MATH 202, Fall 2023 **Name:** ______________________________

In class work 18 **Row and Seat**:______________________________

"The universe is a big place, perhaps the biggest." KURT VONNEGUT

In class work **18** has questions **1** through **2** with a total of **6** points. Turn in your work at the end of class *on paper*. This assignment is due *Tuesday 31 October 13:20*.

1. The zero order Bessel function J_0 is defined by its power series.[1] The power series is

$$\mathrm{J}_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(-\frac{x^2}{4}\right)^k = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} - \frac{x^{10}}{14745600} + \cdots$$

The radius of convergence of this power series is infinity.

2 (a) Find the numerical values of $\mathrm{J}_0(0), \mathrm{J}_0'(0)$, and $\mathrm{J}_0''(0)$.

Solution: We have two representations for the the function J_0; one involves the summation and the other the ellipses (the ...). To find what we need, we only need to consider the first few terms of the series, so let's do the calculation with just the first few terms. Differentiating $\mathrm{J}_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} - \frac{x^{10}}{14745600}$ termwise gives

$$\mathrm{J}_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \ldots,$$
$$\mathrm{J}_0'(x) = -\frac{2x}{4} + \frac{4x^3}{64} - \frac{6x^5}{2304} + \ldots,$$
$$\mathrm{J}_0'(x) = -\frac{2}{4} + \frac{4 \times 3x^2}{64} - \frac{6 \times 5x^4}{2304} + \ldots,$$

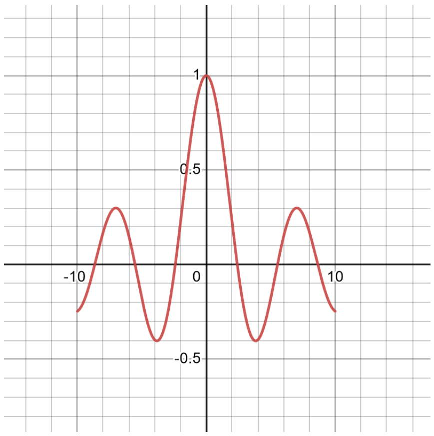
And evaluating at $x = 0$ yields

$$\mathrm{J}_0(0) = 1,$$
$$\mathrm{J}_0'(0) = 0,$$
$$\mathrm{J}_0'(0) = -\frac{1}{2}.$$

1 (b) When x has a modest magnitude, say $-10 < x < 10$, the summand $\frac{1}{(k!)^2}\left(-\frac{x^2}{4}\right)^k$ converges to zero *very quickly*. For example, when $x = 10$ and $k = 100$, the numeric value of the summand is about 7.1×10^{-177}. So it's reasonable to conjecture that $\mathrm{J}_0(x) \approx \sum_{k=0}^{100} \frac{1}{(k!)^2}\left(-\frac{x^2}{4}\right)^k$. Use Desmos to graph $y = \sum_{k=0}^{100} \frac{1}{(k!)^2}\left(-\frac{x^2}{4}\right)^k$ for $-10 < x < 10$ and reproduce the graph here.

[1] This function is named in honor of Friedrich Bessel, a German mathematician and scientist who lived from 1784–1846. Bessel functions arise in mechanical vibrations of a drum and electromagnetic waves in a coaxial cable, to mention just a few applications.

Solution: We'd need some theory to show that the graph is accurate–with a bit of faith, the graph is



2. A function $\mathscr{E}$ is defined by a power series; specifically $\mathscr{E}(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$. The radius of convergence of this power series is infinity.

$\boxed{1}$ (a) Find the numerical value of $\mathscr{E}(0)$.

Solution:

$$\mathscr{E}(0) = \sum_{k=0}^{\infty} \frac{1}{k!} 0^k = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} = 1.$$

$\boxed{1}$ (b) Find a power series for $\mathscr{E}'$ and show that for all real numbers x, we have $\mathscr{E}'(x) = \mathscr{E}(x)$.

Solution: We have

$$\mathscr{E}'(x) = \sum_{k=1}^{\infty} \frac{k}{k!} x^{k-1} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \mathscr{E}(x).$$

$\boxed{1}$ (c) The only solution to the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ and $y|_{x=0} = 1$ is $y = \exp(x)$. What is a simple formula for the function $\mathscr{E}$?

Solution: Since $\mathscr{E}' = \mathscr{E}$ and $\mathscr{E}(0) = 1$, we've shown that $\mathscr{E} = \exp$.