MATH 202, Spring 2024 **Name:** ____________________

Exam 4 Practice **Row and Seat**:____________________

"If people sat outside and looked at the stars each night, I'll bet they'd live a lot differently."

CALVIN (BILL WATTERSON)

1. For the parametricaly defined curve $\begin{cases} x = 3t \\ y = 9t^2 + t \end{cases}$ $-\infty < t < \infty$, eliminate the parameter t. Sketch the resulting curve in the xy cartesian coordinate system.

> **Solution:** Solving $x = 3t$ for t yields $t = x/3$. Substituting this into $y = 9t^2 + t$ gives $y = x^2 + x/3$. The graph is an upward facing parabola with vertex $(x = -\frac{1}{6}, y = -\frac{1}{36})$ and x-intercepts $x = -\frac{1}{3}$ and $x = 0$.

2. For the parametricaly defined curve $\begin{cases} x = 3\cos(t) \\ y = 9\sin(t) \end{cases}$ $0 \le t \le 2\pi$, eliminate the parameter t. Sketch the resulting curve in the xy cartesian coordinate system.

> **Solution:** $(x/3)^2 + (y/9)^2 = 1$. The graph is an ellipse with vertices $(x = -3, y = 0)$,$(x = 3, y = 0)$, $(x = 0, y = -9)$, and $(x = 0, y = 9)$.

3. For the parametrically defined curve $\begin{cases} x = -\sqrt{1+t} \\ y = \sqrt{3t} \end{cases}$ $0 \le t < \infty$, find the numerical values of $\left.\frac{dy}{dx}\right|_{t=3}$ and $\left.\frac{d^2y}{dx^2}\right|_{t=3}$.

> **Solution:** When I make cornbread, I like to gather all my ingredients before I start combining and mixing them. The same is true for math. Before we start,

let's gather our four ingredients; we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=3} = \left.\frac{\sqrt{3}}{2\sqrt{t}}\right|_{t=3} = \frac{1}{2},$$
$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right|_{t=3} = -\left.\left(\frac{\sqrt{3}}{4t^{\frac{3}{2}}}\right)\right|_{t=3} = -\frac{1}{12},$$
$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=3} = -\left.\left(\frac{1}{2\sqrt{t+1}}\right)\right|_{t=3} = -\frac{1}{4},$$
$$\left.\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right|_{t=3} = \left.\frac{1}{4(t+1)^{\frac{3}{2}}}\right|_{t=3} = \frac{1}{32}.$$

We're now ready to make cornbread:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=3} = \frac{\frac{1}{2}}{-\frac{1}{4}} = -2,$$
$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=3} = \frac{\left(-\frac{1}{4}\right)\times\left(-\frac{1}{12}\right) - \frac{1}{32}\times\frac{1}{2}}{\left(-\frac{1}{4}\right)^3} = -\frac{1}{3}$$

If you are unlike me (the laziest boy in Buffalo County), you might like to calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ fully symbolically, simplify it, and last paste in the given value of t. But that's the long row to hoe and it's terribly error prone too. I think that finding the numeric values of all needed derivatives with respect to t is better than finding symbolic values for the x derivatives and then pasting in the given value for t.

4. Represent the arc length of the polar curve $r = a(1-\sin(\theta))$ as a definite integral. Here a is a positive real number.

Solution: A quick Desmos picture shows that the parameter space is $[0, 2\pi]$. The integrand is the square root of $r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$. Specifically

$$r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = a^2(1-\sin(\theta))^2 + a^2\cos(\theta)^2 = 2a^2(1-\sin(\theta)). \quad (1)$$

$$\sqrt{2}\,|a| \int_0^{2\pi} \sqrt{1-\sin(\theta)}\,\mathrm{d}\theta. \quad (2)$$

5. Find all points on the polar curve $r = 3 - 2\sin(\theta)$ that have a horizontal tangent line.

Solution: $\theta = \frac{\pi}{2}, \theta = \arcsin\left(\frac{3}{4}\right), \theta = \pi - \arcsin\left(\frac{3}{4}\right)$

6. Find area of the region bounded by the polar curve $r = 3 - 2\sin(\theta)$

Solution: A quick Desmos picture shows that the parameter space is $[0, 2\pi]$. $\frac{11}{2}\pi$.

7. Find the area bounded by the polar curve $r = \cos(4\theta)$

Solution: A quick Desmos picture shows that the parameter space is $[0, 2\pi]$ and the curve looks like an eight petal flower. (The chocolate flower (*Berlandiera lyrata*) has eight petals). So $\frac{\pi}{2}$