MATH 460 **Name:**

Homework 1, Fall 2024

Homework 1 has questions 1 through 8 with a total of 80 points. When I record your grade, I will scale it to twenty points. For details of the grading scheme for this assignment, please see the section 'Grading rubric' of our syllabus.

Revise, proofread, revise again (and again), *neatly* hand write your solutions, digitize your work, and upload the converted PDF of your work to Canvas. This work is due **Saturday 26 August** at 11:59 PM.

$\boxed{10}$ 1. For the statement $(\exists M \in \mathbf{R})\,(\forall x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} \leq M\right)$, explain why the following proof is abject rubbish:

Proof. Choose $M = \frac{5x}{x+1}$. Let $x \in \mathbf{R}_{\geq 0}$. We have

$$\begin{aligned} \left[\frac{5x}{x+1} \leq M\right] &\equiv \left[\frac{5x}{x+1} \leq \frac{5x}{x+1}\right], && \text{(substitution for } M\text{)} \\ &\equiv \text{True}. && \text{(syntactic equality)}. \end{aligned}$$

□

The proof is abject rubbish because it violates what I call the *left-to-right rule for quantified statements.* Specifically, since M is qualified before x, the value we choose for M is not allowed to depend on x. From the first sentence, "Choose $M = \frac{5x}{x+1}$," the proof is wrong, and the only cure is to start from scratch.

Our work is, however, a *correct* proof of the statement

$$(\forall x \in \mathbf{R}_{\geq 0})\,(\exists M \in \mathbf{R})\left(\frac{5x}{x+1} \leq M\right), \tag{1}$$

but these two statements are *not* logically equivalent.

$\boxed{10}$ 2. Write a correct proof of $(\exists M \in \mathbf{R})\,(\forall x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} \leq M\right)$.

Proof. Choose $M = 5$. Let $x \in \mathbf{R}_{\geq 0}$. We have

$$\begin{aligned}
\left[\frac{5x}{x+1} \leq M\right] &\equiv \left[\frac{5x}{x+1} \leq 5\right], && \text{(substitution for } M\text{)}\\
&\equiv [5x \leq 5(x+1)], && \text{(multiply by } x+1\text{)}\\
&\equiv [0 \leq 5], && \text{(subtract } 5x\text{)}\\
&\equiv \text{True}. && \square
\end{aligned}$$

Notes:

- Since M is qualified before x, what we choose for M cannot depend on x. Our choice of $M = 5$ satisfies this requirement.
- How did I know to choose $M = 5$? One approach is to sketch a graph (try Desmos) of the equation $y = \frac{5x}{x+1}$ for $x > 0$. You will see that the maximum of y coordinate on the graph is 5. That's how.
- Instead of making the choice $M = 5$, we could choose M to be any real number that is greater than 5.
- We've expressed the proof as a sequence of logical equivalences. A more traditional way to write the proof is a sequence of inequalities that starts with $\frac{5x}{x+1}$ and that ends with 5. Each inequality must be either $<, \leq$, or $=$. You might like to write the proof this way. Which is more natural? Which proof is easier to understand? Which proof is easier to invent? Why?
- In the third line, we multiplied an inequality by a variable (specifically $x + 1$). We need to be careful with such operations, but it is justified because $x \in \mathbf{R}_{\geq 0}$.

$\boxed{10}$ 3. Without explicitly using negation (either $\neg$ or anything equivalent to negation), write the negation of the statement

$$(\exists M \in \mathbf{R}_{<5})\,(\forall x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} < M\right).$$

Unlike the previous questions, the number M in this question must be *less* than five. Also, the final inequality is now a strict inequality (equality is not allowed). These differences are *not* typos.

For a refresher course on how to negate qualified statements, please see our course quick reference sheet. Using these rules, we have:

$$(\forall M \in \mathbf{R}_{<5})\,(\exists x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} \geq M\right).$$

$\boxed{10}$ 4. Show that the statement

$$(\exists M \in \mathbf{R}_{<5})\,(\forall x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} < M\right).$$

is *false* by showing that its negation is true.

Proof. We need to show that

$$(\forall M \in \mathbf{R}_{<5})\,(\exists x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} \geq M\right).$$

Let $M \in \mathbf{R}_{<5}$. Choose $x = \begin{cases} 0 & M < 0 \\ \frac{M}{5-M} & M \geq 0 \end{cases}$. Then $x \in \mathbf{R}_{\geq 0}$ as required. We will consider the cases $M < 0$ and $M \geq 0$ separately. First, for $M < 0$, we have

$$\begin{aligned} \left[\frac{5x}{x+1} \geq M\right] &\equiv [0 \geq M], \\ &\equiv \text{True.} \end{aligned}$$

And second for $M \geq 0$, we have

$$\begin{aligned} \left[\frac{5x}{x+1} \geq M\right] &\equiv [M \geq M], && \text{(algebra)} \\ &\equiv \text{True.} && \square \end{aligned}$$

Unassigned questions

10 1. Show that for every real number b there is a real number m such that

$$\{x \in \mathbf{R} \mid x^2 - b^2 = m(x-b)\} = \{b\}.$$

This shows that there is exactly one line of the form $y - b^2 = m(x-b)$ that intersects the curve $y = x^2$ in exactly one point. Apparently, the line $y - b^2 = m(x-b)$ is the tangent line. But "touching at one point" is *not* a requirement of a tangent line.

Hint Solving (either by factoring or by quadratic formula) $x^2 - b^2 = m(x-b)$ for x gives the solution set $\{b, m-b\}$.

Our proof can either include or exclude the way we discover a correct value for m. Either way is correct. If you can find the value of m by divine intervention and then write a proof that chooses the value of m from thin air, that's fine–a proof doesn't need to explain the logic of how we make choices.

For this solution, let's exclude the method we use for discovering the value of m in the body of the proof, but instead include this logic outside the proof. To find m, we solve the equation $x^2 - b^2 = m(x-b)$ for x. The solution(s) are $x = b$ and $x = m - b$. But the statement says there is only one solution, so we must choose $m = 2b$, making each solution $x = b$. With this choice, our proof is:

Proof. Let $b \in \mathbf{R}$. Choose $m = 2b$. We have

$$\begin{aligned}\{x \in \mathbf{R} \mid x^2 - b^2 = m(x-b)\} &= \{x \in \mathbf{R} \mid x^2 - b^2 = 2b(x-b)\},\\ &= \{x \in \mathbf{R} \mid x^2 - 2bx + b^2 = 0\},\\ &= \{x \in \mathbf{R} \mid (x-b)^2 = 0\},\\ &= \{b\}.\end{aligned}$$

□

Notes:

- The problem statement allows m to depend on b. That is because the qualification on m follows the qualification on b.

- Our initial work in determining the value of m shows that the choice is actually unique. So if you flub and choose m to not equal $2b$, you will not be able to write a proof.
- Tangent lines are often described as “touching in one spot,” but that’s not true. The tangent line to $y = \cos(x)$ at the point $(x = 0, y = 1)$ is $y = 1$. But the curves $y = \cos(x)$ and $y = 1$ touch in infinitely many locations.

$\boxed{10}$ 2. Without explicitly using negation ($\neg$), write the negation of the statement

$$(\exists M \in \mathbf{R}) \left(\forall x \in \mathbf{R}_{\neq 0}\right) \left(\frac{1}{x^2} < M\right).$$

We have

$$(\forall M \in \mathbf{R}) \left(\exists x \in \mathbf{R}_{\neq 0}\right) \left(\frac{1}{x^2} \geq M\right).$$

$\boxed{10}$ 3. Show that the statement

$$(\exists M \in \mathbf{R}) \left(\forall x \in \mathbf{R}_{\neq 0}\right) \left(\frac{1}{x^2} < M\right).$$

is false. To do this, show that its negation is true. This shows that the function $x \in \mathbf{R}_{\neq 0} \mapsto \frac{1}{x^2}$ is not bounded above.

Proof. We will show that

$$(\forall M \in \mathbf{R}) \left(\exists x \in \mathbf{R}_{\neq 0}\right) \left(\frac{1}{x^2} \geq M\right).$$

Let $M \in \mathbf{R}$. Choose $x = \begin{cases} 1 & M \leq 0 \\ \sqrt{\frac{1}{M}} & M > 0 \end{cases}$. Then $x \in \mathbf{R}_{\neq 0}$ as required. We consider the cases $M \leq 0$ and $M > 0$ separately. For $M \leq 0$, we have

$$\left[\frac{1}{x^2} \geq M\right] \equiv [1 \geq M] \equiv \text{True}.$$

And for $M > 0$, we have

$$\left[\frac{1}{x^2} \geq M\right] \equiv [M \geq M] \equiv \text{True}.$$

$\square$

$\boxed{10}$ 4. Find functions $P, Q \in \mathbf{Z} \mapsto \{\text{false}, \text{true}\}$ that makes the statement

$$(\forall k \in \mathbf{Z})\,(P(k) \vee Q(k)) \equiv (\forall k \in \mathbf{Z})\,(P(k)) \vee (\forall k \in \mathbf{Z})\,(Q(k))$$

false. This shows that the universal qualifier does not distribute over the disjunction. One notational subtly is the fact that the various variables k on the left and right sides of this equivalence are actually different. This means that the given statement is logically equivalent to

$$(\forall k \in \mathbf{Z})\,(P(k) \vee Q(k)) \equiv (\forall \ell \in \mathbf{Z})\,(P(\ell)) \vee (\forall m \in \mathbf{Z})\,(Q(m)).$$

Hint: Choose $P = k \in \mathbf{Z} \mapsto [x \text{ is even}]$ and $Q = k \in \mathbf{Z} \mapsto [x \text{ is odd}]$. Explain why this choice shows that the given statement is false.

Proof. Choose $P = k \in \mathbf{Z} \mapsto [x \text{ is even}]$ and $Q = k \in \mathbf{Z} \mapsto [x \text{ is odd}]$. We will choose integers k, ℓ, and m such that

$$P(k) \vee Q(k) \neq P(\ell) \vee Q(m).$$

Choose $k = 1, \ell = 1$ and $m = 2$. We have

$$[P(k) \vee Q(k)] \equiv [P(1) \vee Q(1)] \equiv \text{True}.$$

But

$$[P(\ell) \vee Q(m)] \equiv [P(1) \vee Q(2)] \equiv \text{False}.$$

$\square$