Advanced Calculus **Name:**______________________________

Exam II Review, October 6, 2023 **Row and Seat:**________________________

"All students are enjoined in the strongest possible terms to eschew proofs by contradition! H.L ROYDEN *Real Analysis*

1. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto 8 - \frac{1}{k}$ is bounded above.

2. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{(-1)^k}{k^2}$ converges.

3. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \begin{cases} k! & k < 10^9 \\ \frac{(-1)^k}{k^2} & k \geq 10^9 \end{cases}$ converges.

4. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{3k+1}{2k+8}$ converges.

5. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto k - 3\lfloor \frac{k}{3} \rfloor$ does not converge to 1.

6. Using the definition from the QRS, show that the interval $(-\infty, 8)$ is open.

7. Let $A \subset \mathbf{R}$. Using the definition of an open set in the QRS, write the undefintion of an open set. That is, complete the statement:

   A is not open $\equiv$

8. Using the undefition from the previous question, show that the set $(-\infty, 8) \cup \{9\}$ is not open.

9. Let $A \subset \mathbf{R}$. Using the definition of a limit point in the QRS, write the undefintion of limit point. That is, complete the statement:

   $x \notin \operatorname{lp}(A) \equiv$

10. Use your undefinition from the previous question to show that $5 \notin \operatorname{lp}(\mathbf{Z})$.

11. Use the QRS defintion of a *boundary point* to show that $12 \in \operatorname{bp}((0, 12))$.

12. Use the result of the previous question to show that $(0, 12)$ is not closed.

13. Show that the set $\mathbf{R}$ is not compact by showing that there is an open cover of $\mathbf{R}$ that has no finite subcover.

14. Show that the set $\mathbf{Z}$ is not compact by showing that there is an open cover of $\mathbf{Z}$ that has no finite subcover.

15. Let F be a convergent sequence, and let $\alpha \in \mathbf{R}$. Show that αF is a convergent sequence.

16. Let F be a convergent sequence and suppose range$(F) \subset ([0, \infty))$. Show that $\sqrt{F}$ converges. You may use the fact that $\left(\forall x, y \in \mathbf{R}_{\geq 0}\right) (|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|})$

17. For the sequence $F = k \in \mathbf{Z}_{\geq 0} \mapsto k - 3\lfloor \frac{k}{3} \rfloor$, give three examples of a convergent subsequence.

18. Give an example of a sequence F and a real number α such that αF converges and F diverges.
19. Give an example of sequences F and G such that both F and G diverge, but $F+G$ converges.
20. Give an example of sequences F and G such that both F and G diverge, but FG converges.
21. Show that $\left(\forall x, y \in \mathbf{R}_{\geq 0}\right)(\sqrt{x+y} \leq \sqrt{x} + \sqrt{y})$.