MATH 460 **Name:**

Homework 2, Fall 2023

Homework 2 has questions 1 through 3 with a total of 70 points. When I record your grade, I will scale it to twenty points. For details of the grading scheme for this assignment, please see the section 'Grading rubric' of our syllabus.

Revise, proofread, revise again (and again), typeset your work using Overleaf, and upload the converted PDF of your compiled file work to Canvas. This work is due **Saturday 2 September** at 11:59 PM.

For Question 1, I will compile the class work into a single document. To allow me to do this without retyping your work, copy a link to your Overleaf file (either read only or read and write) here: (insert an URL for your Overleaf work).

1. Define a set of sets $\mathscr{C}$ by $\mathscr{C} = \{\{\pi\}, \{\pi, \infty\}, \{\pi, \infty, \sqrt{3}\}\}$ Enumerate the members of each of the following sets:

$\boxed{10}$ (a) $\bigcap_{x \in \mathscr{C}} x$

Solution: Set intersection is associative, so intersections of two or more sets can be expressed unambiguously without using parenthesis. We have

$$\begin{aligned} \bigcap_{x \in \mathscr{C}} x &= \{\pi\} \cap \{\pi, \infty\} \cap \{\pi, \infty, \sqrt{3}\}, \\ &= \{\pi\}. \end{aligned}$$

$\boxed{10}$ (b) $\bigcup_{x \in \mathscr{C}} x$

Solution: Set union is associative, so intersections of two or more sets can be expressed unambiguously without using parenthesis. We have

$$\begin{aligned} \bigcup_{x \in \mathscr{C}} x &= \{\pi\} \cup \{\pi, \infty\} \cup \{\pi, \infty, \sqrt{3}\}, \\ &= \{\pi, \infty, \sqrt{3}\}. \end{aligned}$$

The order of the set members doesn't matter, so $\{\pi, \sqrt{3}, \infty\}$, for example, is also an answer.

10 (c) $\bigcup_{x\in\mathscr{C}} x \setminus \bigcap_{x\in\mathscr{C}} x$

> **Solution:** Using the previous two parts, we have
>
> $$\bigcup_{x\in\mathscr{C}} x \setminus \bigcap_{x\in\mathscr{C}} x = \{\pi, \infty, \sqrt{3}\} \setminus \{\pi\} = \{\infty, \sqrt{3}\}.$$

10 2. Let X and Y be sets and let $F \in X \to Y$. For all subsets A and B of X, show that $F(A \cap B) \subset F(A) \cap F(B)$.

> **Solution:** The conclusion of this proposition is set inclusion, so we need to use pick-and-show.
>
> *Proof.* Suppose $y \in F(A \cap B)$. We'll show that $y \in F(A) \cap F(B)$. Since $y \in F(A \cap B)$, there is $x \in A \cap B$ such that $y = F(x)$. But $x \in A \cap B$, so $x \in A$ and $x \in \cap B$. So $y \in F(A)$ and $y \in F(B)$; therefore, $y \in F(A) \cap F(B)$. $\square$

3. Define $F = x \in \mathbf{R} \mapsto x^2$. For these problems, freely assume all College Algebra facts about real numbers and square roots.

10 (a) Show that $F((-\infty, 0)) = (0, \infty)$.

> **Solution:** The conclusion of this proposition is set equality, so we'll prove two set inclusions. Each proof of a set inclusion requires a pick-and-show proof. First, we'll show that $F((-\infty, 0)) \subset (0, \infty)$
>
> *Proof.* Suppose $y \in F((-\infty, 0))$. We'll show that $y \in (0, \infty)$. Since $y \in F((-\infty, 0))$, there is $x \in (-\infty, 0)$ such that $y = x^2$. But $-\infty < x < 0$ implies $0 < x^2 < \infty$. So $y \in (0, \infty)$. $\square$
>
> Second, we'll show that $(0, \infty) \subset F((-\infty, 0))$. To do this, we'll assume that all nonnegative numbers have a square root.
>
> *Proof.* Suppose $y \in (0, \infty)$. We'll show that $y \in F((-\infty, 0))$. We have $-\sqrt{y} \in (-\infty, 0)$ and $F(-\sqrt{y}) = y$, so indeed $y \in F((-\infty, 0))$. $\square$

10 (b) Show that $F((0, \infty)) = (0, \infty)$.

Solution: The conclusion of this proposition is set equality, so we'll prove two set inclusions. Each proof of a set inclusion requires a pick-and-show proof. First, we'll show that $F((0,\infty)) \subset (0,\infty)$.

Proof. Suppose $y \in F((0,\infty))$. We'll show that $y \in (0,\infty)$. Since $y \in F((0,\infty))$, there is $x \in (0,\infty)$ such that $y = x^2$. But $x < 0$ implies $0 < x^2$. So $y \in (0,\infty)$ □

Second, we'll show that $(0,\infty) \subset F((0,\infty))$.

Proof. Suppose $y \in (0,\infty)$. We'll show that $y \in F((0,\infty))$. We have $\sqrt{y} \in (0,\infty)$ and $F(\sqrt{y}) = y$. So $y \in F((0,\infty))$. □

10 (c) Find an example of subsets A and B of $\mathbf{R}$ such that $F(A) \cap F(B) \not\subset F(A \cap B)$

Solution: Our example is hiding in plain sight. Choose $A = (-\infty,0)$ and $B = (0,\infty)$. We have

$$F(A \cap B) = F((-\infty,0)) \cap F((0,\infty)) = F(\varnothing) = \varnothing.$$

And we have

$$F(A) \cap F(B) = (0,\infty) \cap (0,\infty) = (0,\infty).$$

Certainly, $(0,\infty) \not\subset \varnothing$.

How can it be the case that $F(A \cap B)$ is "bigger" than is $F(A) \cap F(B)$? Here is how: we have

$$\left[y \in F(A) \cap F(B)\right] \equiv (\exists a \in A)\,(\exists b \in B)\left(y = F(a) \wedge y = F(b)\right).$$

And

$$\left[y \in F(A \cap B)\right] \equiv \left(\exists p \in A \cap B\right)\left(y = F(p)\right).$$

For this condition, we might have, for example $p \in A$ such that $y = F(p)$, but there is no member b in B such that $y = F(b)$.