

**Advanced Calculus****Name:** \_\_\_\_\_**Exam II Review, October 10, 2023****Row and Seat:** \_\_\_\_\_

1. Show that the sequence  $F = k \in \mathbf{Z}_{\geq 1} \mapsto 8 - \frac{1}{k}$  is bounded above.

**Solution:**

*Proof.* To show that  $F$  is bounded, we need to show that

$$(\exists M \in \mathbf{R}) (\forall k \in \mathbf{Z}_{\geq 1}) (F_k < M).$$

Choose  $M = 8$ . Let  $k \in \mathbf{Z}_{\geq 1}$ . We have

$$\begin{aligned} \left[ 8 - \frac{1}{k} < 8 \right] &\equiv \left[ 0 < \frac{1}{k} \right], && \text{(add } 1/k - 8) \\ &\equiv [0 < 1], && \text{(multiply by } k) \\ &\equiv \text{True.} && \square \end{aligned}$$

2. Show that the sequence  $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{(-1)^k}{k^2}$  converges.

**Solution:** We'll show that

$$(\exists L \in \mathbf{R}) (\forall \varepsilon \in \mathbf{R}) (\exists N \in \mathbf{Z}_{>0}) (\forall k \in \mathbf{Z}_{>N}) (|F_k - L| < \varepsilon).$$

Choose  $L = 0$ . Let  $\varepsilon \in \mathbf{R}$ . Choose  $N = \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil$ . Then  $N \in \mathbf{Z}_{>0}$  as required. Let  $k \in \mathbf{Z}_{>N}$ . We have

$$\begin{aligned} |F_k - L| &= \left| \frac{(-1)^k}{k^2} \right|, \\ &= \frac{1}{k^2}, \\ &< \frac{1}{N^2}, \\ &= \frac{1}{\left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil^2}, \\ &\leq \frac{1}{\sqrt{\frac{1}{\varepsilon}}^2}, \\ &= \varepsilon. \end{aligned}$$

3. Show that the sequence  $F = k \in \mathbf{Z}_{\geq 1} \mapsto \begin{cases} k! & k < 10^9 \\ \frac{(-1)^k}{k^2} & k \geq 10^9 \end{cases}$  converges.

**Solution:** This is just like the previous problem, but we need to choose  $N = \max(10^9, \lceil \sqrt{\frac{1}{\varepsilon}} \rceil)$ .

4. Show that the sequence  $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{3k+1}{2k+8}$  converges.

**Solution:**

*Proof.* We'll show that  $F$  converges to  $\frac{3}{2}$ . Specifically, we'll show that

$$(\forall \varepsilon \in \mathbf{R}_{>0}) (\exists N \in \mathbf{Z}_{>0}) (\forall k \in \mathbf{Z}_{>N}) \left( \left| \frac{3k+1}{2k+8} - \frac{3}{2} \right| < \varepsilon \right).$$

Let  $\varepsilon \in \mathbf{R}_{>0}$ . Choose  $N = \lceil \frac{11}{2\varepsilon} \rceil$ . Then as required,  $N \in \mathbf{Z}_{>0}$ . Let  $k \in \mathbf{Z}_{>N}$ . We have

$$\begin{aligned} \left| \frac{3k+1}{2k+8} - \frac{3}{2} \right| &= \frac{11}{2(k+4)}, & (\text{ algebra}) \\ &< \frac{11}{2N}, & (k+4 > N) \\ &= \frac{11}{2\lceil \frac{11}{2\varepsilon} \rceil}, & (\text{ substitution for } N) \\ &\leq \frac{11}{2\frac{11}{2\varepsilon}}, & (\text{ ceiling function property}) \\ &= \varepsilon. & (\text{ algebra}) \end{aligned} \quad \square$$

5. Show that the sequence  $F = k \in \mathbf{Z}_{\geq 1} \mapsto k - 3\lfloor \frac{k}{3} \rfloor$  does not converge to 1.

**Solution:**

*Proof.* We'll show that

$$(\exists \varepsilon \in \mathbf{R}_{>0}) (\forall N \in \mathbf{Z}_{>0}) (\exists k \in \mathbf{Z}_{>N}) (|F_k - 1| \geq \varepsilon).$$

Choose  $\varepsilon = \frac{1}{2}$ . Let  $N \in \mathbf{Z}_{>0}$ . Choose  $k = 3N$ . Then  $k \in \mathbf{Z}_{>N}$  as required. We have  $|F_k - 1| = |F_{3N} - 1| = 1 > \varepsilon$   $\square$

6. Using the definition from the QRS, show that the interval  $(-\infty, 8)$  is open.

**Solution:**

*Proof.* We'll show that

$$(\forall x \in (-\infty, 8)) (\exists r \in \mathbf{R}_{>0}) (\text{ball}(x, r) \subset (-\infty, 8))$$

Let  $x \in (-\infty, 8)$ . Choose  $r = 4 - \frac{x}{2}$ . Since  $x < 8$ , we have  $r > 0$ , as required. We have

$$x + r = x + 4 - \frac{x}{2} = \frac{x}{2} + 4 < \frac{8}{2} + 4 = 8.$$

So  $\text{ball}(x, r) \subset (-\infty, 8)$ . □

7. Let  $A \subset \mathbf{R}$ . Using the definition of an open set in the QRS, write the undefinition of an open set. That is, complete the statement:

$A$  is not open  $\equiv$

**Solution:**

$$A \text{ is not open} \equiv (\exists a \in A) (\forall r \in \mathbf{R}_{>0}) (\text{ball}(a, r) \not\subset A).$$

8. Using the undefinition from the previous question, show that the set  $(-\infty, 8) \cup \{9\}$  is not open.

**Solution:**

*Proof.* Choose  $a = 9$ . Let  $r \in \mathbf{R}_{>0}$ . Then  $\text{ball}(9, r) \not\subset (-\infty, 8) \cup \{9\}$ . □

9. Let  $A \subset \mathbf{R}$ . Using the definition of a limit point in the QRS, write the undefinition of limit point. That is, complete the statement:

$x \notin \text{lp}(A) \equiv$

**Solution:**

$$x \notin \text{lp}(A) \equiv (\exists r \in \mathbf{R}_{>0}) (\text{ball}'(x, r) \cap A = \emptyset).$$

10. Use your undefinition from the previous question to show that  $5 \notin \text{lp}(\mathbf{Z})$ .

**Solution:** We'll show that

$$(\exists r \in \mathbf{R}_{>0}) (\text{ball}'(5, r) \cap \mathbf{Z} = \emptyset).$$

Choose  $r = \frac{1}{2}$ . We have  $\text{ball}'(5, r) = \text{ball}'(5, 1/2) = (5 - \frac{1}{2}, 5) \cup (5, 5 + \frac{1}{2})$ . So  $\text{ball}'(5, 1/2) \cap \mathbf{Z} = \emptyset$ .

11. Use the QRS definition of a *boundary point* to show that  $12 \in \text{bp}((0, 12))$ .

**Solution:** Let  $r$  be a positive number, and let  $x^* = \begin{cases} 12 - r/2 & r < 24 \\ 6 & r \geq 24 \end{cases}$ . Then  $x^* \in (12 - r, 12 + r)$  and  $x^* \in (0, 12)$ . Further we have  $12 + r/2 \notin (0, 12)$  and  $12 + r/2 \in (12 - r, 12 + r)$ .

12. Use the result of the previous question to show that  $(0, 12)$  is not closed.

**Solution:** A closed set contains all of its boundary points. We showed that 12 is a boundary point  $(0, 12)$ , but  $12 \notin (0, 12)$ ; therefore  $(0, 12)$  is not closed.

13. Show that the set  $\mathbf{R}$  is not compact by showing that there is an open cover of  $\mathbf{R}$  that has no finite subcover.

**Solution:** See classnotes for Monday 9 October.

14. Show that the set  $\mathbf{Z}$  is not compact by showing that there is an open cover of  $\mathbf{Z}$  that has no finite subcover.

**Solution:** Define  $\mathcal{C} = \{(-k, k) \subset \mathbf{R} \mid k \in \mathbf{Z}_{>0}\}$ . Since  $\bigcup_{x \in \mathcal{C}} x = (0, \infty)$ , the set  $\mathcal{C}$  is a cover for  $\mathbf{N}$ . Let  $\mathcal{C}'$  be any finite subset of  $\mathcal{C}$ . Then  $\bigcup_{x \in \mathcal{C}'} x$  is bounded because it's a finite union of bounded sets. But  $\mathbf{Z}$  isn't bounded, so  $\mathbf{Z} \not\subset \bigcup_{x \in \mathcal{C}'} x$ ; therefore  $\mathbf{Z}$  isn't compact.

15. Let  $F$  be a convergent sequence, and let  $\alpha \in \mathbf{R}$ . Show that  $\alpha F$  is a convergent sequence.

**Solution:** Let  $\varepsilon > 0$ . Since  $f$  converges, there is a number  $L$  and  $M \in \mathbf{N}$  such that for all  $k > M$ , we have  $|f_k - L| < \frac{\varepsilon}{1 + |\alpha|}$ . For all  $k > M$ , we have

$$|\alpha f_k - \alpha L| = |\alpha| |f_k - L| < \frac{|\alpha|}{1 + |\alpha|} \varepsilon < \varepsilon.$$

16. Let  $F$  be a convergent sequence and suppose  $\text{range}(F) \subset ([0, \infty))$ . Show that  $\sqrt{F}$  converges. You may use the fact that  $(\forall x, y \in \mathbf{R}_{\geq 0}) (|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|})$

**Solution:** See classnotes for Monday 9 October.

17. For the sequence  $F = k \in \mathbf{Z}_{\geq 0} \mapsto k - 3 \lfloor \frac{k}{3} \rfloor$ , give three examples of a convergent subsequence.

**Solution:** Define  $\phi = \mathbb{Z}_{\geq 0} k \mapsto 3k$ . Then  $F \circ \phi = k \in \mathbb{Z}_{\geq 0} \mapsto 0$ .

Define  $\phi = \mathbb{Z}_{\geq 0} k \mapsto 3k + 1$ . Then  $F \circ \phi = k \in \mathbb{Z}_{\geq 0} \mapsto 1$ .

Define  $\phi = \mathbb{Z}_{\geq 0} k \mapsto 3k + 2$ . Then  $F \circ \phi = k \in \mathbb{Z}_{\geq 0} \mapsto 2$ .

18. Give an example of a sequence  $F$  and a real number  $\alpha$  such that  $\alpha F$  converges and  $F$  diverges.

**Solution:** Choose  $F = k \in \mathbb{Z}_{\geq 0} \mapsto k$  and choose  $\alpha = 0$ . Then  $F$  diverges but  $0F$  converges.

19. Give an example of sequences  $F$  and  $G$  such that both  $F$  and  $G$  diverge, but  $F + G$  converges.

**Solution:** Choose  $F = k \in \mathbb{Z}_{\geq 0} \mapsto k$  and  $G = k \in \mathbb{Z}_{\geq 0} \mapsto -k$ . Both  $F$  and  $G$  diverge, but  $F + G$  converges.

20. Give an example of sequences  $F$  and  $G$  such that both  $F$  and  $G$  diverge, but  $FG$  converges.

**Solution:** Choose  $F = k \in \mathbb{Z}_{\geq 0} \mapsto (-1)^k$  and  $G = k \in \mathbb{Z}_{\geq 0} \mapsto (-1)^k$ . Both  $F$  and  $G$  diverge, but  $FG = k \in \mathbb{Z}_{\geq 0} \mapsto 1$  converges.

21. Show that  $(\forall x, y \in \mathbb{R}_{\geq 0}) (\sqrt{x+y} \leq \sqrt{x} + \sqrt{y})$ .

**Solution:** We begin by proving that  $(\forall x, y \in \mathbb{R}_{\geq 0}) (\sqrt{x^2 + y^2} \leq x + y)$ . Let  $x, y \in \mathbb{R}_{\geq 0}$ . We have

$$\begin{aligned} \sqrt{x^2 + y^2} &= \sqrt{x^2 + 2xy + y^2 - 2xy}, & (\text{add and subtract}) \\ &= \sqrt{(x+y)^2 - 2xy}, & (\text{algebra}) \\ &\leq \sqrt{(x+y)^2}, & (\text{square root function is increasing}) \\ &= x + y. & (\text{algebra}) \end{aligned}$$

To finish the proof, we replace  $x \rightarrow \sqrt{x}$  and  $y \rightarrow \sqrt{y}$  in the above. This yields  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ .