MATH 202 Spring 2010 **Name:**____________________________

Exam 3–Practice **Row and Seat:**________________________

1. Use power series to find the numerical value of each limit. You might like to use the facts

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \cdots$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \ldots$$

(a) $\displaystyle\lim_{x\to 0} \frac{\log\left(x^3+1\right)}{\sin\left(2x^3\right)}$

Solution:

$$\lim_{x\to 0} \frac{\log\left(x^3+1\right)}{\sin\left(2x^3\right)} = \frac{1}{2}$$

(b) $\displaystyle\lim_{x\to 0} \frac{\arctan(x) - \sin(x)}{x^3}$

Solution:

$$\lim_{x\to 0} \frac{\arctan(x) - \sin(x)}{x^3} = -\frac{1}{6}.$$

(c) $\displaystyle\lim_{x\to 0} \frac{\arctan(x) - \sin(x)}{x^4}$

Solution:

$$\lim_{x\to 0^+} \frac{\arctan(x) - \sin(x)}{x^4} = -\infty.$$

$\boxed{1}$ 2. Use the *ratio* test to determine of the series $\sum_{k=0}^{\infty} \frac{\left(\frac{k}{3}\right)^k}{k!}$ converges or diverges.

Solution: Algebra is supposed to be easier than calculus, so let's start by simplifying

$$\frac{(\frac{k+1}{3})^{k+1}}{(k+1)!} \times \frac{k!}{\left(\frac{k}{3}\right)^k} = \frac{1}{3}(\frac{k+1}{k})^k = \frac{1}{3}\left(1+\frac{1}{k}\right)^k.$$

So $\lim\limits_{x\to\infty} \frac{(\frac{k+1}{3})^{k+1}}{(k+1)!} \times \frac{k!}{(\frac{k}{3})^k} = \frac{1}{3}$. And so $\sum\limits_{k=0}^{\infty} \frac{\left(\frac{k}{3}\right)^k}{k!}$ converges.

$\boxed{5}$ 3. Find the numerical value of $\sum\limits_{k=4}^{\infty}\left(\frac{1}{2}\right)^k$.

Solution:

$$\sum_{k=4}^{\infty}\left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+4} = \frac{1}{2^4}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{2}{2^4} = \frac{1}{8}.$$

4. The de Jonquiére function Li_2 can be defined by $\mathrm{Li}_2(x) = \sum\limits_{k=1}^{\infty}\frac{x^k}{k^2}$ and $\mathrm{dom}(\mathrm{Li}_2) = (-1,1)$

$\boxed{5}$ (a) Find the *numerical value* of $\mathrm{Li}_2(0)$.

Solution:
$$\mathrm{Li}_2(0) = \sum_{k=1}^{\infty}\frac{0^k}{k^2} = 0.$$

$\boxed{5}$ (b) Find the *radius of convergence* for the power series $\sum_{k=1}^{\infty}\frac{x^k}{k^2}$.

Solution: We need to use the absolute ratio test; for $x \neq 0$, we have
$$\lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)^2}\frac{k^2}{x^k}\right| = |x|.$$
So the sum converges for $|x| < 1$; that is, the radius of convergence is 1.

$\boxed{5}$ (c) Find the *numerical value* of $\mathrm{Li}_2'(0)$. **Hint:** $\mathrm{Li}_2(x) = x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots$.

Solution:

$$\mathrm{Li}_2(0) = 1.$$

5. Determine convergence or divergence of each series. Fully justify your work. This doesn't mean you need to use our definition of convergent; instead use the theorems we've estiblished.

$\boxed{5}$ (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

Solution: This is the harmonic series. It diverges. Alternatively, the integral test applies and tells us that the series diverges; the ratio test gives no information.

$\boxed{5}$ (b) $\displaystyle\sum_{k=0}^{\infty} (-1)^k$

Solution: The sequence $k \mapsto (-1)^k$ does not converge to zero; therefore the series $\sum_{k=0}^{\infty} (-1)^k$ diverges. The alternating series test, the integral test, and the ratio test either do not apply or they give no information.

$\boxed{5}$ (c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$

Solution: This is a convergent alternating series. We need to check three things.

- ✔ Is $k \to \frac{1}{1+k}$ a positive sequence? **Yes**,
- ✔ Is $k \to \frac{1}{1+k}$ a decreasing sequence? **Yes**
- ✔ does $k \to \frac{1}{1+k}$ converge to zero? **Yes**

$\boxed{5}$ (d) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$

Solution: This is a series of positive terms; we'll try the ratio test. We have

$$\lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{1}{1+k} = 0.$$

Therefore, the series converges.

$\boxed{5}$ 6. My friend Milhous claims that the sum $\sum_k f_k$ converges provided that the sequence f converges to zero. Show Milhous an example of a seqence f that converges to zero, but the sum $\sum_k f_k$ diverges.

Solution: The sequence $k \in \mathbf{Z}_{>0} \mapsto \frac{1}{k}$ converges to zero, but the harmonic sum $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

7. For all real numbers x, we have $\sin(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

$\boxed{2}$ (a) Find the power series representation for $\sin(x) - x$ centered at zero. **Hint:** When you don't know where to start, go to your happy place: write the first few terms of the Taylor series for sine centered at zero. Then subtract x.

Solution: Let's go to our happy place; for all real x, we have

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots.$$

Thus

$$\sin(x) - x = -\frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots.$$

Arguably this answer is OK, but the ellipsis (that is the $\cdots$) leaves too much to the imagination. An explicit answer is

$$\sin(x) - x = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

$\boxed{2}$ (b) For $x \neq 0$, find the *first two nonzero terms* in a power series representation for $\frac{\sin(x)-x}{x^3}$. Again, try visiting your happy place.

Solution: Here is my happy place:

$$\sin(x) - x = -\frac{1}{6}x^3 + \frac{1}{120}x^5 \cdots$$

For $x \neq 0$, we have

$$\frac{\sin(x) - x}{x^3} = -\frac{1}{6} + \frac{1}{120}x^2 \cdots$$

And explicitly, we have

$$\frac{\sin(x)-x}{x^3}=\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k-2}$$

It's optional to do, but changing $k \to k+1$ gives

$$\frac{\sin(x)-x}{x^3}=\sum_{k=0}^{\infty}-\frac{(-1)^k}{(2k+3)!}x^{2k}$$

2 (c) Use the above result to find the *numerical value* of the limit

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}.$$

Solution: We have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x) - x}{x^3} &= \lim_{x\to 0} \sum_{k=0}^{\infty} -\frac{(-1)^k}{(2k+3)!} x^{2k}, \\ &= \sum_{k=0}^{\infty} -\frac{(-1)^k}{(2k+3)!} 0^{2k}, \\ &= -\frac{1}{6}.\end{aligned}$$

1 8. Define a function erf by the definite integral $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$. Find a power series representation for erf. Find the radius of convergence of this power series.

Solution: We have

$$\begin{aligned}\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t, \\ &= \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k}\,\mathrm{d}t, \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k}{k!} t^{2k}\,\mathrm{d}t, \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^x t^{2k}\,\mathrm{d}t, \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k+1}}{2k+1}.\end{aligned}$$

We could use the ratio test, but the theory of power series tells us that the termwise integration does not change the radius of convergence. So the radius of convergence is infinity.