{[0,\infty)}$.

Theorems

(a) $C_A^2 \subset C_A^1$.

In words this says that if a function has a continuous second derivative on an interval $[a,b]$, it has a continuous first derivative on the interval $[a,b]$. Similarly, we have $\cdots C_A^4 \subset C_A^3 \subset C_A^2 \subset C_A^1$.

(b) $F \in C_{[a,b]} \wedge G \in C_{[a,b]} \implies F+G \in C_{[a,b]}$.

In words, this says that the sum of functions that are continuous on an interval $[a,b]$ is is continuous on the interval $[a,b]$.

(c) $F \in C_{[a,b]} \wedge G \in C_{[a,b]} \implies FG \in C_{[a,b]}$.

In words, this says that the product of functions that are continuous on an interval $[a,b]$ is is continuous on the interval $[a,b]$.

(d) $F \in C_{[a,b]} \wedge G \in C_{[a,b]} \wedge 0 \notin G([a,b]) \implies F/G \in C_{[a,b]}$.

In words, this says that if functions F and G are continuous on an interval $[a,b]$ and $G(x) \neq 0$ for all $x \in [a,b]$, then F/G is continuous on $[a,b]$.