

A Let/choose example

August 20, 2022

Can we make sense of this?

Proposition

We have

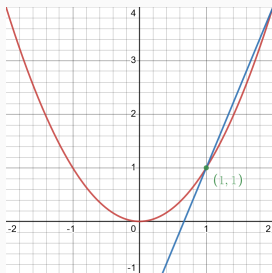
$$(\forall a \in \mathbf{R}) (\exists m \in \mathbf{R}) (\forall x \in \mathbf{R}) (x^2 \geq a^2 + m(x - a)) .$$

- Puzzled? Good.
- Let's try GNAT (= Geometric, Numerical, Algebraic, and Thinking).
- But how to do that?

G is for geometric

The main event of our proposition is the inequality $x^2 \geq a^2 + m(x - a)$.

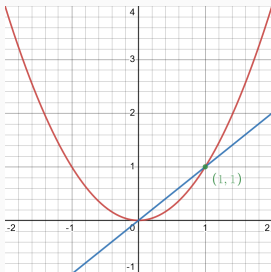
- For given values of m and a , geometrically this says the parabola $y = x^2$ is above or touching the line $y = a^2 + m(x - a)$.
- Let's try $a = 1$ and $m = 3$. A graph is



- To the left of one, the line is below the parabola, where it should be,
- but just to the right of one, the line is *above* the parabola, where it shouldn't be.
- Apparently, the line is too steep.

Let's be shallow

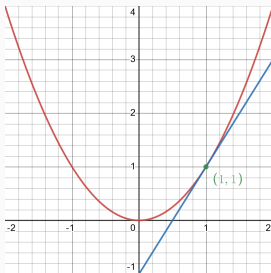
- Again, let's try $a = 1$. Since $m = 3$ was too big, let's try $m = 1$. A graph is






- To the left of one, the line is above the parabola, where it shouldn't be,
- but to the right of one, the line is *below the parabola*, where it should be.
- OK: $m = 1$ is too small and $m = 3$ is too big. What do we choose?

Let's not be shallow

You guessed it: Choose $m = 2$. The graph is



Now the parabola is uniformly above or touching the line. Apparently, the magic choice for m is two.

-  Choices are hard.
-  Like a Jeep, for m there is only one (possible value).
-  Choose $m = 2 + \frac{1}{10^{100}}$, the inequality will not be satisfied for some value of x or a .

G is also for Generalization

Question OK, when $a = 1$, we need to choose $m = 2$. What's the story for other values of a ?

Answer Sure. We need to choose the line to be the *tangent line*. Thus in the general case, we need to choose $m = 2a$.

A is for algebraic

Proposition


We have

$$(\forall a \in \mathbf{R}) (\exists m \in \mathbf{R}) (\forall x \in \mathbf{R}) (x^2 \geq a^2 + m(x - a)) .$$





Proof

Let $a \in \mathbf{R}$. Choose $m = 2a$. Let $x \in \mathbf{R}$. We have

$$\begin{aligned} [x^2 \geq a^2 + m(x - a)] &\equiv [x^2 \geq a^2 + 2a(x - a)] , && \text{(substitution)} \\ &\equiv [x^2 \geq 2ax - a^2] , && \text{(expand)} \\ &\equiv [x^2 - 2ax + a^2 \geq 0] , && \text{(subtraction)} \\ &\equiv [(x - a)^2 \geq 0] , && \text{(factor)} \\ &\equiv \text{True} . && \text{(inequality fact)} \end{aligned}$$

 Since a precedes m in the statement $(\forall a \in \mathbf{R}) (\exists m \in \mathbf{R}) \dots$, it's OK for m to depend on a .

D is for Details

-  The main part of the proof is a string of *logical equivalences* of the form $\text{blob}_1 \equiv \text{blob}_2 \equiv \dots \equiv \text{True}$.
-  The transitive property of equivalence allows us to conclude that $\text{blob}_1 \equiv \text{True}$. And that was what we wanted to prove.
-  For a proof of this form to be valid, it's vital that *every* pair of statements be connected by equivalence, not implication.
-  If we have instead $\text{blob}_1 \implies \text{blob}_2 \implies \dots \implies \text{True}$, congratulations! You just proved that

$$\text{blob}_1 \implies \text{True}$$

is true. But $P \implies \text{True}$ is a true statement regardless of the truth value of P .

Proposition

We have

$$(\forall a \in \mathbf{R}) (\exists m \in \mathbf{R}) (\forall x \in \mathbf{R}) (x^2 \geq a^2 + m(x - a)) .$$

Fake Proof

Let $a \in \mathbf{R}$. Choose $m = a + x$. Let $x \in \mathbf{R}$. We have

$$\begin{aligned} [x^2 \geq a^2 + m(x - a)] &\equiv [x^2 \geq a^2 + (x + a)(x - a)] , && \text{(substitution)} \\ &\equiv [x^2 \geq x^2] , && \text{(expand)} \\ &\equiv \text{True} . && \text{(equality fact)} \end{aligned}$$

R is for Rubbish

Proposition

We have

$$(\forall a \in \mathbf{R}) (\exists m \in \mathbf{R}) (\forall x \in \mathbf{R}) (x^2 \geq a^2 + m(x - a)) .$$

Fake Proof

Let $a \in \mathbf{R}$. Choose $m = a + x$. Let $x \in \mathbf{R}$. We have

 Since x follows m in the statement

$$(\forall a \in \mathbf{R}) (\exists m \in \mathbf{R}) (\forall x \in \mathbf{R})$$

the value of m *cannot* depend on x .

 The choice $m = a + x$ violates this.

 So this is a “fake proof.”