

## Greek characters

Name	Symbol	Typical use(s)
alpha	$\alpha$	angle, constant
beta	$\beta$	angle, constant
gamma	$\gamma$	angle, constant
delta	$\delta$	limit definition
epsilon	$\epsilon$ or $\varepsilon$	limit definition
theta	$\theta$ or $\vartheta$	angle
pi	$\pi$ or $\pi$	circular constant
phi	$\phi$ or $\varphi$	angle, constant

## Named sets

empty set	$\emptyset$	integers	$\mathbf{Z}$
real numbers	$\mathbf{R}$	positive integers	$\mathbf{Z}_{>0}$
ordered pairs	$\mathbf{R}^2$	positive reals	$\mathbf{R}_{>0}$

## Set symbols

Meaning	Symbol	Meaning	Symbol
is a member	$\in$	union	$\cup$
subset	$\subset$	complement	superscript <sup>C</sup>
intersection	$\cap$	set minus	$\setminus$

## Intervals

For numbers  $a$  and  $b$ , we define the intervals:

$$\begin{aligned}(a, b) &= \{x \in \mathbf{R} \mid a < x < b\} \\ [a, b) &= \{x \in \mathbf{R} \mid a \leq x < b\} \\ (a, b] &= \{x \in \mathbf{R} \mid a < x \leq b\} \\ [a, b] &= \{x \in \mathbf{R} \mid a \leq x \leq b\}\end{aligned}$$

## Logic symbols

Meaning	Symbol	Meaning	Symbol
negation	$\neg$	equivalent	$\equiv$
and	$\wedge$	iff	$\iff$
or	$\vee$	for all	$\forall$
implies	$\implies$	there exists	$\exists$

## Tautologies

$$\begin{aligned}\neg(P \wedge Q) &\equiv \neg P \vee \neg Q \\ (P \implies Q) &\equiv (\neg Q \implies \neg P) \\ P \not\implies Q &\equiv P \wedge \neg Q \\ ((P \implies Q) \wedge (Q \implies R)) &\implies (P \implies R) \\ (P \iff Q) &\equiv ((P \implies Q) \wedge (Q \implies P)) \\ \neg(\forall x \in A)(P(x)) &\equiv (\exists x \in A)(\neg P(x)) \\ \neg(\exists x \in A)(P(x)) &\equiv (\forall x \in A)(\neg P(x))\end{aligned}$$

## Function notation

$\text{dom}(F)$	domain of function $F$
$\text{range}(F)$	range of function $F$
$C_A$	set of continuous functions on set $A$
$C_A^1$	set of differentiable functions on set $A$
$A \rightarrow B$	set of functions from $A$ to $B$

## Generalized set operators

Each member of a set  $\mathcal{C}$  is a set:

$$\begin{aligned}\bigcup_{A \in \mathcal{C}} A &= \{z \mid (\exists B \in \mathcal{C})(z \in B)\} \\ \bigcap_{A \in \mathcal{C}} A &= \{z \mid (\forall B \in \mathcal{C})(z \in B)\}\end{aligned}$$

$$\text{Theorem: } \bigcup_{A \in \mathcal{C}} A^C = \left( \bigcap_{A \in \mathcal{C}} A \right)^C$$

## Functions applied to sets

Let  $A \subset \text{dom}(F)$  and  $B \subset \text{range}(F)$ :

$$\begin{aligned}F(A) &= \{F(x) \mid x \in A\} \\ F^{-1}(B) &= \{x \in \text{dom}(F) \mid F(x) \in B\}\end{aligned}$$

## Triangle inequalities

For all  $x, y \in \mathbf{R}$ , we have

$$\begin{aligned}|x + y| &\leq |x| + |y| \\ ||x| - |y|| &\leq |x - y|\end{aligned}$$

## Floor and ceiling

Definitions:

$$\begin{aligned}\lfloor x \rfloor &= \max\{k \in \mathbf{Z} \mid k \leq x\} \\ \lceil x \rceil &= \min\{k \in \mathbf{Z} \mid k \geq x\}\end{aligned}$$

Properties:

$$\begin{aligned}(\forall x \in \mathbf{R}, n \in \mathbf{Z})(x < n \iff \lfloor x \rfloor < n) \\ (\forall x \in \mathbf{R}, n \in \mathbf{Z})(n < x \iff n < \lceil x \rceil)\end{aligned}$$

## Bounded sets

**Bounded below** A set  $A$  is *bounded below* provided  $(\exists M \in \mathbf{R})(\forall x \in A)(M \leq x)$ .

**Bounded above** The set  $A$  is *bounded above* provided  $(\exists M \in \mathbf{R})(\forall x \in A)(x \leq M)$ .

**Bounded** A set is *bounded* if it is bounded below and bounded above.

## Elementary function properties

**Increasing**  $(\forall x, y \in A)(x < y \implies F(x) \leq F(y))$ . For strictly increasing, replace  $F(x) \leq F(y)$  with  $F(x) < F(y)$ .

**Decreasing**  $(\forall x, y \in A)(x < y \implies F(x) \geq F(y))$ . For strictly decreasing, replace  $F(x) \geq F(y)$  with  $F(x) > F(y)$ .

**One-to-one**  $(\forall x, y \in \text{dom}(F))(F(x) = F(y) \implies x = y)$

**Subadditive**  $(\forall x, y \in \text{dom}(F))(F(x + y) \leq F(x) + F(y))$

**Bounded above**  $(\exists M \in \mathbf{R})(\forall x \in \text{dom}(F))(F(x) \leq M)$

**Bounded below**  $(\exists M \in \mathbf{R})(\forall x \in \text{dom}(F))(M \leq F(x))$

## Topology

**Open ball**  $\text{ball}(a, r) = \{x \in \mathbf{R} \mid -r + a < x < r + a\}$

**Punctured ball**  $\text{ball}'(a, r) = \text{ball}(a, r) \setminus \{a\}$

**Open set** A subset  $A$  of  $\mathbf{R}$  is *open* provided  $(\forall x \in A)(\exists r \in \mathbf{R}_{>0})(\text{ball}(x, r) \subset A)$ .

**Closed set** A subset  $A$  of  $\mathbf{R}$  is *closed* provided  $\mathbf{R} \setminus A$  is open.

**Limit point** A number  $a$  is a *limit point* of a set  $A$  provided  $(\forall r \in \mathbf{R}_{>0})(\text{ball}'(a, r) \cap A \neq \emptyset)$ .

**Boundary point** A number  $a$  is a *boundary point* of a set  $A$  provided

$$(\forall r \in \mathbf{R}_{>0}) (\text{ball}(a, r) \cap A \neq \emptyset \wedge \text{ball}(a, r) \cap A^c \neq \emptyset).$$

**Set closure**  $\bar{A} = A \cup \text{LP}(A)$ , where  $\text{LP}(A)$  is the set of limit points of  $A$ .

**Open cover** A set  $\mathcal{C}$  is an open cover of a set  $A$  provided

- (a) every member of  $\mathcal{C}$  is an open set
- (b)  $A \subset \bigcup_{B \in \mathcal{C}} B$

**Compact** A set  $A$  is compact provided for every open cover  $\mathcal{C}$  of  $A$ , there is a finite subset  $\mathcal{C}'$  of  $\mathcal{C}$  such that  $\mathcal{C}'$  is an open cover of  $A$ .

## Least and greatest bounds

For any subset  $A$  of  $\mathbf{R}$ :

**glb**  $z = \text{glb}(A)$  provided

- (a)  $z$  is an lower bound for  $A$
- (b) if  $x$  is a lower bound for  $A$  then  $x \leq z$

**lub**  $z = \text{lub}(A)$  provided

- (a)  $z$  is an upper bound for  $A$
- (b) if  $x$  is a upper bound for  $A$  then  $z \leq x$

## Sequences

**Bounded** A sequence  $F$  is bounded if  $\text{range}(F)$  bounded.

**Monotone** A sequence is monotone if it either increases or decreases.

**Cauchy** A sequence  $F$  is Cauchy provided

- (a) for every  $\varepsilon \in \mathbf{R}_{>0}$
- (b) there is  $n \in \mathbf{Z}$
- (c) such that for all  $k, \ell \in \mathbf{Z}_{>n}$
- (d)  $|F_k - F_\ell| < \varepsilon$

**Converges** A sequence  $F$  converges provided

- (a) there is  $L \in \mathbf{R}$
- (b) and  $n \in \mathbf{Z}$
- (c) such that for all  $k \in \mathbf{Z}_{>n}$
- (d)  $|F_k - L| < \varepsilon$ .

## Functions

**Continuous** A function  $F$  is continuous at  $a$  provided

- (a)  $a \in \text{dom}(F)$  and
- (b) for every  $\varepsilon \in \mathbf{R}_{>0}$
- (c) there is  $\delta \in \mathbf{R}_{>0}$
- (d) such that for all  $x \in \text{ball}(a, \delta) \cap \text{dom}(F)$
- (e) we have  $F(x) \in \text{ball}(F(a), \varepsilon)$ .

**Uniformly continuous** A function  $F$  is uniformly continuous on a set  $A$  provided

- (a)  $A \subset \text{dom}(F)$ ; and
- (b) for every  $\varepsilon \in \mathbf{R}_{>0}$
- (c) there is  $\delta \in \mathbf{R}_{>0}$
- (d) such that for all  $x, y \in A$  and  $|x - y| < \delta$
- (e) we have  $|F(x) - F(y)| < \varepsilon$ .

**Limit** A function  $F$  has a limit toward  $a$  provided

- (a)  $a$  is a limit point of  $\text{dom}(F)$ ; and
- (b) there is  $L \in \mathbf{R}$
- (c) such that for every  $\varepsilon \in \mathbf{R}_{>0}$
- (d) there is  $\delta \in \mathbf{R}_{>0}$
- (e) such that for all  $x \in \text{ball}'(a, \delta)$
- (f) we have  $F(x) \in \text{ball}(L, \varepsilon)$ .

**Differentiable** A function  $F$  is differentiable at  $a$  provided

- (a)  $a \in \text{dom}(F)$ ; and
- (b) there is  $\phi \in \text{dom}(F) \rightarrow \mathbf{R}$
- (c) such that  $\phi$  is continuous at  $a$  and
- (d)  $(\forall x \in \text{dom}(F))(F(x) = F(a) + (x - a)\phi(x))$ .

## Riemann sums

**Partition** A set  $\mathcal{P}$  is a partition of an interval  $[a, b]$  provided

- (a) the set  $\mathcal{P}$  is finite
- (b) every member of  $\mathcal{P}$  is an open interval
- (c) the members of  $\mathcal{P}$  are pairwise disjoint
- (d)  $\bigcup_{I \in \mathcal{P}} I = [a, b]$

Let  $F$  be a bounded function on an interval  $[a, b]$  and let  $\mathcal{P}$  be a partition of  $[a, b]$ .

**Lower sum**  $\underline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{glb}(F(I)) \times \text{length}(I)$

**Upper sum**  $\bar{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{lub}(F(I)) \times \text{length}(I)$

**Riemann sum**  $\sum_{I \in \mathcal{P}, x^* \in I} F(x^*) \times \text{length}(I)$

## Axioms

**Completeness** Every nonempty subset  $A$  of  $\mathbf{R}$  that is bounded above has a least upper bound.

**Well-ordering** Every nonempty set of positive integers contains a least element.

**Induction**  $(\forall n \in \mathbf{Z}_{\geq 0})(P(n))$  if and only if  $P(0) \wedge (\forall n \in \mathbf{Z}_{\geq 0})(P(n) \implies P(n+1))$ .

## Named theorems

**Archimedean**  $(\forall x \in \mathbf{R})(\exists n \in \mathbf{Z})(n > x) \equiv \text{true}$ .

**Bolzano–Weierstrass** Every bounded real valued sequence has a convergent subsequence.

**Heine–Borel** A subset of  $\mathbf{R}$  is compact iff it is closed and bounded.

**Cauchy completeness** Every Cauchy sequence in  $\mathbf{R}$  converges.

**Monotone convergence** Every bounded monotone sequence converges.

**Intermediate value theorem** If  $F \in C_{[a,b]}$ , then for all  $y \in [\min(F(a), F(b)), \max(F(a), F(b))]$  there is  $x \in [a, b]$  such that  $F(x) = y$ .

**Mean Value** If  $F \in C_{[a,b]} \cap C^1_{(a,b)}$ , there is  $\xi \in (a, b)$  such that  $(b - a)F'(\xi) = F(b) - F(a)$ .