# Trouble Makers

October 15, 2020

# The most important theorem that isn't in our book

## Theorem

If a function $F$ is differentiable at $a$ there is a function $\Phi$ that is continuous at $a$ and

$$F(x) = F(a) + (x - a)\Phi(x).$$

Further $F'(a) = \Phi(a)$.

1. A formula for $\Phi$ is

$$\Phi(x) = \begin{cases} \frac{F(x)-F(a)}{x-a} & x \neq a \\ F'(a) & x = a \end{cases}.$$

2. Although $F'(a) = \Phi(a)$, generally $F'(x) \neq \Phi(x)$.

# The L' Hôpital Rule

Suppose $F$ and $G$ are differentiable at $a$ and that $F(a) = 0$ and $G(a) = 0$, but $F'(a) \neq 0$ and $G'(a) \neq 0$ There are functions $\Phi$ and $\Psi$ that are continuous at $a$ such that

$$\begin{aligned} F(x) &= (x-a)\Phi(x), \\ G(x) &= (x-a)\Psi(x). \end{aligned}$$

Thus

$$\lim_{x \to a} \frac{F(x)}{G(x)} = \lim_{x \to a} \frac{(x-a)\Phi(x)}{(x-a)\Psi(x)} = \lim_{x \to a} \frac{\Phi(x)}{\Psi(x)} = \frac{\Phi(a)}{\Psi(a)} = \frac{F'(a)}{G'(a)}.$$

1. The L'Hôpital rule is more general than this–see page 247.
2. Specifically if $F$ and $G$ are differentiable in a neighborhood of $a$, the derivative of $G$ doesn't vanish in this neighborhood, and $\lim_{x \to a} \frac{F'(x)}{G'(x)}$ exists, then

$$\lim_{x \to a} \frac{F(x)}{G(x)} = \lim_{x \to a} \frac{F'(x)}{G'(x)}.$$