Let/Show/Suppose Proof Examples

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1 Introduction

Many propositions in analysis consist of concatenated 'for all' and 'there exist' qualifiers followed by something that has a boolean value. The part with the boolean value is called the *predicate*. We'll see that expressing the proposition in symbolic form gives us a road map for constructing a proof. To follow the road map, we to adhere to some rules, and it's best to follow some traditions. Following the rules is needed for correctness, and using the traditions makes our work easier to understand.

For every qualification of the form $\forall x \in A$, where A is a set, our proof includes the sentence 'Let $x \in A$.' Synonyms for 'let' are 'permit' and 'allow,' but in mathematics the tradition is to use 'let.' In the context of mathematics, 'Let $x \in \mathbf{R}$' means that we can only assume that x is real number, but we cannot assume anything more about x. It would be wrong, for example, to write in a proof 'Let $x \in \mathbf{R}$. Assume $x = 0$.'

For every qualification of the form $\exists x \in A$, it's just like fourth grade show and tell. We must choose a *specific* value for the variable x. Choices are, of course, sometimes difficult, so we need to be careful in making them. When we make a bad choice, we might not be able to complete our proof. And when we make a bad choice, we need to backtrack and re-do our work from where we made the choice.

For proofs in analysis, often the choice isn't unique, but when there is more than one possible choice, we still give the reader exactly one choice that allows the proof to proceed, we don't clog the logic by giving more than one. Additionally, when multiple choices, we use the most natural choice we can think of. For example if we need to chose a number that is between given numbers, a natural choice is to use the arithmetic average. Don't complicate your work by making a correct, but obscure choice.

The final thing we need to know about our symbolic road map is that when we make a choice for a variable, our choice can depend on all previously introduced variables (that is, variables to the left), but it cannot depend on variables that are yet to be introduced (variables to the right). For example, for the statement

$$(\forall x \in \mathbf{R})\,(\exists M \in \mathbf{R})\,\big(\forall y \in \mathbf{R}\big)$$

what we choose for M can depend on x, but it *cannot* depend on y. To explicitly show this dependence, some writers use functional notion, for example 'there is $M(x) \in \mathbf{R}$.' If this functional notation helps you, use it. But arguably $M(x) \in \mathbf{R}$ conflates a real number (in this case M) with a function.

Finally, what about 'suppose?' When a proof involves 'suppose,' generally it means that a hypothesis follows. For example, if we are proving a conditional 'if $x > 0$, then $x > -1$,' our proof will start with 'Suppose $x > 0$.' This language alerts the reader that what follows 'suppose' is a hypothesis. When a proof involves special cases, a proof might also use 'suppose' to introduce each case.

Now we give some fairly simple examples of what we will call a 'let/show/suppose' proof.

2 Examples

Proposition 1. *For all $x, y \in \mathbf{R}$, there is $a \in \mathbf{R}$ such that $x < y$ implies $x < a < y$.*

In symbolic form, the Proposition 1 is $(\forall x, y \in \mathbf{R})\,(\exists a \in \mathbf{R})\,(x < a < y)$. Since a follows both x and y, what we choose for a is allowed to depend on both x and y. The most natural choice for a is almost surely the arithmetic average. With this, our proof is:

Proof. (BW) Let $x, y \in \mathbf{R}$. Suppose $x < y$. Choose $a = \frac{x+y}{2}$. Then $a \in \mathbf{R}$ as required. We have

$$\begin{aligned}
[x < a < y] &\equiv \left[x < \frac{x+y}{2} < y\right], && \text{(substitution for } a\text{)}\\
&\equiv \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(subtract } {}^{(x+y)}/_{2}\text{)}\\
&\equiv \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \text{(simplification)}\\
&\equiv \text{True}. && \text{(hypothesis } x < y\text{)}
\end{aligned}$$

□

Each step in this series has a parenthetical remark clarifying the reason for the equivalence. Such remarks can make a proof much easier to follow.

If the fact that the arithmetic average is known to be "in the middle," our proof could end at the second line. It's sometimes a judgment call on whether a fact can be used in a proof. When in doubt, include, don't assume. It is, of course, always wrong to use a result in a proof that is way outside the context of our class.

Notice that starting with the conclusion of Proposition 1, we expressed our proof as a series of logical *equivalences*. Logical equivalence has the transitive property, so indeed our proof shows that $x < a < y$ is true. When we start with the conclusion as we did in this proof, it's vitally important that we use a series of logical equivalences, *not* logical implications. If we expressed this work as

$$\begin{aligned}
[x < a < y] &\implies \left[x < \frac{x+y}{2} < y\right], && \text{(substitution)}\\
&\implies \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(subtract } {}^{(x+y)}/_{2}\text{)}\\
&\implies \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \text{(simplification)}\\
&\implies \text{True}. && \text{(hypothesis } x < y\text{)}
\end{aligned}$$

we have shown that $(x < a < y) \implies \text{True}$. But $P \implies \text{True}$ is true *regardless* of the truth value of P (that is $P \implies \text{True}$ is a tautology). So the above work proves *nothing*. By reversing the order of the statements, we could fix this effort; for example,

$$\begin{aligned}
x < y &\implies \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \big((y-x>0) \wedge (x-y<0)\big)\\
&\implies \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(algebraic equivalence)}\\
&\implies \left[x < \frac{x+y}{2} < y\right], && \text{(add } {}^{(x+y)}/_{2}\text{)}\\
&\implies [x < a < y]. && \text{(substitution for } a\text{).}
\end{aligned}$$

Now we have proved that $x < y \implies x < a < y$. And since $x < y$ is a hypothesis, we've indeed shown that $x < a < y$ is true, as we intended to do. Compared to our proof that used a series of logical equivalences, the first logical implication in this proof (that is $x < y \implies \frac{x-y}{2} < 0 < \frac{y-x}{2}$) seems a bit like a rabbit pulled out of a magician's hat, making the proof somewhat mysterious. When it's possible to write a proof as a series of logical equivalences that starts with the conclusion and ends with true, it's often the clearest way to write the proof.

Proposition 2. *For all* $r \in \mathbf{R}_{>0}$ *there is* $x \in [0,1)$ *such that* $1 - r < x$.

In symbolic form, Proposition 2 is $(\forall r \in \mathbf{R}_{>0})\,(\exists x \in [0,1))\,(1 - r < x)$. And since $x \in [0,1)$, additionally we know that $x < 1$. So the predicate $1 - r < x$ is equivalent to $1 - r < x < 1$. Now Proposition 2 looks a great deal like Proposition 1. Maybe we can simply choose x to be the arithmetic average of $1 - r$ and 1. That is choose $x = 1 - \frac{r}{2}$. But the requirement that $0 < x$ spoils this choice. If $r = 4$, for example, we are choosing $x = -1$. And that's not allowed.

One way to fix this is to choose x to be the arithmetic average of $1 - r$ and 1 when $r < 1$ and choose $x = \frac{1}{2}$ when $r \geq 1$. Actually, when $r \geq 1$, we could choose x to be any member of $[0,1)$, but choosing the midpoint is arguably the most natural. Our proof breaks into two cases:

Proof. (BW) Let $r \in \mathbf{R}_{>0}$. Choose $x = \begin{cases} 1 - \frac{r}{2} & r < 1 \\ \frac{1}{2} & r \geq 1 \end{cases}$. For $r < 1$, we have

$$\begin{aligned}
[1 - r < x] &\equiv \left[1 - r < 1 - \frac{r}{2}\right], && \text{(substitution for } x\text{)} \\
&\equiv \left[0 < \frac{r}{2}\right], && \text{(add } r - 1\text{)} \\
&\equiv \text{True}. && (0 < r < 1)
\end{aligned}$$

And for $r \geq 1$, we have

$$\begin{aligned}
[1 - r < x] &\equiv \left[1 - r < \frac{1}{2}\right], && \text{(substitution for } x\text{)} \\
&\equiv \left[\frac{1}{2} < r\right], && \text{(add } r - 1\text{)} \\
&\equiv \text{True}. && (r \geq 1)
\end{aligned}$$

□

Another way to discover this proof is to

Proposition 3. *For all* $x \in \mathbf{R}_{>0}$ *there is* $y \in \mathbf{R}_{>0}$ *such that* $y < x$.

Generically, Proposition 3 looks like another choose the arithmetic average proof: we need to choose a number between zero and y. Our proof is

Proof. (BW) Let $x \in \mathbf{R}_{>0}$. Choose $y = x/2$. Then $y \in \mathbf{R}_{>0}$ as required. We have

$$\begin{aligned}
[y < x] &\equiv \left[\frac{x}{2} < x\right], && \text{(substitution for } y\text{)} \\
&\equiv \left[0 < \frac{x}{2}\right], && \text{(subtract } x/2\text{)} \\
&\equiv \text{True}. && (\mathrm{x} > 0)
\end{aligned}$$

□

Abstracted, Proposition 3 says that there is no smallest positive number. That is, given any positive number, there is a positive number that is smaller. This is an important property of the field of real numbers.

Proposition 4. *For every* $y \in \mathbf{R}_{>0}$ *there is* $x \in \mathbf{R}_{>0}$ *such that* $y \geq x$.

Paraphrased, Proposition 4 says that given any positive number, there is a positive number that is greater. Alternatively, we might say that there is no largest positive number. Adding one is a good strategy for constructing a larger number, but this proposition allows equality, not strict inequality. When possible, I like to choose equality. Our proof

Proof. (BW) Let $y \in \mathbf{R}_{>0}$. Choose $x = y$. Then $x \in \mathbf{R}_{>0}$, as required. We have

$$\begin{aligned} [y \geq x] &\equiv [y \geq y], && \text{(substitution for } x\text{)} \\ &\equiv [0 \geq 0], && \text{(subtract } y\text{)} \\ &\equiv \text{True}. \end{aligned}$$

□

Proposition 5. *For all $x \in \mathbf{R}_{>0}$, there is $M \in \mathbf{R}$ such that $\frac{1}{x} + 1 > M$.* *(SB)*

Proposition 6. *There is $M \in \mathbf{R}$ such that for all $x \in \mathbf{R}_{>0}$, we have $\frac{1}{x} + 1 > M$.* *(DD)*

Proposition 7. *There is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $1 + m(x-1) \leq x^2$.* *(TK)*

Proposition 8. *For every $a \in \mathbf{R}$, there is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $a^2 + m(x-a) \leq x^2$.* *(AK)*

Proposition 9. *For all $x, y \in \mathbf{R}$, we have $(x^2 = y^2) \implies (x = y)$.* *(DM)*

Proposition 10. *For all $x, y \in \mathbf{R}$, we have $(x^3 = y^3) \implies (x = y)$.* *(CR)*

Proposition 11. *For all $r \in \mathbf{R}_{>0}$ there is $x \in \mathbf{R}$ such that $1 < x < 1 + r$.* *(AA)*