A Let/choose example

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Can we make sense of this?

Proposition

We have

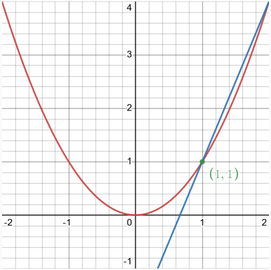
$$(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R})\,(\forall\ x \in \mathbf{R})\,\left(x^2 \geq a^2 + m(x-a)\right).$$

- Puzzled? Good.
- Let's try GNAT (= Geometric, Numerical, Algebraic, and Thinking).
- But how to do that?

G is for geometric

The main event of our proposition is the inequality $x^2 \geq a^2 + m(x - a)$.

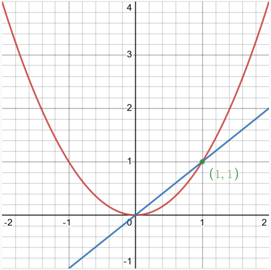
- For given values of m and a, geometrically this says the parabola $y = x^2$ is above or touching the line $y = a^2 + m(x - a)$.
- Let's try $a = 1$ and $m = 3$. A graph is



- To the left of one, the line is below the parabola, where is should be,
- but just to the right of one, the line is *above* the parabola, where it shouldn't be.
- Apparently, the line is too steep.

Let’s be shallow

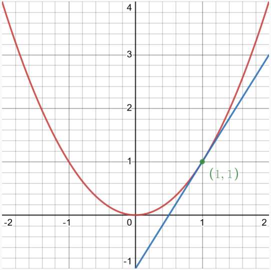
- Again, let’s try $a = 1$. Since $m = 3$ was too big, let’s try $m = 1$. A graph is



- To the left of one, the line is above the parabola, where it shouldn’t be,
- but to the right of one, the line is *below the parabola*, where it should be.
- OK: $m = 1$ is too small and $m = 3$ is too big. What do we choose?

Let's not be shallow

You guessed it: Choose $m = 2$. The graph is



Now the parabola is uniformly above or touching the line. Apparently, the magic choice for m is two.

- ✎ Choices are hard.
- ✎ Like a Jeep, for m there is only one (possible value).
- ✎ Choose $m = 2 + \frac{1}{10^{100}}$, the inequality will not be satisfied for some value of x or a.

G is also for Generalization

Question OK, when $a = 1$, we need to choose $m = 2$. What's the story for other values of a?

Answer Sure. We need to choose the line to be the *tangent line*. Thus in the general case, we need to choose $m = 2a$.

✎ In words, $(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R})\,(\forall\ x \in \mathbf{R})\,\big(x^2 \geq a^2 + m(x - a)\big)$ means that *every* tangent line to the parabola is below or touching the parabola.

A is for algebraic

Proposition

We have

$$(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R})\,(\forall x \in \mathbf{R})\,\left(x^2 \geq a^2 + m(x-a)\right).$$

Proof

Let $a \in \mathbf{R}$. *Choose* $m = 2a$. *Let* $x \in \mathbf{R}$. *We have*

$$\begin{aligned}
\left[x^2 \geq a^2 + m(x-a)\right] &\equiv \left[x^2 \geq a^2 + 2a(x-a)\right], && \textit{(substitution)}\\
&\equiv \left[x^2 \geq 2ax - a^2\right], && \textit{(expand)}\\
&\equiv \left[x^2 - 2ax + a^2 \geq 0\right], && \textit{(subtraction)}\\
&\equiv \left[(x-a)^2 \geq 0\right], && \textit{(factor)}\\
&\equiv \textit{True}. && \textit{(inequality fact)}
\end{aligned}$$

✎ Since a precedes m in the statement $(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R}) \ldots$, it's OK for m to depend on a.

D is for Details

- ✎ The main part of the proof is a string of *logical equivalences* of the form $\text{blob}_1 \equiv \text{blob}_2 \equiv \cdots \equiv \text{True}$.
- ✎ The transitive property of equivalence allows us to conclude that $\text{blob}_1 \equiv \text{True}$. And that was what we wanted to prove.
- ✎ For a proof of this form to be valid, it's vital that *every* pair of statements be connected by equivalence, not implication.
- ✎ If we have instead $\text{blob}_1 \implies \text{blob}_2 \implies \cdots \implies \text{True}$, congratulations! You just proved that

  $$\text{blob}_1 \implies \text{True}$$

  is true. But $P \implies \text{True}$ is a true statement regardless of the truth value of P.

Grade my work

Proposition

We have

$$(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R})\,(\forall\ x \in \mathbf{R})\left(x^2 \geq a^2 + m(x-a)\right).$$

Fake Proof

Let $a \in \mathbf{R}$. Choose $m = a + x$. Let $x \in \mathbf{R}$. We have

$$\begin{aligned}
\left[x^2 \geq a^2 + m(x-a)\right] &\equiv \left[x^2 \geq a^2 + (x+a)(x-a)\right], && \textit{(substitution)}\\
&\equiv \left[x^2 \geq x^2\right], && \textit{(expand)}\\
&\equiv \textit{True}. && \textit{(equality fact)}
\end{aligned}$$

R is for Rubbish

Proposition

We have

$$(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R})\,(\forall\ x \in \mathbf{R})\,\left(x^2 \geq a^2 + m(x-a)\right).$$

Fake Proof

Let $a \in \mathbf{R}$. Choose $m = a + x$. Let $x \in \mathbf{R}$. We have

- Since x follows m in the statement

$$(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R})\,(\forall\ x \in \mathbf{R})$$

  the value of m *cannot* depend on x.
- The choice $m = a + x$ violates this.
- So this is a “fake proof.”

T is for Thinking

Question What can we say about functions F that have the property

$$(\forall\ a \in \mathbf{R})\,(\exists\ m \in \mathbf{R})\,(\forall\ x \in \mathbf{R})\,(F(x) \geq F(a) + m(x-a))\,.$$

Specifically:

- ✎ Does this function property have a standard name?
- ✎ Does the sum, difference, product, or quotient of functions with this property also have this property?
- ✎ Is the number m always unique?
- ✎ Must the function F be differentiable? If so, must $m = F'(a)$?
- ✎ Is there an easy way to check if a function has this property?