MATH 202, Fall 2023 **Name:** ______________________

In class work 22 **Row and Seat**: ______________________

1. Use power series to find the numerical value of each limit. You might like to use the facts

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots,$$
$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots,$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \cdots$$
$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots$$

$\boxed{1}$ (a) $\lim_{x\to 0} \dfrac{\sin(x) - x + x^3/6}{x^5}$

Solution:

$$\lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{x^5} = \lim_{x\to 0} \frac{x^5/5! + \cdots}{x^5} = \frac{1}{120}.$$

$\boxed{1}$ (b) $\lim_{x\to 0} \dfrac{\sin(3x^2)}{1 - \cos(2x)}$

Solution:

$$\lim_{x\to 0} \frac{\sin(3x^2)}{1 - \cos(2x)} = \lim_{x\to 0} = \lim_{x\to 0} \frac{3x^2 + \cdots}{(2x)^2 + \cdots} = \frac{3}{4}.$$

2. Use the *ratio* test to determine the radius of convergence of each power series.

$\boxed{1}$ (a) $\displaystyle\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} x^k$

Solution:

$$\lim_{k\to\infty} \left| \frac{((k+1)!)^2 x^{k+1}}{(2k+2)! x^k} \right| = \lim_{k\to\infty} \frac{k+1}{4k+2} |x| = \frac{|x|}{4}$$

The radius of convergence is 4.

$\boxed{1}$ (b) $\displaystyle\sum_{k=0}^{\infty} k! x^k$

Solution:

$$\lim_{k\to\infty} \left| \frac{(k+1)! x}{k!} \right| = \lim_{k\to\infty} (k+1)|x| = \begin{cases} \infty & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The radius of convergence is zero.

$\boxed{1}$ 3. Find the numerical value of $\displaystyle\sum_{k=3}^{\infty} \left(\frac{1}{10}\right)^k$.

Solution:

$$\sum_{k=3}^{\infty} \left(\frac{1}{10}\right)^k = \sum_{k=3}^{\infty} \left(\frac{1}{10}\right)^{k+3} = \frac{1}{10^3} \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k = \frac{1}{10^3} \frac{1}{1-\frac{1}{10}} = \frac{1}{900}$$

4. The de Jonquiére function Li_4 can be defined by $\mathrm{Li}_4(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^4}$ and $\mathrm{dom}(\mathrm{Li}_4) = (-1, 1)$

1 (a) Find the *numerical value* of $\mathrm{Li}_4(0)$.

Solution:

$$\mathrm{Li}_4(0) = \sum_{k=1}^{\infty} \frac{0^k}{k^4} = 0.$$

$\boxed{1}$ (b) Find the *numerical value* of $\mathrm{Li}_4'(0)$.

Solution: We have $\mathrm{Li}_4(x) = x + \frac{x^2}{2^4} + \cdots$, so $\mathrm{Li}_4'(0) = 1$.

$\boxed{1}$ (c) Find the *numerical value* of $\mathrm{Li}_4''(0)$.

Solution: We have $\mathrm{Li}_4(x) = x + \frac{x^2}{2^4} + \cdots$, so $\mathrm{Li}_4''(0) = \frac{1}{8}$.

5. Determine convergence or divergence of each series. Fully justify your work quoting theorems from our class.

$\boxed{1}$ (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{8k+2}$

Solution: Since $\int_1^{\infty} \frac{1}{8x+2}\,\mathrm{d}x$ divereges, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{8k+2}$ diverges.

$\boxed{1}$ (b) $\displaystyle\sum_{k=0}^{\infty} (2k+1)(-1)^k$

Solution: The sequence $k \mapsto (-1)^k(2k+1)$ does not converge to zero; therefore the series $\sum_{k=0}^{\infty}(2k+1)(-1)^k$ diverges. The alternating series test, the integral test, and the ratio test either do not apply or they give no information.

$\boxed{1}$ (c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k+3^k}$

Solution: This is a convergent alternating series. We need to check three things.

- ✔ Is $k \to \frac{1}{2^k+3^k}$ a positive sequence? **Yes**,
- ✔ Is $k \to \frac{1}{2^k+3^k}$ a decreasing sequence? **Yes**
- ✔ does $k \to \frac{1}{2^k+3^k}$ converge to zero? **Yes**

6. For all real numbers x, we have $\cos(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$. Define a wild and crazy function Q as $Q(x) = \int_0^x \cos\left(t^4\right) \mathrm{d}t$.

$\boxed{1}$ (a) Find a power series centered at zero for Q.

Solution:

$$Q(x) = \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} \,\mathrm{d}t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(2k)} x^{2k+1}$$

$\boxed{1}$ (b) Find the radius of convergence for the power series you found in the previous question.

> **Solution:** Since the radius of convergence of $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ is infinty, so is the radius of convergence for the PS for Q.

$\boxed{1}$ 7. For all $x \in (-1,1)$, we have $\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k$. Find the numerical value of

$$\sum_{k=0}^{42} \binom{\frac{1}{2}}{k} \binom{\frac{1}{2}}{42-k}.$$

> **Solution:** Let's define $a_k = \binom{\frac{1}{2}}{k}$. We have $\sqrt{1+x}\sqrt{1+x} = 1+x = \sum_{k=0}^{\infty} c_k x^k$, where $c_k = \sum_{\ell=0}^{k} a_\ell a_{k-\ell}$. So $\sum_{k=0}^{42} \binom{\frac{1}{2}}{k}\binom{\frac{1}{2}}{42-k} = c_{42}$. But $1+x = 1+x+0x^2+0x^3+\cdots$, so $c_{42} = 0$.