Review for Exam I

1. Let A and B be subsets of $\mathbf{R}$. Show that if A and B are bounded above, then $A \cup B$ is bounded above. You may use the fact that for real numbers a and b, we have $a \leq \max(a,b)$ and $b \leq \max(a,b)$.

Solution:

Proof. Let A and B be subsets of $\mathbf{R}$ that are bounded above. We'll show that $A \cup B$ is bounded above. Since A and B are bounded above, there are $p, q \in \mathbf{R}$ such that for all $a \in A$, we have $a \leq p$ and for all $b \in A$, we have $b \leq q$. Define $M = \max(p,q)$. We'll show that M is an upper bound for $A \cup B$. Let $x \in A \cup B$. Either $x \in A$ or $x \in B$. If $x \in A$, we have $x \leq p \leq M$. Similarly, if If $x \in B$, we have $x \leq q \leq M$. □

2. Give an example of a subset of $\mathbf{R}$ that does not have a least upper bound.

Solution: The set $\mathbf{R}$ is a subset of $\mathbf{R}$ that does not have a least upper bound.

3. Give an example of a subset A of $\mathbf{R}$ such that $\text{lub}(A) \in A$.

Solution: We have $\text{lub}((0,1]) = 1$ and $1 \in (0,1])$.

4. Give an example of a subset A of $\mathbf{R}$ such that $\text{lub}(A) \notin A$.

Solution: We have $\text{lub}((0,1)) = 1$ and $1 \notin (0,1])$.

5. Show that $\text{lub}((-\infty,2)) = 2$.

Solution: We'll show that 2 is an upper bound for $(-\infty,2)$ and that for all $r \in \mathbf{R}_{>0}$, there is $a \in (-\infty,2)$ such that $2 - r < a$.

The fact that 2 is an upper bound for $(-\infty,2)$ is apparent. Let $r \in \mathbf{R}_{>0}$. Choose $a = 2 - r/2$. Then $a \in (-\infty,2)$ as required. Further since $r > 0$, we have $2 - r < 2 - r/2$.

6. Show that $\text{lub}([0,2)) = 2$.

Solution: We'll show that 2 is an upper bound for $([0,2)$ and that for all $r \in \mathbf{R}_{>0}$, there is $a \in ([0,2)$ such that $2 - r < a$.

The fact that 2 is an upper bound for $[0,2)$ is apparent. Let $r \in \mathbf{R}_{>0}$. Choose $a = \max(1, 2 - r/2)$. Then $a \in [0,2)$ as required. For $r < 2$, we have $a = 2 - r/2$. Since $r > 0$, we have $a < 2$, For $r \geq 2$, we have $a = 1$. We have $1 < 2$ as required.

7. Let A be a subset of $\mathbf{R}$. Show that A has at most one least upper bound.

Solution: See class notes.

8. Write a proof for

Proposition 1. *For all $x, y \in \mathbf{R}$, there is $a \in \mathbf{R}$ such that $x < y$ implies $x < a < y$.*

Solution: See class notes.

9. Write a proof for

Proposition 2. *For all $x \in \mathbf{R}_{>0}$ there is $y \in \mathbf{R}_{>0}$ such that $y < x$.*

Solution: See class notes.

10. Without explicitly using negation, write the negation of

Proposition 3. *There are $x, y \in \mathbf{R}$ such that* $\sin(x) = \sin(y) \implies x = y$.

11. Either write a proof of

Proposition 4. *There are $x, y \in \mathbf{R}$ such that* $\sin(x) = \sin(y) \implies x = y$.

or write a proof of its negation.

12. Let $(\mathscr{F}, +, \times)$ be a field and let O be the additive identity and I be the multiplicative identity. Given that $O = I$, show that $\mathscr{F} = \{O\}$.

Solution: See class notes.

13. Let $(\mathscr{F}, +, \times)$ be a field. Show that for all $a, b \in \mathscr{F}$, we have $a \times b = a \times (-b)$.

Solution: See class notes.

14. Let $(\mathscr{F}, +, \times)$ be an ordered field. For all $a, b, c \in \mathscr{F}$, show that $a < b$ and $c < 0$ implies $a \times c > b \times c$.

Solution: See class notes.

15. Show that

$$(\forall k \in \mathbf{Z}_{>1}) \left(\frac{1}{k^2} \leq \frac{1}{k-1} - \frac{1}{k} \right).$$

Solution: We'll write our solution as a sequence of logical equivalences. Let $k \in \mathbf{Z}_{>1}$. We have

$$\begin{aligned}
\left[\frac{1}{k^2} \le \frac{1}{k-1} - \frac{1}{k}\right] &\equiv \left[\frac{1}{k^2} - \frac{1}{k-1} + \frac{1}{k} \le 0\right], && \text{(algebra)}\\
&\equiv \left[-\frac{1}{(k-1)k^2} \le 0\right], && \text{(factor)}\\
&\equiv \text{true}. && (k-1>0 \text{ and } k^2>0)
\end{aligned}$$

16. Show that

$$(\forall x \in (-\infty, 1))\,(\exists r \in \mathbf{R}_{>0})\,((x-r, x+r) \subset (-\infty, 1))\,.$$

Solution: We need to choose a number r such that $x+r<1$ and $0<r$. Thus $0<r<1-x$. One choice is $r = \frac{1-x}{2}$. Since $x<1$, this choice does satisfy the condition $r>0$.

Proof Let $x \in (-\infty, 1)$. Choose $r = \frac{1-x}{2}$. Since $x<1$, it follows that $r \in \mathbf{R}_{>0}$ as required. Since $r>0$, the condition $(x-r, x+r) \subset (-\infty, 1)$ is equivalent to $x+r<1$. We have

$$[x+r<1] \equiv \left[x + \frac{1-x}{2} < 1\right] \equiv \left[\frac{1+x}{2} < 1\right] \equiv [1+x<2] \equiv [x<1] \equiv \text{true}.$$

17. Let A, B be subsets of $\mathbf{R}$ and let A be bounded above. Show that $A \setminus B$ is bounded above.

Solution: Since A is bounded above, there is $M \in \mathbf{R}$ such that $(\forall x \in A)(x \le M)$. We will show that

$$(\exists M' \in \mathbf{R})(\forall x \in A \setminus B)(x \le M').$$

Choose $M' = M$. Let $x \in A \setminus B$. Then $x \in A$; thus we have

$$[x \le M'] \equiv [x \le M] \equiv \text{true}.$$

18. Give an example of subsets A, B of $\mathbf{R}$ such that $A \setminus B$ is bounded above, but A is not bounded above.

Solution: One (of many) example is $A = \mathbf{R}$ and $B = \mathbf{R}$. Then A is not bounded above, but $A \setminus B = \varnothing$, so $A \setminus B$ is bounded above (because the empty set is bounded above).

19. Define $F = x \in \mathbf{R} \mapsto x^2$. Enumerate the members of the set

$$F(\{-4, -1, 0, 1, 4\}).$$

Solution:

$$F(\{-4,-1,0,1,4\}) = \{F(-4), F(-1), F(0), F(1), F(4)\} = \{0,1,16\}.$$

20. Show that

$$(\forall a \in \mathbf{R})\,(\exists m \in \mathbf{R})\,(\forall x \in \mathbf{R})\,\big(x^2 - a^2 \geq m(x-a)\big).$$

Solution: We will write our proof as a sequence of logical equivalences. Let $a \in \mathbf{R}$. Choose $m = 2a$. Let $x \in \mathbf{R}$. We have

$$\begin{aligned}
\left[x^2 - a^2 \geq m(x-a)\right] &= \left[x^2 - a^2 \geq 2a(x-a)\right], && \text{(substitution for } m\text{)}\\
&= \left[x^2 - 2a(x-a) - a^2 \geq 0\right], && \text{(algebra)}\\
&= \left[x^2 - 2a + a^2 \geq 0\right], && \text{(algebra)}\\
&= \left[(x-a)^2 \geq 0\right], && \text{(factor)}\\
&= \text{true}.
\end{aligned}$$