Advanced Calculus, Fall 2023 **Name:**______________________________

Practice Exam 3 **Row and Seat:**________________________

$\boxed{10}$ 1. Define the *derivative* as the limit of a *Newton quotient*.

> **Solution:** Let $F \in \mathbf{R} \to \mathbf{R}$. We say a function F is *differentiable at a* provided (i) $a \in \operatorname{dom}(F)$, (ii) $a \in \operatorname{LP}(\operatorname{dom}(F))$, (iii) and $\lim\limits_{x \to a} \dfrac{F(x) - F(a)}{x - a}$ is a real number.

$\boxed{10}$ 2. Show that the function $x \in (-\infty, 0) \cup 5 \mapsto x$ is not differentiable at 5.

> **Solution:** Since $5 \notin \operatorname{LP}((-\infty, 0) \cup 5)$, the given function isn't differentiable at 5.

$\boxed{10}$ 3. Use the definition you gave in question 1 to find the derivative of $x \in \mathbf{R} \mapsto x^2 + x$ at 2.

> **Solution:** We have
>
> $$\begin{aligned} \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} &= \lim_{x \to 2} \frac{(x-2)(x+3)}{x - 2}, \\ &= \lim_{x \to 2} (x + 3), \\ &= 5. \end{aligned}$$

$\boxed{10}$ 4. Use the definition you gave in question 1 to find the derivative of $x \in \mathbf{R} \mapsto x^2 + x$ at a, where a is any real number.

$\boxed{10}$ 5. Use the definition you gave in question 1 to find the derivative of $x \in \mathbf{R} \mapsto \sqrt{x}$ at 3.

$\boxed{10}$ 6. Use the QRS definition of uniformly continuity to show that that $x \in [-1, 1] \mapsto x^2$ is *uniformly continuous* on its domain.

Solution:

Proof. Let ε be a positive number, and let $x, y \in [-1,1]$. If $|x-y| < \varepsilon/2$, we have

$$\begin{aligned}
|x^2 - y^2| &= |x-y||x+y|, \\
&< \frac{\varepsilon}{2}\left(|x| + |y|\right), \\
&= \frac{\varepsilon}{2}(1+1), \\
&= \varepsilon.
\end{aligned}$$

$\square$

$\boxed{10}$ 7. Use the undefinition of uniformly continuity to show that the function $x \in \mathbf{R} \mapsto 8x^2$ is not uniformly continuous on its domain.

Solution:

Proof. For every positive number δ we'll find $x, y \in \mathbf{R}$ such that $|x-y| < \delta$ yet $|8x^2 - 8y^2| > 1/2$.

Choose $x = \frac{1}{8\delta} - \frac{\delta}{4}$ and $y = \frac{1}{8\delta} + \frac{\delta}{4}$. Then $|x-y| = \dfrac{\delta}{2} < \delta$. Further

$$\begin{aligned}
|8x^2 - 8y^2| &= 8|x-y||x+y|, \\
&< 8 \times \frac{\delta}{2} \times \frac{1}{4\delta}, \\
&= 1, \\
&> \frac{1}{2}.
\end{aligned}$$

$\square$

$\boxed{10}$ 8. Show that the function $x \in \mathbf{R} \mapsto \begin{cases} x\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is *continuous* at 0. You may use the fact that $|\cos(x)| \leq 1$ for all real x without proving it.

Solution:

Proof. Let F be the given function, and let ε be a positive number. For $x \in \operatorname{ball}'(0,\varepsilon)$ we have

$$|F(x) - F(0)| = |x\cos(1/x) - 0| = |x||\cos(1/x)| \le |x| < \varepsilon.$$

Also, $|F(0) - F(0)| = 0 < \varepsilon$. So for all $x \in \operatorname{ball}(0,\varepsilon)$ we have $|F(x) - F(0)| < \varepsilon$. □

10 9. Show that the function $x \in \mathbf{R} \mapsto \begin{cases} x^2\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at 0. Depending on your method, the result of the previous question might be useful.

Solution:

Proof. Define functions F and G by $F(x) = \begin{cases} x^2\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ and $G(x) = \begin{cases} x\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$. The function G is continuous at 0; further we have $F(x) = F(0) + (x-0)G(x)$. This shows that F is differentiable at 0. □

10 10. Let $F : \mathbf{R} \to \mathbf{R}$ be continuous at a. If $F(a) > 0$, show that there is a positive number δ such that $F(x) > 0$ for all $x \in \operatorname{ball}(a,\delta) \cap \operatorname{dom}(F)$.

10 11. Show that the function $x \in \mathbf{R}_{>0} \mapsto \frac{1}{x}$ is not uniformly continuous on its domain.

10 12. Let $F : \mathbf{R} \to \mathbf{R}$ be differentiable at a and suppose $F'(a) > 0$. Is it true that F is increasing on a neighborhood of a? If so, prove it.

10 13. Give an example of a function $F : [-1,1] \to \mathbf{R}$ such that $\sup\big(\operatorname{range}(F)\big) \notin \operatorname{range}(F)$.

Solution: $F = x \in [-1,1] \mapsto \begin{cases} x & x \neq 1 \\ 0 & x = 1 \end{cases}.$

$\boxed{10}$ 14. Give an example of a function $F : (-1,1) \to \mathbf{R}$ such that $\sup\big(\operatorname{range}(F)\big) \notin \operatorname{range}(F)$ and F is continuous on $(-1,1)$.

Solution: $F = x \in (-1,1) \mapsto x$.

15. Let $F : \mathbf{R} \to \mathbf{R}$ satisfy the inequality $|F(x) - F(y)| \leq |x - y|$ for all $x, y \in \mathbf{R}$.

$\boxed{10}$ (a) Show that F is *continuous at zero.*

$\boxed{10}$ (b) Show that F is *uniformly continuous* on $\mathbf{R}$.

$\boxed{10}$ 16. Show that $x \in \mathbf{R} \mapsto x^3$ is continuous at 10.

Solution:

Proof. Let ε be a positive number. Define $\delta = \min\left(1, \frac{\varepsilon}{331}\right)$. For $x \in \operatorname{ball}(10;\delta)$, we have $9 \leq x \leq 11$. Further, we have

$$\begin{aligned}
|x^3 - 10^3| &= |x-10||x^2 + 10x + 100|, \\
&< |x-10|\left(|x|^2 + 10|x| + 100\right), \\
&\leq |x-10|\left(11^2 + 110 + 100\right), \\
&= 331\,|x-10|, \\
&\leq \varepsilon.
\end{aligned}$$

□

$\boxed{10}$ 17. Show that the function with signature $F : \mathbf{R} \to \mathbf{R}$ and formula $F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

Solution:

Proof. We'll show that for every positive number δ there is $x \in \text{ball}(0;\delta)$ such that $|F(x) - F(0)| > \frac{1}{2}$. Let δ be a positive real number. There is an integer n such that $0 < \dfrac{2}{(n+1)\pi} < \delta$. Further we have

$$\left| F\left(\frac{2}{(n+1)\pi}\right) - F(0) \right| = |1 - 0| > \frac{1}{2}.$$

□

$\boxed{10}$ 18. Either prove or disprove: Let $F, G : \mathbf{R} \to \mathbf{R}$, and let $a \in \text{dom}(FG)$. If FG is continuous at a, then both F and G are continuous at a.

Solution: The statement is false. Let $F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ and let $G(x) = x$. In question 1, we showed that FG is continuous at 0, but in question 17, we showed that F is *not* continuous at 0.

$\boxed{10}$ 19. Let $F : \mathbf{R} \to \mathbf{R}$ be continuous at a. Show that $|F|$ is continuous at a.

Solution:

Proof. Let ε be a positive number. Since F is continuous at a, there is a positive number δ such that for all $x \in \text{ball}(a, \delta) \cap \text{dom}(F)$, we have

$$|F(x) - F(a)| < \varepsilon.$$

For all $x \in \text{ball}(a, \delta) \cap \text{dom}(F)$, we have

$$\big||F(x)| - |F(a)|\big| \leq |F(x) - F(a)| < \varepsilon.$$

Therefore $|F|$ is continuous at a. □

$\boxed{10}$ 20. Use the inequality $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$, for $a, b > 0$ to show that the square root function is uniformly continuous on $[0, \infty)$.

Solution:

Proof. Let ε be a positive number. For $x, y \in [0, \infty)$ with $|x-y| < \varepsilon^2$, we have $\left|\sqrt{x}-\sqrt{y}\right| \le \sqrt{|x-y|} < \varepsilon$. $\square$

$\boxed{10}$ 21. Show that the function $x \in \mathbf{R} \mapsto x^2$ is not uniformly continuous on $\mathbf{R}$.

Solution: Let δ be a positive number. Define $x = \frac{1}{\delta} + \frac{\delta}{4}$ and $y = \frac{1}{\delta} - \frac{\delta}{4}$. Then $|x-y| < \delta$; however,

$$|x^2 - y^2| = |x-y||x+y| = \frac{\delta}{2}\frac{2}{\delta} = 1.$$

The definition of uniform continuity: Let $F : E \to \mathbf{R}$. We say F is *uniformly* continuous on E provided

(a) For *every* $\varepsilon > 0$

(b) *there is* a $\delta > 0$

(c) for *all* $x, y \in \mathrm{dom}(F)$ such that $|x-y| < \delta$

(d) we have $|F(x) - F(y)| < \varepsilon$.

And the "undefinition" of uniform continuity:

(a) For *some* $\varepsilon > 0$

(b) and *every* $\delta > 0$

(c) *there are* $x, y \in \mathrm{dom}(F)$ and $|x-y| < \delta$

(d) such that $|F(x) - F(y)| > \varepsilon$.

$\boxed{10}$ 22. Show that $x \in \mathbf{R} \mapsto x^2|x|$ is differentiable at 0. (The absolute value function isn't differentiable at 0, so the product rule *isn't* an option!)

$\boxed{10}$ 23. Use the MVT to show that for all $x, y \in \mathbf{R}$, we have $|\cos(x) - \cos(y)| \le |x-y|$. You may use the facts (i) $\cos' = \sin$ and (ii) $|\sin(x)| \le 1$ for all real x.