MATH 202, Fall 2023 **Name:** ________________________

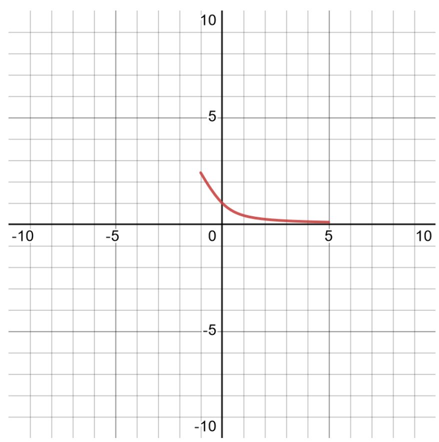
In class work 13 **Row and Seat:** ____________________

"Money buys everything except love, personality, freedom, immortality, silence, peace." CARL SANDBURG

In class work **13** has questions **1** through **4** with a total of **10** points. Turn in your work at the end of class *on paper*. This assignment is due *Tuesday October 10 13:20.*

$\boxed{2}$ 1. Use Desmos to graph $y = \sqrt{x^2+1} - x$. Reproduce the graph here. Based on the graph, what is your guess for the numeric value of $\lim_{x\to\infty} \sqrt{x^2+1} - x$?

Solution:



The graph indicates that $\lim_{x\to\infty} \sqrt{x^2+1} - x = 0$. Since $\lim_{x\to\infty} \sqrt{x^2+1} = \infty$ and $\lim_{x\to\infty} -x = -\infty$, the limit problem $\lim_{x\to\infty} \sqrt{x^2+1} - x$ gives an indeterminate form of the type $\infty - \infty$.

Knowing that we're dealing with an indeterminate form of the type $\infty - \infty$ might tell us something about how to find the limit, but it tells us nothing about the numerical value of the limit.

$\boxed{2}$ 2. Show that the sequence whose formula is $a_k = \sqrt{k^2+1} - k$ converges. Show all of your work.

Solution: We'll use the trick of moving the radicals to the denominator. We have

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2+1}-x &= \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\times\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x},\\ &= \lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x},\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x},\end{aligned}$$

We have $\lim\limits_{x\to\infty}\sqrt{x^2+1}+x=\infty$, so we have

$$= 0$$

Although the calculation $\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x^2+1}+x}=\dfrac{1}{\infty}=0$ is based on a theorem, it's not considered to be something you should do in public. Specifically, the theorem is: If $\lim\limits_{a} F\in\mathbf{R}$ and $\lim\limits_{a} G\in\mathbf{R}=\infty$, then $\lim\limits_{a}\dfrac{F}{G}=0$. Be careful with this–the limit of the numerator must be a real number.

$\boxed{2}$ 3. Determine if the sequence whose formula is $b_k = k \ln\left(1+\frac{8}{k}\right)$ converges. If it does, find its limit. As always, show your work.

Solution: We need the limit of a product. Since $\lim\limits_{k\to\infty} k = \infty$ and $\lim\limits_{k\to\infty} \left(1+\dfrac{8}{k}\right) = 0$, we have an indeterminate form of the type $0 \times \infty$. By rearranging this to $\frac{\left(1+\frac{8}{k}\right)}{\frac{1}{k}}$, we have an indeterminate form of the type $\frac{0}{0}$. And maybe the l'Hôpital rule will help. We have

$$\begin{aligned}
\lim_{k\to\infty} \ln\left(1+\frac{8}{k}\right) &= \lim_{k\to\infty} \frac{\left(1+\frac{8}{k}\right)}{\frac{1}{k}}, \\
&= \lim_{k\to\infty} \frac{-\frac{8}{\left(\frac{8}{k}+1\right)k^2}}{-\frac{1}{k^2}} \\
&= \lim_{k\to\infty} \frac{8}{\frac{8}{k}+1}, \\
&= 8.
\end{aligned}$$

Actually, until we've determined that the limit of the quotient of derivatives is a real number, equality between the first and second lines is not certain. Notionally, we could use $\stackrel{?}{=}$ To mean equality as long as the right side is a real number, but that notation (I didn't invent it) is a bit weird.

4. A sequence c is defined recursively by

$$c_n = \begin{cases} 2 & n = 0 \\ 5 & n = 1 \\ 5c_{n-1} - 6c_{n-2} & n = 2, 3, 4, \ldots \end{cases}$$

$\boxed{2}$ (a) Find the numeric values of c_2, c_3, and c_4.

Solution:

$$\begin{aligned} c_2 &= 5c_1 - 6c_0 = 5 \times 5 - 6 \times 2 = 25 - 12 = 13, \\ c_3 &= 5c_2 - 6c_1 = 5 \times 13 - 6 \times 5 = 65 - 30 = 35, \\ c_4 &= 5c_3 - 6c_2 = 5 \times 35 - 6 \times 13 = 25 - 12 = 97. \end{aligned}$$

$\boxed{2}$ (b) Show that $c_n = 2^n + 3^n$ is a solution to the equation $c_n = 5c_{n-1} - 6c_{n-2}$. To do this, show that $c_n = 5c_{n-1} - 6c_{n-2}$ simplifies to an identity using $c_n = 2^n + 3^n$.

Solution: We have

$$\begin{aligned} [c_n = 5c_{n-1} - 6c_{n-2}] &= \left[2^n + 3^n = 5(2^{n-1} + 3^{n-1}) - 6(2^{n-2} + 3^{n-2})\right], \\ &= \left[2^n + 3^n = \frac{5}{2} \times 2^n + \frac{3}{2} \times 3^n - \frac{6}{4} \times 2^n + \frac{6}{9} \times 3^n\right], \\ &= \left[2^n + 3^n = 2^n + 3^n\right], \\ &= \text{True}. \end{aligned}$$