Some linear algebra

Definition 1. A set of vectors β is a **basis** for a K-vector space V if

(a) The elements of β are linearly independent, meaning

$$\sum_i a_i \vec{b}_i = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0$$

where $|\beta| = n$ and $a_i \in K$

(b) V is spanned by β, meaning for all $\vec{v} \in V$

$$\vec{v} = \sum_i a_i \vec{b}_i$$

for some $a_i \in K$.

Theorem 1. (Basis and minimal spanning set) Given a vector space V, if a spanning set S is minimal with respect to the number of elements, then S is a basis for V.

Proof We must show that the vectors in a minimal spanning set are linearly independent. Seeking a contradiction, suppose they are not linearly independent. In this case

$$a_1 \vec{s}_1 + \cdots + a_n \vec{s}_n = 0$$

Without loss of generality (WLOG), say $a_n \neq 0$. Now we may write

$$a_1 \vec{s}_1 + \cdots + a_{n-1} \vec{s}_{n-1} = -a_n \vec{s}_n$$

so

$$-\frac{a_1}{a_n} \vec{s}_1 - \cdots - \frac{a_{n-1}}{a_n} \vec{s}_{n-1} = \vec{s}_n$$

Hence, $\vec{s}_1, \ldots, \vec{s}_{n-1}$ spans V, and this is a contradiction as S was assumed to be minimal. $\blacksquare$

Learning outcomes:
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