

# No More, No Less Than Sum of its Parts: The Algebra of Graphics, Statistics, and Interaction

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October 19, 2024

## Abstract

Interactive data visualization has become a staple of modern data presentation. Yet, despite its growing popularity, there still remains a lack of a formal framework for turning raw data into summary statistics that can then be displayed by interactive graphics. This gap may stem from a subtle yet profound issue: while we would often like treat statistics, graphics, and interaction as independent entities, they are in fact deeply connected. This paper aims to shed light on this interdependence by relating it to two fundamental concepts from category theory: groups and monoids. Specifically, if we want our graphics to support interactive features which break the data into multiple parts and then combine them back together (such as in the case of linked selection), then we need the statistics in our plots to fulfill certain algebraic properties. Ultimately, understanding this hidden structure may empower data visualization experts to build more flexible and expressive interactive data visualization systems.

*Keywords:* Interactive data visualization, category theory, abstract algebra, the Gestalt Principles, descriptive statistics

# 1 Introduction

“Effective graphical analysis makes things seem obvious, the effort involved in making the graphical analysis effective is not so obvious.” (Unwin, 2018)

The essence of data visualization is comparison. When we visualize, we split our data into parts, summarize each part by a set of one or more statistics, and then encode these summaries into visual channels such as position, size, or color that we then use to draw geometric objects (Bertin, 1983; Wilkinson, 2012; for a recent review, see Franconeri et al., 2021; Wilke, 2019). For example, when drawing a typical barplot, we split our data based on the levels of some categorical variable, count or sum up the values of some other variable, and finally represent these summaries by drawing bars of the corresponding heights. In this way, each bar becomes a visual representation of its own “small data set”, a subset of the original data.

To ground this idea in a more concrete example, one that will be used throughout the rest of this paper, imagine going through your bookshelf, sorting the books by genre, and stacking them on top of each other. You would construct a physical version of the barplot, forming one “bar” (stack) per genre: one bar for sci-fi novels, one bar for romance, one bar for detective fiction, and so on. Even from this very simple example, interesting mathematical properties jump out: for example, a single book cannot go into two stacks (the subsets of the data are *disjoint*) and if all of the stacks are combined back together, we should end up with the bookshelf we started with (the subsets are *surjective* and cover the entirety of the data set, see Ziemkiewicz and Kosara, 2009).

Now, just as we can learn about the contents of our library by organizing books in the way we just described, we can learn about our data by splitting it into parts. A natural question that arises is: can we learn more by subdividing our data into even smaller parts

and comparing these? For instance, in our book example, we could split each stack of books into two smaller stacks based on the gender of the author. Each smaller stack will then represent the unique combination (Cartesian product) of the two categorical variables (genre-gender). Moreover, there arises a hierarchical relationship between the stacks: by combining two genre-gender stacks (child components) together, we can recover the original genre stack (parent component). Similar situation often arises in interactive graphics, where we use techniques such as linked selection to highlight specific subsets of our data. Either way, by forming these finer partitions, we can represent our data with greater granularity and discover trends that may otherwise hide in the aggregate.

However, once we have subdivided each category into its child components, we need a way of representing this information that visually preserves the hierarchical nature of the relationship. In data visualization, there are two popular methods for doing this, known as “stacking” and “dodging” (see Figure 1). These methods have existed at least since the time of William Playfair (1801; 1822). Let’s briefly illustrate them on the example of the classical barplot. In a stacked barplot, bar segments representing the product of the two factor variables are plotted vertically on top of each other (“stacked”; the operation can be also thought of as “highlighting” parts of the bar). In contrast, in a dodged barplot, bars are plotted side-by-side, as clusters of bars. Finally, there is also a third popular method, known as layering, where we plot bar segments in separate graphical layers and use partial transparency to mitigate overplotting (this technique may have been a bit challenging for William Playfair but has been made simple with computers).

Much has been written about the relative merits of stacking, dodging, and layering. For example, layering is only useful with few categories, as blending many colors can make it difficult to tell the categories apart (Franconeri et al., 2021; Wilke, 2019). Further,

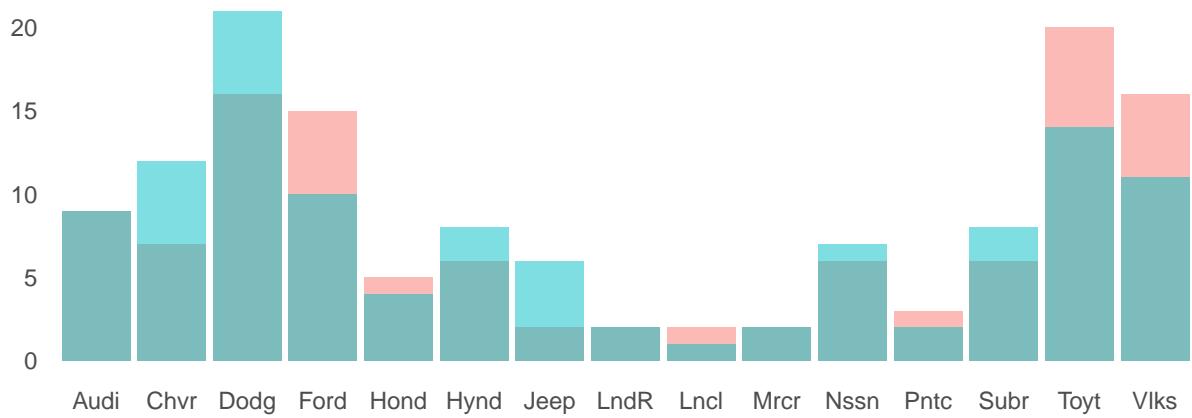
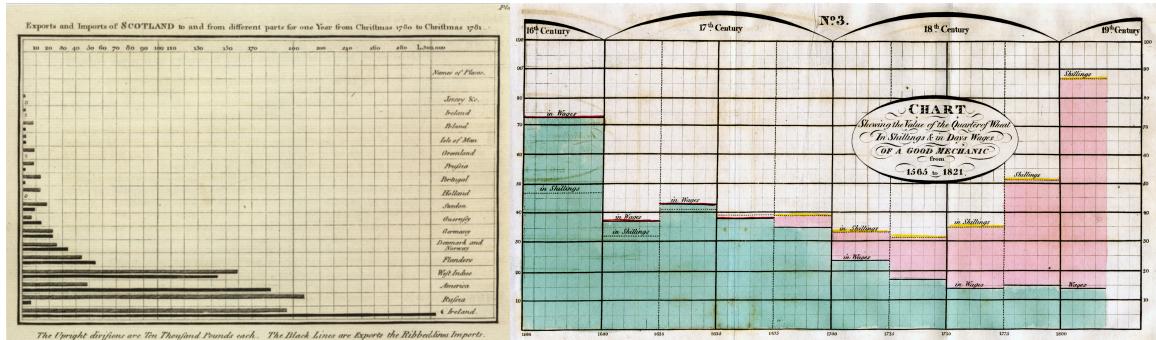


Figure 1: Examples of the three methods for dealing with bars that have subdivided into segments. Top row: dodging (left) and stacking/highlighting (right); both by William Playfair (1801, 1822). Bottom row: layering (created with ggplot2; Wickham, 2010).

in a landmark study, Cleveland and McGill (1984) showed that people tend to be less accurate when reading information from stacked bar charts as opposed to dodged bar charts. Specifically, since the lower y-axis coordinate of a stacked segment is pushed up by the cumulative height of the segments below it, it becomes difficult to accurately compare segments' length, both within and across bars (Cleveland and McGill, 1984). Subsequent research has independently validated and expanded upon these findings (see e.g. Heer and Bostock, 2010; Thudt et al., 2016; Quadri and Rosen, 2021). Ultimately, the suboptimal statistical legibility of stacking has led some authors to advise against its use altogether (Kosara, 2016; Wilke, 2019), or at least urge caution (see e.g. Byron and Wattenberg, 2008; Cairo, 2014; Franconeri et al., 2021).

However, despite the mixed reception of stacked plots in static visualization, they remain a popular choice in interactive data visualization. Among popular interactive data visualization libraries, stacked plots such as bar charts and histograms make frequent appearance (see e.g. Forbes et al., 2023; Murray, 2017; The Vega Project, 2022; Vanderplas et al., 2018; Sievert, 2020; Swayne et al., 2003; Xie et al., 2014; Theus, 2002; Urbanek and Theus, 2003; Urbanek, 2011; Wills, 2008; Ware, 2019). What is the reason for this? Has the interactive data visualization community simply failed to catch up with the current best practices? Or is there perhaps more going on?

We argue that not only is stacking a useful technique, but that it can in fact offer valuable insights about the mathematical structure underlying our visualizations. To make the strongest case possible, we bring forward evidence from several levels of the visualization process. We start by discussing the properties of visual perception and how they apply to both static and interactive data visualizations. Next, we explore the process of computing statistical summaries that form the basis of our plots. Finally, we dive into the algebraic

properties of the functions underlying these statistical summaries. Throughout the text, we use the example of the humble barplot to ground our discussion.

Our goal is to convince you that, in order to produce coherent and efficient interactive graphics, we simply cannot treat graphics, statistics, and interaction as independent components. Instead we need to consider them holistically, within the context of their respective visual and mathematical properties. Understanding this deep web of interdependence may in turn empower us to innovate and build new kinds of robust interactive visualizations.

## 2 Visual Perception and Interactivity

### 2.1 Visual perception and the Gestalt Principles

Over the course of our evolutionary history, the human brain has evolved a sophisticated pipeline for processing visual information. This pipeline has been the subject of a great body of research in cognitive neuroscience (Goebel et al., 2004; Knudsen, 2020; for a brief overview, see Ware, 2019, 20). Many of the findings from this research are directly applicable to data visualization. For example, certain particularly salient visual stimuli are perceived automatically, rapidly (<10 ms), in parallel, and at a constant speed regardless of the number of distractors (Treisman, 1985). This phenomenon, known as pre-attentive processing, is key for data visualization, as the right use of visual attributes such as color, contrast, or shape can help us design figures which communicate insights much more effectively (Ware, 2019).

However, many of the findings from visual perception research are quite specific; to guide effective visualization, we also need some general, high-level rules. One such set

of high-level rules are the Gestalt principles of visual perception (see e.g. Cairo, 2012; Ware, 2019; Rosli and Cabrera, 2015; Vanderplas et al., 2020; Todorovic, 2008; Pinker, 1990). These principles formulate a set of fundamental “laws” related to the perception of patterns, primarily visual. They were discovered by the founders of the Gestalt school of psychology early in the 20th century, primarily by means of self-observation, and, despite not being grounded in sound understanding of the neural mechanisms of the brain, have nevertheless been later independently supported by findings from cognitive neuroscience (Guberman, 2017; Ware, 2019; Wertheimer, 1938).

Among the Gestalt principles, there are a few which are particularly relevant to data visualization. First, there are the two principles of proximity and similarity. These principles state that objects that appear in close together and/or similar in appearance tend to be perceived as part of single group. Second, the closure and common region principles state that regions with closed contours tend to be perceived as single, unified objects. Finally, the figure and ground principle states we tend to see one part of a scene as the central object (the “figure”) while the rest is perceived as the background (the “ground”). This can be modulated by other Gestalt principle: for example, smaller, darker, and symmetric objects with closed contours are more likely to be perceived as the figure (Ware, 2019).

Whether we intend them to or not, the Gestalt principles how we perceive data visualizations. As such, it is a good idea to try to use them to our advantage. Sometimes, little effort is needed. Such is the case, for example, when identifying clusters in a scatterplot (Rosli and Cabrera, 2015): the principles of proximity and similarity ensure that clouds of similarly shaped and colored points will be perceived as discrete groups by the observer. In other cases, however, our design choices can have a significant impact. For instance, another way to have objects to be perceived as part of a group is to draw a closed shape

around them (Cairo, 2012). Alternatively, if we want to highlight a particular part of a data set, we can leverage the figure and ground principle and use attributes such as color and saturation to make it stand out (for specific examples, see Franconeri et al., 2021; Wilke, 2019, 34).

Finally, ignoring or violating the Gestalt principles can make our graphics less legible or even misleading. Tufte (2001) had documented many such violations, such as deceptive 3D effects, inadvertent optical illusions caused by dense stripe patterns, and the overuse of superfluous graphical elements he termed “chartjunk”. Similarly, Cairo (2019) highlights many cases of graphics that violate the Gestalt principles. Finally, there is also the issue of well-designed graphics that communicate no statistical information at all, or even information that is blatantly false (Cairo, 2014; Franconeri et al., 2021; Cairo, 2019; Gelman and Unwin, 2013; Tufte, 2001). Either way, whether intentional or not, poor design can have a significant impact on the legibility and effectiveness of statistical graphics.

## 2.2 Gestalt and Interactivity

The Gestalt principles apply to both static and interactive visualizations, however, the way they apply is different. Static data visualizations can be designed well or poorly, but this design quality does not change over time. In contrast, interactive visualizations change over time, with user interaction. As such, an interactive visualization may conform to the Gestalt principles at one instance of time and violate them in another. In fact, interactive visualizations can even differ in how well they leverage design principles throughout time: the animations that interactions induce may themselves conform to good or poor design (see Hullman et al., 2013; Ware, 2019).

Thus, interactivity inherently places further constraints the design of figures. If we

want well-designed interactive figures, we need to ensure that they conform to good design principles across all states induced by user-interaction. This may not always be easy. We will investigate this idea in the following section, on the example of linked selection.

### 2.3 Case study: Linked Selection

Linked selection, also known as linked brushing or linked highlighting, has been consistently ranked as one of the most useful interactive features in data visualization (see e.g. Buja et al., 1996; Heer and Shneiderman, 2012; Wilhelm, 2003; Ware, 2019; Ward et al., 2015). By clicking or clicking-and-dragging over objects in one plot, users can highlight parts of objects corresponding to the selected cases across all other “linked” plots. This allows the user to rapidly “drill-down” (Dix and Ellis, 1998; Theus, 2002) and explore trends across different dynamically-generated subsets of the data (similar to interactive filtering, see e.g. Heer and Shneiderman, 2012). The ability to rapidly materialize different views of the data makes linked selection a particularly versatile tool for data exploration.

Yet, despite its versatility, linked selection also imposes some fundamental constraints on the visualization. Specifically, all objects we visualize need to be able to:

1. **Be selected**
2. **Represent selection**
3. (both in a visually and statistically sound way).

While these constraints may not seem too restrictive, we argue that they pose significant challenges for certain types of plots and visualization styles. Specifically, some kinds of plots may produce undesirable visual behavior when combined with linked selection. In fact, this may explain why stacking is preferred over dodging within the interactive data visualization

community. Let's explore this idea in more depth by comparing linked selection with a stacked vs. dodged barplot.

First, we propose that stacking is the superior method for *displaying* selection (see Figure 2). In a stacked barplot, each stacked bar presents itself as a single visual entity, thanks to the four Gestalt principles: proximity, similarity, closure, and common region. Since the stacked segments are placed right on top of each other, and share the same width and closed border, they are perceived as part of a unified whole (the bar). This remains true throughout selection - no matter how the heights of the highlighted segments within a stacked bar change, the height of the whole bar remains constant, and so does the overall outline of the plot.

Dodged barplots lack the degree of visual consistency found in stacked barplots. In a dodged barplot, the clusters of segments placed side-by-side lack a shared border (common region), leading to several design challenges. First, when selection occurs, the jagged contour of the dodged barplot can change dramatically as more cases are added or removed. This creates distracting visual clutter that makes it difficult to track changes over time. Second, selection may alter the height of the tallest bar segment in the plot. Since the height of the bar segments is not bound by the height of the tallest bar (as is the case in stacked barplot), this presents us with a dilemma. Either either we make the upper y-axis limit dynamic, meaning that the bar segments will always fit the plotting region, but then the context that the upper y-axis limit provides is lost with each time selection happens. Alternatively, we keep the upper y-axis limit static, but then we risk the bar segments growing outside of the bounds of the plotting region.

Stacked barplots offer several other advantages over dodged bars within the context of linked selection. First, we propose that stacking also provides a simpler mental model

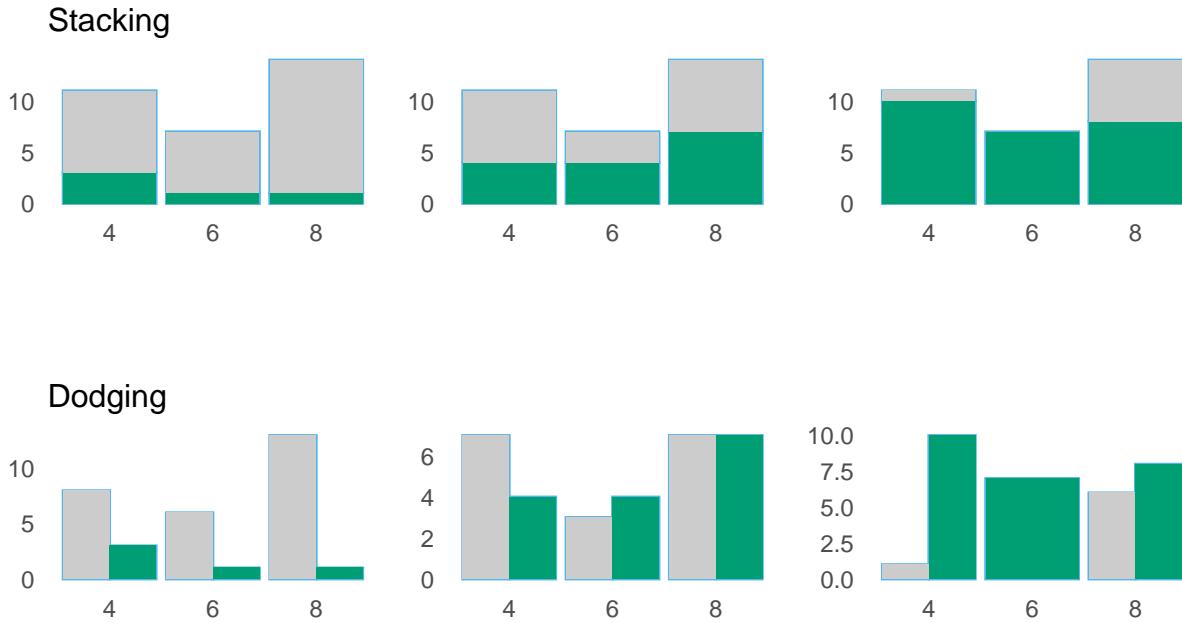


Figure 2: The visual coherence of stacking vs. dodging during linked selection on the `mtcars` data set (Henderson and Velleman, 1981). Bars show the numbers of cars in the 4-, 6-, and 8-cylinder categories. Dark segments represent selected cases, light segments represent unselected cases, and the plots' contour is shown via a light outline. Plots left to right show more cars being selected. Top row: the contour of a stacked barplot remains constant throughout selection (three bars of the same height). Bottom row: the contour of a dodged barplot changes dramatically with selection. Note the changes to the upper y-axis limit and the absence of the lighter segment among the 6-cylinder cars in the rightmost plot (since all 6-cylinder cars are selected, the darker segment has been stretched to fill the available space; `ggplot2` default for handling empty subcategories).

for *selection*. Since each stacked bar is a single visual entity, it makes intuitive sense that clicking it will select the entire category corresponding to it. Indeed, this is how many interactive data visualization systems implement linked selection: clicking on part of an object selects the category corresponding to the entire object (see e.g. Forbes et al., 2023; The Vega Project, 2022; Vanderplas et al., 2018; Xie et al., 2014; Theus, 2002; Urbanek and Theus, 2003; Urbanek, 2011; Swayne et al., 2003). In dodged barplots, on the other hand, the expectation is less clear: when we click on a bar segment in a dodged barplot, will we select the entire category or just the cases corresponding to the segment? Second, in a stacked barplot, the bar width and inter-bar gap are constant throughout selection. In a dodged barplot, it is not clear how to handle the gaps when some selection groups are empty. For example, when no cases are selected, should we leave gaps for the (currently absent) highlighted segments, or draw wide bars and only shrink these in response to selection? Finally, by presenting fewer visual entities on the screen, stacked bars may also reduce cognitive load (Sweller et al., 2019; for a non-technical overview, see Knafllic, 2015).

To summarize, stacked bars respond gracefully to selection. In contrast, dodging produces several undesirable behaviors when combined with selection: it produces jagged objects with non-uniform outline that changes dramatically with selection. These difference may explain why stacking is still popular in the interactive data visualization community.

However, a key question remains: when is stacking appropriate? So far, we have focused on the visual aspect of stacking. But stacking goes beyond aesthetics. This will be the subject of the next section.



Figure 3: An image is not the thing it represents. Left: The Treachery of Images by René Magritte (1929) makes the point that an image of a thing is different from the thing itself. Right: similarly, an image containing a few coloured rectangles is not a barplot, unless the rectangles accurately represent some underlying data.

### 3 Statistics

#### 3.1 From Data to Representation

While data visualizations are made up of geometric objects such as points, lines, or areas, these objects alone are not enough to constitute data visualizations. A coloured rectangle holds no meaning in and of itself; a canvas filled with such rectangles is just an image (see Figure 3). To turn an image of rectangles into a plot, we need to make sure that each rectangle accurately represents some underlying data.

However, we can rarely just “throw” data at the computer screen, in its raw form. Instead, we often first need to translate it into a format that the graphics device can understand. In most data visualization packages, this is done in two steps: aggregation and scaling.

First, the data needs to be aggregated or “wrangled” into sets of statistical summaries.

These summaries may be very simple: for instance, in a typical scatterplot, we assign each point the raw values of the x- and y-variables (the summary function is then just identity,  $f(x) = x$ ). More often, however, the data is grouped and aggregated in some way. For example, as was discussed previously, in a barplot, data points are grouped by the levels of the x-axis variable, and each group is then summarized by some descriptive statistic, such as count, sum, mean, or maximum. Similarly, in a typical histogram, the x-axis variable is divided into bins, and each bin is then summarized by the number of cases that fall inside it.

Second, the computed statistics have to be encoded into graphical attributes such as position, length, size, or colour (see e.g. Franconeri et al., 2021; Wilkinson, 2012; Pinker, 1990). This is the job of specialized functions known as scales or coordinate systems (see e.g. Murrell, 2005, 2007; Wickham, 2010, 2016; Wilkinson, 2012). For instance, in a typical scatterplot, the values of the x-axis variable are mapped to the x-position, such that points corresponding to small values of the x-axis variable are placed near the left-hand side of the plot (and larger values are placed near the right-hand side). These mappings may be subject to non-linear transformations, such as log or square-root, and may also apply to discrete values such as barplot categories. Scales and coordinate systems are a rich subject and we have barely scratched the surface here; interested reader is advised to see Wilkinson (2012), Ziemkiewicz and Kosara (2009), or Hotz et al. (2020).

## 3.2 Combining Statistics

What does stacking mean in the context of the statistics we are trying to represent? To a novice, stacking may appear as a purely graphical operation. If they attempt to implement a data visualization pipeline from scratch, they may leave stacking to the very end, after

the statistical summaries have already been translated into screen coordinates.

This view of stacking as a purely graphical operation has a certain appeal. Many modern data visualization systems follow the grammar-based model of visualization (McNutt, 2022; Kim et al., 2022; Vanderplas et al., 2020; Wickham, 2010; Satyanarayan et al., 2014, 2016; Wilkinson, 2012). This model emphasizes compositionality, allowing plots to be built from independent, modular components, which the users can combine freely. For example, in the popular `ggplot2` library (Wickham, 2010), plots are built by combining components such as geometric objects (called `geoms`) and statistical summaries (`stats`). These grammar-based approach offers a great deal of expressiveness, and has gained widespread popularity as a result.

Thus, it may be tempting to treat stacking just as another modular component of a visualization, such that we may be able to stack any geometric objects and statistics we like. Unfortunately, this approach is deeply flawed. Many data visualization experts have issued explicit warnings about this:

“Stacking is useful when the sum of the amounts represented by the individual stacked bars is in itself a meaningful amount” (Wilke, 2019, 52).

“Because this gives the visual impression of one element that is the sum of several others, it is very important that if the element’s size is used to display a statistic, then that statistic must be summable. Stacking bars that represent counts, sums, or percentages are fine, but a stacked bar chart where bars show average values is generally meaningless.” (Wills, 2011, 112).

“[...] We do this to ensure that aggregate statistics are always computed over the input data, and so users do not inadvertently compute e.g., averages of averages, which can easily lead to misinterpretation.” (Wu, 2022)

The message is clear: stacking some statistics such as counts, sums, and percentages is acceptable, but stacking others is not. As outlined by Wills, while the sum of grouped counts produces a valid overall count, the sum of group means does not yield a meaningful statistic. Even averaging group means, as noted by Wu (2022), does not produce a valid summary - the mean of group means differs from the grand mean (unless by coincidence). This problem is extends beyond barplots; for example, concerns about the statistical validity of stacked density plots have also been raised recently (Pu and Kay, 2020).

What exactly makes stacking sums acceptable but stacking other statistics such as means not? Wilke, Wills, and Wu et al. all argue that the aggregated value should be “meaningful”. In the case of sums, perhaps another way to phrase this is: “the whole is equal to the sum of its parts”. Or, more specifically:

“the *sum* of the *sums* on the parts is equal to the *sum* of all values”

But what if we replace word “sum” with some placeholder (such as “foo”):

“the *foo* of *foos* on the parts is equal to the *foo* of all values”

Then, we find there are other mathematical operators that also behave this way. For example, the product of products or the maximum of maximums are also valid overall statistics, see Figure 4. In fact, there is an entire class of mathematical objects with this exact behavior: monoids. We will explore monoids in the following section.

### 3.3 Monoids

Monoids are a simple yet powerful concept that plays a key role in several areas of mathematics, including abstract algebra and category theory. The short treatment here draws primarily from Fong and Spivak (2019), Lawvere and Schanuel (2009), Baez (2023), and

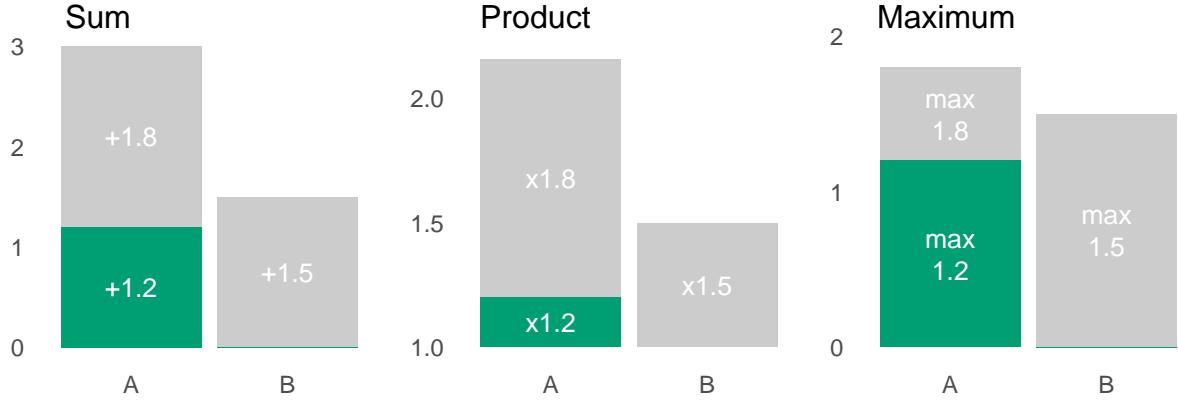


Figure 4: Different operators can be used for stacking. The values 1.2, 1.8, NA, and 1.5 were assigned to a 2-by-2 table and stacked using three different operators. Left: stacking using the sum operator. Middle: stacking using the product operator (note that the y-axis starts at 1). Right: stacking using the max operator.

Milewski (2018). For a particularly accessible introduction, readers are encouraged to seek out Fong and Spivak (2019) and Lawvere and Schanuel (2009).

While the term “monoid” may sound intimidating, the concept is really quite simple. A monoid represents the idea of a “whole equal to the sum of its parts”, if we relax our idea about what it means to “sum”. More formally, a monoid is a tuple  $(M, e, \otimes)$  consisting of:

- A collection (set) of objects  $M$
- A neutral element  $e \in M$  called the *monoidal unit*
- A binary operation  $\otimes : M \times M \rightarrow M$  called the *monoidal product*

Conforming to two rules:

- Unitality:*  $x \otimes e = e \otimes x = x$
- Associativity:*  $x \otimes (y \otimes z) = (x \otimes y) \otimes z = x \otimes y \otimes z$

(for all  $x, y, z \in M$ )

In plain words, when we have a monoid, we have some set of elements  $m \in M$  and a way to combine them  $\otimes$ , such that the order in which we do the combining does not matter. Further, among the set of elements  $M$ , we also have a special element  $e$ , called the “neutral” or “identity” element, that, when combined with any other element  $m \in M$ , does nothing (and simply yields back  $m$ ). This identity element is always unique (the proof is straightforward, for reference see e.g. Fong and Spivak, 2019; Lawvere and Schanuel, 2009).

Let’s return to the book example from Section 1. Is it a monoid? First of all, what is our set  $M$ ? This could be the set of all book stacks we can form by combining books in our bookshelf. Second, do we have a neutral element? If we are a bit creative, we can imagine an “empty” stack of zero books. Let’s use that as our identity element. Third, what is our binary operation? This is just the operation of taking one stack of books and putting it on top of another stack.

Now we just need to verify that our structure conforms to the two rules. First, is the operation associative? Clearly the answer is “yes”, since a stack of detective novels will be the same height regardless of whether we put the novels by Agatha Christie on top of Jo Nesbø novels or vice versa. Second, is the operation unital? Again, this is self-evidently true, since adding zero books on top of a stack does not change its height. Therefore, we do have a monoid.

We can also describe the book example a bit more formally. Specifically, summation on natural numbers  $(\mathbb{N}, 0, +)$  is a typical example of a monoid, because, as we have shown, it is associative and unital. Other examples of monoids include the previously mentioned products of real numbers  $(\mathbb{R}, 1, \times)$ , and the min and max operators  $(\mathbb{R}, \infty, \min)$  and  $(\mathbb{R}, -\infty, \max)$ . As a counterexample, exponentiation is not a monoid, since it is not associative:  $x^{(yz)} \neq (x^y)^z$ .

The definition of a monoid is quite broad however, and applies to more exotic structures than just operations on numbers. For example, the set of booleans  $\mathbb{B} = \{\text{True}, \text{False}\}$  equipped with either of the logical operators **AND** or **OR** is also a monoid, and so is the multiplication of  $n \times n$  square matrices. Likewise, the operation of concatenating vectors and computing the euclidean distance is also a monoid (Stepanov and McJones, 2009):

$$\|(\|(x, y)\|_2, z)\|_2 = \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2} = \|(x, y, z)\|_2$$

Even very “non-number-like” operations such as concatenation of strings can be a monoid:

`"hello" + "" = "" + "hello" = "hello"`

`("quick" + "brown") + "fox" = "quick" + ("brown" + "fox") = "quick brown fox"`

### 3.4 Groups

Groups are another fundamental concept in mathematics, representing the idea of symmetry and reversible transformations. While introducing two algebraic concepts in quick succession may seem like a big ask, a group is really just a monoid with the additional property. Specifically, for all  $x, y, z \in M$ , we have:

- 3. *Inverse operator*: There exists  $\otimes^{-1}$  such that if  $x \otimes y = z$  then  $z \otimes^{-1} x = y$

In simpler terms, a group embodies the idea that “the whole is equal to the sum of its parts, *in a non-destructive way*”. That is, when we combine two elements, we can always recover either one by “subtracting” the contribution of the other. Of note, like the identity element for monoids, the inverse operator in a group is also always unique (see e.g. Fong and Spivak, 2019; Lawvere and Schanuel, 2009).

Turning back to the book example, we can remove Agatha Christie novels off the detective novel stack and recover the height of the rest of the detective novels stack. Thus, summation on natural numbers is a group:  $2 + 3 = 5 \implies 5 - 2 = 3$  (technically speaking, we now have to identify the underlying set as integers  $\mathbb{Z}$ ). But, for example, the min and max operators lack an inverse and so do not constitute groups: if  $\max(x, 6) = 6$ , then there is no way to recover  $x$  from the combined result (6) alone.

### 3.5 Groups, Monoids, and Stacking

We now arrive at the core idea of the present paper. Suppose that we have some data set  $D$ , a way of summarizing this data via some associative binary operation  $\otimes$ , and a neutral value with respect to this operation  $e$ . In other words,  $(D, \otimes, e)$  forms a monoid. Define  $F(A)$  as the result of summarizing or “folding” some subset  $A \subseteq D$  using the binary operation:

$$F(A) = a_1 \otimes a_2 \otimes a_3 \dots \otimes a_n$$

Note that we could have written this as  $F(A) = (a_1 \otimes (a_2 \otimes (a_3 \otimes (\dots \otimes a_n))))$ , however, since the operation is associative, we do not have to worry about brackets. Also, if  $A$  is empty, then clearly:

$$F(\emptyset) = e$$

Now comes the most important idea: combining summaries of two disjoint subsets of the data set gives us the same result as summarizing the union of the original subsets:

$$\begin{aligned}
F(A) \otimes F(B) &= (a_1 \otimes a_2 \otimes \dots \otimes a_n) \otimes (b_1 \otimes b_2 \otimes \dots \otimes b_n) \\
&= a_1 \otimes a_2 \otimes \dots \otimes a_n \otimes b_1 \otimes b_2 \otimes \dots \otimes b_n \quad (\text{by associativity}) \\
&= F(A \cup B)
\end{aligned}$$

Another way to show the same idea is the following commutative diagram:

$$\begin{array}{ccccc}
& & A & & \\
& \swarrow -\cup B & \downarrow & \searrow -\cup(B \cup C) & \\
A \cup B & \xrightarrow{-\cup C} & & \xrightarrow{\quad} & A \cup B \cup C \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
& & F(A) & & \\
& \swarrow -\otimes F(B) & \searrow -\otimes F(B \cup C) & & \\
F(A \cup B) & \xrightarrow{-\otimes F(C)} & & \xrightarrow{\quad} & F(A \cup B \cup C)
\end{array}$$

The top and bottom triangles in the diagram both indicate associativity: in the top triangle, we can either first take the union of  $A$  and  $B$  and then union that with  $C$ , or immediately union  $A$  with  $B \cup C$  and the result will be identical. The same applies for the monoidal summary in the bottom triangle. By saying that the diagram commutes, we mean that any two parallel paths (the same start and endpoint) always produce the same result. Practically, this means we can either first summarize each set and then combine the summaries, or first union the sets and then summarize the union, or do some mix of the two, and we will always end up with a valid summary. As a sidenote, the diagram above actually corresponds to something called a functor or monoid homomorphism, however, that is beyond the scope of the present paper (interested reader should see Fong and Spivak, 2019; Lawvere and Schanuel, 2009).

But this is precisely the idea of a stackable statistic. If our statistic conforms to the diagram, then the combined summary of subsets of the data is always a valid summary of the whole data set. In Wilke's words, we have discovered what makes the combined

summary a “meaningful amount”!

It might seem that we are done. However, there is still one subtle issue left. Suppose we summarize three parts of the data  $A$ ,  $B$ , and  $C$ , and combine these summaries by repeatedly applying the operation, as a kind of running total  $F^*$ :

$$\begin{aligned} F^*(F(A), F(B), F(C)) &= (F(A), F(A) \otimes F(B), F(A) \otimes F(B) \otimes F(C)) \\ &= (F(A), F(A \cup B), F(A \cup B \cup C)) \end{aligned}$$

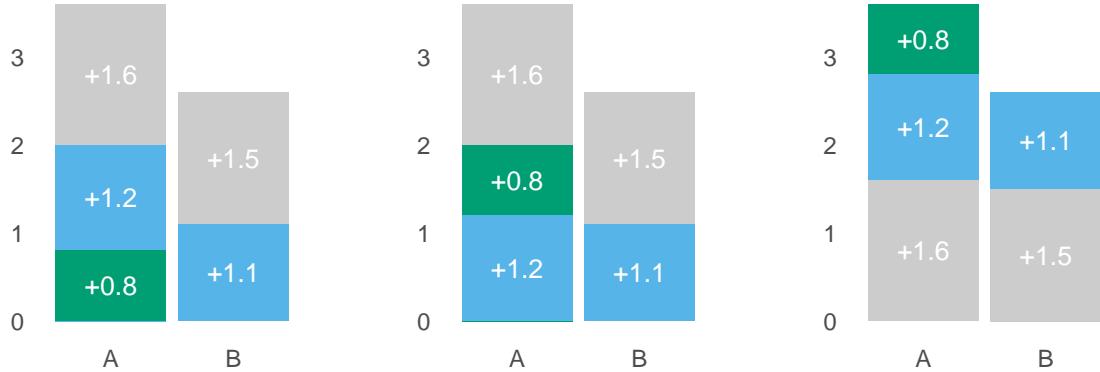
Notice that we end up with summaries on nested subsets of the data, i.e.  $F(A)$ ,  $F(A \cup B)$ , and  $F(A \cup B \cup C)$ . This is fine, for example, when we do linked selection with a single group and compare the summary on the highlighted subset ( $A$ ) vs. the entirety the data ( $A \cup B$ ).

However, what if we want to compare the subsets  $A$  and  $B$  directly? For example, suppose we do linked selection and with two highlighted groups ( $A$  and  $B$ ). Unfortunately, the properties of monoids do not guarantee that we will be able to recover  $F(B)$  from  $F(A \cup B)$  using  $F(A)$ . Put plainly, the monoidal operation may “collapse” the information contained in the disjoint parts, see Figure 5. To ensure that  $F(B)$  is recoverable from  $F(A \cup B)$ , we also need the inverse operator  $\otimes^{-1}$ :

$$F(B) = F(A \cup B) \otimes^{-1} F(A)$$

Importantly,  $F(A) \otimes F(B)$  is still a valid summary of  $A \cup B$  even when the inverse does not exist. It is just that  $F(A) \otimes F(B)$  may “forget” the information contained in either of its two parts. Note also that this is not a problem of different permutations of parts producing different sequences of accumulated summaries - this will happen regardless of whether or not there is an inverse.

**Sum**



**Maximum**

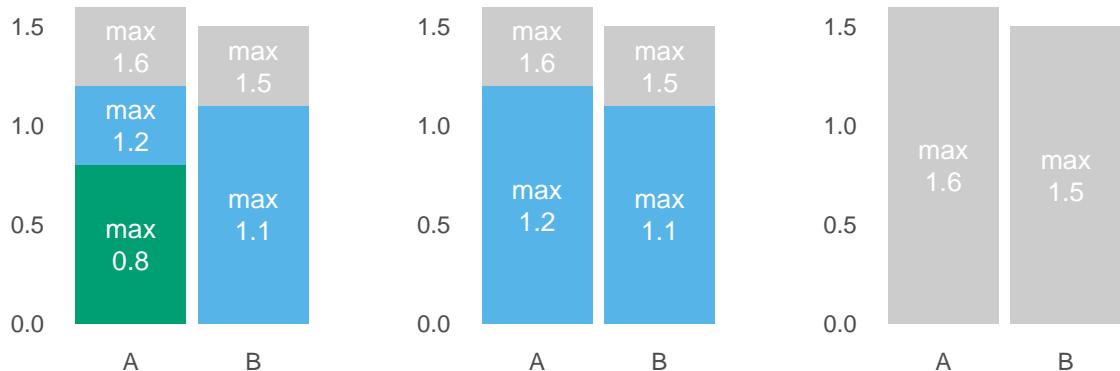


Figure 5: Monoids can “collapse” information about disjoint parts. Top: since sum is a group with an inverse (minus), we can recover the value of each disjoint part by subtracting the heights of the preceding segments (i.e. the part order does not matter). Bottom: since maximum lacks an inverse, we may lose information about the individual parts under certain permutations (note the absence of segments in the middle and rightmost plot).

To summarize, whether our statistic is a monoid or a group determines what kinds of comparisons we can make on stacked objects:

- Monoids: comparing nested subsets, part vs. whole comparisons
- Groups: comparing disjoint subsets, part vs. part comparisons

This becomes important when implementing interactive data visualization features such as linked selection. Specifically, if we want to compare a single selected group vs. the rest of the data, then it's enough for our summary statistic to be a monoid. If we want to implement multiple selection groups, then our statistic needs to have an inverse as well. If we fail either of these conditions, then we may produce data representations which are either inaccurate (in the case of a non-monoidal statistic), or accurate but not “full” (in the case of a monoid but no inverse).

### 3.6 Groups, Monoids, and Graphics

The connection between summary statistics and algebraic structures is important because it allows us to predict which new combinations of summaries and figures will “work”. For example, while not generally thought of as a summary statistic, the convex hull algorithm (see e.g. Barber et al., 1996) provides a way to reduce a set of points to a simpler “summary”, see Figure 6. We can recast the operation of finding the convex hull as a monoid: starting with an empty set, we iterate through the data, adding points if they lie outside of the current hull (and removing any points in the interior of the new hull). This may not be the most computationally efficient method for finding convex hulls, however, since it does meet the definition of a monoid, this tells us *what convex hulls can be used for*. If we combine two convex hulls, we know that the new hull is a valid summary for the union of the underlying data points. In turn, this allows us to implement interactive features such

as single-group highlighting/selection, safe in the knowledge that the underlying structure is preserved.

However, does the operation of adding a point to a convex hull have an inverse? No. Once we add a point to the hull, we cannot recover the old hull by “subtracting” that point - we do not know how many points from the interior of the old hull we removed. This tells us that merging two convex hulls may irrevocably collapse information. As such, we cannot use convex hulls to display multiple selections, without resorting to some ad-hoc methods (e.g. keeping separate representation of the hulls and using partial transparency).

### 3.7 Groups, Monoids, and the Neutral Element

There are also other advantages that groups and monoids offer for visualizing data. One of these is the existence of the neutral element  $e$  (see Section 3.3). While seemingly inconspicuous, the neutral element can actually be quite useful.

Specifically, the neutral element ensures that we can always represent empty subsets of our data. Turning once more to the book example from Section 1, suppose we split the books into stacks based on the book’s genre and the author’s gender, and then summarize each stack by its total and average word-count. What if there are no books by male sci-fi authors? For total word-count, we have a meaningful default value - zero - as in, there are zero words across the zero books by male sci-fi authors. However, what about the average word-count? Here we run into trouble, since the mean of an empty set is not defined. It simply does not make sense to talk about the “average word-count across zero books”.

A common approach is to omit the objects representing empty subsets. However, with non-monoidal summaries, this approach is flawed and leads ambiguity. Suppose we are drawing the bars of averages. If we omit bars representing the empty categories, then the

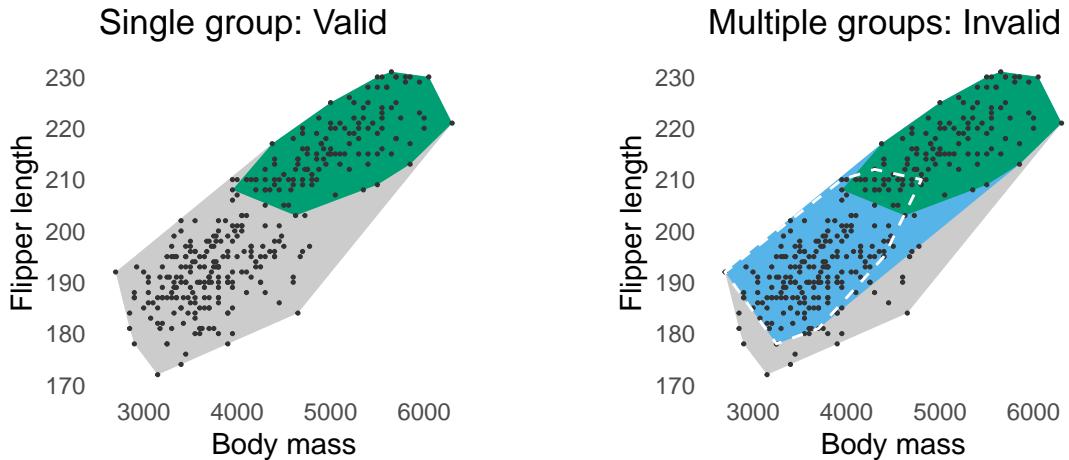


Figure 6: Convex hull can be used as a valid representation of single-group selection but not multi-group selection. Both plots show data from Palmer penguins data set (Horst 2020). The light polygon shows the convex hull of the whole dataset, the dark polygon shows the hull of the Gentoo species, the medium-brightness polygon shows the combined hull of Gentoo and Chinstrap species, and the white dashed line shows the outline the hull of the Chinstrap penguins alone. Points show individual cases. Left: since convex hull is a monoid, we can use it to show single group selection/highlighting. Right: because of the lack of an inverse, we cannot guarantee that hull of the Chinstrap penguins (white-dashed line) will be recoverable from the combined hull of Gentoo and Chinstrap - we do not know how many interior points were “forgotten”.

missingness of a bar can indicate either of two things: either there are *no* cases in the category, or there *are* cases and their average is equal to the lower y-axis limit. There is just no way to tell from the graphic alone. In contrast, with monoidal summaries such as sums, the message of an object's absence is unambiguous. Regardless of whether there are zero or hundred cases in a category, if the bar representing them is missing, this means that their sum is zero.

The neutral element can also provide a new perspective on a perennial debate in data visualization: should the base of a barplot always start at zero? Some authors argue that starting bars from a non-zero base makes the lengths of the bars meaningless (Cleveland, 1985). However, Wilkinson (2012), for example, suggests that the bar base is really a statement about scales rather than about graphics, and as such there is no compulsion for starting it at zero. We offer the following new perspective: often, it might make sense for the base of the barplot to start at  $e$ , the neutral element, since this represents the default state of doing nothing. In other words, the bar base is tied to the properties of the *statistic*, rather than the just the graphic or scale.

In a barplot of sums or counts, this approach aligns with the traditional prescription, since the neutral element is zero. However, in a barplot of products, the neutral element is one. Thus, we may should our bars at one. One might object that this invalidates the y-axis as a ratio scale (Stevens, 1946), since a bar representing the value 4 will be no longer twice as tall as the bar representing the value 2. And indeed, if we start the base at one, the bar representing the value 2 will be only  $2 - 1 = 1$  units tall, whereas the bar representing the value 4 will be  $4 - 1 = 3$  units tall. However, we need to think about what the values represent. The bar representing the value 2 shows a 200% change or 100% increase; the bar representing 4 shows 400% change or 300% increase. Therefore, it makes sense that

the bar representing 4 is three times taller. Also, this scale still fits the Stevens (1946) definition of a ratio scale, since we can still determine equality of ratios: 1.5 is to 2 (100% / 50% = 2, double the increase) as 2 is to 3 (200% / 100% = 2, double the increase).

This example provides further evidence to support the claim that we cannot separate statistics from other data visualization components such as graphics and scales. Even before we apply any transformations, the method we have used to derive our summary statistics determines important properties of our visualization. We need to be mindful of this.

### 3.8 Groups, Monoids, and Computation

Groups and monoids also come with some inherent computational advantages. Specifically, if a summary statistic is a monoid, associativity guarantees that it can be computed in a single pass through the data (i.e.  $O(n)$  complexity), and in a distributed/parallel fashion (see e.g. Lämmel, 2008; Lin, 2013). This is why monoids are popular in functional programming, where they are used alongside the so-called `fold` or `reduce` functions (see e.g. Milewski, 2018; Abelson and Sussman, 2022). To be fair, computational efficiency is usually not the primary concern in data visualization: most data visualization systems are designed to work in a single-threaded environment, and rely on summary statistics that are fairly cheap to compute. Nevertheless, the guarantee of inherent efficiency can still be useful when visualizing larger data sets.

The computational advantages of groups and monoids become significant in interactive data visualization, however. Specifically, when designing an interactive data visualization, we may be able to exploit the fact that summaries on fine-grained subsets of the data “add up” to the summaries on coarse-grained subsets (their unions) to avoid unnecessary com-

putation. Consider a linked histogram with reactive axes and interactive manipulation of binwidth and anchor. When the user engages in selection, the counts within the individual highlighted segments may change but the counts across the whole stacked bars will not. Thus, there is no need to recompute and update the upper y-axis limit - we know this will stay constant throughout selection. However, if the user interactively changes the binwidth or anchor, then both the counts within the whole bars *and* the highlighted segments may change and so we need to update the graphic appropriately.

In other words, associativity allows us to implement strategic updating. Fine-grained summaries (individual segments) need to be updated when the more coarse-grained summaries (whole stacked bars) change, but not vice versa. By pre-computing summaries on different levels of aggregation, we may be able to reduce computational effort.

In fact, if our summary statistic has an inverse, we can be even more computationally efficient. Suppose that one data point out of a thousand is removed from selection. If our summary has an inverse, we can use it to subtract the contribution of that single data point, instead of having to recompute the whole summary from scratch. This is, in fact, the principle behind the fast updates in the popular JavaScript library Crossfilter, which is unfortunately no longer actively maintained (Crossfilter Organization, 2023; see also the Github discussion initiated by Monfera, 2017)<sup>1</sup>.

However, while the inverse does bring advantages, it is also the case that fewer summary statistics have it. For instance, as was mentioned before, while maximum operator is a valid monoid, it lacks an inverse and as such does not meet the definition of a group. Once we summarize some data with the maximum, we cannot retrieve the second or third highest value - that information is gone forever. And there are many other operations

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<sup>1</sup>Note that, as the present date, we are not aware of any explicit discussion of monoids or groups in the Crossfilter documentation

(such as the convex hull in Figure 6) which lack an inverse. As such, when designing and interactive data visualization library, we need to carefully weigh the ability to support different kinds of summary statistics (general, monoids, groups) versus supporting different kinds of interactions (e.g. single-group selection vs. multi-group selection).

## 4 Conclusion

We hope we have demonstrated that, when designing interactive graphics, algebraic thinking is invaluable. To support specific interactive features, the statistics underlying our plots must possess appropriate algebraic properties. For instance, if we want to display the result of splitting our data into parts and combining these parts back together, then we need either of two classes of mathematical structures. If we want to compare nested subsets of our data (for example, via single-group linked selection) then our summaries need to be monoids. If we want to compare disjoint parts of our data (multi-group linked selection), then our summaries need to be (algebraic) groups. The properties of these structures ensure that our figures are statistically sound, computationally efficient, and adhere to good design principles.

We have focused on unitality, associativity, and inverses, however, it may be useful to consider other algebraic properties as well. For instance, many of the visual attributes that we use to represent our data can only *grow* in one direction. Such is the case with, for example, width, height, and area (there is no such thing as “negative area”). If the visual attribute can only grow in one direction, then our summary statistic should be *monotonic*, such that if  $x_1 \leq y_1$  and  $x_2 \leq y_2$  then  $x_1 \otimes x_2 \leq y_1 \otimes y_2$ . Similarly, in many types of plots, the order of the data points does not matter, however, in some, such as time-series plots, it does. Thus, it may be important to determine whether or not the statistics

underlying our plots are *commutative*, such that  $x \otimes y = y \otimes x$ . Further exploration of these properties in relation to interactive graphics may yield useful insights, however, that is outside of the scope of the present paper.

We do not advocate for exclusively visualizing groups and monoids. Rather, we want to highlight the trade-off present in the choice of the summary statistic. Few summary statistics meet the definition of a group, but, if we choose such a statistic, then we can do a lot of things with it. With a monoid, our options are more limited, and with a non-monoidal statistic, even more so. Thus, like many past authors (such as Tufte, 2001; Tukey et al., 1977; Wilkinson, 2012), we simply want to point to the importance of thinking critically about the mathematical structure underlying our graphics. Geometric objects, statistics, and interaction are all deeply connected, and a full elaboration on this web of interdependence may require a lot more work. Groups and monoids merely allow us to carve out a small space of well-behaved graphics.

On a more practical note, we also need a framework for efficiently transforming raw data into summary statistics that can be displayed by interactive graphics. A decade ago, Wickham et al. (2009) noted the lack of such a formal pipeline (or “plumbing”), and, despite some attempts at implementing such pipelines (see e.g. Lawrence et al., 2009), that same sentiment was echoed again recently (Vanderplas et al., 2020). Ideally, such a framework would minimize recomputation during user interaction. The computational “shortcuts” that groups and monoids provide may be invaluable here. We envision a visualization pipeline structured as a hierarchy of reactive partitions, where only the downstream (finer) partitions need to be recomputed when the upstream (coarser) partitions change, but not vice versa. We are currently exploring this concept in an R package called **plotscaper**<sup>2</sup> (the project still has far to go to be truly generic).

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<sup>2</sup><https://github.com/bartonicek/plotscaper>

Data visualization, in practice, is rarely about mathematical purity. Real world data is often flawed or incomplete, and it takes cunning and experience to dredge up valuable insights from it. Nevertheless, we believe that simple algebraic thinking offers a valuable foundation. Concepts like monoids and groups provide a rigorous framework for reasoning about graphics, interactive or otherwise. Ultimately, these concepts may empower us to build a broader class of robust interactive data visualizations.

## 5 Disclosure Statement

The authors report there are no competing interests to declare.

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