

Sets

Sets

Quasi-definition

We don't attempt to define a set, but we describe a set as a collection of things, often called *members*. The members of a set can be numbers, ordered pairs, functions, or sets themselves. In a bit, we'll learn that there are some things that might appear to be valid sets, but really are not sets.

Here are some examples of sets:

- Ⓐ $\{46, 107\}$ is a set with two members, namely 46 and 107. We reserve the curly braces to delineate a set.
- Ⓑ $\{46, \{46, 107\}\}$ is a set with two members, namely 46 and $\{46, 107\}$. One member of this set is an integer, but the other is a set with two members—that's OK.
- Ⓒ $\{0, 1, 2, 3, \dots\}$ is apparently the set of all nonnegative integers. I say apparently because the ellipses (the \dots) isn't entirely clear.
- Ⓓ $\{\}$ is a set with no members.

Named Sets

We'll use the following names for subsets of real numbers:

\mathbf{R} = the set of real numbers,

\mathbb{R} = the set of real numbers for handwritten text,

$\mathbf{R}_{>0} = \{x \in \mathbf{R} \mid x > 0\}$,

$\mathbf{R}_{\neq 0} = \{x \in \mathbf{R} \mid x \neq 0\}$, (and similarly for other subscripts)

\mathbf{Z} = the set of integers,

\mathbb{Z} = the set of integers for handwritten text,

\mathbf{Q} = the set of rational numbers,

\mathbb{Q} = the set of rational numbers for handwritten text,

\emptyset = A set with no members, that is the empty set

Membership

For a set A , we define a predicate (boolean valued function) as

$$x \in A = \begin{cases} \text{T} & \text{if } x \text{ is a member of } A \\ \text{F} & \text{if } x \text{ is not member of } A \end{cases}$$

For example:

- Ⓐ $107 \in \{46, 107\} = \text{T}$
- Ⓑ $\{107\} \in \{46, \{107\}\} = \text{T}$
- Ⓒ $\{107\} \in \{46, 107\} = \text{F}$

Set Operators

Definition

Let A and B be sets. Define the set *union*, *intersection*, and *difference*

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\},$$

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\},$$

$$A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\},$$

respectively.

Set (an) example

Example

We have

$$\{6, 107\} \cap \{28, 107\} = \{107\},$$

$$\{6, 107\} \cup \{28, 107\} = \{6, 28, 107\},$$

$$\{6, 107\} \setminus \{28, 107\} = \{6\},$$

$$\{28, 107\} \setminus \{6, 107\} = \{28\}.$$

- ✓ The last two examples show that in general $A \setminus B \neq B \setminus A$.
- ✓ The set difference is so much like real number subtraction, sometimes the symbol "-" is used instead of \setminus .

Set predicates

Definition

Let A and B be sets. Define

$$A \subset B \equiv (\forall x \in A)(x \in B),$$

$$A = B \equiv (A \subset B) \wedge (B \subset A).$$

Specializing $A \subset B$ to $A = \emptyset$ gives

$$[\emptyset \subset B] \equiv (\forall x \in \emptyset)(x \in B) \equiv \text{true}.$$

We've shown that:

Proposition

Thus for all sets A and for any empty set \emptyset , we have $\emptyset \subset A$.

Set equality

To show that sets A and B are equal, we almost always prove that $A \subset B$ and $B \subset A$. If a proposition has the form

Proposition

If H_1, H_2, \dots , and H_n , then $A = B$.

where H_1, H_2, \dots, H_n is the hypothesis, a template for proving the theorem is

Proof

Suppose $x \in A$. We'll show that $x \in B$. Since $x \in A, H_1, H_2, \dots$ and H_n , we have ...; therefore $x \in B$.

Suppose $x \in B$. We'll show that $x \in A$. Since $x \in B, H_1, H_2, \dots$ and H_n , we have ...; therefore $x \in A$.

- 1 Notice how in the first case we append $x \in A$ to the hypothesis; and in the second case, we append $x \in B$.

Establish notation

Proposition

The set union is associative.

Proof

Let A , B , and C be sets. We'll show that $A \cup (B \cup C) = (A \cup B) \cup C$. Our proof uses the fact that the disjunction is associative; we have

$$\begin{aligned} A \cup (B \cup C) &= \{x \mid (x \in A) \vee (x \in B \cup C)\}, \\ &= \{x \mid (x \in A) \vee (x \in B) \vee (x \in C)\}, \\ &= \{x \mid ((x \in A) \vee (x \in B)) \vee x \in C\}, \\ &= (A \cup B) \cup C. \end{aligned}$$

- ✓ The statement of the proposition doesn't introduce notation, so the proof must do so.
- ✓ Alternatively, we can show that $A \cup (B \cup C) \subset (A \cup B) \cup C$ and $(A \cup B) \cup C \subset A \cup (B \cup C)$.

Alternative proofs

Proof

Let A , B , and C be sets. We'll show that $A \cup (B \cup C) = (A \cup B) \cup C$. We have

$$\begin{aligned}x \in A \cup (B \cup C) &\implies (x \in A) \vee (x \in B \cup C), \\&\implies (x \in A) \vee ((x \in B) \vee (x \in C)), \\&\implies ((x \in A) \vee (x \in B)) \vee (x \in C), \\&\implies x \in (A \cup B) \cup C.\end{aligned}$$

Similarly, we can show that $x \in (A \cup B) \cup C \implies x \in A \cup (B \cup C)$.

The uniqueness of emptiness

Proposition

There is at most one empty set.

Proof

Let O and O' be empty sets. Since O is empty, we have $O \subset O'$. Similarly since O' is empty, we have $O' \subset O$. We have shown that $O \subset O'$ and $O' \subset O$; therefore $O = O'$.

- ① With impunity, we can now refer to **the** empty set.
- ② A clumsy way to prove this is by contradiction. The proof assumes that there are empty sets O and O' , but $O \neq O'$.

Conflation

Question: True or false: $\emptyset = \{\emptyset\}$.

Answer: It's false. We have $\emptyset \in \{\emptyset\}$, but $\emptyset \notin \emptyset$, so $\emptyset \neq \{\emptyset\}$.

A unique template

If a proposition has the form

Proposition

If H_1, H_2, \dots , and H_n , there is at most one object X .

A template for its proof is

Proof

Let X and X' be such objects. Since H_1, H_2, \dots , and H_n , we have \dots ; therefore $X = X'$.

- 1 When X and X' are real numbers, we might prove $X = X'$ by showing that both $X \leq X'$ and $X' \leq X$ are true. Together, these inequalities prove that $X = X'$.

Generalized unions

Let I be a set. And suppose that every member of I is a set. Since every member of I is a set, we can find the union of all of its members. We define

$$\bigcup_{x \in I} x = \{a \mid (\exists x \in I)(a \in x)\}.$$

In this context, we say that the set I is an *index set*.

Example

Define $I = \{\{1, 2\}, \{107\}\}$. Then I is a set and each member of I is a set. We have

$$\begin{aligned}\bigcup_{x \in I} x &= \{a \mid (\exists x \in I)(a \in x)\}, \\ &= \{a \mid a \in \{1, 2\} \vee a \in \{107\}\}, \\ &= \{1, 2\} \cup \{107\}, \\ &= \{1, 2, 107\}.\end{aligned}$$

Finite unions

Proposition

Let A_1, A_2, \dots, A_n be sets. Define an index set I by $I = \{A_1, A_2, \dots, A_n\}$. Then

$$\bigcup_{x \in I} x = A_1 \cup A_2 \cup \dots \cup A_n.$$

- 1 The set union is associative and commutative, so the meaning of $A_1 \cup A_2 \cup \dots \cup A_n$ unambiguous.
- 2 An index set needn't be finite.

Nonfinite unions

Example

The index set needn't be finite—here is an example. For $x \in \mathbf{R}$, define $I = \{(-\infty, x) \mid x \in \mathbf{R}\}$. Our index set is a set of open intervals. We claim that

$$\bigcup_{x \in I} x = \mathbf{R}.$$

Proof

Suppose $a \in \bigcup_{x \in I} x$. We'll show that $a \in \mathbf{R}$. Since $a \in \bigcup_{x \in I} x$, there is $z \in I$ such that $a \in z$. But $z \subset \mathbf{R}$, so $a \in \mathbf{R}$; we've shown that $\bigcup_{x \in I} x \subset \mathbf{R}$.

Suppose $a \in \mathbf{R}$. We'll show that $a \in \bigcup_{x \in I} x$. We have $a \in (-\infty, a + 1)$. Further $(-\infty, a + 1) \in I$; therefore $a \in \bigcup_{x \in I} x$.

- 1 Notice that $a \notin (-\infty, a)$. But it is true that $a \in (-\infty, a + 1)$.
- 2 It's also true that $a \in (-\infty, a + 107\pi^2)$.

Generalized intersections

Let I be a set. And suppose that every member of I is a set. Since every member of I is a set, we can find the intersection of all of its members. We define

$$\bigcap_{x \in I} x = \{a \mid (\forall x \in I)(a \in x)\}.$$

Example

Define $I = \{\{1, 2\}, \{107\}\}$. Then I is a set and each member of I is a set. We have

$$\begin{aligned}\bigcap_{x \in I} x &= \{a \mid (\forall x \in I)(a \in x)\}, \\ &= \{a \mid a \in \{1, 2\} \wedge a \in \{107\}\}, \\ &= \{1, 2\} \cap \{107\}, \\ &= \emptyset.\end{aligned}$$

Finite intersections

Proposition

Let A_1, A_2, \dots, A_n be sets. Define an index set I by $I = \{A_1, A_2, \dots, A_n\}$. Then

$$\bigcap_{x \in I} x = A_1 \cap A_2 \cap \dots \cap A_n.$$

- 1 The set intersection is associative and commutative, so the meaning of $A_1 \cap A_2 \cap \dots \cap A_n$ unambiguous.
- 2 An index set needn't be finite.

Nonfinite intersections

Example

For $x \in \mathbf{R}$, define $I = \{(-\infty, x) \mid x \in \mathbf{R}\}$. Our index set is a set of open intervals. We claim that

$$\bigcap_{x \in I} x = \emptyset.$$

Proof

We'll prove this using contradiction. Suppose $\bigcap_{x \in I} x$ has at least one member; say $a \in \bigcap_{x \in I} x$. We have

$$(\forall x \in \mathbf{R})(a \in (-\infty, x)).$$

In particular, we have $a \in (-\infty, a)$. But $a \in (-\infty, a)$ is false; therefore $\bigcap_{x \in I} x$ cannot have a member, so $\bigcap_{x \in I} x$ is the empty set.

Alternative notation

Sometimes we take the index set to be a subset of \mathbf{R} and we denote the sets members by subscripts. Say $I \subset \mathbf{R}$ and A_x is a set for each $x \in I$. This notation is particularly popular when $I = \mathbf{Z}_{>0}$. For example

$$\bigcap_{k \in \mathbf{Z}_{>0}} A_k = \{a \mid (\forall n \in \mathbf{Z}_{>0})(a \in A_n)\}$$

And

$$\bigcup_{k \in \mathbf{Z}_{>0}} A_k = \{a \mid (\exists n \in \mathbf{Z}_{>0})(a \in A_n)\}$$

When the index set is uncountable, maybe it's just me, but definitions such as

$$A_x = (-\infty, x) \text{ for all } x \in \mathbf{R}$$

are semi-bazaar looking. For such cases, I think it's more clear to define the index set to be a set of sets:

$$I = \{(-\infty, x) \mid x \in \mathbf{R}\}.$$

Functions

To define a function F with domain A and formula blob, we can write

$$F = x \in A \mapsto \text{blob}.$$

In the rare cases that it's important to give the function a codomain, we can write

$$F = x \in A \mapsto \text{blob} \in B,$$

where $\text{codomain}(F) = B$. Generically for a function F with domain A and codomain B , we say that F is a function from A to B .

Example

The notation

$$F = x \in [-1, 1] \mapsto 2x + 1$$

is our compact way of writing: Define $F(x) = 2x + 1$, for $-1 \leq x \leq 1$.

Function signature

The notation $F : A \rightarrow B$ means

- ① F is a function.
- ② $\text{dom}(F) = A$.
- ③ $\text{codomain}(F) = B$.

We'll say that $A \rightarrow B$ is the *signature* of a function. The signature of a function doesn't tell us its formula. It does tell us the domain of a function and it indicates what the outputs of the function can be.

Range

Definition

For any function, we define

$$\text{range}(F) = \{F(x) | x \in \text{dom}(F)\}.$$

Thus $\text{range}(F)$ is the set of all outputs.

Fact

Let F be a function. Then

$$[y \in \text{range}(F)] \equiv (\exists x \in \text{dom}(F)) (y = F(x)).$$

Example

Define $F = x \in [-1, 1] \mapsto 2x + 1$. Then $\frac{3}{2} \in \text{range}(F)$ because $\frac{1}{4} \in \text{dom}(F)$ and $F(\frac{1}{4}) = \frac{3}{2}$.

Onto-ness

The codomain of a function tells us something about its outputs, but remember that the range and the codomain of a function need not be the same. For all functions F , we have

$$\text{range } F \subset \text{codomain}(F).$$

Definition

A function is *onto* if its range and codomain are equal.

Example

Question: Is the sine function onto? **Answer** It is if its codomain is $[-1, 1]$. But if its codomain is \mathbf{R} , then no it's not onto. There is no standard value for the codomain of the trigonometric functions, so the asking "Is the sine function onto?" is rubbish.

Equality

Definition

Functions F and G are *equal* $\text{dom}(F) = \text{dom}(G)$ and for all $x \in \text{dom}(F)$, we have $F(x) = G(x)$. Equivalently

$$(F = G) \equiv (\text{dom}(F) = \text{dom}(G)) \wedge (\forall x \in \text{dom}(F))(F(x) = G(x)).$$

- 1 The definition of function equality does not involve the codomain of the function. Thus two functions can be equal, but have unequal codomains.

Example

The functions $F = x \in [-1, 1] \mapsto x \in [-1, 1]$ and $G = x \in [-1, 1] \mapsto x \in \mathbf{R}$ are equal, but F is onto and G is not onto. Thus onto-ness isn't a property of a function.

Apply a function to a set

Definition

Let $F : A \rightarrow B$. For any subset A' of A define

$$F(A') = \{F(x) | x \in A'\}.$$

Equivalently, we have

$$y \in F(A') \equiv (\exists x \in A')(y = F(x)).$$

Proposition

For all functions F , we have $F(\text{dom } F) = \text{range}(F)$. Further $F(\emptyset) = \emptyset$.

Inverse image

Definition

Let $F : A \rightarrow B$. For any subset B' of B define

$$F^{-1}(B') = \{x \in A \mid F(x) \in B'\}.$$

Equivalently, we have

$$x \in F^{-1}(B') \equiv F(x) \in B'.$$