MATH 202, Fall 2023 **Name:** ______________________________

In class work 14 **Row and Seat**: ______________________________

"You are never dedicated to something you have complete confidence in."

ROBERT M. PIRSIG

In class work **14** has questions **1** through **3** with a total of **8** points.Turn in your work at the end of class *on paper*. This assignment is due *Thursday 12 October 13:20.*

$\boxed{2}$ 1. When Morwenna graduates from UNK and starts her first job, she expects to earn a starting annual salary of \$42,000. She plans to work for 42 years and she expects to earn a 3% raise each year. Thus, in her n^{th} year of work, her salary is $42,000 \times 1.03^{n-1}$. During Morwenna's 42 years of labor, how much will she earn?

Solution: In this problem it's easy to get tangled up with errors between 41 and 42, and between a sum index that starts at zero or that starts at one. To navigate these problems, I suggest expressing the lifetime amount Morwenna earns as simply as possible; for example[1]

$$42,000 + 42,000 \times 1.03 + 42,000 \times 1.03^2 + \cdots + 42,000 \times 1.03^{41}.$$

Expressed this way, it's most natural to make the lower sum index be zero, not one, I think. And a bonus to using a lower sum index of zero is that way, the sum explicitly matches the geometric sum identity in the QRS.

In summation notation, the lifetime earnings for Morwenna is

$$\sum_{n=0}^{41} 42,000 \times 1.03^n.$$

To express this explicitly as a number, we need to use outativity along with the geometric sum identity; we have

$$\begin{aligned}
\sum_{n=0}^{41} 42,000 \times 1.03^n &= 42,000 \sum_{n=0}^{41} 1.03^n, && \text{(outative property)} \\
&= 42,000 \frac{1 - 1.03^{42}}{1 - 1.03}, && \text{(geometric sum identity)} \\
&= 3,444,974.25. && \text{(round to the nearest penny)}
\end{aligned}$$

2. Given a formula for a sequence b, find its limit. Show all of your work.

$\boxed{2}$ (a) $b_n = \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k$.

Solution: We don't have any rules for finding the limit of a sequence whose formula is a sum. For us to have any chance of solving this problem, we need to find an alternative formula for the sequence b that doesn't involve a summation.

Fortunately, the formula for b is a geometric sum. And we have a nifty way to simplify to summation; we have

$$\sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = \frac{1 - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}} = 3\left(1 - \left(\frac{2}{3}\right)^{n+1}\right)$$

I know what you are thinking: The QRS formula tells us that for for all positive integers n and for all $x \neq 1$, we have $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$, but we need $\sum_{k=0}^{n} x^k$. Are we toast? No way. The upper sum index in $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$ can be replaced by any positive integer. But we have to remember that the mathematician standard: what we do to the left, we need to do to the right. So we can replace every n by $n+1$ in the identity $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$ to determine that $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$.

Finding the limit now is possible; we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 3\left(1 - \left(\frac{2}{3}\right)^{n+1}\right) = 3. \tag{1}$$

The base of the exponential term $\left(\frac{2}{3}\right)^{n+1}$ is in the interval $(-1, 1)$, so this is a decaying exponential term. And its limit is zero.

2 (b) $b_n = \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k$.

Solution: The strategy for this problem is much the same as before; we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1-\left(\frac{3}{2}\right)^{n+1}}{1-\frac{3}{2}} = \infty. \tag{2}$$

The base of the exponential term $\left(\frac{3}{2}\right)^{n+1}$ is outside the interval $[-1,1]$, so this is a growing exponential term. And its limit is infinity.

2 3. The Newton–Raphson method[2] is a way to find an approximate solution to an equation $F(x)=0$. Specifically, the method starts with a guess for the solution, call it a_1, and then refines the guess with values $a_2, a_3, \ldots$. This sequence is called a *Newton sequence*. If all goes well, the sequence a converges to solution to $F(x)=0$. Specifically for $F(x) = x^2-2$ and an initial guess of 1, the Newton sequence is defined recursively by

$$a_{n+1} = \begin{cases} 1 & n = 0 \\ a_n - \frac{a_n^2-2}{2a_n} & n > 0 \end{cases}.$$

Assuming that the sequence a converges to a positive number, find the numerical value of $\lim_{n\to\infty} a_n$. Use the fact that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+1} = L$.

Many of you will have the urge to"simplify" $a_n - \frac{a_n^2-2}{2a_n}$ to $\frac{a_n^2+2}{2a_n}$ or possibly to $\frac{a_n}{2}+\frac{1}{a_n}$. Doing so is an OK thing to do, but I suggest doing a bit more 'T' from GNAT[3] before you give into your urge to simplify.

Solution: Equal things have equal limits. So we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(a_n - \frac{a_n^2-2}{2a_n}\right).$$

The usual rules of limits tells us that this equation is equivalent to

$$L = L - \frac{L^2-2}{2L}$$

[2]Raphson invented the method before Newton. If you didn't learn the Newton–Raphson method in Calculus I, I should tell the Office of Student Records to expunge your MATH 115 credit.

[3]GNAT = Graphical, Numerical, Algebraic, Think. I possibly invented the acronym, but the concept was invented by Deborah Hughes Hallett.

So assuming the sequence a converges to a positive number, the sequence a converges to $\sqrt{2}$.

Incidentally: Every term in the sequence a is a rational number; specifically, the first few terms are

$$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \frac{886731088897}{627013566048}, \frac{1572584048032918633353217}{1111984844349868137938112}, \dots$$

The fact that sequence whose range is in the set of rational number converges to an irrational number is a deep idea in mathematics. Also rounded to 15 decimal digits, we have

$$\frac{1572584048032918633353217}{1111984844349868137938112} \approx 1.414213562373095$$

And 1.414213562373095 is very close to $\sqrt{2}$. We don't have a measure of convergence rate of a sequence, but this is a sequence that we might say converges very quickly.