

NAMED SETS

empty set	\emptyset
real numbers	\mathbf{R}
ordered pairs	\mathbf{R}^2
integers	\mathbf{Z}
positive integers	$\mathbf{Z}_{>0}$
positive real numbers	$\mathbf{R}_{>0}$

EXPONENTS

For $a, b \in \mathbf{R}_{>0}$, $x \in \mathbf{R}$, and $m, n \in \mathbf{R}$,

$$\begin{aligned} a^0 &= 1, & 0^a &= 0 \\ 1^x &= 1, & a^n a^m &= a^{n+m} \\ a^n / a^m &= a^{n-m}, & (a^n)^m &= a^{n \cdot m} \\ a^{-m} &= 1/a^m, & (a/b)^m &= a^m / b^m \\ \sqrt{x^2} &= |x| \end{aligned}$$

TRIGONOMETRIC IDENTITIES

We define $\text{dom}(\text{arccot}) = (0, \pi)$.

$$\begin{aligned} (\cos(x))^2 + (\sin(x))^2 &= 1 \\ 2(\cos(x))^2 &= 1 + \cos(2x) \\ 2(\sin(x))^2 &= 1 - \cos(2x) \\ (\cos(x))^2 - (\sin(x))^2 &= \cos(2x) \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \text{arccot}(x) &= \pi/2 - \arctan(x) \\ \text{arccsc}(x) &= \arcsin(1/x) \\ \text{arcsec}(x) &= \arccos(1/x) \\ \arcsin(x) + \arccos(x) &= \pi/2 \\ \text{arcsec}(x) + \text{arccsc}(x) &= \pi/2 \end{aligned}$$

HYPERBOLIC FUNCTIONS

$$\begin{aligned} 2 \cosh(x) &= \exp(x) + \exp(-x) \\ 2 \sinh(x) &= \exp(x) - \exp(-x) \\ \tanh(x) &= \sinh(x)/\cosh(x) \\ \cosh(x)^2 - \sinh(x)^2 &= 1 \end{aligned}$$

LOGARITHMS

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

DERIVATIVES

Specific cases

$F(x)$	$F'(x)$
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec(x)^2$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\cot(x)\csc(x)$
$\cot(x)$	$-\csc(x)^2$
$\arccos(x)$	$-1/\sqrt{1-x^2}$
$\arcsin(x)$	$1/\sqrt{1-x^2}$
$\arctan(x)$	$1/(x^2+1)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\tanh(x)$	$1/\cosh(x)^2$
$\text{arccosh}(x)$	$1/\sqrt{x^2-1}$
$\text{arcsinh}(x)$	$1/\sqrt{1+x^2}$
$\text{arctanh}(x)$	$1/(1-x^2)$
$\exp(x)$	$\exp(x)$
$\ln(x)$	$1/x$

General Cases

$F(x)$	$F'(x)$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$1/g(x)$	$-g'(x)/g(x)^2$
$f(x)/g(x)$	$(g(x)f'(x) - f(x)g'(x))/g(x)^2$
$f(g(x))$	$g'(x)f'(g(x))$
$f^{-1}(x)$	$1/f'(f^{-1}(x))$

ANTIDERIVATIVES

$$\begin{aligned} \int a \, dx &= ax \\ \int x^a \, dx &= \frac{1}{a+1} x^{a+1}, \quad \text{if } a \neq -1 \\ \int \frac{1}{x} \, dx &= \ln|x| \\ \int \cos(x) \, dx &= \sin(x) \\ \int \sin(x) \, dx &= -\cos(x) \\ \int \tan(x) \, dx &= \ln|\sec(x)| \\ \int \sec(x) \, dx &= \ln|\tan(x) + \sec(x)| \\ \int \csc(x) \, dx &= -\ln|\csc(x) + \cot(x)| \\ \int \cot(x) \, dx &= \ln|\sin(x)| \\ \int |x| \, dx &= x|x|/2 \end{aligned}$$

SUMS

For $n \in \mathbf{Z}_{>0}$

$$\begin{aligned} \sum_{k=0}^{n-1} 1 &= n, & \sum_{k=0}^{n-1} k &= (n-1)n/2 \\ \sum_{k=0}^{n-1} k^2 &= (n-1)n(2n-1)/6, \\ \sum_{k=0}^{n-1} x^k &= \sum_{k=1}^n x^{k-1} = \frac{1-x^n}{1-x}, \quad x \neq 1 \\ \sum_{k=0}^{\infty} x^k &= \begin{cases} \frac{1}{1-x} & x \in (-1, 1) \\ \infty & x \in [1, \infty] \end{cases} \end{aligned}$$

When $x \in (-\infty, -1]$, the series $\sum_{k=0}^{\infty} x^k$ diverges.

APPLICATIONS

Arc length of curve $y = f(x)$ with $a \leq x \leq b$

$$= \int_a^b \sqrt{1 + f'(x)^2} \, dx$$

For the region Q of the xy plane given by

$$Q = \{(x, y) \mid f(x) \leq y \leq g(x) \wedge a \leq x \leq b\},$$

we have

$$\text{Area}(Q) = \int_a^b g(x) - f(x) \, dx$$

Assuming $0 \leq f(x)$ and rotating about the x -axis

$$\text{Vol}(Q) = \pi \int_a^b g(x)^2 - f(x)^2 \, dx$$

Assuming $0 \leq a < b$ and rotating about the y -axis

$$\text{Vol}(Q) = 2\pi \int_a^b x(g(x) - f(x)) \, dx$$

Centroid

$$\text{Area}(Q) \times \bar{x} = \int_a^b x(g(x) - f(x)) \, dx$$

$$\text{Area}(Q) \times \bar{y} = \frac{1}{2} \int_a^b (g(x)^2 - f(x)^2) \, dx$$

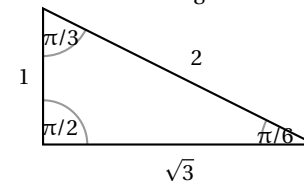
For the region Q of the xy plane given by

$$Q = \{(x, y) \mid f(y) \leq x \leq g(y) \wedge a \leq y \leq b\},$$

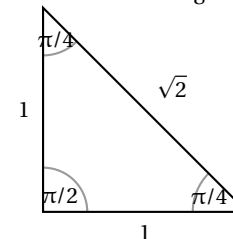
interchange x and y in *all* the previous formulas.

FAMOUS TRIANGLES

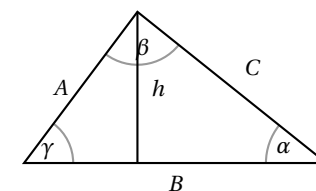
The 30-60-90 triangle



The 45-45-90 triangle



LAWS OF COSINE & SINE



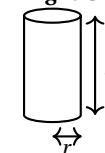
Law of cosine: $c^2 = a^2 + b^2 - 2ab \cos(\gamma)$

Law of sines: $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$

Area: $\text{Area} = 1/2 hb = 1/2 ab \sin(\gamma)$

VOLUMES

Right Circular Cylinder



Volume: $V = \pi r^2 h$

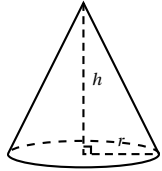
Area: (not including circular ends) $A = 2\pi r h$

Sphere with radius r

Area: $A = 4\pi r^2$

Volume: $V = \frac{4\pi}{3} r^3$

Cone



Volume: $V = \pi r^2 h/3$

Area

$A = \pi r \sqrt{r^2 + h^2}$ (not including circular base).

P-SERIES, DIVERGENCE TEST, RATIO TEST, COMPARISON, & AST

The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p \in (1, \infty)$; otherwise it diverges.

If $\lim_{k \rightarrow \infty} a_k \neq 0$, the series $\sum a_k$ diverges.

Let a be a sequence with $0 \notin \text{range}(a)$. Define $L = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$.

- $L \in [0, 1) \Rightarrow \sum |a_k|$ converges.
- $L \in (1, \infty] \Rightarrow \sum a_k$ diverges.

Let a and b be positive sequences. Define $L = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$.

- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ converges, then $\sum b_k$ converges.
- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ diverges, then $\sum b_k$ diverges.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Let a be a positive and eventually decreasing sequence. Then $\sum (-1)^k a_k$ converges if and only if $\lim_{k \rightarrow \infty} a_k = 0$.

TAYLOR AND MACLAURIN SERIES

If a function F is infinitely differentiable at a , its Taylor series centered at a is

$$\sum_{k=0}^{\infty} \frac{F^{(k)}(a)}{k!} (x-a)^k.$$

When a is zero, the Taylor series is also known as a MacLaurin Series.

POLAR TO CARTESIAN

$$x = r \cos(\theta) \quad y = r \sin(\theta)$$

For $r > 0$ and $0 \leq \theta < 2\pi$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} 2\pi - \arccos(x/r) & \text{if } y < 0 \\ \arccos(x/r) & \text{if } y \geq 0 \end{cases}$$

INTEGRATE POWERS OF TRIG

Let $m, n \in \mathbf{Z}_{\geq 0}$. Then

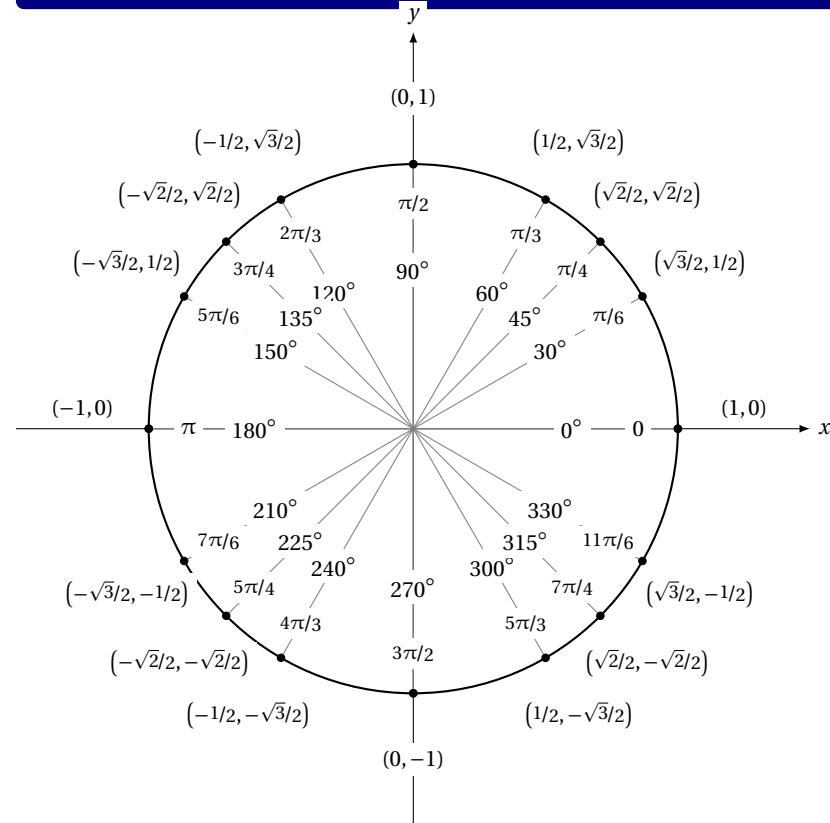
- $\int \cos(x)^{2m} \sin(x)^{2n} dx = \int \left(\frac{1 + \cos(2x)}{2} \right)^m \left(\frac{1 - \cos(2x)}{2} \right)^n dx$
- $\int \cos(x)^{2m+1} \sin(x)^n dx = \int (1 - z^2)^m z^n dz$, where $z = \sin(x)$
- $\int \cos(x)^m \sin(x)^{2n+1} dx = - \int z^m (1 - z^2)^n dz$, where $z = \cos(x)$

- $\int \sec(x)^n dx = \frac{1}{n-1} \sec(x)^{n-2} \tan(x) + \frac{n-2}{n-1} \int \sec(x)^{n-2} dx$, provided $n \neq 1$.
- $\int \tan(x)^{2m+1} \sec(x)^n dx = \int (z^2 - 1)^m z^{n-1} dz$, where $z = \sec(x)$
- $\int \tan(x)^{2m} \sec(x)^n dx = \int (\sec(x)^2 - 1)^m \sec(x)^n dx$.

TRIG SUBSTITUTIONS

- $\int F\left(x, (1-x^2)^{n/2}\right) dx$, use $x = \sin(\vartheta)$, where $\vartheta \in [-\pi/2, \pi/2]$, then integrate $\int F(\sin(\vartheta), \cos(\vartheta)^n) \cos(\vartheta) d\vartheta$
- $\int F\left(x, (1+x^2)^{n/2}\right) dx$, use $x = \sinh(\vartheta)$, where $\vartheta \in \mathbf{R}$, then integrate $\int F(\sinh(\vartheta), \cosh(\vartheta)^n) \cosh(\vartheta) d\vartheta$
- $\int F\left(x, (x^2-1)^{n/2}\right) dx$, use $x = \sec(\vartheta)$, then integrate $\int F(\sec(\vartheta), \tan(\vartheta)^n) \sec(\vartheta) \tan(\vartheta) d\vartheta$

UNIT CIRCLE



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