Some linear algebra

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(We'll prove that a basis is a minimal spanning set)

Definition A set of vectors $\mathcal{B}$ is a basis for a K-vector space V if

① The elements of $\mathcal{B}$ are linearly independent meaning

$$\sum_{\vec{b}_i \in \mathcal{B}} a_i \vec{b}_i = 0 \Rightarrow a_1 = a_2 = \cdots = a_n$$

where $|\mathcal{B}| = n$ and $a_i \in K$

② V is spanned by $\mathcal{B}$, meaning for all $\vec{v} \in V$

$$\vec{v} = \sum_{\vec{b}_i \in \mathcal{B}} a_i \vec{b}_i$$

for some $a_i \in K$.

Theorem (Basis and minimal spanning set) Given a vector space V if a spanning set S is minimal with respect to the number of elements, then S is a basis for V.

Proof We must show that the vectors in a minimal spanning set are linearly independent. Seeking a contradiction, suppose they are not linearly independent. In this case

$$a_1 \vec{s}_1 + \cdots + a_n \vec{s}_n = 0 \qquad |S| = n$$

WLOG, say $a_n \neq 0$. Now we may write

$$a_1 \vec{s}_1 + \cdots + a_{n-1} \vec{s}_{n-1} = -a_n \vec{s}_n$$

so

$$\frac{-a_1}{a_n} \vec{s}_1 - \cdots - \frac{a_{n-1}}{a_n} \vec{s}_{n-1} = \vec{s}_n$$

Hence $\{s_1, \ldots, s_{n-1}\}$ spans V, and this is a contradiction as S was assumed to be minimal.