MATH 460 **Name:**
Homework 5, Fall 2022

I have neither given nor received unauthorized assistance on this assignment.

Homework 5 has questions 1 through 3 with a total of 30 points. Neatly **handwrite your solutions**, digitize your work, and turn it into Canvas. You do **not** need to use LaTeX for this assignment. This work is due *Saturday 1 October at 11:59* PM.

5 1. Show that $(\forall x \in \mathbf{R})\left(\max(0,x) = \frac{x+|x|}{2}\right)$. Likely you will want to consider the cases $x < 0$ and $x \ge 0$ separately.

Solution: We'll consider the cases $x < 0$ and $x \ge 0$ separately. For $x < 0$, we have

$$\begin{aligned} \frac{x+|x|}{2} &= \frac{x-x}{2}, && \text{(simplify absolute value)} \\ &= 0, && \text{(algebra)} \\ &= \max(0,x). && (x < 0) \end{aligned}$$

For $x \ge 0$, we have

$$\begin{aligned} \frac{x+|x|}{2} &= \frac{x+x}{2}, && \text{(simplify absolute value)} \\ &= x, && \text{(algebra)} \\ &= \max(0,x). && (x \ge 0) \end{aligned}$$

5 2. Let F be a convergent sequence. Show that the sequence $k \in \mathbf{Z}_{\ge 0} \mapsto \max(F_k, 0)$ converges. **Hint:** Use the result of Question 1.

Solution: Let's define $G = k \in \mathbf{Z}_{\ge 0} \mapsto \max(F_k, 0)$. Alternatively $G = \frac{F+|F|}{2}$. Since F converges, so does $|F|$. That makes G a linear combination of convergent sequences, so G converges.

3. Define $H = n \in \mathbf{Z}_{>0} \mapsto \displaystyle\sum_{k=1}^{n} \frac{1}{k}$.

5 (a) Use the fact that $(\forall x \in \mathbf{R}_{\ge 1})\left(\ln(x+1) - \ln(x) \le \dfrac{1}{x}\right)$ to show that the sequence H is not bounded above (and consequently does not converge). To do this, you will use some standard calculus facts about telescoping sums and about the natural

logarithm. One fact that you might use is the fact that the natural logarithm is not bounded above.

Solution: Let $n \in \mathbf{Z}_{>0}$. We have

$$\begin{aligned} H_n &= \sum_{k=1}^{n} \frac{1}{k}, && \text{(definiton)} \\ &\geq \sum_{k=1}^{n} \ln(k+1) - \ln(k), && \text{(given fact)} \\ &= \ln(n+1) - \ln(1), && \text{(telescoping sum)} \\ &= \ln(n+1). && \text{(simplifcation)} \end{aligned}$$

Since the natural logarithm isn't bounded above, neither is the sequence H. Thus H diverges.

5 (b) Show that $(\forall\, \varepsilon \in \mathbf{R}_{>0})\, (\exists\, n \in \mathbf{Z})\, (\forall\, k \in \mathbf{Z}_{>n})\, (|H_{k+1} - H_k| < \varepsilon)$.

Solution: Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $n = \lceil \frac{1}{\varepsilon} \rceil$. Thus $n \in \mathbf{Z}$ as required. Let $k \in \mathbf{Z}_{>n}$. We have

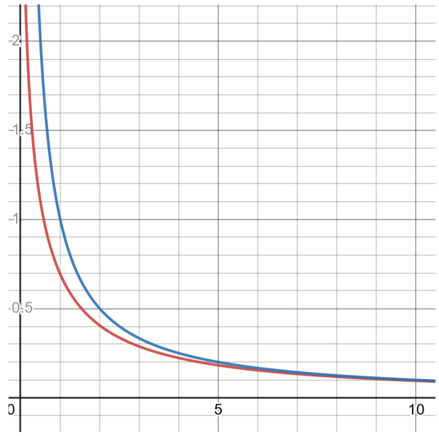
$$|H_{k+1} - H_k| = |\frac{1}{k+1}| < \frac{1}{n} = \frac{1}{\lceil \frac{1}{\varepsilon} \rceil} \leq \frac{1}{\frac{1}{\varepsilon}}. = \varepsilon.$$

5 (c) Show that the sequence H is not Cauchy.

Solution: Since H diverges, it is not Cauchy.

5 (d) Draw a graph that shows that the fact $(\forall x \in \mathbf{R}_{\geq 1}) \left(\ln(x+1) - \ln(x) \leq \frac{1}{x} \right)$ is due to the fact that the natural logarithm function is concave down. **Hint:** For the graph $y = \ln(x)$ and a positive number k, show the tangent line at the point $(x = k, y = \ln(k))$ and the secant line through the points $(x = k, y = \ln(k))$ and $(x = k+1, y = \ln(k+1))$.

Solution: It's not what I had in mind, but one graphical method is to graph both $y = \ln(x+1) - \ln(x)$ and $y = 1/x$ together and compare them. Here the red graph is $y = \ln(x+1) - \ln(x)$ and the blue graph is $y = 1/x$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