Review for Exam I

1. Let A and B be subsets of $\mathbf{R}$. Show that if A and B are bounded above, then $A \cup B$ is bounded above. You may use the fact that for real numbers a and b, we have $a \leq \max(a, b)$ and $b \leq \max(a, b)$.

2. Give an example of a subset of $\mathbf{R}$ that does not have a least upper bound.

3. Give an example of a subset A of $\mathbf{R}$ such that $\operatorname{lub}(A) \in A$.

4. Give an example of a subset A of $\mathbf{R}$ such that $\operatorname{lub}(A) \notin A$.

5. Show that $\operatorname{lub}((-\infty, 2)) = 2$.

6. Show that $\operatorname{lub}([0, 2)) = 2$.

7. Let A be a subset of $\mathbf{R}$. Show that A has at most one least upper bound.

8. Write a proof for

   Proposition 1 *For all $x, y \in \mathbf{R}$, there is $a \in \mathbf{R}$ such that $x < y$ implies $x < a < y$.*

9. Write a proof for

   Proposition 2 *For all $x \in \mathbf{R}_{>0}$ there is $y \in \mathbf{R}_{>0}$ such that $y < x$.*

10. Without explicitly using negation, write the negation of

    Proposition 3 *There are $x, y \in \mathbf{R}$ such that* $\sin(x) = \sin(y) \implies x = y$.

11. Either write a proof of

    Proposition 4 *There are $x, y \in \mathbf{R}$ such that* $\sin(x) = \sin(y) \implies x = y$.

    or write a proof of its negation.

12. Let $(\mathscr{F}, +, \times)$ be a field and let O be the additive identity and I be the multiplicative identity. Given that $O = I$, show that $\mathscr{F} = \{O\}$.

13. Let $(\mathscr{F}, +, \times)$ be a field. Show that for all $a, b \in \mathscr{F}$, we have $a \times b = a \times (-b)$.

14. Let $(\mathscr{F}, +, \times)$ be an ordered field. For all $a, b, c \in \mathscr{F}$, show that $a < b$ and $c < 0$ implies $a \times c > b \times c$.

15. Show that

    $$(\forall k \in \mathbf{Z}_{>1}) \left(\frac{1}{k^2} \leq \frac{1}{k-1} - \frac{1}{k} \right).$$

16. Show that

$$(\forall x \in (-\infty, 1))\,(\exists r \in \mathbf{R}_{>0})\,((x - r, x + r) \subset (-\infty, 1))\,.$$

17. Let A, B be subsets of $\mathbf{R}$ and let A be bounded above. Show that $A \setminus B$ is bounded above.

18. Give an example of subsets A, B of $\mathbf{R}$ such that $A \setminus B$ is bounded above, but A is not bounded above.

19. Define $F = x \in \mathbf{R} \mapsto x^2$. Enumerate the members of the set

$$F(\{-4, -1, 0, 1, 4\}).$$

20. Show that

$$(\forall a \in \mathbf{R})\,(\exists m \in \mathbf{R})\,(\forall x \in \mathbf{R})\,\big(x^2 - a^2 \geq m(x - a)\big)\,.$$