Can I do X?

MATH 202

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"The law is reason unaffected by desire." ARISTOTLE

Everything which is not forbidden is allowed

In the legal system, generally everything not forbidden is legal.

- ☛ In K-Town, city code prohibits backyard chickens, making chickens illegal,
- ☛ but painting your house purple is legal because it's not mentioned in city code.

However, I'm not a lawyer, so check before buying the purple house paint.

Math is different

- In math, most things not explicitly allowed are forbidden.
- In math, a snarky answer to the question "Can I do X?" is "If there is an explicit rule for it, sure; if not, likely no."

Undistracted by the question

- To remain in power, politicians need to be *undistracted by the question.*
- Math teachers need this skill too.
- A nonsnarky, but evasive, answer to the question "Can I do X?" is "Maybe, but let's think about a strategy first."
- We'll return to strategy in a bit.

Exhaustive rules

In algebra, we attempt to enumerate everything that is allowed. If something isn't listed as a rule, likely it's not true.

- The list of rules of exponents in our QRS aims to be exhaustive.
- Often in math, we try to condense rules to a minimal set,
- so sometimes something might be true, but not explicitly stated as a rule.

Nonenumeratiuon of nonrules

In the section on antideriatives, our book **doesn't** list

$$\int f(x)g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \times \int g(x)\,\mathrm{d}x$$

as a rule. Why not? Because it's abject rubbish:

$$\int x^2x^3\,\mathrm{d}x = \int x^5\,\mathrm{d}x = \frac{1}{6}x^6$$

But

$$\int x^2\mathrm{d}x \times \int x^3\mathrm{d}x = \frac{1}{12}x^7.$$

What's a rule?

Theorem (multiplicative cancellation)
We have

$$(\forall a, c \in \mathbf{R}_{\neq 0}, b \in \mathbf{R}) \left(\frac{ab}{ac} = \frac{b}{c} \right) \equiv \mathit{True}.$$

- In words, this says that a common nonzero *multiplicative* factor in a numerator and denominator can be "canceled."
- Notationally, we write $\frac{\cancel{a}b}{\cancel{a}c} = \frac{b}{c}$.
- Replacing $\frac{ab}{ac}$ by $\frac{b}{c}$ is generally regarded as a *simplification.*

Avoid slang

- The verb "cancel" is mathematical slang.
- Slang is poorly defined, so it often gets misused. An example of misuse is the **bogus cancellation**:

$$\frac{a+b}{a+c} = \frac{\cancel{a}+b}{\cancel{a}+c} = \frac{1+b}{1+c}.$$

- Our rule says that a common *multiplicative factor* in the numerator and denominator can be canceled, but in this example, the common term is additive, not multiplicative.

One bad apple

Theorem

$$(\exists a, b, c \in \mathbf{R}_{>0}) \left(\frac{a+b}{a+c} \neq \frac{1+b}{1+c} \right) \equiv \mathit{True}.$$

Proof.
Choose $a = 2, b = 3$, and $c = 4$. We have

$$\left[\frac{a+b}{a+c} \neq \frac{1+b}{1+c} \right] \equiv \left[\frac{2+3}{2+4} \neq \frac{1+3}{1+4} \right] \equiv \left[\frac{5}{6} \neq \frac{4}{5} \right] \equiv \text{True!}$$

$\square$

One bad apple redux

Theorem

$$(\forall a, b, c \in \mathbf{R}_{>0}) \left(\frac{a+b}{a+c} = \frac{1+b}{1+c} \right) \equiv \textit{False}.$$

Proof.
The given statement is logically equivalent to

$$(\exists a, b, c \in \mathbf{R}_{>0}) \left(\frac{a+b}{a+c} \neq \frac{1+b}{1+c} \right) \equiv \text{True}.$$

$\square$

Semantic matching

To apply the multiplicative cancellation rule

$$(\forall a, c \in \mathbf{R}_{\neq 0}, b \in \mathbf{R}) \left(\frac{ab}{ac} = \frac{b}{c} \right) \equiv \text{True}.$$

we don't need a literal match (a syntactic match) with $\frac{ab}{ac}$; rather a, b, and c can match with any 'blob,' as long as the 'blobs' a and c are nonvanishing.

For example

$$\frac{\cos(\pi x)\cancel{(z^2+1)}}{\cancel{z^2+1}} = \cos(\pi x)$$

is legitimate. We matched a with $z^2 + 1$, matched b with $\cos(\pi x)$, and matched c with 1.

For every means for every

Consider the statement:

$$(\forall a \in \mathbf{R}_{\neq 0}) \left(\frac{a+b}{a} = 1 + b \right).$$

This statement is false. For instance, if we choose $a = 2$ and $b = 5$, then

$$\left[\frac{2+5}{2} = 1 + 5 \right] \equiv \left[\frac{7}{2} = 6 \right] \equiv [7 = 12] \equiv \text{False}.$$

☛ Checking a special case for a "for every" statement is a powerful way to possibly show that it is false.

Referential transparency

Referential transparency is a fancy term that means that we can substitute like for like without changing meaning.

- Since for all real x, we have $x(x-1) = x^2 - x$, it's also the case that for all real x, we have

$$\cos(x(x-1)) = \cos(x^2 - x).$$

- Generally, mathematical notation is referentially transparent. **But**
- the question "Does $\sum_{k=0}^{\infty} \frac{1}{2^k}$ converge?" violates referential transparency. That's because the question "Does 1 converge?" is silly, yet $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1$ is true.

Referential transparency minutia

Since ‘people’ and ‘human’ are synonyms, the statement

All people are created equal.

has the same meaning as

All humans are created equal.

But the statement

Larry said, “All people are created equal.”

could be true while

Larry said, “All humans are created equal.”

could be false.

Mathematical taxonomy

Imagine that you are a turkey. If you can recognize that a tree is an oak (taxonomy), you have located a source of tasty acorns. It doesn't matter if the tree is a red or white oak–regardless it's food.

Organizing mathematical concepts into categories (taxonomy), helps to suggest a solution strategy. Examples

- If you can recognize an integrand is the product of a polynomial and an exponential function (taxonomy), use IBP.
- To find a limit of an indeterminate form (taxonomy), use the L'Hôpital rule.