MATH 460 **Name:**

Homework 3, Fall 2023

Homework 3 has questions 1 through 2 with a total of 50 points. When I record your grade, I will scale it to twenty points. For details of the grading scheme for this assignment, please see the section 'Grading rubric' of our syllabus.

Revise, proofread, revise again (and again), typeset your work using Overleaf, and upload the converted pdf of your compiled file work to Canvas. This work is due **Saturday 9 September** at 11:59 PM.

$\boxed{10}$ 1. Let $(\mathscr{F}, +, \times)$ be a field with θ as its additive identity. And let $a \in \mathscr{F}_{\neq\theta}$ and $b \in \mathscr{F}$. Show that there is a unique $x \in \mathscr{F}$ such that $a \times x = b$.

2. Define $\mathbf{Q}$ to be the set of rational numbers. Thus $\frac{2}{3} \in \mathbf{Q}$ and $\sqrt{2} \notin \mathbf{Q}$, for example. For $(a,b),(a',b') \in \mathbf{Q} \times \mathbf{Q}$, define the binary operators $+$ and $\times$ on $\mathbf{Q} \times \mathbf{Q}$ by

$$(a,b) + (a',b') = (a+a', b+b'),$$
$$(a,b) \times (a',b') = (aa' + 2bb', ab' + ba').$$

These definitions make $(\mathbf{Q} \times \mathbf{Q}, +, \times)$ into a field. If you don't believe me, I invite you to check. But you can take my word.

$\boxed{10}$ (a) Find the additive identity for this field. You don't need to show the algebra you used to find the additive identity, but you do need to prove that your putative answer is correct.

Solution: We'll show that the additive identity for this field is $(0,0)$.

Proof. Let $(a,b) \in \mathscr{F}$, We have

$$(a,b) + (0,0) = (a+0, b+0) = (a,b).$$

So $(0,0)$ is the additive identity for $\mathscr{F}$. $\square$

$\boxed{10}$ (b) Find the multiplicative identity for this field. You don't need to show the algebra you used to find the multiplication identity, but you do need to prove that your putative answer is correct.

Solution: We'll show that the multiplicative identity for this field is $(1,0)$.

Proof. Let $(a,b) \in \mathscr{F}$, We have

$$(a,b) \times (1,0) = (a \times 1 + b \times 0, a \times 0 + b \times 1) = (a,b).$$

So $(1,0)$ is the multiplicative identity for $\mathscr{F}$. □

10 (c) Find the multiplicative inverse of $(1,2)$. You don't need to show the algebra you used to find the answer, but you do need to prove that your putative answer is correct.

Solution: We'll show that $(1,2)^{-1} = \left(-\frac{1}{7}, \frac{2}{7}\right)$.

Proof. We have

$$(1,2) \times \left(-\frac{1}{7}, \frac{2}{7}\right) = \left(-\frac{1}{7} + \frac{8}{7}, \frac{2}{7} - \frac{2}{7}\right) = (1,0).$$

So $(1,2)^{-1} = \left(-\frac{1}{7}, \frac{2}{7}\right)$. □

10 (d) For $a, b \in \mathbf{Q}$, a calculation gives

$$(a,b)^{-1} = \left(-\frac{a}{2b^2 - a^2}, \frac{b}{2b^2 - a^2}\right). \tag{1}$$

But wait! What's the story if $2b^2 - a^2 = 0$? Either, I flubbed and $(\mathbf{Q} \times \mathbf{Q}, +, \times)$ isn't a field, or the above formula for the inverse is rubbish, or something else. What is the something else?

Solution: Since $\sqrt{2} \notin \mathbf{Q}$, for $a, b \in \mathbf{Q}$, we have $2b^2 - a^2 = 0$ iff $a = 0$ and $b = 0$. Although the denominator looks like it might vanish, it doesn't vanish unless both $a = 0$ and $b = 0$.