Advanced Calculus, Fall 2023 **Name:**____________________________

Practice Exam 3 **Row and Seat:**________________________

$\boxed{10}$ 1. Define the *derivative* as the limit of a *Newton quotient.*

$\boxed{10}$ 2. Show that the function $x \in (-\infty, 0) \cup 5 \mapsto x$ is not differentiable at 5.

$\boxed{10}$ 3. Use the definition you gave in question 1 to find the derivative of $x \in \mathbf{R} \mapsto x^2 + x$ at 2.

$\boxed{10}$ 4. Use the definition you gave in question 1 to find the derivative of $x \in \mathbf{R} \mapsto x^2 + x$ at a, where a is any real number.

$\boxed{10}$ 5. Use the definition you gave in question 1 to find the derivative of $x \in \mathbf{R} \mapsto \sqrt{x}$ at 3.

$\boxed{10}$ 6. Use the QRS definition of uniformly continuity to show that that $x \in [-1, 1] \mapsto x^2$ is *uniformly continuous* on its domain.

$\boxed{10}$ 7. Use the undefinition of uniformly continuity to show that the function $x \in \mathbf{R} \mapsto 8x^2$ is not uniformly continuous on its domain.

$\boxed{10}$ 8. Show that the function $x \in \mathbf{R} \mapsto \begin{cases} x\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is *continuous* at 0. You may use the fact that $|\cos(x)| \leq 1$ for all real x without proving it.

$\boxed{10}$ 9. Show that the function $x \in \mathbf{R} \mapsto \begin{cases} x^2\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at 0. Depending on your method, the result of the previous question might be useful.

$\boxed{10}$ 10. Let $F : \mathbf{R} \to \mathbf{R}$ be continuous at a. If $F(a) > 0$, show that there is a positive number δ such that $F(x) > 0$ for all $x \in \text{ball}(a, \delta) \cap \text{dom}(F)$.

$\boxed{10}$ 11. Show that the function $x \in \mathbf{R}_{>0} \mapsto \frac{1}{x}$ is not uniformly continuous on its domain.

$\boxed{10}$ 12. Let $F : \mathbf{R} \to \mathbf{R}$ be differentiable at a and suppose $F'(a) > 0$. Is it true that F is increasing on a neighborhood of a? If so, prove it.

$\boxed{10}$ 13. Give an example of a function $F : [-1, 1] \to \mathbf{R}$ such that $\sup\big(\text{range}(F)\big) \notin \text{range}(F)$.

$\boxed{10}$ 14. Give an example of a function $F : (-1,1) \to \mathbf{R}$ such that $\sup\big(\text{range}(F)\big) \not\in \text{range}(F)$ and F is continuous on $(-1,1)$.

15. Let $F : \mathbf{R} \to \mathbf{R}$ satisfy the inequality $|F(x) - F(y)| \le |x - y|$ for all $x, y \in \mathbf{R}$.

$\boxed{10}$ (a) Show that F is *continuous at zero.*

$\boxed{10}$ (b) Show that F is *uniformly continuous* on $\mathbf{R}$.

$\boxed{10}$ 16. Show that $x \in \mathbf{R} \mapsto x^3$ is continuous at 10.

$\boxed{10}$ 17. Show that the function with signature $F : \mathbf{R} \to \mathbf{R}$ and formula $F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

$\boxed{10}$ 18. Either prove or disprove: Let $F, G : \mathbf{R} \to \mathbf{R}$, and let $a \in \text{dom}(FG)$. If FG is continuous at a, then both F and G are continuous at a.

$\boxed{10}$ 19. Let $F : \mathbf{R} \to \mathbf{R}$ be continuous at a. Show that $|F|$ is continuous at a.

$\boxed{10}$ 20. Use the inequality $|\sqrt{a} - \sqrt{b}| \le \sqrt{|a - b|}$, for $a, b > 0$ to show that the square root function is uniformly continuous on $[0, \infty)$.

$\boxed{10}$ 21. Show that the function $x \in \mathbf{R} \mapsto x^2$ is not uniformly continuous on $\mathbf{R}$.

$\boxed{10}$ 22. Show that $x \in \mathbf{R} \mapsto x^2|x|$ is differentiable at 0. (The absolute value function isn't differentiable at 0, so the product rule *isn't* an option!)

$\boxed{10}$ 23. Use the MVT to show that for all $x, y \in \mathbf{R}$, we have $|\cos(x) - \cos(y)| \le |x - y|$. You may use the facts (i) $\cos' = \sin$ and (ii) $|\sin(x)| \le 1$ for all real x.