# Derivative divides integration

AKA the U substitution

# Chain rule swapperoo

Recall the chain rule; for differentiable functions $F$ and $g$ in function notation, we have

$$(F \circ g)' = g' F' \circ g.$$

Equivalently in formula notation, the chain rule is

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(g(x)) = g'(x)F'(g(x)).$$

**Fact**

Up to an additive constant, the antiderivative "undoes" the derivative; thus

$$\int g'(x)F'(g(x))\,\mathrm{d}x = F(g(x)) + c,$$

where $c \in \mathbf{R}$.

# Match this

If we can match an integrand to $g'(x)F'(g(x))$ and we know an antiderivative of $F$, **we win.** An example

$$\int 2x \sin(x^2)\,\mathrm{d}x.$$

The match isn't particularly hidden:

- $g(x) = x^2 \implies g'(x) = 2x,$
- $F' = \sin \implies F = -\cos.$

Thus

$$\int 2x \sin(x^2)\,\mathrm{d}x = F(g(x)) + c = -\cos(x^2) + c.$$

# Out with the old, in with the new

Let's re-do the problem $\int 2x\sin(x^2)\,\mathrm{d}x$, but organize our work differently.

- The argument of $\sin$ is $x^2$. Let's define a new variable $u = x^2$. Then
$$\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = 2x\,\mathrm{d}x.$$
- We now need to write $2x\sin(x^2)\,\mathrm{d}x$ *entirely* in terms of the new variable $u$.
- When I say entirely, I mean *entirely*. This includes expressing $\mathrm{d}x$ in terms of $\mathrm{d}u$ .
- Grouping the factor of $2x$ together with $\mathrm{d}x$, we have
$$2x\sin(x^2)\,\mathrm{d}x = \sin(x^2)(2x\,\mathrm{d}x) = \sin(u)\,\mathrm{d}u.$$
- So
$$\int 2x\sin(x^2)\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u = -\cos(u) = -\cos(x^2).$$
- For the step $-\cos(u) = -\cos(x^2)$ we reverted to the "original" integration variable.

# Example redux

Let's try $\int x \exp(x^2)\,\mathrm{d}x$

- The argument of $\exp$ is $x^2$. Let's define a new variable $u = x^2$. Then

$$\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = 2x\,\mathrm{d}x.$$

- Unlike the previous problem, the integrand is missing a factor of $2$ for a complete matching. No big deal; we have

$$[\mathrm{d}u = 2x\,\mathrm{d}x] = \left[\frac{1}{2}\mathrm{d}u = x\,\mathrm{d}x\right]$$

- So

$$x\exp(x^2)\,\mathrm{d}x = \exp(x^2)(x\,\mathrm{d}x) = \frac{1}{2}\exp(u)\,\mathrm{d}u.$$

- So

$$\int x\exp(x^2)\,\mathrm{d}x = \int \frac{1}{2}\exp(u)\,\mathrm{d}u = \frac{1}{2}\exp(u) = \frac{1}{2}\exp(x^2).$$

- In step $\exp(u) = \exp(x^2)$ we reverted to the "original" integration variable.

# Derivative divides

- When we match $u = x^2$ to find $\int x \exp(x^2)\,\mathrm{d}x$, the derivative of $u$, that is $\frac{\mathrm{d}u}{\mathrm{d}x}$, doesn't exactly match the remaining factor of $x$, but it does match up to a multiplicative factor of 2.
- Since the quotient of $\frac{\mathrm{d}u}{\mathrm{d}x}$ divided by the the remaining factor of $x$ is a constant, the method is called *derivative divides*.
- But calculus books call the method *U substitution*
- If you don't like my explanations, no problem; watch `https://www.youtube.com/watch?v=8B31SAk1nD8`.

# Finding You

## Heuristic

- The integrand should be a product.
- Choose $u$ to be an expression that is "inside" a function with a known antiderivative.
- The derivative of $u$ times a constant should match the remaining factors of the integrand.

- A heuristic is a guiding principle that often works, but occasionally fails.

# Matching examples

$\int x^2 \cos(x^3)\,\mathrm{d}x$. The cosine has a known antiderivative. The expression inside cosine is $x^3$. So choose $u = x^3$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ matches the remaining factor $x^2$ in the integrand up to a multiplicative factor. **We win**.

$$[u = x^3] = [\mathrm{d}u = 3x^2 \mathrm{d}x] = \left[x^2 \mathrm{d}x = \frac{1}{3}\mathrm{d}u\right].$$

So

$$\int x^2 \cos(x^3)\,\mathrm{d}x = \int \frac{1}{3}\cos(u)\,\mathrm{d}u = \frac{1}{3}\sin(u) = \frac{1}{3}\sin(x^3).$$

# Actually, we are losers

Actually, we are **losers**. Why? Because we didn't check our work:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\sin(x^3)\right) = \frac{1}{3}3x^2\cos(x^3) = x^2\cos(x^3)$$

The integrand is $x^2\cos(x^3)$. So our answer is OK.

- Anytime we fail to check our work, we **are losers.**
- *When you check your work, you only cry once. But when we don't, we cry many times.*

# Example

$\int x \exp(-x^2)\,\mathrm{d}x$. The function $\exp$ has a known antiderivative. The expression inside $\exp$ is $-x^2$. So choose $u = -x^2$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x$ matches the remaining factor $x$ in the integrand up to a multiplicative factor. **We win**.

$$[u = -x^2] = [\mathrm{d}u = -2x\mathrm{d}x] = \left[x\mathrm{d}x = -\frac{1}{2}\mathrm{d}u\right].$$

So

$$\int x \exp(-x^2)\,\mathrm{d}x = \int -\frac{1}{2}\exp(u)\mathrm{d}u = -\frac{1}{2}\exp(u) = -\frac{1}{2}\exp(-x^2).$$

# Example

$\int \sqrt{5x+7}\,\mathrm{d}x$. Oops! The integrand isn't a product. Are we losers? No way. The integrand is $\int 1 \times \sqrt{5x+7}\,\mathrm{d}x$.
The square root has a known antiderivative, so choose $u = 5x+7$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 5$ matches the remaining factor $1$ in the integrand up to a multiplicative factor. **We win**.

$$[u = 5x + 7] = [\mathrm{d}u = 5\mathrm{d}x] = \left[\mathrm{d}x = \frac{1}{5}\mathrm{d}u\right].$$

So

$$\int \sqrt{5x+7}\,\mathrm{d}x = \int \frac{1}{5}\sqrt{u}\,\mathrm{d}u = \frac{1}{5} \times \frac{2}{3}u^{3/2} = \frac{2}{15}(5x+7)^{3/2}.$$

# Fake matches

Let's try the problem

$$\int x \exp(x^4)\,\mathrm{d}x.$$

Since $\exp$ has a known antiderivative, for $u$ choose what is inside $\exp$. Thus

$$[u = x^4] = [\mathrm{d}u = 4x^3\mathrm{d}x] = \left[\mathrm{d}x = \frac{1}{4x^3}\mathrm{d}u\right]$$

So far, Okie-dokie.

$$\int x \exp(x^4)\,\mathrm{d}x = \int x \exp(u)\frac{1}{4x^3}\mathrm{d}u = \int \exp(u)\frac{1}{4x^2}\mathrm{d}u$$

- Since $\int \exp(u)\frac{1}{4x^2}\mathrm{d}u$ depends on both the old variable $x$ and the new variable $u$, we **have not completed our task of out with the old and in with the new**.
- We could push on this a bit using $x = u^{1/4}$ to eliminate the remaining terms involving $x$. This gives

$$\int \frac{1}{4\sqrt{u}} \exp(u)\,\mathrm{d}u.$$

# Occult (that is hidden) matches

$$\int \frac{x}{\sqrt{1-x^4}}\,\mathrm{d}x$$

- We know the antiderivative of reciprocial square root. That suggests

  $$u = 1 - x^4$$

  But then

  $$\mathrm{d}u = -4x^3\,\mathrm{d}x.$$

- The substitution yields

  $$\int \frac{x}{\sqrt{1-x^4}}\,\mathrm{d}x = \int -\frac{x}{\sqrt{u}}\frac{\mathrm{d}u}{4x^3} = \int -\frac{1}{\sqrt{u}}\frac{\mathrm{d}u}{4x^2}$$

- As yet, we have a yucky mixture of old and new. This doesn't look promising. Should we bail out, or try some thing new?

# The rule of holes [1]

- Let's bailout.
- We also know the antiderivative of $\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x$.
- This suggests choosing $1 - x^4 = 1 - u^2$; equivalently $x^2 = u$. Then

$$\mathrm{d}u = 2x\mathrm{d}x.$$

- So

$$\int \frac{x}{\sqrt{1-x^4}}\, \mathrm{d}x = \int \frac{1}{2} \frac{1}{\sqrt{1-u^2}}\, \mathrm{d}u = \frac{1}{2}\sin^{-1}(u) = \frac{1}{2}\sin^{-1}(x^2)$$

[1] When you are in a hole, digging faster isn't the answer.