MATH 460 **Name:**
Homework 3, Fall 2022

I have neither given nor received unauthorized assistance on this assignment.

Homework 3 has questions 1 through 4 with a total of 20 points. Edit this file and append you answers using LaTeX. Be sure to fill in your name. Upload the converted pdf of your work to Canvas. This assignment is due *Saturday 10 September at 11:59* PM.

Link to your Overleaf work: `XXX`

[5] 1. Show that $(\forall x \in (-1,1))\,(\exists r \in \mathbf{R}_{>0})\,((x-r, x+r) \subset (-1,1))$.

Solution:

[5] 2. Define $S = \{(-k,k) | k \in \mathbf{Z}_{>0}\}$. Show that $\bigcup_{q \in S} q = \mathbf{R}$.

Solution:

[5] 3. On $\mathbf{R}^2$ define the binary operators $+$ and $\times$ by

$$(a,b) + (c,d) = (a+c, b+d),$$
$$(a,b) \times (c,d) = (ac + 2bd, ad + bc).$$

These operators are commutative and associative. Further, the additive identity is $(0,0)$ and the multiplicative identity is $(1,0)$. Given these facts, show that $(\mathbf{R}^2, +, \times)$ is a field.

Solution:

[5] 4. Show that the complex field is not ordered. Hint: Suppose it is. Let P be its positive set. Since $\mathrm{i} \neq 0$, either $\mathrm{i} \in P$ or $-\mathrm{i} \in P$. Show that both $\mathrm{i} \in P$ or $-\mathrm{i} \in P$ are contradictions.

Solution: We will prove this by contradiction. Suppose the complex field is ordered, and let P be its positive set. Since $\mathrm{i} \neq 0$, either $\mathrm{i} \in P$ or $-\mathrm{i} \in P$. If $\mathrm{i} \in P$, closer of P under multiplication implies that $\mathrm{i}^2 \in P$ and $\mathrm{i}^4 \in P$. But this says that $-1 \in P$ and $1 \in P$. That violates tricotomy.

Similarly, the assumption $-\mathrm{i} \in P$ violates tricotomy; therefore the assumption the complex field is ordered is false.

Fun Fact It is possible to define $<$ on the complex field that has the properties

(a) for all $a, b \in \mathbf{C}$ exactly one of the following is true: $a < b$ or $a = b$ or $b < a$.

(b) for all $a, b, c \in \mathbf{C}$, we have $a < b$ and $b < c$ implies $a < c$.

But the set $\{z \in \mathbf{C} | 0 < z\}$ does not have the properties required by an ordered field to be a positive set.