

Greek characters

Name	Symbol	Typical use(s)
alpha	α	angle, constant
beta	β	angle, constant
gamma	γ	angle, constant
delta	δ	limit definition
epsilon	ϵ or ε	limit definition
theta	θ or ϑ	angle
pi	π or π	circular constant
phi	ϕ or φ	angle, constant

Named sets

empty set	\emptyset
real numbers	\mathbf{R}
ordered pairs	\mathbf{R}^2

integers	\mathbf{Z}
positive integers	$\mathbf{Z}_{>0}$
positive reals	$\mathbf{R}_{>0}$

Set symbols

Meaning	Symbol
is a member	\in
subset	\subset
intersection	\cap

Meaning	Symbol
union	\cup
complement	superscript \mathbf{C}
set minus	\setminus

Intervals

For numbers a and b , we define the intervals:

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$$

$$[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$$

$$(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$$

$$[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$

Logic symbols

Meaning	Symbol
negation	\neg
and	\wedge
or	\vee
implies	\implies

Meaning	Symbol
equivalent	\equiv
iff	\iff
for all	\forall
there exists	\exists

Tautologies

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$P \not\Rightarrow Q \equiv P \wedge \neg Q$$

$$(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P))$$

$$\neg(\forall x \in A)(P(x)) \equiv (\exists x \in A)(\neg P(x))$$

$$\neg(\exists x \in A)(P(x)) \equiv (\forall x \in A)(\neg P(x))$$

Function notation

$\text{dom}(F)$	domain of function F
$\text{range}(F)$	range of function F
C_A	set of continuous functions on set A
C_A^1	set of differentiable functions on set A
$A \rightarrow B$	set of functions from A to B

Generalized set operators

Each member of a set \mathcal{C} is a set:

$$\bigcup_{A \in \mathcal{C}} A = \{z \mid (\exists B \in \mathcal{C})(z \in B)\}$$

$$\bigcap_{A \in \mathcal{C}} A = \{z \mid (\forall B \in \mathcal{C})(z \in B)\}$$

$$\text{Theorem: } \bigcup_{A \in \mathcal{C}} A^{\mathbf{C}} = \left(\bigcap_{A \in \mathcal{C}} A \right)^{\mathbf{C}}$$

Functions applied to sets

Let $A \subset \text{dom}(F)$ and $B \subset \text{range}(F)$:

$$F(A) = \{F(x) \mid x \in A\}$$

$$F^{-1}(B) = \{x \in \text{dom}(F) \mid F(x) \in B\}$$

Triangle inequalities

For all $x, y \in \mathbf{R}$, we have

$$|x + y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

Floor and ceiling

Definitions:

$$\lfloor x \rfloor = \max\{k \in \mathbf{Z} \mid k \leq x\}$$

$$\lceil x \rceil = \min\{k \in \mathbf{Z} \mid k \geq x\}$$

Properties:

$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(x < n \iff \lfloor x \rfloor < n)$$

$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(n < x \iff n < \lceil x \rceil)$$

Bounded sets

Bounded below A set A is *bounded below* provided $(\exists M \in \mathbf{R})(\forall x \in A)(M \leq x)$.

Bounded above The set A is *bounded above* provided $(\exists M \in \mathbf{R})(\forall x \in A)(x \leq M)$.

Bounded A set is *bounded* if it is bounded below and bounded above.

Elementary function properties

Increasing $(\forall x, y \in A)(x < y \implies F(x) \leq F(y))$. For strictly increasing, replace $F(x) \leq F(y)$ with $F(x) < F(y)$.

Decreasing $(\forall x, y \in A)(x < y \implies F(x) \geq F(y))$. For strictly decreasing, replace $F(x) \geq F(y)$ with $F(x) > F(y)$.

One-to-one

$$(\forall x, y \in \text{dom}(F))(F(x) = F(y) \implies x = y)$$

Subadditive

$$(\forall x, y \in \text{dom}(F))(F(x + y) \leq F(x) + F(y))$$

Bounded above $(\exists M \in \mathbf{R})(\forall x \in \text{dom}(F))(F(x) \leq M)$

Bounded below $(\exists M \in \mathbf{R})(\forall x \in \text{dom}(F))(M \leq F(x))$

Topology

Open ball $\text{ball}(a, r) = \{x \in \mathbf{R} \mid -r + a < x < r + a\}$

Punctured ball $\text{ball}'(a, r) = \text{ball}(a, r) \setminus \{a\}$

Open set A subset A of \mathbf{R} is *open* provided $(\forall x \in A)(\exists r \in \mathbf{R}_{>0})(\text{ball}(x, r) \subset A)$

Closed set A subset A of \mathbf{R} is *closed* provided $\mathbf{R} \setminus A$ is open.

Limit point A number a is a *limit point* of a set A provided $(\forall r \in \mathbf{R}_{>0})(\text{ball}'(a, r) \cap A \neq \emptyset)$.

Set closure $\bar{A} = A \cup \text{LP}(A)$, where $\text{LP}(A)$ is the set of limit points of A .

Open cover A set \mathcal{C} is a cover of a set A provided

- (a) every member of \mathcal{C} is a set
- (b) $A \subset \bigcup_{B \in \mathcal{C}} B$

Compact A set A is compact provided for every open cover \mathcal{C} of A , there is a finite subset \mathcal{C}' of \mathcal{C} such that \mathcal{C}' is an open cover of A .

Least and greatest bounds

For any subset A of \mathbf{R} :

glb $z = \text{glb}(A)$ provided

- (a) z is an lower bound for A
- (b) x is a lower bound for A implies $x \leq z$

lub $z = \text{lub}(A)$ provided

- (a) z is an upper bound for A
- (b) x is a upper bound for A implies $z \leq x$

Sequences

Bounded A sequence F is bounded if $\text{range}(F)$ bounded.

Monotone A sequence is monotone if it either increases or decreases.

Cauchy A sequence F is Cauchy provided

- (a) for every $\varepsilon \in \mathbf{R}_{>0}$
- (b) there is $n \in \mathbf{Z}$
- (c) such that for all $k, \ell \in \mathbf{Z}_{>n}$
- (d) $|F_k - F_\ell| < \varepsilon$

Converges A sequence F converges provided

- (a) there is $L \in \mathbf{R}$
- (b) and $n \in \mathbf{Z}$
- (c) such that for all $k \in \mathbf{Z}_{>n}$
- (d) $|F_k - L| < \varepsilon$.

Functions

Continuous A function F is continuous at a provided

- (a) $a \in \text{dom}(F)$ and
- (b) for every $\varepsilon \in \mathbf{R}_{>0}$
- (c) there is $\delta \in \mathbf{R}_{>0}$
- (d) such that for all $x \in \text{ball}(a, \delta) \cap \text{dom}(F)$
- (e) we have $F(x) \in \text{ball}(F(a), \varepsilon)$.

Uniformly continuous A function F is uniformly continuous on a set A provided

- (a) $A \subset \text{dom}(F)$; and
- (b) for every $\varepsilon \in \mathbf{R}_{>0}$
- (c) there is $\delta \in \mathbf{R}_{>0}$
- (d) such that for all $x, y \in A$ and $|x - y| < \delta$
- (e) we have $|F(x) - F(y)| < \varepsilon$.

Limit A function F has a limit toward a provided

- (a) a is a limit point of $\text{dom}(F)$; and
- (b) there is $L \in \mathbf{R}$
- (c) such that for every $\varepsilon \in \mathbf{R}_{>0}$
- (d) there is $\delta \in \mathbf{R}_{>0}$
- (e) such that for all $x \in \text{ball}'(a, \delta)$
- (f) we have $F(x) \in \text{ball}(L, \varepsilon)$.

Differentiable A function F is differentiable at a provided

- (a) $a \in \text{dom}(F)$; and
- (b) there is $\phi \in \text{dom}(F) \rightarrow \mathbf{R}$
- (c) such that ϕ is continuous at a and
- (d) $(\forall x \in \text{dom}(F))(F(x) = F(a) + (x - a)\phi(x))$.

Riemann sums

Partition A set \mathcal{P} is a partition of an interval $[a, b]$ provided

- (a) the set \mathcal{P} is finite
- (b) every member of \mathcal{P} is an open interval
- (c) the members of \mathcal{P} are pairwise disjoint
- (d) $\bigcup_{I \in \mathcal{P}} I = [a, b]$

Let F be a bounded function on an interval $[a, b]$ and let \mathcal{P} be a partition of $[a, b]$.

Lower sum $\underline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{glb}(F(I)) \times \text{length}(I)$

Upper sum $\bar{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{lub}(F(I)) \times \text{length}(I)$

Riemann sum $\sum_{I \in \mathcal{P}, x^* \in I} F(x^*) \times \text{length}(I)$

Axioms

Completeness Every nonempty subset A of \mathbf{R} that is bounded above has a least upper bound.

Well-ordering Every nonempty set of positive integers contains a least element.

Induction $(\forall n \in \mathbf{Z}_{\geq 0})(P(n))$ if and only if $P(0) \wedge (\forall n \in \mathbf{Z}_{\geq 0})(P(n) \implies P(n+1))$.

Named theorems

Archimedean $(\forall x \in \mathbf{R})(\exists n \in \mathbf{Z})(n > x) \equiv \text{true}$.

Bolzano-Weirstrass Every bounded real valued sequence has a convergent subsequence.



Heine-Borel A subset of \mathbf{R} is compact iff it is closed and bounded.

Cauchy completeness Every Cauchy sequence in \mathbf{R} converges.

Monotone convergence Every bounded monotone sequence converges.

Intermediate value theorem If $F \in C_{[a,b]}$, then for all $y \in [\min(F(a), F(b)), \max(F(a), F(b))]$ there is $x \in [a, b]$ such that $F(x) = y$.

Mean Value If $F \in C_{[a,b]} \cap C^1_{(a,b)}$, there is $\xi \in (a, b)$ such that $(b - a)F'(\xi) = F(b) - F(a)$.

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