Advanced Calculus **Name:**____________________________

Exam II Review, October 12, 2023 **Row and Seat:**________________________

1. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto 8 - \frac{1}{k}$ is bounded above.

Solution:

Proof. To show that F is bounded, we need to show that

$$(\exists M \in \mathbf{R})\,(\forall k \in \mathbf{Z}_{\geq 1})\,(F_k < M).$$

Choose $M = 8$. Let $k \in \mathbf{Z}_{\geq 1}$. We have

$$\begin{aligned}
\left[8 - \frac{1}{k} < 8\right] &\equiv \left[0 < \frac{1}{k}\right], && \text{(add } 1/k - 8\text{)}\\
&\equiv [0 < 1], && \text{(multiply by } k\text{)}\\
&\equiv \text{True.} && \square
\end{aligned}$$

Of course, when they exist, upper bounds are never unique. To complete this proof, you could choose M to be any real number greater than eight. But if you choose M to be less than eight, you will not be able to complete the proof.

2. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{(-1)^k}{k^2}$ converges.

Solution: We'll show that

$$(\exists L \in \mathbf{R})\,(\forall \varepsilon \in \mathbf{R})\,(\exists N \in \mathbf{Z}_{>0})\,(\forall k \in \mathbf{Z}_{>N})\,(|F_k - L| < \varepsilon).$$

Choose $L = 0$. Let $\varepsilon \in \mathbf{R}$. Choose $N = \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil$. Then $N \in \mathbf{Z}_{>0}$ as required. Let $k \in \mathbf{Z}_{>N}$. We have

$$\begin{aligned}
|F_k - L| &= \left|\frac{(-1)^k}{k^2}\right|, && \text{(substitute for } F \text{ and } L\text{)}\\
&= \frac{1}{k^2}, && \text{(algebra)}\\
&< \frac{1}{N^2}, && (k < N)\\
&= \frac{1}{\left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil^2}, && \text{(ceiling property)}\\
&\leq \frac{1}{\sqrt{\frac{1}{\varepsilon}}^2}, && \text{(substitute for } N\text{)}\\
&= \varepsilon. && \text{(algebra)}
\end{aligned}$$

3. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \begin{cases} k! & k < 10^9 \\ \frac{(-1)^k}{k^2} & k \geq 10^9 \end{cases}$ converges.

Solution: This is just like the previous problem, but we need to choose $N = \max(10^9, \left\lceil \sqrt{\frac{1}{\varepsilon}} \right\rceil)$.

4. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto \frac{3k+1}{2k+8}$ converges.

Solution:

Proof. We'll show that F converges to $\frac{3}{2}$. Specifically, we'll show that

$$(\forall \varepsilon \in \mathbf{R}_{>0})\,(\exists N \in \mathbf{Z}_{>0})\,(\forall k \in \mathbf{Z}_{>N})\left(\left|\frac{3k+1}{2k+8} - \frac{3}{2}\right| < \varepsilon\right).$$

Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $N = \left\lceil \frac{11}{2\varepsilon} \right\rceil$. Then as required, $N \in \mathbf{Z}_{>0}$. Let $k \in \mathbf{Z}_{>N}$. We have

$$\begin{aligned}
\left|\frac{3k+1}{2k+8} - \frac{3}{2}\right| &= \frac{11}{2\,(k+4)}, && \text{(algebra)} \\
&< \frac{11}{2N}, && (k+4 > N) \\
&= \frac{11}{2\lceil \frac{11}{2\varepsilon} \rceil}, && \text{(substitution for } N\text{)} \\
&\leq \frac{11}{2\frac{11}{2\varepsilon}}, && \text{(ceiling function property)} \\
&= \varepsilon. && \text{(algebra)}
\end{aligned}$$

□

5. Show that the sequence $F = k \in \mathbf{Z}_{\geq 1} \mapsto k - 3\lfloor \frac{k}{3} \rfloor$ does not converge to 1.

Solution:

Proof. We'll show that

$$(\exists \varepsilon \in \mathbf{R}_{>0})\,(\forall N \in \mathbf{Z}_{>0})\,(\exists k \in \mathbf{Z}_{>N})\,(|F_k - 1| \geq \varepsilon)\,.$$

Choose $\varepsilon = \frac{1}{2}$. Let $N \in \mathbf{Z}_{>0}$. Choose $k = 3N$. Then $k \in \mathbf{Z}_{>N}$ as required. We have $|F_k - 1| = |F_{3N} - 1| = 1 > \varepsilon$. □

6. Using the definition from the QRS, show that the interval $(-\infty, 8)$ is open.

Solution:

Proof. We'll show that

$$(\forall x \in (-\infty, 8))\,(\exists r \in \mathbf{R}_{>0})\,(\text{ball}(x,r) \subset (-\infty, 8)).$$

Let $x \in (-\infty, 8)$. Choose $r = 4 - \frac{x}{2}$. Since $x < 8$, we have $r > 0$, as required. We have

$$x + r = x + 4 - \frac{x}{2} = \frac{x}{2} + 4 < \frac{8}{2} + 4 = 8.$$

So $\text{ball}(x,r) \subset (-\infty, 8)$. □

7. Let $A \subset \mathbf{R}$. Using the definition of an open set in the QRS, write the undefintion of an open set. That is, complete the statement:

   A is not open $\equiv$

Solution:

$$A \text{ is not open} \equiv (\exists a \in A)\,(\forall r \in \mathbf{R}_{>0})\,(\text{ball}(a,r) \not\subset A)\,.$$

8. Using the undefition from the previous question, show that the set $(-\infty, 8) \cup \{9\}$ is not open.

Solution:

Proof. Choose $a = 9$. Let $r \in \mathbf{R}_{>0}$. Then $\text{ball}(9,r) \not\subset (-\infty, 8) \cup \{9\}$. When $r < 1$, we have $9 - r/2 \in \text{ball}(9,r)$, but $9 - r/2 \notin (-\infty, 8) \cup \{9\}$. When $r \geq 1$ we have $9 - 1/2 \in \text{ball}(9,r)$, but $9 - 1/2 \notin (-\infty, 8) \cup \{9\}$; so again, $\text{ball}(9,r) \not\subset (-\infty, 8) \cup \{9\}$. □

9. Let $A \subset \mathbf{R}$. Using the definition of a limit point in the QRS, write the undefintion of limit point. That is, complete the statement:

   $x \notin \text{lp}(A) \equiv$

Solution:

$$x \notin \text{lp}(A) \equiv (\exists r \in \mathbf{R}_{>0})\,\big(\text{ball}'(x,r) \cap A = \varnothing\big)\,.$$

10. Use your undefinition from the previous question to show that $5 \notin \text{lp}(\mathbf{Z})$.

Solution: We'll show that

$$(\exists r \in \mathbf{R}_{>0})\left(\text{ball}'(5,r) \cap \mathbf{Z} = \varnothing\right).$$

Choose $r = \frac{1}{2}$. We have $\text{ball}'(5,r) = \text{ball}'(5,1/5) = (5-\frac{1}{2},5) \cup (5,5+\frac{1}{2})$. So $\text{ball}'(5,1/2) \cap \mathbf{Z} = \varnothing$.

11. Use the QRS defintion of a *boundary point* to show that $12 \in \text{bp}((0,12))$.

Solution: Let r be a positive number, and let $x^\star = \begin{cases} 12 - r/2 & r < 24 \\ 6 & r \geq 24 \end{cases}$. Then $x^\star \in (12-r, 12+r)$ and $x^\star \in (0,12)$. Further we have $12 + r/2 \notin (0,12)$ and $12 + r/2 \in (12-r, 12+r)$.

12. Use the result of the previous question to show that $(0,12)$ is not closed.

Solution: A closed set contains all of its boundary points. We showed that 12 is a boundary point $(0,12)$, but $12 \notin (0,12)$; therefore $(0,12)$ is not closed.

13. Show that the set $\mathbf{R}$ is not compact by showing that there is an open cover of $\mathbf{R}$ that has no finite subcover.

Solution: See classnotes for Monday 9 October.

14. Show that the set $\mathbf{Z}$ is not compact by showing that there is an open cover of $\mathbf{Z}$ that has no finite subcover.

Solution: Define $\mathscr{C} = \{(-k,k) \subset \mathbf{R} \mid k \in \mathbf{Z}_{>\mathbf{0}}$. Since $\bigcup\limits_{x \in \mathscr{C}} x = (0,\infty)$, the set $\mathscr{C}$ is a cover for $\mathbf{N}$. Let $\mathscr{C}'$ be any finite subset of $\mathscr{C}$. Then $\bigcup\limits_{x \in \mathscr{C}'} x$ is bounded because it's a finite union of bounded sets. But $\mathbf{Z}$ isn't bounded, so $\mathbf{Z} \not\subset \bigcup\limits_{x \in \mathscr{C}'} x$; therefore $\mathbf{Z}$ isn't compact.

15. Let F be a convergent sequence, and let $\alpha \in \mathbf{R}$. Show that αF is a convergent sequence.

Solution: Let $\varepsilon > 0$. Since f converges, there is a number L and $M \in \mathbf{N}$ such that for all $k > M$, we have $|F_k - L| < \frac{\varepsilon}{1+|\alpha|}$. For all $k > M$, we have

$$|\alpha F_k - \alpha L| = |\alpha||F_k - L| < \frac{|\alpha|}{1+|\alpha|}\varepsilon < \varepsilon.$$

16. Let F be a convergent sequence and suppose $\text{range}(F) \subset ([0,\infty))$. Show that $\sqrt{F}$ converges. You may use the fact that $\left(\forall x, y \in \mathbf{R}_{\geq 0}\right)(|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|})$

Solution: See classnotes for Monday 9 October.

17. For the sequence $F = k \in \mathbf{Z}_{\geq 0} \mapsto k - 3\lfloor \frac{k}{3} \rfloor$, give three examples of a convergent subsequence.

Solution: Define $\phi = \mathbf{Z}_{\geq 0} k \mapsto 3k$. Then $F \circ \phi = k \in \mathbf{Z}_{\geq 0} \mapsto 0$.

Define $\phi = \mathbf{Z}_{\geq 0} k \mapsto 3k + 1$. Then $F \circ \phi = k \in \mathbf{Z}_{\geq 0} \mapsto 1$.

Define $\phi = \mathbf{Z}_{\geq 0} k \mapsto 3k + 2$. Then $F \circ \phi = k \in \mathbf{Z}_{\geq 0} \mapsto 2$.

18. Give an example of a sequence F and a real number α such that αF converges and F diverges.

Solution: Choose $F = k \in \mathbf{Z}_{\geq 0} \mapsto k$ and choose $\alpha = 0$. Then F diverges but $0F$ converges.

19. Give an example of sequences F and G such that both F and G diverge, but $F + G$ converges.

Solution: Choose $F = k \in \mathbf{Z}_{\geq 0} \mapsto k$ and $F = k \in \mathbf{Z}_{\geq 0} \mapsto -k$. Both F and G diverge, but $F + G$ converges.

20. Give an example of sequences F and G such that both F and G diverge, but FG converges.

Solution: Choose $F = k \in \mathbf{Z}_{\geq 0} \mapsto (-1)^k$ and $G = k \in \mathbf{Z}_{\geq 0} \mapsto (-1)^k$. Both F and G diverge, but $FG = k \in \mathbf{Z}_{\geq 0} \mapsto 1$ converges.

21. Show that $\left(\forall x, y \in \mathbf{R}_{\geq 0}\right)(\sqrt{x+y} \leq \sqrt{x} + \sqrt{y})$.

Solution: We begin by proving that $\left(\forall x, y \in \mathbf{R}_{\geq 0}\right)(\sqrt{x^2 + y^2} \leq x + y)$. Let $x, y \in \mathbf{R}_{\geq 0}$. We have

$$
\begin{aligned}
\sqrt{x^2 + y^2} &= \sqrt{x^2 + 2xy + y^2 - 2xy}, && \text{(add and subtract)} \\
&= \sqrt{(x+y)^2 - 2xy}, && \text{(algebra)} \\
&\leq \sqrt{(x+y)^2}, && \text{(squre root function is increasing} \\
&= x + y. && \text{(algebra)}
\end{aligned}
$$

To finish the proof, we replace $x \to \sqrt{x}$ and $y \to \sqrt{y}$ in the above. This yields $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.