Proposition 1. *For all $x, y \in \mathbf{R}$, there is $a \in \mathbf{R}$ such that $x < y$ implies $x < a < y$.*

Proof. (BW) Let $x, y \in \mathbf{R}$. Suppose $x < y$. Choose $a = \frac{x+y}{2}$. Then $a \in \mathbf{R}$ as required. We have

$$\begin{aligned}
[x < a < y] &\equiv \left[x < \frac{x+y}{2} < y\right], && \text{(substitution)}\\
&\equiv \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(subtract } (x+y)/2\text{)}\\
&\equiv \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \text{(simplification)}\\
&\equiv \text{True.} && \big((y-x>0) \wedge (x-y<0)\big)
\end{aligned}$$

□

Proposition 2. *For all $r \in \mathbf{R}_{>0}$ there is $x \in [0,1)$ such that $1 - r < x$.*

Proof. (BW) Let $r \in \mathbf{R}_{>0}$. Choose $x = \begin{cases} 1 - \frac{r}{2} & r < 1 \\ \frac{1}{2} & r \geq 1 \end{cases}$. For $r < 1$, we have

$$\begin{aligned}
[1 - r < x] &\equiv \left[1 - r < 1 - \frac{r}{2}\right], && \text{(substitution for } x\text{)}\\
&\equiv \left[0 < \frac{r}{2}\right], && \text{(add } r-1\text{)}\\
&\equiv \text{True.} && (0 < r < 1)
\end{aligned}$$

And for $r \geq 1$, we have

$$\begin{aligned}
[1 - r < x] &\equiv \left[1 - r < \frac{1}{2}\right], && \text{(substitution for } x\text{)}\\
&\equiv \left[\frac{1}{2} < r\right], && \text{(add } r-1\text{)}\\
&\equiv \text{True.} && (r \geq 1)
\end{aligned}$$

□

Proposition 3. *For all $x \in \mathbf{R}_{>0}$ there is $y \in \mathbf{R}_{>0}$ such that $y < x$.*

Proof. (BW) Let $x \in \mathbf{R}_{>0}$. Choose $y = x/2$. Then $y \in \mathbf{R}_{>0}$ as required. We have

$$\begin{aligned}
[y < x] &\equiv \left[\frac{x}{2} < x\right], && \text{(substitution for } y\text{)}\\
&\equiv \left[0 < \frac{x}{2}\right], && \text{(subtract } x/2\text{)}\\
&\equiv \text{True.} && (\mathrm{x} > 0)
\end{aligned}$$

□

Proposition 4. *For every $y \in \mathbf{R}_{>0}$ there is $x \in \mathbf{R}_{>0}$ such that $y \geq x$.*

Proof. (BW) Let $y \in \mathbf{R}_{>0}$. Choose $x = y$. Then $x \in \mathbf{R}_{>0}$, as required. We have

$$\begin{aligned} [y \geq x] &\equiv [y \geq y], && \text{(substitution for } x\text{)} \\ &\equiv [0 \geq 0], && \text{(subtract } y\text{)} \\ &\equiv \text{True}. \end{aligned}$$

□

Proposition 5. *For all $x \in \mathbf{R}_{>0}$, there is $M \in \mathbf{R}$ such that $\frac{1}{x} + 1 > M$.* (SB)

Proposition 6. *There is $M \in \mathbf{R}$ such that for all $x \in \mathbf{R}_{>0}$, we have $\frac{1}{x} + 1 > M$.* (DD)

Proposition 7. *There is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $1 + m(x-1) \leq x^2$.* (TK)

Proposition 8. *For every $a \in \mathbf{R}$, there is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $a^2 + m(x-a) \leq x^2$.* (AK)

Proposition 9. *For all $x, y \in \mathbf{R}$, we have $(x^2 = y^2) \implies (x = y)$.* (DM)

Proposition 10. *For all $x, y \in \mathbf{R}$, we have $(x^3 = y^3) \implies (x = y)$.* (CR)

Proposition 11. *For all $r \in \mathbf{R}_{>0}$ there is $x \in \mathbf{R}$ such that $1 < x < 1 + r$.* (AA)