MATH 460 **Name:**

Homework 9, Fall 2023

Homework 9 has questions 1 through 5 with a total of 50 points. This work is due **Saturday 4 November** at 11:59 PM.

$\boxed{10}$ 1. Let $F \in \mathbf{R} \to \mathbf{R}_{\geq 0}$ have a limit toward 5. Use the QRS definition of a limit to show that $\sqrt{F}$ has a limit toward 5. Almost surely, you will want to use the fact that the square root function is subadditive; that is

$$\left(\forall x, y \in \mathbf{R}_{>0}\right) (|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}).$$

Solution:

Proof. □

$\boxed{10}$ 2. Use the fact that the absolute value function is continuous on $\mathbf{R}$ to show that the function $F = x \in \mathbf{R} \mapsto x|x|$ is differentiable at zero. (Actually, you'll only use the fact that the absolute value function is continuous at zero. But it's true that it is continuous on $\mathbf{R}$.)

Solution:

Proof. □

$\boxed{10}$ 3. Show that the function $F = x \in \mathbf{R} \mapsto \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ has a limit toward zero.

To do this, use the fact that for all $x \in \mathbf{R}_{\neq 0}$, we have $|x \sin(\frac{1}{x})| \leq |x|$. From this work, show that F is continuous at zero.

Solution:

Proof. □

$\boxed{10}$ 4. Show that the function $G = x \in \mathbf{R} \mapsto \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at zero. One way to proceed is to use the result of the previous question.

Solution:

Proof. $\square$

$\boxed{10}$ 5. Again, define $G = x \in \mathbf{R} \mapsto \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Away from zero, the product rule, the chain rule, and the rule for the derivative of sine, gives $G'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$; and at zero, we know that $G'(0) = 0$. Thus G is differentiable on $\mathbf{R}$ and

$$G' = x \in \mathbf{R} \mapsto \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Show that G' does not have a limit toward 0. Consequently, $G'(0) \neq \lim_0 G'$

Hints Let $\delta \in \mathbf{R}_{>0}$. Archimedes tells us that there is a positive integer n such that $\frac{1}{n} < \delta$. Also for all positive integers n, we have $G'(\frac{1}{2\pi n}) = -1$ and $G'(\frac{1}{\pi(2n+1)}) = 1$.

Solution:

Proof. $\square$