# Set complements

Lesson 6

The whole problem with the world is that fools and fanatics are always so certain of themselves, and wiser people so full of doubts.

Bertrand Russel

# The set complement

## Definition

Let $U$ be the universal set. For any set $A$, we define its complement $A^c$ by $A^c = U \setminus A$.

- We have
$$(x \in A^c) \equiv (x \in U) \wedge (x \notin A).$$
- Assisted by Mr. DeMorgan, we have
$$(x \notin A^c) \equiv (x \notin U) \vee (x \in A).$$

# Every definition deserves a theorem

## Theorem

Let $A$ be a set. We have

$$\begin{aligned} \varnothing^c &= U, \\ U^c &= \varnothing, \\ (A^c)^c &= A, \\ A \cap A^c &= \varnothing, \\ A \cup A^c &= U. \end{aligned}$$

## Proof ( $U = \varnothing^c$)

We have

$$\varnothing^c = \{u \in U \mid x \notin \varnothing\} = \{u \in U \mid \text{true}\} = U.$$

- We used the fact that $(\forall x)(x \notin \varnothing) \equiv \text{true}$.
- The proof is a string of equalities. The conclusion of the proof is to compare the far left to the far right of the string.
- Alternatively, we could show that $U \subset \varnothing^c$ and $\varnothing^c \subset U$. But I think this proof is more clear.

## Proof ($(A^c)^c = A$)

Let $A$ be a set. We have

$$(A^c)^c = \{u \in U \mid u \notin A^c\} = \{u \in U \mid u \in A\} = A.$$

- Arguably this proof makes too large a jump in logic from $u \notin A^c$ to $u \in A$.
- Here is a fix:

$$\begin{aligned}\{u \in U \mid u \notin A^c\} &= \{u \in U \mid (u \notin U) \vee (u \in A)\},\\ &= \{u \in U \mid \text{false} \vee (u \in A)\},\\ &= \{u \in U \mid u \in A\},\\ &= A.\end{aligned}$$

- We have

$$x \in A \setminus B \equiv (x \in A) \wedge (x \notin B)$$

- So

$$x \notin A \setminus B \equiv \neg((x \in A) \wedge (x \notin B)) = (x \notin A) \vee (x \in B).$$

**Proof ($\varnothing^c = U$)**

We've already shown that $\varnothing = U^c$. Using this fact, we have

$$\varnothing^c = (U^c)^c = U.$$

- Our proof uses the fact that for all sets $A$, we have $(A^c)^c = A$. Since $U$ is a set, we have $(U^c)^c = U$.
- With malice aforethought, we proved the statements in the proposition in a different order than they were presented.
- We switched up the order to make use of $(A^c)^c = A$ in a later proof.

# Looks like homework

**Proof ( $A \cap A^c = \varnothing$)**

(This looks like it should be homework–it's all up to you. You are certainly allowed to use all the results we have proved so far.)

**Proof ( $A \cup A^c = U$)**

(This looks like it should be homework–it's all up to you. You are certainly allowed to use all the results we have proved so far.)