Sets

Lesson 3

Named Sets

We'll use the following names for subsets of real numbers:

$$\mathbf{R} = \text{the set of real numbers,}$$
$$\mathbb{R} = \text{the set of real numbers for handwritten text,}$$
$$\mathbf{R}_{>0} = \{x \in \mathbf{R} \mid x > 0\},$$
$$\mathbf{R}_{\neq 0} = \{x \in \mathbf{R} \mid x \neq 0\}, \text{(and similarly for other subscripts)}$$
$$\mathbf{Z} = \text{the set of integers,}$$
$$\mathbb{Z} = \text{ the set of integers for handwritten text,}$$
$$\mathbf{Q} = \text{the set of rational numbers,}$$
$$\mathbb{Q} = \text{ the set of rational numbers for handwritten text,}$$
$$\varnothing = \text{A set with no members, that is the empty set}$$

Set Operators

Definition

Let A and B be sets. Define the set *union*, *intersection*, and *difference*

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\},$$
$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\},$$
$$A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\},$$

respectively.

Set (an) example

Example

We have

$$\begin{aligned}
\{6, 107\} \cap \{28, 107\} &= \{107\}, \\
\{6, 107\} \cup \{28, 107\} &= \{6, 28, 107\}, \\
\{6, 107\} \setminus \{28, 107\} &= \{6\}, \\
\{28, 107\} \setminus \{6, 107\} &= \{28\}.
\end{aligned}$$

- The last two examples show that in general $A \setminus B \neq B \setminus A$.
- The set difference is so much like real number subtraction, sometimes the symbol "-" is used instead of $\setminus$.

Set predicates

Definition

Let A and B be sets. Define

$$A \subset B \equiv (\forall x \in A)(x \in B),$$
$$A = B \equiv (A \subset B) \wedge (B \subset A).$$

Specializing $A \subset B$ to $A = \varnothing$ gives

$$[\varnothing \subset B] \equiv (\forall x \in \varnothing)(x \in B) \equiv \text{true}.$$

We've shown that:

Theorem

Thus for all sets A and for any empty set $\varnothing$, we have $\varnothing \subset A$.

Set equality

To show that sets A and B are equal, we almost always prove that $A \subset B$ and $B \subset A$. If a proposition has the form

Theorem

If $H_1, H_2, \ldots$, and H_n, then $A = B$.

where $H_1, H_2, \ldots H_n$ is the hypothesis, a template for proving the theorem is

Proof

Suppose $x \in A$. We'll show that $x \in B$. Since $x \in A, H_1, H_2, \ldots$ and H_n, we have $\ldots$; therefore $x \in B$.

Suppose $x \in B$. We'll show that $x \in A$. Since $x \in B, H_1, H_2, \ldots$ and H_n, we have $\ldots$; therefore $x \in A$.

1. Notice how in the first case we append $x \in A$ to the hypothesis; and in the second case, we append $x \in B$.

Establish notation

Theorem

The set union is associative.

Proof

Let A, B, and C be sets. We'll show that $A \cup (B \cup C) = (A \cup B) \cup C$. Our proof uses the fact that the disjunction is associative; we have

$$\begin{aligned} A \cup (B \cup C) &= \{x \mid (x \in A) \vee (x \in B \cup C)\}, \\ &= \{x \mid (x \in A) \vee (x \in B) \vee (x \in C)\}, \\ &= \{x \mid ((x \in A) \vee (x \in B)) \vee x \in C)\}, \\ &= (A \cup B) \cup C. \end{aligned}$$

- The statement of the proposition doesn't introduce notation, so the proof must do so.
- Alternatively, we can show that $A \cup (B \cup C) \subset (A \cup B) \cup C$ and $(A \cup B) \cup C \subset A \cup (B \cup C)$.

Alternative proofs

Proof

Let A, B, and C be sets. We'll show that $A \cup (B \cup C) = (A \cup B) \cup C$. We have

$$\begin{aligned} x \in A \cup (B \cup C) &\implies (x \in A) \vee (x \in B \cup C), \\ &\implies (x \in A) \vee ((x \in B) \vee (x \in C)), \\ &\implies ((x \in A) \vee (x \in B)) \vee (x \in C), \\ &\implies x \in (A \cup B) \cup C. \end{aligned}$$

Similarly, we can show that $x \in (A \cup B) \cup C \implies x \in A \cup (B \cup C)$.

The uniqueness of emptiness

Theorem

There is at most one empty set.

Proof

Let O and O' be empty sets. Since O is empty, we have $O \subset O'$. Similarly since O' is empty, we have $O' \subset O$. We have shown that $O \subset O'$ and $O' \subset O$; therefore $O = O'$.

1. With impunity, we can now refer to **the** empty set.
2. A clumsy way to proof this is by contradiction. The proof assumes that there are empty sets O and O', but $O \neq O'$.

Conflation

Question: True or false: $\varnothing = \{\varnothing\}$.

Answer: It's false. The set $\{\varnothing\}$ is a set that has (exactly) one member, namely its member is the empty set. But the empty set has no members, so $\varnothing \neq \{\varnothing\}$

We can write this as

Counterexample ($\varnothing = \{\varnothing\}$)

We have $\varnothing \in \{\varnothing\}$, but $\varnothing \notin \varnothing$; therefore $\varnothing \neq \{\varnothing\}$.

- A *counterexample* is a proof of the negation of some statement. Usually to be considered a counter example, the proof examines one particular case.

A unique template

If a proposition has the form

Theorem

If $H_1, H_2, \ldots$, and H_n, there is at most one object X.

A template for its proof is

Proof

Let X and X' be such objects. Since $H_1, H_2, \ldots$, and H_n, we have $\ldots$; therefore $X = X'$.

1. When X and X' are real numbers, we might prove $X = X'$ by showing that both $X \leq X'$ and $X' \leq X$ are true. Together, these inequalities prove that $X = X'$.

Generalized unions

Let I be a set. And suppose that every member of I is a set. Since every member of I is a set, we can find the union of all of its members. We define

$$\bigcup_{x \in I} x = \{a \mid (\exists x \in I)(a \in x)\}.$$

In this context, we say that the set I is an *index set.*

Example

Define $I = \{\{1,2\},\{107\}\}$. Then I is a set and each member of I is a set. We have

$$\begin{aligned}
\bigcup_{x \in I} x &= \{a \mid (\exists x \in I)(a \in x)\}, \\
&= \{a \mid a \in \{1,2\} \vee a \in \{107\}\}, \\
&= \{1,2\} \cup \{107\}, \\
&= \{1,2,107\}.
\end{aligned}$$

Finite unions

Theorem

Let $A_1, A_2, \ldots, A_n$ be sets. Define an index set I by $I = \{A_1, A_2, \ldots, A_n\}$. Then

$$\bigcup_{x \in I} x = A_1 \cup A_2 \cup \cdots \cup A_n.$$

1. The set union is associative and commutative, so the meaning of $A_1 \cup A_2 \cup \cdots \cup A_n$ unambiguous.
2. An index set neededn't be finite.

Nonfinite unions

Example

The index set needn't be finite–here is an example. For $x \in \mathbf{R}$, define $I = \{(-\infty, x) \mid x \in \mathbf{R}\}$. Our index set is a set of open intervals. We claim that

$$\bigcup_{x \in I} x = \mathbf{R}.$$

Proof

Suppose $a \in \bigcup_{x \in I} x$. We'll show that $a \in \mathbf{R}$. Since $a \in \bigcup_{x \in I} x$, there is $z \in I$ such that $a \in z$. But $z \subset \mathbf{R}$, so $a \in \mathbf{R}$; we've shown that $\bigcup_{x \in I} x \subset \mathbf{R}$.

Suppose $a \in \mathbf{R}$. We'll show that $a \in \bigcup_{x \in I} x$. We have $a \in (-\infty, a+1)$. Further $(-\infty, a+1) \in I$; therefore $a \in \bigcup_{x \in I} x$.

1. Notice that $a \notin (-\infty, a)$. But it is true that $a \in (-\infty, a+1)$.
2. It's also true that $a \in (-\infty, a + 107\,\pi^2)$.

Generalized intersections

Let I be a set. And suppose that every member of I is a set. Since every member of I is a set, we can find the union of all of its members. We define

$$\bigcap_{x \in I} x = \{a \mid (\forall x \in I)(a \in x)\}.$$

Example

Define $I = \{\{1, 2\}, \{107\}\}$. Then I is a set and each member of I is a set. We have

$$\begin{aligned}
\bigcap_{x \in I} x &= \{a \mid (\forall x \in I)(a \in x)\}, \\
&= \{a \mid a \in \{1, 2\} \wedge a \in \{107\}\}, \\
&= \{1, 2\} \cap \{107\}, \\
&= \varnothing.
\end{aligned}$$

Finite intersections

Theorem

Let $A_1, A_2, \ldots, A_n$ be sets. Define an index set I by $I = \{A_1, A_2, \ldots, A_n\}$. Then

$$\bigcap_{x \in I} x = A_1 \cap A_2 \cap \cdots \cap A_n.$$

1. The set intersection is associative and commutative, so the meaning of $A_1 \cap A_2 \cap \cdots \cap A_n$ unambiguous.
2. An index set needn't be finite.

Nonfinite intersections

Example

For $x \in \mathbf{R}$, define $I = \{(-\infty, x) \mid x \in \mathbf{R}\}$. Our index set is a set of open intervals. We claim that

$$\bigcap_{x \in I} x = \varnothing.$$

Proof

We'll prove this using contradiction. Suppose $\bigcap_{x \in I} x$ has at least one member; say $a \in \bigcap_{x \in I} x$. We have

$$(\forall x \in \mathbf{R})(a \in (-\infty, x)).$$

In particular, we have $a \in (-\infty, a)$. But $a \in (-\infty, a)$ is false; therefore $\bigcap_{x \in I} x$ cannot have a member, so $\bigcap_{x \in I} x$ is the empty set.

Alternative notation

Sometimes we take the index set to be a subset of $\mathbf{R}$ and we denote the sets members by subscripts. Say $I \subset \mathbf{R}$ and A_x is a set for each $x \in I$. This notation is particularly popular when $I = \mathbf{Z}_{>0}$. For example

$$\bigcap_{k \in \mathbf{Z}_{>0}} A_k = \{a \mid (\forall n \in \mathbf{Z}_{>0})(a \in A_n)\}$$

And

$$\bigcup_{k \in \mathbf{Z}_{>0}} A_k = \{a \mid (\exists n \in \mathbf{Z}_{>0})(a \in A_n)\}$$

When the index set is uncountable, maybe it's just me, but definitions such as

$$A_x = (-\infty, x) \text{ for all } x \in \mathbf{R}$$

are semi-bazaar looking. For such cases, I think it's more clear to define the index set to be a set of sets:

$$I = \{(-\infty, x) \mid x \in \mathbf{R}\}.$$

Functions

To define a function F with domain A and formula blob, we can write

$$F = x \in A \mapsto \text{blob}.$$

In the rare cases that it's important to give the function a codomain, we can write

$$F = x \in A \mapsto \text{blob} \in B,$$

where $\mathrm{codomain}(F) = B$. Generically for a function F with domain A and codomain B, we say that F is a function from A to B.

Example

The notation

$$F = x \in [-1, 1] \mapsto 2x + 1$$

is our compact way of writing: Define $F(x) = 2x + 1$, for $-1 \leq x \leq 1$.

Function signature

The notation $F : A \to B$ means

1. F is a function.
2. $\mathrm{dom}(F) = A$.
3. $\mathrm{codomain}(F) = B$.

We'll say that $A \to B$ is the *signature* of a function. The signature of a function doesn't tell us its formula. It does tell us the domain of a function and it indicates what the outputs of the function can be.

Range

Definition

For any function, we define

$$\mathrm{range}(F) = \{F(x) | x \in \mathrm{dom}(F)\}.$$

Thus $\mathrm{range}(F)$ is the set of all outputs.

Fact

Let F be a function. Then

$$[y \in \mathrm{range}(F)] \equiv (\exists x \in \mathrm{dom}(F))\,(y = F(x)).$$

Example

Define $F = x \in [-1, 1] \mapsto 2x + 1$. Then $\frac{3}{2} \in \mathrm{range}(F)$ because $\frac{1}{4} \in \mathrm{dom}(F)$ and $F(\frac{1}{4}) = \frac{3}{2}$.

Ontoness

The codomain of a function tells us something about its outputs, but remember that the range and the codomain of a function need not be the same. For all functions F, we have

$$\operatorname{range} F \subset \operatorname{codomain}(F).$$

Definition

A function is *onto* if its range and codomain are equal.

Example

Question: Is the sine function onto? **Answer** It is if its codomain is $[-1, 1]$. But if its codomain is $\mathbf{R}$, then no it's not onto. There is no standard value for the codomain of the trigonometric functions, so the asking "Is the sine function onto?" is rubbish.

Equality

Definition

Functions F and G are *equal* $\mathrm{dom}(F) = \mathrm{dom}(G)$ and for all $x \in \mathrm{dom}(F)$, we have $F(x) = G(x)$. Equivalently

$$(F = G) \equiv (\mathrm{dom}(F) = \mathrm{dom}(G)) \wedge (\forall x \in \mathrm{dom}(F))(F(x) = G(x)).$$

1. The definition of function equality does not involve the codomain of the function. Thus two functions can be equal, but have unequal codomains.

Example

The functions $F = x \in [-1, 1] \mapsto x \in [-1, 1]$ and $G = x \in [-1, 1] \mapsto x \in \mathbf{R}$ are equal, but F is onto and G is not onto. Thus ontoness isn't a property of a function.

Apply a function to a set

Definition

Let $F : A \to B$. For any subset A' of A define

$$F(A') = \{F(x) | x \in A'\}.$$

Equivalently, we have

$$y \in F(A') \equiv (\exists x \in A')(y = F(x)).$$

Theorem

For all functions F, we have $F(\operatorname{dom} F) = \operatorname{range}(F)$. Further $F(\varnothing) = \varnothing$.

Inverse image

Definition

Let $F : A \to B$. For any subset B' of B define

$$F^{-1}(B') = \{x \in A | F(x) \in B\}.$$

Equivalently, we have

$$x \in F^{-1}(B') \equiv F(x) \in B.$$