Can I do X?

MATH 202

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"The law is reason unaffected by desire." Aristotle

Everything which is not forbidden is allowed

In the legal system, generally, everything not explicitly forbidden is legal.

- For example, in Kearney, backyard chickens are prohibited by city code, making them illegal,
- but painting your house purple is legal because it's not mentioned in city code.

However, I'm not a lawyer, so ...

But math is different

- In math, most things not explicitly allowed are forbidden.
- In math, a pretty good answer to the answer to the question "Can I do X?" is "if there is a rule for it, sure; if not, no way."

Exhaustive rules

In algebra, we attempt to enumerate everything that is allowed. If something isn't listed as a rule, likely it's not true.

- The enumeration of rules of exponents in our QRS aims to be exhaustive.
- But algebra tries to condense rules to a minimal set, so sometimes something might be provably true from a set of rules but not explicitly stated.

Can I do . . .

Theorem (multiplicative cancellation)
We have

$$(\forall a, c \in \mathrm{R}_{\neq 0}, c \in \mathrm{R})\left(\frac{ab}{ac} = \frac{b}{c}\right) \equiv True.$$

- In words, this says that a common nonzero *multiplicative* factor in a numerator and denominator can be "canceled."
- Notationally, we write $\frac{\cancel{a}b}{\cancel{a}c} = \frac{b}{c}$.
- Provided that a and c are nonzero, replacing $\frac{ab}{ac}$ by $\frac{b}{c}$ in any statement doesn't change its meaning (or truth value).

Avoiding slang

- The verb "canceled" is mathematical slang—its use is convenient but subject to abuse.
- A problem with slang is that it is often poorly defined and misused. An example of misuse is the **bogus cancellation**:

$$\frac{a+b}{a+c} = \frac{\cancel{a}+b}{\cancel{a}+c} = \frac{b}{c}.$$

- Our rule says that a common *multiplicative factor* in the numerator and denominator can be canceled, but in this example, the common term is additive, not multiplicative.

For every means for every

Consider the statement:

$$(\forall a \in \mathrm{R}_{\neq 0})\left(\frac{a+b}{a} = 1 + b\right) \equiv \text{True}$$

This statement is false. For instance, if we choose $a = 2$ and $b = 5$, then

$$\left[\frac{2+5}{2} = 1+5\right] \equiv \left[\frac{7}{2} = 6\right] \equiv [7 = 12] \equiv \text{False}.$$

☛ Checking a special case for a "for every" statement is a powerful way to possibly show that it is false.