Advanced Calculus I

MATH 460—01

Fall 2023

Instructor: Barton Willis, PhD, Professor of Mathematics
Office: Discovery Hall, Room 368
Office Hours: Monday, Wednesday, and Friday 10:00 AM—11:00 AM, Tuesday and Thursday 9:30 AM—11:00 AM, and by appointment.
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`https://github.com/barton-willis/advanced-calculus`

Important Dates

- **First Homework due** .... 26 August
- **Exam 1** .... 15 September
- **Exam 2** .... 13 October
- **Exam 3** .... 10 November
- **Final exam** .... 13 December 8:00 AM—10:00 AM

Grading

Your course grade will be based on twelve homework sets, three midterm exams, and a comprehensive final exam; specifically:

- **Weekly Homework:** *12 twenty point assignments* .... 240 (total)
- **Mid-term exams 1, 2, and 3:** *100 points each* .... 300 (total)
- **Comprehensive Final exam** .... 150 (total)

If we end the term with less than 240 points for homework, your homework point total will be scaled to 240 points.

The following table shows the *minimum* number of points (out of 690) that are required for each of the twelve letter grades D- through A+. For example, a point total of 598 points will earn you a grade of B+, and a point total of 621 points will earn you a grade of A-. If you earn a point total of 413 or less, you will a failing course grade.

D-	414	B-	552
D	437	B	575
D+	460	B+	598
C-	483	A-	621
C	506	A	644
C+	529	A+	683

Class meeting time and place

This class meets Monday, Wednesday, and Friday from 9:05 AM– 9:55 AM in Discovery Hall, room 386.

Course Resources

Our textbook is *An Introduction to Analysis*, 2nd edition, Waveland Press, Prospect Heights, Illinois, 2002 (ISBN 13: 978-1-57766-232-7) by James Kirkwood. The book by the same author and title, but published by PWS Publishing Company (Boston, 1995, ISBN 13: 0-534-94422-1) is identical.

Some homework assignments for this course will need to be typeset. To do this, you will need to create a *no cost* account on Overleaf (`https://www.overleaf.com/`). For a tutorial for using Overleaf, see `https://www.overleaf.com/tutorial`.

Course Calendar

Generally, we'll adhere to the scheduled exam dates even if we are ahead or behind with coursework. When we are ahead or behind, the topics on the exams will be appropriately adjusted.

Notices:

(a) The three midterm exams will be given on Friday of the week they are assigned.

(b) Homework (labelled **HW**) will be due one minute before midnight on Saturday of the week they are assigned.

(c) The final exam will be given on 13 December 8:00 AM—10:00 AM.

Week	Week Starting	Section(s)	Topic(s)	Assessments
1	21 August		Logic, Proof methods, and Overleaf	**HW 1**
2	28 August	§1.1 – §1.3	Sets, Functions, Real numbers, Completeness	**HW 2**
3	4 September	§2.1–§2.2	Sequences & Subsequences	**HW 3**
4	11 September	§2.1–§2.2	Sequences & Subsequences	**Exam 1**
5	18 September	§2.3	Bolzano-Weierstrass	**HW 4**
6	25 September	§3.1	Topology	**HW 5**
7	2 October	§3.1	Topology	**HW 6**
8	9 October	§4.1	Limits and Continuity	**Exam 2**
9	16 October	§4.1	Limits and Continuity	**HW 7**
10	23 October	§5.1	Derivatives	**HW 8**
11	30 October	§5.1	Derivatives	**HW 9**
12	6 November	§5.2	Some Mean Value Theorems	**Exam 3**
13	13 November	§6.1	Some Mean Value Theorems	**HW 10**
14	20 November	§6.2	The Riemann Integral	**HW 11**
15	27 November	§6.2	The Riemann Integral	**HW 12**
16	4 December	§1.1–§6.2	Catch up or review	
17	11 December			**Final Exam**

Learning Outcomes

On completion of this class, students will

(a) be able to prove basic propositions that involve the fundamentals of point set topology, including the concepts of open sets, closed sets, boundary points, and limit points.

(b) be able to prove basic propositions that involve the concept of the infimum and supremum.

(c) demonstrate competence with basic properties of sequences including determining convergence and proving results involving the sum, difference, product, and quotient of sequences.

(d) be able to use the definitions of continuity, uniform continuity, the limit, and the derivative to prove basic propositions involving these concepts as well as be able to prove facts about specific functions.

(e) demonstrate the ability to use the Mean Value Theorem to prove theorems.

(f) be able to define and evaluate the lower, upper, and general Riemann sums.

(g) demonstrate a solid understanding of the fundamental theorem of calculus.

Class Policies

Unless an assessment is *explicitly* stated to be a group project, *all work you turn in for a grade must be your own*. Each homework assignment you turn in for a grade must include the statement:

> I have neither given nor received unauthorized assistance on this assignment.

If two assignments are so similar that only collaboration could explain their similarities, both assignments will receive a grade of zero.

If you need assistance completing a homework assignment, please ask me for help. Googling for answers, asking chatGPT to do your work, seeking help from the Learning Commons or other faculty members, or using a solution key from a previous term (either from UNK or other universities) violates our class academic integrity policy.

If your homework involves concepts that are far more advanced than our textbook or class work (for example, a Hausdorff space or the BLT theorem), I will take that as evidence of using unauthorized materials and you will earn a score of zero for that assessment. Additional policies:

1. Generally, if you are ill, injured, or absent for any reason (including athletics), you must turn in your in class work on time. Permission to turn in work late must be made in advance, otherwise work turned in late will earn a score of zero.

2. During class time, please refrain from using electronic devices. If your device usage distracts your classmates, I will ask you to put it away. If it's my impression that you are often not paying attention in class, I reserve the right to decline to help you with homework assignments.

3. Using unauthorized materials or communication devices while taking a test will earn you a grade of zero on that assessment.

4. It is essential and expected for you to attend class regularly. If you miss class, please ask a classmate for class notes. You may ask me for a copy of my class notes, but I might decline–it's unlikely my notes will be of any value you.

5. For examinations and homework, show your work. *No credit will be given for multistep problems without the necessary work. Your solution must contain enough detail so that I am convinced that you could correctly solve any similar problem.* Also erase or clearly mark any work you want me to ignore; otherwise, I'll grade it.

6. The work you turn in for a grade must be *accurate, complete, concise, neat*, and *well-organized. You will not earn full credit on work that falls short of these expectations.*

7. Class cancellations due to weather, illness, or other unplanned circumstances may require that we make adjustments to the course calendar, exam dates, or due dates for course assessments.

8. I will *decline* all requests for extra credit or for redoing an assignment or examination to earn a higher grade.

9. For examinations, you may use a teacher provided quick reference sheet, but no other reference materials. You may also use a pencil, an eraser, and a scientific calculator. For examinations, your phone and all such devices must be turned off and *out of sight.*

10. The final examination will be *comprehensive* and it will be given during the time scheduled by the University. Except for *extraordinary circumstances* you must take the exam at this time.

11. If you have questions about how your work has been graded, make an appointment with me immediately.

12. Please regularly check Canvas to verify that your scores have been recorded correctly. If I made a mistake in recording one of your grades, I'll correct it provided you saved your paper.

University Policies[1]

Student Attendance Policy Statement Students are expected to attend all meetings of classes for which they are registered, including the first and last scheduled meetings and the final examination period. Instructors hold the right and responsibility to establish attendance policies for their courses. Each instructor must inform all classes at the beginning of each semester concerning their attendance policies.

Participation in official University activities, serious health concerns, personal emergencies, and religious observances are valid reasons for absence from classes. Students are responsible for informing their instructors prior to their absence(s) from class and for completing assignments missed during their absence(s). No adverse or prejudicial effects shall result to any student with a documented, excused absence.

Questions may be directed to the Dean of Student Affairs office or to Student Health & Counseling.

Academic Integrity Policy The maintenance of academic honesty and integrity is a vital concern of the University community. Any student found in violation of the standards of academic integrity may be subject to both academic and disciplinary sanctions. Academic dishonesty includes, but is not limited to, the following:

- **Cheating** Copying or attempting to copy from an academic test or examination of another student; using or attempting to use unauthorized materials, information, notes, study aids or other devices for an academic test, examination or exercise; engaging or attempting to engage the assistance of another individual in misrepresenting the academic performance of a student; or communicating information in an unauthorized manner to another person for an academic test, examination or exercise.

- **Fabrication and falsification** Falsifying or fabricating any information or citation in any academic exercise, work, speech, test or examination. Falsification is the alteration of information, while fabrication is the invention or counterfeiting of information.

- **Plagiarism** Presenting the work of another as one's own (i.e., without proper acknowledgment of the source) and submitting examinations, theses, reports, speeches, drawings, laboratory notes or other academic work in whole or in part as one's own when such work has been prepared by another person or copied from another person.

- **Abuse of academic materials and/or equipment** Destroying, defacing, stealing, or making inaccessible library or other academic resource material.

- **Complicity in academic dishonesty** Helping or attempting to help another student to commit an act of academic dishonesty.

- **Falsifying grade reports** Changing or destroying grades, scores or markings on an examination or in an instructor's records.

- **Misrepresentation to avoid academic work** Misrepresentation by fabricating an otherwise justifiable excuse such as illness, injury, accident, etc., in order to avoid or delay timely submission of academic work or to avoid or delay the taking of a test or examination.

- **Other Acts of Academic Dishonesty** Academic units and members of the faculty may prescribe and give students prior written notice of additional standards of conduct for academic honesty in a particular course, and violation of any such standard shall constitute a violation of the Code.

[1] These polices are copied from `https://catalog.unk.edu/undergraduate/academics/academic-regulations/` and from `https://www.unk.edu/academic_affairs/asa_forms/course-policies-and-resources.php`. As of 29 May 2023, these polices are current.

Under Section 2.9 of the Bylaws of the Board of Regents of the University of Nebraska, the respective colleges of the University have responsibility for addressing student conduct solely affecting the college. Just as the task of inculcating values of academic honesty resides with the faculty, the college faculty are entrusted with the discretionary authority to decide how incidents of academic dishonesty are to be resolved. For more information, please visit UNK's Procedures and Sanctions for Academic Integrity and the Student Code of Conduct.

Reporting Student Sexual Harassment, Sexual Violence or Sexual Assault Reporting allegations of rape, domestic violence, dating violence, sexual assault, sexual harassment, and stalking enables the University to promptly provide support to the impacted student(s), and to take appropriate action to prevent a recurrence of such sexual misconduct and protect the campus community. Confidentiality will be respected to the greatest degree possible. Any student who believes they may be the victim of sexual misconduct is encouraged to report to one or more of the following resources:

- Local Domestic Violence, Sexual Assault Advocacy Agency 308-237-2599
- Campus Police (or Security) 308-865-8911
- Title IX Coordinator 308-865-8655

Retaliation against the student making the report, whether by students or University employees, will not be tolerated.

Students with Disabilities It is the policy of the University of Nebraska at Kearney to provide flexible and individualized reasonable accommodation to students with documented disabilities. To receive accommodation services for a disability, students must be registered with the UNK Disabilities Services for Students (DSS) office, 175 Memorial Student Affairs Building, 308-865-8214 or by email unkdso@unk.edu

Students Who are Pregnant It is the policy of the University of Nebraska at Kearney to provide flexible and individualized reasonable accommodation to students who are pregnant. To receive accommodation services due to pregnancy, students must contact the Student Health office at 308-865-8218. The following links provide information for students and faculty regarding pregnancy rights:

- `https://thepregnantscholar.org/title-ix-basics/`
- `https://nwlc.org/resource/faq-pregnant-and-parenting-college-graduate-students-rights/`

UNK Statement of Diversity & Inclusion UNK stands in solidarity and unity with our students of color, our Latinx and international students, our LGBTQIA+ students and students from other marginalized groups in opposition to racism and prejudice in any form, wherever it may exist. It is the job of institutions of higher education, indeed their duty, to provide a haven for the safe and meaningful exchange of ideas and to support peaceful disagreement and discussion. In our classes, we strive to maintain a positive learning environment based upon open communication and mutual respect. UNK does not discriminate on the basis of race, color, national origin, age, religion, sex, gender, sexual orientation, disability or political affiliation. Respect for the diversity of our backgrounds and varied life experiences is essential to learning from our similarities as well as our differences. The following link provides resources and other information regarding D&I: `https://www.unk.edu/about/equity-access-diversity.php`.

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Blocks marked “Appointments” (yellow highlight) means *usually* available to make appointments. Blocks with gray highlights mean not available.

Fall 2022

Time	Monday	Tuesday	Wednesday	Thursday	Friday
9:00					
9:15	**Advanced Calculus I** 9:05-9:55, DSCH 386	**Not Available** 9:05-9:30	**Advanced Calculus I** 9:05-9:55, DSCH 386	**Not Available** 9:05-9:30	**Advanced Calculus I** 9:05-9:55, DSCH 386
9:30		**Office Hours** 9:30–11:00, DSCH 368		**Office Hours** 9:30–11:00, DSCH 368	
9:45					
10:00	**Office Hours** 10:00–11:00, DSCH 368		**Office Hours** 10:00–11:00, DSCH 368		**Office Hours** 10:00–11:00, DSCH 368
10:15					
10:30					
10:45					
11:00	**Lunch**	**Lunch**	**Lunch**	**Lunch**	**Lunch**
11:15					
11:30					
11:45					
12:00					
12:15	**Calculus I** 12:20-13:10, DSCH 383		**Calculus I** 12:20-13:10, DSCH 383		**Calculus I** 12:20-13:10, DSCH 383
12:30		**Calculus I** 12:30-13:20, DSCH 383		**Calculus I** 12:30-13:20, DSCH 383	
12:45					
13:00					
13:15	**Appointments** 13:15-15:00, DSCH 368		**Appointments** 13:15-15:00, DSCH 368		**Appointments** 13:15-15:00, DSCH 368
13:30					
13:45					
14:00		**Departental Meeting** 14:00-15:00		**Departental Meeting** 14:00-15:00	
14:15					
14:30					
14:45					
15:00					
15:15	**Not Available** 15:15-17:00		**Not Available** 15:15-17:00	**Not Available** 15:15-17:00	
15:30					
15:45					
16:00					
16:15					
16:30					
16:45					
17:00					

Greek characters

Name	Symbol	Typical use(s)
alpha	α	angle, constant
beta	β	angle, constant
gamma	γ	angle, constant
delta	δ	limit definition
epsilon	ϵ or ε	limit definition
theta	θ or ϑ	angle
pi	π or ϖ	circular constant
phi	ϕ or φ	angle, constant

Named sets

empty set	$\varnothing$
real numbers	$\mathbf{R}$
ordered pairs	$\mathbf{R}^2$

integers	$\mathbf{Z}$
positive integers	$\mathbf{Z}_{>0}$
positive reals	$\mathbf{R}_{>0}$

Set symbols

Meaning	Symbol
is a member	$\in$
subset	$\subset$
intersection	$\cap$

Meaning	Symbol
union	$\cup$
complement	superscript $^{\mathrm{C}}$
set minus	$\setminus$

Intervals

For numbers a and b, we define the intervals:

$$(a,b) = \{x \in \mathbf{R} \mid a < x < b\}$$
$$[a,b) = \{x \in \mathbf{R} \mid a \le x < b\}$$
$$(a,b] = \{x \in \mathbf{R} \mid a < x \le b\}$$
$$[a,b] = \{x \in \mathbf{R} \mid a \le x \le b\}$$

Logic symbols

Meaning	Symbol
negation	$\neg$
and	$\wedge$
or	$\vee$
implies	$\implies$

Meaning	Symbol
equivalent	$\equiv$
iff	$\iff$
for all	$\forall$
there exists	$\exists$

Tautologies

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$
$$(P \implies Q) \equiv (\neg Q \implies \neg P)$$
$$P \not\implies Q \equiv P \wedge \neg Q$$
$$(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P))$$
$$\neg(\forall x \in A)(P(x)) \equiv (\exists x \in A)(\neg P(x))$$
$$\neg(\exists x \in A)(P(x)) \equiv (\forall x \in A)(\neg P(x))$$

Function notation

$\mathrm{dom}(F)$	domain of function F
$\mathrm{range}(F)$	range of function F
C_A	set of continuous functions on set A
C^1_A	set of differentiable functions on set A
$A \to B$	set of functions from A to B

Generalized set operators

Each member of a set $\mathcal{C}$ is a set:

$$\bigcup_{A \in \mathcal{C}} A = \{z \mid (\exists B \in \mathcal{C})(z \in B)\}$$
$$\bigcap_{A \in \mathcal{C}} A = \{z \mid (\forall B \in \mathcal{C})(z \in B)\}$$

Theorem: $\displaystyle \bigcup_{A \in \mathcal{C}} A^{\mathrm{C}} = \left(\bigcap_{A \in \mathcal{C}} A\right)^{\mathrm{C}}$

Functions applied to sets

Let $A \subset \mathrm{dom}(F)$ and $B \subset \mathrm{range}(F)$:

$$F(A) = \{F(x) \mid x \in A\}$$
$$F^{-1}(B) = \{x \in \mathrm{dom}(F) \mid F(x) \in B\}$$

Triangle inequalities

For all $x, y \in \mathbf{R}$, we have

$$|x+y| \le |x| + |y|$$
$$\big||x| - |y|\big| \le |x - y|$$

Floor and ceiling

Definitions:

$$\lfloor x \rfloor = \max\{k \in \mathbf{Z} \mid k \le x\}$$
$$\lceil x \rceil = \min\{k \in \mathbf{Z} \mid k \ge x\}$$

Properties:

$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(x < n \iff \lfloor x \rfloor < n)$$
$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(n < x \iff n < \lceil x \rceil)$$

Bounded sets

Bounded below A set A is *bounded below* provided $(\exists M \in \mathbf{R})(\forall x \in A)(M \le x)$.

Bounded above The set A is *bounded above* provided $(\exists M \in \mathbf{R})(\forall x \in A)(x \le M)$.

Bounded A set is *bounded* if it is bounded below and bounded above.

Elementary function properties

Increasing $(\forall x, y \in A)(x < y \implies F(x) \le F(y))$. For strictly increasing, replace $F(x) \le F(y)$ with $F(x) < F(y)$.

Decreasing $(\forall x, y \in A)(x < y \implies F(x) \ge F(y))$ For strictly decreasing, replace $F(x) \ge F(y)$ with $F(x) > F(y)$.

One-to-one $(\forall x, y \in \mathrm{dom}(F))(F(x) = F(y) \implies x = y)$

Subadditive $(\forall x, y \in \mathrm{dom}(F))(F(x+y) \le F(x) + F(y))$

Bounded above $(\exists M \in \mathbf{R})(\forall x \in \mathrm{dom}(F))(F(x) \le M)$

Bounded below $(\exists M \in \mathbf{R})(\forall x \in \mathrm{dom}(F))(M \le F(x))$

Topology

Open ball $\mathrm{ball}(a,r) = \{x \in \mathbf{R} \mid -r + a < x < r + a\}$

Punctured ball $\mathrm{ball}'(a,r) = \mathrm{ball}(a,r) \setminus \{a\}$

Open set A subset A of $\mathbf{R}$ is *open* provided

$$(\forall x \in A)\,(\exists r \in \mathbf{R}_{>0})(\mathrm{ball}(x,r) \subset A)\,.$$

Closed set A subset A of $\mathbf{R}$ is *closed* provided $\mathbf{R} \setminus A$ is open.

Limit point A number a is a *limit point* of a set A provided $(\forall r \in \mathbf{R}_{>0})\big(\mathrm{ball}'(a,r) \cap A \neq \varnothing\big)$.

Boundary point A number a is a *boundary point* of a set A provided

$$(\forall r \in \mathbf{R}_{>0})\Big(\text{ball}(a,r) \cap A \neq \varnothing \wedge \text{ball}(a,r) \cap A^{\mathrm{C}} \neq \varnothing\Big).$$

Set closure $\overline{A} = A \cup \text{LP}(A)$, were $\text{LP}(A)$ is the set of limit points of A.

Open cover A set $\mathcal{C}$ is an open cover of a set A provided

(a) every member of $\mathcal{C}$ is an open set
(b) $A \subset \bigcup_{B \in \mathcal{C}} B$

Compact A set A is compact provided for every open cover $\mathcal{C}$ of A, there is a finite subset $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathcal{C}'$ is an open cover of A.

Least and greatest bounds

For any subset A of $\mathbf{R}$:

glb $z = \text{glb}(A)$ provided

(a) z is an lower bound for A
(b) if x is a lower bound for A then $x \le z$

lub $z = \text{lub}(A)$ provided
(a) z is an upper bound for A
(b) if x is a upper bound for A then $z \le x$

Sequences

Bounded A sequence F is bounded if $\text{range}(F)$ bounded.

Monotone A sequence is monotone if it either increases or decreases.

Cauchy A sequence F is Cauchy provided
(a) for every $\varepsilon \in \mathbf{R}_{>0}$
(b) there is $n \in \mathbf{Z}$
(c) such that for all $k, \ell \in \mathbf{Z}_{>n}$
(d) $|F_k - F_\ell| < \varepsilon$

Converges A sequence F converges provided
(a) there is $L \in \mathbf{R}$
(b) and $n \in \mathbf{Z}$
(c) such that for all $k \in \mathbf{Z}_{>n}$
(d) $|F_k - L| < \varepsilon$.

Functions

Continuous A function F is continuous at a provided

(a) $a \in \text{dom}(F)$ and
(b) for every $\varepsilon \in \mathbf{R}_{>0}$
(c) there is $\delta \in \mathbf{R}_{>0}$
(d) such that for all $x \in \text{ball}(a,\delta) \cap \text{dom}(F)$
(e) we have $F(x) \in \text{ball}(F(a), \epsilon)$.

Uniformly continuous A function F is uniformly continuous on a set A provided

(a) $A \subset \text{dom}(F)$; and
(b) for every $\varepsilon \in \mathbf{R}_{>0}$
(c) there is $\delta \in \mathbf{R}_{>0}$
(d) such that for all $x, y \in A$ and $|x - y| < \delta$
(e) we have $|F(x) - F(y)| < \epsilon$.

Limit A function F has a limit toward a provided

(a) a is a limit point of $\text{dom}(F)$; and
(b) there is $L \in \mathbf{R}$
(c) such that for every $\varepsilon \in \mathbf{R}_{>0}$
(d) there is $\delta \in \mathbf{R}_{>0}$
(e) such that for all $x \in \text{ball}'(a,\delta)$
(f) we have $F(x) \in \text{ball}(L, \epsilon)$.

Differentiable A function F is differentiable at a provided

(a) $a \in \text{dom}(F)$; and
(b) there is $\phi \in \text{dom}(F) \to \mathbf{R}$
(c) such that ϕ is continuous at a and
(d) $(\forall x \in \text{dom}(F))(F(x) = F(a) + (x-a)\phi(x))$.

Riemann sums

Partition A set $\mathcal{P}$ is a partition of an interval $[a,b]$ provided

(a) the set $\mathcal{P}$ is finite
(b) every member of $\mathcal{P}$ is an open interval
(c) the members of $\mathcal{P}$ are pairwise disjoint
(d) $\bigcup_{I \in \mathcal{P}} \overline{I} = [a,b]$

Let F be a bounded function on an interval $[a,b]$ and let $\mathcal{P}$ be a partition of $[a,b]$.

Lower sum $\underline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{glb}\big(F(\overline{I})\big) \times \text{length}(I)$

Upper sum $\overline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{lub}\big(F(\overline{I})\big) \times \text{length}(I)$

Riemann sum $\sum_{I \in \mathcal{P}, x^\star \in \overline{I}} F(x^\star) \times \text{length}(I)$

Axioms

Completeness Every nonempty subset A of $\mathbf{R}$ that is bounded above has a least upper bound.

Well-ordering Every nonempty set of positive integers contains a least element.

Induction $(\forall n \in \mathbf{Z}_{\ge 0})(P(n))$ if and only if $P(0) \wedge (\forall n \in \mathbf{Z}_{\ge 0})(P(n) \implies P(n+1))$.

Named theorems

Archimedean $(\forall x \in \mathbf{R})(\exists n \in \mathbf{Z})(n > x) \equiv \text{true}$.

Bolzano–Weirstrass Every bounded real valued sequence has a convergent subsequence.

Heine–Borel A subset of $\mathbf{R}$ is compact iff it is closed and bounded.

Cauchy completeness Every Cauchy sequence in $\mathbf{R}$ converges.

Monotone convergence Every bounded monotone sequence converges.

Intermediate value theorem If $F \in \mathrm{C}_{[a,b]}$, then for all $y \in [\min(F(a), F(b)), \max(F(a), F(b))]$ there is $x \in [a,b]$ such that $F(x) = y$.

Mean Value If $F \in \mathrm{C}_{[a,b]} \cap \mathrm{C}^1_{(a,b)}$, there is $\xi \in (a,b)$ such that $(b-a)F'(\xi) = F(b) - F(a)$.