1. Show that for all real x and y, we have

$$\left(\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}\right).$$

Solution: Let $x, y \in \mathbf{R}$. Starting with the inequality we want to prove, we'll write our proof as a sequence of logical equivalences. We'll end with an equality that is known to be true. We have

$$\begin{aligned}
\left[\frac{|x+y|}{1+|x+y|} \le \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}\right] &\equiv \left[\frac{|x+y|}{1+|x+y|} - \frac{|x|}{1+|x|} - \frac{|y|}{1+|y|} \le 0\right], && \text{(subtract right side)} \\
&\equiv \left[\frac{|x+y| - |x| - |y| - |x||y|(2+|x+y|)}{(1+|x+y|)(1+|x|)(1+|y|)} \le 0\right], && \text{(lots of algebra)} \\
&\equiv \text{True.} && \text{(triangle inequality)}
\end{aligned}$$

To justify the last line, we need the triangle inequality (that makes $|x+y| - |x| - |y|$ negative) along with the fact that the term $-|x||y|(2+|x+y|)$ is also negative. Finally, the denominator is positive because it's a product of positive terms–that makes the denominator positive and the numerator negative, so the quotient is negative.

If we instead attempt to prove this using a string of less than or equal equalities starting with the triangle inequality, likely we'll get stuck. Try it:

$$\begin{aligned}
\frac{|x+y|}{1+|x+y|} &\le \frac{|x|+|y|}{1+|x+y|} && \text{(triangle inequality)} \\
&\le \text{What?}
\end{aligned}$$