Calculus II with Analytic Geometry

MATH 202—01

Spring 2024

Instructor: Barton Willis, PhD, Professor of Mathematics
Office: Discovery Hall, Room 368
Office Hours: Monday, Wednesday, and Friday 9:00 AM—11:00 AM, Tuesday and Thursday 9:30 AM—11:00 AM, and by appointment.
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https://github.com/barton-willis/math-202-spring-2024

Important Dates

Exam I	16 February
Exam II	29 March
Exam III	26 April
Exam IV	13 May 1:00 PM– 3:00 PM
Final exam	16 May 10:30 AM—12:30 PM

Grading

Your course grade will be based on twenty-five in class assignments, four midterm exams, three online homework assignments, and a comprehensive final exam; specifically:

In class work *Twenty-five six point assignments*	150 (total)
Mid-term exams I, II, III *100 points each*	300 (total)
Mid-term exam IV	50 (total)
Online homework *Three twenty point assignments*	60 (total)
Comprehensive Final exam	150 (total)

If we end the term with less than 150 points for in class work, your inclass work point total will be scaled to 150 points. There will be a five point bonus that is due the first week of the term.

The following table shows the *minimum* number of points (out of 710) that are required for each of the twelve letter grades D- through A+. For example, a point total of 615 points will earn you a grade of B+, and a point total of 639 points will earn you a grade of A-. If you earn a point total of 425 or less, you will a failing course grade.

D-	426	B-	568
D	449	B	591
D+	473	B+	615
C-	497	A-	639
C	520	A	662
C+	544	A+	703

Class meeting time and place

This class meets Monday, Wednesday, and Friday from 12:20 PM—1:10 PM and Tuesday and Thursday 12:30 PM—1:45 PM in Discovery Hall, room 386.

Course Resources

Our textbook is *University Calculus: Early Transcendentals*, by Joel Hass, Christopher Heil, Maurice Weir, Przemyslaw Bogacki, and George Thomas, 4$^{\text{th}}$ edition. You *must* purchase access to the online homework.

For online homework, you will need an internet connected computer. I don't recommend attempting to do your homework on a phone. For class you will need a way to take class notes–either paper for a tablet with pen input (not a keyboard). If you use paper, colored pencils or pens are nice to have.

Course Calendar

Generally, we'll adhere to the scheduled exam dates even if we are ahead or behind with coursework. When we are ahead or behind, the topics on the exams will be appropriately adjusted.

Notices:

(a) Midterm exams I–III will be given on Friday of the week they are assigned.

(b) Pursuant to UNK policy for a five credit class, Midterm exam IV will be given during final exam week.

(c) In class work (labelled **IC**) will generally be given on Tuesday and Thursday of the week they are assigned.

(d) For Exam weeks, we'll have just one in class work that will generally be given on Thursday.

(e) The final exam will be given on 16 May 10:30 AM—12:30 PM.

Week	Week Starting	Section(s)	Topic(s)	Assessments
1	22 January	§6.1 – §6.3	Volumes using cross-sections, Shells, Arclength	IC 1
2	29 January	§6.4 – §6.5	Areas, Work	IC 2, IC 3
3	5 February	§6.6 –§7.1	Center of mass, Logarithms	IC 4, IC 5
4	12 February	§7.2 – §7.3	Separable DEs, Hyperbolic functions	IC 6, OL 1, **Exam I**
5	19 February	§8.1– §8.2	Integration by parts, Trigonometric integrals	IC 7, IC 8
6	26 February	§8.3 – §8.5	Trigonometric substitutions, Rational functions, Tables	IC 9, IC 10
7	4 March	§8.6 – §8.7	Numerical integration, Improper integrals	IC 11, IC 12
8	18 March	§9.1 – §9.3	Sequences, Infinite series, Integral test	IC 13, IC 14
9	25 March	§9.4 – §9.5	Comparison Test, Absolute convergence	IC 15, OL 2, **Exam II**
10	1 April	§9.6	Alternating series	IC 16, IC 17
11	8 April	§9.7	Power series	IC 18, IC 19
12	15 April	§9.8–§9.9	Taylor and Maclaurin series, Convergence of Taylor series	IC 20, IC 21
13	22 April	§9.10–§10.1	Applications of Taylor series, Plane curves	IC 22, OL 3, **Exam III**
14	29 April	§10.2 – §10.3	Calculus with parametric curves, Polar Coordinates	IC 23, IC 24
15	6 May	§10.4–§10.5	Graphing polar equations, Areas & lengths of polar curves	IC 25
16	13 May			**Exam IV**, **Final Exam**

Grading rubric

Generally each exam question will be worth five points. These points will be assigned according to the scheme

All major steps are shown and only one correct answer .... 5 points
Initial part of work is correct, but incomplete solution .... 3 points
Work is mostly correct, but a minor correctable error .... 3 or 4 points
Answer (either correct or not) with no work shown .... 0 points
Work contains a major error .... 0 points
Work (correct or not) that answers something other than the given question .... 0 points

I will subtract one or more points for work that is work is messy, hard to read, or poorly organized. Minor errors include arithmetic errors, sign errors, and recopying errors. Major errors include using bogus identities, such as $(1+x)^2 = 1+x^2$, $\frac{a+b}{a} = 1+b$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, and $\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) = f'(x)g'(x)$.

Learning Outcomes

On completion of this course, students will be able to

(a) use definite integrals to solve problems involving volume, arc length, surface area, work, and center of mass.

(b) use integration by parts, trigonometric substitution, and partial fractions to evaluate definite and indefinite integrals.

(c) apply the concepts of limits, convergence, and divergence to evaluate improper integrals.

(d) determine convergence or divergence of sequences and series.

(e) use Taylor and MacLaurin series to represent functions and integrate functions.

(f) use parametrizations and polar coordinates to find areas and arc lengths.

Class Policies

1. Generally, if you are ill, injured, incarcerated, or absent for any reason (including athletics), you must turn in your in class work on time. Permission to turn in work late must be made in advance, otherwise work turned in late will earn a score of zero.

2. During class, you may use a tablet with a pen (not a keyboard) to take class notes, but please refrain from using other electronic devices. If your device usage distracts your classmates, I will ask you to put it away. If it's my impression that you are often not paying attention in class, I reserve the right to decline to help you with homework assignments.

3. Using unauthorized materials or communication devices while taking an exam will earn you a grade of zero on that assessment.

4. It is essential and expected for you to attend class regularly. If you miss class, please ask a classmate for class notes. You may ask me for a copy of my class notes, but I might decline–it's unlikely my notes will be of any value you.

5. Examinations must be taken in class, not by Zoom.

6. If you arrive to class a bit late, please enter and take your seat. If you are habitually late for no good reason, I will ask you to make changes to your schedule.

7. For examinations and homework, show your work. *No credit will be given for multistep problems without the necessary work. Your solution must contain enough detail so that I am convinced that you could correctly solve any similar problem.* Also erase or clearly mark any work you want me to ignore; otherwise, I'll grade it.

8. The work you turn in for a grade must be *accurate, complete, concise, neat*, and *well-organized. You will not earn full credit on work that falls short of these expectations.*

9. Class cancellations due to weather, illness, or other unplanned circumstances may require that we make adjustments to the course calendar, exam dates, or due dates for course assessments. It is your responsibility to attend class and to be aware of changes the class calendar.

10. I will *decline* all requests for extra credit or for redoing an assignment or examination to earn a higher grade.

11. For examinations, you may use a teacher provided quick reference sheet, but no other reference materials. You may also use a pencil, an eraser, and a scientific calculator. For examinations, your phone and all such devices must be turned off and *out of sight.*

12. The final examination will be *comprehensive* and it will be given during the time scheduled by the University. Except for *extraordinary circumstances* you must take the exam at this time.

13. If you have questions about how your work has been graded, make an appointment with me immediately.

14. Please regularly check Canvas to verify that your scores have been recorded correctly. If I made a mistake in recording one of your grades, I'll correct it provided you saved your paper.

University Policies[1]

Student Attendance Policy Statement Students are expected to attend all meetings of classes for which they are registered, including the first and last scheduled meetings and the final examination period. Instructors hold the right and responsibility to establish attendance policies for their courses. Each instructor must inform all classes at the beginning of each semester concerning their attendance policies.

Participation in official University activities, serious health concerns, personal emergencies, and religious observances are valid reasons for absence from classes. Students are responsible for informing their instructors prior to their absence(s) from class and for completing assignments missed during their absence(s). No adverse or prejudicial effects shall result to any student with a documented, excused absence.

Questions may be directed to the Dean of Student Affairs office or to Student Health & Counseling.

Academic Integrity Policy The maintenance of academic honesty and integrity is a vital concern of the University community. Any student found in violation of the standards of academic integrity may be subject to both academic and disciplinary sanctions. Academic dishonesty includes, but is not limited to, the following:

- **Cheating** Copying or attempting to copy from an academic test or examination of another student; using or attempting to use unauthorized materials, information, notes, study aids or other devices for an academic test, examination or exercise; engaging or attempting to engage the assistance of another individual in misrepresenting the academic performance of a student; or communicating information in an unauthorized manner to another person for an academic test, examination or exercise.

- **Fabrication and falsification** Falsifying or fabricating any information or citation in any academic exercise, work, speech, test or examination. Falsification is the alteration of information, while fabrication is the invention or counterfeiting of information.

- **Plagiarism** Presenting the work of another as one's own (i.e., without proper acknowledgment of the source) and submitting examinations, theses, reports, speeches, drawings, laboratory notes or other academic work in whole or in part as one's own when such work has been prepared by another person or copied from another person.

- **Abuse of academic materials and/or equipment** Destroying, defacing, stealing, or making inaccessible library or other academic resource material.

[1]These polices are copied from `https://catalog.unk.edu/undergraduate/academics/academic-regulations/` and from `https://www.unk.edu/academic_affairs/asa_forms/course-policies-and-resources.php`. As of 29 May 2024, these polices are current.

- **Complicity in academic dishonesty** Helping or attempting to help another student to commit an act of academic dishonesty.
- **Falsifying grade reports** Changing or destroying grades, scores or markings on an examination or in an instructor's records.
- **Misrepresentation to avoid academic work** Misrepresentation by fabricating an otherwise justifiable excuse such as illness, injury, accident, etc., in order to avoid or delay timely submission of academic work or to avoid or delay the taking of a test or examination.
- **Other Acts of Academic Dishonesty** Academic units and members of the faculty may prescribe and give students prior written notice of additional standards of conduct for academic honesty in a particular course, and violation of any such standard shall constitute a violation of the Code.

Under Section 2.9 of the Bylaws of the Board of Regents of the University of Nebraska, the respective colleges of the University have responsibility for addressing student conduct solely affecting the college. Just as the task of inculcating values of academic honesty resides with the faculty, the college faculty are entrusted with the discretionary authority to decide how incidents of academic dishonesty are to be resolved. For more information, please visit UNK's Procedures and Sanctions for Academic Integrity and the Student Code of Conduct.

Reporting Student Sexual Harassment, Sexual Violence or Sexual Assault Reporting allegations of rape, domestic violence, dating violence, sexual assault, sexual harassment, and stalking enables the University to promptly provide support to the impacted student(s), and to take appropriate action to prevent a recurrence of such sexual misconduct and protect the campus community. Confidentiality will be respected to the greatest degree possible. Any student who believes they may be the victim of sexual misconduct is encouraged to report to one or more of the following resources:

- Local Domestic Violence, Sexual Assault Advocacy Agency 308-237-2599
- Campus Police (or Security) 308-865-8911
- Title IX Coordinator 308-865-8655

Retaliation against the student making the report, whether by students or University employees, will not be tolerated.

Students with Disabilities It is the policy of the University of Nebraska at Kearney to provide flexible and individualized reasonable accommodation to students with documented disabilities. To receive accommodation services for a disability, students must be registered with the UNK Disabilities Services for Students (DSS) office, 175 Memorial Student Affairs Building, 308-865-8214 or by email unkdso@unk.edu

Students Who are Pregnant It is the policy of the University of Nebraska at Kearney to provide flexible and individualized reasonable accommodation to students who are pregnant. To receive accommodation services due to pregnancy, students must contact the Student Health office at 308-865-8218. The following links provide information for students and faculty regarding pregnancy rights:

- `https://thepregnantscholar.org/title-ix-basics/`
- `https://nwlc.org/resource/faq-pregnant-and-parenting-college-graduate-students-rights/`

UNK Statement of Diversity & Inclusion UNK stands in solidarity and unity with our students of color, our Latinx and international students, our LGBTQIA+ students and students from other marginalized groups in opposition to racism and prejudice in any form, wherever it may exist. It is the job of institutions of higher education, indeed their duty, to provide a haven for the safe and meaningful exchange of ideas and to support peaceful disagreement and discussion. In our classes, we strive to maintain a positive learning environment based upon open communication and mutual respect. UNK does not discriminate on the basis of race, color, national origin, age, religion, sex, gender, sexual orientation, disability or political affiliation. Respect for the diversity of our backgrounds and varied life experiences is essential to learning from our similarities as well as our differences. The following link provides resources and other information regarding D&I: `https://www.unk.edu/about/equity-access-diversity.php`.

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Blocks marked “Appointments” (yellow highlight) means *possibly* available to make appointments. Blocks with gray highlights mean not available.

Spring 2024

	Monday	Tuesday	Wednesday	Thursday	Friday
8:00	**CYBR 304-01** 8:00-8:55, DSCH 116	**Not Available** 8:00-9:30	**CYBR 304-01** 8:00-8:55, DSCH 116	**Not Available** 8:00-9:30	**CYBR 304-01** 8:00-8:55, DSCH 116
9:00	**Office Hours** 9:00–11:00, DSCH 368		**Office Hours** 9:00–11:00, DSCH 368		**Office Hours** 9:00–11:00, DSCH 368
9:30		**Office Hours** 9:30–11:00, DSCH 368		**Office Hours** 9:30–11:00, DSCH 368	
11:00	**Lunch (not available)**	**Lunch (not available)**	**Lunch (not available)**	**Lunch (not available)**	**Lunch (not available)**
12:15	**Calculus II** 12:20-13:10, DSCH 386		**Calculus II** 12:20-13:10, DSCH 386		**Calculus II** 12:20-13:10, DSCH 386
12:30		**Calculus II** 12:30-13:20, DSCH 386		**Calculus II** 12:30-13:20, DSCH 386	
13:15	**Appointments** 13:15-15:00, DSCH 368	**Appointments** 13:20-14:00, DSCH 368	**Appointments** 13:15-15:00, DSCH 368	**Appointments** 13:20-14:00, DSCH 368	**Appointments** 13:15-15:00, DSCH 368
14:00		**Departmental Meeting** 14:00-15:00		**Departmental Meeting** 14:00-15:00	
15:00	**Not Available** 15:00-17:00	**Not Available** 15:00-17:00	**Not Available** 15:00-17:00	**Not Available** 15:00-17:00	**Not Available** 15:00-17:00

Named Sets

empty set	$\varnothing$
real numbers	$\mathbf{R}$
ordered pairs	$\mathbf{R}^2$
integers	$\mathbf{Z}$
positive integers	$\mathbf{Z}_{>0}$
positive real numbers	$\mathbf{R}_{>0}$

Exponents

For $a, b \in \mathbf{R}_{>0}, x \in \mathbf{R}$, and $m, n \in \mathbf{R}$,

$$a^0 = 1, \qquad 0^a = 0$$
$$1^x = 1, \qquad a^n a^m = a^{n+m}$$
$$a^n/a^m = a^{n-m}, \qquad (a^n)^m = a^{n\cdot m}$$
$$a^{-m} = 1/a^m, \qquad (a/b)^m = a^m/b^m$$
$$\sqrt{x^2} = |x|$$

Trigonometric Identities

We define $\mathrm{dom}(\mathrm{arccot}) = (0, \pi)$.

$$(\cos(x))^2 + (\sin(x))^2 = 1$$
$$2(\cos(x))^2 = 1 + \cos(2x)$$
$$2(\sin(x))^2 = 1 - \cos(2x)$$
$$(\cos(x))^2 - (\sin(x))^2 = \cos(2x)$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\mathrm{arccot}(x) = \pi/2 - \arctan(x)$$
$$\mathrm{arccsc}(x) = \arcsin(1/x)$$
$$\mathrm{arcsec}(x) = \arccos(1/x)$$
$$\arcsin(x) + \arccos(x) = \pi/2$$
$$\mathrm{arcsec}(x) + \mathrm{arccsc}(x) = \pi/2$$

Hyperbolic Functions

$$2\cosh(x) = \exp(x) + \exp(-x)$$
$$2\sinh(x) = \exp(x) - \exp(-x)$$
$$\tanh(x) = \sinh(x)/\cosh(x)$$
$$\cosh(x)^2 - \sinh(x)^2 = 1$$

Logarithms

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Derivatives

Specific cases

$F(x)$	$F'(x)$
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec(x)^2$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\cot(x)\csc(x)$
$\cot(x)$	$-\csc(x)^2$
$\arccos(x)$	$-1/\sqrt{1-x^2}$
$\arcsin(x)$	$1/\sqrt{1-x^2}$
$\arctan(x)$	$1/(x^2+1)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$
$\tanh(x)$	$1/\cosh(x)^2$
$\mathrm{arccosh}(x)$	$1/\sqrt{x^2-1}$
$\mathrm{arcsinh}(x)$	$1/\sqrt{1+x^2}$
$\mathrm{arctanh}(x)$	$1/(1-x^2)$
$\exp(x)$	$\exp(x)$
$\ln(x)$	$1/x$

General Cases

$F(x)$	$F'(x)$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$1/g(x)$	$-g'(x)/g(x)^2$
$f(x)/g(x)$	$(g(x)f'(x) - f(x)g'(x))/g(x)^2$
$f(g(x))$	$g'(x)f'(g(x))$
$f^{-1\prime}(x)$	$1/f'(f^{-1}(x))$

Antiderivatives

$$\int a\,\mathrm{d}x = ax$$
$$\int x^a\,\mathrm{d}x = \frac{1}{1+a}x^{a+1}, \quad \text{if } a \neq -1$$
$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|$$
$$\int \cos(x)\,\mathrm{d}x = \sin(x)$$
$$\int \sin(x)\,\mathrm{d}x = -\cos(x)$$
$$\int \tan(x)\,\mathrm{d}x = \ln|\sec(x)|$$
$$\int \sec(x)\,\mathrm{d}x = \ln|\tan(x) + \sec(x)|$$
$$\int \csc(x)\,\mathrm{d}x = -\ln|\csc(x) + \cot(x)|$$
$$\int \cot(x)\,\mathrm{d}x = \ln|\sin(x)|$$
$$\int |x|\,\mathrm{d}x = x|x|/2$$

Sums

For $n \in \mathbf{Z}_{>0}$

$$\sum_{k=0}^{n-1} 1 = n, \quad \sum_{k=0}^{n-1} k = (n-1)n/2$$
$$\sum_{k=0}^{n-1} k^2 = (n-1)n(2n-1)/6,$$
$$\sum_{k=0}^{n-1} x^k = \sum_{k=1}^{n} x^{k-1} = \frac{1-x^n}{1-x}, \quad x \neq 1$$
$$\sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} & x \in (-1,1) \\ \infty & x \in [1,\infty] \end{cases}.$$

When $x \in (-\infty, -1]$, the series $\sum_{k=0}^{\infty} x^k$ diverges.

Applications

Arc length of curve $y = f(x)$ with $a \le x \le b$

$$= \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

For the region Q of the xy plane given by

$$Q = \{(x,y) \mid f(x) \le y \le g(x) \wedge a \le x \le b\},$$

we have

$$\mathrm{Area}(Q) = \int_a^b g(x) - f(x)\,\mathrm{d}x$$

Assuming $0 \le f(x)$ and rotating about the x-axis

$$\mathrm{Vol}(Q) = \pi \int_a^b g(x)^2 - f(x)^2\,\mathrm{d}x$$

Assuming $0 \le a < b$ and rotating about the y-axis

$$\mathrm{Vol}(Q) = 2\pi \int_a^b x(g(x) - f(x))\,\mathrm{d}x$$

Centroid

$$\mathrm{Area}(Q) \times \overline{x} = \int_a^b x\left(g(x) - f(x)\right)\mathrm{d}x$$
$$\mathrm{Area}(Q) \times \overline{y} = \frac{1}{2}\int_a^b \left(g(x)^2 - f(x)^2\right)\mathrm{d}x$$

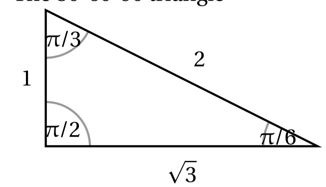
For the region Q of the xy plane given by

$$Q = \{(x,y) \mid f(y) \le x \le g(y) \wedge a \le y \le b\},$$

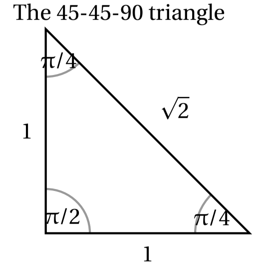
interchange x and y in *all* the previous formulas.

Famous Triangles

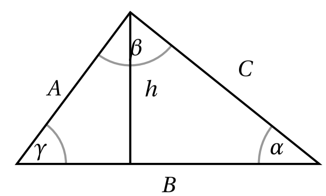
The 30-60-90 triangle



The 45-45-90 triangle



Laws of Cosine & Sine



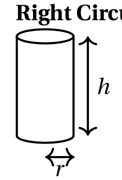
Law of cosine: $C^2 = A^2 + B^2 - 2AB\cos(\gamma)$

Law of sines: $\dfrac{\sin(\alpha)}{A} = \dfrac{\sin(\beta)}{B} = \dfrac{\sin(\gamma)}{C}$

Area: $\mathrm{Area} = 1/2\,hB = 1/2\,AB\sin(\gamma)$

Volumes

Right Circular Cylinder



Volume: $V = \pi r^2 h$

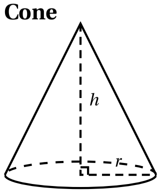
Area: (not including circular ends) $A = 2\pi r h$

Sphere with radius r

Area: $A = 4\pi r^2$

Volume: $V = \dfrac{4\pi}{3} r^3$

Cone



Volume: $V = \pi r^2 h/3$
Area
$A = \pi r\sqrt{r^2+h^2}$ (not including circular base).

P-Series, Divergence Test, Ratio Test, Comparison, & AST

The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p \in (1,\infty)$; otherwise it diverges.

If $\lim_{k\to\infty} a_k \neq 0$, the series $\sum a_k$ diverges.

Let a be a sequence with $0 \notin \text{range}(a)$. Define $L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$.

- $L \in [0,1) \implies \sum |a_k|$ converges.
- $L \in (1,\infty] \implies \sum a_k$ diverges.

Let a and b be positive sequences. Define $L = \lim_{k\to\infty} \frac{a_k}{b_k}$.

- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ converges, then $\sum b_k$ converges.
- If $L \in \mathbf{R}_{>0}$ and $\sum a_k$ diverges, then $\sum b_k$ diverges.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Let a be a positive and eventually decreasing sequence. Then $\sum (-1)^k a_k$ converges if and only if $\lim_{k\to\infty} a_k = 0$.

Taylor and MacLaurin Series

If a function F is infinitely differentiable at a, its Taylor series centered at a is

$$\sum_{k=0}^{\infty} \frac{F^{(k)}(a)}{k!}(x-a)^k.$$

When a is zero, the Taylor series is also known as a MacLaurin Series.

Polar to Cartesian

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

For $r > 0$ and $0 \le \theta < 2\pi$

$$r = \sqrt{x^2+y^2}, \quad \theta = \begin{cases} 2\pi - \arccos(x/r) & \text{if } y < 0 \\ \arccos(x/r) & \text{if } y \ge 0 \end{cases}$$

Integrate Powers of Trig

Let $m, n \in \mathbf{Z}_{\ge 0}$. Then

- $\int \cos(x)^{2m}\sin(x)^{2n}\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)^m \left(\frac{1-\cos(2x)}{2}\right)^n dx$
- $\int \cos(x)^{2m+1}\sin(x)^n\,dx = \int (1-z^2)^m z^n\,dz$, where $z = \sin(x)$
- $\int \cos(x)^m \sin(x)^{2n+1}\,dx = -\int z^m(1-z^2)^n\,dz$, where $z = \cos(x)$
- $\int \sec(x)^n\,dx = \frac{1}{n-1}\sec(x)^{n-2}\tan(x) + \frac{n-2}{n-1}\int \sec(x)^{n-2}\,dx$, provided $n \neq 1$.
- $\int \tan(x)^{2m+1}\sec(x)^n\,dx = \int (z^2-1)^m z^{n-1}\,dz$, where $z = \sec(x)$
- $\int \tan(x)^{2m}\sec(x)^n\,dx = \int (\sec(x)^2-1)^m \sec(x)^n\,dx$.

Trig Substitutions

- $\int F\left(x, \left(1-x^2\right)^{n/2}\right) dx$, use $x = \sin(\vartheta)$, where $\vartheta \in [-\pi/2, \pi/2]$, then integrate
  $\int F\left(\sin(\vartheta), \cos(\vartheta)^n\right)\cos(\vartheta)\,d\vartheta$
- $\int F\left(x, \left(1+x^2\right)^{n/2}\right) dx$, use $x = \sinh(\vartheta)$, where $\vartheta \in \mathbf{R}$, then integrate
  $\int F\left(\sinh(\vartheta), \cosh(\vartheta)^n\right)\cosh(\vartheta)\,d\vartheta$
- $\int F\left(x, \left(x^2-1\right)^{n/2}\right) dx$, use $x = \sec(\vartheta)$, then integrate
  $\int F\left(\sec(\vartheta), \tan(\vartheta)^n\right)\sec(\vartheta)\tan(\vartheta)\,d\vartheta$

Unit Circle