Guidelines for Writing Proofs
Barton Willis, PhD

> *I have made this letter longer than usual, only because I have not had the time to make it shorter.*
> Blaise Pascal

1 Write Sentences

Write proofs using sentences. Do not write proofs as sentence fragments connected by arrows.

Replace

> n, m integers $\sqrt{2} = n/m \longrightarrow 2m^2 = n^2$.

with

> Let n and m be integers. If $\sqrt{2} = n/m$, then $2\,m^2 = n^2$.

2 Format proofs as regular text

Poetry is often printed using centered lines and wide margins. Mathematics isn't poetry; don't format mathematics as if it were poetry.

Replace

Let $\varepsilon > 0$.
Choose $\delta = \varepsilon/3$.
For $|x-1| < \delta$ we have $x < 1 + \delta$.

with

> Let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. For $|x-1| < \delta$ we have $x < 1 + \delta$.

3 Do not use the first person

In a proof, do not use the first person.

Replace

> First, I consider the case $x < 0$.

with

> First, consider $x < 0$.

Outside a proof, use the first person when it is natural.

4 Use “we”

In mathematical text, “we” means the author and the reader. Use “we” instead of the passive voice when it makes the text shorter.

> **Replace** (passive voice)
>
> > It has been shown that x is prime.
>
> **with** (active voice)
>
> > We’ve shown that x is prime.
>
> Alternatively, write this as
>
> > The number x is prime.

5 Refer to theorems and axioms by name

It is usually better to write

> The completeness of the real numbers implies that ...

then it is to write

> It follows from Axiom (A9) that ...

The meaning of the first is clear, but the second won’t be clear to most readers until they look up Axiom (A9).

6 Use consistent terminology

Mathematics is full of synonyms; for example, *set* and *collection* are synonyms. In a proof, it’s best to stay with the same word to express a concept.

> **Replace**
>
> > Let A be a set and let B be a nonempty collection.
>
> **with**
>
> > Let A be a set and let B be a nonempty set.

7 Use words with clear meanings

A well-written proof can be hard to follow. Do not make it more difficult to understand by using imprecise language. Informally I might say that a quantity is “tiny.” Don’t use such language in a proof.

8 Mathematical writing should be readable

Some mathematical writing looks OK in print, but it is nonsense when read aloud. Read your writing out loud to find such mistakes.

> **Replace**
>
> > Let $k > 0$ be an integer.
>
> **with**
>
> > Let k be a positive integer.
>
> Alternatively, this can be expressed as
>
> > Let $k \in \mathbf{Z}_{\geq 0}$.

Spoken, "Let $k > 0$ be an integer" is equivalent to "Let k is greater than zero be an integer."

9 Write succinctly

Extraneous facts make it difficult for the reader to follow a logical argument. Re-read your work and make certain that each thread of reasoning is needed.

10 Include sufficient detail

The proofs you write should contain enough detail so that a classmate could follow it; however, most purely algebraic steps may be omitted.

> **Replace**
>
> > Subtracting 8 from both sides of the inequality (1) and dividing by 42, we deduce that $x < 1$.
>
> **with**
>
> > Inequality (1) implies that $x < 1$.

11 Be careful with "it"

Make sure the meaning of every pronoun is unambiguous.

> **Replace**
>
> > Let $a = 1$. To find b, add 5 to it and then multiply it by 8.
>
> **with**
>
> > Let $a = 1$. Define $b = 8(a + 5)$.

12 Define symbols before you use them

With few exceptions (π, for example), every symbol you use must be defined before, or near, the place you first use it.

> **Replace**
>
> > The volume is $\pi r^2 h$.
>
> **with**
>
> > The volume of a right circular cylindrical with height h and radius r is $\pi r^2 h$.

13 Qualify identifiers

Some symbols have more than one meaning; for example, $(0,1)$ can represent an open interval in $\mathbf{R}$, an element of $\mathbf{R}^2$, or a complex number. Do not leave it up to the reader to use conventions or context (n is an integer, z is complex number, F is a function, etc.) to guess the type of each identifier; rather, tell the reader the meaning of symbols. If F is a function, for example, immediately let the reader know:

> Let F be a function ...

Unless a proof is very long, the reader needn't be reminded that F is a function.

> **Replace**
>
> > Let F be a function defined on $\mathbf{R}$. If the function F is continuous on the open interval $(0,1)$, ...
>
> **with**
>
> > Let F be a function defined on $\mathbf{R}$. If F is continuous on the open interval $(0,1)$, ...

14 Mathematics is case–sensitive

Lower and upper case identifiers are distinct. This means, for example that a and A are different symbols.

15 Avoid unnecessary notation

Introduce notation and identifiers only when needed.

> For example, **Replace**
>
> > **Proposition** The number $x = \sqrt{2}$ is irrational.
>
> **with**
>
> > **P**roposition The number $\sqrt{2}$ is irrational.

In addition to being verbose, the first statement of the proposition has several other problems. First, $x = \sqrt{2}$ isn't a number, it's an *equation.* Second, only a number can be irrational, but the first proposition asserts that an *equation* can be irrational.

16 Don't confuse a function with its formula

If F is a function, don't write $F(x)$ when you mean F.

Replace

If $F(x)$ and $G(x)$ are continuous, $F(x)+G(x)$ is continuous.

with

If F and G are continuous, $F+G$ is continuous.

17 Use function signature notation

When a new function is first introduced, you should immediately tell the reader the *name, domain*, and *co-domain* of a function by giving its signature.[1]

Replace

Let $F(x,y)$ be a real-valued function.

with

Let $F:\mathbf{R}^2 \to \mathbf{R}$.

Writing $F(x,y)$ to *imply* that F is defined on $\mathbf{R}^2$ is a poor substitute for a clear statement of the function signature. If the exact domain of a function isn't important, use the notation

Let $F:\subset \mathbf{R}^2 \to \mathbf{R}$.

This tells the reader that the domain of F is a subset of $\mathbf{R}^2$, but it doesn't give its actual domain.

18 Don't confuse "let" with "therefore"

To introduce a new object, use "let." Use "therefore" to state a logical consequence of the facts that precede it.

Replace

Let n be an integer. Let $n(n+1)$ be even.

with

Let n be an integer. Therefore $n(n+1)$ is even.

Alternatives to "therefore" are "so" and "thus."

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[1] The terminology *function signature* isn't standard.

19 Omit empty words and phrases

It doesn't help the reader to say that something is "easy to show" or "obvious." Omit or replace the empty words and phrases listed in the following table.

Word or Phrase	**Replacement**
It is trivial to show	(omit)
It is easy to show	(omit)
Clearly	(omit)
Obviously	(omit)
It immediately follows	It follows
For the purpose of contradiction	Suppose
By definition	Thus

Also, expunge weasel words from your text. For a list of weasel words, see, for example, `http://en.wikipedia.org/wiki/Weasel_word`.

20 Do not be wishy-washy

A proof is no place to be indefinite. Do not use phrases similar to "I believe," "should be" or "I think that."

21 Use related theorems

A short proof that uses several powerful theorems is more likely to be correct than is a long proof that starts from scratch. Whenever possible, base your proofs on propositions that have been proved. Of course, using a result that we have not yet proved in class is not allowed.

22 Let the reader know where the argument is heading

It often helps the reader to know in advance where a proof is heading. Depending on the audience, it may be better to write

> Let $x \in A$. To show that $A \subset B$, we'll show that $x \in B$. Since, ... Thus $x \in B$; therefore $A \subset B$.

than to write

> Let $x \in A$. Since, ... Thus $x \in B$; therefore $A \subset B$.

23 Use idioms

Some patterns enter into proofs so frequently that we express them using an idiom.[2] Think of an idiom as a template for a proof. When a proof conforms to a template, the reader has a good idea of what will follow. This can make a proof easier to read. We'll learn a few templates for proofs.

24 Use correct spelling, punctuation, and grammar

Misspelled words, incorrect punctuation, and poor grammar distract the reader. Always carefully check your work for these errors. Additionally, displayed formulas should be punctuated as regular text. For example,

> Repeated use of the double angle formula gives
>
> $$\sin^4(x) = \frac{\cos(4x)}{8} - \frac{\cos(2x)}{2} + \frac{3}{8}.$$

The displayed formula ends the sentence; thus it terminates with a period.

[2] Idiom (noun): A style or manner of expression peculiar to a given people. (From dictionary.com)

25 Begin each sentence with a word

A sentence that starts with a symbol is difficult to read; every sentence must begin with a word, not a symbol.

> **Replace**
>
> A is continuous on $\mathbf{R}$; we have ...
>
> **with**
>
> Since A is continuous on $\mathbf{R}$, we have ...

26 Place a word between adjacent formulas

When one formula follows another, place a word in between the formulas.

> **Replace**
>
> When $u > n$ $k < 0$.
>
> **with**
>
> When $u > n$, we also have $k < 0$.

27 Don't over use the colon

Do not use a colon immediately before a formula when the formula completes the sentence.

> **Replace**
>
> For all integers n, the number: $n(n+1)$ is even.
>
> **with**
>
> For all integers n, the number $n(n+1)$ is even.

If the phrase that follows a colon is a complete sentence, capitalize the first word following a colon; otherwise, do not capitalize the first word following a colon.

28 Don't use unnecessary commas

Do not surround an appositive with commas.

> **Replace**
>
> The revenue, R, is an increasing function of price, p.
>
> **with**
>
> The revenue R is an increasing function of price p.

Further when a sentence starts with "further," "therefore," or "thus," a comma isn't needed after the first word.

29 If there is a "first," there must be a "second"

If you enumerate ideas using starting with "first," you must identify the remaining ideas using second, third, etc.

Replace

First, we'll show that $x \leq 0$. Next, we'll show that $x \geq 0$.

with

First, we'll show that $x \leq 0$. Second, we'll show that $x \geq 0$.

30 Capitalize proper names and theorems

Capitalize the words *axiom, definition, proposition*, and *theorem* when they refer to something specific; otherwise, do not capitalize them.

Replace

Today, Mary proved theorem 1. Yesterday, she proved two Theorems.

with

Today, Mary proved Theorem 1. Yesterday, she proved two theorems.

Also capitalize proper names; for example, *the Euler constant.*

31 Don't confuse it's and its

The word "it's" always means it is. Don't confuse it's with its.[3]

Replace

Memorize Definition 1-1; its the most important thing you will learn this term.

with

Memorize Definition 1-1; it's the most important thing you will learn this term.

32 Do scratch work

By all means, if it helps you construct a proof, draw pictures and diagrams filled with lines and arrows. But do not include your scratch work in the final copy.

33 Proofread your work

When you have finished a proof, put it aside for a while. After that, proofread it several more times. Imagine that it is someone else's work. Does it still make sense? Read your proof one line at a time (cover up all the other lines). Finally, try reading your work out loud. Mistakes that are easy to skip often manifest when spoken.

[3] I suppose you could name your dog "It." Then "It's doghouse has two stories." is correct, and "It's" does not mean "It is."

34 Check that you used every hypothesis

After your first draft, check that you have used every hypothesis. If you did not, it doesn't mean that your proof is wrong. But it does mean that it is suspect. Generally mathematicians state theorems without unnecessary hypotheses.

35 Prove, show, and demonstrate all mean prove

If you are asked to show that something is true, you need to supply a proof. The words prove, show, and demonstrate all mean the same thing.

36 Prove what is given–not a special case

It's not fair to append additional conditions to a proposition and then use these additional conditions to prove the proposition. Put differently, proving a *special case* of a proposition does not prove the proposition.

Replace

Proposition Let A and B be sets. Show that $A \subset A \cup B$.

Proof Let $A = \{2,3,4\}$ and $B = \{3,4,5\}$. So $A \cup B = \{2,3,4,5\}$. Since $\{2,3,4\} \subset \{2,3,4,5\}$, we've shown that $A \subset A \cup B$

with

Proof If $x \in A$, we have $x \in A \cup B$; therefore $A \subset A \cup B$.

37 Prove the conclusion, not the hypothesis

Make certain that the logic of your proof flows from the hypothesis to the conclusion. I've seen proofs that start with the conclusion and end with either the same statement or with one or more of the hypothesis.

38 If objects aren't the same type, they aren't equal.

In this class, we mostly work with *functions, sets,* or *numbers.* A set can never equal a number and a number can never equal a function. Carefully check your work and make sure that you have equality written only between objects with the same type.

The *range* of a function is a *set*; it is not a number. If your proof contains

$\text{range}(F) = 1,$

possibly you need to replace it with

$\text{range}(F) = \{1\}$

or with

$1 \in \text{range}(F)$

Of course, equality shouldn't be written between things with the same type that aren't equal; however, this isn't the point that is being made here.

39 The symbol "=" means equal

Use "=" between things that are equal. Do not use "=" to mean "the next step is" or "implies."

Replace

$$F(x) = x^2 = \frac{d}{dx}(x^2) = 2x = F'(x)$$

with

$$F'(x) = (x^2)' = 2x.$$

Also, do not practice what I call non–committal math. This is when mathematics is written as a stream of nouns without any connections between them.

Replace

$$x(x+1) \quad (x(x+1))' \quad (x)'(x+1)+x(x+1)' \quad x+1+x \quad 2x+1$$

with

$$(x(x+1))' = (x)'(x+1)+x(x+1)' = x+1+x = 2x+1$$

Finally, do not use an arrow when you mean equality.

Replace

$$(x+1)^2 \to x^2+2x+1$$

with

$$(x+1)^2 = x^2+2x+1$$

40 Avoid arrows

The double arrow $\Rightarrow$ means *therefore*, not equal. When you want to express the fact that expressions are *equal*, use the equal sign, not the double arrow. Although we do not generally use symbols for logical connectives, a correct usage of the double arrow is: *Chocolate is made from the bean of the cacao tree* $\Rightarrow$ *Chocolate contains caffeine*. An *incorrect* usage is: $\left(1+x^2\right)' \Rightarrow 2x$. A single arrow (either left or right pointing) generally means "**Replace** by." A correct usage of the single arrow is: *Substituting* $x \to 5$ *in* $y = 6+x$ *yields* $y = 11$. An *incorrect* usage is: $x(1-x) = 0 \to x = 0, 1$.

41 Check for dangling silent variables

Nothing should depend in any significant way on a silent variable. If your work has a silent variable, make sure it is qualified in some way.

In the following i is a silent variable. In the last sentence, the silent variable is unqualified:

If $x \in \left(\bigcap_{i\in I} A_i\right)^C$, then $x \in \mathscr{U}$ and $x \notin \bigcap_{i\in I} A_i$. So $x \notin A_i$.

Replace this with

If $x \in \left(\bigcap_{i\in I} A_i\right)^C$, then $x \in \mathscr{U}$ and $x \notin \bigcap_{i\in I} A_i$. So $x \notin A_i$ for some $i \in I$.

42 A statement must follow "such that"

What follows "such that" must be a statement (something that must either be true or false). In

> Let A and B be sets such that $A \cap B$.

A noun $A \cap B$ follows the "such that"; it's not clear what the writer intended. One possibility is

> Let A and B be sets such that $A \cap B$ is nonempty.

43 Say what you mean

Consider

> If $x \in A \cap B$, then $x \in A$ and $x \in B$. So $x \in A \cap B \subset A$.

Spoken the last sentence is *So x is a member of A intersect B is a subset of A.* To fix this, end the proof with *So* $x \in A$.

1 Proof idioms

1 The let-choose idiom

To show that the quantified statement such as $(\exists x_o \in \mathbf{R})\,(\forall x > x_o)\,(\exists M \in \mathbf{R})\,\big(|7 + 5x| \le M x^2\big)$, use the let-choose idiom. For each $\forall$, use the word 'let,' and for each $\exists$ use the word 'choose.' For example

Proof Choose $x_o = 1$ and let $x > 1$. Choose $M = 12$. For all $x > 1$, we have

$$\begin{aligned}
|7 + 5x| &\le |7| + 5|x|, && \text{(triangle inequality)}\\
&\le 7x + 5x, && \text{(using } x > 1)\\
&= 12x, && \text{(arithmetic)}\\
&\le 12x^2, && \text{(using } x^2 > x)\\
&= Mx^2. && \text{(substitution)}
\end{aligned}$$

2 The one–bad–apple idiom

You can show that a proposition is *false* by displaying just one example that shows that it is false. You don't need two examples or infinitely many examples; just one "bad apple" is enough.

3 The pick-and-show idiom

Anytime you need to show one set is a subset of another, you should use the "pick-and-show" idiom; it looks like this

> **P**roposition Let A and B be sets and suppose $H_1, H_2, \ldots,$ and H_n. Then $A \subset B$.
>
> **P**roof If $x \in A$, we have (deductions made using the facts H_1 through H_n); therefore $x \in B$.

Here, the statements H_1 through H_n are the hypothesis of the proposition. To demonstrate set equality, use the pick-and-show idiom twice. Here is and example of using pick-and-show.

> **P**roposition Let A and B be nonempty sets and suppose $A \times B = B \times A$. Then $A = B$.

The conclusion of the proposition is $A = B$; we need to use the pick-and-show idiom twice. The proof starts with

> **P**roof First we show that $A \subset B$. If $a \in A$, we have ....

We need a consequence of $a \in A$ that somehow involves the hypothesis $A \times B = B \times A$. Since B is nonempty, it has an element b. Thus we have $(a, b) \in A \times B$. It's downhill from here. For our proof, it might be best to explain that B has an element and give it a name before we start the pick-and-show idiom. Here's a proof.

> **P**roof First we show that $A \subset B$. Since B is nonempty, it has at least one element, call it b. If $x \in A$, we have $(x, b) \in A \times B$. But $A \times B = B \times A$; thus $(x, b) \in B \times A$. Therefore $x \in B$; consequently $A \subset B$.
>
> Second we show that $B \subset C$. Since A is nonempty, it has at least one element, call it a. If $x \in B$, we have $(a, x) \in A \times B$. But $A \times B = B \times A$; thus $(a, x) \in B \times A$. Therefore $x \in A$; consequently $A \subset B$.

It's tempting to write the proof as

> **P**roof If $(a, b) \in A \times B$, we have $(a, b) \in B \times A$. Thus $a \in B$ and $b \in A$; therefore $A \subset B$ and $B \subset A$.

What's wrong with this? Plenty: First it uses pick-and-show on $A \times B$ and $B \times A$; it should use pick-and-show on A and B. Second, the proof never makes any use of the hypothesis that A and B are nonempty; this doesn't mean that the proof is wrong, just highly suspect. Third, and most importantly, the proof is logically flawed. To see the flaw, look at it carefully and decide what has really been proved. Since we've used pick-and-show on $A \times B$, we're aiming toward a proof that $A \times B = B \times A$ implies $B \times A \subset A \times B$. We don't need to do any work to conclude that! (For any set A, we have $A \times \varnothing = \varnothing$. Think about that.)

2 How to get started

Students often tell me that "I know how to prove it, but I can't get started." Here are a few suggestions on how to start.

1 Know the definitions

Have you ever tried to translate an article written in a foreign language by looking up the definition of each word? If you have, you'll agree that the method doesn't work. If you don't thoroughly understand the definition of every word in the statement of a theorem, you won't be able to prove it. When you thoroughly understand a mathematical definition, you should be able to:

(a) write a definition without peeking at the textbook,
(b) give examples (and supply proofs) of things that satisfy the definition,
(c) give examples (and supply proofs) of things that don't satisfy the definition.

2 Use an idiom

Does the proof follow a standard pattern? If it does, look at a proof that follows the same pattern and try to use the template.

3 Look at related theorems

Look at all theorems that have been previously proved that involve one or more of the hypothesis of the theorem you are trying to prove. These related theorems might help you prove your theorem.

4 Checked that you have used every hypothesis

If you are stuck part way through a proof, check that you have used *every* hypothesis. If you have ignored one fact, it's likely that it holds the key to finishing the proof.

5 Try to disprove the proposition

Sometimes if you try to disprove a proposition, you'll discover why it is true.

6 Take a walk

It's easy to get flustered or stuck on a bad idea. When that happens, put the work away and do some else for a while. When you return, you may be unstuck.

Further Reading

(1) Leslie Lamport, "How to Write a Proof," American Mathematical Monthly, **1**02 (1993) 600-608.

(2) Donald Knuth, Tracy Larrabee, and Paul Roberts, *Mathematical Writing,* MAA Notes Series, 1989.

(3) Leonard Gillman, *Writing Mathematics Well,* Mathematical Association of America, 1987.

(4) Ashley Reiter, "Writing a Research Paper in Mathematics," `web.mit.edu/jrickert/www/mathadvice.html`.

(5) `www.mit.edu/afs/athena.mit.edu/course/other/mathp2/www/piil.html`.

(6) For an introduction to formal proofs, see `en.wikipedia.org/wiki/Proof_theory` and `www.economist.com/s`