Class notes for sequences

MATH 202

March 21, 2024

"It's the sides of the mountain that sustain life, not the top."

Robert Pirsig

Definitions

Definition A function from either $Z_{\geq 0}$ or $Z_{>0}$ to R is a *real-valued sequence*.

- Our textbook modifies sequence with *infinite*.
- The identifiers $i, j, k, \ldots, n$ are traditional names for the *independent* variable of a sequence.
- The identifiers a, b, and c are traditional names for a sequence.
- An integer subscript on a sequence means function evaluation. Thus, if a is a sequence, we have

$$a_1 = a(1), \quad a_2 = a(2), \quad a_k = a(k).$$

- If the domain of a sequence a is $\mathrm{Z}_{>0}$, we say a_1 is its *first term*, a_2 is its *second term*, and .... If the domain is $\mathrm{Z}_{\geq 0}$, its first term is a_0.
- Our textbook (sometimes) surrounds a sequence with curly braces. This is fru-fru clutter. And I don't like fru-fru clutter.

Example Each of the following functions are sequences:

(a) $a(k) = (-1)^k$ and $\mathrm{dom}(a) = \mathbb{Z}_{\geq 0}$.

(b) $b(k) = \begin{cases} 100 & \text{if } k < 10 \\ \frac{1}{k} & \text{if } k \geq 10 \end{cases}$ and $\mathrm{dom}(b) = \mathbb{Z}_{\geq 0}$.

(c) $c(k) = \dfrac{1}{k!}$ and $\mathrm{dom}(c) = \mathbb{Z}_{\geq 0}$.

(d) $d(k) = \displaystyle\sum_{\ell=1}^{k} \frac{1}{\ell}$ and $\mathrm{dom}(d) = \mathbb{Z}_{\geq 1}$. We have $d_1 = 1$, $d_2 = 3/2$, and $d_3 = 1 + 1/2 + 1/3 = 11/6$.

Abject silliness

If you *only* know the first few terms of a sequence, you know **nothing** about its subsequent terms. To illustrate, the first five terms of the sequences $a_n = 2n+1$ and $b_n = \begin{cases} 2n+1 & n \leq 5 \\ \sqrt{5} & n \geq 6. \end{cases}$ are identical, but $a \neq b$.

Our textbook has some questions that give the first few terms of a sequence and asks you to guess the formula. Such questions are **abject silliness**.

Convergence

Definition A sequence a converges provided:

(a) there is number L such that
(b) for *each* positive number ε
(c) there is an integer N such that
(d) for all $k \in \mathbb{Z}_{\geq N}$, we have $L - \varepsilon < a_k$ and $a_k < L + \varepsilon$.

An alternative to 'd' is $|a_k - L| < \varepsilon$ for all $k \in \mathbb{Z}_{>N}$.

When this is the case, we say the sequence a converges to L. This is expressed as either

$$\lim_{\infty} a = L \text{ or } \lim_{n \to \infty} a_n = L.$$

In logician speak, we have

$$(\exists L \in \mathbb{R})(\forall \varepsilon \in \mathbb{R}_{>0})(\exists N \in \mathbb{Z})(\forall k \in \mathbb{Z}_{>N})(|a_k - L| < \varepsilon).$$

G is for graphical

- A sequence converges provided its graph has a horizontal asymptote.
- A definition of a horizontal asymptote would hardly differ from the definition of convergence.

Undefinition of convergence

Definition A sequence a diverges provided:

(a) for every number L

(b) there is a positive number ε such that

(c) for every integer N

(d) there is an integer $k \in \mathbf{Z}_{>N}$ such that either $a_k < L - \varepsilon$ or $L + \varepsilon < a_k$.

Fact

Theorem
If a sequence converges, its limit is unique.

This proposition is about the only fact that our book doesn't stuff into §9.1!

Since limits of sequences are unique, we can write things like

$$\lim_{\infty} a = 42$$

without disrespecting equality. (Suppose we could prove that a sequence a converges to both 42 and to 107. Then we would write

$$\lim_{\infty} a = 42 \text{ and } \lim_{\infty} a = 107.$$

And that is a proof that the truth value $42 = 107$ is true.