MATH 202, Fall 2023 **Name:**____________________________

Practice Exam **Row and Seat:**________________________

Warning: For the most part, I've only given answers, not solutions–this is BOB (back-of-book) fashion. Of course, for your exam you will need to **show your work**.

1. Define a region Q of the xy-plane by $Q = \{(x, y) | 0 \le y \le x\sin(x), 0 \le x \le \pi\}$

   (a) Find area(Q).

   Solution:

   $$\text{area}(Q) = \int_0^{\pi} x\sin(x)\,\mathrm{d}x = \pi.$$

   (b) Find the x coordinate of the centroid of Q.

   Solution:

   $$\text{area}(Q)\overline{x} = \int_0^{\pi} x^2\sin(x)\,\mathrm{d}x = \pi^2 - 4.$$

   So $\overline{x} = \frac{\pi^2-4}{\pi} \approx 1.868353108854631$.

2. Find the value of each indefinite or definite integral.

   (a) $\displaystyle\int xe^{x^2}\,\mathrm{d}x =$

   Solution:

   $$\int xe^{x^2}\,\mathrm{d}x = \frac{e^{x^2}}{2}$$

   (b) $\displaystyle\int_0^1 \frac{x}{\left(1+x^2\right)^{3/2}}\,\mathrm{d}x =$

   Solution:

   $$\int_0^1 \frac{x}{\left(1+x^2\right)^{3/2}}\,\mathrm{d}x = 1 - \frac{1}{\sqrt{2}}$$

(c) $\int x\sqrt{1-x^2}\,\mathrm{d}x =$

Solution:

$$\int x\sqrt{1-x^2}\,\mathrm{d}x = -\frac{\left(1-x^2\right)^{\frac{3}{2}}}{3}$$

(d) $\int \tan^{-1}(x)\,\mathrm{d}x =$

Solution:

$$\int \tan^{-1}(x)\,\mathrm{d}x = x\tan^{-1}(x) - \frac{\ln\left(x^2+1\right)}{2}$$

(e) $\int x\ln(x)\,\mathrm{d}x =$

Solution:

$$\int x\ln(x)\,\mathrm{d}x = \frac{x^2\ln(x)}{2} - \frac{x^2}{4}$$

(f) $\int_0^1 xe^{-x}\,\mathrm{d}x =$

Solution:

$$\int_0^1 xe^{-x}\,\mathrm{d}x = 1 - 2\mathrm{e}^{-1}$$

(g) $\int \frac{1}{(x+5)(x+9)}\,\mathrm{d}x =$

Solution:

$$\int \frac{1}{(x+5)(x+9)}\,\mathrm{d}x = \frac{\ln|x+5|}{4} - \frac{\ln|x+9|}{4}$$

(h) $\int \cos^2(x)\,\mathrm{d}x =$

Solution:

$$\int \cos^2(x)\,\mathrm{d}x = \frac{\sin(2x)}{4} + \frac{x}{2}$$

(i) $\displaystyle\int \cos^3(x)\sin(x)\,\mathrm{d}x =$

Solution: The easy way is to substitute $z = \cos(x)$. Doing so, an antiderivative is

$$\int \cos^3(x)\sin(x)\,\mathrm{d}x = -\frac{\cos(x)^4}{4}$$

If you peel off one factor of $\cos(x)$ and use the Pythagorean identity on the rest, then make the substitution $z = \sin(x)$, your answer will be

$$\int \cos^3(x)\sin(x)\,\mathrm{d}x = \frac{1}{2}\sin(x)^2 - \frac{1}{4}\sin(x)^4.$$

Replacing $\sin(x)^2$ by $1 - \cos(x)^2$ in this gives (after some algebra)

$$\int \cos^3(x)\sin(x)\,\mathrm{d}x = \frac{1}{4} - \frac{\cos(x)^4}{4}.$$

Of course, antiderivatives are undetermined up to an additive constant, so this is equivalent to

$$\int \cos^3(x)\sin(x)\,\mathrm{d}x = -\frac{\cos(x)^4}{4}.$$

3. Find the numerical value of each improper integral.

(a) $\int_0^\infty x\mathrm{e}^{-x^2}\,\mathrm{d}x$

Solution:

$$\int_0^\infty x\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{2}.$$

(b) $\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x$

Solution:

$$\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x = 1.$$

(c) $\int_{-\infty}^\infty \frac{1}{x^2+9}\,\mathrm{d}x$

Solution:

$$\int_{-\infty}^\infty \frac{1}{x^2+9}\,\mathrm{d}x = \frac{\pi}{3}.$$

(d) $\int_0^1 \frac{1}{x^{9/10}}\,\mathrm{d}x$

Solution:

$$\int_0^1 \frac{1}{x^{9/10}}\,\mathrm{d}x = 10.$$

$\boxed{2}$ 4. When Morwenna graduates from UNK and starts her first job, she expects to earn a starting annual salary of $42,000. She plans to work for 42 years and she expects to earn a 3% raise each year. Thus, in her n$^{\text{th}}$ year of work, her salary is $42,000 \times 1.03^{n-1}$. During Morwenna's 42 years of labor, how much will she earn?

Solution:

$$42000 \sum_{k=0}^{41} 1.03^k = 3,444,974.25$$

5. Given a formula for a sequence b, find its limit. Show all of your work.

$\boxed{2}$

(a) $b_n = \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k$.

Solution:

$$\lim_{n\to\infty} b_n = 3. \tag{1}$$

$\boxed{2}$ (b) $b_n = \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k$.

Solution:

$$\lim_{n\to\infty} b_n = \infty.$$

6. Show that the sequence whose formula is $a_k = \sqrt{k^2+3k+1} - k$ converges. Show all of your work.

Solution: We'll use the trick of moving the radicals to the denominator. We have

$$\begin{aligned}
\lim_{k\to\infty}\left(\sqrt{k^2+3k+1}-k\right) &= \lim_{k\to\infty}\left(\sqrt{k^2+3k+1}-k\right) \times \frac{\sqrt{k^2+1}+k}{\sqrt{k^2+1}+k}, \\
&= \lim_{k\to\infty}\frac{(k^2+3k+1)-k^2}{\sqrt{k^2+1}+k}, \\
&= \lim_{k\to\infty}\frac{3k+1}{\sqrt{k^2+3k+1}+k}, \\
&= \lim_{k\to\infty}\frac{3k}{2k}, \\
&= \frac{3}{2}
\end{aligned}$$

$\boxed{2}$ 7. Determine if the sequence whose formula is $b_k = k \ln\left(1 + \frac{8}{k}\right)$ converges. If it does, find its limit. As always, show your work.

Solution: We need the limit of a product. Since $\lim_{k\to\infty} k = \infty$ and $\lim_{k\to\infty} \ln\left(1 + \dfrac{8}{k}\right) = 0$, we have an indeterminate form of the type $0 \times \infty$. By rearranging this to $\frac{\ln\left(1+\frac{8}{k}\right)}{\frac{1}{k}}$, we have an indeterminate form of the type $\frac{0}{0}$. And maybe the l'Hôpital rule will help. We have

$$\begin{aligned}
\lim_{k\to\infty} \ln\left(1 + \frac{8}{k}\right) &= \lim_{k\to\infty} \frac{\ln\left(1 + \frac{8}{k}\right)}{\frac{1}{k}}, \\
&= \lim_{k\to\infty} \frac{-\frac{8}{\left(\frac{8}{k}+1\right) k^2}}{-\frac{1}{k^2}} \\
&= \lim_{k\to\infty} \frac{8}{\frac{8}{k} + 1}, \\
&= 8.
\end{aligned}$$

Actually, until we've determined that the limit of the quotient of derivatives is a real number, equality between the first and second lines is not certain. Notionally, we could use $\stackrel{?}{=}$ To mean equality as long as the right side is a real number, but that notation (I didn't invent it) is nonstandard.

8. A sequence c is defined recursively by

$$c_n = \begin{cases} 2 & n = 0 \\ 5 & n = 1 \\ 5c_{n-1} - 6c_{n-2} & n = 2, 3, 4, \ldots \end{cases}$$

$\boxed{2}$ (a) Find the numeric values of c_2, c_3, and c_4.

Solution:

$$\begin{aligned} c_2 &= 5c_1 - 6c_0 = 5 \times 5 - 6 \times 2 = 25 - 12 = 13, \\ c_3 &= 5c_2 - 6c_1 = 5 \times 13 - 6 \times 5 = 65 - 30 = 35, \\ c_4 &= 5c_3 - 6c_2 = 5 \times 35 - 6 \times 13 = 25 - 12 = 97. \end{aligned}$$

$\boxed{2}$ 9. Find the *numeric value* of the integral $\int_0^\infty \frac{x}{1+x^4}\,\mathrm{d}x$. **Hint:** To find an antiderivative of $\int \frac{x}{1+x^4}\,\mathrm{d}x$, use the substitution $z = x^2$.

Solution: Let's begin by finding an antiderivative; once we found it, we'll use the FTC along with a limit to find the value of the improper integral. We have

$$\begin{aligned} \int \frac{x}{1+x^4}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{1+(x^2)^2}\,\mathrm{d}x^2, && \left(x\mathrm{d}x = \frac{1}{2}\mathrm{d}x^2\right), \\ &= \frac{1}{2}\int \frac{1}{1+z^2}\,\mathrm{d}z, && \text{(replace } x^2 \text{ by } z\text{)} \\ &= \frac{1}{2}\arctan(z), && \text{(standard antiderivative)} \\ &= \frac{1}{2}\arctan(x^2) && \text{(replace} z \text{ by } x^2\text{).} \end{aligned}$$

Second, we take on the improper integral:

$$\begin{aligned}\int_0^\infty \frac{x}{1+x^4}\,\mathrm{d}x &= \lim_{a\to\infty}\int_0^a \frac{x}{1+x^4}\,\mathrm{d}x,\\ &= \lim_{a\to\infty}\left(\frac{1}{2}\arctan(x^2)\Big|_0^a\right),\\ &= \lim_{a\to\infty}\left(\frac{1}{2}\arctan(a^2)-\frac{1}{2}\arctan(0)\right),\\ &= \lim_{a\to\infty}\left(\frac{1}{2}\arctan(a^2)\right),\\ &= \frac{\pi}{4}\end{aligned}$$

$\boxed{1}$ 10. Show that $\int_0^\infty \frac{28+\cos(x)}{1+x^2}\,\mathrm{d}x$ converges. To do this, use a comparison test with $\frac{\alpha}{1+x^2}$, where α is a number that you cleverly choose.

Solution: For all real numbers x, we have $27 \le 28+\cos(x) \le 29$. Let's (cleverly) choose α to be 29. Then for all real numbers x, we have

$$0 \le \frac{28+\cos(x)}{1+x^2} \le \frac{29}{1+x^2}. \tag{2}$$

But $\int_0^\infty \frac{29}{1+x^2}\,\mathrm{d}x$ converges, so $\int_0^\infty \frac{28+\cos(x)}{1+x^2}\,\mathrm{d}x$ converges.
Be careful We only know that $\int_0^\infty \frac{28+\cos(x)}{1+x^2}\,\mathrm{d}x$ is a real number, but the comparison test **doesn't** tell us its value. We'll it does tell us that

$$\int_0^\infty \frac{28+\cos(x)}{1+x^2}\,\mathrm{d}x \le \int_0^\infty \frac{29}{1+x^2}\,\mathrm{d}x = \frac{29\pi}{2} \approx 45.553093477052.$$

Numerical integration gives us the approximation $\int_0^\infty \frac{28+\cos(x)}{1+x^2}\,\mathrm{d}x \approx 44.560$

$\boxed{1}$ 11. Show that $\int_1^\infty \frac{107+\mathrm{e}^{-x}}{1+x^2}\,\mathrm{d}x$ converges. To do this, use a limit comparison test.

Solution: We know that $\int_1^\infty \frac{1}{1+x^2}\,\mathrm{d}x$ converges. And for all real $x \geq 1$ we have $\frac{107+\mathrm{e}^{-x}}{1+x^2} > 0$ and $\frac{1}{1+x^2} > 0$. Finally, everything in sight is continuous; so look at

$$\begin{aligned}\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{107+\mathrm{e}^{-x}}{1+x^2}} &= \lim_{x\to\infty} \frac{1}{107+\mathrm{e}^{-x}},\\ &= \frac{1}{107}.\end{aligned}$$

So $\int_1^\infty \frac{107+\mathrm{e}^{-x}}{1+x^2}\,\mathrm{d}x$ converges.

12. Use the integral test to show that the series $\sum_{k=0}^\infty \frac{1}{1+k^2}$ converges.