Sets

Sets

Quasi-definition

We don't attempt to define a set, but we describe a set as a collection of things, often called *members*. The members of a set can be numbers, ordered pairs, functions, or sets themselves. In a bit, we'll learn that there are some things that might appear to be valid sets, but really are not sets.

Here are some examples of sets:

a. $\{46, 107\}$ is a set with two members, namely 46 and 107. We reserve the curly braces to delineate a set.
b. $\{46, \{46, 107\}\}$ is a set with two members, namely 46 and $\{46, 107\}$. One member of this set is an integer, but the other is a set with two members–that's OK.
c. $\{0, 1, 2, 3, \ldots\}$ is apparently the set of all nonnegative integers. I say apparently because the ellipses (the $\ldots$) isn't entirely clear.
d. $\{\}$ is a set with no members.

Named Sets

We'll use the following names for subsets of real numbers:

$$\mathbf{R} = \text{the set of real numbers,}$$

$$\mathbb{R} = \text{the set of real numbers for handwritten text,}$$

$$\mathbf{R}_{>0} = \{x \in \mathbf{R} \mid x > 0\},$$

$$\mathbf{R}_{\neq 0} = \{x \in \mathbf{R} \mid x \neq 0\}, \text{(and similarly for other subscripts)}$$

$$\mathbf{Z} = \text{the set of integers,}$$

$$\mathbb{Z} = \text{ the set of integers for handwritten text,}$$

$$\mathbf{Q} = \text{the set of rational numbers,}$$

$$\mathbb{Q} = \text{ the set of rational numbers for handwritten text,}$$

$$\varnothing = \text{A set with no members, that is the empty set}$$

Membership

For a set A, we define a predicate (boolean valued function) as

$$x \in A = \begin{cases} \mathrm{T} & \text{if } x \text{ is a member of } A \\ \mathrm{F} & \text{if } x \text{ is not member of } A \end{cases}$$

For example:

a. $107 \in \{46, 107\} = \mathrm{T}$
b. $\{107\} \in \{46, \{107\}\} = \mathrm{T}$
c. $\{107\} \in \{46, 107\} = \mathrm{F}$

Set Operators

Definition

Let A and B be sets. Define the set *union*, *intersection*, and *difference*

$$
\begin{aligned}
A \cap B &= \{x \mid (x \in A) \wedge (x \in B)\}, \\
A \cup B &= \{x \mid (x \in A) \vee (x \in B)\}, \\
A \setminus B &= \{x \mid (x \in A) \wedge (x \notin B)\},
\end{aligned}
$$

respectively.

Set (an) example

Example

We have

$$\begin{aligned}
\{6, 107\} \cap \{28, 107\} &= \{107\},\\
\{6, 107\} \cup \{28, 107\} &= \{6, 28, 107\},\\
\{6, 107\} \setminus \{28, 107\} &= \{6\},\\
\{28, 107\} \setminus \{6, 107\} &= \{28\}.
\end{aligned}$$

- The last two examples show that in general $A \setminus B \neq B \setminus A$.
- The set difference is so much like real number subtraction, sometimes the symbol "-" is used instead of $\setminus$.

Set predicates

Definition

Let A and B be sets. Define

$$A \subset B \equiv (\forall x \in A)(x \in B),$$
$$A = B \equiv (A \subset B) \wedge (B \subset A).$$

Specializing $A \subset B$ to $A = \varnothing$ gives

$$[\varnothing \subset B] \equiv (\forall x \in \varnothing)(x \in B) \equiv \text{true}.$$

We've shown that:

Proposition

Thus for all sets A and for any empty set $\varnothing$, we have $\varnothing \subset A$.

Set equality

To show that sets A and B are equal, we almost always prove that $A \subset B$ and $B \subset A$. If a proposition has the form

Proposition

If $H_1, H_2, \ldots$, and H_n, then $A = B$.

where $H_1, H_2, \ldots H_n$ is the hypothesis, a template for proving the theorem is

Proof

Suppose $x \in A$. We'll show that $x \in B$. Since $x \in A, H_1, H_2, \ldots$ and H_n, we have $\ldots$; therefore $x \in B$.

Suppose $x \in B$. We'll show that $x \in A$. Since $x \in B, H_1, H_2, \ldots$ and H_n, we have $\ldots$; therefore $x \in A$.

1. Notice how in the first case we append $x \in A$ to the hypothesis; and in the second case, we append $x \in B$.

Establish notation

Proposition

The set union is associative.

Proof

Let A, B, and C be sets. We'll show that $A \cup (B \cup C) = (A \cup B) \cup C$. Our proof uses the fact that the disjunction is associative; we have

$$\begin{aligned} A \cup (B \cup C) &= \{x \mid (x \in A) \vee (x \in B \cup C)\}, \\ &= \{x \mid (x \in A) \vee (x \in B) \vee (x \in C)\}, \\ &= \{x \mid ((x \in A) \vee (x \in B)) \vee x \in C)\}, \\ &= (A \cup B) \cup C. \end{aligned}$$

- The statement of the proposition doesn't introduce notation, so the proof must do so.
- Alternatively, we can show that $A \cup (B \cup C) \subset (A \cup B) \cup C$ and $(A \cup B) \cup C \subset A \cup (B \cup C)$.

Alternative proofs

Proof

Let A, B, and C be sets. We'll show that $A \cup (B \cup C) = (A \cup B) \cup C$. We have

$$\begin{aligned} x \in A \cup (B \cup C) &\implies (x \in A) \vee (x \in B \cup C), \\ &\implies (x \in A) \vee ((x \in B) \vee (x \in C)), \\ &\implies ((x \in A) \vee (x \in B)) \vee (x \in C), \\ &\implies x \in (A \cup B) \cup C. \end{aligned}$$

Similarly, we can show that $x \in (A \cup B) \cup C \implies x \in A \cup (B \cup C)$.

The uniqueness of emptiness

Proposition

There is at most one empty set.

Proof

Let O and O' be empty sets. Since O is empty, we have $O \subset O'$. Similarly since O' is empty, we have $O' \subset O$. We have shown that $O \subset O'$ and $O' \subset O$; therefore $O = O'$.

1. With impunity, we can now refer to **the** empty set.
2. A clumsy way to proof this is by contradiction. The proof assumes that there are empty sets O and O', but $O \neq O'$.

Conflation

Question: True or false: $\varnothing = \{\varnothing\}$.

Answer: It's false. The set $\{\varnothing\}$ is a set that has (exactly) one member, namely its member is the empty set. But the empty set has no members, so $\varnothing \neq \{\varnothing\}$

We can write this as

Counterexample ($\varnothing = \{\varnothing\}$)

We have $\varnothing \in \{\varnothing\}$, but $\varnothing \notin \varnothing$; therefore $\varnothing \neq \{\varnothing\}$.

- A *counterexample* is a proof of the negation of some statement. Usually to be considered a counter example, the proof examines one particular case.