

1. Use power series to find the numerical value of each limit. You might like to use the facts

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots,$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \cdots,$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots$$

(a) $\lim_{x \rightarrow 0} \frac{\log(x^3 + 1)}{\sin(2x^3)}$

(b) $\lim_{x \rightarrow 0} \frac{\arctan(x) - \sin(x)}{x^3}$

(c) $\lim_{x \rightarrow 0} \frac{\arctan(x) - \sin(x)}{x^4}$

- 1 2. Use the *ratio* test to determine if the series $\sum_{k=0}^{\infty} \frac{\left(\frac{k}{3}\right)^k}{k!}$ converges or diverges.

- 5 3. Find the numerical value of $\sum_{k=4}^{\infty} \left(\frac{1}{2}\right)^k$.

4. The de Jonqui re function Li_2 can be defined by $\text{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ and $\text{dom}(\text{Li}_2) = (-1, 1)$

5 (a) Find the *numerical value* of $\text{Li}_2(0)$.

5 (b) Find the *radius of convergence* for the power series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$.

5 (c) Find the *numerical value* of $\text{Li}'_2(0)$. **Hint:** $\text{Li}_2(x) = x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots$.

5. Determine convergence or divergence of each series. Fully justify your work.

This doesn't mean you need to use our definition of convergent; instead use the theorems we've established.

5 (a) $\sum_{k=1}^{\infty} \frac{1}{k}$

5 (b) $\sum_{k=0}^{\infty} (-1)^k$

5 (c) $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$

5 (d) $\sum_{k=0}^{\infty} \frac{1}{k!}$

- 5 6. My friend Milhous claims that the sum $\sum_k f_k$ converges provided that the sequence f converges to zero. Show Milhous an example of a sequence f that converges to zero, but the sum $\sum_k f_k$ diverges.

7. For all real numbers x , we have $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

- 2 (a) Find the power series representation for $\sin(x) - x$ centered at zero.
Hint: When you don't know where to start, go to your happy place: write the first few terms of the Taylor series for sine centered at zero. Then subtract x .

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- (b) For $x \neq 0$, find the *first two nonzero terms* in a power series representation for $\frac{\sin(x)-x}{x^3}$. Again, try visiting your happy place.

- 2 (c) Use the above result to find the *numerical value* of the limit

$$\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3}.$$

- 1 8. Define a function erf by the definite integral $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Find a power series representation for erf. Find the radius of convergence of this power series.