Greek Characters

Name	Symbol	Typical use(s)
alpha	α	angle, constant
beta	β	angle, constant
gamma	γ	angle, constant
delta	δ	limit definition
epsilon	ϵ or ε	limit definition
theta	θ or ϑ	angle
pi	π or ϖ	circular constant
phi	ϕ or φ	angle, constant

Named Sets

empty set	$\varnothing$
real numbers	$\mathbf{R}$
ordered pairs	$\mathbf{R}^2$
integers	$\mathbf{Z}$
positive integers	$\mathbf{Z}_{>0}$
positive real numbers	$\mathbf{R}_{>0}$

Set Symbols

Meaning	Symbol
is a member	$\in$
subset	$\subset$
intersection	$\cap$
union	$\cup$
set minus	$\setminus$

Intervals

For numbers a and b, we define the intervals:

$$(a,b) = \{x \in \mathbf{R} \mid a < x < b\}$$

$$[a,b) = \{x \in \mathbf{R} \mid a \le x < b\}$$

$$(a,b] = \{x \in \mathbf{R} \mid a < x \le b\}$$

$$[a,b] = \{x \in \mathbf{R} \mid a \le x \le b\}$$

Logic Symbols

Meaning	Symbol
negation	$\neg$
and	$\wedge$
or	$\vee$
implies	$\implies$
equivalent	$\equiv$
iff	$\iff$
for all	$\forall$
there exists	$\exists$

Exponents

For $a, b > 0, x \in \mathbf{R}$, and m, n real:

$$a^0 = 1, \qquad 0^a = 0$$

$$1^a = 1, \qquad a^n a^m = a^{n+m}$$

$$a^n/a^m = a^{n-m}, \qquad (a^n)^m = a^{n\cdot m}$$

$$a^{-m} = 1/a^m, \qquad (a/b)^m = a^m/b^m$$

$$\sqrt{x^2} = |x|$$

Trigonometric Identities

$$\cos(x)^2 + \sin(x)^2 = 1$$

$$2\cos(x)^2 = 1 + \cos(2x)$$

$$2\sin(x)^2 = 1 - \cos(2x)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\operatorname{arccot}(x) = \pi/2 - \arctan(x)$$

$$\operatorname{arccsc}(x) = \pi/2 - \arcsin(1/x)$$

$$\operatorname{arcsec}(x) = \arccos(1/x)$$

Limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to\infty} \ln(x) = \infty \qquad \lim_{x\to 0^+} \ln(x) = -\infty$$

Derivatives

Specific cases

$F(x)$	$F'(x)$
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec(x)^2$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\cot(x)\csc(x)$
$\cot(x)$	$-\csc(x)^2$
$\arccos(x)$	$-1/\sqrt{1-x^2}$
$\arcsin(x)$	$1/\sqrt{1-x^2}$
$\arctan(x)$	$1/\left(x^2+1\right)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\tanh(x)$	$1/\cosh(x)^2$
$\operatorname{arccosh}(x)$	$1/\sqrt{x^2-1}$
$\operatorname{arcsinh}(x)$	$1/\sqrt{1+x^2}$
$\operatorname{arctanh}(x)$	$1/(1-x^2)$
$\exp(x)$	$\exp(x)$
$\ln(x)$	$1/x$

General Cases

$F(x)$	$F'(x)$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$1/g(x)$	$-g'(x)/g(x)^2$
$f(x)/g(x)$	$(g(x)f'(x) - f(x)g'(x))/g(x)^2$
$f(g(x))$	$g'(x)f'(g(x))$
$f^{-1\prime}(x)$	$1/f'(f^{-1}(x))$

Antiderivatives

$$\int a\,\mathrm{d}x = ax$$

$$\int x^a\,\mathrm{d}x = \frac{1}{1+a}x^{a+1}, \quad \text{if } a \neq -1$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|$$

$$\int \cos(x)\,\mathrm{d}x = \sin(x)$$

$$\int \sin(x)\,\mathrm{d}x = -\cos(x)$$

$$\int \tan(x)\,\mathrm{d}x = \ln|\sec(x)|$$

$$\int \sec(x)\,\mathrm{d}x = \ln|\tan(x) + \sec(x)|$$

$$\int \csc(x)\,\mathrm{d}x = -\ln|\csc(x) + \cot(x)|$$

$$\int \cot(x)\,\mathrm{d}x = \ln|\sin(x)|$$

$$\int |x|\,\mathrm{d}x = x|x|/2$$

Sums

For $k, m, n \in \mathbf{Z}_{>0}$

$$\sum_{k=0}^{n-1} 1 = n$$

$$\sum_{k=0}^{n-1} k = \frac{(n-1)\,n}{2}$$

$$\sum_{k=0}^{n-1} k^2 = \frac{(n-1)\,n\,(2n-1)}{6}$$

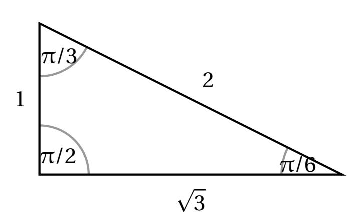
$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}, \quad x \neq 1$$

Logarithms

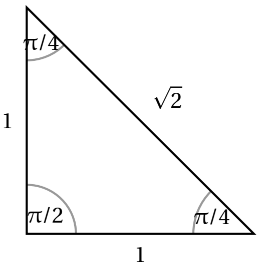
$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Famous Triangles

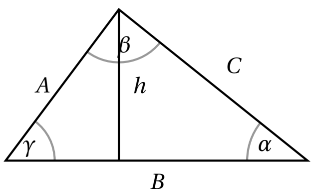
The 30-60-90 triangle



The 45-45-90 triangle



Laws of Cosine & Sine



Law of cosine: $C^2 = A^2 + B^2 - 2AB\cos(\gamma)$

Law of sines: $\dfrac{\sin(\alpha)}{A} = \dfrac{\sin(\beta)}{B} = \dfrac{\sin(\gamma)}{C}$

Area: $\text{Area} = \dfrac{1}{2}hB = \dfrac{1}{2}AB\sin(\gamma)$

Hyperbolic Functions

$$2\cosh(x) = \exp(x) + \exp(-x)$$

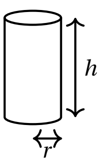
$$2\sinh(x) = \exp(x) - \exp(-x)$$

$$\tanh(x) = {}^{\cosh(x)}/_{\sinh(x)}$$

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

Volumes

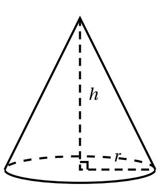
Right Circular Cylinder



Volume: $V = \pi r^2 h$

Area: (not including circular ends) $A = 2\pi r h$

Cone



Volume: $V = \dfrac{1}{3}\pi r^2 h$

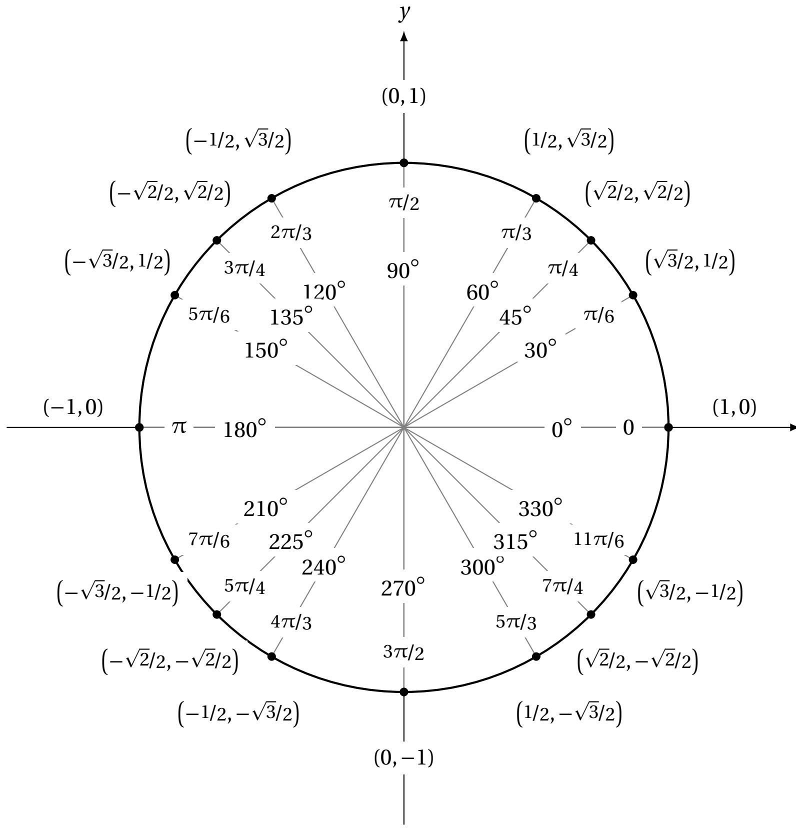
Area (not including circular base)
$A = \pi r\sqrt{r^2 + h^2}$

Sphere

Area: $A = 4\pi r^2$

Volume: $V = \dfrac{4\pi}{3} r^3$

Unit Circle



Applications

Arclength of curve $y = f(x)$ with $a \le x \le b$

$$= \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

For the region Q of the xy plane given by

$$Q = \{(x, y) \mid f(x) \le y \le g(x) \wedge a \le x \le b\},$$

we have

$$\text{Area}(Q) = \int_a^b g(x) - f(x)\,\mathrm{d}x$$

Assuming $0 \le f(x)$ and rotating about the x-axis

$$\text{Vol}(Q) = \pi \int_a^b g(x)^2 - f(x)^2\,\mathrm{d}x$$

Assuming $0 \le a < b$ and rotating about the y-axis

$$\text{Vol}(Q) = 2\pi \int_a^b x(g(x) - f(x))\,\mathrm{d}x$$

Centroid

$$\text{Area}(Q) \times \overline{x} = \int_a^b x\big(g(x) - f(x)\big)\,\mathrm{d}x$$

$$\text{Area}(Q) \times \overline{y} = \frac{1}{2}\int_a^b \Big(g(x)^2 - f(x)^2\Big)\,\mathrm{d}x$$

For the region Q of the xy plane given by

$$Q = \{(x, y) \mid f(y) \le x \le g(y) \wedge a \le y \le b\},$$

interchange x and y in *all* the previous formulas.

Geometric series

$$\sum_{k=0}^{\infty} z^k = \begin{cases} \frac{1}{1-z} & z \in (-1,1) \\ \infty & z \in [1,\infty] \end{cases}.$$

When $z \in (-\infty,-1]$, the series $\sum_{k=0}^{\infty} z^k$ diverges.

Divergence Test

If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum a_k$ diverges.

P-series

The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p \in (1,\infty)$; otherwise it diverges.

Ratio Test

Let a be a sequence with $0 \notin \mathrm{range}(a)$. Define $L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$.

- $L \in [0,1) \implies \sum |a_k|$ converges.
- $L \in (1,\infty) \implies \sum a_k$ diverges.

Limit Comparison Test

Let a and b be positive sequences. Define $L = \lim_{k\to\infty} \frac{a_k}{b_k}$.

- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ converges then $\sum b_k$ converges.
- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ diverges then $\sum a_k$ diverges.
- If $L =$ and $\sum b_k$ converges, then $\sum a_k$ converges
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Alternating Series Test

Let a be a positive and eventually decreasing sequence. Then $\sum (-1)^k a_k$ converges iff $\lim_{k\to\infty} a_k = 0$.

Taylor and MacLaurin Series

If a function F is infinitely differentiable at a, its Taylor series centered at a is

$$\sum_{k=0}^{\infty} \frac{F^{(k)}(a)}{k!}(x-a)^k.$$

When a is zero, the Taylor series is also known as the MacLaurin Series.

Polar to Cartesian

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

For $r > 0$ and $0 \le \theta < 2\pi$

$$r = \sqrt{x^2+y^2}$$
$$\theta = \begin{cases} 2\pi - \arccos(x/r) & \text{if } y < 0 \\ \arccos(x/r) & \text{if } y \ge 0 \end{cases}$$

Integrate Powers of Trig

Let $m, n \in \mathbf{Z}_{\ge 0}$. Then

- $\int \cos(x)^{2m}\sin(x)^{2n}\,\mathrm{d}x = \int \left(\frac{1+\cos(2x)}{2}\right)^m \left(\frac{1-\cos(2x)}{2}\right)^n \mathrm{d}x$
- $\int \cos(x)^{2m+1}\sin(x)^n\,\mathrm{d}x = \int (1-z^2)^m z^n\,\mathrm{d}z$, where $z = \cos(x)$
- $\int \cos(x)^m \sin(x)^{2n+1}\,\mathrm{d}x = \int z^m(1-z^2)^n\,\mathrm{d}z$, where $z = \sin(x)$
- $\int \sec(x)^n\,\mathrm{d}x = \frac{1}{n-1}\sec(x)^{n-2}\tan(x) + \frac{n-2}{n-1}\int \sec(x)^{n-2}\,\mathrm{d}x$
- $\int \tan(x)^{2m+1}\sec(x)^n\,\mathrm{d}x = \int (z^2-1)^n z^{n-1}\,\mathrm{d}z$, where $z = \tan(x)$
- $\int \tan(x)^{2m}\sec(x)^n\,\mathrm{d}x = \int (\sec(x)^2-1)^m \sec(x)^n\,\mathrm{d}x$

Trig Substitutions

- $\int F\left(x, \left(1-x^2\right)^{n/2}\right)\mathrm{d}x$, use $x = \sin(\vartheta)$, where $\vartheta \in [-\pi/2, \pi/2]$, then integrate $\int F\left(\sin(\vartheta), \cos(\vartheta)^n\right)\cos(\vartheta)\,\mathrm{d}\vartheta$
- $\int F\left(x, \left(1+x^2\right)^{n/2}\right)\mathrm{d}x$, use $x = \sinh(\vartheta)$, where $\vartheta \in \mathbf{R}$, then integrate $\int F\left(\sinh(\vartheta), \cosh(\vartheta)^n\right)\cosh(\vartheta)\,\mathrm{d}\vartheta$
- $\int F\left(x, \left(x^2-1\right)^{n/2}\right)\mathrm{d}x$, use $x = \sec(\vartheta)$, then integrate $\int F(\sec(\vartheta), \tan(\vartheta)^n)\sec(\vartheta), \tan(\vartheta)\,\mathrm{d}\vartheta)$

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