Can I do X?

MATH 202

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"The law is reason unaffected by desire." — Aristotle

Everything which is not forbidden is allowed

In the legal system, generally, everything not forbidden is legal.

- In K-Town, city code prohibits yard chickens, making yard birds illegal,
- but painting your house purple is legal because it's not mentioned in city code.

However, I'm not a lawyer, so . . .

Math is different

- In math, most things not explicitly allowed are forbidden.
- In math, a snarky answer to the question "Can I do X?" is "If there is an explicit rule for it, sure; if not, no way."

Undistracted by the question

- To remain in power, politicians need to be *undistracted by the question.*
- Math teachers need this skill too.
- A nonsnarky, but evasive, answer to the question "Can I do X?" is "Maybe, but let's think about a strategy first."
- We'll return to strategy in a bit.

Exhaustive rules

In algebra, we attempt to enumerate everything that is allowed. If something isn't listed as a rule, likely it's not true.

- The list of rules of exponents in our QRS aims to be exhaustive.
- Often in math, we try to condense rules to a minimal set,
- so sometimes something might be true, but not explicitly stated as a rule.

What's a rule?

Theorem (multiplicative cancellation)
We have

$$(\forall a, c \in \mathbf{R}_{\neq 0}, b \in \mathbf{R}) \left(\frac{ab}{ac} = \frac{b}{c} \right) \equiv \mathit{True}.$$

- In words, this says that a common nonzero *multiplicative* factor in a numerator and denominator can be "canceled."
- Notationally, we write $\frac{\cancel{a}b}{\cancel{a}c} = \frac{b}{c}$.
- Replacing $\frac{ab}{ac}$ by $\frac{b}{c}$ is generally regarded as a *simplification.*

Avoid slang

- The verb "canceled" is mathematical slang.
- Slang is poorly defined, so it often gets misused. An example of misuse is the **bogus cancellation**:

$$\frac{a+b}{a+c} = \frac{\cancel{a}+b}{\cancel{a}+c} = \frac{b}{c}.$$

- Our rule says that a common *multiplicative factor* in the numerator and denominator can be canceled, but in this example, the common term is additive, not multiplicative.

For every means for every

To apply the multiplicative cancellation rule

$$(\forall a, c \in \mathbf{R}_{\neq 0}, b \in \mathbf{R}) \left(\frac{ab}{ac} = \frac{b}{c} \right) \equiv \text{True}.$$

we don't need a literal match (a syntactic match) with $\frac{ab}{ac}$; rather a, b, and c can match with any 'blob,' as long as the 'blobs' a and c are nonvanishing.

For example

$$\frac{\cos(\pi x)\cancel{(z^2+1)}}{\cancel{z^2+1}} = \cos(\pi x)$$

is legitimate. We matched a with $z^2 + 1$, matched b with $\cos(\pi x)$, and matched c with 1.

For every means for every

Consider the statement:

$$(\forall a \in \mathbf{R}_{\neq 0}) \left(\frac{a+b}{a} = 1 + b \right) \equiv \text{True}$$

This statement is false. For instance, if we choose $a = 2$ and $b = 5$, then

$$\left[\frac{2+5}{2} = 1 + 5 \right] \equiv \left[\frac{7}{2} = 6 \right] \equiv [7 = 12] \equiv \text{False}.$$

☛ Checking a special case for a "for every" statement is a powerful way to possibly show that it is false.

Referential transparency

Referential transparency is a fancy term that means that we can substitute like for like without changing meaning.

- Since for all real x, we have $x(x-1) = x^2 - x$, it's also the case that for all real x, we have

$$\cos(x(x-1)) = \cos(x^2 - x).$$

- Generally, mathematics is referentially transparent. **But**
- the question "Does $\sum_{k=0}^{\infty} \frac{1}{2^k}$ converge?" violates referential transparency. That's because the question "Does 1 converge?" is silly, yet $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1$ is true.

Mathematical taxonomy

Imagine that you are a turkey. If you can recognize that a tree is an oak (taxonomy), you've located a source of tasty acorns.

Mathematical taxonomy (categorization and classification) is a useful way to build an effective problem solving strategy.
Examples

- If you can recognize an integrand is the product of a polynomial and an exponential function, use IBP.
- To find a limit of an indeterminate form, use the L'Hôpital rule.