

**Let / Choose Proofs**

# The Let/Choose Template

Many propositions have the form

(string of  $\forall \exists$  qualifiers in involving  $x_1$  thru  $x_n$ ) ( $P(x_1, \dots, x_n)$ ),

where  $P$  is a predicate. It behooves us to have a template for proving such propositions. Let's try the example

## proposition

For every  $x \in \mathbf{R}$  there is  $y \in \mathbf{R}$  such that  $x < y$ .

# 0 Write the proposition in symbolic form:

$$(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})(x < y).$$

#1 Write the proposition in the form of a question:

*Given a real number  $x$ , can I find a number  $y$  such that  $x < y$ ?*

#2 Answer your question.

*Sure—a number that is greater than  $x$  is  $x + 1$ .*

#3 Using the symbolic form of the proposition, strictly moving from left to right, replace  $\forall$  with “Let,” and  $\exists$  with “Choose.” After each “choose” make a box to fill in. Finish with the predicate:

Let  $x \in \mathbf{R}$ . Choose  $y = \boxed{\phantom{0}}$ . We have

$$[x < y] =$$

#4 Fill in the boxes with the answers you chose, and attempt to show that the predicate is true: Let  $x \in \mathbf{R}$ . Choose  $y = \boxed{x+1}$ . We have

$$\begin{aligned}[x < y] &\equiv [x < x + 1], && \text{( substitute for } y \text{)} \\ &\equiv [0 < 1], && \text{( subtract } x \text{ from both sides )} \\ &\equiv \text{true.}\end{aligned}$$

#5 Erase the boxes:

## Proof

Let  $x \in \mathbf{R}$ . Choose  $y = x + 1$ . We have

$$\begin{aligned}[x < y] &\equiv [x < x + 1], && \text{( substitute for } y \text{)} \\ &\equiv [0 < 1], && \text{( subtract } x \text{ from both sides )} \\ &\equiv \text{true.}\end{aligned}$$

## → Pedantic

- ➊ Proof construction is a creative activity—there is no step of steps that will always generate a proof.
- ➋ But having patterns to follow and knowing techniques is useful for all creative endeavors.

## Respecting order

The order of qualifiers matters. To show this, let's reverse the order of qualifiers in the previous proposition:

### proposition

There is  $y \in \mathbf{R}$  such that for every  $x \in \mathbf{R}$  we have  $x < y$ .

**Question** Can I find a real number  $y$  such that for every real number  $x$ , we have  $x < y$ ?

**Answer:** No I don't think so—the number we choose has to be larger than  $10^{10}$ , larger than  $10^{10^{10}}$  and larger than every number.

**Tip** Proving things that are wrong take too much time. So try to avoid attempting.

Let's show that the proposition is false by showing that its negation is true; the negation of the proposition is

### proposition

For all  $y \in \mathbf{R}$  there is  $x \in \mathbf{R}$  such that  $x \geq y$ .

### Proof

Let  $y \in \mathbf{R}$ . Choose  $x = y$ . Then  $[x \geq y] \equiv [x \geq x] \equiv \text{true}$ .

- 1 We could choose  $x = y + 1$ , but we only need  $x \geq y$ , so we can choose  $x = y$ .

## Later, rinse, repeat

### proposition

For all  $x \in \mathbf{R}_{>0}$  there is  $y \in \mathbf{R}_{>0}$  such that  $y < x$ .

- 1 Write the proposition in the form of a question:  
*Given a positive real number  $x$ , can I find a positive  $y$  such that  $y$  is smaller than  $x$ ?*
- 2 Answer your question.  
*Sure—a positive number that is smaller than  $x$  is the average of zero and  $x$ ; that is  $x/2$ .*
- 3 I'm ready, I think:

### Proof

Let  $x \in \mathbf{R}_{>0}$ . Choose  $y = x/2$ . Then  $y \in \mathbf{R}_{>0}$ . Further

$$\begin{aligned} [y < x] &\equiv [x/2 < x] && \text{( substitute for } y \text{)} \\ &\equiv [1/2 < 1] && \text{( divide inequality by positive number } x \text{)} \\ &\equiv \text{true.} \end{aligned}$$