

# Well ordering

## Lesson 11

“The only way to learn mathematics is to do mathematics.”

Paul Halmos

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## Definition

Let  $A$  be a subset of  $\mathbb{R}$ . We say that  $A$  is *bounded above* provided

$$(\exists M \in \mathbb{R}) (\forall x \in A) (x \leq M).$$

The number  $M$  is an *upper bound* for the set  $A$ . We say that the set  $A$  is *bounded below* provided

$$(\exists M \in \mathbb{R}) (\forall x \in A) (M \leq x).$$

The number  $M$  is a *lower bound* for the set  $A$ . If  $A$  is bounded below and bounded above, we say  $A$  is *bounded*.

- ① If  $M$  is an upper bound for a set  $A$ , and  $M \leq M'$ , then  $M'$  is an upper bound for  $A$ .
- ② So we need to say a *(not the) upper bound*.
- ③ Similarly, lower bounds are not unique.

## Bounded details

- ① Notice that we do *not* require that an upper bound for a set  $A$  to be a member of  $A$ .
- ② Same for a lower bound.
- ③ Since we require that an upper bound be a *real number*, we disallow infinity from being an upper bound. If we did, every set would be bounded.
- ④ Although infinity is a number, it isn't a *real* number.

# Bounded and unbounded examples

## Example

- ① The empty set is bounded above by 0.
- ② Actually every real number is an upper bound for the empty set.
- ③ Every real number is a lower bound for the empty set.
- ④ The interval  $[0, 1]$  is bounded above by 1.
- ⑤ The interval  $[0, 1]$  is bounded above by 107.
- ⑥ The interval  $[0, \infty)$  is bounded below by 0.
- ⑦ The interval  $[0, \infty)$  is not bounded above.

# Being least

## Definition

Let  $A$  be a subset of  $R$ . The set  $A$  has a *least member* provided

$$(\exists a^* \in A) (\forall a \in A) (a^* \leq a).$$

We say that  $a^*$  is a least member. The set  $A$  has a *greatest member* provided

$$(\exists a^* \in A) (\forall a \in A) (a \leq a^*).$$

- ① Unlike a lower bound, we require that a least member of a set  $A$  be a member of the set.
- ② The same for a greatest member.

# Uniqueness of being least

## Theorem

If a subset of the reals has a least member, it is unique.

## Proof

Let  $A \subset \mathbb{R}$ . Suppose  $x$  and  $x'$  are least members of  $A$ . Since  $x$  is a least member of  $A$  we have  $x \in A$ . But  $x'$  is a least member, so  $x' \leq x$ . Interchanging the roles of  $x$  and  $x'$ , we have  $x \leq x'$ ; therefore  $x = x'$ .

- ① Equality is hard, inequality is easier.
- ② We proved equality by proving two inequalities.

# Well ordering principle

## Axiom

Let  $A$  be a nonempty subset of  $\mathbb{Z}$  that is *bounded below*. Then  $A$  has a least member.

- ① This is an axiom—we'll take it on faith.
- ② Again, a least member of a set  $A$  *must* be a member of  $A$ .
- ③ Thus the empty set does *not* have a least member.
- ④ The qualification that the set be nonempty for it to have a least member is crucial.

## Theorem

Let  $A \subset \mathbb{Z}$  be (i) *nonempty* and (ii) *bounded above*. Then  $A$  has a greatest member.

## Well ordering principle for the reals?

**Question** Are the real numbers well ordered? That is, does every nonempty subset of  $\mathbb{R}$  that is bounded below have a least member?

**Answer** No. The interval  $(0, 1)$  is nonempty and bounded below, but it doesn't have a least member. Although zero is less than every member of  $(0, 1)$ , since zero isn't a member of  $(0, 1)$ , it is not a least member.



## Existence of the floor

Let  $x \in \mathbb{R}_{\geq 0}$ . Define the set  $M$  by  $M = \{k \in \mathbb{R} \mid k \leq x\}$ .

- ① Since  $x \geq 0$ , it follows that  $0 \in M$ .
- ② So, the set  $M$  is nonempty.
- ③ Further the set  $M$  is bounded above by  $x$ .
- ④ The well ordering principle tells us that  $M$  has a least member.
- ⑤ Actually, the least member is unique.
- ⑥ Of course the least member depends on  $x$ .
- ⑦ Something that (i) depends on  $x$  and (ii) is unique defines a function!
- ⑧ We've used the well ordering principle to define the *floor function* for nonnegative inputs.