**MATH 460** **Name:**

**Homework 3, Fall 2022**

I have neither given nor received unauthorized assistance on this assignment.

Homework 3 has questions 1 through 4 with a total of 20 points. Edit this file and append you answers using LaTeX. Be sure to fill in your name. Upload the converted pdf of your work to Canvas. This assignment is due *Saturday 10 September at 11:59* PM.

**Link to your Overleaf work:** XXX

$\boxed{5}$ 1. Show that $(\forall x \in (-1,1))\,(\exists r \in \mathbf{R}_{>0})\,((x-r, x+r) \subset (-1,1))$.

**Solution:** We need $-1 < x - r$ and $x + r < 1$; solving for $r$ we need $r < x + 1$ and $r < 1 - x$. Thus $r < \min(1+x, 1-x)$. Specifically, we'll choose $r = \frac{1}{2}\min(1+x, 1-x)$. We haven't yet checked that $r > 0$. This follows from the fact that $-1 < x < 1$. Adding one to $-1 < x$ gives $0 < x + 1$; similarly, subtracting one from $[x < 1] \equiv [-x > -1] \equiv [1 - x > 0]$. So indeed, $\frac{1}{2}\min(1+x, 1-x) > 0$

$\boxed{5}$ 2. Define $S = \{(-k, k) | k \in \mathbf{Z}_{>0}\}$. Show that $\underset{q \in S}{\cup} q = \mathbf{R}$.

**Solution:**

**Claim** $\underset{q \in S}{\cup} q \subset \mathbf{R}$.

Suppose $x \in \underset{q \in S}{\cup} q$; we'll show that $x \in \mathbf{R}$. For some $q' \in S$, we have $x \in q'$. But $q' \subset \mathbf{R}$, so $x \in \mathbf{R}$.

$\boxed{5}$ 3. On $\mathbf{Z}^2$ define the binary operators $+$ and $\times$ by

$$(a,b) + (c,d) = (a + c, b + d),$$
$$(a,b) \times (c,d) = (ac + 2bd, ad + bc).$$

These operators are commutative and associative. Additionally, the additive identity is $(0,0)$, the multiplicative identity is $(1,0)$, every member of $\mathbf{Z}^2$ has an additive identity, and multiplication distributes over addition. Given these facts, show that $(\mathbf{Z}^2, +, \times)$ is a field. The only thing left to show is that every member $\mathbf{Z}^2$ except for the additive identity has a multiplicative inverse. To prove this, you might like to use the fact that $\sqrt{2}$ is not rational.

**Solution:** We only need to show that every member $\mathbf{Z}^2$ has a multiplicative inverse. Thus given $(a,b) \in \mathbf{Z}^2_{\neq 0}$, we need to find $(x,y) \in \mathbf{Z}^2$ such that $(a,b) \times (x,y) = (1,0)$ Thus we must solve

$$\begin{aligned} ax + 2by &= 1 \\ ay + bx &= 0 \end{aligned}$$

These are linear equations for $x$ and $y$; in matrix form, the equations are

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

These equations have a unique solution provided $a^2 - 2b^2 \neq 0$ for all $(a,b) \in \mathbf{Z}^2_{\neq 0}$. Since $\sqrt{2}$ is not rational, we have $a^2 - 2b^2 \neq 0$. So the equations have a solution.

$\boxed{5}$ 4. Show that the complex field is not ordered. Hint: Suppose it is. Let $P$ be its positive set. Since $\mathrm{i} \neq 0$, either $\mathrm{i} \in P$ or $-\mathrm{i} \in P$. Show that both $\mathrm{i} \in P$ or $-\mathrm{i} \in P$ are contradictions.

**Solution:**

**Solution:** We will prove this by contradiction. Suppose the complex field is ordered, and let $P$ be its positive set. Since $\mathrm{i} \neq 0$, either $\mathrm{i} \in P$ or $-\mathrm{i} \in P$. If $\mathrm{i} \in P$, closer of $P$ under multiplication implies that $\mathrm{i}^2 \in P$ and $\mathrm{i}^4 \in P$. But this says that $-1 \in P$ and $1 \in P$. That violates tricotomy.

Similarly, the assumption $-\mathrm{i} \in P$ violates tricotomy; therefore the assumption the complex field is ordered is false.

**Fun Fact** It is possible to define $<$ on the complex field that has the properties

(a) for all $a, b \in \mathbf{C}$ exactly one of the following is true: $a < b$ or $a = b$ or $b < a$.

(b) for all $a, b, c \in \mathbf{C}$, we have $a < b$ and $b < c$ implies $a < c$.

But the set $\{z \in \mathbf{C} | 0 < z\}$ does not have the properties required by an ordered field to be a positive set.