Covers

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Covers

Definition

Let A be a subset of $\mathbf{R}$. We say $\mathcal{C}$ is a *cover of the set A* provided

1. $\mathcal{C}$ is a set,
2. every member of $\mathcal{C}$ is an open set,
3. we have $A \subset \bigcup_{x \in \mathcal{C}} x$.

- A cover is a bit like a quilt–each square of the quilt is too small to cover the bed, but collectively (that is their union) covers the bed.
- When a set has a cover, it is *not* unique.

Examples and nonexamples

Example

1. The set $\mathbf{R}$ is not a cover of $\mathbf{R}$. Why? The members of $\mathbf{R}$ are real numbers, not open sets.
2. The set $\{\mathbf{R}\}$ is a cover of $\mathbf{R}$. Why?
    1. $\{\mathbf{R}\}$ is a set–sure, it's a set with one member that is a set.
    2. every member of $\{\mathbf{R}\}$ is an open set–sure, the only member is $\mathbf{R}$ and we know that $\mathbf{R}$ is open.
    3. We have

$$\bigcup_{x \in \{\mathbf{R}\}} x = \mathbf{R}.$$

3. The set $\varnothing$ is a cover of itself.
    1. $\varnothing$ is a set–sure, it's a set.
    2. every member of $\varnothing$ is an open set–sure, it's vacuously true.
    3. We have

$$\bigcup_{x \in \varnothing} x = \varnothing.$$

More Examples, less nonexamples

Example

1. Define $\mathcal{C} = \{\mathrm{B}(0,k) \mid k \in \mathbf{Z}_{>0}\}$. Then $\mathcal{C}$ is a cover of $\mathbf{R}$.
   1. $\mathcal{C}$ is a set–sure, it's a set.
   2. every member of $\mathcal{C}$ is an open set–sure, every member is an open ball.
   3. We have

      $$\bigcup_{x\in\mathcal{C}} x = \mathbf{R}.$$

      Claim: $\mathbf{R} \subset \bigcup_{x\in\mathcal{C}} x$.
      Proof: Let $z \in \mathbf{R}$. Then $z \in \mathrm{B}(0, \lceil |z| \rceil + 1)$. But $\mathrm{B}(0, \lceil |z| \rceil + 1) \in \mathcal{C}$; therefore $z \in \bigcup_{x\in\mathcal{C}} x$.

Subcovers

Definition

Let $\mathcal{C}$ be a cover of a set A. Any subset $\mathcal{C}'$ of $\mathcal{C}$ is a *subcover* of $\mathcal{C}$ provided $\mathcal{C}'$ is a cover of A. If $\mathcal{C}'$ is a finite set, it's called a *finite subcover of* A.

1. A set is finite if either it is empty or its members can be uniquely labeled using the integers 1 to n, for some integer n.
2. The set $\{\mathbf{R}\}$ is finite; the set $\mathbf{R}$ is not finite.
3. The set $\{\infty\}$ is finite.

Examples of subcovers

Example

1. The set $\{\mathrm{B}(0, x) \mid x \in \mathbf{R}_{>0}\}$ is a cover of $[0, 1]$. The set $\{\mathrm{B}(0, 2)\}$ is a finite subcover.
2. The set $\{\mathrm{B}(0, x) \mid x \in \mathbf{R}_{>0}\}$ is a cover of $\mathbf{R}$. This cover has no finite subcover.
   Why Every member of t $\{\mathrm{B}(0, x) \mid x \in \mathbf{R}_{>0}\}$ is bounded. The finite union of bounded sets is bounded. Thus regardless of what finite subset of $\{\mathrm{B}(0, x) \mid x \in \mathbf{R}_{>0}\}$ we choose, its union will be bounded. But $\mathbf{R}$ is unbounded, so is is not contained in any bounded set.