Let/Show/Suppose Proof Examples

Fall 2023, Advanced Calculus Class

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1 Introduction

Many propositions in analysis consist of concatenated 'for all' and 'there exist' qualifiers followed by something that has a boolean value. The part with the boolean value is called the *predicate*. We'll see that expressing the proposition in symbolic form gives us a road map for constructing a proof. To follow the road map, we to adhere to some rules, and it's best to follow some traditions. Following the rules is needed for correctness, and using the traditions makes our work easier to understand.

For every qualification of the form $\forall x \in A$, where A is a set, our proof includes the sentence 'Let $x \in A$.' Synonyms for 'let' are 'permit' and 'allow,' but in mathematics the tradition is to use 'let.' In the context of mathematics, 'Let $x \in \mathbf{R}$' means that we can only assume that x is real number, but we cannot assume anything more about x. It would be wrong, for example, to write in a proof 'Let $x \in \mathbf{R}$. Assume $x = 0$.'

For every qualification of the form $\exists x \in A$, it's just like fourth grade show and tell. We must choose a *specific* value for the variable x. Choices are, of course, sometimes difficult, so we need to be careful in making them. When we make a bad choice, we might not be able to complete our proof. And when we make a bad choice, we need to backtrack and re-do our work from where we made the choice.

For proofs in analysis, often the choice isn't unique, but when there is more than one possible choice, we still give the reader exactly one choice that allows the proof to proceed, we don't clog the logic by giving more than one. Additionally, when there are multiple choices, we use the most natural choice we can think of. For example, if we need to choose a number that is between given numbers, a natural choice is to use the arithmetic average. Don't complicate your work by making a correct, but obscure choice.

The final thing we need to know about our symbolic road map is that when we make a choice for a variable, our choice can depend on all previously introduced variables (that is, variables to the left), but it cannot depend on variables that are yet to be introduced (variables to the right). For example, for the statement

$$(\forall x \in \mathbf{R})\,(\exists M \in \mathbf{R})\,\big(\forall y \in \mathbf{R}\big)$$

what we choose for M can depend on x, but it *cannot* depend on y. To explicitly show this dependence, some writers use functional notion, for example 'there is $M(x) \in \mathbf{R}$.' If this functional notation helps you, use it. But arguably $M(x) \in \mathbf{R}$ conflates a real number (in this case M) with a function.

Finally, what about 'suppose?' When a proof involves 'suppose,' generally it means that a hypothesis follows. For example, if we are proving a conditional 'if $x > 0$, then $x > -1$,' our proof will start with 'Suppose $x > 0$.' This language alerts the reader that what follows 'suppose' is a hypothesis. When a proof involves special cases, a proof might also use 'suppose' to introduce each case.

Now we give some fairly simple examples of what we will call a 'let/show/suppose' proof.

2 Examples

Proposition 1. *For all $x, y \in \mathbf{R}$, there is $a \in \mathbf{R}$ such that $x < y$ implies $x < a < y$.*

In symbolic form, the Proposition 1 is $\big(\forall x, y \in \mathbf{R}\big)\big(x < y \implies \exists a \in \mathbf{R}\big)\big(x < a < y\big)$. Since $(\exists a \in \mathbf{R})$ follows $\big(\forall x, y \in \mathbf{R}\big)$, what we choose for a is allowed to depend on both x and y. The most natural choice for a is the arithmetic average. With this, our proof is:

Proof. Let $x, y \in \mathbf{R}$. Suppose $x < y$. Choose $a = \frac{x+y}{2}$. Then $a \in \mathbf{R}$, as required. We have

$$
\begin{aligned}
\big[x < a < y\big] &\equiv \left[x < \frac{x+y}{2} < y\right], && \text{(substitution for } a\text{)} \\
&\equiv \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(subtract } {}^{(x+y)}/{}_{2}\text{)} \\
&\equiv \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \text{(simplification)} \\
&\equiv \text{True}. && \text{(hypothesis } x < y\text{)} \qquad \square
\end{aligned}
$$

In our proof, the 'Suppose' in 'Suppose $x < y$' alerts the reader that what follows is a hypothesis.

Each step in this series has a parenthetical remark clarifying the reason for the equivalence. Such remarks can make a proof easier to follow. You are encouraged to include such remarks in your proofs.

If the fact that the arithmetic average is known to be "in the middle," our proof could end at the second line. It's sometimes a judgment call on whether a fact can be used in a proof. When in doubt, include, don't assume. But it is, of course, always wrong to use a result in a proof that is way outside the context of our class.

Notice that starting with the conclusion of Proposition 1, we expressed our proof as a series of logical *equivalences*. Logical equivalence has the transitive property, so indeed our proof shows that $x < a < y$ is true. When we start with the conclusion as we did in this proof, it's vitally important that we use a series of logical equivalences, *not* logical implications. If we expressed this work as

$$
\begin{aligned}
\big[x < a < y\big] &\implies \left[x < \frac{x+y}{2} < y\right], && \text{(substitution)} \\
&\implies \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(subtract } {}^{(x+y)}/{}_{2}\text{)} \\
&\implies \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \text{(simplification)} \\
&\implies \text{True}. && \text{(hypothesis } x < y\text{)}
\end{aligned}
$$

we have shown that $(x < a < y) \implies$ True. But $P \implies$ True is true *regardless* of the truth value of P (that is $P \implies$ True is a tautology). So the above work proves *nothing*. By reversing the order of the statements, we could fix this effort; for example,

$$
\begin{aligned}
x < y &\implies \left[\frac{x-y}{2} < 0 < \frac{y-x}{2}\right], && \big((y - x > 0) \land (x - y < 0)\big) \\
&\implies \left[x - \frac{x+y}{2} < 0 < y - \frac{x+y}{2}\right], && \text{(algebraic equivalence)} \\
&\implies \left[x < \frac{x+y}{2} < y\right], && \text{(add } {}^{(x+y)}/{}_{2}\text{)} \\
&\implies \big[x < a < y\big]. && \text{(substitution for } a\text{).}
\end{aligned}
$$

Now we have proved that $x < y \implies x < a < y$. And since $x < y$ is a hypothesis, we've indeed shown that $x < a < y$ is true, as we intended to do. Compared to our proof that used a series of logical equivalences, the first logical implication in this proof (that is $x < y \implies \frac{x-y}{2} < 0 < \frac{y-x}{2}$) seems a bit like a rabbit pulled out of a magician's hat, making the proof somewhat mysterious. When it's possible to write a proof as a series of logical equivalences that starts with the conclusion and ends with true, it's often the clearest way to write the proof.

Proposition 2. *For all $r \in \mathbf{R}_{>0}$ there is $x \in [0,1)$ such that $1 - r < x$.*

In symbolic form, Proposition 2 is $(\forall r \in \mathbf{R}_{>0})\,(\exists x \in [0,1))\,(1 - r < x)$. And since $x \in [0,1)$, additionally we know that $x < 1$. So the predicate $1 - r < x$ is equivalent to $1 - r < x < 1$. Now Proposition 2 looks a great deal like Proposition 1. Maybe we can simply choose x to be the arithmetic average of $1 - r$ and 1. That is choose $x = 1 - \frac{r}{2}$. But the requirement that $0 < x$ spoils this choice. If $r = 4$, for example, we are choosing $x = -1$. And that's not allowed.

One way to fix this is to choose x to be the arithmetic average of $1 - r$ and 1 when $r < 1$ and choose $x = \frac{1}{2}$ when $r \geq 1$. Actually, when $r \geq 1$, we could choose x to be any member of $[0,1)$, but choosing the midpoint is arguably the most natural. Our proof breaks into two cases:

Proof. Let $r \in \mathbf{R}_{>0}$. Choose $x = \begin{cases} 1 - \frac{r}{2} & r < 1 \\ \frac{1}{2} & r \geq 1 \end{cases}$. For $r < 1$, we have

$$\begin{aligned}
[1 - r < x] &\equiv \left[1 - r < 1 - \frac{r}{2}\right], && \text{(substitution for } x\text{)} \\
&\equiv \left[0 < \frac{r}{2}\right], && \text{(add } r - 1\text{)} \\
&\equiv \text{True.} && (0 < r < 1)
\end{aligned}$$

And for $r \geq 1$, we have

$$\begin{aligned}
[1 - r < x] &\equiv \left[1 - r < \frac{1}{2}\right], && \text{(substitution for } x\text{)} \\
&\equiv \left[\frac{1}{2} < r\right], && \text{(add } r - 1\text{)} \\
&\equiv \text{True.} && (r \geq 1)
\end{aligned}$$

□

If you prefer to condense this proof into one case, try choosing $r = \max(\frac{1}{2}, 1 - \frac{1}{2})$.

Proposition 3. *For all $x \in \mathbf{R}_{>0}$ there is $y \in \mathbf{R}_{>0}$ such that $y < x$.*

Generically, Proposition 3 looks like another choose the arithmetic average proof: we need to choose a number between zero and y. Our proof is

Proof. (BW) Let $x \in \mathbf{R}_{>0}$. Choose $y = x/2$. Then $y \in \mathbf{R}_{>0}$ as required. We have

$$\begin{aligned}
[y < x] &\equiv \left[\frac{x}{2} < x\right], && \text{(substitution for } y\text{)} \\
&\equiv \left[0 < \frac{x}{2}\right], && \text{(subtract } x/2\text{)} \\
&\equiv \text{True.} && (\text{x} > 0)
\end{aligned}$$

□

Abstracted, Proposition 3 says that there is no smallest positive number. That is, given any positive number, there is a positive number that is smaller. This is an important property of the field of real numbers.

Proposition 4. *For every $y \in \mathbf{R}_{>0}$ there is $x \in \mathbf{R}_{>0}$ such that $y \geq x$.*

Paraphrased, Proposition 4 says that given any positive number, there is a positive number that is greater. Alternatively, we might say that there is no largest positive number. Adding one is a good strategy for constructing a larger number, but this proposition allows equality, not strict inequality. When possible, I like to choose equality. Our proof

Proof. (BW) Let $y \in \mathbf{R}_{>0}$. Choose $x = y$. Then $x \in \mathbf{R}_{>0}$, as required. We have

$$
\begin{aligned}
[y \geq x] &\equiv [y \geq y], && \text{(substitution for } x\text{)} \\
&\equiv [0 \geq 0], && \text{(subtract } y\text{)} \\
&\equiv \text{True.}
\end{aligned}
$$

□

Proposition 5. *For all $x \in \mathbf{R}_{>0}$, there is $M \in \mathbf{R}$ such that $\frac{1}{x} + 1 > M$.*

Proof. (SB) Let $x \in \mathbf{R}_{>0}$. Choose $M = 1$. We have

$$
\begin{aligned}
\left[\frac{1}{x} + 1 > M\right] &\equiv \left[\frac{1}{x} + 1 > 1\right], && \text{(substitution for } M\text{)} \\
&\equiv \left[\frac{1}{x} > 0\right], && \text{(subtract 1)} \\
&\equiv \text{True.} && \text{(Ordered Field Axioms)}
\end{aligned}
$$

□

Proposition 6. *There is $M \in \mathbf{R}$ such that for all $x \in \mathbf{R}_{>0}$, we have $\frac{1}{x} + 1 > M$.*

A quick graph of $y = \frac{1}{x} + 1$ for $x \in \mathbf{R}_{>0}$ shows that its graph is above the horizontal line $y = 1$. For M, we can choose any number that is one or less.

Proof. Choose $M = 1$. Let $x \in \mathbf{R}_{>0}$, we have

$$
\begin{aligned}
\left[\frac{1}{x} + 1 > M\right] &\equiv \left[\frac{1}{x} + 1 > 1\right], && \text{(substitution)} \\
&\equiv \left[\frac{1}{x} > 0\right], && \text{(subtract one)} \\
&\equiv \text{True.} && (x > 0)
\end{aligned}
$$

□

Proposition 7. *There is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $1 + m(x-1) \leq x^2$.*

A good way of thinking about this proposition is geometrically as two graphs. In the x-y plane, the graph of $y = 1 + m(x-1)$, is a line and the graph of $y = x^2$ is a parabola. Both the line and the parabola contain the point $(x = 1, y = 1)$. And the condition $1 + m(x-1) \leq x^2$ tells us that the line $y = 1 + m(x-1)$ must be below or touching the parabola $y = x^2$. How can this be? A few quick sketches of lines that contain the point $(x = 1, y = 1)$ shows that the line $y = 1 + m(x-1)$ must be a tangent line to the parabola at the point $(x = 1, y = 1)$. If the line isn't tangent to the parabola, the line will be below the parbola on one side of $x = 1$ and above on the other. We need the line to be on or below the parabola. So elementary calculus tells us that the slope of the line must be two. With that choice our proof is

Proof. Choose $m = 2$. Let $x \in \mathbf{R}$. We have

$$\begin{aligned}
\left[1 + m(x-1) \le x^2\right] &\equiv \left[1 + 2(x-1) \le x^2\right], && \text{(substitution for } m\text{)} \\
&\equiv \left[2x - 1 \le x^2\right], && \text{(Algebra)} \\
&\equiv \left[0 \le x^2 - 2x + 1\right], && \text{(Algebra)} \\
&\equiv \left[0 \le (x-1)^2\right], && \text{(Factor)} \\
&\equiv \text{True}. && \text{(Ordered field axioms}
\end{aligned}$$

□

If you enjoy crabbed proofs, you can rearrange this as a sequence of implications starting from $0 \le (x-1)^2$.

Proposition 8. *For every $a \in \mathbf{R}$, there is $m \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, we have $a^2 + m(x-a) \le x^2$.*

This proposition generalizes the previous proposition to a point of tangency ($x = a$, $y = a^2$). Given that we choose $m = 2a$.

Proof. Let $a \in \mathbf{R}$. Choose $m = 2a$. We have

$$\begin{aligned}
\left[a^2 + m(x-a) \le x^2\right] &\equiv \left[a^2 + 2a(x-a) \le x^2\right] && \text{(substitution)} \\
&\equiv \left[0 \le x^2 - 2ax + a^2\right] && \text{(algebra)} \\
&\equiv \left[0 \le (x-a)^2\right] && \text{factor}
\end{aligned}$$

□

Proposition 9. *For all $x, y \in \mathbf{R}$, we have $(x^2 = y^2) \implies (x = y)$.*

This says that the function that squares is one-to-one. We know this is false. It's certainly possible for outputs of the squaring function to be equal and the inputs distinct. An example is $(-1)^2 = 1^2$ and $-1 \ne 1$. Although this example is a counterexample to the proposition, for practice negating qualified statements and conditionals, we'll explicitly negate the proposition and write a let/show/suppose proof that it's negation is true. To do this we need to remember the tautology

$$(P \implies Q) \equiv (P \land \neg Q)$$

This we will prove

Proposition 10. *There are $x, y \in \mathbf{R}$ such that $(x^2 = y^2)$ and $x \ne y$).*

Proof. Choose $x = -1$ and $y = 1$. We have $(-1)^2 = 1^2$ and $-1 \ne 1$. □

Proposition 11. *For all $x, y \in \mathbf{R}$, we have $(x^3 = y^3) \implies (x = y)$.*

Distilled, this proposition says that the real-valued cubing function is one-to-one. Graphically this is easy to verify, but algebraically, it requires the tricky factorization

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) = (x-y)\left((x + y/2)^2 + 3y^2/4\right).$$

The factor $\left(x + y/2\right)^2 + 3y^2/4$ is a sum of squares (SOS) that only vanishes when $x = 0$ and $y = 0$. This fact is the core of our proof.

Proof. Let $x, y \in \mathbf{R}$. Suppose $x^3 = y^3$; we'll show that $x = y$. We have

$$\begin{aligned}
\left[x^3 = y^3\right] &\equiv \left[x^3 - y^2 = 0\right], && \text{(algebra)} \\
&\equiv \left[(x-y)\left((x+y/2)^2 + 3y^2/4\right) = 0\right], && \text{(tricky algebra)} \\
&\equiv \left[(x-y) = 0 \vee (x+y/2)^2 + 3y^2/4) = 0\right], && \text{(ordered field properties)} \\
&\equiv \left[x = y \vee \left(x = 0 \wedge y = 0\right)\right], && \text{(SOS fact)} \\
&\equiv \left[x = y\right]. && \text{(disjunction condense)}
\end{aligned}$$

□

Proposition 12. *For all* $r \in \mathbf{R}_{>0}$ *there is* $x \in \mathbf{R}$ *such that* $1 < x < 1 + r$.

To prove this, we only need to choose a number that is between 1 and $1 + r$. Again, we can choose the arithmetic average.

Proof. Let $r \in \mathbf{R}_{>0}$. Choose $x = 1 + r/2$. Then indeed $x \in \mathbf{R}_{>0}$ as required. We have

$$\begin{aligned}
[1 < x < 1 + r] &\equiv [1 < 1 + r/2 < 1 + r], && \text{(substitution for } x\text{)} \\
&\equiv [-r/2 < 0 < r/2], && \text{(subtraction)} \\
&\equiv \text{True}. && (r > 0).
\end{aligned}$$

□