

Greek Characters

Name	Symbol	Typical use(s)
alpha	α	angle, constant
beta	β	angle, constant
gamma	γ	angle, constant
delta	δ	limit definition
epsilon	ϵ or ε	limit definition
theta	θ or ϑ	angle
pi	π or π	circular constant
phi	ϕ or φ	angle, constant

Named Sets

empty set	\emptyset
real numbers	\mathbf{R}
ordered pairs	\mathbf{R}^2
integers	\mathbf{Z}
positive integers	$\mathbf{Z}_{>0}$
positive real numbers	$\mathbf{R}_{>0}$

Set Symbols

Meaning	Symbol
is a member	\in
subset	\subset
intersection	\cap
union	\cup
set minus	\setminus

Intervals

For numbers a and b , we define the intervals:

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$$

$$[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$$

$$(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$$

$$[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$

Logic Symbols

Meaning	Symbol
negation	\neg
and	\wedge
or	\vee
implies	\Rightarrow
equivalent	\equiv
iff	\Leftrightarrow
for all	\forall
there exists	\exists

Exponents

For $a, b > 0$, $x \in \mathbf{R}$, and m, n real:

$$a^0 = 1, \quad 0^a = 0$$

$$1^a = 1, \quad a^n a^m = a^{n+m}$$

$$a^n / a^m = a^{n-m}, \quad (a^n)^m = a^{n \cdot m}$$

$$a^{-m} = 1/a^m, \quad (a/b)^m = a^m / b^m$$

$$\sqrt{x^2} = |x|$$

Trigonometric Identities

$$(\cos(x))^2 + (\sin(x))^2 = 1$$

$$2(\cos(x))^2 = 1 + \cos(2x)$$

$$2(\sin(x))^2 = 1 - \cos(2x)$$

$$(\cos(x))^2 - (\sin(x))^2 = \cos(2x)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\operatorname{arccot}(x) = \pi/2 - \arctan(x), \quad \operatorname{dom}(\operatorname{arccot}) = (0, \pi)$$

$$\operatorname{arccsc}(x) = \arcsin(1/x)$$

$$\operatorname{arcsec}(x) = \arccos(1/x)$$

$$\arcsin(x) + \arccos(x) = \pi/2$$

$$\operatorname{arcsec}(x) + \operatorname{arccsc}(x) = \pi/2$$

Limits

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0$$

$$\lim_{x \rightarrow \infty} e^x = \infty \quad \lim_{x \rightarrow -\infty} e^x = 0$$

$$\lim_{x \rightarrow \infty} \ln(x) = \infty \quad \lim_{x \rightarrow 0^+} \ln(x) = -\infty$$

Derivatives

Specific cases

$F(x)$	$F'(x)$
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec(x)^2$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\cot(x)\csc(x)$
$\cot(x)$	$-\csc(x)^2$
$\arccos(x)$	$-1/\sqrt{1-x^2}$
$\arcsin(x)$	$1/\sqrt{1-x^2}$
$\arctan(x)$	$1/(x^2+1)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\tanh(x)$	$1/\cosh(x)^2$
$\operatorname{arccosh}(x)$	$1/\sqrt{x^2-1}$
$\operatorname{arsinh}(x)$	$1/\sqrt{1+x^2}$
$\operatorname{arctanh}(x)$	$1/(1-x^2)$
$\exp(x)$	$\exp(x)$
$\ln(x)$	$1/x$

General Cases

$F(x)$	$F'(x)$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$1/g(x)$	$-g'(x)/g(x)^2$
$f(x)/g(x)$	$(g(x)f'(x) - f(x)g'(x))/g(x)^2$
$f(g(x))$	$g'(x)f'(g(x))$
$f^{-1}(x)$	$1/f'(f^{-1}(x))$

Antiderivatives

$$\int a \, dx = ax$$

$$\int x^a \, dx = \frac{1}{1+a} x^{a+1}, \quad \text{if } a \neq -1$$

$$\int \frac{1}{x} \, dx = \ln|x|$$

$$\int \cos(x) \, dx = \sin(x)$$

$$\int \sin(x) \, dx = -\cos(x)$$

$$\int \tan(x) \, dx = \ln|\sec(x)|$$

$$\int \sec(x) \, dx = \ln|\tan(x) + \sec(x)|$$

$$\int \csc(x) \, dx = -\ln|\csc(x) + \cot(x)|$$

$$\int \cot(x) \, dx = \ln|\sin(x)|$$

$$\int |x| \, dx = x|x|/2$$

Sums

For $k, m, n \in \mathbf{Z}_{>0}$

$$\sum_{k=0}^{n-1} 1 = n$$

$$\sum_{k=0}^{n-1} k = \frac{(n-1)n}{2}$$

$$\sum_{k=0}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6}$$

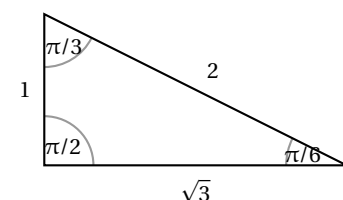
$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}, \quad x \neq 1$$

Logarithms

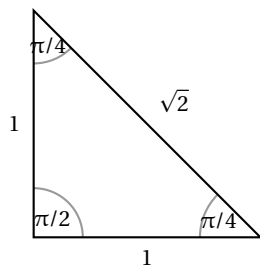
$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Famous Triangles

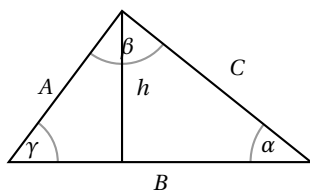
The 30-60-90 triangle



The 45-45-90 triangle



Laws of Cosine & Sine



Law of cosine: $C^2 = A^2 + B^2 - 2AB \cos(\gamma)$

Law of sines: $\frac{\sin(\alpha)}{A} = \frac{\sin(\beta)}{B} = \frac{\sin(\gamma)}{C}$

Area: $\text{Area} = \frac{1}{2}hB = \frac{1}{2}AB \sin(\gamma)$

Hyperbolic Functions

$$2 \cosh(x) = \exp(x) + \exp(-x)$$

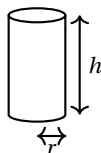
$$2 \sinh(x) = \exp(x) - \exp(-x)$$

$$\tanh(x) = \cosh(x)/\sinh(x)$$

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

Volumes

Right Circular Cylinder

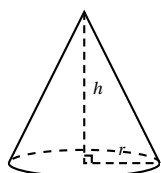


Volume: $V = \pi r^2 h$

Area: (not including circular ends)

$$A = 2\pi r h$$

Cone



Volume: $V = \frac{1}{3}\pi r^2 h$

Area (not including circular base)

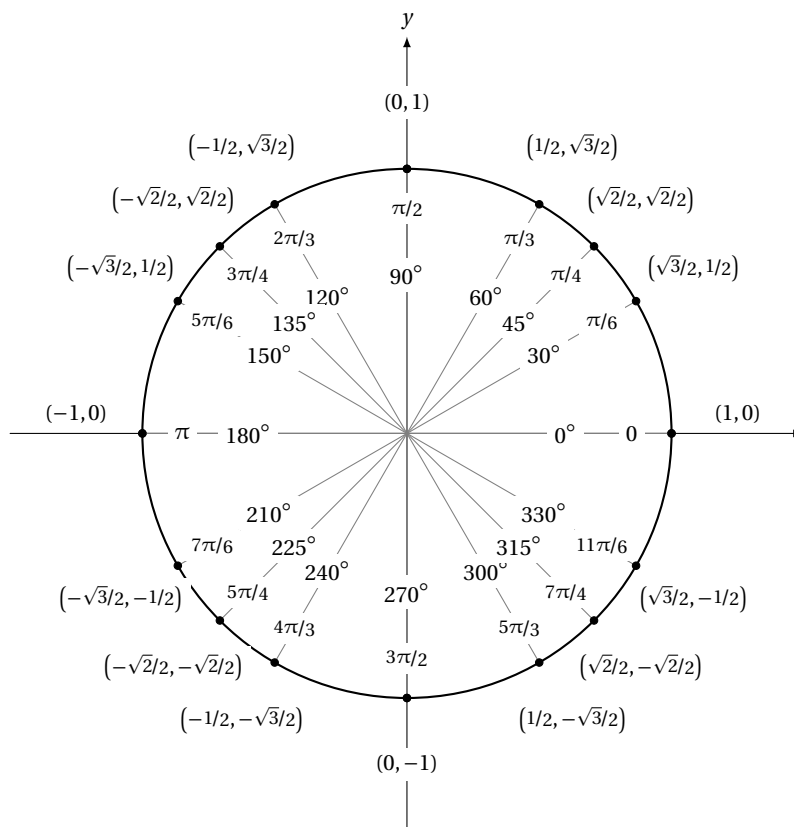
$$A = \pi r \sqrt{r^2 + h^2}$$

Sphere

Area: $A = 4\pi r^2$

Volume: $V = \frac{4\pi}{3} r^3$

Unit Circle



Applications

Arclength of curve $y = f(x)$ with $a \leq x \leq b$

$$= \int_a^b \sqrt{1 + f'(x)^2} dx$$

For the region Q of the xy plane given by

$$Q = \{(x, y) \mid f(x) \leq y \leq g(x) \wedge a \leq x \leq b\},$$

we have

$$\text{Area}(Q) = \int_a^b g(x) - f(x) dx$$

Assuming $0 \leq f(x)$ and rotating about the x -axis

$$\text{Vol}(Q) = \pi \int_a^b g(x)^2 - f(x)^2 dx$$

Assuming $0 \leq a < b$ and rotating about the y -axis

$$\text{Vol}(Q) = 2\pi \int_a^b x(g(x) - f(x)) dx$$

Centroid

$$\text{Area}(Q) \times \bar{x} = \int_a^b x(g(x) - f(x)) dx$$

$$\text{Area}(Q) \times \bar{y} = \frac{1}{2} \int_a^b (g(x)^2 - f(x)^2) dx$$

For the region Q of the xy plane given by

$$Q = \{(x, y) \mid f(y) \leq x \leq g(y) \wedge a \leq y \leq b\},$$

interchange x and y in *all* the previous formulas.

Geometric series

$$\sum_{k=0}^{\infty} z^k = \begin{cases} \frac{1}{1-z} & z \in (-1, 1) \\ \infty & z \in [1, \infty) \end{cases}.$$

When $z \in (-\infty, -1]$, the series $\sum_{k=0}^{\infty} z^k$ diverges.

Divergence Test

If $\lim_{k \rightarrow \infty} a_k \neq 0$, then the series $\sum a_k$ diverges.

P-series

The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p \in (1, \infty)$; otherwise it diverges.

Ratio Test

Let a be a sequence with $0 \notin \text{range}(a)$. Define $L = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$.

- $L \in [0, 1) \Rightarrow \sum |a_k|$ converges.
- $L \in (1, \infty) \Rightarrow \sum a_k$ diverges.

Limit Comparison Test

Let a and b be positive sequences. Define $L = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$.

- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ converges then $\sum b_k$ converges.
- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ diverges then $\sum b_k$ diverges.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Alternating Series Test

Let a be a positive and eventually decreasing sequence. Then $\sum (-1)^k a_k$ converges iff $\lim_{k \rightarrow \infty} a_k = 0$.

Taylor and MacLaurin Series

If a function F is infinitely differentiable at a , its Taylor series centered at a is

$$\sum_{k=0}^{\infty} \frac{F^{(k)}(a)}{k!} (x-a)^k.$$

When a is zero, the Taylor series is also known as the MacLaurin Series.

Polar to Cartesian

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

For $r > 0$ and $0 \leq \theta < 2\pi$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} 2\pi - \arccos(x/r) & \text{if } y < 0 \\ \arccos(x/r) & \text{if } y \geq 0 \end{cases}$$

Integrate Powers of Trig

Let $m, n \in \mathbf{Z}_{\geq 0}$. Then

- $\int \cos(x)^{2m} \sin(x)^{2n} dx = \int \left(\frac{1 + \cos(2x)}{2} \right)^m \left(\frac{1 - \cos(2x)}{2} \right)^n dx$
- $\int \cos(x)^{2m+1} \sin(x)^n dx = \int (1 - z^2)^m z^n dz$, where $z = \sin(x)$
- $\int \cos(x)^m \sin(x)^{2n+1} dx = \int z^m (1 - z^2)^n dz$, where $z = \cos(x)$
- $\int \sec(x)^n dx = \frac{1}{n-1} \sec(x)^{n-2} \tan(x) + \frac{n-2}{n-1} \int \sec(x)^{n-2} dx$, provided $n \neq 1$.
- $\int \tan(x)^{2m+1} \sec(x)^n dx = \int (z^2 - 1)^m z^{n-1} dz$, where $z = \sec(x)$
- $\int \tan(x)^{2m} \sec(x)^n dx = \int (\sec(x)^2 - 1)^m \sec(x)^n dx$.

Trig Substitutions

- $\int F\left(x, \left(1 - x^2\right)^{n/2}\right) dx$, use $x = \sin(\theta)$, where $\theta \in [-\pi/2, \pi/2]$, then integrate $\int F(\sin(\theta), \cos(\theta)^n) \cos(\theta) d\theta$
- $\int F\left(x, \left(1 + x^2\right)^{n/2}\right) dx$, use $x = \sinh(\theta)$, where $\theta \in \mathbf{R}$, then integrate $\int F(\sinh(\theta), \cosh(\theta)^n) \cosh(\theta) d\theta$
- $\int F\left(x, \left(x^2 - 1\right)^{n/2}\right) dx$, use $x = \sec(\theta)$, then integrate $\int F(\sec(\theta), \tan(\theta)^n) \sec(\theta), \tan(\theta) d\theta$

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