MATH 460 **Name:**

This past week, I reread a book of my youth (the actual copy): *Introduction to Mathematical Logic*, by Flora Dinkines. Here are some fun questions that are adapted from this textbook. This is *not* an assignment, bonus or otherwise. But if you would like a mid-July logic exercise that is more educational than solving another Sudoku puzzle, give these problems a try. And if you have questions about them, please let me know.

1. Define the predicate

$$Q = (x, y) \in \mathbf{R}^2 \mapsto [x + 2y = 4].$$

For example, we have $Q(1,1) = [3 = 4] = \text{False}$ and $Q(2,1) = [4 = 4] = \text{True}$. Decide on the truth value of each the following statements; if the statement is true, prove it; if the statement is false, prove that its negation is true.

(a) $(\forall x \in \mathbf{R})\left(\exists y \in \mathbf{R}\right)\left(Q(x, y)\right)$.

Proof. Let $x \in \mathbf{R}$. Choose $y = \frac{4-x}{2}$; then $y \in \mathbf{R}$ as required. We have

$$\begin{aligned}
\left[x + 2y = 4\right] &= \left[x + 2 \times \frac{4-x}{2} = 4\right], && \text{(substitution)} \\
&= [4 = 4], && \text{(algebra)} \\
&= \text{True}. && \text{(syntactic equality)}
\end{aligned}$$

□

(b) $(\exists x \in \mathbf{R})\left(\forall y \in \mathbf{R}\right)\left(Q(x, y)\right)$.

We'll show that the statement is false by showing that its negation is true; its negation is

$$(\forall x \in \mathbf{R})\left(\exists y \in \mathbf{R}\right)\left(x + 2y \neq 4\right)$$

Proof. Let $x \in \mathbf{R}$. Choose $y = -\frac{x}{2}$. Then $y \in \mathbf{R}$ as required. We have

$$\begin{aligned}
\left[x + 2y \neq 4\right] &= \left[x + 2 \times -\frac{x}{2} \neq 4\right], && \text{(substitution)} \\
&= [0 \neq 4], && \text{(algebra)} \\
&= \text{True}. && \text{(syntactic inequality)}
\end{aligned}$$

□

(c) $(\forall x \in \mathbf{R})\left(\exists y \in \mathbf{R}\right)\left(\neg Q(x,y)\right)$.

Proof. (See part‘b.’)

□

(d) $(\forall x \in \mathbf{R})\neg\left(\forall y \in \mathbf{R}\right)\left(Q(x,y)\right)$. To start, let’s rewrite the statement as

$$(\forall x \in \mathbf{R})\left(\exists y \in \mathbf{R}\right)\left(x + 2y \neq 4\right).$$

Proof. (See part‘b.’)

□

2. Find examples of predicates P and Q such that the statement

$$(\forall x)\,(P(x) \vee Q(x)) \equiv (\forall x)\,(P(x)) \vee (\forall x)\,(Q(x))$$

is false. This shows that the existential qualifier does not distribute over the disjunction.

Proof. Define predicates P and Q as

$$P = k \in \mathbf{Z} \mapsto k \text{ is even},$$
$$Q = k \in \mathbf{Z} \mapsto k \text{ is odd}.$$

Since $Q = \neg P$, for all integers k, we have $(\forall k \in \mathbf{Z})\,(P(k) \vee Q(k))$ is true. But both the statements $(\forall x)\,(P(x))$ and $(\forall x)\,(Q(x))$ are false, so the statement $(\forall x)\,(P(x)) \vee (\forall x)\,(Q(x))$ is false. □

3. Does the existential qualifier distribute over the conjunction?

4. For any predicate P, show that

$$(\exists x)\left(\exists y\right)\left(P(x,y)\right) \equiv \left(\exists y\right)(\exists x)\left(P(x,y)\right).$$

Proof. First, we’ll show that $(\exists x)\left(\exists y\right)\left(P(x,y)\right) \implies \left(\exists y\right)(\exists x)\left(P(x,y)\right)$. Suppose $\left(\exists y\right)\left(P(x,y)\right)$ Thus there are a and b such that $P(a,b)$ is true. To show that $\left(\exists y\right)(\exists x)\left(P(x,y)\right)$ is true, we only need to choose $y = a, x = b$.

□

5. For any predicate P, show that

$$(\forall x)\left(\forall y\right)\left(P(x,y)\right) \equiv \left(\forall y\right)(\forall x)\left(P(x,y)\right).$$

6. For any predicate P, show that

$$(\exists x)\left(\forall y\right)\left(P(x,y)\right) \implies \left(\forall y\right)(\exists x)\left(P(x,y)\right).$$

7. Show there is a predicate P such that

$$\left(\forall y\right)(\exists x)\left(P(x,y)\right) \implies (\exists x)\left(\forall y\right)\left(P(x,y)\right).$$

is false.