$$\mathrm{Fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) & \text{otherwise} \end{cases}$$

By induction prove that:

$$\mathrm{Fib}(n) = \frac{(\varphi^n - \psi^n)}{\sqrt{5}}$$

1 Base case

For $n = 0$

$$\mathrm{Fib}(0) = 0$$
$$\frac{\varphi^0 - \psi^0}{\sqrt{5}} = 0$$

For $n = 1$

$$\mathrm{Fib}(1) = 1$$
$$\begin{aligned} \frac{\varphi^1 - \psi^1}{\sqrt{5}} &= \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} \\ &= \frac{2\sqrt{5}}{2\sqrt{5}} \\ &= 1 \end{aligned}$$

For $n = 2$

$$\mathrm{Fib}(2) = 1$$
$$\begin{aligned} \frac{\varphi^2 - \psi^2}{\sqrt{5}} &= \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} \\ &= \frac{1 + 2\sqrt{5} + \sqrt{5}^2 - (1 - 2\sqrt{5} + \sqrt{5}^2)}{4\sqrt{5}} \\ &= \frac{2\sqrt{5} + 2\sqrt{5}}{4\sqrt{5}} \\ &= \frac{4\sqrt{5}}{4\sqrt{5}} \\ &= 1 \end{aligned}$$

True.

2 Inductive step:

Assume

$$\mathrm{Fib}(n) = \frac{(\varphi^n - \psi^n)}{\sqrt{5}}$$

$$\mathrm{Fib}(n-1) = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$$

Is it true that:

$$\mathrm{Fib}(n-1) = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}$$

$$\begin{aligned}
\mathrm{Fib}(n+1) &= \frac{(\varphi^n - \psi^n)}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\
&= \frac{\varphi^n - \psi^n + \varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\
&= \frac{\varphi^n + \varphi^{n-1} - \psi^n - \psi^{n-1}}{\sqrt{5}} \\
&= \frac{\varphi^{n-1}(\varphi + 1) - (\psi^{n-1}(\psi + 1))}{\sqrt{5}}
\end{aligned}$$

We would like $\varphi + 1 = \varphi^2$, is it so?

$$\begin{aligned}
\varphi + 1 &= \frac{1+\sqrt{5}}{2} + 1 \\
&= \frac{2(1+\sqrt{5})}{4} + \frac{4}{4} \\
&= \frac{2 + 2\sqrt{5} + 4}{4} \\
&= \frac{1 + 1 + 2\sqrt{5} + 4}{4} \\
&= \frac{1 + 2\sqrt{5} + 5}{4} \\
&= \frac{(1+\sqrt{5})^2}{2^2} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^2 \\
&= \varphi^2
\end{aligned}$$

By analogy we see that $\psi + 1$ will be equal to ψ^2.

$$
\begin{aligned}
\mathrm{Fib}(n+1) &= \frac{\varphi^{n-1}(\varphi+1) - (\psi^{n-1}(\psi+1))}{\sqrt{5}} \\
&= \frac{\varphi^{n-1}\varphi^2 - \psi^{n-1}\psi^2}{\sqrt{5}} \\
&= \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}
\end{aligned}
$$

QED

3 Proove that $\mathrm{Fib}(n)$ is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$

We know that:

$$\mathrm{Fib}(n) = \frac{\varphi^n + \psi^n}{\sqrt{5}}$$

We need to prove that the distance between $\mathrm{Fib}(n)$ and $\frac{\varphi^n}{\sqrt{5}}$ is less or equal to $\frac{1}{2}$ (if it's more then there exists an integer that s closer).

$$
\begin{aligned}
\left|\mathrm{Fib}(n) - \frac{\varphi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\left|\frac{\varphi^n + \psi^n}{\sqrt{5}} - \frac{\varphi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
\left|\frac{\psi^n}{\sqrt{5}}\right| &\leq \frac{1}{2} \\
|\psi^n| &\leq \frac{\sqrt{5}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sqrt{5}}{2} &\approx 1.12 > 1 \\
\left|\frac{1-\sqrt{5}}{2}\right| &\approx 0.62 < 1
\end{aligned}
$$

From exponentation we know what if $x < 1$ then $x^n < 1 \; \forall n > 0$

$$|\phi^n| < \frac{\sqrt{5}}{2}$$

Is true.