Notation

We'll use some standard notation that isn't in our book.

Notation	Meaning
$\mathrm{dom}(F)$	The *domain* of a function F; thus $\mathrm{dom}(F)$ is the set of all *inputs* to F
$\mathrm{range}(F)$	The *range* of a function F is the set $\mathrm{range}(F)$; thus $\mathrm{range}(F)$ is the set of all *outputs* to F.
$F : A \to B$	This means that (i) F is a function, (ii) $\mathrm{dom}(F) = A$, and $\mathrm{range}(F) \subset B$.
$x \in A \mapsto F(x)$	Defines a function whose domain is the set A and whose formula is $F(x)$.
C_A	The set of functions that are continuous on the set A.
C_A^n	The set of functions F whose zeroth through n^{th} derivatives are continuous on the set A.

Examples

(a) Every real number is a valid input to the natural exponential function exp; thus $\mathrm{dom}(\exp) = \mathbf{R}$.

(b) The set of outputs to the natural exponential function ln is $(0, \infty)$; thus $\mathrm{range}(\exp) = (0, \infty)$.

(c) The domain of the sine function is $\mathbf{R}$ and every output of sine is in $\mathbf{R}$; thus $\sin : \mathbf{R} \to \mathbf{R}$.

(d) The domain of the natural logarithm ln is the interval $(0, \infty)$ and every output of ln is a real number; thus $\ln : (0, \infty) \to \mathbf{R}$.

(e) The domain of the sine function is $\mathbf{R}$ and every output of sine is in $[-1, 1]$; thus $\sin : \mathbf{R} \to [-1, 1]$. It's somewhat confusing that both $\sin : \mathbf{R} \to \mathbf{R}$ and $\sin : \mathbf{R} \to [-1, 1]$, but remember that the notation $F : A \to B$ means that $B \subset \mathrm{range}(F)$. Specifically, the notation $F : A \to B$ tells us the domain of F, but it doesn't tell us the range of F or its formula.

(f) $x \in [-1, 1] \mapsto x^2$ defines a function whose domain is the set $[-1, 1]$ and whose output is the square of the input. This notation allows us to define a function without giving it a name. It also gives a way to combine defining the domain of the function with its formula.

(g) Since the sine function is continuous on $\mathbf{R}$, we have $\sin \in C_{\mathbf{R}}$.

(h) Since the sine function is continuous on $[0, 2\pi]$, we have $\sin \in C_{[0,2\pi]}$. Notice that the zeroth derivative of a function is itself.

(i) $F \in C^2_{[-1,1]}$ is equivalent to $F \in C_{[-1,1]}$ and $F' \in C_{[-1,1]}$.

(j) The square root function $\sqrt{\ }$ is continuous on $[0, \infty)$, but its derivative is not; thus we have $\sqrt{\ } \in C_{[0,\infty)}$ and $\sqrt{\ } \notin C^1_{[0,\infty)}$.

Theorems

(a) $F, G \in C_{[a,b]} \implies F + G \in C_{[a,b]}$.

In words, this says that the sum of functions that are continuous on an interval $[a, b]$ is continuous on the interval $[a, b]$.

(b) $F, G \in C_{[a,b]} \implies FG \in C_{[a,b]}$.

In words, this says that the product of functions that are continuous on an interval $[a, b]$ is continuous on the interval $[a, b]$.

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