Let / Choose Proofs

The Let/Choose Template

Many propositions have the form

$$(\text{string of } \forall\ \exists \text{ qualifiers in involving } x_1 \text{ thru } x_n)\ (P(x_1, \ldots x_n)),$$

where P is a predicate. It behooves us to have a template for proving such propositions. Let's try the example

Proposition

For every $x \in \mathbf{R}$ there is $y \in \mathbf{R}$ such that $x < y$.

\# 0 Write the proposition in symbolic form:

$$(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})(x < y).$$

\#1 Write the proposition in the form of a question:

Given a real number x, can I find a number y such that $x < y$?

\#2 Answer your question.

Sure–a number that is greater than x is $x + 1$.

#3 Using the symbolic form of the proposition and *strictly moving from left to right*, replace $\forall$ with "Let," and $\exists$ with "Choose." After each "choose" make a box to fill in. Finish with the predicate:
Let $x \in \mathbf{R}$. Choose $y = \boxed{}$. We have

$$[x < y] =$$

#4 Fill in the boxes with the answers you chose, and attempt to show that the predicate is true: Let $x \in \mathbf{R}$. Choose $y = \boxed{x{+}1}$. We have

$$\begin{aligned} [x < y] &\equiv [x < x + 1], && (\text{ substitute for } y) \\ &\equiv [0 < 1], && (\text{ subtract } x \text{ from both sides }) \\ &\equiv \text{true}. \end{aligned}$$

#5 Erase the boxes:

Proof

Let $x \in \mathbf{R}$. Choose $y = x + 1$. We have

$$\begin{aligned} [x < y] &\equiv [x < x + 1], && (\text{ substitute for } y) \\ &\equiv [0 < 1], && (\text{ subtract } x \text{ from both sides }) \\ &\equiv \text{true}. \end{aligned}$$

$\neg$ Pedantic

1. Proof construction is a creative activity–there is no step of steps that will always generate a proof.
2. But having patterns to follow and knowing techniques is useful for all creative endeavors.

Respecting order

The order of qualifiers matters. To show this, let's reverse the order of qualifiers in the previous proposition:

Proposition

There is $y \in \mathbf{R}$ such that for every $x \in \mathbf{R}$ we have $x < y$.

Question Can I find a real number y such that for every real number x, we have $x < y$?

Answer: No I don't think so–the number we choose has to be larger than 10^{10}, larger than $10^{10^{10}}$ and larger than every number. The statement requires that y be a *real number*, so choosing $y = \infty$ isn't an option.

Tip Proving things that are wrong take too much time. So try to avoid attempting.

Let's show that the proposition is false by showing that its negation is true; the negation of the proposition is

Proposition

For all $y \in \mathbf{R}$ there is $x \in \mathbf{R}$ such that $x \geq y$.

Proof

Let $y \in \mathbf{R}$. Choose $x = y$. Then $[x \geq y] \equiv [x \geq x] \equiv \text{true}$.

1. We could choose $x = y + 1$, but we only need $x \geq y$, so we can choose $x = y$.

Later, rinse, repeat

Proposition

For all $x \in \mathbf{R}_{>0}$ there is $y \in \mathbf{R}_{>0}$ such that $y < x$.

1. Write the proposition in the form of a question:
   Given a positive real number x, can I find a positive y such that y is smaller than x?
2. Answer your question.
   Sure–a positive number that is smaller than x is the average of zero and x; that is $x/2$.
3. I'm ready, I think:

Proof

Let $x \in \mathbf{R}_{>0}$. Choose $y = x/2$. Then $y \in \mathbf{R}_{>0}$. Further

$$
\begin{aligned}
[y < x] &\equiv [x/2 < x] && (\text{ substitute for } y) \\
&\equiv [1/2 < 1] && (\text{ divide inequality by positive number } x) \\
&\equiv \text{true}.
\end{aligned}
$$