MATH 460 **Name:**

Homework 1, Fall 2023

Homework 1 has questions 1 through 4 with a total of 40 points. When I record your grade, I will scale it to twenty points. For details of the grading scheme for this assignment, please see the section 'Grading rubric' of our syllabus.

Revise, proofread, revise again (and again), *neatly* hand write your solutions, digitize your work, and up load the converted pdf of your work to Canvas. This work is due **Saturday 26 August** at 11:59 PM.

$\boxed{10}$ 1. For the statement $(\exists M \in \mathbf{R})\, (\forall x \in \mathbf{R}_{\geq 0})\, \left(\frac{5x}{x+1} \leq M\right)$, explain why

Proof. Choose $M = \frac{5x}{x+1}$. Let $x \in \mathbf{R}_{\geq 0}$. We have

$$\begin{aligned} \left[\frac{5x}{x+1} \leq M\right] &\equiv \left[\frac{5x}{x+1} \leq \frac{5x}{x+1}\right], && \text{(substitution for } M\text{)} \\ &\equiv \text{True}. && \text{(syntactic equality)} \ \square \end{aligned}$$

is *abject rubbish.*

Solution: The proof is abject rubbish because it violates the left-to-right rule. Since M is qualified before x, the value we choose for M is not allowed to depend on x.

$\boxed{10}$ 2. Write a correct proof of $(\exists M \in \mathbf{R})\, (\forall x \in \mathbf{R}_{\geq 0})\, \left(\frac{5x}{x+1} \leq M\right)$.

Proof. Choose $M = 5$. Let $x \in \mathbf{R}_{\geq 0}$. We have

$$\begin{aligned} \left[\frac{5x}{x+1} \leq M\right] &\equiv \left[\frac{5x}{x+1} \leq 5\right] && \text{(substitution)} \\ &\equiv [5x \leq 5(x+1)] && \text{(multiply by } x+1\text{)} \\ &\equiv [0 \leq 5] && \text{(subtract } 5x\text{)} \quad \square \\ &\equiv \text{True}. \end{aligned}$$

$\boxed{10}$ 3. Without explicitly using negation (either $\neg$ or anything equivalent to negation), write the negation of the statement

$$(\exists M \in \mathbf{R}_{<5})\, (\forall x \in \mathbf{R}_{\geq 0}) \left(\frac{5x}{x+1} < M\right).$$

Unlike the previous questions, the number M in this question must be *less* than five. Also, the final inequality is now a strict inequality (equality is not allowed). These differences are *not* typos.

Solution:

$$(\forall M \in \mathbf{R}_{<5})\,(\exists x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} \geq M\right).$$

$\boxed{10}$ 4. Show that the statement

$$(\exists M \in \mathbf{R}_{<5})\,(\forall x \in \mathbf{R}_{\geq 0})\left(\frac{5x}{x+1} < M\right).$$

is *false* by showing that its negation is true.

Proof. Let $M \in \mathbf{R}_{<5}$. Choose $x = \begin{cases} 0 & M < 0 \\ \frac{M}{5-M} & M \geq 0 \end{cases}$. Then $x \in \mathbf{R}_{\geq 0}$ as required. For $M < 0$, we have

$$\begin{aligned} \left[\frac{5x}{x+1} \geq M\right] &= [0 \geq M], \\ &\equiv \text{True} \end{aligned}$$

And for $M \geq 0$, we have

$$\begin{aligned} \left[\frac{5x}{x+1} \geq M\right] &= [M \geq M], && \text{(algebra)} \\ &\equiv \text{True} \end{aligned}$$

$\square$