Boolean Logic

Lesson 1

Statements

Quasi-definition

A *statement*, also known as a *proposition*, is a sentence that has a truth value of either true or false. A *theorem* is a statement that has a truth value of true.

1. Boolean logic is named in honor of **George Boole** (1815 – 1864).
2. In boolean logic, the truth values are either **true** or **false**.
3. A statement is a concept that we can describe, but don't define.
4. An *axiom* is a statement that is *assumed* to have a truth value of true. Generally, the truth value of an axiom cannot be determined by the truth value of other theorems.

Example

Examples of statements:

1. $1 = 1$.
2. Every square is a rectangle.
3. Some integers are divisible by 42.

Examples of non-statements:

1. Square houses are boring.
2. Please make your bed, brush your teeth, and take out the garbage.

Logical notation

We'll use the ISO standard names for logical functions. These names are

negation	$\neg$
and	$\wedge$
or	$\vee$
implies	$\implies$
equivalent	$\equiv$
for all	$\forall$
there exists	$\exists$

1. For a quick review of these functions, see `https://en.wikipedia.org/wiki/Boolean_algebra`.
2. For additional ISO math symbols, see `https://en.wikipedia.org/wiki/ISO_31-11`.
3. In mathematics, for statements P and Q, the statement $P \vee Q$ is true when both P and Q are true; that is, we use the disjunction inclusive.

Negation

Definition

For a statement P, we define its *logical negation*, denoted by $\neg P$, with the *truth table*

P	$\neg P$
T	F
F	T

.

1. We'll use the ISO symbols for logical functions; see `https://en.wikipedia.org/wiki/ISO_31-11`.

Equality

Definition

Let P and Q be statements. We define *equivalence* $P \equiv Q$ by the truth table

P	Q	$P \equiv Q$
T	T	T
T	F	F
F	T	F
F	F	T

.

1. Statements P and Q are equivalent provided the statements have the same truth value.
2. Since both P and Q have two possible values, the truth table has $4(= 2 \times 2$ rows.
3. $P \equiv Q$ is an example of a *compound statement*. Its constituent parts are the statements P and Q.

Disjunctions

Definition

Let P and Q be statements. The *disjunction* of P with Q, denoted by $P \vee Q$, is a statement whose truth value is given by

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

.

1. That is $P \vee Q$ is false when both P and Q are false; otherwise $P \vee Q$ is true.
2. $P \vee Q$ is another example of a *compound statement*.
3. In mathematical logic, notice that True $\vee$ True has a truth value of true.

Conjunctions

Definition

Let P and Q be a statements. The *conjunction* of P with Q, denoted by $P \wedge Q$, is a statement whose truth value is given by

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

.

1. That is $P \wedge Q$ is true provided both P and Q are true; otherwise $P \wedge Q$ is false.

Tautologies

Definition

A compound statement that has a truth value of true for all possible truth values of its constituent parts is a *tautology*.

Example

Each of the following are tautologies:

1. $P \vee \neg P$,
2. $P \equiv P$,
3. $P \equiv \neg\neg P$,
4. $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$.

Example

Example

Let's show that $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is a tautlogy. There are two contistuent parts, so we need a truth table with four rows. How many columns it has depends on how many steps we are willing to skip.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	T

.

The last column shows that regardless of the truth values for P and Q, the statement $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is true; therefore $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is a tautalogy.

1. Possibly the truth table should have columns for $\neg P$ and $\neg Q$.
2. The tautalogy $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ is due to De Morgan, and is known as *De Morgan's law.*

Conditionals

The conditional is a logical connective that allows us to form a compound statement with the meaning "if P, then Q." Specifically:

Definition

Let P and Q be a statements. We define $P \implies Q$ with the truth table

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

.

.

1. In the conditional $P \implies Q$, we say that P is the *hypothesis* and Q is the *conclusion*.
2. A conditional is false when the hypothesis is true, but the conclusion is false; otherwise, a conditional is true.

Converse

Definition

The *converse* of the conditional $P \implies Q$ is the conditional $Q \implies P$.

Fact

A truth table shows that $(P \implies Q) \equiv (Q \implies P)$ is not a tautology. Specifically, $T \implies F$ is false, but $F \implies T$ is true.

Example

Consider the statement

If $x < 5$, then $x < 7$

and its converse

If $x < 7$, then $x < 5$.

The first statement is true, but its converse is false (because, for example, x could be six, making $x < 7$ true, but $x < 5$ false.

Contrapositive

Definition

The *contrapositive* of the conditional $P \implies Q$ is the conditional $\neg Q \implies \neg P$.

Fact

A truth table shows that $(P \implies Q) \equiv (\neg Q \implies \neg P)$ is a tautology.

Example

Consider the statements:

If $x < 5$, *then* $x < 7$

and its contrapositive

If $x \geq 7$, *then* $x \geq 5$

These statements are logically equivalent.

Extra conditional

Fact

A truth table shows that $(P \implies Q) \equiv \neg P \vee Q$ is a tautology. This makes $P \not\Longrightarrow Q$ equivalent to $P \wedge \neg Q$.

Predicates

Definition

A function whose range is a subset of $\{\mathrm{true}, \mathrm{false}\}$ is a *predicate*. Alternatively, a boolean valued function is a predicate.

Example

The function

$$F = x \in (-\infty, \infty) \mapsto \begin{cases} \text{true} & \text{if } x \text{ is rational} \\ \text{false} & \text{if } x \text{ is irrational} \end{cases}$$

is a predicate. We have, for example

$$F(2/3) = \text{true}, \quad F(\sqrt{2}) = \text{false}, \quad F(\pi) = \text{false}, \quad F(\mathrm{e}) = \text{false},$$

Last I checked, nobody knows the value of $F(\pi - \mathrm{e})$.

Universal quantification

Quasi-definition

Let P be a predicate defined on a set A. The statement

$$(\forall x \in A)\,(P(x))$$

is true provided for all $x \in A$, the statement $P(x)$ is true; the statement is false if for some $x \in A$, the statement $P(x)$ is false.

1. The symbol $\forall$ is the *universal quantifier*.
2. To show that $(\forall x \in A)\,(P(x))$ is true, we cannot simply show that $P(x)$ is true for one specific member of the set A.

Existential quantification

Quasi-definition

Let P be a predicate defined on a set A. The statement

$$(\exists x \in A)\,(P(x))$$

is true provided there is $x \in A$ such that the statement $P(x)$ is true; the statement is false if for all $x \in A$, the statement $P(x)$ is false.

1. The symbol $\exists$ is the *existential quantifier*.
2. To show that a statement of the form $(\exists x \in A)\,(P(x))$ is true, the task is to choose a specific member x of the set A that makes $P(x)$ true.
3. Since it's impossible to choose a specific member of the empty set $\varnothing$, regardless of the predicate P, the statement $(\exists x \in \varnothing)\,(P(x))$ is false.

Negative practice

For each member x of a set A, let $T(x)$ be a statement. Each of the following are tautologies:

$$\neg(\forall x \in A)(T(x)) \equiv (\exists x \in A)(\neg T(x)),$$
$$\neg(\exists x \in A)(T(x)) \equiv (\forall x \in A)(\neg T(x)).$$

1. We don't negate the set membership–the following is rubbish:

$$\neg(\forall x \in A)(T(x)) \equiv (\exists x \notin A)(\neg T(x)).$$

For $x \notin A$, the predicate T might not even be defined.

Negative experiences

Consider the statement "For all $x \in \mathbf{R}$, we have $x \in (-1,1) \implies x^2 < 1$."
Symbolically, the statement is

$$(\forall x \in \mathbf{R})(x \in (-1,1) \implies x^2 < 1).$$

Its negation is (in general $a \not< b) \equiv (a \geq b)$

$$(\exists x \in \mathbf{R})(x \notin (-1,1) \vee x^2 \geq 1).$$

In English, the negation is "There is $x \in \mathbf{R}$ such that either $x \in (-1,1) \vee x^2 \geq 1$. "

More Famous Tautologies

Let P and Q be statements. Each of the following are tautologies:

1. $(P \equiv Q) \equiv (P \implies Q) \wedge (Q \implies P)$
2. $\neg (P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ (De Morgan's rule)
3. $(P \implies Q) \equiv (\neg Q \implies \neg P)$ (Rule of contrapositive)

Logical tips

Tip Any time you have trouble proving $P \implies Q$, try proving $\neg Q \implies \neg P$ instead.

Tip Generally to prove $P \equiv Q$, you should prove both $P \implies Q$ and $Q \implies P$. See tautology one of the previous slide. Students often refer to this process as "proving it both ways."

Tip In general, $Q \implies P$ is **not** equivalent to $P \implies Q$. Accidentally (on purpose) proving $Q \implies P$ instead of $P \implies Q$ will almost surely earn you zero points.

Baby steps

Let P, Q, and R be statements. The following is a tautology:

$$((P \implies Q) \wedge (Q \implies R)) \implies (P \implies R).$$

Thus we can show that $P \implies R$ is true by finding a statement Q such that both $P \implies Q$ is true and $Q \implies R$ is true.

1. Think of proving $P \implies Q$ and $Q \implies R$ as *baby steps* in proving $P \implies R$.
2. Generally, we can make multiple baby steps; thus

$$((P \implies Q_1) \wedge (Q_1 \implies Q_2) \wedge \cdots \wedge (Q_n \implies R)) \implies (P \implies R).$$

   is a tautology.