Sets as ordered pairs

Ordered pairs

An ordered pair is a familiar object–the Cartesian coordinates of a point in a plane, for example, is an ordered pair of real numbers.

Example

Examples of ordered pairs of real numbers:

$$(0,0), \quad (2,-6), \quad (2,\sqrt{42}), \quad (107,28).$$

- We say that a is the *first coordinate* of the ordered pair (a,b), and b is its *second coordinate*.
- We efine equality of ordered pairs using $[(a,b)=(a',b')] \equiv [a=a'] \wedge [b=b']$.
- Of course, the symbols a through b' need to be objects for which equality is defined.

As nice this may be

Question Ordered pairs are somewhat like sets, but the order matters. Can we define an ordered pair as a set?
Answer Sure. To an ordered pair (a, b) we associate it with the set $\{\{a\}, \{a, b\}\}$.

- Since $\{\{a\}, \{a, b\}\}$ is a set of sets, we can form the intersection of its members. Define $I = \{\{a\}, \{a, b\}\}$. Then

  $$\bigcap_{x \in I} x = \{a\} \cap \{a, b\} = \{a\}.$$

  So the intersection of the member of I gives the first coordinate of the ordered pair (a, b).
- How do we extract the second coordinate?

  $$\bigcup_{x \in I} x \setminus \bigcap_{x \in I} x = \{a, b\} \setminus \{a\} = \{b\}.$$

Theorem

If $\{\{a\},\{a,b\}\} = \{\{a'\},\{a',b'\}\}$, then $a = a'$ and $b = b'$.

- A proof uses the ingredients: If $\{a\} = \{a'\}$, then $a = a'$. It also uses

$$\bigcap_{x \in I} x = \{a\} \cap \{a,b\} = \{a\},$$

and

$$\bigcup_{x \in I} x \setminus \bigcap_{x \in I} x = \{a,b\} \setminus \{a\} = \{b\}.$$

where $I = \{\{a\},\{a,b\}\}$.