1. For each statement, do the following: (a) Write the statement in symbolic form (b) Without explicitly using negation, write the negation of the statement in symbolic form. (c) Decide if the statement is true or false. (d) Write a proof of the statement that is true.

   (a) For all $x, y \in \mathbf{R}$, there is $a \in \mathbf{R}$ such that $x < y$ implies $x < a < y$

   Symbolically, the statement is

   $$\big(\forall x, y \in \mathbf{R}\big)\,(\exists a \in \mathbf{R})\,\big((x < y) \implies x < a < y\big).$$

   This statement says that between any two real numbers, there is a real number. One such number is the arithmetic average. So let's write a proof. But for completeness, its negation is

   $$\big(\exists x, y \in \mathbf{R}\big)\,(\forall a \in \mathbf{R})\,\big((x < y) \wedge (x \geq a \vee a \geq y\big).$$

   Proof. Let $x, y \in \mathbf{R}$. Suppose $x < y$. Choose $a = \frac{x+y}{2}$. We have $a \in \mathbf{R}$ as required. We have

   $$\begin{aligned}
   \big[x < a < y\big] &\equiv \left[x < \frac{x+y}{2} < y\right], && \text{(substitution)}\\
   &\equiv \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(subtraction)}\\
   &\equiv \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \text{(algebra)}\\
   &\equiv \text{true} && (y - x > 0 \wedge x - y < 0).
   \end{aligned}$$

   A key ingredient to the proof is the fact that $(\forall a, x, y \in \mathbf{R})((x < y) \equiv (x - a < y - a))$. (BW) □

   (b) For all $r \in \mathbf{R}_{>0}$ there is $x \in [0, 1)$ such that $1 - r < x$. (BW)

   (c) For all $x \in \mathbf{R}_{>0}$ there is $y \in \mathbf{R}_{>0}$ such that $y < x$. (BW)

   (d) There is $y \in \mathbf{R}_{>0}$ such that for all $x \in \mathbf{R}_{>0}$ we have $y < x$.

   (e) For all $x \in \mathbf{R}_{>0}$, there is $M \in \mathbf{R}$ such that $\frac{1}{x} + 1 > M$. (SB)

   (f) There is $M \in \mathbf{R}$ such that for all $x \in \mathbf{R}_{>0}$, we have $\frac{1}{x} + 1 > M$. (DD)

   (g) There is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $1 + m(x-1) \leq x^2$. (TK)

   (h) For every $a \in \mathbf{R}$, there is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $a^2 + m(x-a) \leq x^2$. (AK)

   (i) For all $x, y \in \mathbf{R}$, we have $(x^2 = y^2) \implies (x = y)$. (DM)

   (j) For all $x, y \in \mathbf{R}$, we have $(x^3 = y^3) \implies (x = y)$. (CR)