Greek Characters

Name	Symbol	Typical use(s)
alpha	α	angle, constant
beta	β	angle, constant
gamma	γ	angle, constant
delta	δ	limit definition
epsilon	ϵ or ε	limit definition
theta	θ or ϑ	angle
pi	π or ϖ	circular constant
phi	ϕ or φ	angle, constant

Named Sets

empty set	$\varnothing$
real numbers	$\mathbf{R}$
ordered pairs	$\mathbf{R}^2$

integers	$\mathbf{Z}$
positive integers	$\mathbf{Z}_{>0}$
positive real	$\mathbf{R}_{>0}$

Set Symbols

Meaning	Symbol
is a member	$\in$
subset	$\subset$
intersection	$\cap$

Meaning	Symbol
union	$\cup$
complment	superscript$^{\mathrm{C}}$
set minus	$\setminus$

Intervals

For numbers a and b, we define the intervals:

$$(a,b) = \{x \in \mathbf{R} \mid a < x < b\}$$
$$[a,b) = \{x \in \mathbf{R} \mid a \le x < b\}$$
$$(a,b] = \{x \in \mathbf{R} \mid a < x \le b\}$$
$$[a,b] = \{x \in \mathbf{R} \mid a \le x \le b\}$$

Logic Symbols

Meaning	Symbol
negation	$\neg$
and	$\wedge$
or	$\vee$
implies	$\implies$

Meaning	Symbol
equivalent	$\equiv$
if and only if	$\iff$
for all	$\forall$
there exists	$\exists$

Tautologies

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$
$$P \not\Longrightarrow Q \equiv P \wedge \neg Q$$
$$(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P))$$
$$\neg(\forall x \in A)(P(x)) \equiv (\exists x \in A)(\neg P(x))$$
$$\neg(\exists x \in A)(P(x)) \equiv (\forall x \in A)(\neg P(x))$$

Generalized set operators

Each member of a set $\mathcal{C}$ is a set:

$$\bigcup_{A \in \mathcal{C}} A = \{z \mid (\exists B \in \mathcal{C})(z \in B)\}$$
$$\bigcap_{A \in \mathcal{C}} A = \{z \mid (\forall B \in \mathcal{C})(z \in B)\}$$

Functions applied to sets

Let $A \subset \mathrm{dom}(F)$ and $B \subset \mathrm{codom}(F)$:

$$F(A) = \{F(x) \mid x \in A\}$$
$$F^{-1}(B) = \{x \in \mathrm{dom}(F) \mid F(x) \in B\}$$

Function Notation

$\mathrm{dom}(F)$	domain of function F
$\mathrm{range}(F)$	range of function F
C_A	set of continuous functions on set A
C^1_A	set of differentiable functions on set A
$A \to B$	set of functions from$A$ to B

Triangle inequalities

For all $x, y \in \mathbf{R}$, we have

$$|x+y| \le |x| + |y|$$
$$||x| - |y|| \le |x-y|$$

Floor and ceiling

Definitions:

$$\lfloor x \rfloor = \max\{k \in \mathbf{Z} \mid k \le x\}$$
$$\lceil x \rceil = \min\{k \in \mathbf{Z} \mid k \ge x\}$$

Properties:

$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(x < n \iff \lfloor x \rfloor < n)$$
$$(\forall x \in \mathbf{R}, n \in \mathbf{Z})(n < x \iff n < \lceil x \rceil)$$

Bounded sets

Bounded below A set A is *bounded below* provided $(\exists M \in \mathbf{R})(\forall x \in A)(M < x)$.

Bounded above The set A is *bounded above* provided $(\exists M \in \mathbf{R})(\forall x \in A)(x < M)$.

Bounded A set is *bounded* if it is bounded below and bounded above.

Elementary function properties

Increasing $(\forall x, y \in A)(x < y \implies F(x) < F(y))$

Decreasing $(\forall x, y \in A)(x < y \implies F(x) > F(y))$

One-to-one
$(\forall x, y \in \mathrm{dom}(F))(F(x) = F(y) \implies x = y)$

Subadditive
$(\forall x, y \in \mathrm{dom}(F))(F(x+y) \le F(x) + F(y))$

Bounded above $(\exists M \in \mathbf{R})(\forall x \in \mathrm{dom}(F))(F(x) < M)$

Bounded below
$(\exists M \in \mathbf{R})\,(\forall x \in \mathrm{dom}(F))(M < F(x))$

Topology

Open ball $\text{ball}(a,r) = \{x \in \mathbf{R} \mid -r < x - a < r\}$

Punctured ball $\text{ball}'(a,r) = \text{ball}(a,r) \setminus \{a\}$

Open set A subset A of $\mathbf{R}$ is *open* provided $(\forall x \in A)\,(\exists r \in \mathbf{R}_{>0})(\text{ball}(x,r) \subset A)$

Closed set A subset A of $\mathbf{R}$ is *closed* provided $\mathbf{R} \setminus A$ is open.

Limit point A number a is a *limit point* of a set A provided $(\forall r \in \mathbf{R}_{>0})\big(\text{ball}'(a,r) \cap A \neq \varnothing\big)$

Open cover A set $\mathcal{C}$ is a cover of a set A provided

(a) every member of $\mathcal{C}$ is a set
(b) $A \subset \bigcup_{B \in S} B$

Compact A set A is compact provided for every open cover $\mathcal{C}$ of A, there is a finite subset $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathcal{C}'$ is an open cover of A.

Least and greatest bounds

For any subset A of $\mathbf{R}$:

glb $z = \text{glb}(A)$ provided

(a) z is an lower bound for A
(b) x is a lower bound for A implies $z \leq x$

lub $z = \text{lub}(A)$ provided

(a) z is an upper bound for A
(b) x is a upper bound for A implies $z \leq x$

Sequences

Bounded A sequence F is bounded if $\text{range}(F)$ bounded.

Cauchy A sequence F is Cauchy provided

(a) for every $\varepsilon \in \mathbf{R}_{>0}$
(b) there is $n \in \mathbf{Z}$
(c) such that for all $k, \ell \in \mathbf{Z}_{>n}$
(d) $|F_k - F_\ell| < \varepsilon$

Converges A sequence F converges provided

(a) there is $L \in \mathbf{R}$
(b) and $n \in \mathbf{Z}$
(c) such that for all $k \in \mathbf{Z}_{>n}$
(d) $|F_k - L| < \varepsilon$.

Functions

Continuous A function F is continuous at a provided

(a) $a \in \text{dom}(F)$; and
(b) for every $\varepsilon \in \mathbf{R}_{>0}$
(c) there is $\delta \in \mathbf{R}_{>0}$
(d) such that for all $x \in \text{ball}(a,\delta) \cap \text{dom}(F)$
(e) we have $F(x) \in \text{ball}(F(a), \epsilon)$.

Uniformly continuous A function F is uniformly continous on a set A provided

(a) $A \subset \text{dom}(F)$; and
(b) for every $\varepsilon \in \mathbf{R}_{>0}$
(c) there is $\delta \in \mathbf{R}_{>0}$
(d) such that for all $x, y \in A \wedge |x - y| < \delta$
(e) we have $|F(x) - F(y)| < \epsilon$.

Limit A function F has a limit toward a provided

(a) a is a limit point of $\text{dom}(F)$; and
(b) there is $L \in \mathbf{R}$
(c) such that for every $\varepsilon \in \mathbf{R}_{>0}$
(d) there is $\delta \in \mathbf{R}_{>0}$
(e) such that for all $x \in \text{ball}'(a,\delta)$
(f) we have $F(x) \in \text{ball}(L, \epsilon)$.

Differentiable A function F is differentiable at a provided

(a) $a \in \text{dom}(F)$; and
(b) there is $\phi \in \text{dom}(F) \to \mathbf{R}$
(c) such that ϕ is continuous at a and
(d) $(\forall x \in \text{dom}(F))(F(x) = F(a) + (x-a)\phi(x))$.

Riemann sums

Partition A set $\mathcal{P}$ is a partition of an interval $[a,b]$ provided

(a) the set $\mathcal{P}$ is finite
(b) every member of $\mathcal{P}$ is a closed interval
(c) the members of $\mathcal{P}$ are pairwise disjoint
(d) $\bigcup_{I \in \mathcal{P}} I = [a,b]$

Let F be a bounded function on an interval $[a,b]$ and let $\mathcal{P}$ be a partition of $[a,b]$.

Lower sum $\underline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{glb}(F(A)) \times \text{length}(I)$

Upper sum $\overline{S}(\mathcal{P}) = \sum_{I \in \mathcal{P}} \text{lub}(F(A)) \times \text{length}(I)$

Riemann sum $\sum_{I \in \mathcal{P}, x^\star \in I} F(x^\star) \times \text{length}(I)$

Axioms

Completeness Every nonempty subset A of $\mathbf{R}$ that is bounded above has a least upper bound.

Well-ordering Every non-empty set of positive integers contains a least element.

Induction

$$(\forall n \in \mathbf{Z}_{\geq 0}) \iff P(0) \wedge (\forall n \in \mathbf{Z}_{\geq 0} P(n) \implies P(n+1))$$

Named Theorems

Bolzano-Weirstrass Every bounded real valued sequence has a convergent subsequence.

Heine–Borel A subset of $\mathbf{R}$ is compact iff it is closed and bounded.

Intermediate value theorem If $F \in \mathrm{C}_{[a,b]}$, then $F(I) \subset [\min(F(a), F(b)), \max(F(a), F(b))]$.

Mean Value If $F \in \mathrm{C}_{[a,b]} \cap \mathrm{C}^1_{(a,b)}$ there is $\xi \in (a,b)$ such that $(b-a)F'(\xi) = F(b) - F(a)$.