

Advanced Calculus**Name:** _____**Exam II Review, October 9, 2023****Row and Seat:** _____

1. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto 8 - \frac{1}{k}$ is bounded above.

Solution: We'll show that F is bounded above by 8. Let $k \in \mathbf{Z}_{\geq 1}$. We have

$$\begin{aligned} \left[8 - \frac{1}{k} < 8 \right] &\equiv \left[0 < \frac{1}{k} \right], && \text{(add } \frac{1}{k} - 8) \\ &\equiv [0 < 1], && \text{(multiply by } k) \\ &\equiv \text{True.} \end{aligned}$$

2. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{(-1)^k}{k^2}$ converges.

Solution: We'll show that

$$(\exists L \in \mathbf{R}) (\forall \varepsilon \in \mathbf{R}) (\exists N \in \mathbf{Z}) (\forall k \in \mathbf{Z}_{>N}) (|F_k - L| < \varepsilon). \quad (1)$$

Choose $L = 0$. Let $\varepsilon \in \mathbf{R}$. Choose $N = \lceil \sqrt{\frac{1}{\varepsilon}} \rceil$. Then $N \in \mathbf{Z}$ as required. Let $k \in \mathbf{Z}_{>N}$. We have

$$\begin{aligned} |F_k - L| &= \left| \frac{(-1)^k}{k^2} \right|, \\ &= \frac{1}{k^2}, \\ &< \frac{1}{N^2}, \\ &= \frac{1}{\lceil \sqrt{\frac{1}{\varepsilon}} \rceil^2}, \\ &\leq \frac{1}{\sqrt{\frac{1}{\varepsilon}}^2} = \varepsilon. \end{aligned}$$

3. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \begin{cases} k! & k < 10^9 \\ \frac{(-1)^k}{k^2} & k \geq 10^9 \end{cases}$ converges.

Solution: This is just like the previous problem, but we need to choose $N = \max(10^9, \lceil \sqrt{\frac{1}{\varepsilon}} \rceil)$.

4. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{3k+1}{2k+8}$ converges.

5. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto k - 3\lfloor \frac{k}{3} \rfloor$ does not converge to 1.
6. Using the definition from the QRS, show that the interval $(-\infty, 8)$ is open.
7. Let $A \subset \mathbf{R}$. Using the definition of an open set in the QRS, write the undefinition of an open set. That is, complete the statement:

A is not open \equiv

8. Using the undefinition from the previous question, show that the set $(-\infty, 8) \cup \{9\}$ is not open.
9. Let $A \subset \mathbf{R}$. Using the definition of a limit point in the QRS, write the undefinition of limit point. That is, complete the statement:

$x \notin \text{lp}(A) \equiv$

10. Use your undefinition from the previous question to show that $5 \notin \text{lp}(\mathbf{Z})$.

Solution:

11. Use the QRS definition of a *boundary point* to show that $12 \in \text{bp}((0, 12))$.

Solution: Let δ be a positive number, and let $x^* = \begin{cases} 12 - \delta/2 & \delta < 24 \\ 6 & \delta \geq 24 \end{cases}$. Then $x^* \in (12 - \delta, 12 + \delta)$ and $x^* \in (0, 12)$. Further we have $12 + \delta/2 \notin (0, 12)$ and $12 + \delta/2 \in (12 - \delta, 12 + \delta)$.

Alternative For every positive number δ , we have $(0, 12) \cap B(12, \delta) = (\max\{0, 12 - \delta\}, 12) \neq \emptyset$. Further $(0, 12)^C \cap B(12, \delta) = (12, 12 + \delta) \neq \emptyset$.

12. Use the result of the previous question to show that $(0, 12)$ is not closed.

Solution: A closed set contains all of its boundary points. We showed that 12 is a boundary point $(0, 12)$, but $12 \notin (0, 12)$; therefore $(0, 12)$ is not closed.

13. Show that the set \mathbf{R} is not compact by showing that there is an open cover of \mathbf{R} that has no finite subcover.
14. Show that the set \mathbf{Z} is not compact by showing that there is an open cover of \mathbf{Z} that has no finite subcover.

Solution: Define $\mathcal{C} = \{(0, k) \subset \mathbf{R} \mid k \in \mathbf{N}\}$. Since $\bigcap_{x \in \mathcal{C}} x = (0, \infty)$, the set \mathcal{C} is a cover for \mathbf{N} . Let \mathcal{C}' be any finite subset of \mathcal{C} . Then $\bigcap_{x \in \mathcal{C}'} x$ is bounded because it's a finite union of bounded sets. But \mathbf{N} isn't bounded, so $\mathbf{N} \not\subset \bigcap_{x \in \mathcal{C}'} x$; therefore \mathbf{N} isn't compact.

15. Let F be a convergent sequence, and let $\alpha \in \mathbf{R}$. Show that αF is a convergent sequence.

Solution: Let $\varepsilon > 0$. Since f converges, there is a number L and $M \in \mathbf{N}$ such that for all $k > M$, we have $|f_k - L| < \frac{\varepsilon}{1+|\alpha|}$. For all $k > M$, we have

$$|\alpha f_k - \alpha L| = |\alpha| |f_k - L| < \frac{|\alpha|}{1+|\alpha|} \varepsilon < \varepsilon.$$

16. Let F be a convergent sequence and suppose $\text{range}(F) \subset ([0, \infty))$. Show that \sqrt{F} converges. You may use the fact that $(\forall x, y \in \mathbf{R}_{\geq 0}) (|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|})$

Solution: Let $\varepsilon > 0$. Define $\delta = \min\{1, \varepsilon^2/5\}$. For $x \in B'(2, \delta)$, we have $x - 2 \in [-1, 1]$. Thus $x + 2 \in [3, 5]$; consequently $|x + 2| \leq 5$. Again, for $x \in B'(2, \delta)$, we have

$$\begin{aligned} |\sqrt{1+x^2} - \sqrt{5}| &\leq \sqrt{|x^2 - 4|}, \\ &= \sqrt{|x+2|} \sqrt{|x-2|}, \\ &< \sqrt{5\delta}, \\ &< \varepsilon. \end{aligned}$$

17. For the sequence $F = k \in \mathbf{Z}_{\geq 0} \mapsto k - 3\lfloor \frac{k}{3} \rfloor$, give three examples of a convergent subsequence.
18. Give an example of a sequence F and a real number α such that αF converges and F diverges.

Solution: Choose $F = k \in \mathbf{Z}_{\geq 0} \mapsto k$ and choose $\alpha = 0$. Then F diverges but $0F$ converges.

19. Give an example of sequences F and G such that both F and G diverge, but $F + G$ converges.

Solution: Choose $F = k \in \mathbf{Z}_{\geq 0} \mapsto k$ and $G = k \in \mathbf{Z}_{\geq 0} \mapsto -k$. Both F and G diverge, but $F + G$ converges.

20. Give an example of sequences F and G such that both F and G diverge, but FG converges.

Solution: Choose $F = k \in \mathbf{Z}_{\geq 0} \mapsto (-1)^k$ and $G = k \in \mathbf{Z}_{\geq 0} \mapsto (-1)^k$. Both F and G diverge, but $FG = k \in \mathbf{Z}_{\geq 0} \mapsto 1$ converges.

21. Show that $(\forall x, y \in \mathbf{R}_{\geq 0}) (\sqrt{x+y} \leq \sqrt{x} + \sqrt{y})$.

Solution: We begin by proving that $(\forall x, y \in \mathbf{R}_{\geq 0})(\sqrt{x^2 + y^2} \leq x + y)$. Let $x, y \in \mathbf{R}_{\geq 0}$. We have

$$\begin{aligned}\sqrt{x^2 + y^2} &= \sqrt{x^2 + 2xy + y^2 - 2xy}, && \text{(add and subtract)} \\ &= \sqrt{(x + y)^2 - 2xy}, && \text{(algebra)} \\ &\leq \sqrt{(x + y)^2}, && \text{(square root function is increasing)} \\ &= x + y. && \text{(algebra)}\end{aligned}$$

To finish the proof, we replace $x \rightarrow \sqrt{x}$ and $y \rightarrow \sqrt{y}$ in the above. This yields $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$.