

**Proposition 1.** For all  $x, y \in \mathbf{R}$ , there is  $a \in \mathbf{R}$  such that  $x < y$  implies  $x < a < y$ .

*Proof.* (BW) Let  $x, y \in \mathbf{R}$ . Suppose  $x < y$ . Choose  $a = \frac{x+y}{2}$ . Then  $a \in \mathbf{R}$  as required. We have

$$\begin{aligned}
 [x < a < y] &\equiv \left[ x < \frac{x+y}{2} < y \right], && \text{(substitution for } a) \\
 &\equiv \left[ x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2} \right], && \text{(subtract } \frac{x+y}{2}) \\
 &\equiv \left[ \frac{x-y}{2} < 0 < \frac{y-x}{2} \right], && \text{(simplification)} \\
 &\equiv \text{True.} && ((y-x > 0) \wedge (x-y < 0))
 \end{aligned}$$

□

**Proposition 2.** For all  $r \in \mathbf{R}_{>0}$  there is  $x \in [0, 1)$  such that  $1 - r < x$ .

*Proof.* (BW) Let  $r \in \mathbf{R}_{>0}$ . Choose  $x = \begin{cases} 1 - \frac{r}{2} & r < 1 \\ \frac{1}{2} & r \geq 1 \end{cases}$ . For  $r < 1$ , we have

$$\begin{aligned}
 [1 - r < x] &\equiv \left[ 1 - r < 1 - \frac{r}{2} \right], && \text{(substitution for } x) \\
 &\equiv \left[ 0 < \frac{r}{2} \right], && \text{(add } r - 1) \\
 &\equiv \text{True.} && (0 < r < 1).
 \end{aligned}$$

And for  $r \geq 1$ , we have

$$\begin{aligned}
 [1 - r < x] &\equiv \left[ 1 - r < \frac{1}{2} \right], && \text{(substitution for } x) \\
 &\equiv \left[ \frac{1}{2} < r \right], && \text{(algebra)} \\
 &\equiv \text{True.} && (r \geq 1).
 \end{aligned}$$

□

**Proposition 3.** For all  $x \in \mathbf{R}_{>0}$  there is  $y \in \mathbf{R}_{>0}$  such that  $y < x$ .

*Proof.* (BW) Let  $x \in \mathbf{R}_{>0}$ . Choose  $y = x/2$ . Then  $y \in \mathbf{R}_{>0}$  as required. We have

$$\begin{aligned} [y < x] &\equiv \left[ \frac{x}{2} < x \right], && \text{(substitution for } y) \\ &\equiv \left[ 0 < \frac{x}{2} \right], && \text{(algebra)} \\ &\equiv \text{True.} && (x > 0) \end{aligned}$$

□

**Proposition 4.** *For every  $y \in \mathbf{R}_{>0}$  there is  $x \in \mathbf{R}_{>0}$  such that  $y \geq x$ .*

*Proof.* (BW) Let  $y \in \mathbf{R}_{>0}$ . Choose  $x = y$ . Then  $x \in \mathbf{R}_{>0}$ , as required. We have

$$\begin{aligned} [y \geq x] &\equiv [y \geq y], && \text{(substitution for } x) \\ &\equiv [0 \geq 0], && \text{(algebra)} \\ &\equiv \text{True.} \end{aligned}$$

□

**Proposition 5.** *For all  $x \in \mathbf{R}_{>0}$ , there is  $M \in \mathbf{R}$  such that  $\frac{1}{x} + 1 > M$ .* (SB)

**Proposition 6.** *There is  $M \in \mathbf{R}$  such that for all  $x \in \mathbf{R}_{>0}$ , we have  $\frac{1}{x} + 1 > M$ .* (DD)

**Proposition 7.** *There is  $m \in \mathbf{R}$  such that for all  $x \in \mathbf{R}$ , we have  $1 + m(x - 1) \leq x^2$ .* (TK)

**Proposition 8.** *For every  $a \in \mathbf{R}$ , there is  $m \in \mathbf{R}$  such that for all  $x \in \mathbf{R}$ , we have  $a^2 + m(x - a) \leq x^2$ .* (AK)

**Proposition 9.** *For all  $x, y \in \mathbf{R}$ , we have  $(x^2 = y^2) \implies (x = y)$ .* (DM)

**Proposition 10.** *For all  $x, y \in \mathbf{R}$ , we have  $(x^3 = y^3) \implies (x = y)$ .* (CR)