# History of Mathematics

## Math 2168: Spring 2013

# Contents

# 1 Food for Thought

Feel free to do these problems in any order.

**1)** What is a number? List out some qualities that you think a number has.

**2)** How do you know when something is not a number?

**3)** What are different ways we could represent numbers?

**4)** What if we used the following system:

$$1 = a, \quad 2 = aa, \quad 3 = aaa, \quad \text{and so on.}$$

What does the word *concatenation* mean? How does it apply to this system? How would we add? How would we multiply? How could we represent negative numbers?

**5)** Once Oscar wondered what the number $\pi$ was. So he typed it into his calculator and found:

$$\pi = 3.1415926\ldots$$

Oscar then exclaimed, "Ah now I know what number $\pi$ is." Can you explain Oscar's thoughts on numbers?

## 2 When in Rome...

**1)** Count from 10–20 in the Hindu-Arabic, Roman, Mayan, Babylonian, and Egyptian systems.

**2)** Count from 100–200 *by tens* in the Hindu-Arabic, Roman, Mayan, Babylonian, and Egyptian systems.

**3)** Count from 60–600 *by sixties* in the Hindu-Arabic, Roman, Mayan, Babylonian, and Egyptian systems.

**4)** Discuss the advantages and disadvantages of the various systems. In particular, can you distinguish between *concatenation* systems and *place-value* systems?

**5)** Adding and subtracting Roman numerals is perhaps easier than you may think. For each of the problems above, give three examples of addition problems using Roman numerals from the various ranges. See if you can explain why Roman numerals were used in bookkeeping well into the 1700s. Big hint: Align like-letters when you can.

# 3 Picture Yourself Dividing

We want to understand how to visualize

$$\frac{a}{b} \div \frac{c}{d}$$

Let's see if we can ease into this like a cold swimming pool.

**1)** Draw a picture that shows how to compute:

$$6 \div 3$$

Explain how your picture could be redrawn for other similar numbers.

**2)** Try to use a similar process to the one you used in the first problem to draw a picture that shows how to compute:

$$\frac{1}{4} \div 3$$

Explain how your picture could be redrawn for other similar numbers.

**3)** Try to use a similar process to the one you used in the first two problems to draw a picture that shows how to compute:

$$3 \div \frac{1}{4}$$

Explain how your picture could be redrawn for other similar numbers.

**4)** Try to use a similar process to the one you used in the first three problems to draw a picture that shows how to compute:

$$\frac{5}{3} \div \frac{1}{4}$$

Explain how your picture could be redrawn for other similar numbers.

**5)** Explain how to draw pictures to visualize:

$$\frac{a}{b} \div \frac{c}{d}$$

**6)** Use pictures to explain why:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

## 4 Friendly Pairs

In this activity, we are going to play with pairs of numbers. We'll say that the pair $(a, b)$ is a "friend" to the pair $(c, d)$ if $a \cdot d = b \cdot c$.

**1)** Give 5 examples of pairs that are friends. Give 5 examples of pairs that are not friends.

**2)** Is this a good rule? That is, if $(a_1, b_1)$ is a friend to $(c, d)$ and $(a_2, b_2)$ is a friend to $(c, d)$, is it true that either $(a_1, b_1)$ is a friend to $(a_2, b_2)$ or $(a_2, b_2)$ is a friend to $(a_1, b_1)$? If yes, justify your answer. If no, give a counterexample.

Any counting number $a$ can be turned into a pair by the following rule:

$$a = (a, 1)$$

**3)** Suppose you can "multiply" pairs by the following rule:

$$(a, b) \cdot (c, d) = (ac, bd)$$

Is this a good rule, meaning do counting numbers still multiply the correct way? Give three additional examples of pairs (that are not integers) multiplying to make new pairs.

**4)** Using the rules in Problem 2 and Problem 3 where now the $a$, $b$, $c$, $d$, can themselves be pairs, simplify:

$$\big((11, 16), (3, 5)\big)$$

That is, show that $\big((11, 16), (3, 5)\big)$ is a friend to a simple pair.

**5)** What does any of this have to do with fractions? What does this have to do with the "invert and multiply" rule for dividing fractions?

# 5 Estimating the Area of a Circle

Draw a (fairly large) circle on a blank sheet of paper. We'll think of this as a unit circle.

**1)** Divide the unit circle into $2^2 = 4$ equal wedges each with its vertex at the center of the circle $O$. On each wedge, call the two corners of the wedge that lie on the circle $A$ and $B_2$. Let $\mathcal{A}_2$ denote the area of the triangle $\triangle OAB_2$ and let $\theta_2$ denote the measure of the angle at $O$. Explain how to estimate the area of the circle with triangle $\triangle OAB_2$. What is your estimate?

**2)** Divide the unit circle into $2^3 = 8$ equal wedges each with its vertex at the center of the circle $O$. On each wedge, call the two corners of the wedge that lie on the circle $A$ and $B_3$. Let $\mathcal{A}_3$ denote the area of the triangle $\triangle OAB_3$ and let $\theta_3$ denote the measure of the angle at $O$. Explain how to estimate the area of the circle with triangle $\triangle OAB_3$. What information do you need to know to actually do this computation?

**3)** Given an angle $\theta$, explain the relation of $\sin(\theta)$ and $\cos(\theta)$ to the unit circle. How could these values help with the calculation described above?

**4)** Divide the unit circle into $2^n$ equal wedges each with its vertex at the center of the circle $O$. On each wedge, call the two corners of the wedge that lie on the circle $A$ and $B_n$. Let $\mathcal{A}_n$ denote the area of the triangle $\triangle OAB_n$ and let $\theta_n$ denote the measure of the angle at $O$. Explain why someone would be interested in the value of:

$$\sin\left(\frac{\theta_n}{2}\right)$$

**5)** Recalling that:

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{2}} \qquad \text{and} \qquad \cos(\theta)^2 + \sin(\theta)^2 = 1$$

Explain why:

$$2\mathcal{A}_{n+1} = \sqrt{\frac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}}$$

**6)** Let's fill out the following table (a calculator will help!):

| $n$ | $\mathcal{A}_n$ | Approx. Area | $\sqrt{1-(2\mathcal{A}_n)^2}$ | $\frac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}$ | $2\mathcal{A}_{n+1} = \sqrt{\frac{1-\sqrt{1-(2\mathcal{A}_n)^2}}{2}}$ |
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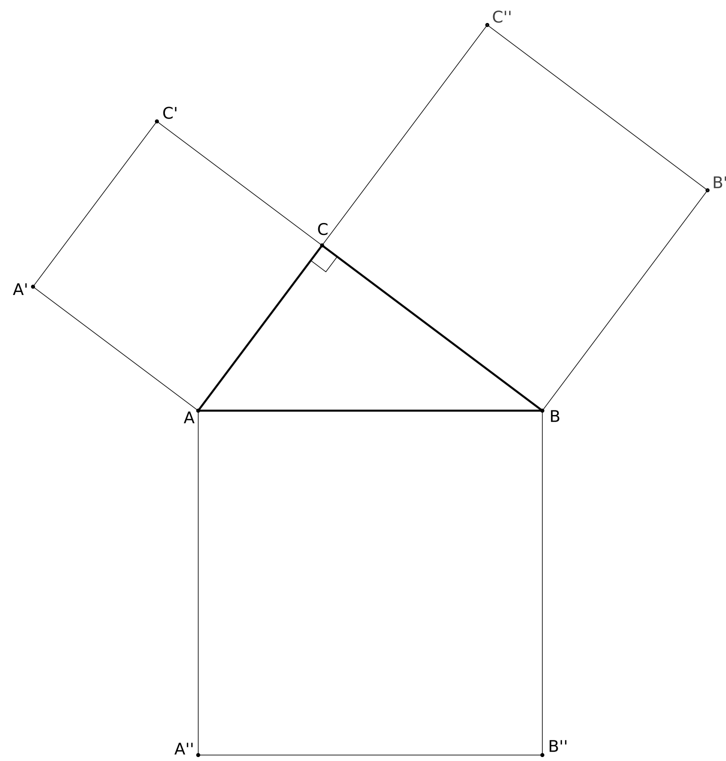
What do you notice?

# 6 Euclid's Proof of Someone's Theorem

**1)** Remind us, what is the most famous theorem of all and what exactly does it assert?

**2)** What would one need to prove about the following diagram to prove the "most famous theorem of all?"




Let's see if we can do this!

**3)** Draw a line perpendicular to $\overline{AB}$ that passes though both $C$ and $\overline{A''B''}$. Call the intersection between this line and $\overline{AB}$, point $E$; call the intersection point between this line and $\overline{A''B''}$, point $E'$. Explain why $\triangle ACA''$ has half the area of rectangle $AEE'A''$.

**4)** Explain why $\triangle ABA'$ has half the area of square $ACC'A'$.

**5)** Explain why $\triangle ACA''$ is congruent to $\triangle ABA'$.

**6)** Explain why area of square $ACC'A'$ is equal to the area of rectangle $AEE'A''$.

**7)** Use similar ideas to complete a proof the "most famous theorem of all."

## 7 Of Course All Right Angles are Congruent...

**1)** Put a dot at the center of a blank sheet of paper and call it $O$. Use a protractor to draw an angle of $50^\circ$ with vertex at the point $O$ and sides extending all the way out to the edge of the paper. Cut the paper along one side of the angle and one side only. Make a cone by moving the cut edge to the other side of the angle you drew. This cone (extended infinitely) is your universe.

**2)** Does Euclid's 1st Postulate hold in your universe? Explain your reasoning.

**3)** Does Euclid's 2nd Postulate hold in your universe? Explain your reasoning.

**4)** Does Euclid's 3rd Postulate hold in your universe? Explain your reasoning.

**5)** You measure angles on your universe by laying the paper out flat and measuring the angles on the paper. Draw a line that passes through $O$ along the cut edge. Now lay your paper flat and bisect the angle formed by the other edge and your line. What does this say about Euclid's 4th Postulate? Explain your reasoning.

**6)** Does Euclid's 5th Postulate hold in your universe? Explain your reasoning. Big hint, how few sides can a $n$-gon have?

# 8 Triangles on a Cone

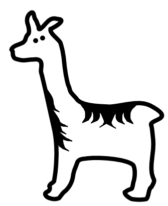
**1)** Put a dot at the center of a blank sheet of paper and call it $O$. Use a protractor to draw an angle of $50^\circ$ with vertex at the point $O$ and sides extending all the way out to the edge of the paper. Cut the paper along one side of the angle and one side only. Make a cone by moving the cut edge to the other side of the angle you drew. This cone (extended infinitely) is your universe.

**2)** Make a triangle in your universe that surrounds $O$. To do this, unfold your universe and lay it out flat on the desk and make the sides with your ruler. When a side gets to the cut side of your angle, put the other side of the angle on top and keep going.

**3)** You measure angles on your universe by laying the paper out flat and measuring the angles on the paper. Measure the angles in your triangle, what do they sum to?

**4)** Repeat the problems above, but this time cut an angle of $40^\circ$ to make your cone. What do you notice?

Let's see if we can explain this. Do you know who is eager to help you? That's right: Louie Llama.



**5)** Take your triangle and denote the measure of its angles as $a$, $b$, and $c$. We would like to parade Louie around the triangle. There is only one catch: What happens to Louie when he passes over the "cut?" Draw some pictures and see if you can figure it out.

Start Louie Llama out along a side adjacent to the angle of measure $a$. He should be on the outside of the triangle, his feet should be pointing toward the triangle, and his face should be pointing toward the angle of measure $b$. Continue this process and walk him all around the triangle. When he gets to the "cut" put the paper together, and let him continue his walk.

**6)** Through what angle does Louie rotate when he strolls around a vertex?

**7)** How many degrees did the "cut" rotate Louie?

**8)** All in all, how many degrees did Louie Llama rotate in his walk?

**9)** If a cone is made on a sheet of paper with a cut of $\theta$ degrees, and a triangle is made surrounding the point of the cone, what is the sum of the degrees of this triangle?

**10)** Explain why not all of Euclid's postulates could hold in this universe. Exactly which postulates don't hold?

## 9 Classy Work

**1)** What is a regular tessellation? Classify all regular tessellations and explain how you know you have found all of them.

**2)** What is a Platonic solid? Classify all Platonic solids and explain how you know you have found all of them.

**3)** What is the connection between the first two questions?

**4)** Build the Platonic solids.

**5)** Take two tetrahedrons. Glue them together along one of the triangular faces. Is this a *sixth* Platonic solid? Explain your reasoning.

# 10 Polyhedral Planets

In this activity, we will suppose in turn that you live on a planet that is shaped like each of the regular convex polyhedra—not unlike *Le Petit Prince* who lives on an asteroid.

**1)** Considering each of the five regular convex polyhedra in turn, what fraction of the planet's surface could you see if you stood:

(a) In the middle of a face?

(b) In the middle of an edge?

(c) On a vertex?

In each case, explain your reasoning.

**2)** Now suppose that you wish to go on a walk, surveying your polyhedral planet. For each platonic solid, give the shortest path you can (draw it on a net) that would allow you to observe the entire planet. In each case, explain your reasoning.

## 11 Symmetry is the Key

It is believed by some physicists that the fundamental forces of the universe are related via symmetry. In this activity, we will see how the symmetries of even the simplest 3D solid are in fact quite complex. **You'll need a real-life tetrahedron for this activity.**

**1)** Make a group table of all the rotational symmetries of your tetrahedron.

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**2)** Is the group commutative?

**3)** What elements of the group commute with all other elements? Do they themselves form a group?

**4)** Can you find a subgroup containing exactly $n$ elements when $n = 2, 3, 4, 5, \ldots, 12$? Which integers work and which don't?

# 12 Owing Numbers

You are an Indian mathematician back in the year 650 working with Brahmagupta and Bhaskara. You know only the counting numbers and 0. You learned in school how to add and multiply in this number system.

commutative.

**1)** In school you learned:

- Both addition and multiplication were associative and commutative. Explain what this means.
- The number 0 is an identity element for addition and 1 is an identity element for multiplication. Explain what this means.
- The distributive law holds. Explain what this means.
- The cancellation property holds, meaning:

$$\begin{aligned} a+b &= a+c \\ \Rightarrow b &= c \end{aligned} \qquad\qquad \begin{aligned} a\cdot b &= a\cdot c \\ \Rightarrow b &= c \end{aligned}$$

  Give examples of the cancellation properties.

**2)** Suppose there were some new numbers $e$ and $f$ such that

$$e+n=n \qquad \text{and} \qquad f\cdot n=n$$

for every counting number $n$. Explain why $e$ must in fact be equal to 0 and $f$ must be in fact equal to 1.

**3)** Explain why given any counting number $n$

$$0\cdot n=0$$

Big hint: Use the fact that $0+0=0$.

You think the number system should be enlarged to incorporate *owing* an amount as well as *having* an amount. That's easy, just adjoin the numbers $\{\widetilde{1},\widetilde{2},\dots\}$. Adding in this bigger number system, that you call the integers, is easy, and you still have that addition is associative and commutative, and that 0 is still the identity element for addition. Defining multiplication of two numbers $a\cdot b$ is not so easy when at least one of the two numbers is negative. You want to define it so that multiplication is still associative and commutative, that 1 is still the identity for multiplication, and so that the distributive law continues to hold.

**4)** Explain why

$$m\cdot\widetilde{n}=\widetilde{m\cdot n}$$

where $m$ and $n$ are counting numbers (or 0)?

**5)** Use the problem above to explain why this forces you to define

$$\widetilde{m} \cdot \widetilde{n} = m \cdot n$$

when $m$ and $n$ are counting numbers (or 0)?

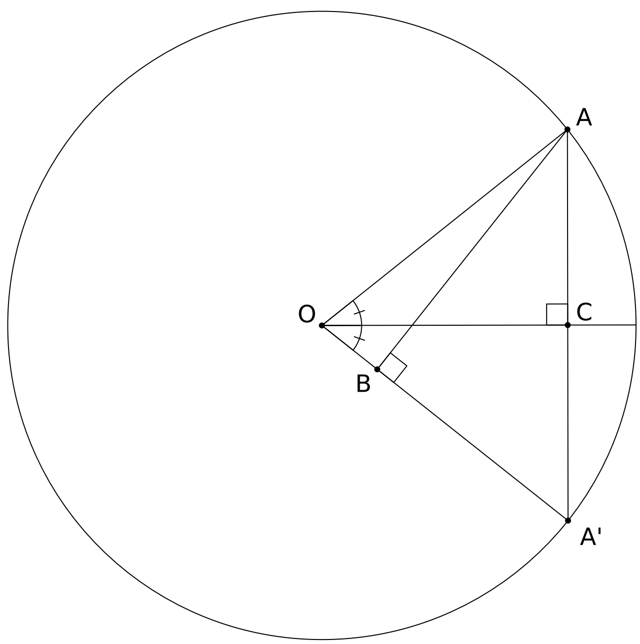
**6)** What facts about negative numbers are we illustrating?

# 13 Trigonometry Before Sine and Cosine

In this activity, we are going to explore how someone could *understand* that

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1-\cos(\theta)}{2}} \\ &= \sqrt{\frac{1-\sqrt{1-\sin(\theta)^2}}{2}},\end{aligned}$$

*before* we had the notions of sine and cosine. Consider the following unit circle:




**1)** Explain why $\triangle OCA$ is similar to $\triangle ABA'$.

**2)** Explain why $|BA'| = |CA| \cdot |AA'| = 2|CA|^2$.

**3)** Explain why $|OB|^2 = (1 - 2|CA|^2)^2$.

**4)** Explain why $|BA|^2 = 1 - (1 - 2|CA|^2)^2$.

**5)** Solve for $|CA|$ in terms of $|BA|$.

**6)** Explain how we have done what we set out to do.

# 14 Geometry and Quadratic Equations

In ancient and Medieval times the discussion of quadratic equations was often broken into three cases:

(a) $x^2 + bx = c$

(b) $x^2 = bx + c$

(c) $x^2 + c = bx$

where $b$ and $c$ are positive numbers.

**1)** Create real-world word problems involving length and area for each case above.

**2)** Solve each of the three cases above by actually "completing the square" using a **real** square.

**3)** Is this a complete list of cases? If not, what is missing and why is it (are they) missing? Explain your reasoning.

# 15 State of the Art Circa 1550

Somewhere deep in your brain is a sleeping technique... AWAKE! We want to solve:

$$x^3 + 9x - 26 = 0$$

We're going to have to use the Ferro-Tartaglia method, but all I can tell you are these three steps:

(a) Replace $x$ with $u + v$.

(b) Set $uv$ so that all of the terms are eliminated except for $u^3$, $v^3$, and constant terms.

(c) Clear denominators and use the quadratic formula.

**1)** Use the Ferro-Tartaglia method to solve $x^3 + 9x - 26 = 0$.

**2)** How many solutions should our equation above have? Where/what are they? Hint: Make use of an old forgotten foe...

## 16 A Whole New World

**1)** List Euclid's five postulates. Draw pictures representing these postulates.

**2)** Here are other statements closely related to Euclid's fifth postulate:

(5A) Exactly one line can be drawn through any point not on a given line parallel to the given line.

(5B) The sum of the interior angles of every triangle is equal to $180°$.

(5C) If two lines $\ell_1$ and $\ell_2$ are both perpendicular to some third line, then $\ell_1$ and $\ell_2$ do not meet.

draw pictures depicting these statements. Can you explain why Euclid's fifth postulate is sometimes called the *parallel postulate*?

There is a very natural geometry where the sum of the angles in every triangle is *greater than* $180°$ and the essences of the first four also still hold. Instead of working with a plane, we now work on a sphere. We call this sort of geometry *Spherical Geometry*. Points, circles, angles, and distances are exactly what we would expect them to be. But what do we mean by lines on a sphere? Lines are supposed to be extended indefinitely. In Spherical Geometry, the lines are the *great circles*.

**Definition** A **great circle** is a circle on the sphere with the same center as the sphere.

**3)** Draw some pictures of great circles and see if you can explain what the definition above is saying.

It is a theorem of Euclidean Geometry that the shortest path between any two points on a plane is given by a line segment. We have a similar theorem in Spherical Geometry.

**Theorem 1** *The shortest path between any two points on a sphere is given by an arc of a great circle.*

**4)** In Spherical Geometry, what is the difference between a great circle and a regular Spherical Geometry circle? Try to come up with a definition of a *circle* that will be true in both Euclidean and Spherical Geometry.

**5)** Explain why the following proposition from Euclidean Geometry does not hold in Spherical Geometry: A triangle has at most one right angle. (Can you find a triangle in Spherical Geometry with three right angles?)

**6)** Explain why the following result from Euclidean Geometry does not hold in Spherical Geometry: When the radius of a circle increases, its circumference also increases.

**7)** Define distinct lines to be **parallel** if they do not intersect. Can you have parallel lines in Spherical Geometry? Explain why or why not.

**8)** Come up with a definition of a *polygon* that will be true in both Euclidean and Spherical Geometry.

**9)** A mathematician goes camping. She leaves her tent, walks one mile due south, then one mile due east. She then sees a bear before walking one mile north back to her tent. What color was the bear?

**10)** The great German mathematician Gauss measured the angles of the triangle formed by the mountain peaks of Hohenhagen, Inselberg, and Brocken. What reasons might one have for doing this?

# 17 Disk World

Once humans became comfortable with Euclidean Geometry, we took the next step: Could the postulates and proofs be improved? In particular, the Parallel Postulate seemed overly complicated. Could it be *proven* from the previous four, or was it really necessary? It was not until the early 19th century that the mystery about the role of the Parallel Postulate was solved and its absolute necessity was shown. This was done by constructing geometries in which all of Euclid's other postulates are true but the Parallel Postulate is **false**. Here is a construction that depends on two things:

(a) Algebraic formulas about the distances between points on a circle.

(b) A geometric construction.

Start with a circle of radius of 5 cm, center $O$ and two points $A$ and $B$ inside it—place $A$ and $B$ near the edges. The inside of this circle will be called the **hyperbolic plane**.

**1)** Use a ruler and calculator to calculate:

$$\frac{5^2}{|OA|}.$$

**2)** Extend the ray $\overrightarrow{OA}$ far enough outside the circle to be able to mark the point $A'$ such that

$$|OA'| = \frac{5^2}{|OA|}.$$

**3)** Construct the circle passing through $A$, $A'$, and $B$. Call the two points where our two circles intersect $P$ and $Q$.

The points $P$ and $Q$ divide the newly constructed circle into two arcs, the arc inside the hyperbolic plane and the arc outside the hyperbolic plane. Call the inside arc the **hyperbolic line** passing through $A$ and $B$.

We need to have notions of *distance* and *angle* to have a geometry. The **angle** between two hyperbolic lines that cross at a point $X$ is just the angle between the tangent lines to the two pieces of circle. The **hyperbolic distance** between two points, for example the two points $A$ and $B$ we started with, is found via some complicated algebra:

$$d_H(A, B) = \left| \ln \left( \frac{|AP|}{|BP|} \cdot \frac{|BQ|}{|AQ|} \right) \right|$$

**4)** Does $d_H(A, B) = d_H(B, A)$?

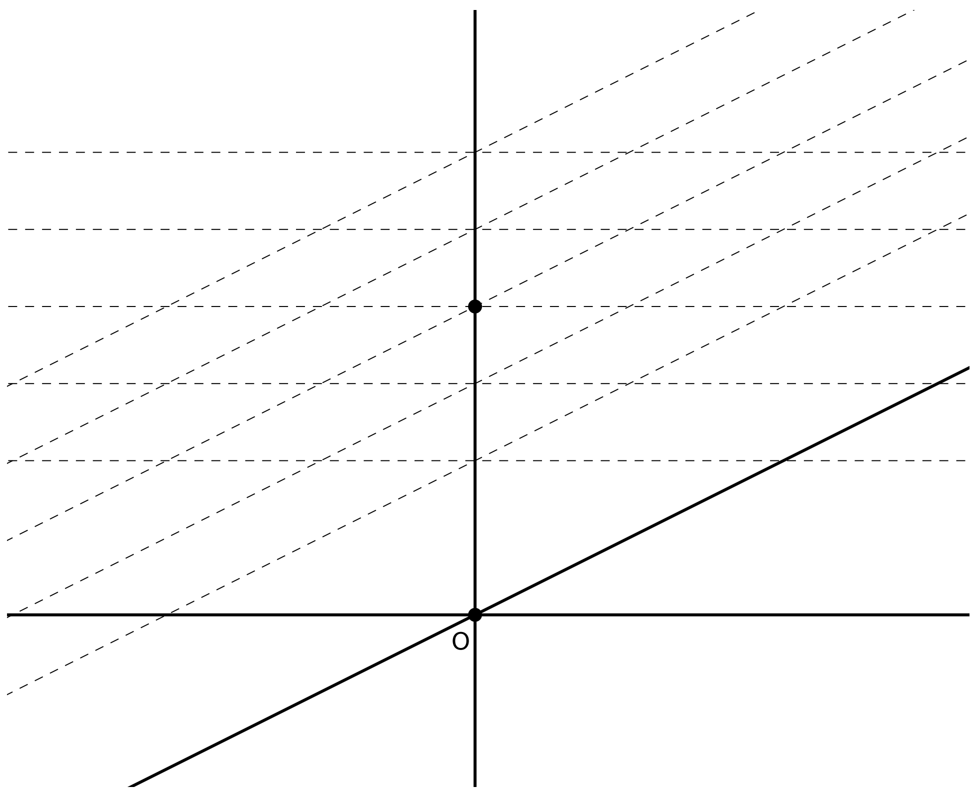
**5)** What happens to $d_H(A, B)$ when $A$ is fixed and $B$ gets closer and closer to the edge of the circle?

**6)** If $A$, $B$, and $C$ are on the same hyperbolic line, with $B$ between $A$ and $C$, can you explain why $d_H(A, C) = d_H(A, B) + d_H(B, C)$?

# 18 Projective Geometry

While people were still struggling with Euclid's postulate (until the 19th Century) the Renaissance was happening in Europe (15th Century). In Italy this brought with it an interest in exploration and realism in painting. Realism demanded the understanding of perspective, that is, how to faithfully represent the eye's view of a (3-dimensional) landscape or other object extending off into the distance (two rails of a train-track converging in the distance—small problem: There were no railroads during the Renaissance). Logical conclusion: There should be a point infinitely far away where the two (parallel) rails meet. In fact the geometers made those infinitely far away points. Let's see how.

The solid lines are the axes of 3-dimensional Euclidean space with origin $O$. The dotted plane is our canvas.




**1)** Put a point $A$ on the canvas. Draw the line through $O$ and $A$ (and extend it infinitely in both directions).

**2)** Put a point $B$ on the canvas. Draw the line through $O$ and $B$ (and extend it infinitely in both directions).

**3)** Does each point on the canvas determine a line through $O$? Do different points determine different lines through $O$?

**4)** Are there some lines through $O$ that get missed, that is, lines that don't correspond to points on the canvas? Which ones? We will say that each of those lines corresponds to a 'canvas point at infinity.'

**5)** Draw some train tracks (parallel lines $l$ and $m$) on the canvas. Show that each rail (line) determines a plane through $O$.

**6)** Let $L$ and $M$ stand for the planes through $O$ corresponding to the lines $l$ and $m$ respectively. How do $L$ and $M$ intersect?

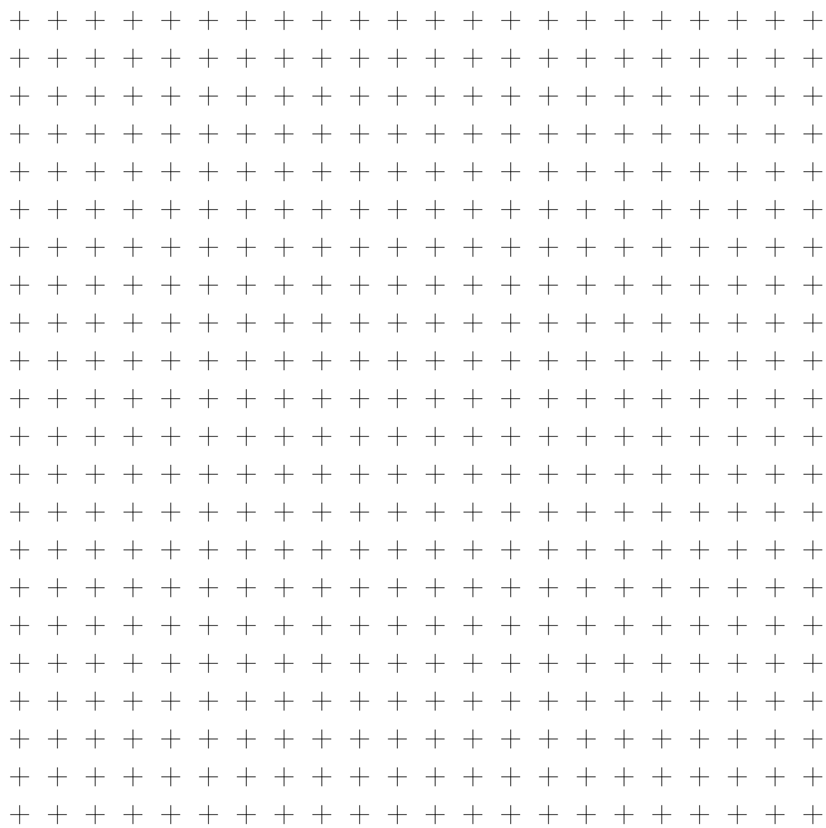
**7)** How do you find the line at which the planes $L$ and $M$ intersect?

**8)** Why do Problems 4 through 6 above justify the statement that "parallel lines on the canvas also meet at exactly one point, it's just that point is a canvas point at infinity?"

# 19 Coordinate Geometry

Mathematicians in Europe in the 17th Century were just beginning to come to terms with representing algebraic relations geometrically using a coordinate plane, and conversely, representing geometric loci in the (coordinate) plane by algebraic relations. The solution to solving this problem for one particular kind of locus was to have spectacular consequences in astronomy and physics within a century, and was all tied up with the discovery of calculus by Newton and Leibniz.

In this activity we will explore the locus of points $(x, y)$ such that the distance between $(x, y)$ and $(5, 0)$ plus the the distance between $(x, y)$ and $(-4, 0)$ is equal to 15.



**1)** Pick two points, one inside your locus and another outside.

**2)** Find four points that are exactly on the locus. Hint: You can do two in your head, but you may need the Pythagorean theorem and the quadratic formula to get the other two.

**3)** Write "the distance between $(x, y)$ and $(5, 0)$ plus the the distance between $(x, y)$ and $(-4, 0)$ is equal to 15" as an algebraic equation.

Now we will attempt to put the equation above into "standard form:"

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

**4)** To start, isolate one of the square-roots, and square both sides.

**5)** Next, isolate the other square-root, and square both sides.

**6)** Complete the square, and get it into standard form! Can you figure out what $a$, $b$, $h$, and $k$ represent?

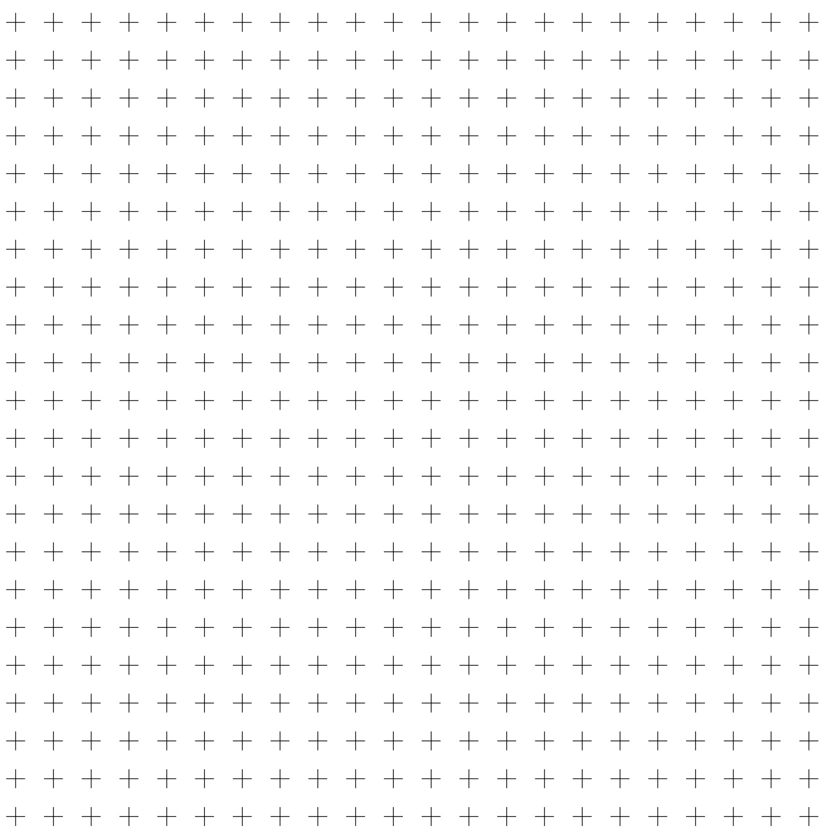
# 20 Complex Numbers From Different Angles

In this activity we will investigate complex multiplication.

**1)** Consider the innocent little equation:

$$x^3 - 1 = 0$$

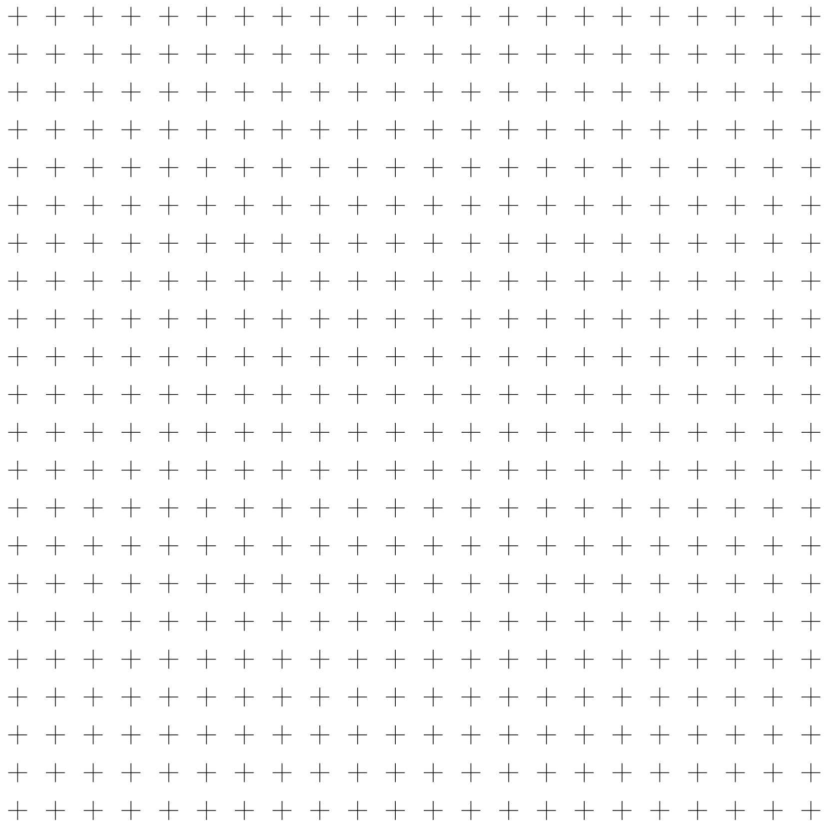
How many solutions does it have? What are they? Plot them on the complex plane below.



**2)** Thinking about your work above, see if you can solve:

$$x^4 - 1 = 0$$

Plot the solutions on the complex plane below.



**3)** Suppose I told you that:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

Explain why we say:

$$e^{x \cdot i} = \cos(x) + i\sin(x)$$

**4)** This is Euler's famous formula:

$$e^{\pi \cdot i} + 1 = 0$$

Use the problem above to explain why it is true.

**5)** What does all this have to do with De Moivre's Theorem?

**6)** How can you use this to take the $n$th root of a complex number?

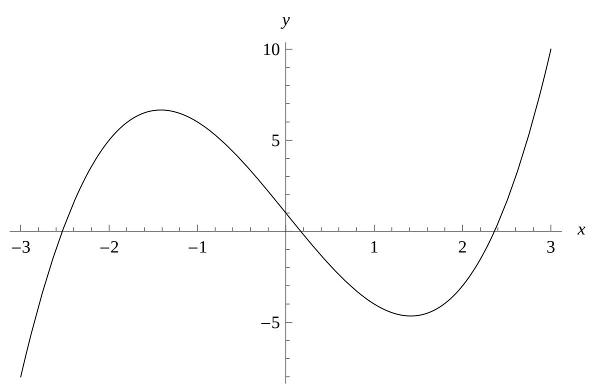
# 21 De Moivre Saves the Day!

The rules for solving second and third degree equations led to a "sticky wicket." Namely the rule for solving a quadratic equation forced its user to take the square root of a number that was sometimes negative. Worse yet, the formula for solving a cubic equation forced the user to take the cube root of a number that resulted from taking square roots! Despite misgivings, mathematicians were eventually forced to expand the number system to one where square roots can be found for any of its numbers.

You may recall that the Ferro-Tartaglia method gives

$$\sqrt[3]{\frac{-1+\sqrt{-31}}{2}}+\frac{2}{\sqrt[3]{\frac{1}{2}(-1+\sqrt{-31})}}$$

as a solution to: $y = x^3 - 6x + 1$. Moreover, as this plot of $y = x^3 - 6x + 1$ shows

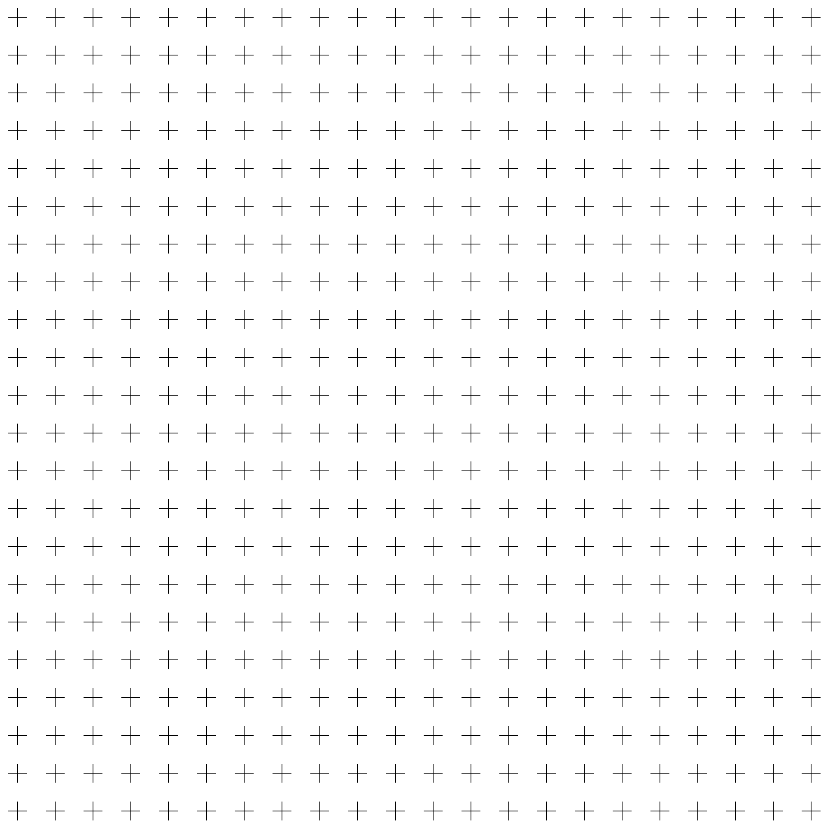



this must be a real solution... But how? There is a square-root of a negative number in the expression above!

We're going to investiagate incredible connection between adding and multiplying complex numbers and plane geometry. It's called *De Moivre's rule*—it is here to save the day. Please fasten your seat belts and place your trays in the upright position.

You will need a scientific calculator and a straight-edge and compass.

**1)** Plot the point $(4, 3)$ in the plane below—let each square be $1/2$ unit. We will think of this point as the complex number $4 + 3i$.



(a) Draw the ray from the origin through your chosen complex number.

(b) Measure its distance to the origin (called the *absolute value* of the complex number $4 + 3i$).

(c) Use inverse tangent to measure the angle between your complex number and the positive real axis.

**2)** We're going to solve $(x + yi)^3 = 4 + 3i$ for $x$ and $y$ using geometry!

(a) Compute the cube root of the absolute value you found before.

(b) Compute one-third of the angle you found before.

(c) Use sine and cosine to finish it off!

(d) How can you check your work? Do it.

**3)** Now use De Moivre's method to take the cube roots found in:

$$\sqrt[3]{\frac{-1+\sqrt{-31}}{2}}+\frac{2}{\sqrt[3]{\frac{1}{2}(-1+\sqrt{-31})}}$$

## 22 Tic-Tac-D'oh

In the 17th and 18th centuries, finally mathematics was coming into its own. The codification of algebra, the introduction of the complex numbers, the discovery of the calculus, all these formed the analogy in their day of the introduction of the personal computer and the Internet in modern times. There was, if you will, an explosion of mathematics (although among a very small community—unlike the recent technological revolution).

The sense was that mathematics was this huge new tool that could be applied to understand many, many things, from the movement of the planets to the workings of business and industry. And it wasn't long before it was applied to the ancient human activity of gambling. The key was knowing more than your opponent about how the game would come out. Let's do an experiment with the game tic-tac-toe:

**1)** Consider the following set-up:

| | | |
|---|---|---|
| o | | x |
| x | x | o |
| o | | |

It is currently x's turn. What is the probablity that x wins assuming that the rest of the moves are random?

**2)** Consider the following set-up:

| | | |
|---|---|---|
| | | |
| o | x | |
| | | |

It is currently x's turn. What is the probablity that x wins in the next three moves assuming that the rest of the moves are random?

# 23 Bertrand's Paradox

In this activity we are going to investigate the following question:

> Given a circle, find the probability that a chord chosen at random is longer than the side of an inscribed equilateral triangle.

### Method 1: Random Endpoints

**1)** We want to define our chord via "random endpoints." Explain why, without loss of generality, we may assume that one of the endpoints is on the vertex of the triangle.

**2)** Draw a picture of the situation and use the arcs of the circle defined by the triangle to help you determine the probability in question.

### Method 2: Random Radius

**3)** We want to define our chord via a random point on a "random radius." Explain why, without loss of generality, we may assume that any chord is a random point on some radius of the circle.

**4)** Draw a picture of the situation and use the length of the radius (in relationship to the inscribed triangle) to help you determine the probability in question.

### Method 3: Random Midpoint

**5)** We want to define our chord via its midpoint. Explain why a midpoint will (almost!) always determine a unique chord of a circle. When does it fail?

**6)** Give a compass and straightedge construction for a chord given its midpoint.

**7)** Draw a picture of the situation along with the incircle of the equilateral triangle to help you determine the probability in question.

### Conclusion?!

**8)** What did you find? How do we resolve the paradox?

## 24 Tenths are Best

Again in the 17th and 18th centuries, the algebra of the number system was being established. Numbers were associated to lengths, areas, weights, etc., so that putting things together corresponded to adding their associated numbers.

**1)** Give three examples of assigning a number to measure the size or shape or position of a geometric or physical object so that putting the objects together corresponds to adding the numbers.

**2)** Give three examples of assigning a number to measure the size or shape or position of a geometric or physical object so that putting the objects together does *not* correspond to adding the numbers.

**3)** One example of the first problem is to assign a number to a line segment (called its length).

(a) I want my system to be such that, by telling the number to someone half-way round the world, they will be able to make a line segment of the same length, that is, a line segment congruent to mine. What do I need to do to make such an assignment?

(b) If I want to have a number for every possible line segment, what kind of a number system will need to have? Is the system of rational numbers enough? That is, can you always construct a segment whose length is not a fraction? Why or why not?

**4)** The previous discussion led us to the decimal number system. Decimals can be finite or infinite.

(a) Describe how to add two infinite decimals.

(b) Describe how to multiply

$$(0.111111\ldots)\cdot(0.333333\ldots)$$

(c) Describe how to multiply any two (infinite) decimals.

**5)** What is the relationship between $0.999999\ldots$ and 1? Can you explain this?

# 25 Whose Intimidating Who Now?

In this activity, we'll investigate problems related to length, area, and volume. To start, let me tell you a bit about myself. I'm about $6'$ tall and I weigh about 160 pounds.

**1)** Imagine if you will, that one day I "divide by zero" and I am fantastically made 100 times taller. Let's call this bigger me *monster-me.*

(a) How tall is monster-me?

(b) Relative to my eyeball, how much surface area does monster-me's eyeball have?

(c) How much does monster-me weigh?

**2)** While you are probably imagining me as some sort of "car-eating monster," the reality is much different. I claim that monster-me would have a hard time moving. Can you explain why this is true?

**3)** I also claim that monster-me would suffocate. Can you explain why this is true? Hint: How do lungs work? How does oxygen get to the brain? How much blood would the monster-me have?

**4)** Now imagine if you will, that one day I "divide by infinity" and I am fantastically made 100 times smaller. Let's call this smaller me *mini-me.*

(a) How tall is mini-me?

(b) Relative to my eyeball, about how much surface area does mini-me's eyeball have?

(c) How much does mini-me weigh?

**5)** I claim that mini-me would be able to fall from a great (relative) height and be just fine. Can you explain why this is true?

**6)** When I was little, I used to want to fold a giant paper airplane and actually fly around in it. Would this work? Why or why not?

## 26 Diophantus—Fermat—Wiles

Long before the invention of algebra, the Greek mathematician constructed and solved riddles with whole numbers, several of which are described in today's sketch. Fermat had algebra at his disposal and so was so taken with the number-riddles of Diophantus that he recorded (and solved) many of them as equations where the solution numbers should all be whole numbers (integers). These have come to be known as Diophantine equations.

**1)** One problem was to give all solutions to the Diophantine equation

$$X^2 + Y^2 = Z^2$$

remembering that $X$, $Y$, and $Z$ all have to be integers. Diophantus would have asked "Make a list of all the square numbers that are the sum of two square numbers." Explain why this is the same problem as solving

$$x^2 + y^2 = 1$$

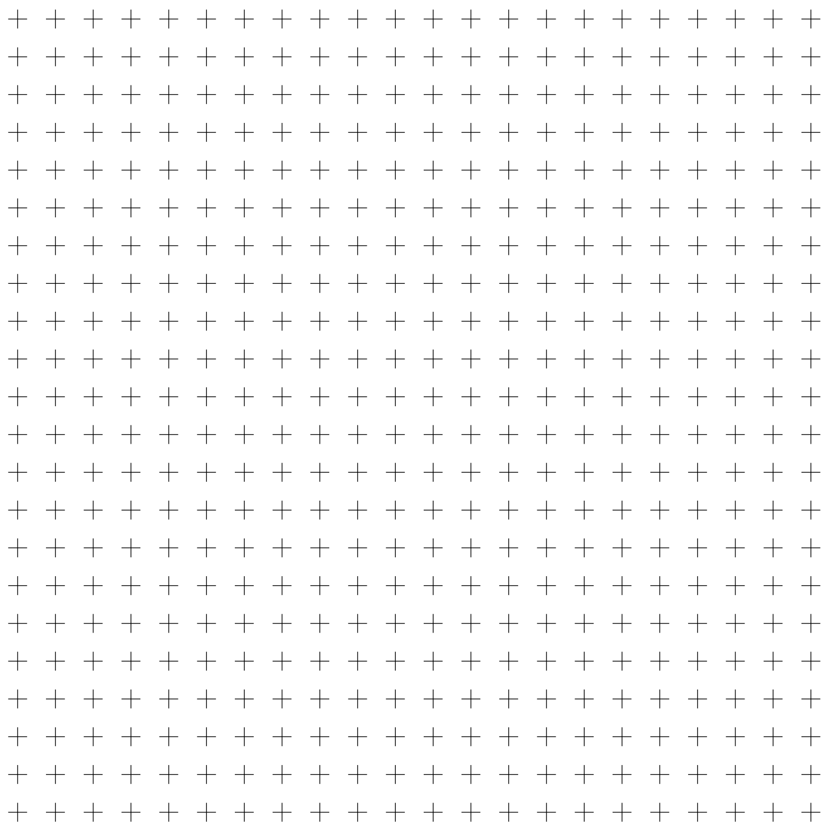
where $x$ and $y$ are rational numbers.

**2)** In the plane below, draw the real-number solution set of $(x, y)$ so that $x^2 + y^2 = 1$.

(a) Find one rational-number solution $C = (x, y)$ to the equation such that neither $x$ nor $y$ is 0. Plot your solution $C$ in the plane below.

(b) Draw the line through your solution and the solution $A = (0, 1)$. Mark the point $B$ at which your line crosses the $x$-axis.

(c) Find the $x$-coordinate of $B$. Is it a rational number? Why or why not?



**3)** Now on the grid below,

(a) Draw the real-number solution set $x^2 + y^2 = 1$, and mark the point $A' = (0, 1)$.

(b) Mark any point $B' = (r, 0)$ inside the circle on the $x$-axis such that $r$ is a rational number.

(c) Draw the line through $A'$ and $B'$.

(d) Find the coordinates of the other point $C'$ at which your line cuts the solution set $x^2 + y^2 = 1$. Are they rational numbers? Why or why not?

**4)** Turn your solution $C'$ in the problem above into an integer solution to the equation $X^2 + Y^2 = Z^2$.

## 27 Cantor Can!

It took until the 1700's to get algebra and number systems in place in a workable way. But there was still trouble understanding what infinity was. Was the set of counting numbers really infinite, or was it only as big as the highest number that anyone had ever counted, or as big as the number of atoms in the universe, or...? But even if the set of counting numbers was infinite, then the set of real numbers was also infinite. But then again, were they the same infinity? Some math grad student in Germany around 1850 shocked the math world by saying 'no.'

**1)** Here is a table of rational numbers:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\cdots$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $\cdots$ |
| $\cdots$ | $\frac{-5}{2}$ | | $\frac{-3}{2}$ | | $\frac{-1}{2}$ | | $\frac{1}{2}$ | | $\frac{3}{2}$ | | $\frac{5}{2}$ | $\cdots$ |
| $\cdots$ | $\frac{-5}{3}$ | $\frac{-4}{3}$ | | $\frac{-2}{3}$ | $\frac{-1}{3}$ | | $\frac{1}{3}$ | $\frac{2}{3}$ | | $\frac{4}{3}$ | $\frac{5}{3}$ | $\cdots$ |
| $\cdots$ | $\frac{-5}{4}$ | | $\frac{-3}{4}$ | | $\frac{-1}{4}$ | | $\frac{1}{4}$ | | $\frac{3}{4}$ | | $\frac{5}{4}$ | $\cdots$ |
| $\cdots$ | | $\frac{-4}{5}$ | $\frac{-3}{5}$ | $\frac{-2}{5}$ | $\frac{-1}{5}$ | | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | | $\cdots$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |

(a) What does the 12th row of the table look like?

(b) Name three different rational numbers. Will they (eventually) appear on the table?

(c) Will every rational number eventually appear in the table above?

(d) Can you figure out how to "enumerate" the rationals?

**2)** The question: Are the set of counting numbers and the set of real numbers between 0 and 1 the same size?

Cantor's answer: Suppose they were, then you could make a one-to-one, onto match-up:

$$\begin{aligned}
&1 : 0.22343798784\ldots\\
&2 : 0.85984759348\ldots\\
&3 : 0.11290293980\ldots\\
&4 : 0.03432340563\ldots\\
&5 : 0.93928498239\ldots\\
&6 : 0.79788937833\ldots\\
&\ \vdots
\end{aligned}$$

So, you think you did it, eh? I will find a real number between zero and one that is not on your list. How will I do it?

**3)** Explain why the same argument does *not* show that the rationals cannot be enumerated.

# 28 Can You Repeat the Question?

Explain why the following “joke” is “funny.”

> If you choose an answer to this question at random, what is the chance you will be correct?
>
> (a) 25%
>
> (b) 50%
>
> (c) 75%
>
> (d) 25%

## 29 The Applied Side—Stat!

While Cantor was doing very abstract and pure mathematics in Germany, statistics was making its first appearance as a mathematical subject in England. In this activity, we are going to investigate the *mean*, *median*, and *mode* of a set of data.

**1)** Explain what is meant by the mean, median, and mode of a set.

**2)** Give a set of data consisting of 6 values (not all equal!) such that the mean, median, and mode are all equal. Plot your data in some reasonable way.

**3)** Give a set of data consisting of 6 values such that the mean is larger than both the median and mode which are equal. Plot your data in some reasonable way.

**4)** Give a set of data consisting of 6 values such that the mode is larger than both the mean and median. Plot your data in some reasonable way.

**Simpson's Paradox** In the 1970's a scandal arose at the University of California, Berkley. There was a distinct gender bias against women in the number of graduate students admitted among the 6 largest departments:

| | Applicants | Admitted |
|---|---|---|
| Men | 2590 | 46% |
| Women | 1835 | 30% |

Data from individual departments was collected for further investigation:

| Department | Men | | Women | |
|---|---|---|---|---|
| | Applicants | Admitted | Applicants | Admitted |
| A | 825 | 62% | 108 | 82% |
| B | 560 | 63% | 25 | 68% |
| C | 325 | 37% | 593 | 34% |
| D | 417 | 33% | 375 | 35% |
| E | 191 | 28% | 393 | 24% |
| F | 272 | 6% | 341 | 7% |

**5)** Examine this data critically. What seems to be the case? Does this jive with your intuition? What is actually happening? Can you explain when this will happen?

# 30 ~~Deci~~mals, *Bi*-mals

We use the decimal system to denote real numbers, but computers don't. They use a "sophisticated" elaboration, *bi-mals*. That is, they would denote the number 101 as:

$$1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = \text{"1100101"}$$

Let's pretend like we are computers and work exclusively with bi-mals!

**1)** Calculate $1100 + 110$ by hand.

**2)** Calculate $1100 \cdot 110$ by hand.

**3)** Calculate $1100101^{10}$ by hand. Remember that the exponent is a bi-mal number too!

Computers use the "bi-mal point" to write all real numbers in bi-mal. For example:

$$1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 0 \cdot 2^{-3} + 1 \cdot 2^{-4} = \text{"110.0101"}$$

**4)** Do the following bi-mal long division problem

$$11 \overline{)1100101}$$

**5)** Express $\dfrac{1}{7}$ as a bi-mal.

**6)** Which bi-mal fractions will terminate? Which will repeat?

**7)** Calculate $110.0101^{10}$ by hand.

## 31 Roll Play This

Long before the *World of Warcraft*, but not as long ago as *The Lord of the Rings*, there was a game called *Dungeons&Dragons*. In this game you played a "character" with abilities such as strength, dexterity, and so on. You computed these abilities by rolling a 6-sided die three times and adding the numbers up (denoted 3d6). To give you a little feel for this, suppose you were making your own character, and you were finding their "Math-Class" ability. Fill out the form below:

Name:________________________________________

(make up a creative name for yourself)

Math-Class Ability: $\boxed{\phantom{00}}$

(Roll 3d6)

Let's see how your character does in a made-up math class: (roll a 20-sided die 4 times and count how many times it rolls **below** your "Math-Class" ability.)

4: A congratulations!

3: B nice!

2: C no worries!

1: D is for diploma!

0: E umm...

As you can see, the game was thrilling. My question is this: Why do they use the sum of rolling a 6-sided die 3 times to determine the abilities? Let's see if we can figure this out.

**1)** Roll a 3d6 5 times and write down what you get. Demand that your results are shared with the class and see what you get.

**2)** Compute the *relative frequency* of obtaining different results for "math ability."

**3)** Compute the *probability* of obtaining different results for "math ability."

**4)** Can you explain why 3d6 was chosen to determine ability scores in the game?

# 32 It's How You Play The Game

Let's play a game! Here is how it will work: Taking turns, each of you will pick a square on the chart below to be "yours." Continue until every square is chosen.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 8 | 9 | 10 | 11 | 12 |

Then you will roll some dice. If your number comes up, you get a point! First to 10 points wins.

**1)** Play the game with a 12-sided die. Does it matter which square you chose? Explain your thoughts on this.

**2)** Play the game with two 6-sided dice, adding their values together. Does it matter which square you chose? Explain your thoughts on this.

**3)** Play the game with three 4-sided dice, adding their values together. Does it matter which square you chose? Explain your thoughts on this.

**4)** Compute the probability of rolling different squares with a 12-sided die, two 6-sided dice, and three 4-sided dice. Make three charts to help you out. What do you notice?

Now let's change the rules of the game. In this game, you get the number of points of the square you are on. So if you choose "square 3" and you roll a 3, you get 3 points! First to 80 wins!

**5)** Now which is the best square to choose if you are playing with:

- One 12-sided die?
- Two 6-sided dice?
- Three 4-sided dice?

Note, if you enjoyed this game, check out *Settlers of Catan*.