Derivative divides integration

Lesson 7

All things start with the chain rule

Recall the chain rule; for differentiable functions F and g, in pure function notation, we have

$$(F \circ g)' = g' F' \circ g.$$

Equivalently in formula notation, the chain rule is

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(g(x)) = g'(x)F'(g(x)).$$

Fact

Up to an additive constant, the antiderivative "undoes" the derivative; thus

$$\int g'(x)F'(g(x))\,\mathrm{d}x = F(g(x)) + c,$$

where $c \in \mathbf{R}$.

Match this

If we can match an integrand to $g'(x)F'(g(x))$ and we know an antiderivative of F, we win. An example

$$\int 2x \sin(x^2)\,\mathrm{d}x.$$

Maybe you see the match isn't particularly hidden:

- $g(x) = x^2 \implies g'(x) = 2x$,
- $F' = \sin \implies F = -\cos$.

Thus

$$\int 2x \sin(x^2)\,\mathrm{d}x = -\cos(x^2) + c.$$

Out with the old, in with the new

Let's re-do the problem $\int 2x\sin(x^2)\,\mathrm{d}x$, but organize our work differently.

- The argument of $\sin$ is x^2. Let's define a new variable $u = x^2$. Then

$$\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = 2x\,\mathrm{d}x.$$

- We now need to write $2x\sin(x^2)\,\mathrm{d}x$ *entirely* in terms of the new variable u.
- When I say entirely, I mean *entirely*. This includes expressing $\mathrm{d}x$ in terms of $\mathrm{d}u$
- Grouping the factor of $2x$ together with $\mathrm{d}x$, we have

$$2x\sin(x^2)\,\mathrm{d}x = \sin(x^2)(2x\,\mathrm{d}x) = \sin(u)\,\mathrm{d}u.$$

- So

$$\int 2x\sin(x^2)\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u = -\cos(u) = -\cos(x^2).$$

- In step $-\cos(u) = -\cos(x^2)$ we reverted to the "original" integration variable.

Example redux

Let's try $\int x \exp(x^2)\,\mathrm{d}x$

- The argument of $\exp$ is x^2. Let's define a new variable $u = x^2$. Then

$$\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = 2x\,\mathrm{d}x.$$

- Unlike the previous problem, the integrand is missing a factor of 2 for a complete matching. No big deal; we have

$$[\mathrm{d}u = 2x\,\mathrm{d}x] = \left[\frac{1}{2}\mathrm{d}u = x\,\mathrm{d}x\right]$$

- So

$$x\exp(x^2)\,\mathrm{d}x = \exp(x^2)(x\,\mathrm{d}x) = \frac{1}{2}\exp(u)\,\mathrm{d}u.$$

- So

$$\int x\exp(x^2)\,\mathrm{d}x = \int \exp(u)\,\mathrm{d}u = \exp(u) = \exp(x^2).$$

- In step $\exp(u) = \exp(x^2)$ we reverted to the "original" integration variable.

Derivative divides

- When we match $u = x^2$ to find $\int x \exp(x^2)\, \mathrm{d}x$, the derivative of u, that is $\frac{\mathrm{d}u}{\mathrm{d}x}$ doesn't exactly match the remaining factor of x, but it does match up to a multiplicative factor of 2.
- Since the quotient of the derivative of u divided by the the remaining factor of x is a constant, the method is called *derivative divides*.
- But calculus books call the method *U substitution*
- I you don't like my explanations, see `https://www.youtube.com/watch?v=8B31SAk1nD8`.

Finding You

Heuristic

- The integrand should be a product.
- Choose u to be an expression that is “inside” a function with a known antiderivative.
- The derivative of u times a constant should match the remaining factors of the integrand.

Matching examples

$\int x^2 \cos(x^3)\,\mathrm{d}x$. The cosine has a known antiderivative. The expression inside cosine is x^3. So choose $u = x^3$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ matches the remaining factor x^2 in the integrand up to a multiplicative factor. **We win**.

$$[u = x^3] = [\mathrm{d}u = 3x^2\mathrm{d}x] = \left[x^2\mathrm{d}x = \frac{1}{3}\mathrm{d}u\right].$$

So

$$\int x^2 \cos(x^3)\,\mathrm{d}x = \int \frac{1}{3}\cos(u)\,\mathrm{d}u = \frac{1}{3}\sin(u) = \frac{1}{3}\sin(x^3).$$

Actually, we are losers

Actually, we are **losers**. Why? Because we didn't check our work:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\sin(x^3)\right) = \frac{1}{3}3x^2\cos(x^3) = x^2\cos(x^3)$$

The integrand is $x^2\cos(x^3)$. So we aren't losers.

$\int x \exp(-x^2)\,\mathrm{d}x$. The function $\exp$ has a known antiderivative. The expression inside $\exp$ is $-x^2$. So choose $u = -x^2$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x$ matches the remaining factor x in the integrand up to a multiplicative factor. **We win**.

$$[u = -x^2] = [\mathrm{d}u = -2x\mathrm{d}x] = \left[x\mathrm{d}x = -\frac{1}{2}\mathrm{d}u\right]$$

So

$$\int x \exp(-x^2)\,\mathrm{d}x = \int -\frac{1}{2}\exp(u)\mathrm{d}u = -\frac{1}{2}\exp(u) = -\frac{1}{2}\exp(-x^2).$$

$\int \sqrt{5x+7}\,\mathrm{d}x$. Oops! The integrand isn't a product. Are we loosers? No way. The integrand is $\int 1 \times \sqrt{5x+7}\,\mathrm{d}x$.
The square root has a known antiderivative, so choose $u = 5x + 7$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 5$ matches the remaining factor 1 in the integrand up to a multiplicative factor. **We win**.

$$[u = 5x + 7] = [\mathrm{d}u = 5\mathrm{d}x] = \left[\mathrm{d}x = \frac{1}{5}\mathrm{d}u\right].$$

So

$$\int \sqrt{5x+7}\,\mathrm{d}x = \int \frac{1}{5}\sqrt{u}\,\mathrm{d}u = \frac{1}{5}\frac{2}{3}u^{3/2} = \frac{2}{15}(5x+7)^{3/2}.$$

Fake matches

Let's try the problem

$$\int x \exp(x^4)\,\mathrm{d}x.$$

Since $\exp$ has a known antiderivative, for u choose what is inside $\exp$. Thus

$$[u = x^4] = [\mathrm{d}u = 4x^3 \mathrm{d}x] = \left[\mathrm{d}x = \frac{1}{4x^3}\mathrm{d}u\right]$$

So far, Okie-dokie.

$$\int x \exp(x^4)\,\mathrm{d}x = \int x \exp(u) frac14x^3 \mathrm{d}u = \int \exp(u) \frac{1}{4x^2} \mathrm{d}u$$

Hidden matches

Let's try the related antiderivative

$$\int x \sin(x^2)\,\mathrm{d}x.$$

If we match $g'(x) = x$, we have $g(x) = \frac{1}{2}x^2$ The match isn't particularly hidden:

- $g(x) = x^2 \implies g'(x) = 2x$,
- $F' = \sin \implies F = -\cos$.

Thus

$$\int 2x \sin(x^2)\,\mathrm{d}x = -\cos(x^2) + c.$$