Classnotes
MATH 202
March 20, 2024

"It's the sides of the mountain that sustain life, not the top."

Robert Pirsig

Definitions

Definition A function from either $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{>0}$ to $\mathbb{R}$ is a *real-valued sequence*.

- Our textbook modifies sequence with *infinite*.
- The identifiers $i, j, k, \ldots, n$ are traditional names for the *independent* variable of a sequence.
- The identifiers a, b, and c are traditional names for a sequence.
- An integer subscript on a sequence means function evaluation. Thus if a is a sequence, we have

$$a_1 = a(1), \quad a_2 = a(2), \quad a_k = a(k).$$

- If the domain of a sequence a is $Z_{>0}$, we say a_1 is its *first term*, a_2 is its *second term*, and .... If the domain is $Z_{\geq 0}$, its first term is a_0.
- Our textbook (sometimes) surrounds a sequence with curly braces. This is fru-fru clutter. And I don't like fru-fru clutter.

Example

Each of the following functions are sequences:

(a) $a(k) = (-1)^k$ and $\operatorname{dom}(a) = \mathbb{Z}_{\geq 0}$.

(b) $b(k) = \begin{cases} 100 & \text{if } k < 10 \\ \frac{1}{k} & \text{if } k \geq 10 \end{cases}$ and $\operatorname{dom}(b) = \mathbb{Z}_{\geq 0}$.

(c) $c(k) = \dfrac{1}{k!}$ and $\operatorname{dom}(c) = \mathbb{Z}_{\geq 0}$.

(d) $d(k) = \displaystyle\sum_{\ell=1}^{k} \frac{1}{\ell}$ and $\operatorname{dom}(d) = \mathbb{Z}_{\geq 1}$. We have $d_1 = 1$, $d_2 = 3/2$, and $d_3 = 1 + 1/2 + 1/3 = 11/6$.

Abject silliness

If you *only* know the first few terms of a sequence, you know **nothing** about its subsequent terms. To illustrate, the first five terms of the sequences $a_n = 2n+1$ and $b_n = \begin{cases} 2n+1 & n \leq 5 \\ \sqrt{5} & n \geq 6. \end{cases}$ are identical, but $a \neq b$.

Our textbook has some questions that give the first few terms of a sequence and asks you to guess the formula. Such questions are **abject silliness**.

Convergence

Definition A sequence a converges provided:

(a) there is number L such that
(b) for *each* positive number ε
(c) there is an integer N such that
(d) for all $k \in \mathbb{Z}_{\geq N}$, we have $L - \varepsilon < a_k$ and $a_k < L + \varepsilon$.

An alternative to 'd' is $|a_k - L| < \varepsilon$ for all $k \in \mathbb{Z}_{\geq N}$. When this is the case, we say the sequence a converges to L. This is expressed as either

$$\lim_{\infty} a = L \text{ or } \lim_{n \to \infty} a_n = L.$$

In logician speak, we have

$$(\exists L \in \mathbb{R})(\forall \varepsilon \in \mathbb{R}_{>0})(\exists N \in \mathbb{Z})(\forall k \in \mathbb{Z}_{>N})(|a_k - L| < \varepsilon).$$

- An alternative to ‘d’ is $|a_k - L| < \varepsilon$ for all $k \in \mathbb{Z}_{\geq N}$.
- Graphically, a sequence *converges* if its graph has a horizontal asymptote towards infinity.

G is for graphical

- A sequence converges provided its graph has a horizontal asymptote.
- A definition of a horizontal asymptote would hardly differ from the definition of convergence.

Undefinition of convergence

Definition A sequence a diverges provided:

(a) for every number L

(b) there is a positive number ε such that

(c) for every integer N

(d) there is an integer $k \in \mathbb{Z}_{\geq N}$ such that either $a_k < L - \varepsilon$ or $L + \varepsilon < a_k$.

Facts

Theorem If a sequence converges, its limit is unique.

1. This proposition is about the only fact that our book doesn't stuff into §9.1!