**Advanced Calculus, Fall 2022**
*Review for Exam II*

1. Show that the sequence $k \in \mathbf{Z}_{>0} \mapsto \frac{k+1}{k+5}$ converges.

> **Solution:** We'll show that
>
> $$(\exists\, L \in \mathbf{R})(\forall\, \varepsilon \in \mathbf{R}_{>0})(\exists\, n \in \mathbf{Z})\,(\forall\, k \in \mathbf{Z}_{>n})\left(\left|\frac{k+1}{k+5} - L\right| < \varepsilon\right).$$
>
> Choose $L = 1$. Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $n = \left\lceil \frac{4}{\varepsilon} \right\rceil$. Let $k \in \mathbf{Z}_{>n}$. We have
>
> $$\begin{aligned} \left|\frac{k+1}{k+5} - L\right| &= \left|\frac{k+1}{k+5} - 1\right|, && \text{(substitution)} \\ &= \frac{4}{k+5}, && \text{(algebra)} \\ &< \frac{4}{n}, && (k+5 > n) \\ &= \frac{4}{\left\lceil \frac{4}{\varepsilon} \right\rceil}, && \text{(substitution)} \\ &\le \frac{4}{\frac{4}{\varepsilon}}, && \text{(ceiling property)} \\ &= \varepsilon. && \text{(algebra)} \end{aligned}$$

2. Give an example of a convergent subsequence of $F = k \in \mathbf{Z}_{>0} \mapsto (-1)^k$.

> **Solution:** Define $\phi = n \in \mathbf{Z} \mapsto 2n$. Then $F \circ \phi = k \in \mathbf{Z} \mapsto (-1)^{2k}$. This is a constant sequence; it converges.

3. Show that sequence $k \in \mathbf{Z}_{>0} \mapsto \begin{cases} k! & k < 1000 \\ \frac{k+1}{k+5} & k \ge 1000 \end{cases}$ converges.

> **Solution:** We'll show that
>
> $$(\exists\, L \in \mathbf{R})(\forall\, \varepsilon \in \mathbf{R}_{>0})(\exists\, n \in \mathbf{Z})(\forall\, k \in \mathbf{Z}_{>n})\left(\left|\frac{k+1}{k+5} - L\right| < \varepsilon\right).$$

Choose $L = 1$. Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $n = \max(1000, \lceil \frac{4}{\varepsilon} \rceil)$. Let $k \in \mathbf{Z}_{>n}$. We have

$$\begin{aligned}
\left| \frac{k+1}{k+5} - L \right| &= \left| \frac{k+1}{k+5} - L \right|, && (k > 1000) \\
&= \left| \frac{k+1}{k+5} - 1 \right|, && \text{(substitution)} \\
&= \frac{4}{k+5}, && \text{(algebra)} \\
&< \frac{4}{n}, && (k+5 > n) \\
&= \frac{4}{\lceil \frac{4}{\varepsilon} \rceil}, && \text{(substitution)} \\
&\le \frac{4}{\frac{4}{\varepsilon}}, && \text{(ceiling property)} \\
&= \varepsilon. && \text{(algebra)}
\end{aligned}$$

4. Use the QRS definition of an open set to show that interval $(0,1)$ is open.

**Solution:** We'll show that

$$(\forall\, x \in (0,1))(\exists\, r \in \mathbf{R}_{>0})(\mathrm{ball}(x,r) \subset (0,1)).$$

Let $x \in (0,1)$. Choose $r = \frac{1}{2}\min(1-x, x)$. Since $x > 0$ and $x < 1$, we have $r \in \mathbf{R}_{>0}$ as required. We need to show that $0 < x - r$ and $x + r < 1$. We have

$$[0 < x - r] \equiv [r < x] \equiv \text{True}$$

And

$$[x + r < 1] \equiv [x < 1 - r] \equiv \text{True}$$

5. Use the QRS definition of a closed set to show that interval $[0,1]$ is closed.

**Solution:** We need to show that the complement of $[0,1]$ is open. We have $[0,1]^{\mathrm{C}} = (-\infty, 0) \cup (1, \infty)$. But both $(-\infty, 0)$ and $(1, \infty)$ are open. And we know that the union of open sets is open, so $[0,1]^{\mathrm{C}}$ is open; therefore $[0,1]$ is closed.

Arguably we should use the QRS definition to show that both $(-\infty, 0)$ and $(1, \infty)$ are open. But the fact that these sets are open is a book theorem.

6. Use the QRS definitions of open and closed to show that the set $\mathbf{R}$ is open and closed.

**Solution:** Let $x$ be a real number. We have $\operatorname{ball}(x,1) \subset \mathbf{R}$; therefore, $\mathbf{R}$ is open. To show that $\mathbf{R}$ is closed, we'll show that $\mathbf{R}^{\mathrm{C}}$ is open; since $\mathbf{R}^{\mathrm{C}} = \varnothing$, we'll show that $\varnothing$ is open. To show that $\varnothing$ is open, we need to show that

$$(\forall\, x \in \varnothing)\,(\exists\, r \in \mathbf{R}_{>0})\,(\operatorname{ball}(x,r) \subset \varnothing)\,.$$

This is vacuously true.

7. Use the QRS definition of a boundary point to show that $\partial(0,1] = \{0,1\}$. Use this result to explain why $(0,1]$ is not closed.

**Solution:** First, we'll show that $0 \in \partial(0,1]$. Let $\delta$ be a positive real number. We have $-\delta/2 \notin (0,1]$. Further define $x^\star = \begin{cases} \frac{\delta}{2} & \text{if } \delta < 1 \\ \frac{1}{2} & \text{if } \delta \geq 1 \end{cases}$. Then $x^\star \in (0,1]$ and $x^\star \in \operatorname{ball}(0,\delta)$. I'll leave it to you to show that $1 \in (0,1]$.

The set $(0,1]$ is not closed because $0 \in \partial(0,1]$ and $0 \notin (0,1]$. (A closed set must contain all of its boundary points.)

8. Use the QRS definition to show that $0 \not\in \operatorname{LP}(\mathbf{Z})$.

**Solution:** We need to show that $(\exists\, r \in \mathbf{R}_{>0})(\operatorname{ball}'(0,r) \cap \mathbf{Z} = \varnothing)$. Choose $r = 1/2$. Then $\operatorname{ball}'(0,r) \cap \mathbf{Z} = \varnothing$.

9. Show that the function $F(x) = \begin{cases} -1 & \text{if } x < 5 \\ 1 & \text{if } x \geq 5 \end{cases}$ does not have a limit toward 5.

**Solution:** We'll show that

$$(\forall\, L \in \mathbf{R})\,(\exists\, \varepsilon \in \mathbf{R}_{>0})\,(\forall\, \delta \in \mathbf{R}_{>0})\,\big(\exists\, x \in \operatorname{ball}'(5,\delta)\big)\,(|F(x) - L| \ge \varepsilon)\,.$$

Let $L \in \mathbf{R}$. Choose $\varepsilon = 1$. Let $\delta \in \mathbf{R}_{>0}$. Choose $x = \begin{cases} 5+\delta/2 & L \le 0 \\ 5-\delta/2 & L \ge 0 \end{cases}$. When $L \le 0$, we have

$$|F(x) - L| = |F(5+\delta/2) - L| = |1 - L| \ge 1.$$

The case $L > 0$ is similar.

10. Show that the function $F(x) = x^2$ has a limit toward 2.

**Solution:** Let $\varepsilon$ be a positive real number. Choose $\delta = \min\{1, \frac{\delta}{5}\}$. Let $x \in \operatorname{ball}(2,\delta)$. To start, we notice that since $|x-2| \le 1$, we have $1 \le x \le 3$; thus $3 \le x+2 \le 5$. So $|x+2| \le 5$. We have

$$\begin{aligned} |x^2 - 4| &= |x-2||x+2|, && \text{(algebra)} \\ &< |x+2|\delta, && (x \in \operatorname{ball}(2,\delta)) \\ &\le 5\delta, && (|x+2| \le 5) \\ &\le \varepsilon. && (\delta \le \varepsilon/5) \end{aligned}$$

11. Show that the set $(0,\infty)$ is not compact by showing that there is an open cover of $(0,\infty)$ that has no finite subcover.

**Solution:** For $k \in \mathbf{Z}_{>0}$, define $I_k = (-k,k)$. The set $\mathscr{C} = \{I_k | k \in \mathbf{Z}_{>0}\}$ is a cover of $(0,\infty)$. The union of every finite subset of $\mathscr{C}$ is bounded. Thus no finite subset of $\mathscr{C}$ is a cover of $(0,\infty)$; therefore, $(0,\infty)$ is not compact.

12. Show that if a subset of $\mathbf{R}$ is not bounded, it is not compact. Do this using the definition of compact that involves open covers.

**Solution:** **Proof** See your class notes.

13. Show that the union of two compact sets is compact. Do this using the definition of compact that involves open covers.

**Solution:** Let $F_1$ and $F_2$ be compact, and let $\mathscr{C}$ be an open cover of $F_1 \cup F_2$. Then $\mathscr{C}$ is an open cover of $F_1$ and $\mathscr{C}$ is an open cover of $F_2$. Since $F_1$ and $F_2$ are compact, there are finite sets $\mathscr{C}_1 \subset \mathscr{C}$ and $\mathscr{C}_2 \subset \mathscr{C}$ such that $F_1 \subset \bigcup_{x \in \mathscr{C}_1} x$ and $F_2 \subset \bigcup_{x \in \mathscr{C}_2} x$. Thus $F_1 \cup F_2 \subset \bigcup_{x \in \mathscr{C}_1 \cup \mathscr{C}_2} x$. Since $\mathscr{C}_1 \cup \mathscr{C}_2$ is finite, it follows that $F_1 \cup F_2$ is compact.

14. Show that if sets $A$ and $B$ are closed, so is $A \cup B$.

**Solution:** We'll show that $(A \cup B)^{\mathrm{C}}$ is open. We have $(A \cup B)^{\mathrm{C}} = A^{\mathrm{C}} \cap B^{\mathrm{C}}$. But both $A^{\mathrm{C}}$ and $B^{\mathrm{C}}$ are open; since a finite intersection of open sets is open, we have $A^{\mathrm{C}} \cap B^{\mathrm{C}}$ is open.

15. Give an example of open sets $G_1, G_2, G_3, \ldots$ such that the intersection $\bigcap_{k \in \mathbf{Z}_{>0}} G_k$ is not open.

**Solution:** For $k \in \mathbf{Z}_{>0}$, define $I_k = (1 - 1/k, \infty)$. We have $\bigcap_{k \in \mathbf{Z}_{>0}} I_k = [1, \infty)$. The set $[1, \infty)$ isn't open.

16. Define $F = x \in \mathbf{Z} \mapsto \sqrt[3]{x^{14} + 1066} + \sqrt[43]{x^2 + 1776}$. Either prove or disprove: The function $F$ has a limit toward 1.

**Solution:** Since $1 \notin \mathrm{LP}(\mathbf{Z})$, the sequence $F$ doesn't have a limit toward 1.

17. Define $F = x \in \mathbf{Z} \mapsto \sqrt[3]{x^{14} + 1066} + \sqrt[43]{x^2 + 1776}$. Show that $F$ is continuous at 1.

**Solution:** We'll show that

$$(\forall\, \varepsilon \in \mathbf{R}_{>0})(\exists\, \delta \in \mathbf{R}_{>0})(\forall\, x \in \mathrm{ball}(1,\delta) \cap \mathbf{Z})((|F(x) - F(1)| < \varepsilon).$$

Let $\varepsilon \in \mathbf{R}_{>0}$. Choose $\delta = \frac{1}{2}$. Let $x \in \mathrm{ball}(1,\delta) \cap \mathbf{Z}$. We have $x = 1$. Thus

$$|F(x) - F(1)| = |F(1) - F(1)| = 0 < \varepsilon.$$

18. Let $F$ be a convergent sequence and let $\alpha \in \mathbf{R}$. Show that $\alpha F$ is a convergent sequence.

**Solution:** We'll prove this from scratch. Suppose $F$ converges to $L$. We'll show that $\alpha F$ converges to $\alpha L$. Let $\varepsilon \in \mathbf{R}_{>0}$. We have $\frac{\varepsilon}{1+|\alpha|} \in \mathbf{R}_{>0}$. Choose $n \in \mathbf{Z}$ such that $(\forall\, k \in \mathbf{Z}_{>n})(|\alpha F(k) - L| < \frac{\varepsilon}{1+|\alpha|}$. Again for all $k \in \mathbf{Z}_{>n}$, we have

$$|\alpha F(k) - \alpha L| = |\alpha||F(k) - L| \le \frac{\varepsilon|\alpha|}{1+|\alpha|} < \varepsilon.$$

19. Let $|F|$ be a convergent sequence. Show that $|F|$ is a convergent sequence.

**Solution:** We'll prove this from scratch. Suppose $F$ converges to $L$. We'll show that $|F|$ converges to $|L|$. Let $\varepsilon \in \mathbf{R}_{>0}$). Choose $n \in \mathbf{Z}$ such that

$$(\forall\, k \in \mathbf{Z}_{>n})(|F(k) - L| < \varepsilon.$$

Again for all $k \in \mathbf{Z}_{>n}$, we have

$$||F(k)| - |L|| \le |F(k) - L| < \varepsilon.$$

20. Use the inequality $|\sqrt{a} - \sqrt{b}| \le \sqrt{|a-b|}$, for $a, b > 0$ to show that the function $F = x \in [-1,\infty) \mapsto \sqrt{1+x}$, is continuous at 1.

**Solution:** Let $\varepsilon \in \mathbf{R}$. Choose $\delta = \varepsilon^2$. Let $x \in \mathrm{ball}(1,\delta) \cap [-1,\infty)$ We have

$$|\sqrt{1+x} - \sqrt{2}| \le \sqrt{|x-1|} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon.$$