Social Networks & Recommendation Systems

III. Real networks properties and their visualization.

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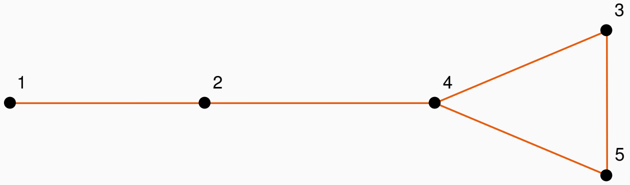
Before classes

Vertex degree

Number of edges connected to vertex

$$k = \{1, 2, 2, 3, 2\},$$



Rememeber: fat tailed distributions

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- Is it possible for the probability distribution to do not have expected value? What with variance?
- What are examples of such distributions?
- What do we know about the support of the distribution with infinite moments?
- What is the interpretation of mean from the sample for distributions without expected value? To what value does it converge as the sample size increases?

Lecture

What real complex networks are?

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Examples?

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Be careful!

The above observations are empirical, not mathematical. It is not that **each** complex network has **each** of the above features.

What are fat tails?

Probability reminder:

$$\mathbb{E}X^p = \int_{-\infty}^{\infty} x^p f(x) dx = \infty^\star$$

analogously for discrete distributions

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The necessary condition for the divergence of integrals is the infinity support of the density function (probability mass function).

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What with real networks?

Finite or infinite?

A digression of mathematical precision

How to deal with this problem?

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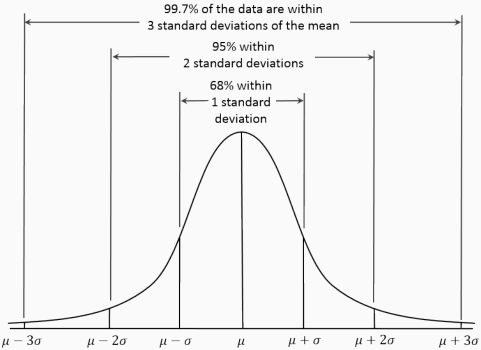
We will be more mathematically precise in Lecture 7.

What are results of the fat tails?



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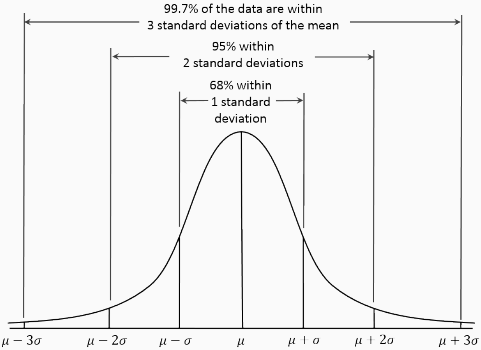

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Pareto Rule (80/20)

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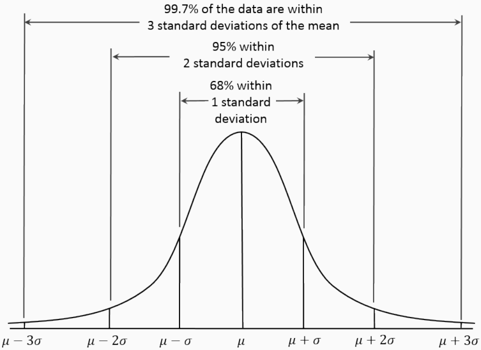

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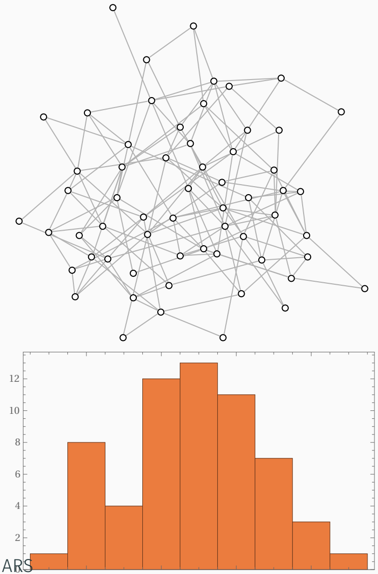
Pareto Rule (80/20)

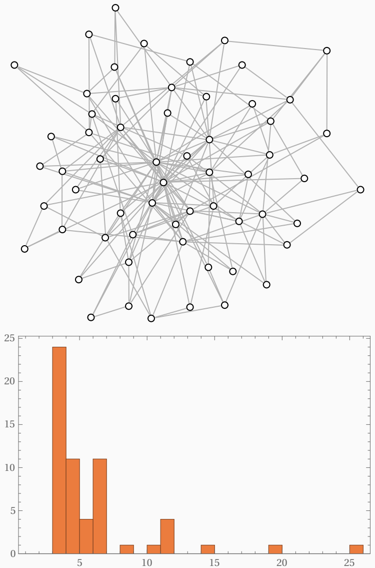
How much of the resources belong to what proportion of the population?

Does it holds for any distribution?

We will check during projects.

Empirical justification





Why power law networks are scale free?

Mathematical justification

Fat-tailed distributions are those for which there is no expected value or one of the higher moments.

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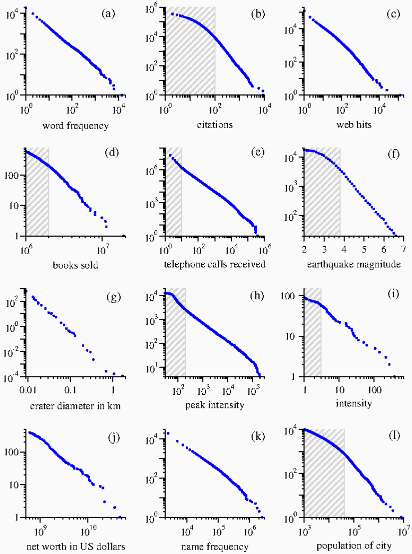
So we have no scale!

How to interpret this in a finite case?

Networks with power law distributions



M.E.J. Newman, *Power laws, Pareto distributions and Zipf's law*, Contemporary Physics, **46**, 323-351 (2005).

Matthew Effect

For to everyone who has will be given, and he will have more: but from him who has not, even what he has will be taken away.

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M. Perc, J. R. Soc. Interface, **11**, (2014).

Is the distribution sufficient?

Question:

Prove or find a counterexample: whether the vertex degree distribution unambiguously characterize a network or a graph.

How can networks with the same distribution differ?

Assortativity vs disassortativity

$$\mathcal{P}(k_i|k_j) = \frac{\mathcal{P}(k_i, k_j)}{k_j \mathcal{P}(k_j)/\langle k \rangle}$$

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- A disassortative network is one where the probability of connecting nodes of very different degrees is high.
- An assortative network is one where the probability of connecting nodes of a similar degree is high.

Question:

What are real examples of such networks?

How can networks with the same distribution differ?

Correlations

$\mathcal{P}(k_i, k_j)$ - probability that randomly chosen edge connectrs vertices of degrees k_i i k_j

$$\mathcal{R}(k_i, k_j) = \frac{\mathcal{P}(k_i, k_j)}{\mathcal{P}_u(k_i, k_j)},$$

and P_u is for uncorrelated network with the same distributon.

Derivation of $\mathcal{P}_u$

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- $\mathcal{P}_u(k_i, k_j) = \frac{k_i k_j \mathcal{P}(k_i)\mathcal{P}(k_j)}{\langle k \rangle^2}$

Other properties of real networks – small worlds

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How to measure the smallness of the world?

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What is the average path length in the selected network models?

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- for scale free networks with $\alpha = 3$ $\ell \sim \frac{\ln N}{\ln \ln N}$,
- for scale free networks with $\alpha \in (2,3)$ $\ell \sim \ln \ln N$.

Other properties of real networks

- fractal distribution networks,

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- fractal distribution networks,
- hierarchical networks,
- networks with community structure.

Question:
What are examples of such networks?

Graph visualization methods

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- appropriate distance between vertices,

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Comment:

Optimization of the above features (especially the last one!) Is algorithmically demanding.

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Conclusion:

Visualization is largely an art ... or the use of ready-made tools.

For R users:

`http://kateto.net/network-visualization`

Summary

Homework

Read

- M.E.J. Newman, *Power laws, Pareto distributions and Zipf's law*, Contemporary Physics, **46**, 323-351 (2005).
  and/or
- chapter 3.1-3.3 in A. Fronczak, P. Fronczak, *Świat sieci złożonych* PWN (2009).

Thank you for your attention!

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