What your calculus textbook doesn't tell you about trigonometric substitution

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Abstract The method of trigonometric substitution for indefinite integration can give impressive looking results, but these results are sometimes poorly suited for numerical evaluation (ill-conditioned sums). For one particular example, we give two workarounds for the ill-conditioned sum problem, one that involves algebraically altering the antiderivative, and the other using the Gauss hypergeometric function.

After making multiple bad starts, wasting ten sheets of engineering paper, and erasing more than you would like to admit, finally you are confident that you have correctly used integration by trigonometric substitution and have deduced an impressive looking answer to your first homework question. Your answer is

$$\int \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x = \frac{\left(x^2-1\right)^{9/2}}{2x^2} - \frac{\left(x^2-1\right)^{7/2}}{2} + \frac{7\left(x^2-1\right)^{5/2}}{10} - \frac{7\left(x^2-1\right)^{3/2}}{6} + \frac{7(x^2-1)^{1/2}}{2} + \frac{7\arcsin\left(\frac{1}{x}\right)}{2}.$$

But wait! There's more. To add to the punishment, your teacher asks for the value of the definite integral $\int_1^{10^9} \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x$. Easy-peasy, you think. Surely after the tortuous trigonometric substitution problem, this should be a breeze. All you need to do is evaluate

$$\frac{\left(10^{18}-1\right)^{9/2}}{2 10^{18}} - \frac{\left(10^{18}-1\right)^{7/2}}{2} + \frac{7\left(10^{18}-1\right)^{5/2}}{10} - \frac{7\left(10^{18}-1\right)^{3/2}}{6} + \frac{7(10^{18}-1)^{1/2}}{2} + \frac{7\arcsin\left(10^{-18}\right)}{2} - \frac{7\arcsin(1)}{2}.$$

You tap this into your nearest computing device, and out pops the answer to an impressive sixteen decimal places. Your final answer is

$$\int_1^{10^9} \frac{\left(x^2-1\right)^{\frac{7}{2}}}{x^3}\,\mathrm{d}x \approx 9.134385233318144 \times 10^{46}.$$

So you circle this answer and turn it in. Good job, you think, and you quickly move on to the next question.

Not so fast. Your ten sheets of wasted engineering paper earned you full credit for your antiderivative, but your decimal approximation earns you a score of zero. Although your teacher says that your numerical value isn't *manifestly wrong*, its just shy of being *obviously wrong*. To see that the numerical value is close to being obviously wrong, some thought shows that for all $x \in \mathbf{R}_{\geq 1}$, we have $0 < \frac{\left(x^2-1\right)^{\frac{7}{2}}}{x^3} < \frac{\left(x^2\right)^{7/2}}{x^3} = x^4$. So it must be true that

$$0 < \int_1^{10^9} \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x < \int_1^{10^9} x^4\,\mathrm{d}x = \frac{10^{45}-1}{5} < 2 \times 10^{44}.$$

Yikes! Using an exact calculation and some arithmetic, you got a value for the definite integral that is nearly one thousand times larger than an easily found upper bound.

What's the story? Pasting in the upper limit of 10^9 into our painstakingly determined antiderivative, we need to sum

$$\begin{aligned}&5.000000000000003\times 10^{62}-5.000000000000002\times 10^{62}+7.000000000000003\times 10^{44}\\&-1.1666666666666669\times 10^{27}+3.5000000000000005\times 10^{9}+3.5000000000000003\times 10^{-9}.\end{aligned}$$

This is an example of what is known as an *ill conditioned sum*, and it is the first two terms of the sum that are the troublemakers. Each term in this sum is properly rounded to sixteen decimal digits. This means that the relative difference between each term in the sum and its true value are no more than about 10^{-16}. Specifically, the true value of the first term can be as small as $5.000000000000003\times 10^{62}\times(1-10^{-16})$ or as large as $5.000000000000003\times 10^{62}\times(1+10^{-16})$. And similarly for the second term.

Putting this together, the sum of the first two terms might be as small as

$$5.000000000000003\times 10^{62}\times(1-10^{-16})-5.000000000000002\times 10^{62}\times(1+10^{-16})\approx -1.1182158029987521\times 10^{46},$$

and as large as

$$5.000000000000003\times 10^{62}\times(1+10^{-16})-5.000000000000002\times 10^{62}\times(1-10^{-16})\approx 1.888178419700126\times 10^{47}.$$

We don't even know if the sum of the first two terms is negative or positive. Using sixteen decimal digits, we cannot accurately evaluate this expression.

Is there a cure? Well, yes there is. It is tedious to show, but an algebraically equivalent form for the antiderivative is

$$\int\frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x=\frac{\sqrt{x^2-1}\left(6x^6-32x^4+116x^2+15\right)+105\arcsin\left(\frac{1}{x}\right)x^2}{30x^2}.$$

Numerically evaluating this expression at 10^9 does not involve subtracting two numbers that are nearly equal. So, unlike our first expression, this form of the answer is well-conditioned. Typing this into the nearest calculator yields

$$\int_1^{10^9}\frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x\approx 1.9999999999999998\times 10^{44}.$$

This decimal approximation is accurate, and it is smaller than the upper bound of 2×10^{44}.

If you are willing to dip into hypergeometric functions, there is another cure. The name "hypergeometric" puts some people off, but if you invest some time learning about them, you'll discover that the hypergeometric functions are useful and not so difficult to understand. As an added bonus, if you know a bit about the hypergeometric functions, you can forget the rules for trigonometric substitution, and use rules such as

$$\int x^a(x-1)^b\,\mathrm{d}x={}_2\mathrm{F}_1\left[\begin{matrix}-a,\ b+1\\ b+2\end{matrix};1-x\right]\frac{(x-1)^{b+1}}{b+1}.$$

Back to our specific problem, making the substitution $z=x^2$, in terms of the Gauss hypergeometric function, we have

$$\int\frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x={}_2\mathrm{F}_1\left[\begin{matrix}2,\ 9/2\\ 11/2\end{matrix};1-x^2\right]\frac{(x^2-1)^{9/2}}{9}.$$

Using this form of the antiderivative, any worthy computer algebra system will give

$$\int_1^{10^9}\frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x={}_2\mathrm{F}_1\left[\begin{matrix}2,\ 9/2\\ 11/2\end{matrix};1-10^{18}\right]\frac{(10^{18}-1)^{9/2}}{9}=1.999999999999999988\cdots\times 10^{44}.$$