# Well ordering

Lesson 10

"If you can't prove what you want to prove, demonstrate something else and pretend that they are the same."

Darrell Huff

*How to Lie with Statistics*

## Definition

Let $A$ be a subset of *reals*. We say that $A$ is *bounded above* provided

$$(\exists M \in \mathrm{R})\,(\forall x \in A)\,(x \leq M).$$

We say that the number $M$ is *an upper bound for the set* $A$. The set $A$ is *bounded below* provided

$$(\exists M \in \mathrm{R})\,(\forall x \in A)\,(M \leq x).$$

We say that the number $M$ is *a lower bound for the set* $A$. If $A$ is bounded below and bounded above, we say $A$ is *bounded*.

# Bounded details

1. Notice that we do *not* require that a upper bound for a set $A$ to be a member of $A$.
2. Same for a lower bound.
3. If $M$ is an upper bound for a set $A$, any number that is greater than $M$ is also an upper bound for $A$.
4. Since we require that an upper bound be a real number, we disallow infinity from being an upper bound. If we did, every set would be bounded.

# Bounded examples

## Example

1. The empty set is bounded above by 0.
2. Actually every real number is an upper bound for the empty set.
3. Every real number is a lower bound for the empty set.
4. The interval $[0, 1]$ is bounded above by 1.
5. The interval $[0, 1]$ is bounded above by 107.
6. The interval $[0, \infty)$ is bounded below by 0.
7. The interval $[0, \infty)$ is not bounded above.

# Being least

## Definition

Let $A$ be a subset of $\mathbb{R}$. The set $A$ has a least member provided

$$(\exists a^\star \in A)\,(\forall a \in A)\,(a^\star \leq a).$$

We say that $a^\star$ is a least member. The set $A$ has a greatest member provided

$$(\exists a^\star \in A)\,(\forall a \in A)\,(a \leq a^\star).$$

1. We require that a least member of a set $A$ be a member of the set.

# Well ordering principle

**Axiom**

Let $A \subset Z$ be (i) *nonempty* and (ii) *bounded below*. Then $A$ has a least member.

1. This is an axiom–we'll take it on faith.
2. Again, a least member of a set $A$ *must* be a member of $A$.
3. Thus the empty set does *not* have a least member.
4. The qualification that the set be nonempty for it to have a least member is crucial.

**Theorem**

Let $A \subset Z$ be (i) *nonempty* and (ii) *bounded above*. Then $A$ has a greatest member.

# Well ordering principle for the reals?

**Question** Are the real numbers well ordered? That is, does every nonempty subset of R that is bounded below have a least member?

**Answer** No. The interval $(0,1)$ is nonempty and bounded below, but it doesn't have a least member. Although zero is less than every member of $(0,1)$, since zero isn't a member of $(0,1)$, it is not a least member.

# Uniqueness of the least

**Theorem**

If a subset of the reals has a least member, it is unique.

**Proof**

Let $A \subset \mathrm{R}$. Suppose $x$ and $x'$ are least members of $A$. Since $x$ is a least member of $A$ we have $x \in A$. But $x'$ is a least member, so $x' \leq x$. Interchanging the roles of $x$ and $x'$, we have $x' \leq x$; therefore $x = x'$.

1. Equality is hard, inequality is easier.
2. We proved equality by proving two inequalities.

# Existence of the floor

Let $x \in \mathrm{R}$. Define the set $M$ by

$$M = \{k \in \mathrm{R} \mid k \geq x\}$$