# What your calculus textbook doesn't tell you about trigonometric substitution

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**Abstract** The method of trigonometric substitution for indefinite integration can give impressive looking results, but these results are sometimes poorly suited for numerical evaluation.

After making multiple bad starts, wasting ten sheets of engineering paper, and erasing more than you would like to admit, finally you are confident that you have correctly used integration by trigonometric substitution and have deduced an impressive looking answer to your first homework question. Your answer is

$$\int \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x = \frac{\left(x^2-1\right)^{9/2}}{2x^2} - \frac{\left(x^2-1\right)^{7/2}}{2} + \frac{7\left(x^2-1\right)^{5/2}}{10} - \frac{7\left(x^2-1\right)^{3/2}}{6} + \frac{7(x^2-1)^{1/2}}{2} + \frac{7\arcsin\left(\frac{1}{x}\right)}{2}.$$

But wait! There's more. To add to the punishment, your teacher asks for the value of the definite integral $\int_1^{10^9} \frac{(x^2-1)^{7/2}}{x^3}\,\mathrm{d}x$. Easy-peasy, you think. Surely after the tortuous trigonometric substitution problem, this should be easy. All you need to do is evaluate

$$\frac{\left(10^{18}-1\right)^{9/2}}{2 10^{18}} - \frac{\left(10^{18}-1\right)^{7/2}}{2} + \frac{7\left(10^{18}-1\right)^{5/2}}{10} - \frac{7\left(10^{18}-1\right)^{3/2}}{6} + \frac{7(10^{18}-1)^{1/2}}{2} + \frac{7\arcsin\left(10^{-18}\right)}{2} - \frac{7\arcsin(1)}{2}$$

You tap this into your nearest computing device, and out pops the answer to an impressive sixteen decimal places. Your final answer is

$$\int_1^{10^9} \frac{(x^2-1)^{\frac{7}{2}}}{x^3}\,\mathrm{d}x \approx 9.134385233318144 \times 10^{46}.$$

So you circle this answer and turn it in. Good job, you think—let's very quickly move on to the next question.

Not so fast. Your ten sheets of wasted engineering paper rewarded you with full credit for your antiderivative, but your decimal approximation earns you a score of zero. Although your teacher says that your numerical value isn't *manifestly wrong*, its just shy of being *obviously wrong*. Some thought shows that $0 < \frac{(x^2-1)^{\frac{7}{2}}}{x^3} < \frac{(x^2)^{7/2}}{x^3} = x^4$. So it must be true that

$$0 < \int_1^{10^9} \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x < \int_1^{10^9} x^4\,\mathrm{d}x = \frac{10^{45}-1}{5} < 2\times 10^{44}$$

Yikes! Using an exact calculation and some arithmetic, your value for the definite integral is nearly one thousand times larger than an easily found upper bound.

What's the story? Pasting in the upper limit of $10^9$ into our painstakingly determined antiderivative, we need to sum

$$\begin{aligned}&5.000000000000003 \times 10^{62} - 5.000000000000002 \times 10^{62} + 7.000000000000003 \times 10^{44}\\&- 1.166666666666669 \times 10^{27} + 3.500000000000005 \times 10^{9} + 3.500000000000003 \times 10^{-9}.\end{aligned}$$

This is an example of what is known as an *ill conditioned sum*, and it is the first two terms of the sum that are the troublemakers. Each term in this sum is properly rounded to sixteen decimal digits. This means that the relative difference between each term in the sum and its true value are no more than about $10^{-16}$. Specifically, the true value of the first term can be as small as $5.000000000000003 \times 10^{62} \times (1 - 10^{-16})$ or as large as $5.000000000000003 \times 10^{62} \times (1 + 10^{-16})$. And similarly for the second term.

Putting this together, the sum of the first two terms might be as small as

$$5.000000000000003 \times 10^{62} \times (1 - 10^{-16}) - 5.000000000000002 \times 10^{62} \times (1 + 10^{-16}) \approx -1.1182158029987521 \times 10^{46},$$

and as large as

$$5.000000000000003 \times 10^{62} \times (1 + 10^{-16}) - 5.000000000000002 \times 10^{62} \times (1 - 10^{-16}) \approx 1.888178419700126 \times 10^{47}$$

We don't even know if the sum of the first two terms is negative or positive. Using sixteen decimal digits, we cannot accurately evaluate this expression.

Is there a cure? Well, yes there is. It is tedious to show, but an algebraically equivalent form for the antiderivative is

$$\int \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x = \frac{\sqrt{x^2-1}\left(6x^6 - 32x^4 + 116x^2 + 15\right) + 105\,\mathrm{asin}\left(\frac{1}{x}\right)x^2}{30x^2}.$$

Numerically evaluating this expression at $10^9$ does not involve subtracting two numbers that are nearly equal. Unlike, our first expression, this form of the answer is well-conditioned. Typing this into the nearest calculator yields

$$\int \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x \approx 1.9999999999999998 \times 10^{44}.$$

This approximation is accurate and it is smaller than the upper bound of $2 \times 10^{44}$.

If you are willing to dip into hypergeometric functions, there is another cure. In terms of the Gauss hypergeometric function, we have

$$\int \frac{\left(x^2-1\right)^{7/2}}{x^3}\,\mathrm{d}x = {}_2\mathrm{F}_1\left[\begin{matrix}2,\ 9/2\\ 11/2\end{matrix};1-x^2\right]\frac{(x^2-1)^{9/2}}{9}.$$

Any worthy computer algebra system will give

$${}_2\mathrm{F}_1\left[\begin{matrix}2,\ 9/2\\ 11/2\end{matrix};1-10^{18}\right]\frac{(10^{18}-1)^{9/2}}{9} = 1.99999999999999999988\cdots \times 10^{44}$$