Real numbers

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Binary operator

Definition

A *binary operator* on a set S is a function from $S \times S$ to S. A binary operator F is commutative provided

$$(\forall a, b \in S)(F(a, b) = F(b, a)).$$

It is associative provided

$$(\forall a, b, c \in S)(F(a, F(b, c)) = F(F(a, b), c)).$$

It has a *left identity element* provided

$$(\exists\, \theta \in S)((\forall a \in S)(F(\theta, a) = a).$$

And it has a *right identity element* provided

$$(\exists\, \theta \in S)((\forall a \in S)(F(a, \theta) = a).$$

1. Addition and multiplication of real numbers are examples of binary operators; these operators are commutative and associative.
2. In this context, binary means that the function takes *two* members of the same set; the use of binary has nothing to do with base two representation of a number.
3. Usually binary operators are expressed in *infix notation*; that is, the operator is in between its arguments.
4. For example, we write $1 + 107 = 108$, not $+(1, 107) = 108$.
5. For a commutative binary operator, every right identity element is a left identity element; so we'll call them collectively an identity element.

Examples

(a) Addition $+$ is a binary operator on $\mathbf{R}$. Since $x + 0 = x$ for all real x, the identity element for addition is zero. Further we know that addition is commutative and associative.

(b) Function composition $\circ$ is a binary operator on the set of functions from $\mathbf{R}$ to $\mathbf{R}$. The function $x \in \mathbf{R} \mapsto x$ is the identity element for function composition. Function composition is associative, but not commutative.

Unique elements

Theorem

Let S be a set and let F be a commutative binary operator on S. Then F has at most one identity element.

Proof

Let θ and θ' be identity elements for F. We'll show that $\theta = \theta'$. We have

$$\begin{aligned} \theta &= F(\theta', \theta), && \text{(because } \theta \text{ is an identity element.)} \\ &= F(\theta, \theta'), && \text{(because } F \text{ is commutative)} \\ &= \theta'. && \text{(because } \theta' \text{ is an identity element.)} \end{aligned}$$

So $\theta = \theta'$.

Fields

We would like to capture the important features of the real numbers and give all such structures a name. This object is a *field*.

Definition

A field is an ordered triple $(\mathcal{F}, +, \times)$ where $\mathcal{F}$ is a set and both $+$ and $\times$ are commutative and associative binary operators on $\mathcal{F}$ that have identity elements; the identity element for $+$ is 0 and the identity element for $\times$ is 1.

1. For all $a, b, c \in \mathcal{F}$, we have $a \times (b + c) = a \times b + a \times c$.
2. For all $a \in \mathcal{F}$ there is $-a \in \mathcal{F}$ such that $a + -a = 0$.
3. For all $a \in \mathcal{F}_{\neq 0}$ there is $a^{-1} \in \mathcal{F}$ such that $aa^{-1} = 1$.

1. We say that $-a$ is an additive inverse of a.
2. We say that a^{-1} is a multiplicative inverse of a.

Unique inverses

Theorem

Let $(\mathcal{F}, +, \times)$ be a field. The additive and multiplicative inverses are unique.

Proof

Let $a \in \mathcal{F}$ and suppose $a + b = 0$ and $a + b' = 0$. We'll show that $b = b'$. We have

$$
\begin{aligned}
b &= b + 0, \\
&= b + (a + b'), \\
&= (b + a) + b', \\
&= (a + b) + b', \\
&= 0 + b', \\
&= b'.
\end{aligned}
$$

The proof for the multiplicative inverse is similar and left as an exercise for the willing.

Famous fields

Let $+$ and $\times$ be ordinary number addition and multiplication, respectively. Then

a. $(\mathbf{R}, +, \times)$ is the real field.
b. $(\mathbf{Q}, +, \times)$ is the rational field. Certainly the sum and product of rational numbers is a rational number so indeed, $+ : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ and similarly for $\times$. The other required conditions are “inherited” from the properties of the real field.
c. $(\mathbf{Z}, +, \times)$ isn't a field because, for example, there is no $x \in \mathbf{Z}$ such that $2x = 1$.

It's true: $x \times \theta = \theta$

Proposition

Let $(\mathcal{F}, +, \times)$ be a field and let θ be its additive identity. Then

$$(\forall a \in \mathcal{F})\,(a \times \theta = \theta) \tag{1}$$

Proof.

Let $x \in \mathcal{F}$. We have $x + \theta = x$. Multiplying this by x and using the distributive property, we have $x^2 + x\theta = x^2$. Adding $-x^2$ to this yields $x \times \theta = \theta$.

$\square$

It's true: $a \times (-b) = -(a \times b)$

Proposition

Let $(\mathcal{F}, +, \times)$ be a field and let $\mathcal{O}$ be its additive identity. Then

$$(\forall a, b \in \mathcal{F})\,(a \times (-b) = -(a \times b))\,. \tag{2}$$

Proof.

Let $a, b \in \mathcal{F}$. We have

$$\theta = a \times \theta = a \times (b + -b) = a \times b + a \times (-b).$$

So $-(a \times b) = a \times (-b)$ $\square$

Something for nothing

We know that $-(a \times b) = a \times (-b)$ is an identity. Replacing the silent variable a by $-a$ gives an syntactically different but semantically equal identity

$$-((-a) \times b) = (-a) \times (-b)$$

But $((-a) \times b) = b \times (-a) = -(a \times b)$. That tells gives us the famous result that

$$(a \times b) = (-a) \times (-b).$$

Let's **not** confuse this mathematical fact with **rubbish** such as "Morwenna deposits -\$10 a total of -5 times. What is Morwenna's account value?"

Ordered Fields

Definition

A field $(\mathcal{F}, +, \times)$ is ordered provided there is a subset P of $\mathcal{F}$ such that

a. If $a, b \in P$, we have $a + b \in P$,
b. If $a, b \in P$, we have $a \times b \in P$,
c. For all $a \in \mathcal{F}$ exactly one of the following is true: (i) $a \in P$, (ii) $-a \in P$, (iii) $a = 0$.