MATH 202 **Name:** ____________________

Calculus Practice I, Fall 2023

Here is an opportunity for you to maintain your calculus skills over the summer. If you complete these problems, digitize your work, and submit your work to Canvas, I will send you my solutions. If you need some help with these questions, email me with your questions (willisb@unk.edu)

Completing this work is optional, and it does not enter into your class grade in any way–this work is not a bonus, extra credit, or anything like that.

1. Find an equation of the tangent line to the curve $y = \sqrt{x^2+1}$ at the point $(x = 1, y = \sqrt{2})$.

Solution:

To find an equation of a line we need to know (a) its slope and (b) a point on the line. We're given a point on the line, so our main task is to find the slope of the tangent line. To do that we need to first find a formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and second we need to evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 1$. The chain rule tell us that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \times 2x(x^2+1)^{-1/2} = x(x^2+1)^{-1/2}. \tag{1}$$

And pasting in $x \to 1$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = x(x^2+1)^{-1/2}\Big|_{x=1} = \frac{1}{\sqrt{2}}. \tag{2}$$

An equation of the given tangent line is

$$y - \sqrt{2} = \frac{1}{\sqrt{2}}(x-1) \tag{3}$$

Traditionally, we would simplify $\frac{1}{\sqrt{2}}$ to $\frac{\sqrt{2}}{2}$ and add $\sqrt{2}$ to solve for y. Doing so results in

$$y = \frac{\sqrt{2}}{2}(x+1). \tag{4}$$

Arguably, $y = \frac{\sqrt{2}}{2}(x+1)$ is more simple than is $y - \sqrt{2} = \frac{\sqrt{2}}{2}(x-1)$. But in this context, I don't see much of an advantage to doing this. In the first formula $y - \sqrt{2} = \frac{1}{\sqrt{2}}(x-1)$ it's clear that $(x = 1, y = \sqrt{2})$ is a point on the line.

In short, all the answers

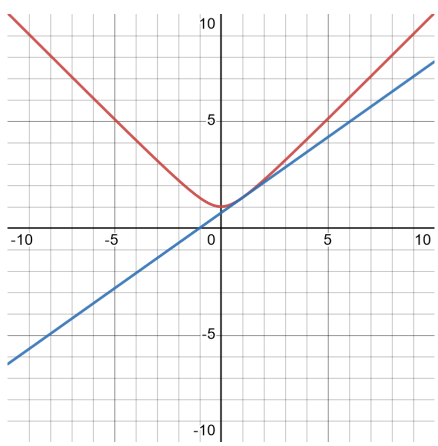
$$y - \sqrt{2} = \frac{1}{\sqrt{2}}(x-1),$$
$$y = \frac{1}{\sqrt{2}}(x+1),$$
$$y = \frac{\sqrt{2}}{2}(x+1),$$

are OK answers to the question "Find an equation of the tangent line." Semantically, all these answers are equivalent and each of them are simplified according to our class standard: (from the class FAQ)

Question Is this OK, or do I need to simplify it?

Answer Maybe. Giving guidelines on what it means to simplified is tough. But if you (a) do all rational arithmetic (b) combine all like terms, (c) simplify all "famous" values of functions, for example $\cos(0) = 1$ and $\sqrt{81} = 9$, and (d) make an effort to simplify all vanishing subexpressions to zero (for example, $\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2}$ simplifies to zero), you'll be OK.

Instead of worrying so much about simplifying expressions according to a murky and unpublished rule, let's shift our main attention on to learning to use various tools to check our work. For this question, a graph is a good way to check that the line and the curve are tangent. Let's ask Desmos to draw graphs of both $y = \sqrt{1+x^2}$ and $y - \sqrt{2} = \frac{1}{\sqrt{2}}(x-1)$. Does the line and the curve appear to be tangent? Sure.



The picture is fairly convincing–we have a (not the) correct answer.

2. Find each antiderivative.

(a) $\int x^2 - x - 2\,\mathrm{d}x$

Solution:

$$\int x^2 - x - 2\,\mathrm{d}x = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x. \tag{5}$$

Some teachers will insist on the $+c$. I say let's just remember that all antiderivatives are undetermined up to an additive constant and implicitly include the additive constant in with the left side of

$$\int x^2 - x - 2\,\mathrm{d}x = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x \tag{6}$$

(and not the right side) and for forget the silly explicit $+c$ rule. It is maybe a bit peculiar, but we do need to remember that

$$\int F(x)\,\mathrm{d}x = \int F(x)\,\mathrm{d}x \tag{7}$$

is *not* an identity. Usually visually equal things are equivalent, but this case is an exception.

Notice that the $+c$ rule is a problem if the variable name is c. Should we add $+c$ to

$$\int c^2 - c - 2\,\mathrm{d}c = \frac{1}{3}c^3 - \frac{1}{2}c^2 - 2c? \tag{8}$$

Unless we want a wrong answer, no we should not.

(b) $\int (x-1)(x+2)\,\mathrm{d}x$

Solution: This one is a freebie. Expanding the integrand, this problem is the same as the previous problem.

It's tempting, I know, but do not make the **error** of thinking that that the anti derivative of a product is the product of the anti derivatives. That is do not do

$$\int (x-1)(x+2)\,\mathrm{d}x = \frac{1}{2}(x-1)^2 \times \frac{1}{2}(x+2)^2 \tag{9}$$

We should practice the good habit of checking anti derivatives by differentiating. Doing so we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(x-1)^2 \times \frac{1}{2}(x+2)^2\right) &= \frac{(x-1)\ (x+2)^2}{2} + \frac{(x-1)^2\ (x+2)}{2}, \\
&= \frac{(x-1)\ (x+2)\ (2x+1)}{2}, \\
&= x^3 + \frac{3x^2}{2} - \frac{3x}{2} - 1.
\end{aligned}$$

This is not equivalent to the integrand $(x-1)(x+2)$, so our work is **wrong**.

(c) $\int \frac{1+x^2}{x^2}\,\mathrm{d}x$

Solution: On any interval on that does not contain zero, we have

$$\begin{aligned}
\int \frac{1+x^2}{x^2}\,\mathrm{d}x &= \int x^{-2} + 1, \mathrm{d}x, \\
&= -\frac{1}{x} + x.
\end{aligned}$$

If you worked this problem with a first step something like

$$\int \frac{1+x^2}{x^2}\,\mathrm{d}x = \frac{x^2 \int 1 + x^2\,\mathrm{d}x - (1+x^2)\int x^2\,\mathrm{d}x}{\left(\int x^2\,\mathrm{d}x\right)^2} \tag{10}$$

let's make this the very last time you make this mistake.

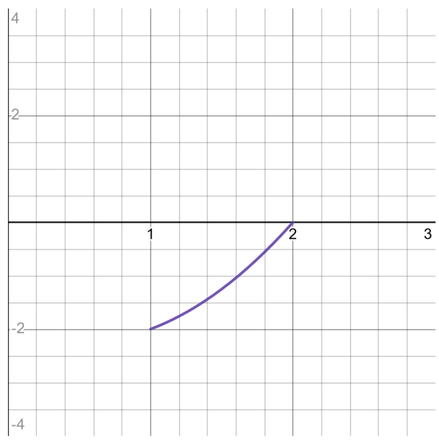
3. Find each definite integral.

(a) $\int_1^2 x^2 - x - 2\,\mathrm{d}x$

Solution: In the previous question, we found the antiderivative; so all we need to do is

$$\int_1^2 x^2 - x - 2\,\mathrm{d}x = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\Big|_{x=1}^{x=2} = -\frac{7}{6}. \tag{11}$$

Graphically, we can tell that this definite integral is negative. Why? The graph of the integrand is entirely below the x-axis, that's why. Here is a graph of the integrand



Actually, the graph of the integrand on the interval $[1,2]$ is pretty well approximated by a line segment joining $(x = 1, y = -2)$ and $(x = 2, y = 0)$. Doing so, we see that $\int_1^2 x^2 - x - 2\,\mathrm{d}x$ is pretty close to the negative of the area of a triangle with base one and height two. So $\int_1^2 x^2 - x - 2\,\mathrm{d}x \approx -1$. And I'd say that $-\frac{7}{6} \approx -1$.

(b) $\int_1^2 (x-1)(x+2)\,\mathrm{d}x$

Solution: Expanding the integrand, we see that we've done this problem before.

(c) $\int_1^4 \frac{1+x^2}{x^2}\,\mathrm{d}x$

Solution: We have

$$\int_1^4 \frac{1+x^2}{x^2}\,\mathrm{d}x = \left(-\frac{1}{x} + x\right)\Big|_{x=1}^{x=2},$$
$$= \frac{3}{2}.$$

4. For each function F, find the solution set of $F'(x) = 0$.

(a) $F(x) = x^2 + x + 3$

Solution: We need to solve

$$[2x + 1 = 0] = [x = -1/2]\,. \tag{12}$$

(b) $F(x) = (x - 3)(x^2 + 3)$

Solution: Using the product rule, we have $F'(x) = 3x^2 - 6x + 3$. So we need to solve

$$\begin{aligned} \left[3x^2 - 6x + 3\right] &= \left[3(x-1)^2\right] && \text{(factor)} \\ &= [x = 1]\,. \end{aligned}$$

(c) $F(x) = 2x + \frac{x}{x-2}$

Solution:

(d) $F(x) = \cos(x)\sin(x)$

Solution:

5. Find the value of each limit.

(a) $\lim_{x\to 4} \dfrac{\cos(x)+1}{x-3}$

Solution: The function is continuous at 4, so we can use DS (direct substitution); we have

$$\lim_{x\to 4} \frac{\cos(x)+1}{x-3} = \cos(4)+1. \tag{13}$$

If somebody wants a decimal approximation to $\cos(4)+1$ to any number of digits, they can do that.

(b) $\lim_{x\to 4} \dfrac{x^2-2x-3}{x-3}$

Solution: Again, the function is continuous at 4, so let's use DS; we have

$$\lim_{x\to 4} \frac{x^2-2x-3}{x-3} = 4^2 - 2\times 4 - 3 = 5. \tag{14}$$

(c) $\lim_{x\to\infty} \dfrac{5x^2+x+1}{7x^2+107}$