Set complements

The whole problem with the world is that fools and fanatics are always so certain of themselves, and wiser people so full of doubts.

Bertrand Russel

The set complement

Definition

Let U be the universal set. For any set A, we define its complement A^c by $A^c = U \setminus A$.

- We have

$$(x \in A^c) \equiv (x \in U) \wedge (x \notin A).$$

- Assisted by Mr. DeMorgan, we have

$$(x \notin A^c) \equiv (x \notin U) \vee (x \in A).$$

Every definition deserves a theorem

Theorem

Let A be a set. We have

$$\begin{aligned} \varnothing^c &= U, \\ U^c &= \varnothing, \\ (A^c)^c &= A, \\ A \cap A^c &= \varnothing, \\ A \cup A^c &= U. \end{aligned}$$

Proof ($U = \varnothing^c$)

We have

$$\varnothing^c = \{u \in U \mid x \notin \varnothing\} = \{u \in U \mid \text{true}\} = U.$$

- We used the fact that $(\forall x)(x \notin \varnothing) \equiv \text{true}$.
- The proof is a string of equalities. The conclusion of the proof is to compare the far left to the far right of the string.
- Alternatively, we could show that $U \subset \varnothing^c$ and $\varnothing^c \subset U$. But I think this proof is more clear.

Proof ($(A^c)^c = A$)

Let A be a set. We have

$$(A^c)^c = \{u \in U \mid u \notin A^c\} = \{u \in U \mid u \in A\} = A.$$

- Arguably this proof makes too large a jump in logic from $u \notin A^c$ to $u \in A$.
- Here is a fix:

$$\begin{aligned}
\{u \in U \mid u \notin A^c\} &= \{u \in U \mid (u \notin U) \vee (u \in A)\}, \\
&= \{u \in U \mid \text{false} \vee (u \in A)\}, \\
&= \{u \in U \mid u \in A\}, \\
&= A.
\end{aligned}$$

- We have

$$x \in A \setminus B \equiv (x \in A) \wedge (x \notin B)$$

- So

$$x \notin A \setminus B \equiv \neg((x \in A) \wedge (x \notin B)) = (x \notin A) \vee (x \in B).$$

Proof ($\varnothing = U^c$)

We've already shown that $\varnothing^c = U$. Using this fact, we have

$$\varnothing = (\varnothing^c)^c = (U)^c.$$

- Our proof uses the fact that for all sets A, we have $(A^c)^c = A$.
- With malice aforethought, we proved the statements in the proposition in a different order than they were presented.
- We switched up the order to make use of $(A^c)^c = A$ in a later proof.

Looks like homework

Proof ($A \cap A^c = \varnothing$)

(This looks like it should be homework–it's all up to you. You are certainly allowed to use all the results we have proved so far.)

Proof ($A \cup A^c = U$)

(This looks like it should be homework–it's all up to you. You are certainly allowed to use all the results we have proved so far.)