Social Networks & Recommendation Systems

VII. Probabilistic aspects of the complex networks.

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MSc program in Data Science has been developed
as a part of task 10 of the project
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Before classes

Reminder

From SNARS_5:

- Properties of the ER graphs.

From SNARS_6:

- Mean-field approach to the BA model.

From other courses:

- generating functions approach in the combinatorics.

To think about:

- What do you think, which of the graphs: Erdős-Rényi or Barábasi-Albert is more vulnerable for intetional attacks and random failures? Why?

Lecture

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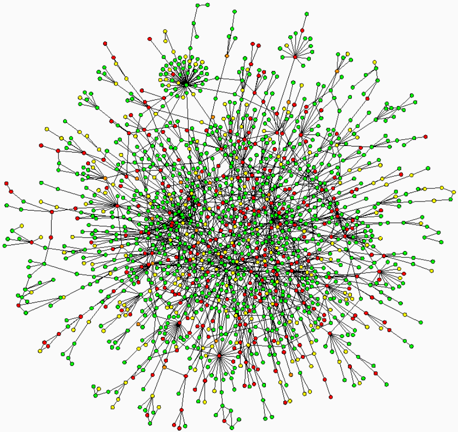
Conclusion:
I stronlgy recommend reading Durret to those who are dissatisfied!



https://services.math.duke.edu/~rtd/RGD/RGD.html

Let's have a look to chapter 4.1.

Master equation

The equation describing the changes in the probability distribution over time

$$\frac{d\mathcal{P}_i}{dt} = \sum_j \mathcal{P}_j T_{j\to i} - \sum_j \mathcal{P}_j T_{i\to j},$$

which in the discrete version takes the following form

$$\mathcal{P}_i(t+1) - \mathcal{P}_i(t) = \sum_j \mathcal{P}_j(t) T_{j\to i} - \sum_j \mathcal{P}_j(t) T_{i\to j},$$

and this is what we will focus on.

Master equation for BA networks

We will follow

- chapter 4.1 in Durret's book,
- S.N. Dorogovstev, J.F.F. Mendes i A.N Samukhin, *Structure of growing networks with preferential linking*, Phys. Rev. Letters. 85, 4633–4636 (2000).

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$$N_k(t+1) - N_k(t) = \frac{m(k-1)}{2mt} N_{k-1}(t) - \frac{mk}{2mt} N_k(t) + \delta_{km}$$

$$N_{m-1}(t) = 0.$$

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Question:

Can you justify the components of this equation?

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Interpretation:

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Question:

Is this solution mathematically exact?

Master equation for BA networks

Inaccuracies in the master equation approach:

- adding an edge does not change preferences (synchronous dynamics, mean-field theory reminiscence), Inaccuracies in the master equation approach:

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Why?

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Why?

Precision vs. simplicity of the method?

i.e. mathematicians vs. physicists...

Master equation for BA networks

$$
\begin{aligned}
N_k(t+1) - N_k(t) &= \frac{k-1}{2t} N_{k-1}(t) - \frac{k}{2t} N_k(t) + \delta_{km} \\
N_{m-1}(t) &= 0.
\end{aligned}
$$

How to solve it?

`goto` Project;

Master equation for BA networks – solution

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Solution:

$$\mathcal{P}(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

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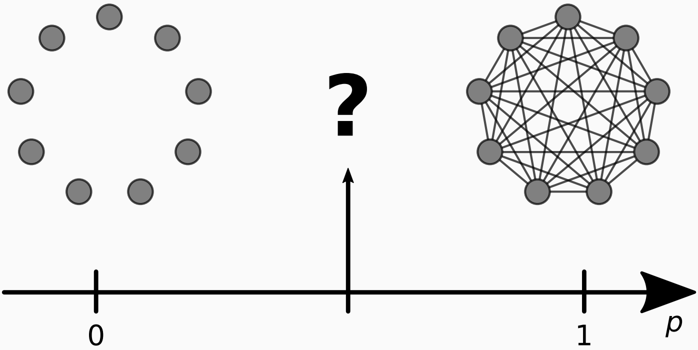
Question:

Argue that for $k \gg 1$ above results agrees with mean-field approach.

Percolation – ER graphs

Let us now return to the ER graphs



What is happening in the middle? We will follow sec. 4.3.2 in Fronczak and Fronczak book.

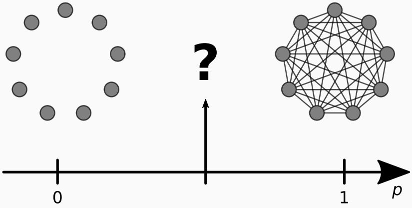
Percolation – ER graphs

What is percolation?

wikipedia

Mathematical model of liquid leakage through porous material.

How does this relate to graphs?



We ask when the graf is percolated

i.e. when percolation claster's size is equal to $N^* \propto N$.

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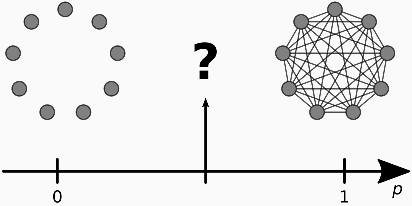

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Attention!

We will approach in the physicist way. For more detailed approaches (sic!) see chapter 2 in Durret's book.

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During Project we will prove that percolation threshold can be defined as

$$\sum_k k\mathcal{Q}(k) \geqslant 2,$$

which is equivalent to $\langle k \rangle_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle} = 2$.

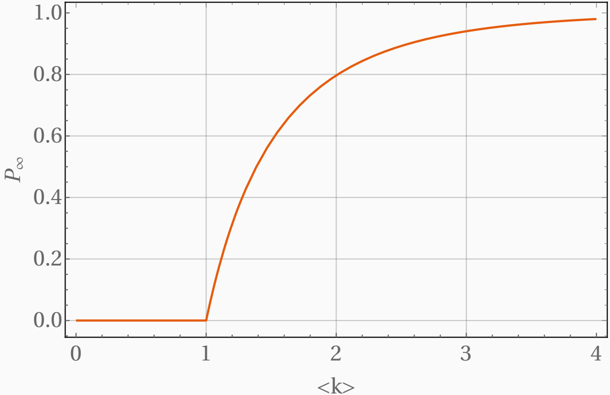
Percolation threshold for ER graphs

For ER graphs we obtain

$$\langle k \rangle = 1,$$

which means

$$p_c = \frac{1}{N}.$$



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