MATH 460 **Name:**

Homework 9, Fall 2023

Homework 9 has questions 1 through 5 with a total of 40 points. This work is due **Saturday 4 November** at 11:59 PM.

$\boxed{10}$ 1. Use the fact that the absolute value function is continous on **R** to show that the function $x \in \mathbf{R} \mapsto x|x|$ is differentiable at zero.

The product rule fails us for the derivative of $x \in \mathbf{R} \mapsto x|x|$ at zero. That's because the absolute value function is not differentiable at zero. Away from zero, the product rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(x|x|) = \frac{\mathrm{d}}{\mathrm{d}x}(x)\,|x| + x\frac{\mathrm{d}}{\mathrm{d}x}(|x|) = |x| + x\begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} = |x| + |x| = 2|x|.$$

It's a happy coincidence that $\frac{\mathrm{d}}{\mathrm{d}x}(x|x|) = 2|x|$ for all real x. But the product rule is not sufficient for this calcuation.

$\boxed{10}$ 2. Show that the function $F = x \in \mathbf{R} \mapsto \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ has a limit toward zero.

To do this, use the fact that for all $x \in \mathbf{R}_{\neq 0}$, we have $|x\sin(\frac{1}{x})| \leq |x|$. From this work, show that F is continuous at zero.

$\boxed{10}$ 3. Show that the function $G = x \in \mathbf{R} \mapsto \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at zero.

4.

$\boxed{10}$ 5. Let f and g be smooth functions. That means that these functions have as many derivatives as we like. Then repeated use of the product rule gives the identities

$$\begin{aligned}
(fg)^{(1)} &= fg^{(1)} + f^{(1)}g, \\
(fg)^{(2)} &= fg^{(2)} + 2f^{(1)}g^{(1)} + f^{(2)}g, \\
(fg)^{(3)} &= fg^{(3)} + 3f^{(1)}g^{(2)} + 3f^{(2)}g^{(1)} + f^{(3)}g.
\end{aligned}$$

It's not too much of a stretch to guess that for any positive integer n that

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

The pattern of the first three derivatives is pretty compelling, but we need a proof. Your task is it proof it. To do this, you'll need a few facts about the binomial coefficients. These facts are

- For all $n \in \mathbf{Z}_{\geq 0}$, we have $\binom{n}{-1} = 0$.
- For all $n \in \mathbf{Z}_{\geq 0}$, we have $\binom{n+1}{n} = 0$.
- For all integers n and k, we have $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Finally, you'll likely need to use the underappreciated facts that $\sum_{k=0}^{n} a_{k+1} = \sum_{k=1}^{n+1} a_k$ and that $\sum_{k=0}^{n} a_{k-1} = \sum_{k=-1}^{n-1} a_k$

6. Let $F \in \mathbf{R} \to \mathbf{R}_{\geq 0}$ have a limit toward 5. Use the QRS definition of a limit to show that $\sqrt{F}$ has a limit toward 5.