CP6: Polynomials with real coefficients

Polynomials with real numbers as coefficients

x is an unknown real number. A polynomial in integral powers of x with real coefficients is an expression of the form

$$a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$

where the a_i are real numbers.

1) Like with the number system of integers, you can add, subtract and multiply polynomials with real coefficients. Give an example of each of the three operations.

2) Like with the number system of integers, you cannot always divide two polynomials and get a polynomial. Show that the quotient

$$x-1 \overline{\big) x^4 - x^3 + x^2 - 1}$$

is a polynomial, but the quotient

$$x-1 \overline{\big) x^4 - x^3 + x^2 + 1}$$

isn't. What polynomial is the remainder when you do this last this division problem?

3) Show that $x-r$ divides $a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$, that is, that there is a polynomial $b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$ such that

$$a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0 = (x-r)\left(b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0\right)$$

if and only if r is a root of $a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$, that is, if and only if

$$a_m r^m + a_{m-1} r^{m-1} + \ldots + a_1 r + a_0 = 0.$$

Polynomials of degree 2 with real numbers as coefficients

How do we find roots of a polynomial of the form

$$a_2 x^2 + a_1 x + a_0$$

where the a_i are real numbers and $a_2 \neq 0$?

1) Write the polynomial as

$$a_2\left(x^2 + \frac{a_1}{a_2} x + \frac{a_0}{a_2}\right).$$

2) "Complete the square," that is, rewrite the polynomial as

$$a_2\left(x^2 + \frac{a_1}{a_2} x + \left(\frac{a_1}{2a_2}\right)^2 + \frac{a_0}{a_2} - \left(\frac{a_1}{2a_2}\right)^2\right)$$

3) Show that this last polynomial is the same as

$$a_2\left(\left(x+\frac{a_1}{2a_2}\right)^2+\frac{4a_2a_0-a_1^2}{(2a_2)^2}\right).$$

4) Show that x is a root of $a_2x^2+a_1x+a_0$ if

$$\left(x+\frac{a_1}{2a_2}\right)^2=\frac{a_1^2-4a_2a_0}{(2a_2)^2},$$

that is, if

$$x+\frac{a_1}{2a_2}=\pm\frac{\sqrt{a_1^2-4a_2a_0}}{2a_2}.$$