MATH 460 **Name:**

Homework 7, Fall 2022

I have neither given nor received unauthorized assistance on this assignment.

Homework 7 has questions 1 through 5 with a total of 35 points. Neatly **hand write your solutions**, digitize your work, and turn it into Canvas. You do **not** need to use LaTeX for this assignment. This work is due *Saturday 8 October at 11:59* PM.

$\boxed{5}$ 1. Show that the set $[1,2)$ is not open. To do this, show that

$$(\exists x \in [1,2))\,(\forall r \in \mathbf{R}_{>0})\,(\mathrm{ball}(x,r) \not\subset [1,2))\,.$$

Solution: Choose $x = 1$. Let $r \in \mathbf{R}_{>0}$. To show that $\mathrm{ball}(x,r) \not\subset [1,2)$, we'll show there is a memember of $\mathrm{ball}(x,r)$ that is not in $[1,2)$. Specifically, We but $1 - r/2 \notin [1,2)$. So $\mathrm{ball}(1,r) \not\subset [1,2)$.

$\boxed{5}$ 2. Let F be a convergent sequence. Show that

$$\mathrm{range}(F) \subset \mathbf{R}_{>0} \implies \lim_{\infty}(F) \in \mathbf{R}_{\geq 0}.$$

Solution: We'll prove the contrapositive. Thus, we'll show that

$$\lim_{\infty} F \in \mathbf{R}_{<0} \implies \mathrm{range}(F) \not\subset \mathbf{R}_{>0}.$$

Define $\lim_{\infty}(F) = L$, where $L < 0$. Since $-L/2 \in \mathbf{R}_{>0}$, there is $n \in \mathbf{Z}$ such that for all $k \in \mathbf{Z}_{>n}$, we have $F_k \in (L + L/2, L - L/2)$; equivalently $F_k \in (3L/2, L/2)$. In particular, $F_{n+1} < L/2 < 0$; therefore $\mathrm{range}(F) \not\subset \mathbf{R}_{>0}$

$\boxed{5}$ 3. Give an example of a sequence G such that $\mathrm{range}(G) \subset \mathbf{R}_{>0}$ but $\lim_{\infty} G \notin \mathbf{R}_{>0}$.

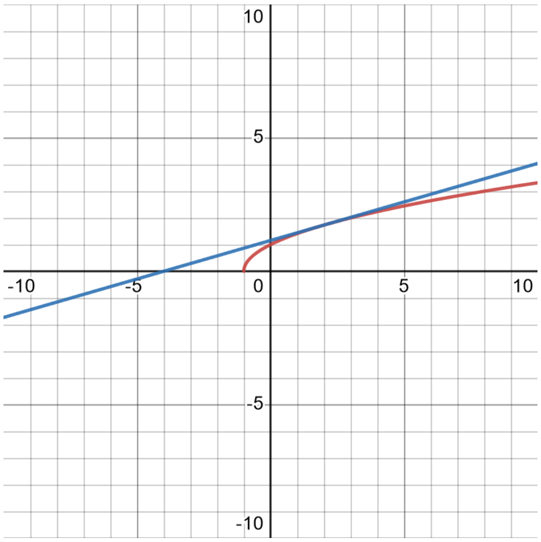
Solution: An example is $F = k \in \mathbf{Z}_{\geq 0} \mapsto 1/k$. The proof that this is an example is a calculus calculation.

$\boxed{5}$ 4. Draw a nicely labeled picture that shows that

$$(\forall a \in \mathbf{R}_{>-1})\,(\forall x \in \mathbf{R}_{>-1})\left(\sqrt{1+x} \leq \sqrt{1+a} + \frac{1}{2\sqrt{1+a}}(x-a)\right).$$

Hint: Draw a graph of $y = \sqrt{1+x}$. On the same graph, draw a graph of the tangent line to $y = \sqrt{1+x}$ at the point $(x = a, y = \sqrt{1+a})$.

Solution:



This picture is specialized to $a = 2$, but the picture for the general case is similar. The general principle is that if a graph is concave down, its tangent lines are above or touching the curve.

5. Define a sequence H partially in terms of itself by

$$H_n = \begin{cases} 2 & n = 0 \\ \sqrt{1 + H_{n-1}} & n \in \mathbf{Z}_{>0} \end{cases}.$$

The first six terms of H are

$$\begin{aligned} H_0 &= 2, \\ H_1 &= \sqrt{3}, \\ H_2 &= \sqrt{\sqrt{3}+1}, \\ H_3 &= \sqrt{\sqrt{\sqrt{3}+1}+1}, \\ H_4 &= \sqrt{\sqrt{\sqrt{\sqrt{3}+1}+1}+1}, \\ H_5 &= \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{3}+1}+1}+1}+1}. \end{aligned}$$

Without proof, you may assume that $\mathrm{range}(H) \subset \mathbf{R}_{>0}$.

5 (a) Show that H decreases. One way to do this is to use Question 4 and induction.

Solution: Define a predicate P by

$$P = n \in \mathbf{Z}_{\geq 0} \mapsto H_n > \frac{\sqrt{5}+1}{2}.$$

We will show that $(\forall k \in \mathbf{Z}_{\geq 0})(P_k)$ by showing that $P(1) \wedge (\forall k \in \mathbf{Z}_{\geq 0})(P_k \implies P_{k+1})$. We have

$$P_1 = \left[H_0 > \frac{\sqrt{5}+1}{2} \right] \equiv \left[2 > \frac{\sqrt{5}+1}{2} \right] \equiv \left[3 > \sqrt{5} \right] = \text{True}.$$

Now suppose that P_k is true. We have

$$\begin{aligned}
\left[H_{k+1} > \frac{\sqrt{5}+1}{2} \right] &\equiv \left[\sqrt{1+H_k} > \frac{\sqrt{5}+1}{2} \right], && \text{(recursive definition)} \\
&\equiv \left[1 + H_k > \frac{6+2\sqrt{5}}{4} \right], && \text{(square)} \\
&\equiv \left[H_k > \frac{1+\sqrt{5}}{2} \right], && \text{(subtract one)} \\
&\equiv \text{True}. && \text{(hypothesis)}
\end{aligned}$$

Since $\sqrt{1+x} - x < 0$ for all $x > \frac{\sqrt{5}+1}{2}$, it follows that H is decreasing.

5 (b) Show that H converges to a nonnegative number.

Solution: The sequence H decreases and it is bounded below by zero; therefore H converges. By the result in Question 2, H converges to a nonnegative number.

5 (c) Show that H converges to the golden ratio. To do this, you may freely use the facts:

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty} \sqrt{1+H_{n-1}} = \sqrt{1 + \lim_{n\to\infty} H_{n-1}} = \sqrt{1 + \lim_{n\to\infty} H_n}.$$

And if you don't know, the golden ratio has an *unearned* celebrity status in mathematics, art, and design. The golden ratio is the number $\dfrac{\sqrt{5}+1}{2}$.

Solution: Define $\lim_{\infty}(H) = L$. We have $\sqrt{1+L} = L$ the only solution to this equation is the golden ratio.