Named Sets

empty set	$\varnothing$
real numbers	$\mathbf{R}$
ordered pairs	$\mathbf{R}^2$
integers	$\mathbf{Z}$
positive integers	$\mathbf{Z}_{>0}$
positive real numbers	$\mathbf{R}_{>0}$

Exponents

For $a, b \in \mathbf{R}_{>0}, x \in \mathbf{R}$, and $m, n \in \mathbf{R}$,

$$a^0 = 1, \qquad 0^a = 0$$
$$1^x = 1, \qquad a^n a^m = a^{n+m}$$
$$a^n/a^m = a^{n-m}, \qquad (a^n)^m = a^{n \cdot m}$$
$$a^{-m} = 1/a^m, \qquad (a/b)^m = a^m/b^m$$
$$\sqrt{x^2} = |x|$$

Trigonometric Identities

We define $\operatorname{dom}(\operatorname{arccot}) = (0, \pi)$.

$$(\cos(x))^2 + (\sin(x))^2 = 1$$
$$2(\cos(x))^2 = 1 + \cos(2x)$$
$$2(\sin(x))^2 = 1 - \cos(2x)$$
$$(\cos(x))^2 - (\sin(x))^2 = \cos(2x)$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\operatorname{arccot}(x) = \pi/2 - \arctan(x)$$
$$\operatorname{arccsc}(x) = \arcsin(1/x)$$
$$\operatorname{arcsec}(x) = \arccos(1/x)$$
$$\arcsin(x) + \arccos(x) = \pi/2$$
$$\operatorname{arcsec}(x) + \operatorname{arccsc}(x) = \pi/2$$

Hyperbolic Functions

$$2\cosh(x) = \exp(x) + \exp(-x)$$
$$2\sinh(x) = \exp(x) - \exp(-x)$$
$$\tanh(x) = \sinh(x)/\cosh(x)$$
$$\cosh(x)^2 - \sinh(x)^2 = 1$$

Logarithms

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Derivatives

Specific cases

$F(x)$	$F'(x)$
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec(x)^2$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\cot(x)\csc(x)$
$\cot(x)$	$-\csc(x)^2$
$\arccos(x)$	$-1/\sqrt{1-x^2}$
$\arcsin(x)$	$1/\sqrt{1-x^2}$
$\arctan(x)$	$1/(x^2+1)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\tanh(x)$	$1/\cosh(x)^2$
$\operatorname{arccosh}(x)$	$1/\sqrt{x^2-1}$
$\operatorname{arcsinh}(x)$	$1/\sqrt{1+x^2}$
$\operatorname{arctanh}(x)$	$1/(1-x^2)$
$\exp(x)$	$\exp(x)$
$\ln(x)$	$1/x$

General Cases

$F(x)$	$F'(x)$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$1/g(x)$	$-g'(x)/g(x)^2$
$f(x)/g(x)$	$(g(x)f'(x) - f(x)g'(x))/g(x)^2$
$f(g(x))$	$g'(x)f'(g(x))$
$f^{-1\prime}(x)$	$1/f'(f^{-1}(x))$

Antiderivatives

$$\int a\,\mathrm{d}x = ax$$
$$\int x^a\,\mathrm{d}x = \frac{1}{1+a}x^{a+1}, \quad \text{if } a \neq -1$$
$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|$$
$$\int \cos(x)\,\mathrm{d}x = \sin(x)$$
$$\int \sin(x)\,\mathrm{d}x = -\cos(x)$$
$$\int \tan(x)\,\mathrm{d}x = \ln|\sec(x)|$$
$$\int \sec(x)\,\mathrm{d}x = \ln|\tan(x) + \sec(x)|$$
$$\int \csc(x)\,\mathrm{d}x = -\ln|\csc(x) + \cot(x)|$$
$$\int \cot(x)\,\mathrm{d}x = \ln|\sin(x)|$$
$$\int |x|\,\mathrm{d}x = x|x|/2$$

Sums

For $n \in \mathbf{Z}_{>0}$

$$\sum_{k=0}^{n-1} 1 = n, \quad \sum_{k=0}^{n-1} k = (n-1)n/2$$
$$\sum_{k=0}^{n-1} k^2 = (n-1)n(2n-1)/6,$$
$$\sum_{k=0}^{n-1} x^k = \sum_{k=1}^{n} x^{k-1} = \frac{1-x^n}{1-x}, \quad x \neq 1$$
$$\sum_{k=0}^{\infty} x^k = \begin{cases} \dfrac{1}{1-x} & x \in (-1,1) \\ \infty & x \in [1,\infty] \end{cases}.$$

When $x \in (-\infty, -1]$, the series $\sum_{k=0}^{\infty} x^k$ diverges.

Applications

Arc length of curve $y = f(x)$ with $a \le x \le b$

$$= \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

For the region Q of the xy plane given by

$$Q = \{(x,y) \mid f(x) \le y \le g(x) \wedge a \le x \le b\},$$

we have

$$\operatorname{Area}(Q) = \int_a^b g(x) - f(x)\,\mathrm{d}x$$

Assuming $0 \le f(x)$ and rotating about the x-axis

$$\operatorname{Vol}(Q) = \pi \int_a^b g(x)^2 - f(x)^2\,\mathrm{d}x$$

Assuming $0 \le a < b$ and rotating about the y-axis

$$\operatorname{Vol}(Q) = 2\pi \int_a^b x(g(x) - f(x))\,\mathrm{d}x$$

Centroid

$$\operatorname{Area}(Q) \times \overline{x} = \int_a^b x\left(g(x) - f(x)\right)\mathrm{d}x$$
$$\operatorname{Area}(Q) \times \overline{y} = \frac{1}{2}\int_a^b \left(g(x)^2 - f(x)^2\right)\mathrm{d}x$$

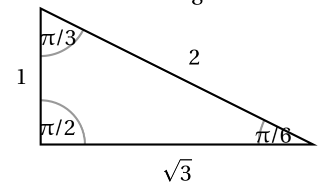
For the region Q of the xy plane given by

$$Q = \{(x,y) \mid f(y) \le x \le g(y) \wedge a \le y \le b\},$$

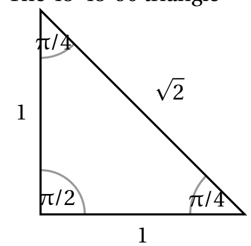
interchange x and y in *all* the previous formulas.

Famous Triangles

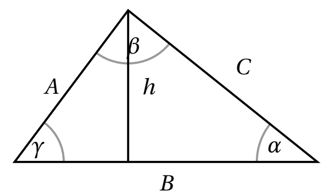
The 30-60-90 triangle



The 45-45-90 triangle



Laws of Cosine & Sine



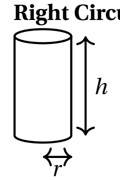
Law of cosine: $C^2 = A^2 + B^2 - 2AB\cos(\gamma)$

Law of sines: $\dfrac{\sin(\alpha)}{A} = \dfrac{\sin(\beta)}{B} = \dfrac{\sin(\gamma)}{C}$

Area: $\operatorname{Area} = 1/2\,hB = 1/2\,AB\sin(\gamma)$

Volumes

Right Circular Cylinder



Volume: $V = \pi r^2 h$

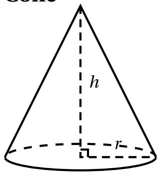
Area: (not including circular ends) $A = 2\pi r h$

Sphere with radius r

Area: $A = 4\pi r^2$

Volume: $V = \dfrac{4\pi}{3} r^3$

Cone



Volume: $V = \pi r^2 h/3$
Area
$A = \pi r\sqrt{r^2+h^2}$ (not including circular base).

P-Series, Divergence Test, Ratio Test, Comparison, & AST

The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p \in (1, \infty)$; otherwise it diverges.

If $\lim_{k\to\infty} a_k \neq 0$, the series $\sum a_k$ diverges.

Let a be a sequence with $0 \notin \mathrm{range}(a)$. Define $L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$.

- $L \in [0,1) \implies \sum |a_k|$ converges.
- $L \in (1, \infty] \implies \sum a_k$ diverges.

Let a and b be positive sequences. Define $L = \lim_{k\to\infty} \frac{a_k}{b_k}$.

- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ converges, then $\sum b_k$ converges.
- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ diverges, then $\sum b_k$ diverges.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Let a be a positive and eventually decreasing sequence. Then $\sum (-1)^k a_k$ converges if and only if $\lim_{k\to\infty} a_k = 0$.

Taylor and MacLaurin Series

If a function F is infinitely differentiable at a, its Taylor series centered at a is

$$\sum_{k=0}^{\infty} \frac{F^{(k)}(a)}{k!} (x-a)^k.$$

When a is zero, the Taylor series is also known as a MacLaurin Series.

Polar to Cartesian

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

For $r > 0$ and $0 \leq \theta < 2\pi$

$$r = \sqrt{x^2+y^2}, \quad \theta = \begin{cases} 2\pi - \arccos(x/r) & \text{if } y < 0 \\ \arccos(x/r) & \text{if } y \geq 0 \end{cases}$$

Integrate Powers of Trig

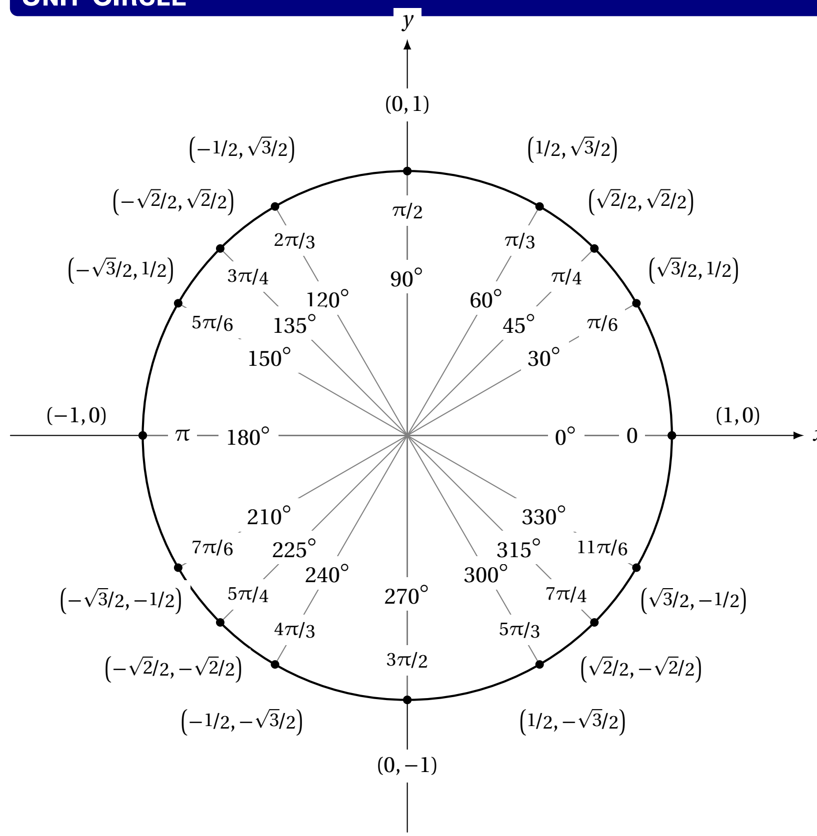
Let $m, n \in \mathbf{Z}_{\geq 0}$. Then

- $\int \cos(x)^{2m} \sin(x)^{2n} \, \mathrm{d}x = \int \left(\frac{1+\cos(2x)}{2}\right)^m \left(\frac{1-\cos(2x)}{2}\right)^n \mathrm{d}x$
- $\int \cos(x)^{2m+1} \sin(x)^n \, \mathrm{d}x = \int (1-z^2)^m z^n \, \mathrm{d}z$, where $z = \sin(x)$
- $\int \cos(x)^m \sin(x)^{2n+1} \, \mathrm{d}x = -\int z^m (1-z^2)^n \, \mathrm{d}z$, where $z = \cos(x)$
- $\int \sec(x)^n \, \mathrm{d}x = \frac{1}{n-1} \sec(x)^{n-2} \tan(x) + \frac{n-2}{n-1} \int \sec(x)^{n-2} \, \mathrm{d}x$, provided $n \neq 1$.
- $\int \tan(x)^{2m+1} \sec(x)^n \, \mathrm{d}x = \int (z^2-1)^m z^{n-1} \, \mathrm{d}z$, where $z = \sec(x)$
- $\int \tan(x)^{2m} \sec(x)^n \, \mathrm{d}x = \int (\sec(x)^2 - 1)^m \sec(x)^n \, \mathrm{d}x$.

Trig Substitutions

- $\int F\left(x, \left(1-x^2\right)^{n/2}\right) \mathrm{d}x$, use $x = \sin(\vartheta)$, where $\vartheta \in [-\pi/2, \pi/2]$, then integrate $\int F\left(\sin(\vartheta), \cos(\vartheta)^n\right) \cos(\vartheta) \, \mathrm{d}\vartheta$
- $\int F\left(x, \left(1+x^2\right)^{n/2}\right) \mathrm{d}x$, use $x = \sinh(\vartheta)$, where $\vartheta \in \mathbf{R}$, then integrate $\int F\left(\sinh(\vartheta), \cosh(\vartheta)^n\right) \cosh(\vartheta) \, \mathrm{d}\vartheta$
- $\int F\left(x, \left(x^2-1\right)^{n/2}\right) \mathrm{d}x$, use $x = \sec(\vartheta)$, then integrate $\int F(\sec(\vartheta), \tan(\vartheta)^n) \sec(\vartheta) \tan(\vartheta) \, \mathrm{d}\vartheta$

Unit Circle



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