MATH 202 Spring $2^3 \times 11 \times 23$ **Name:**____________________________

Exam 3–Practice **Row and Seat:**________________________

"Piglet noticed that even though he had a Very Small Heart, it could hold a rather large amount of Gratitude." WINNIE-THE-POOH

1. Find the *convergence set* for each series
   (a) $\sum_{k=1}^{\infty} \frac{x^k}{k}$
   (b) $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$
   (c) $\sum_{k=1}^{\infty} k! \frac{x^k}{28}$
   (d) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

2. Use power series to find the numerical value of each limit. You might like to use the facts

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots,$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \cdots,$$
$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \ldots.$$

   (a) $\lim\limits_{x\to 0} \dfrac{\log\left(x^3+1\right)}{\sin\left(2x^3\right)}$
   (b) $\lim\limits_{x\to 0} \dfrac{\arctan(x) - \sin(x)}{x^3}$
   (c) $\lim\limits_{x\to 0} \dfrac{\arctan(x) - \sin(x)}{x^4}$

$\boxed{1}$ 3. Use the *ratio* test to determine of the series $\sum_{k=0}^{\infty} \frac{\left(\frac{k}{3}\right)^k}{k!}$ converges or diverges.

$\boxed{5}$ 4. Find the numerical value of $\displaystyle\sum_{k=4}^{\infty} \left(\frac{1}{2}\right)^k$.

5. The de Jonquiére function Li_2 can be defined by $\mathrm{Li}_2(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ and $\mathrm{dom}(\mathrm{Li}_2) = (-1, 1)$

$\boxed{5}$ (a) Find the *numerical value* of $\mathrm{Li}_2(0)$.

$\boxed{5}$ (b) Find the *radius of convergence* for the power series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$.

$\boxed{5}$ (c) Find the *numerical value* of $\mathrm{Li}_2'(0)$. **Hint:** $\mathrm{Li}_2(x) = x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots$.

6. Determine convergence or divergence of each series. Fully justify your work. This doesn't mean you need to use our definition of convergent; instead use the theorems we've estiblished.

$\boxed{5}$ (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

$\boxed{5}$ (b) $\displaystyle\sum_{k=0}^{\infty} (-1)^k$

$\boxed{5}$ (c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$

$\boxed{5}$ (d) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$

$\boxed{5}$ 7. My friend Milhous claims that the sum $\sum_k f_k$ converges provided that the sequence f converges to zero. Show Milhous an example of a seqence f that converges to zero, but the sum $\sum_k f_k$ diverges.

8. For all real numbers x, we have $\sin(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

$\boxed{2}$ (a) Find the power series representation for $\sin(x) - x$ centered at zero. **Hint:** When you don't know where to start, go to your happy place: write the first few terms of the Taylor series for sine centered at zero. Then subtract x.

$\boxed{2}$ (b) For $x \neq 0$, find the *first two nonzero terms* in a power series representation for $\frac{\sin(x)-x}{x^3}$. Again, try visiting your happy place.

$\boxed{2}$ (c) Use the above result to find the *numerical value* of the limit

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3}.$$

$\boxed{1}$ 9. Define a function erf by the definite integral $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, \mathrm{d}t$. Find a power series representation for erf. Find the radius of convergence of this power series.