MATH 202, Fall 2023 **Name:** ____________________

In class work 23 **Row and Seat:** ____________________

In class work **23** has questions **1** through **1** with a total of **4** points. Turn in your work at the end of class *on paper*. This assignment is due at *Tuesday 21 November 13:20.*

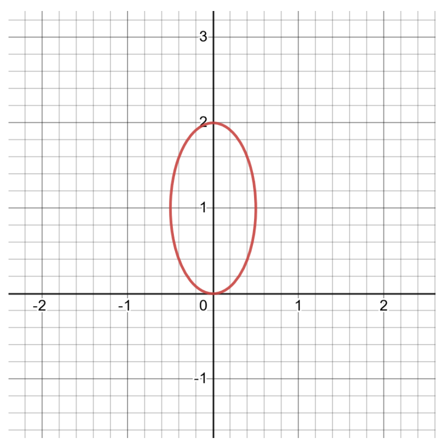
"Piglet noticed that even though he had a very small heart, it could hold a rather large amount of gratitude." A. A. MILNE

1. Consider the parametrically defined curve $\mathscr{C} = \begin{cases} x = \frac{t}{1+t^2}, \\ y = \frac{2t^2}{1+t^2} \end{cases}, -\infty < t < \infty.$

$\boxed{1}$ (a) Use Desmos to draw this curve. Reproduce the curve as best you can on here:

Solution:

Appearances are deceiving, but the curve $\mathscr{C}$ looks like an ellipse.



$\boxed{1}$ (b) Is the point $(x = 0, y = 2)$ on the curve? The picture might indicate that it is, but is it really? To decide, you'll need to solve the equations

$$0 = \frac{t}{1+t^2}, \; 2 = \frac{2t^2}{1+t^2}.$$

Solution: No, the point $(x = 0, y = 2)$ on not on the curve $\mathscr{C}$. To prove this, we need to solve the equations

$$\left[0 = \frac{t}{1+t^2}, 2 = \frac{2t^2}{1+t^2}\right] = \left[0 = t, 2 + 2t^2 = 2t^2\right] = [0 = t, 2 = 0]$$

The solution set is empty, so $(x = 0, y = 2)$ on not on the curve $\mathscr{C}$.

Notice We're solving two equations for one unknown. We need to find a common solution for both equations. The second equation has no solution, so the solution set is empty.

Arguably, $t = \infty$ is a solution of the equations $\left[0 = \frac{t}{1+t^2}, 2 = \frac{2t^2}{1+t^2}\right]$. Indeed $\lim_{t\to\infty} \dfrac{t}{1+t^2} = 0$ and $\lim_{t\to\infty} \dfrac{2t^2}{1+t^2} = 2$. So it's tempting to say that $(x = 0, y = 2) \in \mathscr{C}$, but we'd need to extend the domain to make that true.

$\boxed{1}$ (c) Solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} \neq 0$. Each solution will give a point on the curve with a horizontal tangent line.

Solution:

$$\left[0 = \frac{\mathrm{d}y}{\mathrm{d}t}\right] = \left[0 = \frac{4t}{1+t^2}\right] = [t = 0] \tag{1}$$

A calculation shows that when $t = 0$, we have $\frac{\mathrm{d}x}{\mathrm{d}t}\Big|_{t=0} = 1$. So the only HT happens when $t = 0$. And that makes $x = 0$ and $y = 0$.

The picture might make us think that there is a HT at the point $(x = 0, y = 2)$. But $(x = 0, y = 2) \notin \mathscr{C}$, so I would say that the curve doesn't have an HT at $(x = 0, y = 2)$. Others could quibble with that–to settle this, we'd need a precise definition of HT.

$\boxed{1}$ (d) Substitute $\begin{cases} x = \frac{t}{1+t^2} \\ y = \frac{2t^2}{1+t^2} \end{cases}$ into $4x^2 + y^2 - 2y = 0$. Explain why that shows that the curve $\mathscr{C}$ is a *portion* of an ellipse, but not the entire ellipse.

Solution: Let $x = \frac{t}{1+t^2}$ and let $y = \frac{2t^2}{1+t^2}$ into $4x^2 + y^2 - 2y = 0$ gives

$$4\left(\frac{t}{1+t^2}^{2}\right)^2 + \left(\frac{2t^2}{1+t^2}\right) - 2\frac{2t^2}{1+t^2} = 0. \tag{2}$$

We've shown that if $x = \frac{t}{1+t^2}$ and $y = \frac{2t^2}{1+t^2}$, then $4x^2 + y^2 - 2y = 0$.

We did not show that if $4x^2 + y^2 - 2y = 0$, there is a number t such that $\begin{cases} x = \frac{t}{1+t^2} \\ y = \frac{2t^2}{1+t^2} \end{cases}$. And it's a good thing we didn't prove it because it is false. It's false because $(x = 0, y = 2)$ is a point on the graph of $4x^2 + y^2 - 2y = 0$, but $(x = 0, y = 2) \notin \mathscr{C}$.

Actually the curve $\mathscr{C}$ is the ellipse $4x^2 + y^2 - 2y = 0$ with exactly one point missing. To append this missing point, we could define

$$\mathscr{C}^\star = \begin{cases} x = \begin{cases} \frac{t}{1+t^2} & t \neq \infty \\ 0 & t = \infty \end{cases} \\ y = \begin{cases} \frac{2t^2}{1+t^2} & t \neq \infty \\ 2 & t = \infty \end{cases} \end{cases}, -\infty < t \leq \infty.$$

Finally, I know what you are thinking. How did I eliminate the parameter t

from $\begin{cases} x = \frac{t}{1+t^2} \\ y = \frac{2t^2}{1+t^2} \end{cases}$ to discover $4x^2 + y^2 - 2y = 0$? There is a beautify algorithm for doing this–it involves the polynomial resultant. Ending in the 1950s, I think, such things were taught in a class that was generally called the Theory of Equations.