Well ordering

"The only way to learn mathematics is to do mathematics."

Paul Halmos

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Definition

Let A be a subset of $\mathbf{R}$. We say that A is *bounded above* provided

$$(\exists M \in \mathbf{R})\,(\forall x \in A)\,(x \leq M).$$

The number M is *an upper bound for the set* A. We say that the set A is *bounded below* provided

$$(\exists M \in \mathbf{R})\,(\forall x \in A)\,(M \leq x).$$

The number M is *a lower bound for the set* A. If A is bounded below and bounded above, we say A is *bounded*.

1. If M is an upper bound for a set A, and $M \leq M'$, then M' is an upper bound for A.
2. So we need to say *a (not the) upper bound*.
3. Similarly, lower bounds are not unique.

Bounded details

1. Notice that we do *not* require that an upper bound for a set A to be a member of A.
2. Same for a lower bound.
3. Since we require that an upper bound be a *real number*, we disallow infinity from being an upper bound. If we did, every set would be bounded.
4. Although infinity is a number, it isn't a *real* number.

Bounded and unbounded examples

Example

1. The empty set is bounded above by 0.
2. Actually every real number is an upper bound for the empty set.
3. Every real number is a lower bound for the empty set.
4. The interval $[0, 1]$ is bounded above by 1.
5. The interval $[0, 1]$ is bounded above by 107.
6. The interval $[0, \infty)$ is bounded below by 0.
7. The interval $[0, \infty)$ is not bounded above.

Being least

Definition

Let A be a subset of $\mathbf{R}$. The set A has a *least member* provided

$$(\exists a^\star \in A)\,(\forall a \in A)\,(a^\star \le a).$$

We say that $a^\star$ is a least member. The set A has a *greatest member* provided

$$(\exists a^\star \in A)\,(\forall a \in A)\,(a \le a^\star).$$

1. Unlike a lower bound, we require that a least member of a set A be a member of the set.
2. The same for a greatest member.

Uniqueness of being least

Theorem

If a subset of the reals has a least member, it is unique.

Proof

Let $A \subset \mathbf{R}$. Suppose x and x' are least members of A. Since x is a least member of A we have $x \in A$. But x' is a least member, so $x' \leq x$. Interchanging the roles of x and x', we have $x \leq x'$; therefore $x = x'$.

1. Equality is hard, inequality is easier.
2. We proved equality by proving two inequalities.

Well ordering principle

Axiom

Let A be a nonempty subset of $\mathbf{Z}$ that is *bounded below*. Then A has a least member.

1. This is an axiom–we'll take it on faith.
2. Again, a least member of a set A *must* be a member of A.
3. Thus, the empty set does *not* have a least member.
4. The qualification that the set be nonempty for it to have a least member is crucial.

Theorem

Let $A \subset \mathbf{Z}$ be (i) *nonempty* and (ii) *bounded above*. Then A has a greatest member.

Well ordering principle for the reals?

Question Are the real numbers well-ordered? That is, does every nonempty subset of $\mathbf{R}$ that is bounded below have a least member?

Answer No. The interval $(0, 1)$ is nonempty and bounded below, but it doesn't have a least member. Although zero is less than every member of $(0, 1)$, since zero isn't a member of $(0, 1)$, it is not a least member.

Existence of the floor

Let $x \in \mathbf{R}_{\geq 0}$. Define the set M by $M = \{k \in \mathbf{R} \mid k \leq x\}$.

1. Since $x \geq 0$, it follows that $0 \in M$.
2. So, the set M is nonempty.
3. Further the set M is bounded above by x.
4. The well ordering principle tells us that M as a least member.
5. Actually, the least member is unique.
6. Of course the least member depends on x.
7. Something that (i) depends on x and (ii) is unique defines a function!
8. We've used the well ordering principle to define the *floor function* for nonnegative inputs.
9. Similarly, we can define the ceiling function for nonnegative inputs.
10. The putative identity $\lfloor x \rfloor = -\lceil -x \rceil$ extends the floor and ceiling functions from the nonnegative numbers to the reals.