MATH 202 **Name:** ____________________

Calculus Practice IV, Fall 2023

Here is an opportunity for you to maintain your calculus skills over the summer. If you complete these problems, digitize your work, and submit your work to Canvas, I will send you my solutions. If you need some help with these questions, email me with your questions (willisb@unk.edu)

Completing this work is optional, and it does not enter into your class grade in any way–this work is not a bonus, extra credit, or anything like that.

1. The graph in Figure 1 shows the graph of a wild and crazy function (the red curve) whose domain is $[-4, 4]$ that we'll unimaginatively call F. Define a function G by

$$G(x) = \int_{-4}^{x} F(s)\, \mathrm{d}s.$$

As best you can, draw a graph of G.

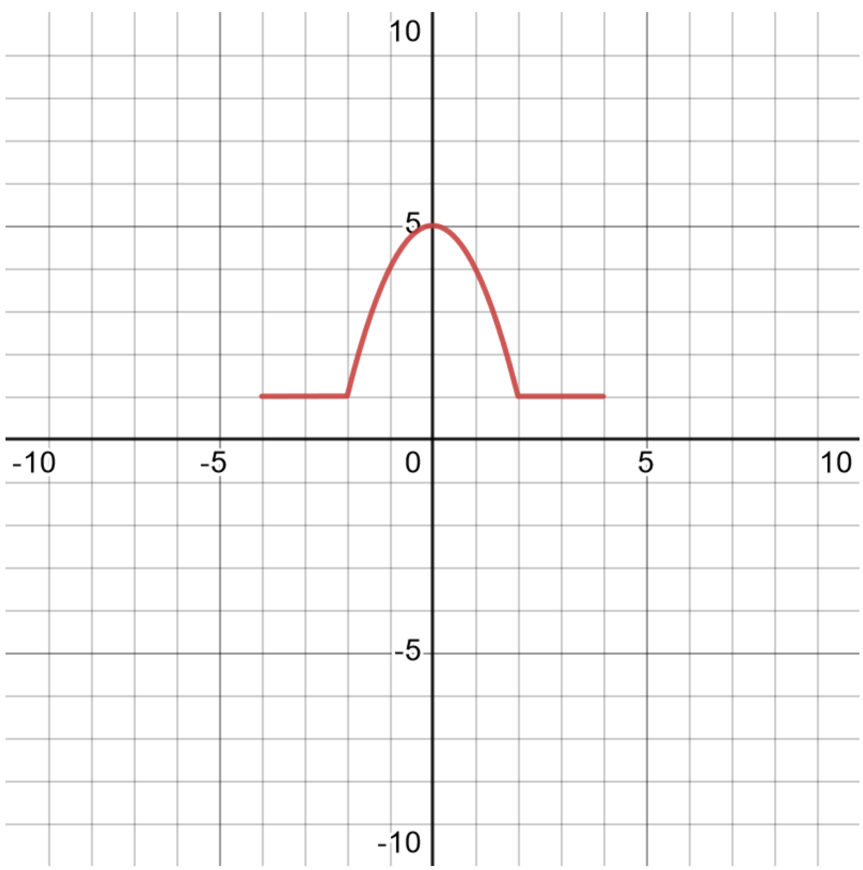


Figure 1: Graph of some wild and crazy function.

Solution:

To start, let's review the properties of a function G that is defined by

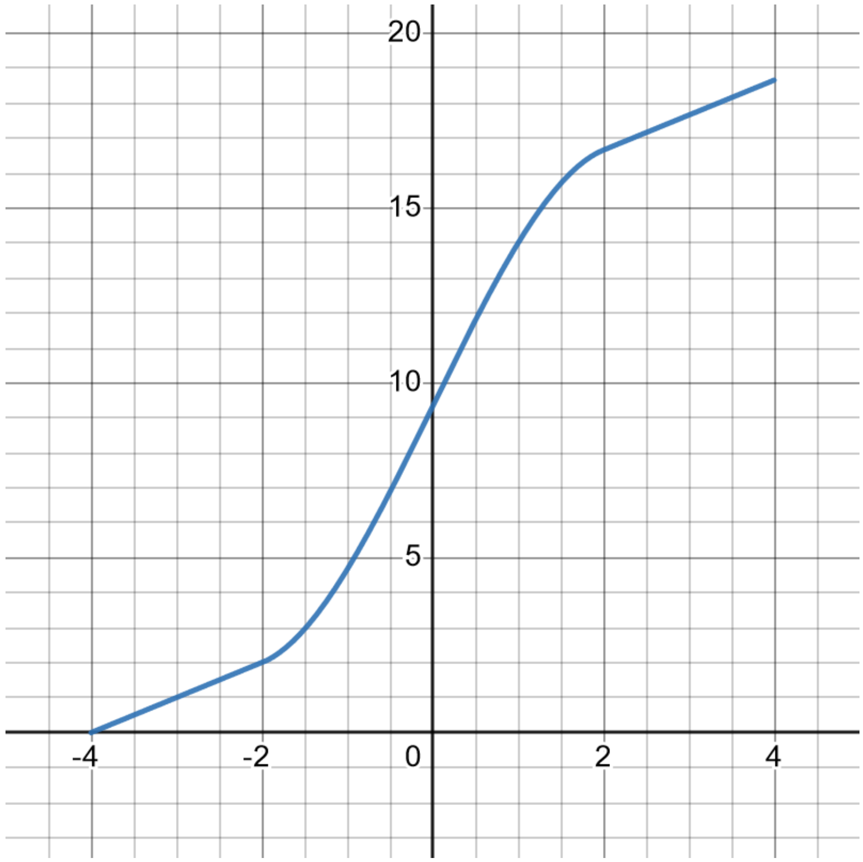
$$G(x) = \int_a^x F(s)\,\mathrm{d}s,$$

where F is integrable on an interval $[a, b]$. We have:

- $G(a) = 0$
- G is continuous on $[a, b]$.
- G is differentiable everywhere that F is continuous.
- Wherever F is continuous, we have $G'(x) = F(x)$.

Specifically for our function F given graphically in Figure 1, we conclude that (a) $G(-4) = 0$, (b) G is continuous on $[-4, 4]$ (c)$G'(x) = F(x)$ on $[-4, 4]$. Further, since $F(x) > 0$, we see that $G'(x) > 0$. This means that G is an increasing function. The steepest part of the graph of G is at 0, where $G'(0) = 5$. Finally, the derivative of G is constant on the interval $[-4, -1]$, so G is a linear function on this interval. The same is true on the interval $[1, 4]$.

Putting all this together, a graph of G looks something like



I know what you are thinking: Isn't the middle part of the graph a downward facing parabola surrounded by two shorter constant portions? Specifically, isn't the formula for F something like

$$F(x) = \begin{cases} 1 & -4 \le x \le -2 \\ 5 - x^2 & -2 < x < 2 \\ 1 & 2 \le x \le 4 \end{cases} \quad ?$$

Sure, the graph of F agrees with this formula. So we can find a formula for F by integrating. For $-4 \le x \le -2$, we have

$$G(x) = \int_{-4}^{x} \mathrm{d}s = x + 4.$$

And for $-2 < x < 2$, we have

$$G(x) = \int_{-4}^{-2} \mathrm{d}s + \int_{-2}^{x} 5 - s^2 \,\mathrm{d}s = 2 + -\frac{x^3}{3} + 5x + \frac{22}{3} = -\frac{x^3}{3} + 5x + \frac{28}{3}$$

And finally for $2 \le x \le 4$, we have

$$G(x) = \int_{-4}^{-2} \mathrm{d}s + \int_{-2}^{2} 5 - s^2 \,\mathrm{d}s + \int_{2}^{x} \mathrm{d}s = x + \frac{44}{3}.$$

Putting this together, a formula for G is

$$G(x) = \begin{cases} x + 4 & -4 \le x \le -2 \\ -\frac{x^3}{3} + 5x + \frac{28}{3} & -2 < x < 2 \\ x + \frac{44}{3} & 2 \le x \le 4 \end{cases}.$$

Another way to do this is to find the antiderivative of each case of the split rule and then to adjust the arbitrary constant for each case to make the function G continuous at -2 and 2. This calculation looks like this

$$G(x) = \begin{cases} x + c_1 & -4 \le x \le -2 \\ 5x - \frac{1}{3}x^3 + c_2 & -2 < x < 2 \\ x + c_3 & 2 \le x \le 4 \end{cases}.$$

To make $G(-4) = 0$, we need $-4 + c_1 = 0$; to make G continuous at -2, we need $c_2 - \frac{22}{3} = c_1 - 2$; and to make G continuous at 2, we need $c_2 + \frac{22}{3} = c_3 + 2$. These are linear equations for c_1, c_2, and c_3. Solving them, we get $c_1 = 4, c_2 = \frac{28}{3}$ and $c_3 = \frac{44}{3}$.

And for something to ponder: a formula for G that is not explicitly a split rule is

$$G(x) = \frac{\operatorname{sgn}(x-2)\,(x-2)^2\,(x+4)\operatorname{sgn}(x+2) + 32\operatorname{sgn}(x+2) - x^3 + 18x + 40}{6},$$

where the sign function sgn is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}.$$

Possibly this all-in-one formula has its appeal, but I'd say that the split rule formula is simpler for me to understand.