Functions

Functions

- There is an elegant way of defining a function from at set A to a set B purely in terms of a subset of $A \times B$.

- And as we have seen, members of $A \times B$ can be defined purely in terms of sets.

- But we don't often think of a function this way.

- Functions usually have a formula (or recipe) for determining the output for every input. But sometimes there is no known recipe–for example

$$F(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ -1 & x \notin \mathbf{Q} \end{cases}$$

Last I checked, nobody knows the value of $F(\pi - \mathrm{e})$. But certainly $F(107) = 1$ and $F(\sqrt{2}) = -1$.

Functions

To define a function F with domain A and formula blob, we can write

$$F = x \in A \mapsto \text{blob}.$$

In the rare cases that it's important to give the function a codomain, we can write

$$F = x \in A \mapsto \text{blob} \in B,$$

where $\mathrm{codomain}(F) = B$. Generically for a function F with domain A and codomain B, we say that F is a function from A to B.

Example

The notation

$$F = x \in [-1, 1] \mapsto 2x + 1$$

is our compact way of writing: Define $F(x) = 2x + 1$ and $\mathrm{dom}(F) = [-1, 1]$.

Function signature

The notation $F : A \to B$ means

1. F is a function.
2. $\mathrm{dom}(F) = A$.
3. $\mathrm{codomain}(F) = B$.

We'll say that $A \to B$ is the *signature* of a function. The signature of a function doesn't tell us its formula. It does tell us the domain of a function and it indicates what the outputs of the function can be.

Range

Definition

For any function, we define

$$\operatorname{range}(F) = \{F(x) \mid x \in \operatorname{dom}(F)\}\,.$$

Thus $\operatorname{range}(F)$ is the set of all outputs.

Fact

Let F be a function. Then

$$[y \in \operatorname{range}(F)] \equiv (\exists x \in \operatorname{dom}(F))\,(y = F(x)).$$

Example

Define $F = x \in [-1, 1] \mapsto 2x + 1$. Then $\frac{3}{2} \in \operatorname{range}(F)$ because $\frac{1}{4} \in \operatorname{dom}(F)$ and $F(\frac{1}{4}) = \frac{3}{2}$.

Ontoness

The codomain of a function tells us something about its outputs, but remember that the range and the codomain of a function need not be the same. For all functions F, we have

$$\operatorname{range} F \subset \operatorname{codomain}(F).$$

Definition

A function is *onto* if its range and codomain are equal.

Example

Question: Is the sine function onto? **Answer** It is if its codomain is $[-1, 1]$. But if its codomain is $\mathbf{R}$, then no it's not onto. There is no standard value for the codomain of the trigonometric functions, so the asking "Is the sine function onto?" is rubbish.

Equality

Definition

Functions F and G are *equal* $\mathrm{dom}(F) = \mathrm{dom}(G)$ and for all $x \in \mathrm{dom}(F)$, we have $F(x) = G(x)$. Equivalently

$$(F = G) \equiv (\mathrm{dom}(F) = \mathrm{dom}(G)) \wedge (\forall x \in \mathrm{dom}(F))(F(x) = G(x)).$$

1. The definition of function equality does not involve the codomain of the function. Thus two functions can be equal, but have unequal codomains.

Example

The functions $F = x \in [-1, 1] \mapsto x \in [-1, 1]$ and $G = x \in [-1, 1] \mapsto x \in \mathbf{R}$ are equal, but F is onto and G is not onto. Thus ontoness isn't a property of a function.

Apply a function to a set

Definition

Let $F : A \to B$. For any subset A' of A define

$$F(A') = \{F(x) | x \in A'\}.$$

Equivalently, we have

$$y \in F(A') \equiv (\exists x \in A')(y = F(x)).$$

Theorem

For all functions F, we have $F(\operatorname{dom} F) = \operatorname{range}(F)$. Further $F(\varnothing) = \varnothing$.

Inverse image

Definition

Let $F : A \to B$. For any subset B' of B define

$$F^{-1}(B') = \{x \in A | F(x) \in B\}.$$

Equivalently, we have

$$x \in F^{-1}(B') \equiv F(x) \in B.$$

Theorem

Let $F : X \to Y$ and let A and B be subsets of X. Then $F(A \cap B) = F(A) \cap F(B)$.

Proof

Suppose $y \in (F(A \cap B)$; we'll show that $y \in F(A) \cap F(B)$. Since $y \in (F(A \cap B)$, there is $x \in A \cap B$ such that $y = F(x)$. But $x \in A \cap B$ implies either $x \in A$ or $x \in B$. If $x \in A$, we have $y \in F(A$; similarly if $x \in B$, we have $y \in F(B)$. So either $y \in F(A)$ or $y \in F(B)$; therefore $y \in F(A) \cap F(B)$.

Suppose $y \in F(A) \cap F(B)$. We'll show that $y \in F(A \cap B)$.