MATH 460 **Name:**
Homework ∞, Fall 2023

Homework ∞ has questions 1 through 1 with a total of 0 points. This work is never due.

1. For first term calculus students, an intimidating function is $F(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$. Finding the derivative of this function will strike fear many students. Using the chain rule twice and not "simplifying" between applications, I claim that a formula for the derivative is

$$F'(x) = \frac{\frac{\frac{1}{2\sqrt{x}}+1}{2\sqrt{x+\sqrt{x}}} + 1}{2\sqrt{\sqrt{x+\sqrt{x}}+x}}.$$

Opinions might vary on a "proper" simplification, but likely most would say that the simplified expression is

$$F'(x) = \frac{4\sqrt{x}\sqrt{x+\sqrt{x}} + 2\sqrt{x} + 1}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{\sqrt{x+\sqrt{x}}+x}}.$$

Although all this calculation is frightening for many calculus students, it's an algorithmic process and fairly straightforward. Teachers need to push the boundaries of the comfort zone of students, and this is a good problem to do that. And just for fun the derivative of $F(x) = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x}}}}$ is the monstrous

$$F'(x) = \frac{\frac{\frac{\frac{1}{2\sqrt{x}}+1}{2\sqrt{x+\sqrt{x}}}+1}{2\sqrt{\sqrt{x+\sqrt{x}}+x}} + 1}{2\sqrt{\sqrt{\sqrt{x+\sqrt{x}}+x}+x}}.$$

As good as these problem is, let's kick it up an notch. Say we infinitely repeat the pattern and "define" a function Q by

$$Q(x) = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}.$$

In mathematics, using an ellipsis is like waving a white flag–use your imagination to continue something in the most natural way, but with no clue of how to define "most natural."

Can we give meaning to the putative function Q without using an ellipsis? Oh, sure. Let's define a sequence of functions whose limit is Q. We can do this recursively

$$G_n(x) = \begin{cases} \sqrt{x} & n = 0 \\ \sqrt{x + G_{n-1}(x)} & n \in \mathbf{Z}_{>0} \end{cases}.$$

Then, for example $G_1(x) = \sqrt{x + \sqrt{x}}$ and $G_2(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$. And it seems like the sequence $k \in \mathbf{Z}_{\geq 0} \mapsto G_k$ converges to Q.

So our question now is does the sequence $k \in \mathbf{Z}_{\geq 0} \mapsto G_k$ converge? Immediately, we have a problem; we're asking a question about the convergence of a function-valued function, not a real-valued function. And we don't know what that means. But one possible meaning is the question: For $x \in \mathbf{R}_{\geq 0}$, does the limit

$$\lim_{n \to \infty} G_n(x)$$

exist? This notion of convergence of a function-valued sequence is known as *pointwise* convergence. But there are other definitions of such convergence (weak convergence and norm convergence, to name a few).

Your challenge is to show that for all $x \in \mathbf{R}_{\geq 0}$, we have

$$\lim_{n \to \infty} G_n(x) = \frac{1 + \sqrt{1 + 4x}}{2}.$$

Amusingly, the natural domain of $\frac{1+\sqrt{1+4x}}{2}$ is $[-1/4, \infty)$, but the domain of each function in the sequence is $[0, \infty)$. This doesn't mean that my answer is rubbish, but it does mean that it's amusing.

If you accept this challenge, don't forget GNAT. Try graphing $y = G_2(x), y = G_3(x), \ldots,$ and $y = \frac{1+\sqrt{1+4x}}{2}$.