## Greek characters

| Name | Symbol | Typical use(s) |
|---|---|---|
| alpha | $\alpha$ | angle, constant |
| beta | $\beta$ | angle, constant |
| gamma | $\gamma$ | angle, constant |
| delta | $\delta$ | limit definition |
| epsilon | $\epsilon$ or $\varepsilon$ | limit definition |
| theta | $\theta$ or $\vartheta$ | angle |
| pi | $\pi$ or $\varpi$ | circular constant |
| phi | $\phi$ or $\varphi$ | angle, constant |

## Named sets

| | |
|---|---|
| empty set | $\varnothing$ |
| real numbers | $\mathbf{R}$ |
| ordered pairs | $\mathbf{R}^2$ |

| | |
|---|---|
| integers | $\mathbf{Z}$ |
| positive integers | $\mathbf{Z}_{>0}$ |
| positive reals | $\mathbf{R}_{>0}$ |

## Set symbols

| Meaning | Symbol |
|---|---|
| is a member | $\in$ |
| subset | $\subset$ |
| intersection | $\cap$ |

| Meaning | Symbol |
|---|---|
| union | $\cup$ |
| complement | superscript $^{\mathrm{C}}$ |
| set minus | $\setminus$ |

## Intervals

For numbers $a$ and $b$, we define the intervals:

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$$
$$[a, b) = \{x \in \mathbf{R} \mid a \le x < b\}$$
$$(a, b] = \{x \in \mathbf{R} \mid a < x \le b\}$$
$$[a, b] = \{x \in \mathbf{R} \mid a \le x \le b\}$$

## Logic symbols

| Meaning | Symbol |
|---|---|
| negation | $\neg$ |
| and | $\wedge$ |
| or | $\vee$ |
| implies | $\implies$ |

| Meaning | Symbol |
|---|---|
| equivalent | $\equiv$ |
| iff | $\iff$ |
| for all | $\forall$ |
| there exists | $\exists$ |

## Tautologies

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$
$$(P \implies Q) \equiv (\neg Q \implies \neg P)$$
$$P \not\Longrightarrow Q \equiv P \wedge \neg Q$$
$$(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P))$$
$$\neg(\forall\, x \in A)(P(x)) \equiv (\exists\, x \in A)(\neg P(x))$$
$$\neg(\exists\, x \in A)(P(x)) \equiv (\forall\, x \in A)(\neg P(x))$$

## Function notation

| | |
|---|---|
| $\mathrm{dom}(F)$ | domain of function $F$ |
| $\mathrm{range}(F)$ | range of function $F$ |
| $\mathrm{C}_A$ | set of continuous functions on set $A$ |
| $\mathrm{C}^1_A$ | set of differentiable functions on set $A$ |
| $A \to B$ | set of functions from $A$ to $B$ |

## Generalized set operators

Each member of a set $\mathcal{C}$ is a set:

$$\bigcup_{A \in \mathcal{C}} A = \{z \mid (\exists\, B \in \mathcal{C})(z \in B)\}$$
$$\bigcap_{A \in \mathcal{C}} A = \{z \mid (\forall\, B \in \mathcal{C})(z \in B)\}$$

Theorem: $\displaystyle \bigcup_{A \in \mathcal{C}} A^{\mathrm{C}} = \left(\bigcap_{A \in \mathcal{C}} A\right)^{\mathrm{C}}$

## Functions applied to sets

Let $A \subset \mathrm{dom}(F)$ and $B \subset \mathrm{range}(F)$:

$$F(A) = \{F(x) \mid x \in A\}$$
$$F^{-1}(B) = \{x \in \mathrm{dom}(F) \mid F(x) \in B\}$$

## Triangle inequalities

For all $x, y \in \mathbf{R}$, we have

$$|x + y| \le |x| + |y|$$
$$\big||x| - |y|\big| \le |x - y|$$

## Floor and ceiling

Definitions:

$$\lfloor x \rfloor = \max\{k \in \mathbf{Z} \mid k \le x\}$$
$$\lceil x \rceil = \min\{k \in \mathbf{Z} \mid k \ge x\}$$

Properties:

$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(x < n \iff \lfloor x \rfloor < n)$$
$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(n < x \iff n < \lceil x \rceil)$$

## Bounded sets

**Bounded below** A set $A$ is *bounded below* provided $(\exists\, M \in \mathbf{R})(\forall\, x \in A)(M \le x)$.

**Bounded above** The set $A$ is *bounded above* provided $(\exists\, M \in \mathbf{R})(\forall\, x \in A)(x \le M)$.

**Bounded** A set is *bounded* if it is bounded below and bounded above.

## Elementary function properties

**Increasing** $(\forall\, x, y \in A)(x < y \implies F(x) \le F(y))$. For strictly increasing, replace $F(x) \le F(y)$ with $F(x) < F(y)$.

**Decreasing** $(\forall\, x, y \in A)(x < y \implies F(x) \ge F(y))$ For strictly decreasing, replace $F(x) \ge F(y)$ with $F(x) > F(y)$.

**One-to-one** $(\forall\, x, y \in \mathrm{dom}(F))(F(x) = F(y) \implies x = y)$

**Subadditive** $(\forall\, x, y \in \mathrm{dom}(F))(F(x + y) \le F(x) + F(y))$

**Bounded above** $(\exists\, M \in \mathbf{R})(\forall x \in \mathrm{dom}(F))(F(x) \le M)$

**Bounded below** $(\exists\, M \in \mathbf{R})(\forall x \in \mathrm{dom}(F))(M \le F(x))$

## Topology

**Open ball** $\mathrm{ball}(a, r) = \{x \in \mathbf{R} \mid -r + a < x < r + a\}$

**Punctured ball** $\mathrm{ball}'(a, r) = \mathrm{ball}(a, r) \setminus \{a\}$

**Open set** A subset $A$ of $\mathbf{R}$ is *open* provided $(\forall x \in A)\,(\exists\, r \in \mathbf{R}_{>0})(\mathrm{ball}(x, r) \subset A)$

**Closed set** A subset $A$ of $\mathbf{R}$ is *closed* provided $\mathbf{R} \setminus A$ is open.

**Limit point** A number $a$ is a *limit point* of a set $A$ provided $(\forall\, r \in \mathbf{R}_{>0})\big(\mathrm{ball}'(a, r) \cap A \ne \varnothing\big)$.

**Boundary point** A number $a$ is a *boundary point* of a set $A$ provided

$$(\forall r \in \mathbf{R}_{>0})\Big(\text{ball}(a,r) \cap A \neq \varnothing \wedge \text{ball}(a,r) \cap A^{\mathrm{C}} \neq \varnothing\Big).$$

**Set closure** $\overline{A} = A \cup \text{LP}(A)$, were $\text{LP}(A)$ is the set of limit points of $A$.

**Open cover** A set $\mathcal{C}$ is an open cover of a set $A$ provided

(a) every member of $\mathcal{C}$ is an open set
(b) $A \subset \bigcup_{B \in \mathcal{C}} B$

**Compact** A set $A$ is compact provided for every open cover $\mathcal{C}$ of $A$, there is a finite subset $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathcal{C}'$ is an open cover of $A$.

## Least and greatest bounds

For any subset $A$ of $\mathbf{R}$:

**glb** $z = \text{glb}(A)$ provided

(a) $z$ is an lower bound for $A$
(b) if $x$ is a lower bound for $A$ then $x \leq z$

**lub** $z = \text{lub}(A)$ provided

(a) $z$ is an upper bound for $A$
(b) if $x$ is a upper bound for $A$ then $z \leq x$

## Sequences

**Bounded** A sequence $F$ is bounded if $\text{range}(F)$ bounded.

**Monotone** A sequence is monotone if it either increases or decreases.

**Cauchy** A sequence $F$ is Cauchy provided

(a) for every $\varepsilon \in \mathbf{R}_{>0}$
(b) there is $n \in \mathbf{Z}$
(c) such that for all $k, \ell \in \mathbf{Z}_{>n}$
(d) $|F_k - F_\ell| < \varepsilon$

**Converges** A sequence $F$ converges provided

(a) there is $L \in \mathbf{R}$
(b) and $n \in \mathbf{Z}$
(c) such that for all $k \in \mathbf{Z}_{>n}$
(d) $|F_k - L| < \varepsilon$.

## Functions

**Continuous** A function $F$ is continuous at $a$ provided

(a) $a \in \text{dom}(F)$ and
(b) for every $\varepsilon \in \mathbf{R}_{>0}$
(c) there is $\delta \in \mathbf{R}_{>0}$
(d) such that for all $x \in \text{ball}(a,\delta) \cap \text{dom}(F)$
(e) we have $F(x) \in \text{ball}(F(a), \epsilon)$.

**Uniformly continuous** A function $F$ is uniformly continuous on a set $A$ provided

(a) $A \subset \text{dom}(F)$; and
(b) for every $\varepsilon \in \mathbf{R}_{>0}$
(c) there is $\delta \in \mathbf{R}_{>0}$
(d) such that for all $x, y \in A$ and $|x - y| < \delta$
(e) we have $|F(x) - F(y)| < \epsilon$.

**Limit** A function $F$ has a limit toward $a$ provided

(a) $a$ is a limit point of $\text{dom}(F)$; and
(b) there is $L \in \mathbf{R}$
(c) such that for every $\varepsilon \in \mathbf{R}_{>0}$
(d) there is $\delta \in \mathbf{R}_{>0}$
(e) such that for all $x \in \text{ball}'(a,\delta)$
(f) we have $F(x) \in \text{ball}(L, \epsilon)$.

**Differentiable** A function $F$ is differentiable at $a$ provided

(a) $a \in \text{dom}(F)$; and
(b) there is $\phi \in \text{dom}(F) \to \mathbf{R}$
(c) such that $\phi$ is continuous at $a$ and
(d) $(\forall x \in \text{dom}(F))(F(x) = F(a) + (x-a)\phi(x))$.

## Riemann sums

**Partition** A set $\mathcal{P}$ is a partition of an interval $[a,b]$ provided

(a) the set $\mathcal{P}$ is finite
(b) every member of $\mathcal{P}$ is an open interval
(c) the members of $\mathcal{P}$ are pairwise disjoint
(d) $\bigcup_{I \in \mathcal{P}} \overline{I} = [a,b]$

Let $F$ be a bounded function on an interval $[a,b]$ and let $\mathcal{P}$ be a partition of $[a,b]$.

**Lower sum** $\underline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{glb}\big(F\big(\overline{I}\big)\big) \times \text{length}(I)$

**Upper sum** $\overline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{lub}\big(F\big(\overline{I}\big)\big) \times \text{length}(I)$

**Riemann sum** $\sum_{I \in \mathcal{P}, x^\star \in \overline{I}} F(x^\star) \times \text{length}(I)$

## Axioms

**Completeness** Every nonempty subset $A$ of $\mathbf{R}$ that is bounded above has a least upper bound.

**Well-ordering** Every nonempty set of positive integers contains a least element.

**Induction** $(\forall n \in \mathbf{Z}_{\geq 0})(P(n))$ if and only if $P(0) \wedge (\forall n \in \mathbf{Z}_{\geq 0})(P(n) \implies P(n+1))$.

## Named theorems

**Archimedean** $(\forall x \in \mathbf{R})(\exists n \in \mathbf{Z})(n > x) \equiv \text{true}$.

**Bolzano–Weirstrass** Every bounded real valued sequence has a convergent subsequence.

**Heine–Borel** A subset of $\mathbf{R}$ is compact iff it is closed and bounded.

**Cauchy completeness** Every Cauchy sequence in $\mathbf{R}$ converges.

**Monotone convergence** Every bounded monotone sequence converges.

**Intermediate value theorem** If $F \in \mathrm{C}_{[a,b]}$, then for all $y \in [\min(F(a), F(b)), \max(F(a), F(b))]$ there is $x \in [a,b]$ such that $F(x) = y$.

**Mean Value** If $F \in \mathrm{C}_{[a,b]} \cap \mathrm{C}^1_{(a,b)}$, there is $\xi \in (a,b)$ such that $(b-a)F'(\xi) = F(b) - F(a)$.