Review for Exam I

1. Show that
$$(\forall k \in \mathbf{Z}_{>1})\left(\frac{1}{k^2} \le \frac{1}{k-1} - \frac{1}{k}\right).$$

Solution: We'll write our solution as a sequence of logical equivalences. Let $k \in \mathbf{Z}_{>1}$. We have
$$\begin{aligned}\left[\frac{1}{k^2} \le \frac{1}{k-1} - \frac{1}{k}\right] &\equiv \left[\frac{1}{k^2} - \frac{1}{k-1} + \frac{1}{k} \le 0\right], && \text{(algebra)}\\ &\equiv \left[-\frac{1}{(k-1)k^2} \le 0\right], && \text{(factor)}\\ &\equiv \text{true}. && (k-1>0 \text{ and } k^2>0)\end{aligned}$$

2. Show that
$$(\forall x \in (-\infty, 1))\,(\exists r \in \mathbf{R}_{>0})\,((x-r, x+r) \subset (-\infty, 1))\,.$$

Solution: We need to choose a number r such that $x + r < 1$ and $0 < r$. Thus $0 < r < 1 - x$. One choice is $r = \frac{1-x}{2}$. Since $x < 1$, this choice does satisfy the condition $r > 0$.

Proof Let $x \in (-\infty, 1)$. Choose $r = \frac{1-x}{2}$. Since $x < 1$, it follows that $r \in \mathbf{R}_{>0}$ as required. Since $r > 0$, the condition $(x-r, x+r) \subset (-\infty, 1)$ is equivalent to $x + r < 1$. We have
$$[x+r<1] \equiv \left[x + \frac{1-x}{2} < 1\right] \equiv \left[\frac{1+x}{2} < 1\right] \equiv \left[\frac{x-1}{2} < 0\right] \equiv \left[x < \frac{1}{2}\right] \equiv \text{true}.$$

3. Let A, B be subsets of $\mathbf{R}$ and let A be bounded above. Show that $A \setminus B$ is bounded above.

Solution: Since A is bounded above, there is $M \in \mathbf{R}$ such that $(\forall x \in A)(x \le M)$. We will show that
$$(\exists M' \in \mathbf{R})(\forall x \in A \setminus B)(x \le M').$$
Choose $M' = M$. Let $x \in A \setminus B$. Then $x \in A$; thus we have
$$[x \le M'] \equiv [x \le M] \equiv \text{true}.$$

4. Give an example of subsets A, B be subsets of $\mathbf{R}$ such that $A \setminus B$ is bounded above, but A is not bounded above.

Solution: One (of many) example is $A = \mathbf{R}$ and $B = \mathbf{R}$. Then A is not bounded above, but $A \setminus B = \varnothing$, so $A \setminus B$ is bounded above (because the empty set is bounded above).

5. Define $F = x \in \mathbf{R} \mapsto x^2$. Enumerate the members of the set

$$F(\{-4,-1,0,1,4\}).$$

Solution:

$$F(\{-4,-1,0,1,4\}) = \{F(-4), F(-1), F(0), F(1), F(4)\} = \{0,1,16\}.$$

6. Define $F = x \in \mathbf{R} \mapsto x^2$. Enumerate the members of the set

$$F^{-1}(\{-4,-1,0,1,4\}).$$

Solution: We need to gather the solution sets of each of the equations $F(x) = -4$, $F(x) = 1$, $F(0) = 0$, $F(x) = 1$, and $F(x) = 4$. Thus

$$F^{-1}(\{-4,-1,0,1,4\}) = \{-2,-1,0,1,2\}.$$

7. Using the definition from the QRS, show that the sequence $k \in \mathbf{Z}_{\geq 0} \mapsto \frac{k-6}{k+28}$ converges.

Solution: We'll show that

$$(\exists L \in \mathbf{R})(\forall \varepsilon \in \mathbf{R}_{>0})(\exists n \in \mathbf{Z})(\forall k \in \mathbf{Z}_{>n})\left(\left|\frac{k-6}{k+28} - L\right| < \varepsilon\right).$$

Choose $L = 1$. Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $n = \lceil \frac{34}{\varepsilon} \rceil$. Let $k \in \mathbf{Z}_{>n}$. We have

$$\begin{aligned}
\left|\frac{k-6}{k+28} - L\right| &= \left|-\frac{34}{k+28}\right|, && \text{(substitution \& algebra)} \\
&= \frac{34}{k+28}, && \text{(absolute value properties)} \\
&< \frac{34}{n}, && (k+28 > n) \\
&= \frac{34}{\lceil \frac{34}{\varepsilon} \rceil}, && \text{(substitution)} \\
&\leq \frac{34}{\frac{34}{\varepsilon}}, && \text{(ceiling function property)} \\
&= \varepsilon. && \text{(algebra)}
\end{aligned}$$

8. Show that the sequence $n \in \mathbf{Z}_{>0} \mapsto \sum_{k=1}^{n} \frac{1}{k^2}$ is bounded above. To do this, use the fact that for all positive integers k, we have $\frac{1}{k^2} \le \frac{1}{k-1} - \frac{1}{k}$.

Solution: Let $n \in \mathbf{Z}_{>1}$. We have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^2} &= 1 + \sum_{k=2}^{n} \frac{1}{k^2}, && \text{(peel off first term of sum)} \\
&\le 1 + \sum_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k}, && \text{(given inequality)} \\
&= 1 + \left(1 - \frac{1}{n}\right), && \text{(telescoping sum)} \\
&= 2 - \frac{1}{n}, && \text{(algebra)} \\
&< 2. && \text{(algebra)}
\end{aligned}$$

9. Either show that the sequence

$$k \in \mathbf{Z} \mapsto \sin(\pi k)$$

converges or that it diverges. For either case, you're proof will must use the definition from the QRS.

Solution: Since $\sin(\pi k) = 0$ for all integers k, an alternative formula for the sequence is

$$k \in \mathbf{Z} \mapsto 0.$$

We'll show that this sequence converges to zero. Specifically, we'll show that

$$(\exists\, L \in \mathbf{R})(\forall\, \varepsilon \in \mathbf{R}_{>0})(\exists\, n \in \mathbf{Z})(\forall\, k \in \mathbf{Z}_{>n})\,(|0 - L| < \varepsilon)\,.$$

Choose $L = 0$. Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $n = 1$. Let $k \in \mathbf{Z}_{>n}$. We have

$$|0 - L| = 0 < \varepsilon.$$

10. Show that the sequence

$$k \in \mathbf{Z} \mapsto \begin{cases} k! & k < 100 \\ \frac{1}{k} & k \ge 100 \end{cases}$$

converges. You must use the definition in the QRS.

Solution: We'll show that

$$(\exists L \in \mathbf{R})(\forall \varepsilon \in \mathbf{R}_{>0})(\exists n \in \mathbf{Z})(\forall k \in \mathbf{Z}_{>n})\left(\left|L - \begin{cases} k! & k < 100 \\ \frac{1}{k} & k \geq 100 \end{cases}\right| < \varepsilon\right).$$

Choose $L = 0$. Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $n = \max(100, \lceil \frac{1}{\varepsilon} \rceil)$. Let $k \in \mathbf{Z}_{>n}$. We have

$$\begin{aligned}
\left|\begin{cases} k! & k < 100 \\ \frac{1}{k} & k \geq 100 \end{cases} - L\right| &= \frac{1}{k}, && (k > 100) \\
&< \frac{1}{n}, && (k > n) \\
&\leq \frac{1}{\lceil \frac{1}{\varepsilon} \rceil}, \\
&\leq \varepsilon.
\end{aligned}$$

11. Show that

$$(\forall a \in \mathbf{R})\,(\exists m \in \mathbf{R})\,(\forall x \in \mathbf{R})\left(x^2 - a^2 \geq m(x-a)\right).$$

Solution: We will write our proof as a sequence of logical equivalences. Let $a \in \mathbf{R}$. Choose $m = 2a$. Let $x \in \mathbf{R}$. We have

$$\begin{aligned}
\left[x^2 - a^2 \geq m(x-a)\right] &= \left[x^2 - a^2 \geq 2a(x-a)\right], && \text{(substitution for } m\text{)} \\
&= \left[x^2 - 2a(x-a) - a^2 \geq 0\right], && \text{(algebra)} \\
&= \left[x^2 - 2a + a^2 \geq 0\right], && \text{(algebra)} \\
&= \left[(x-a)^2 \geq 0\right], && \text{(factor)} \\
&= \text{true}.
\end{aligned}$$