MATH 202, Fall 2023 **Name:** ____________________

In class work 21 **Row and Seat**:____________________

In class work **21** has questions **1** through **1** with a total of **6** points. Turn in your work at the end of class *on paper*. This assignment is due at *Thursday 9 November 13:20*.

"The place to improve the world is first in one's own heart and head and hands, and then work outward from there." ROBERT PIRSIG

1. For all real numbers x, we have $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

$\boxed{2}$ (a) Find the power series representation for $\sin(x) - x$ centered at zero. **Hint:** When you don't know where to start, go to your happy place: write the first few terms of the Taylor series for sine centered at zero. Then subtract x.

Solution: Let's go to our happy place; for all real x, we have

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots.$$

Thus

$$\sin(x) - x = -\frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots.$$

Arguably this answer is OK, but the ellipsis (that is the $\cdots$) leaves too much to the imagination. An explicit answer is

$$\sin(x) - x = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

$\boxed{2}$ (b) For $x \neq 0$, find the *first two nonzero terms* in a power series representation for $\frac{\sin(x)-x}{x^3}$. Again, try visiting your happy place.

Solution: Here is my happy place:

$$\sin(x) - x = -\frac{1}{6}x^3 + \frac{1}{120}x^5 \cdots$$

For $x \neq 0$, we have

$$\frac{\sin(x) - x}{x^3} = -\frac{1}{6} + \frac{1}{120}x^2 \cdots$$

And explicitly, we have

$$\frac{\sin(x) - x}{x^3} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k-2}$$

It's optional to do, but changing $k \to k+1$ gives

$$\frac{\sin(x) - x}{x^3} = \sum_{k=0}^{\infty} -\frac{(-1)^k}{(2k+3)!} x^{2k}$$

$\boxed{2}$ (c) Use the above result to find the *numerical value* of the limit

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}.$$

Solution: We have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x}{x^3} &= \lim_{x\to 0} \sum_{k=0}^{\infty} -\frac{(-1)^k}{(2k+3)!} x^{2k}, \\
&= \sum_{k=0}^{\infty} -\frac{(-1)^k}{(2k+3)!} 0^{2k}, \\
&= -\frac{1}{6}.
\end{aligned}$$