Let/Choose/Suppose Proof Examples

Fall 2023 Advanced Calculus Class

August 22, 2023

1 Introduction

Many statements in analysis consist of concatenated 'for every' and 'there exists' statements that end with an inequality. The inequality part is either true or false–I call it the *predicate*. We'll see that writing a statement in symbolic form gives us a road map to constructing a proof. So, as the first step in constructing a proof, I encourage you to express the proposition symbolically.

For every qualification of the form $\forall x \in A$, our proof will state 'Let $x \in A$.' Other than assuming that $x \in A$, we cannot make any other assumptions about x. A synonym for 'let' is 'allow' and 'permit,' but we'll stick to using 'let' for consistency. This part of a proof is easy.

For every qualification of the form $\exists\, x \in A$, it's just like fourth grade show-and-tell. We must tell the reader *exactly* what member of A we choose for x. Sometimes there will be more than one choice that allows the proof to continue; if so, we stick to exactly one choice instead of *clogging* our logic with extraneous possibilities. This part of a proof is hard–make a bad choice, and you might get stuck.

Moving from left to right, the value x we choose can depend on all previously defined variables (all variables to the left in the statement in symbolic form), but it cannot depend on any variables to the right. So for a statement fragment

$$(\forall x \in \mathbf{R})\,(\exists a \in \mathbf{R})\,(\forall b \in \mathbf{R})$$

the choice for a can depend on x, but it cannot depend on b. To emphasize that a can depend on x, you'll sometimes see this expressed as 'For all $x \in \mathbf{R}$, there is $a(x) \in \mathbf{R}$ such that $\dots$.' This is OK–if it helps you understand, do it. But I tend to not use this notation because I think it's unneeded clutter.

Finally, what about 'suppose?' To alert the reader that the truth of a statement is directly due to a hypothesis, we often introduce the hypothesis with 'suppose.'

2 Examples

Let's practice this skill with some examples. For each given statement, do the following:

(a) Write the statement symbolically.

(b) Without explicitly using negation, write the negation of the statement symbolically.

(c) Decide if the statement is true or false.

(d) Write a proof of the statement that is true.

Proposistion 1. *For all $x, y \in \mathbf{R}$, there is $a \in \mathbf{R}$ such that $x < y$ implies $x < a < y$.*

Solution Symbolically, the statement is

$$\big(\forall x, y \in \mathbf{R}\big)\,(\exists a \in \mathbf{R})\,\big((x < y) \implies x < a < y\big).$$

And its negation is

$$\big(\exists x, y \in \mathbf{R}\big)\,(\forall a \in \mathbf{R})\,\big((x < y) \wedge (x \geq a \vee a \geq y\big).$$

We need to decide if the statement or its negation is true. The statement says that between any two real numbers x and y with $x < y$, there is a real number a that is between x and y. One such number is the arithmetic average. So let's prove the statement, not its negation. A key ingredient to the proof is the fact that subtracting the same real number from both sides of an inequation yields a logically equivalent inequation. Specifically $(\forall a, x, y \in \mathbf{R})((x < y) \equiv (x - a < y - a))$.

Proof. (BW) Let $x, y \in \mathbf{R}$. Suppose $x < y$. Choose $a = \frac{x+y}{2}$. Then $a \in \mathbf{R}$ as required. We have

$$\begin{aligned}
\big[x < a < y\big] &\equiv \left[x < \frac{x+y}{2} < y\right], && \text{(substitution)}\\
&\equiv \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(subtract } \tfrac{x+y}{2}\text{)}\\
&\equiv \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \text{(simplification)}\\
&\equiv \text{True.} && ((y - x > 0) \wedge (x - y < 0))
\end{aligned}$$

□

Notice how the proof uses 'suppose $x < y$' to inform the reader that what follows is something from the hypothesis. Also, if the fact that the arithmetic average of two numbers is between the numbers is an allowed fact, the proof could end after the first displayed line.

The parenthetical remarks about substitution and subtraction can help a reader to understand–I encourage you to use them.

Proposition 2. *For all* $r \in \mathbf{R}_{>0}$ *there is* $x \in [0,1)$ *such that* $1 - r < x$.

Solution Symbolically, the statement is

$$(\forall r \in \mathbf{R}_{>0})\,(\exists x \in [0,1))\,(1 - r < x).$$

And its negation is

$$(\exists r \in \mathbf{R}_{>0})\,(\forall x \in [0,1))\,(1 - r \geq x).$$

The statement says that there is a real number $x \in [0,1)$ that is between $1 - r$ and 1. This seems to be true. When $1 - r > 0$, we can choose x to be the arithmetic average of $1 - r$; when $1 - r \leq 0$, we can choose x to be $\frac{1}{2}$.

Proof. (BW) Let $r \in \mathbf{R}_{>0}$. Choose $x = \begin{cases} 1-\frac{r}{2} & r<1 \\ \frac{1}{2} & r \geq 1 \end{cases}$. For $r<1$, we have

$$\begin{aligned} [1-r<x] &\equiv \left[1-r<1-\frac{r}{2}\right], && \text{(substitution)} \\ &\equiv \left[0<\frac{r}{2}\right], && \text{(algebra)} \\ &\equiv \text{True}. && (0<r<1). \end{aligned}$$

And for $r \geq 1$, we have

$$\begin{aligned} [1-r<x] &\equiv \left[1-r<\frac{1}{2}\right], && \text{(substitution)} \\ &\equiv \left[\frac{1}{2}<r\right], && \text{(algebra)} \\ &\equiv \text{True}. && (r \geq 1). \end{aligned}$$

□

Proposistion 3. *For all $x \in \mathbf{R}_{>0}$ there is $y \in \mathbf{R}_{>0}$ such that $y<x$.*

Solution Symbolically, the statement is

$$(\forall x \in \mathbf{R}_{>0}) \left(\exists y \in \mathbf{R}_{>0}\right) (y<x).$$

And its negation is

$$(\exists x \in \mathbf{R}_{>0}) \left(\forall y \in \mathbf{R}_{>0}\right) (y \geq x).$$

The statement says that for every positive number x, there is a positive number y that is smaller than x. Surely this is true–we can choose, for example, the arithmetic average of zero and x; that is, we can choose $y = x/2$.

Proof. (BW) Let $x \in \mathbf{R}_{>0}$. Choose $y = x/2$. Then $y \in \mathbf{R}_{>0}$ as required. We have

$$\begin{aligned} \left[y<x\right] &\equiv \left[\frac{x}{2}<x\right], && \text{(substitution)} \\ &\equiv \left[0<\frac{x}{2}\right], && \text{(algebra)} \\ &\equiv \text{True}. && \text{(x > 0)} \end{aligned}$$

□

Proposistion 4. *There is $y \in \mathbf{R}_{>0}$ such that for all $x \in \mathbf{R}_{>0}$ we have $y<x$.*

Solution Symbolically, the statement is

$$\left(\exists y \in \mathbf{R}_{>0}\right) (\forall x \in \mathbf{R}_{>0}) \left(y<x\right).$$

This proposition is the same as the previous proposition, but the order of the qualified statements are switched. Usually, order matters in mathematics (and in life), so it's possible that this statement is false. Indeed, we'll show that this is the case.

We might be tempted to prove this proposition as

Proof. Choose $y = \frac{x}{2}$. Let $x \in \mathbf{R}_{>0}$. Then $y \in \mathbf{R}_{>0}$ as required. □

Let's not continue this "proof." What's wrong? Plenty–we violated the strict left to right rule. We allowed the first qualified variable y to depend on the second variable x. And that is not allowed.

Returning to the proposition, it says that there is a positive real number y that is smaller than every positive real number x. Since y is positive, we cannot choose $y = 0$. Surely, the proposition is false; let's hope its negation is true; its negation is

$$\left(\forall y \in \mathbf{R}_{>0}\right)(\exists x \in \mathbf{R}_{>0})\left(y \geq x\right).$$

This is almost surely true–it says for every positive real number y there is a positive real number x such that $y \geq x$. Sure, we can choose $x = y$, for example.

Proposistion 5. *For every $y \in \mathbf{R}_{>0}$ there is $x \in \mathbf{R}_{>0}$ such that $y \geq x$.*

Proof. (BW) Let $y \in \mathbf{R}_{>0}$. Choose $x = y$. Then $x \in \mathbf{R}_{>0}$, as required. We have

$$\begin{aligned} \left[y \geq x\right] &\equiv \left[y \geq y\right], && \text{(substitution)} \\ &\equiv [0 \geq 0], && \text{(algebra)} \\ &\equiv \text{True}. \end{aligned}$$

□

Proposistion 6. *For all $x \in \mathbf{R}_{>0}$, there is $M \in \mathbf{R}$ such that $\frac{1}{x} + 1 > M$.* *(SB)*

Proposistion 7. *There is $M \in \mathbf{R}$ such that for all $x \in \mathbf{R}_{>0}$, we have $\frac{1}{x} + 1 > M$.* *(DD)*

Proposistion 8. *There is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $1 + m(x-1) \leq x^2$.* *(TK)*

Proposistion 9. *For every $a \in \mathbf{R}$, there is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $a^2 + m(x-a) \leq x^2$.* *(AK)*

Proposistion 10. *For all $x, y \in \mathbf{R}$, we have $(x^2 = y^2) \implies (x = y)$.* *(DM)*

Proposistion 11. *For all $x, y \in \mathbf{R}$, we have $(x^3 = y^3) \implies (x = y)$.* *(CR)*