Guidelines for Writing Proofs

Barton Willis

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I have made this letter longer than usual, only because I have not had the time to make it shorter. BLAISE PASCAL

The proofs you write for this class must be logical and they should be written using standard mathematical notation and expressed as English sentences. Other than logical correctness and using English sentences, you are free to write your proofs in a way that makes sense to you and that pleases you.

Our textbook and class notes provides us with plenty of examples of well-written proofs. You may, but are not required to, imitate the proof style of these resources. As you become more comfortable writing proofs, it's good to develop your own style.

1 Write Sentences

Write proofs using sentences. Do not write proofs as sentence fragments connected by arrows.

Replace

n, m integers $\sqrt{2} = n/m \longrightarrow 2m^2 = n^2$.

with

Let n and m be integers. If $\sqrt{2} = n/m$, then $2\,m^2 = n^2$.

2 Format proofs as regular text

Poets frequently format their work with centered lines and wide margins. Mathematics isn't poetry; don't format mathematics as if it were poetry.

Replace

Let $\varepsilon > 0$.
Choose $\delta = \varepsilon/3$.
For $|x-1| < \delta$ we have $x < 1+\delta$.

with

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. For $|x-1| < \delta$ we have $x < 1+\delta$.

3 Do not use the first person

In a proof, do not use the first person.

> **Replace**
>
> > First, I consider the case $x < 0$.
>
> with
>
> > First, consider $x < 0$.

Outside a proof, use the first person when it is natural.

4 Use "we"

In mathematical text, "we" means the author and the reader. Use "we" instead of the passive voice when it makes the text shorter.

> **Replace** (passive voice)
>
> > It has been shown that x is prime.
>
> **with** (active voice)
>
> > We've shown that x is prime.
>
> Alternatively, write this as
>
> > The number x is prime.

5 Refer to theorems and axioms by name

It is usually better to write

> The completeness of the real numbers implies that ...

then it is to write

> It follows from Axiom (A9) that ...

The meaning of the first is clear, but the second won't be clear to most readers until they look up Axiom (A9).

6 Use consistent terminology

Mathematics is full of synonyms; for example, *set* and *collection*. In a proof, it's best to stay with the same word to express a concept.

> **Replace**
>
> > Let A be a set and let B be a nonempty collection.
>
> **with**
>
> > Let A be a set and let B be a nonempty set.

7 Use words with clear meanings

A well-written proof can be hard to follow. Do not make it more difficult to understand by using imprecise language. Informally I might say that a quantity is “tiny.” Don’t use such language in a proof.

8 Mathematical writing should be readable

Some mathematical writing looks OK in print, but it is nonsense when read aloud. Read your writing out loud to find such mistakes.

> **Replace**
>
> > Let $k > 0$ be an integer.
>
> **with**
>
> > Let k be a positive integer.
>
> Alternatively, this can be expressed as
>
> > Let $k \in \mathbf{Z}_{\geq 0}$.

Spoken,“Let $k > 0$ be an integer” is equivalent to “Let k is greater than zero be an integer.”

9 Write succinctly

Extraneous facts make it difficult for the reader to follow a logical argument. Re-read your work and make certain that each thread of reasoning is needed.

10 Include sufficient detail

The proofs you write should contain enough detail so that a classmate could follow it, but you should omit most purely algebraic steps.

> **Replace**
>
> > Subtracting 8 from both sides of the inequality (1) and dividing by 42, we deduce that $x < 1$.
>
> **with**
>
> > Inequality (1) implies that $x < 1$.

11 Be careful with “it”

Make sure the meaning of every pronoun is unambiguous.

> **Replace**
>
> > Let $a = 1$. To find b, add 5 to it and then multiply it by 8.
>
> **with**
>
> > Let $a = 1$. Define $b = 8(a + 5)$.

12 Define symbols before you use them

With few exceptions (π, for example), every symbol you use must be defined before, or near, the place you first use it.

> **Replace**
>
> > The volume is $\pi r^2 h$.
>
> **with**
>
> > The volume of a right circular cylindrical with height h and radius r is $\pi r^2 h$.

13 Qualify identifiers

Some symbols have more than one meaning; for example, $(0,1)$ can represent an open interval in $\mathbf{R}$, an element of $\mathbf{R}^2$, or a complex number. Do not leave it up to the reader to use conventions or context (n is an integer, z is complex number, F is a function, etc.) to guess the type of each identifier; rather, tell the reader the meaning of symbols. If F is a function, for example, immediately let the reader know:

> Let F be a function ...

Unless a proof is very long, the reader needn't be reminded that F is a function.

> **Replace**
>
> > Let F be a function defined on $\mathbf{R}$. If the function F is continuous on the open interval $(0,1)$, ...
>
> **with**
>
> > Let F be a function defined on $\mathbf{R}$. If F is continuous on the open interval $(0,1)$, ...

14 Mathematics is case-sensitive

Lower and upper case identifiers are distinct. This means, for example that a and A are different symbols.

15 Avoid unnecessary notation

Introduce notation and identifiers only when needed.

> For example, **Replace**
>
> > **Proposition** The number $x = \sqrt{2}$ is irrational.
>
> **with**
>
> > **Proposition** The number $\sqrt{2}$ is irrational.

In addition to being verbose, the first statement of the proposition has several other problems. First, $x = \sqrt{2}$ isn't a number, it's an *equation*. Second, only a number can be irrational, but the first proposition asserts that an *equation* can be irrational.

16 Don't confuse a function with its formula

If F is a function, don't write $F(x)$ when you mean F.

Replace

If $F(x)$ and $G(x)$ are continuous, $F(x)+G(x)$ is continuous.

with

If F and G are continuous, $F+G$ is continuous.

17 Use function signature notation

When you introduce a new function, immediately tell the reader its name, domain, and possibly its co-domain. This is the *signature* of a function.

Replace

Let $F(x, y)$ be a real-valued function.

with

Let $F: \mathbf{R}^2 \to \mathbf{R}$.

Writing $F(x, y)$ to *imply* that F is defined on $\mathbf{R}^2$ is a poor substitute for a clear statement of the function signature. If the exact domain of a function isn't important, use the notation

Let $F: \subset \mathbf{R}^2 \to \mathbf{R}$.

This tells the reader that the domain of F is a subset of $\mathbf{R}^2$, but it doesn't give its actual domain.

18 Don't confuse "let" with "therefore"

To introduce a new object, use "let." Use "therefore" to state a logical consequence of the facts that precede it.

Replace

Let n be an integer. Let $n(n+1)$ be even.

with

Let n be an integer. Therefore $n(n+1)$ is even.

Alternatives to "therefore" are "so" and "thus."

19 Omit empty words and phrases

It doesn't help the reader to say that something is "easy to show" or "obvious." Omit or replace the empty words and phrases listed in the following table.

Word or Phrase	Replacement
It is trivial to show	(omit)
It is easy to show	(omit)
Clearly	(omit)
Obviously	(omit)
It immediately follows	It follows
For the purpose of contradiction	Suppose
By definition	Thus

Also, expunge weasel words from your text. For a list of weasel words, see, for example, `http://en.wikipedia.org/wiki/Weasel_word`.

20 Do not be wishy-washy

A proof is no place to be indefinite. Do not use phrases similar to "I believe," "should be" or "I think that."

21 Use related theorems

A short proof that uses several powerful theorems is more likely to be correct than is a long proof that starts from scratch. Whenever possible, base your proofs on propositions that have been proved. Of course, your work should only reference theorems we have covered in class.

22 Let the reader know where the argument is heading

It often helps the reader to know in advance where a proof is heading. Depending on the audience, it may be better to write

> Let $x \in A$. To show that $A \subset B$, we'll show that $x \in B$. Since, ... Thus $x \in B$; therefore $A \subset B$.

then to write

> Let $x \in A$. Since, ... Thus $x \in B$; therefore $A \subset B$.

23 Use idioms

Some patterns enter into proofs so frequently that we express them using an idiom.[1] Think of an idiom as a template for a proof. When a proof conforms to a template, the reader has a good idea of what will follow. This can make a proof easier to read. We'll learn a few templates for proofs.

[1] Idiom (noun): A style or manner of expression peculiar to a given people. (From dictionary.com)

24 Use correct spelling, punctuation, and grammar

Misspelled words, incorrect punctuation, and poor grammar distract the reader. Always carefully check your work for these errors. Additionally, displayed formulas should be punctuated as regular text. For example,

> Repeated use of the double angle formula gives
>
> $$\sin^4(x) = \frac{\cos(4x)}{8} - \frac{\cos(2x)}{2} + \frac{3}{8}.$$

The displayed formula ends the sentence; thus it terminates with a period.

25 Begin each sentence with a word

A sentence that starts with a symbol is difficult to read; every sentence must begin with a word, not a symbol.

Replace

> A is continuous on $\mathbf{R}$; we have ...

with

> Since A is continuous on $\mathbf{R}$, we have ...

26 Place a word between adjacent formulas

When one formula follows another, place a word in between the formulas.

Replace

> When $u > n$ $k < 0$.

with

> When $u > n$, we also have $k < 0$.

27 Don't over use the colon

Do not use a colon immediately before a formula when the formula completes the sentence.

Replace

> For all integers n, the number: $n(n+1)$ is even.

with

> For all integers n, the number $n(n+1)$ is even.

If the phrase that follows a colon is a complete sentence, capitalize the first word following a colon; otherwise, do not capitalize the first word following a colon.

28 Don't use unnecessary commas

Do not surround an appositive with commas.

> **Replace**
>
> The revenue, R, is an increasing function of price, p.
>
> **with**
>
> The revenue R is an increasing function of price p.

Further when a sentence starts with "further," "therefore," or "thus," a comma isn't needed after the first word.

29 If there is a "first," there must be a "second"

If you enumerate ideas using starting with "first," you must identify the remaining ideas using second, third, etc.

> **Replace**
>
> First, we'll show that $x \leq 0$. Next, we'll show that $x \geq 0$.
>
> **with**
>
> First, we'll show that $x \leq 0$. Second, we'll show that $x \geq 0$.

30 Capitalize proper names and theorems

Capitalize the words *axiom, definition, proposition*, and *theorem* when they refer to something specific; otherwise, do not capitalize them.

> **Replace**
>
> Today, Mary proved theorem 1. Yesterday, she proved two Theorems.
>
> **with**
>
> Today, Mary proved Theorem 1. Yesterday, she proved two theorems.

Also capitalize proper names; for example, *the Euler constant.*

31 Don't confuse it's and its

The word "it's" always means it is. Don't confuse it's with its.[2]

> **Replace**
>
> Memorize Definition 1-1; its the most important thing you will learn this term.
>
> **with**
>
> Memorize Definition 1-1; it's the most important thing you will learn this term.

[2] I suppose you could name your dog "It." Then "It's doghouse has two stories." is correct, and "It's" does not mean "It is."

32 Do scratch work

By all means, if it helps you construct a proof, draw pictures and diagrams filled with lines and arrows. But do not include your scratch work in the final copy.

33 Proofread your work

When you have finished a proof, put it aside for a while. After that, proofread it several more times. Imagine that it is someone else's work. Does it still make sense? Read your proof one line at a time (cover up all the other lines). Finally, try reading your work out loud. Mistakes that are easy to skip often manifest when spoken.

34 Check that you used every hypothesis

After your first draft, check that you have used every hypothesis. If you did not, it doesn't mean that your proof is wrong. But it does mean that it is suspect. Generally mathematicians state theorems without unnecessary hypotheses.

35 Prove, show, and demonstrate all mean prove

If you are asked to show that something is true, you need to supply a proof. The words prove, show, and demonstrate all mean the same thing.

36 Prove what is given–not a special case

It's not fair to append additional conditions to a proposition and then use these additional conditions to prove the proposition. Put differently, proving a *special case* of a proposition does not prove the proposition.

Replace

> **Proposition** Let A and B be sets. Show that $A \subset A \cup B$.
>
> **Proof** Let $A = \{2,3,4\}$ and $B = \{3,4,5\}$. So $A \cup B = \{2,3,4,5\}$. Since $\{2,3,4\} \subset \{2,3,4,5\}$, we've shown that $A \subset A \cup B$

with

> **Proof** If $x \in A$, we have $x \in A \cup B$; therefore $A \subset A \cup B$.

37 Prove the conclusion, not the hypothesis

Make certain that the logic of your proof flows from the hypothesis to the conclusion. I've seen proofs that start with the conclusion and end with either the same statement or with one or more of the hypothesis.

38 Don't abuse equality

In this class, we mostly work with *functions, sets*, or *numbers*. A set can never equal a number and a number can never equal a function. Carefully check your work and make sure that you have equality written only between objects with the same type.

The *range* of a function is a *set*; it is not a number. If your proof contains

$$\text{range}(F) = 1,$$

possibly you need to replace it with

$$\text{range}(F) = \{1\}$$

or with

$$1 \in \text{range}(F)$$

Of course, equality shouldn't be written between things with the same type that aren't equal; however, this isn't the point that is being made here.

39 The symbol "=" means equal, not the next step is

Use "=" between things that are equal. Do not use "=" to mean "the next step is" or "implies."

Replace

$$F(x) = x^2 = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x = F'(x)$$

with

$$F'(x) = (x^2)' = 2x.$$

Also, do not practice what I call non-committal math. This is, do not express mathematics as a stream of disconnected nouns.

Replace

$$x(x+1) \quad (x(x+1))' \quad (x)'(x+1) + x(x+1)' \quad x+1+x \quad 2x+1$$

with

$$(x(x+1))' = (x)'(x+1) + x(x+1)' = x+1+x = 2x+1$$

Finally, do not use an arrow when you mean equality.

Replace

$$(x+1)^2 \to x^2 + 2x + 1$$

with

$$(x+1)^2 = x^2 + 2x + 1$$

40 Avoid arrows

The double arrow $\Rightarrow$ means *therefore*, not equal. When you want to express the fact that expressions are *equal*, use the equal sign, not the double arrow. Although we do not generally use symbols for logical connectives, a correct usage of the double arrow is: *Chocolate is made from the bean of the cacao tree* $\Rightarrow$ *Chocolate contains caffeine*. An *incorrect* usage is: $\left(1+x^2\right)' \Rightarrow 2x$. A single arrow (either left or right pointing) generally means "Replace by." A correct usage of the single arrow is: *Substituting* $x \to 5$ *in* $y = 6 + x$ *yields* $y = 11$. An *incorrect* usage is: $x(1-x) = 0 \to x = 0, 1$.

41 Check for dangling silent variables

Nothing should depend in any significant way on a silent variable. If your work has a silent variable, make sure it is qualified in some way.

> In the following i is a silent variable. In the last sentence, the silent variable is unqualified:
>
> If $x \in \left(\bigcap_{i\in I} A_i\right)^C$, then $x \in \mathscr{U}$ and $x \notin \bigcap_{i\in I} A_i$. So $x \notin A_i$.
>
> **Replace** this with
>
> If $x \in \left(\bigcap_{i\in I} A_i\right)^C$, then $x \in \mathscr{U}$ and $x \notin \bigcap_{i\in I} A_i$. So $x \notin A_i$ for some $i \in I$.

42 A statement must follow "such that"

What follows "such that" must be a statement (something that must either be true or false). In

> Let A and B be sets such that $A \cap B$.

A noun $A \cap B$ follows the "such that"; it's not clear what the writer intended. One possibility is

> Let A and B be sets such that $A \cap B$ is nonempty.

43 Say what you mean

Consider

> If $x \in A \cap B$, then $x \in A$ and $x \in B$. So $x \in A \cap B \subset A$.

Spoken the last sentence is *So x is a member of A intersect B is a subset of A.* To fix this, end the proof with *So* $x \in A$.

44 Use correct italics

Most mathematical text is typeset in italic type, but there are some exceptions. Function names that are two or more characters long are typeset in upright text; for example $\sin(x+y)$ is correct, but $sin(x+y)$ is wrong. Other exceptions include the circular constant and the Euler number, both of these symbols should be in upright text, for example $\mathrm{\pi}$ and e, not π and e. Another exception is that differentials should be in upright text, not italics. So $\int x^2\,\mathrm{d}x$ is correct and $\int x^2\,dx$ is nonstandard. But this rule is nearly universally ignored.