1. A function F is increasing on its domain provided

$$\big(\forall x, y \in \operatorname{dom}(F)\big)\big(x < y \implies F(x) \le F(y)\big).$$

$\boxed{5}$ (a) Without using negation (the symbol $\neg$), write the negation of

$$\big(\forall x, y \in \operatorname{dom}(F)\big)\big(x < y \implies F(x) \le F(y)\big).$$

in symbolic form. For assistance with the logic, see the section “Tautologies” in our class QRS.

Solution:

$$\big(\exists x, y \in \operatorname{dom}(F)\big)\big((x < y) \wedge (F(x) > F(y))\big).$$

Alternatively, a solution is

$$\big(\exists x, y \in \operatorname{dom}(F)\big)\big(x < y \not\Longrightarrow F(x) \le F(y)\big).$$

But for most people, the meaning of $(x < y) \wedge (F(x) > F(y))$ is clear and the meaning of $(x < y) \not\Longrightarrow (F(x) \le F(y))$ is less clear.

$\boxed{5}$ (b) Show that the function $x \in [-1, 1] \mapsto |x|$ is not increasing on its domain.

Solution: We'll show that

$$\big(\exists x, y \in [-1, 1]\big)\big((x < y) \wedge (|x| > |y|)\big).$$

Choose $x = -1$ and $y = 0$. We have

$$\big[(x < y) \wedge (|x| > |y|)\big] \equiv [(-1 < 0) \wedge (|-1| > |0|)] \equiv \text{True}.$$

There are infinitely many choices for x and y that yield a proof. We only need one–don't burden the reader with more than one choice.

2. A function F is subadditive on its domain provided

$$\big(\forall x, y \in \operatorname{dom}(F)\big)\big(F(x + y) \le F(x) + F(y)\big).$$

$\boxed{5}$ (a) Without using negation (the symbol $\neg$), write the negation of

$$\big(\forall x, y \in \operatorname{dom}(F)\big)\big(F(x + y) \le F(x) + F(y)\big)$$

in symbolic form.

Solution:

$$\big(\exists x, y \in \operatorname{dom}(F)\big)\big(F(x + y) > F(x) + F(y)\big).$$

$\boxed{5}$ (b) Show that the function $x \in \mathbf{R} \mapsto x^2$ is not subadditive on its domain.

Solution: We'll show that

$$\big(\exists x, y \in \mathbf{R}\big)\big((x + y)^2 > x^2 + y^2\big).$$

Choose $x = 1$ and $y = 1$. We have

$$\big[(x + y)^2 > x^2 + y^2\big] \equiv [4 > 1 + 1] \equiv \text{True}.$$

Again, there are many choices for x and y, but we need only once choice.

$\boxed{5}$ (c) Show that the function $x \in \mathbf{R} \mapsto |x|$ is subadditive on its domain. To do this, you may use the triangle inequality without proving it.

Solution: We'll show that

$$\big(\forall x, y \in \mathbf{R}\big)\big(|x+y| \leq |x| + |y|\big).$$

This is the triangle inequality, a fact we were allowed to use without proof.

The familiar triangle inequality expresses the fact that the absolute value function is subadditive.