Numbers and Algebra

Distributing this Document

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Preface

Three goals of these notes are to:

- Enrich the reader's understanding of both numbers and algebra. In particular, while there is an emphasis on algebraic manipulation of fractions, there is much discussion on the deep connection between algebra and numbers.
- Emphasize problem solving. The reader will be exposed to problems that "fight-back." Worthy minds such as yours deserve worthy opponents. Too often mathematics problems fall after a single "trick." We must understand that some problems take time to solve and cannot be done in a single sitting.
- Challenge the common view that mathematics is a body of knowledge to be memorized and repeated. The art and science of doing mathematics is a process of reasoning and discovery followed by justification and explanation.

In summary—you, the reader, must become a doer of mathematics. To this end, many questions are asked in the text that follows. Sometimes these questions are answered, other times the questions are left for the reader to ponder. To let the reader know which questions are left for cogitation, a large question mark is displayed:

$$?$$

The instructor of the course will address some of these questions. If a question is not discussed to the reader's satisfaction, then I encourage the reader to put on a thinking-cap and think, think, think! If the question is still unresolved, go to the World Wide Web and search, search, search!

Please report corrections, suggestions, gripes, complaints, and criticisms to Bart Snapp at: `snapp@math.ohio-state.edu`

Thanks and Acknowledgments

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Contents

Chapter 1

Numbers

> God created the integers, the rest is the work of man.
>
> —Leopold Kronecker

1.1 The Integers

In this course we will discuss several different sets of numbers. The first set we encounter is called the *integers*.

Definition The set of whole numbers, zero, and negative whole numbers is called the set of **integers**. We use the symbol $\mathbb{Z}$ to denote the integers:

$$\mathbb{Z} = \{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}$$

In case you're wondering, the symbol $\mathbb{Z}$ is used because *Zahlen* is the German word for "numbers."

1.1.1 Division

While there are many properties of the integers we could discuss, we believe it is their multiplicative structure that is of most interest. Read on!

Definition We say that an integer d **divides** an integer n if

$$n = dq$$

in this case we write $d|n$, which is said: "d divides n."

While this may seem easy, it is actually quite tricky. You must always remember the following synonyms for *divides*:

"d **divides** n" $\leftrightsquigarrow$ "d is a **factor** of n" $\leftrightsquigarrow$ "n is a **multiple** of d"

Definition A **prime** number is a positive integer with exactly two positive divisors, namely 1 and itself.

Definition A **composite** number is a positive integer with more than two positive divisors.

I claim that every composite number is divisible by a prime number. Do you believe me? If not, consider this:

> Suppose there was a composite number that was *not* divisible by a prime. Then there would necessarily be a *smallest* composite number that is not divisible by a prime. Since this number is composite, this number is the product of two even smaller numbers, both of which have prime divisors. Hence our original number must have prime divisors.

Question What the heck just happened?! Can you rewrite the above paragraph, drawing pictures and/or using symbols as necessary, making it more clear?

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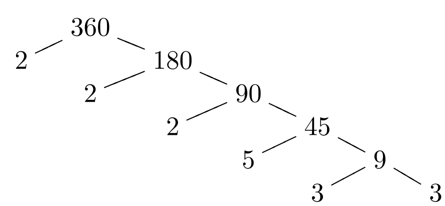
Factoring

At this point we can factor any composite completely into primes. To do this, it is often convenient to make a *factor tree*:



From this tree we see that

$$360 = 2^3 \cdot 3^2 \cdot 5.$$

At each step we simply divided by which ever prime number seemed most obvious, branched off the tree and kept on going. From our factor tree, we can see some of the divisors of the integer in question. However, there are many factors that need to be built up from the prime divisors. One of the most important is the *greatest common divisor*.

Definition The **greatest common divisor** (GCD) of two numbers a and b is a number $g = \gcd(a, b)$ where:

(1) $g|a$ and $g|b$.

(2) If $d|a$ and $d|b$, then $d \leqslant g$.

Question How do you use a factor tree to compute the GCD of two integers?

?

So, to factor an integer or find the GCD, one could use a factor tree. However, when building the factor tree, we had to know what primes to divide by. What if no prime comes to mind? What if you want to factor the integer 391 or 397? This raises a new question:

Question How do you check to see if a given integer is prime[1] ? What prime factors must you check? When can you stop checking?

?

1.1.2 Division with Remainder

We all remember long division, or at least we remember *doing* long division. Sometimes, we need to be reminded of our *forgotten foes*. When aloof old Professor Rufus was trying to explain division to his class, he merely wrote

$$d\,\overline{)\,n}^{\;q \;\; \mathrm{R}r} \qquad \text{where} \qquad \begin{array}{l} d \text{ is the divisor} \\ n \text{ is the dividend} \\ q \text{ is the quotient} \\ r \text{ is the remainder} \end{array}$$

and walked out of the room.

Question Can you give 3 much needed examples of long division with remainders?

?

Question Given positive integers d, n, q, and r how do you know if they leave us with a correct expression above?

?

Question Given positive integers d and n, how many different sets of q and r can you find that will leave us with a correct expression above?

?

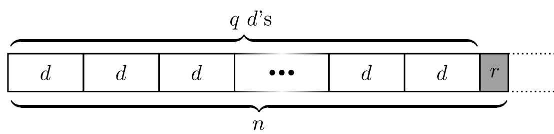
The innocuous questions above can be turned into a theorem. We'll start it for you, but you must finish it off yourself:

[1] Note, if you could come up with a new (and efficient!) way to identify primes or to factor very large integers, then glory-everlasting would be yours (seriously).

Theorem 1 (Division Theorem) *Given any integer n and a nonzero integer d, there exist unique integers q and r such that*

The above space has intentionally been left blank for you to fill in.

Now consider the following picture:



Question How does the picture above "prove" the Division Theorem?

?

Problems for Section 1.1

(1) Describe the set of integers. Give some relevant and revealing examples/nonexamples.

(2) Explain what it means for one integer to *divide* another integer. Give some relevant and revealing examples/nonexamples.

(3) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

- (a) $7|56$
- (b) $55|11$
- (c) $3|40$
- (d) $100|(2^4 \cdot 3^{17} \cdot 5^2 \cdot 7)$
- (e) $5555|(5^{20} \cdot 7^9 \cdot 11^{11} \cdot 13^{23})$
- (f) $3|(3 + 6 + 9 + \cdots 300 + 303)$

(4) *Incognito's Hall of Shoes* is a shoe store that just opened in Myrtle Beach, South Carolina. At the moment, they have 100 pairs of shoes in stock. At their grand opening 100 customers showed up. The first customer tried on every pair of shoes, the second customer tried on every 2nd pair, the third customer tried on every 3rd pair, and so on until the 100th customer, who only tried on the last pair of shoes.

- (a) Which shoes were tried on by only 1 customer?
- (b) Which shoes were tried on by exactly 2 customers?
- (c) Which shoes were tried on by exactly 3 customers?
- (d) Which shoes were tried on by the most number of customers?

Explain your reasoning.

(5) Factor the following integers:

- (a) 111
- (b) 1234
- (c) 2345
- (d) 4567
- (e) 111111

In each case, how large of prime must you check before you can be sure of your answers? Explain your reasoning.

(6) Betsy is factoring the number 24949501. To do this, she divides by successively larger primes. She finds the smallest prime divisor to be 499 with quotient 49999. At this point she stops. Why doesn't she continue? Explain your reasoning.

(7) Find examples of integers a, b, and c such that $a \mid bc$ but $a \nmid b$ and $a \nmid c$. Explain your reasoning.

(8) Can you find 5 composite integers in a row? What about 6 composite integers? Can you find 7? What about n? Explain your reasoning. Hint: Consider $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

(9) How many zeros are at the end of the following numbers:

   (a) $2^4 \cdot 5^3 \cdot 7^3 \cdot 11^5$
   (b) $100!$
   (c) $1000!$
   (d) $28721!$

   Explain your reasoning.

(10) Lisa wants to make a new quilt out of 2 of her favorite sheets. To do this, she is going to cut each sheet into as large of squares as possible while using the entire sheet and using whole inch measurements.

   (a) If the first sheet is 72 inches by 60 inches what size squares should she cut?
   (b) If the second sheet is 80 inches by 75 inches, what size squares should she cut?
   (c) How she might sew these squares together?

   Explain your reasoning.

(11) Deena and Doug like to feed birds. They want to put 16 cups of millet seed and 24 cups of sunflower seeds in their feeder.

   (a) How many total scoops of seed (millet or sunflower) are required if their scoop holds 1 cup of seed?
   (b) How many total scoops of seed (millet or sunflower) are required if their scoop holds 2 cups of seed?
   (c) How large should the scoop be if we want to minimize the total number of scoops?

   Explain your reasoning.

(12) Consider the expression:

$$d\overline{\smash{)}n}^{\,q\ \mathrm{R}r} \qquad \text{where} \qquad \begin{array}{l} d \text{ is the divisor} \\ n \text{ is the dividend} \\ q \text{ is the quotient} \\ r \text{ is the remainder} \end{array}$$

(a) Give 3 relevant and revealing examples of long division with remainders.

(b) Given positive integers d, n, q, and r how do you know if they leave us with a correct expression above?

(c) Given positive integers d and n, how many different sets of q and r can you find that will leave us with a correct expression above?

(d) Give 3 relevant and revealing examples of long division with remainders where some of d, n, q, and r are negative.

(e) Still allowing some of d, n, q, and r to be negative, how do we know if they leave us with a correct expression above?

(13) State the *Division Theorem* for integers. Give some relevant and revealing examples of this theorem in action.

(14) Consider the following:

$$20 \div 8 = 2 \text{ remainder } 4,$$
$$28 \div 12 = 2 \text{ remainder } 4.$$

Is it correct to say that $20 \div 8 = 28 \div 12$? Explain your reasoning.

(15) Give a formula for the nth even number. Show-off your formula with some examples.

(16) Give a formula for the nth odd number. Show-off your formula with some examples.

(17) Give a formula for the nth multiple of 3. Show-off your formula with some examples.

(18) Give a formula for the nth multiple of -7. Show-off your formula with some examples.

(19) Give a formula for the nth number whose remainder when divided by 5 is 1. Show-off your formula with some examples.

(20) Use algebra to help explain the rule:

$$\text{even} + \text{even} = \text{even}$$

(21) Use algebra to help explain the rule:

$$\text{odd} + \text{even} = \text{odd}$$

(22) Use algebra to help explain the rule:

$$\text{odd} + \text{odd} = \text{even}$$

(23) Use algebra to help explain the rule:

$$\text{even} \cdot \text{even} = \text{even}$$

(24) Use algebra to help explain the rule:

$$\text{odd} \cdot \text{odd} = \text{odd}$$

(25) Use algebra to help explain the rule:

$$\text{odd} \cdot \text{even} = \text{even}$$

(26) Let $a \geqslant b$ be positive integers with $\gcd(a, b) = 1$. Compute $\gcd(a+b, a-b)$. Explain your reasoning. Hints:

   (a) Make a chart.
   (b) If $g|x$ and $g|y$ explain why $g|(x + y)$.

(27) Denote the set of all integers that have remainder r when divided by 5 by $[r]_5$. So for example:

$$[0]_5 = \{\dots, -15, -10, -5, 0, 5, 10, 15, \dots\}$$

Write down the sets $[1]_5$, $[2]_5$, $[3]_5$, $[4]_5$. Explain your reasoning.

(28) Plot the sets $[1]_5$, $[2]_5$, $[3]_5$, $[4]_5$ on a number line. Explain your reasoning.

(29) Explain why one could say that $[4]_5 = [9]_5$.

(30) Explain why one could say that $[2]_5 = [-3]_5$.

(31) Explain what you think is meant by the expression:

$$[1]_5 + [2]_5 = [3]_5$$

(32) Explain what you think is meant by the expression:

$$[1]_5 + [4]_5 = [0]_5$$

(33) Stewie decided to count the pennies he had in his piggy bank. He decided it would be quicker to count by fives. However, he ended with two uncounted pennies. So he tried counting by twos but ended up with one uncounted penny. Next he counted by threes and then by fours, each time there was one uncounted penny. Though he knew he had less than a dollars worth of pennies, and more than 50 cents, he still didn't have an exact count. Can you help Stewie out? Explain your reasoning.

1.2 The Euclidean Algorithm

Up to this point, computing the GCD of two integers required you to factor both numbers. This can be difficult to do. The following algorithm, called the *Euclidean algorithm*, makes finding GCD's quite easy. With this said, algorithms can be tricky to explain. Let's try this—study the following calculations, they are examples of the Euclidean algorithm in action:

$$\begin{aligned} 22 &= \mathbf{6}\cdot 3+\mathbf{4} \\ \mathbf{6} &= \mathbf{4}\cdot 1+\boxed{\mathbf{2}} \\ 4 &= 2\cdot 2+0 \qquad \boxed{\therefore \gcd(22,6)=2} \end{aligned}$$

$$\begin{aligned} 33 &= \mathbf{24}\cdot 1+\mathbf{9} \\ \mathbf{24} &= \mathbf{9}\cdot 2+\mathbf{6} \\ \mathbf{9} &= \mathbf{6}\cdot 1+\boxed{\mathbf{3}} \\ 6 &= 3\cdot 2+0 \qquad \boxed{\therefore \gcd(33,24)=3} \end{aligned}$$

$$\begin{aligned} 42 &= \mathbf{16}\cdot 2+\mathbf{10} \\ \mathbf{16} &= \mathbf{10}\cdot 1+\mathbf{6} \\ \mathbf{10} &= \mathbf{6}\cdot 1+\mathbf{4} \\ \mathbf{6} &= \mathbf{4}\cdot 1+\boxed{\mathbf{2}} \\ 4 &= 2\cdot 2+0 \qquad \boxed{\therefore \gcd(42,16)=2} \end{aligned}$$

Question Can you explain how to do the Euclidean algorithm?

?

Question Can you explain why the Euclidean algorithm will always stop? Hint: Division Theorem.

?

Here is how Euclid himself explained his algorithm in Book VII, Proposition 1, of his book *The Elements*:

> Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

Question How does your explanation compare to Euclid's?

?

Study the following calculations:

$$\begin{array}{rclrcl} 22 = 6 \cdot 3 + 4 & \Leftrightarrow & 22 - 6 \cdot 3 = \mathbf{4} & & 6 - 4 \cdot 1 = 2 \\ 6 = 4 \cdot 1 + 2 & \Leftrightarrow & 6 - \mathbf{4} \cdot 1 = 2 & & 6 - (22 - 6 \cdot 3) \cdot 1 = 2 \\ 4 = 2 \cdot 2 + 0 & & & & 6 \cdot 4 + 22(-1) = 2 \end{array}$$

$$\boxed{\therefore 22x + 6y = 2 \text{ where } x = -1 \text{ and } y = 4}$$

$$\begin{array}{rclr} & & & 9 - 6 \cdot 1 = 3 \\ 33 = 24 \cdot 1 + 9 & \Leftrightarrow & 33 - 24 \cdot 1 = \mathbf{9} & 9 - (24 - 9 \cdot 2) \cdot 1 = 3 \\ 24 = 9 \cdot 2 + 6 & \Leftrightarrow & 24 - \mathbf{9} \cdot 2 = \mathbf{6} & 9 \cdot 3 + 24 \cdot (-1) = 3 \\ 9 = 6 \cdot 1 + 3 & \Leftrightarrow & 9 - \mathbf{6} \cdot 1 = 3 & (33 - 24 \cdot 1) \cdot 3 + 24 \cdot (-1) = 3 \\ 6 = 3 \cdot 2 + 0 & & & 33 \cdot 3 + 24 \cdot (-4) = 3 \end{array}$$

$$\boxed{\therefore 33x + 24y = 3 \text{ where } x = 3 \text{ and } y = -4}$$

Question Can you explain how to solve Diophantine equations of the form

$$ax + by = g$$

where $g = \gcd(a, b)$?

?

1.2.1 Fundamental Theorems

The Euclidean algorithm is also useful for theoretical questions.

Question Given nonzero integers a and b, what is the smallest positive integer that can be expressed as

$$ax + by$$

where x and y are also integers?

I'm feeling chatty, so I'll take this one. I claim that $g = \gcd(a, b)$ is the smallest positive integer that can be expressed as

$$ax + by$$

where x and y are integers. How do I know? Well, suppose there was a smaller positive integer, say s where:

$$ax + by = s$$

Hmmm... but we know that $g|a$ and $g|b$. This means that g divides the left-hand-side of the equation. This means that g divides the right-hand-side of the equation. So $g|s$—but this is impossible, as $s < g$. Thus g is the smallest integer that can be expressed as $ax + by$.

Believe it or not, we're going somewhere with all this. The next *lemma* will help us out. What is a lemma, you ask? A lemma is nothing but a little theorem that we have to help us solve another problem. Note that a lemma should not be confused with the more sour *lemon*, as that is something different and unrelated to what we are discussing.

Lemma 2 *If a, b, and c are integers with $\gcd(a, b) = 1$, then*

$$a|bc \qquad \textit{implies that} \qquad a|c.$$

Question Can you use the ideas above to explain why this lemma is true?

?

Now we have set the stage for our fundamental theorem—it is sometimes called the *Fundamental Theorem of Arithmetic*:

Theorem 3 (Unique Factorization) *Every positive integer can be factored uniquely (up to ordering) into primes.*

Proof Well, if an integer is prime, we are done. If an integer is composite, then it is divisible by a prime number. Divide and repeat with the quotient. If our original integer was n, we'll eventually get:

$$n = p_1 p_2 \cdots p_m$$

where some of the p_i's may be duplicates.

How do we know this factorization is unique? Well, suppose that

$$n = p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_l$$

where the p_i's are all prime and none of them equal any of the q_j's which are also prime. So, $\gcd(p_1, q_1) = 1$, and by the definition of "divides"

$$p_1 | q_1 (q_2 \cdots q_l).$$

So by our lemma above, p_1 must divide $(q_2 \cdots q_l)$. Repeat this enough times and you will find that $p_1 = q_j$ for one of the q_j's above. Repeat this process for the p_i's and you see that the factorization is unique. ■

Question Huh?! Can you explain what just happened drawing pictures and/or using symbols as necessary? Could you also give some examples?

?

Let's see the Unique Factorization Theorem for integers in action!

Question If $11|50a$, is it true that $11|a$?

I'll take this one. If $11|50a$, this means that

$$50a = 11 \cdot q \qquad \text{where } q \text{ is some integer.}$$

By the Unique Factorization Theorem for integers, we can factor both sides of the equation above in exactly one way. The upshot is that the primes that appear on the left-hand side of the equation must appear on the right-hand side of the equation. Since

$$2 \cdot 5^2 \cdot a = 11 \cdot q,$$

and I don't see 11 appearing as a factor on the left-hand side, but we know they must be there by the Unique Factorization Theorem! We conclude that 11 must be a factor of a, and hence $11|a$.

Problems for Section 1.2

(1) Explain what a *Diophantine equation* is. Give an example and explain why such a thing has real-world applications.

(2) Explain what the GCD of two integers is. Give some relevant and revealing examples/nonexamples.

(3) Explain what the LCM of two integers is. Give some relevant and revealing examples/nonexamples.

(4) Use the Euclidean algorithm to find: $\gcd(667, 713)$, $\gcd(671, 715)$, $\gcd(671, 713)$, $\gcd(682, 715)$, $\gcd(601, 735)$, and $\gcd(701, 835)$.

(5) Explain the advantages of using the Euclidean algorithm to find the GCD of two integers over factoring.

(6) Find integers x and y satisfying the following Diophantine equations:

(a) $667x + 713y = 92$
(b) $671x + 715y = 11$
(c) $671x + 713y = 41$
(d) $682x + 715y = 11$
(e) $601x + 735y = 4$
(f) $701x + 835y = 15$

(7) Given integers a, b, and c, explain how you know when a solution to a Diophantine equation of the form

$$ax + by = c$$

exists.

(8) Consider the Diophantine equation:

$$6x + 4y = 2$$

(a) Use the Euclidean Algorithm to find a solution to this equation. Explain your reasoning.
(b) Plot the line determined by $6x + 4y = 2$.
(c) Compute the slope of the line $6x + 4y = 2$, writing it in lowest terms. Show your work.
(d) Explain how to find infinitely many more solutions to the Diophantine equation using the slope found above.

(9) Explain why a Diophantine equation

$$ax + by = c$$

has either an infinite number of solutions or zero solutions.

(10) Josh owns a box containing beetles and spiders. At the moment, there are 46 legs in the box. How may beetles and spiders are currently in the box? Explain your reasoning.

(11) At the 7–11 Riverboat Casino, all chips are worth \$7 or \$11 dollars and the dealers only pay out in \$7 and/or \$11 chips.

(a) Explain how to place a bet for \$344.

(b) Suppose the bet of \$344 wins with 3-to-1 odds (meaning a bet of \$1 pays back \$3). Explain how the dealer pays out in \$7 and \$11 chips.

(12) How many different ways can thirty coins (nickles, dimes, and quarters) be worth five dollars? Explain your reasoning.

(13) When Ann is half as old as Mary will be when Mary is three times as old as Mary is now, Mary will be five times as old as Ann is now. Neither Ann nor Mary may vote. How old is Ann? Explain your reasoning.

(14) Lisa collects lizards, beetles and worms. She has more worms than lizards and beetles together. Altogether in the collection there are twelve heads and twenty-six legs. How many lizards does Lisa have? Explain your reasoning.

(15) Can you make exactly \$5 with exactly 100 coins assuming you can only use pennies, dimes, and quarters? If so how, if not why not? Explain your reasoning.

(16) A merchant purchases a number of horses and bulls for the sum of 1770 talers. He pays 31 talers for each bull, and 21 talers for each horse. How many bulls and how many horses does the merchant buy? Solve this problem, explain what a *taler* is, and explain your reasoning—note this problem is an old problem by L. Euler, it was written in the 1700's.

(17) A certain person buys hogs, goats, and sheep, totaling 100 animals, for 100 crowns; the hogs cost him $3\frac{1}{2}$ crowns a piece, the goats $1\frac{1}{3}$ crowns, and the sheep go for $\frac{1}{2}$ crown a piece. How many did this person buy of each? Explain your reasoning—note this problem is an old problem from *Elements of Algebra* by L. Euler, it was written in the 1700's.

(18) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

(a) If $7|13a$, then $7|a$.

(b) If $6|49a$, then $6|a$.

(c) If $10|65a$, then $10|a$.

(d) If $14|22a$, then $14|a$.

(e) $54|931^{21}$.

(f) $54|810^{33}$.

(g) If $a^2|b^2$, then $a|b$.

(19) If $x^2 = 11 \cdot y$, what can you say about y? Explain your reasoning.

(20) If $x^2 = 25 \cdot y$, what can you say about y? Explain your reasoning.

(21) When asked how many people were staying at the *Hotel Chevalier*, the clerk responded "The number you seek is the smallest positive integer such that dividing by 2 yields a perfect square, and dividing by 3 yields a perfect cube." How many people are staying at the hotel? Explain your reasoning.

1.3 Rational Numbers

Once you are familiar with integers, you start to notice something: Given an integer, it may or may not divide into another integer evenly. This property is at the heart of our notions of factoring and primality. Life would be very different if all nonzero integers divided evenly into one another.

Definition A **rational number** is a fraction of integers, where the denominator is nonzero.

The set of all rational numbers is denoted by the symbol $\mathbb{Q}$

$$\mathbb{Q} = \left\{ \frac{a}{b} \text{ such that } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

Fancy folks will replace the words *such that* with a colon ":" to get:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

We call this set the **rational numbers**. The letter $\mathbb{Q}$ stands for the word *quotient*, which should remind us of fractions.

1.3.1 Basic Meaning of Fractions

Like all numbers, fractions have meaning outside of their pure mathematical existence. Let's see if we can get to the heart of some of this meaning.

Question Draw a rectangle. Can you shade $3/8$ of this rectangle? Explain the steps you took to do this.

?

Question Draw a rectangle. Given a fraction a/b where $a \leqslant b$, explain how to shade a/b of this rectangle.

?

Question Draw a rectangle. How could you visualize $8/3$ of this rectangle? Explain the steps you took to do this.

?

Question Draw a rectangle. Given a fraction a/b where $a \geqslant b$, explain how to visualize a/b of this rectangle.

?

Question Draw a rectangle. Can you shade

$$\frac{3/8}{4}$$

of this rectangle? Explain the steps you took to do this.

?

Why do People Hate Fractions?

Why do so many people find fractions difficult? This is a question worth exploring. We'll guide you through some of the tough spots with some questions of our own.

Question Given a fraction a/b, come up with three other different fractions that are all equal to a/b. What confounding feature of fractions are we illustrating?

?

Question Given two fractions a/b and c/d, explain how to tell which fraction is larger. What confounding feature of fractions are we illustrating?

?

Question Given two fractions a/b and c/d with $a/b < c/d$, explain how one might find a fraction between them. What confounding feature of fractions are we illustrating?

?

Question Can you dream up numbers a, b, and c such that

$$\frac{a/b}{c} = \frac{a}{b/c}.$$

Can you dream up other numbers a', b', and c' such that

$$\frac{a'/b'}{c'} \neq \frac{a'}{b'/c'}.$$

What confounding feature of fractions are we illustrating?

?

Question Explain how to add two fractions a/b and c/d. What confounding feature of fractions are we illustrating?

?

Question Can you come up with any other reasons fractions are difficult?

?

1.3.2 Continued Fractions

We're going to use some tricks involving fractions to study numbers that have a nasty form. As an example, consider

$$\sqrt{2} = 1.4142135623\ldots$$

Yuck! That's just some crazy decimal. It would be nice if we could somehow see some order in this chaos! To do this, we'll need some definitions:

Definition A fraction of the form

$$a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}$$

is called a **continued fraction**. If $a_1, a_2, a_3, \ldots$ are all 1, we will call this a **simple continued fraction**.

Definition The **whole-number part** of a number is the largest whole number which is less than or equal to the given number.

Example *The whole-number part of* 2 *is* 2, *while the whole-number part of* 5.32 *is* 5.

Definition The **fractional part** of a number is the number minus its whole-number part.

Example *The fractional part of* 2 *is* 0, *while the fractional part of* 5.32 *is* 0.32.

Question Why don't we just describe the fractional part of a number as the part that is to the right of the decimal point? Hint: Think about $0.99999\ldots$.

?

Given any number, we can write it as a simple continued fraction. Consider 13/5. To start note that

$$3 > \frac{13}{5} > 2.$$

So this means that

$$\frac{13}{5} = 2 + \frac{3}{5}.$$

Here 2 is the whole-number part and 3/5 is the fractional part of 13/5. But in the simple continued fraction, our numerator is 1, not 3. How do we deal with this? Well,

$$\frac{13}{5} = 2 + \frac{3}{5} = 2 + \frac{1}{\frac{5}{3}}.$$

This is an improvement but we only want whole numbers in our simple continued fractions and not 5/3. So we write

$$\frac{5}{3} = 1 + \frac{2}{3}$$

which gives us

$$\frac{13}{5} = 2 + \cfrac{1}{1 + \cfrac{2}{3}}.$$

Again, we want our numerator to be 1, not 2 so we will repeat the steps above to get

$$\frac{13}{5} = 2 + \cfrac{1}{1 + \cfrac{2}{3}} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

and this last expression is the simple continued fraction for 13/5. We could also list our steps as:

$$\begin{aligned} \frac{13}{5} &= \mathbf{2} + \frac{3}{5} \\ \frac{1}{\frac{3}{5}} = \frac{5}{3} &= \mathbf{1} + \frac{2}{3} \\ \frac{1}{\frac{2}{3}} = \frac{3}{2} &= \mathbf{1} + \frac{1}{2} \\ \frac{1}{\frac{1}{2}} &= \mathbf{2} + 0 \end{aligned}$$

These boldface numbers tell us our continued fraction expansion.

We can also find the simple continued fraction of numbers which are not already fractions (otherwise this would all be a bit silly). Consider $\sqrt{2}$, remember how yucky it was?

$$\sqrt{2} = 1.4142135623\ldots$$

To beautify this number, note that $2 > \sqrt{2} > 1$. So this means that

$$\sqrt{2} = 1 + (\sqrt{2} - 1).$$

Where 1 is the whole-number part and $(\sqrt{2} - 1)$ is the fractional part of $\sqrt{2}$. Alright, now look at $1/(\sqrt{2} - 1)$. Again we want to separate the whole-number part and the fractional part. With a little algebra we see that

$$\frac{1}{\sqrt{2} - 1} = \frac{\sqrt{2} + 1}{2 - 1} = \sqrt{2} + 1 = 2 + (\sqrt{2} + 1 - 2) = 2 + (\sqrt{2} - 1).$$

Now don't you get bogged down in the steps. Here it is in fast forward:

$$\begin{aligned}
\sqrt{2} &= \mathbf{1} + (\sqrt{2}-1) \\
\frac{1}{(\sqrt{2}-1)} &= \mathbf{2} + (\sqrt{2}-1) \\
\frac{1}{(\sqrt{2}-1)} &= \mathbf{2} + (\sqrt{2}-1) \\
\frac{1}{(\sqrt{2}-1)} &= \mathbf{2} + (\sqrt{2}-1), \\
&\vdots
\end{aligned}$$

At each step we want:

$$\text{number} = \text{whole-number part} + \text{fractional part}$$

Now from the bold-faced numbers above we will make our continued fraction:

$$\sqrt{2} = \mathbf{1} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \cdots}}}}$$

Beautiful!

Question Can you explain why this works?

?

Question Do you think you could find a regular fraction equal to $\sqrt{2}$?

?

Some Hidden Beauties

Continued fractions allow us to see patterns that are otherwise totally hidden.

Check out $e = 2.718281828459045\ldots$. It turns out that

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cdots}}}}}}}}}$$

Also check out $\pi = 3.14159265358\ldots$. It turns out that we can get a nice continued fraction for π:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cfrac{9^2}{6 + \cfrac{11^2}{6 + \cdots}}}}}}$$

Wow!

Problems for Section 1.3

(1) Describe the set of rational numbers. Give some relevant and revealing examples/nonexamples.

(2) What algebraic properties do the rational numbers enjoy that the integers do not? Explain your reasoning.

(3) What number gives the same result when added to 1/2 as when multiplied by 1/2. Explain your reasoning.

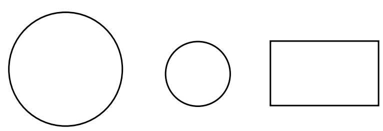
(4) Draw a rectangle to represent a garden. Shade in 3/5 of the garden. Without changing the shading, show why 3/5 of the garden is the same as 12/20 of the garden. Explain your reasoning.

(5) Find yourself a sheet of paper. Now, suppose that this sheet of paper is actually 4/5 of some imaginary larger sheet of paper.

- Shade your sheet of paper so that 3/5 of the larger (imaginary) sheet of paper is shaded in. Explain why your shading is correct.
- Explain how this shows that

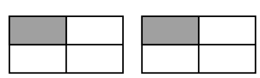
$$\frac{3/5}{4/5} = \frac{3}{4}.$$

(6) Shade in 2/5 of the entire picture below:



Explain your reasoning.

(7) What fractions could the following picture be illustrating?



Explain your reasoning.

(8) Try to find the largest rational number smaller than 3/7. Explain your solution or explain why this cannot be done.

(9) How many rational numbers are there between 3/4 and 4/7? Find 3 of them. Explain your reasoning.

(10) Let a, b, c, and d be positive integers such that

$$a < b < c < d$$

Is it true that

$$\frac{a}{b} < \frac{c}{d}?$$

Explain your reasoning.

(11) Let a, b, c, and d be positive consecutive integers such that

$$a < b < c < d.$$

Is it true that

$$\frac{a}{b} < \frac{c}{d}?$$

Explain your reasoning.

(12) Let a, b, c, and d be positive consecutive integers such that

$$a < b < c < d.$$

Is it true that

$$\frac{a}{b} < \frac{b}{c} < \frac{c}{d}?$$

Explain your reasoning.

(13) Can you generalize Problem (11) and Problem (12) above? Explain your reasoning.

(14) Let a, b, c, and d be positive integers such that

$$\frac{a}{b} < \frac{c}{d}.$$

Is it true that

$$\frac{a}{a+b} < \frac{c}{c+d}?$$

Explain your reasoning.

(15) A youthful Bart loved to eat hamburgers. He ate 5/8 pounds of hamburger meat a day. After testing revealed that his blood consisted mostly of cholesterol, Bart decided to alter his eating habits by cutting his hamburger consumption by 3/4. How many pounds of hamburger a day did Bart eat on his new "low-cholesterol" diet? Explain your reasoning.

(16) Courtney and Paolo are eating popcorn. Unfortunately, 1/3rd of the popcorn is poisoned. If Courtney eats 5/16th of the bowl and Paolo eats 5/13ths of the bowl, did at least one of them eat poisoned kernel? Also, at least how many kernels of popcorn are in the bowl? Explain your reasoning.

(17) Three brothers and a sister won the lottery together and plan to share it equally. If the brothers alone had shared the money, then they would have increased the amount they each received by \$20. How much was won in the lottery? Explain your reasoning.

(18) Best of clocks, how much of the day is past if there remains twice two-thirds of what is gone? Explain what this strange question is asking and answer the question being sure to explain your reasoning—note this is an old problem from the *Greek Anthology* compiled by Metrodorus around the year 500.

(19) Monica, Tessa, and Jim are grading papers. If it would take Monica 2 hours to grade them all by herself, Tessa 3 hours to grade them all by herself, and Jim 4 hours to grade them all by himself how long would it take them to grade the exams if they all work together? Explain your reasoning.

(20) Say quickly, friend, in what portion of a day will four fountains, being let loose together, fill a container which would be filled by the individual fountains in one day, half a day, a third of a day, and a sixth of a day respectively? Explain your reasoning—note this is an old problem from the Indian text *Lilavati* written in the 1200s.

(21) John spent a fifth of his life as a boy growing up, another one-sixth of his life in college, one-half of his life as a bookie, and has spent the last six years in prison. How old is John now? Explain your reasoning

(22) Diophantus was a boy for 1/6th of his life, his beard grew after 1/12 more, he married after 1/7th more, and a son was born five years after his marriage. Alas! After attaining the measure of half his father's full life, chill fate took the child. Diophantus spent the last four years of his life consoling his grief through mathematics. How old was Diophantus when he died? Explain your reasoning—note this is an old problem from the *Greek Anthology* compiled by Metrodorus around the year 500.

(23) Wandering around my home town (perhaps trying to find my former self!), I suddenly realized that I had been in my job for one-quarter of my life. Perhaps the melancholia was getting the best of me, but I wondered: How long would it be until I had been in my job for one-third of my life? Explain your reasoning.

(24) In a certain adult condominium complex, 2/3 of the men are married to 3/5 of the women. Assuming that men are only married to women (and vice versa), and that married residents' spouses are also residents, what portion of the residents are married?

   (a) Before any computations are done, use common sense to guess the solution to this problem.
   (b) Choose numbers for the above problem and solve the problem with these numbers. Use this computation to test your guess—and possibly make a new guess.
   (c) Use algebra to verify your guess.

Explain your reasoning in each step above.

(25) Fred and Frank are two fitness fanatics on a run from A to B. Fred runs half the way and walks the other half. Frank runs for half the time and walks for the other half. They both run at the same speed and they both walk at the same speed. Who finishes first?

(a) Before any computations are done, use common sense to guess the solution to this problem.

(b) Choose numbers for the above problem and solve the problem with these numbers. Use this computation to test your guess—and possibly make a new guess.

(c) Use algebra to verify your guess.

Explain your reasoning in each step above.

(26) Andy and Sandy run a race of a certain distance. Sandy finishes 1/10 of the distance ahead of Andy. After some discussion, Andy and Sandy decide to race the certain distance again, this time Sandy will start 1/10 of the distance behind Andy to "even-up" the competition. Who wins this time?

(a) Before any computations are done, use common sense to guess the solution to this problem.

(b) Choose numbers for the above problem and solve the problem with these numbers. Use this computation to test your guess—and possibly make a new guess.

(c) Use algebra to verify your guess.

Explain your reasoning in each step above.

(27) You have two beakers, one that contains water and another that contains an equal amount of oil. A certain amount of water is transferred to the oil and thoroughly mixed. Immediately, the same amount of the mixture is transferred back to the water. Is there now more water in the oil or is there more oil in the water?

(a) Before any computations are done, use common sense to guess the solution to this problem.

(b) Choose numbers for the above problem and solve the problem with these numbers. Use this computation to test your guess—and possibly make a new guess.

(c) Use algebra to verify your guess.

Explain your reasoning in each step above.

(28) While on a backpacking trip Lisa hiked five hours, first along a level path, then up a hill, then turned round and hiked back to her base camp along the same route. She walks 4 miles per hour on a level trail, 3 uphill, and 6 downhill. Find the total distance traveled. Explain your reasoning.

(29) Three drops of *Monica's XXX Hot Sauce* were mixed with five cups of chili mix to make a spicy treat. Later, two drops of *Monica's XXX Hot Sauce* were mixed with three cups of chili. Which bowl of soup is spicier? Josh suggested the following method to compare the concentrations:

- Remove the second from the first, that is: Take 3 sauce and 5 chili, and remove 2 soup and 3 chili. So we are now comparing

  1 sauce and 2 soup with 2 sauce and 3 soup.

- Now remove the first from the second, that is: Take 2 sauce and 3 soup, and remove 1 sauce and 2 soup. So we are now comparing

  1 sauce and 2 soup with 1 sauce and 1 soup.

Now you can see that the second is more concentrated (and hence hotter!) than the first. Is this correct? Will this strategy always/ever work? Explain your reasoning.

(30) Explain what the **whole-number part** and what the **fractional part** of a number are. Give examples.

(31) Find the simple continued fraction expansion of 5/3. Explain your work.

(32) Find the simple continued fraction expansion of 15/11. Explain your work.

(33) Find the simple continued fraction expansion of 22/17. Explain your work.

(34) Using a calculator, find the first five terms in the simple continued fraction expansion of π. What number do you get by only considering the first term? The first four?

(35) Find the simple continued fraction expansion of $\sqrt{5}$. Explain your work.

(36) Find the simple continued fraction expansion of $\sqrt{10}$. Explain your work.

(37) Find the simple continued fraction expansion of $\sqrt{17}$. Explain your work.

(38) Find the simple continued fraction expansion of $\sqrt{26}$. Explain your work.

(39) Find the simple continued fraction expansion of

$$\frac{1+\sqrt{5}}{2}$$

Explain your work. Note—this is a special number, it is called the *golden ratio.* More on this later..

(40) Courtney Gibbons is someone who has a rather unusual tattoo. She was kind enough to let an unusual person like me take a picture of it. What does her tattoo represent? Explain your reasoning.

(41) Find the simple continued fraction expansion of $1/2$. Explain your work.

(42) Find the simple continued fraction expansion of 11. Explain your work.

(43) What is it about the numbers $2, 5, 10, 17, 26$ that makes it easy to compute the continued fraction expansion of the square-roots of these numbers? Explain your answer.

(44) What is the best rational approximation of $\sqrt{2}$ where the denominator is less than 10? Less than 20? Less than 30? Less than 100?

(45) What is the best rational approximation of $\sqrt{5}$ where the denominator is less than 10? Less than 20? Less than 30? Less than 100?

(46) What is the best rational approximation of $\sqrt{3}$ where the denominator is less than 10? Less than 20? Less than 30? Less than 100?

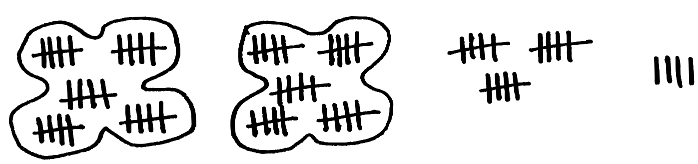
Chapter 2

Arithmetic and Algebra

> As I made my way home, I thought Jem and I would get grown but there wasn't much else left for us to learn, except possibly algebra.
>
> —Harper Lee

2.1 All Your Base Are Belong to Us

Imagine 500 generations past—that's on the order of 10000 years, the dawn of what we would call civilization. This is a long time ago, well before the *Epic of Gilgamesh*. Even then people already knew the need to keep track of numbers. However, they didn't use the numbers we know and love (that's right, *love*!), they used tally-marks. Now what if "a friend" of yours had a time machine? What if they traveled though time and space and they decided to take you back 500 generations? Perhaps you would meet a nice man named Lothar[1] who is trying to keep track of his goats. He has the following written on a clay tablet:



From this picture you discern that Lothar has 69 goats. Lothar is studying the tablet intently when his wife, Gertrude, comes in. She tries in vain to get Lothar to keep track of his goats using another set of symbols:

[1] *Lothar of the Hill People* is his full name

A heated debate between Lothar and Gertrude ensues, the exact details of which are still a mystery. We do glean the following facts:

(1) Under Gertrude's scheme, five goats are denoted by:

10

(2) The total number of Lothar's goats is denoted by:

234

Question Can you explain Gertrude's counting scheme?

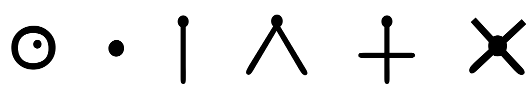
?

Did I mention that "your friend's" time machine is also a spaceship? Oh. . . Well it is. Now you both travel to the planet Omicron Persei 8. There are two things you should know about the inhabitants of Omicron Persei 8:

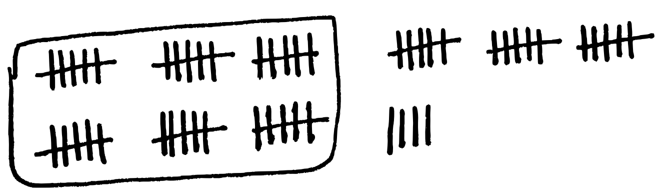
(1) They only have 3 fingers on each hand.

(2) They can eat a human in one bite.

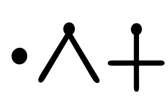
As you can see, there are serious issues that any human visitor to Omicron Persei 8 must deal with. For one thing, since the Omicronians only have 3 fingers on each hand, they've only written down the following symbols for counting:



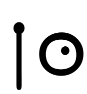
Emperor Lrrr of the Omicronians is tallying how many humans he ate last week



when his wife, Ndnd, comes in and reminds him that he can write this number using their fancy symbols as:



After reading some restaurant menus, you find out that twelve tally-marks are denoted by the symbols:



Question Can you explain the Omicronians' counting scheme?

?

At this point you hop back into "your friend's" space-time ship. "Your friend" kicks off their shoes. You notice that "your friend" has 6 toes on each foot. You strike up a conversation about the plethora of toes. Apparently this anomaly has enabled "your friend" to create their own counting scheme, which they say is based on:

- Toes
- Feets
- Feets of Feets
- and so on

"Your friend" informs you that they would write the number you know as "twenty-six" as 22 or "two feets and two toes." What?! Though you find the conversation to be dull and stinky, you also find out that "your friend" uses two more symbols when they count. "Your friend" uses the letter A to mean what you call "ten," and the letter B to mean what you call "eleven!"

Question Can you explain "your friend's" counting scheme?

?

Problems for Section 2.1

(1) You meet some Tripod aliens, they tally by threes. Thankfully for everyone involved, they use the symbols 0, 1, and 2.

(a) Can you explain how a Tripod would count from 11 to 201? Be sure to carefully explain what number comes after 22.

(b) What number comes before 10? 210? 20110? Explain your reasoning.

(2) You meet some people who tally by sevens. They use the symbols O, A, B, C, D, E, and F.

(a) What do the individual symbols O, A, B, C, D, E, and F mean?

(b) Can you explain how they would count from DD to AOC? Be sure to carefully explain what number comes after FF.

(c) What number comes before AO? ABO? $EOFFO$? Explain your reasoning.

(3) Now, suppose that you meet a hermit who tallies by thirteens. Explain how he might count. Give some relevant and revealing examples.

(4) While visiting Mos Eisley spaceport, you stop by Chalmun's Cantina. After you sit down, you notice that one of the other aliens is holding a discussion on fractions. Much to your surprise, they explain that 1/6 of 30 is 4. You are unhappy with this, knowing that 1/6 of 30 is in fact 5, yet their audience seems to agree with it, not you. Next the alien challenges its audience by asking, "what is 1/4 of 10?" What is the correct answer to this question and how many fingers do the aliens have? Explain your reasoning.

(5) When the first Venusian to visit Earth attended a 6th grade class, it watched the teacher show that

$$\frac{3}{12} = \frac{1}{4}.$$

"How strange," thought the Venusian. "On Venus, $\frac{4}{12} = \frac{1}{4}$." What base do Venusians use? Explain your reasoning.

(6) When the first Martian to visit Earth attended a high school algebra class, it watched the teacher show that the only solution of the equation

$$5x^2 - 50x + 125 = 0$$

is $x = 5$.

"How strange," thought the Martian. "On Mars, $x = 5$ is a solution of this equation, but there also is another solution." If Martians have more fingers than humans, how many fingers do Martians have? Explain your reasoning.

(7) In one of your many space-time adventures, you see the equation

$$\frac{3}{10} + \frac{4}{13} = \frac{21}{20}$$

written on a napkin. How many fingers did the beast who wrote this have? Explain your reasoning.

(8) What is the smallest number of weights needed to produce every integer-value mass from 0 grams to say n grams? Explain your reasoning.

(9) Starting at zero, how high can you count using just your fingers?

(a) Explain how to count to 10.

(b) Explain how to count to 35.

(c) Explain how to count to 1023.

(d) Explain how to count to 59048.

(e) Can you count even higher?

Explain your reasoning.

2.2 Arithmetic

Consider this question:

Question Can you *think* about something if you lack the *vocabulary* required to discuss it?

?

2.2.1 Nomenclature

The numbers and operations we work with have properties whose importance are so fundamental that we have given them names. Each of these properties is surly well known to you; however, the importance of the name is that it gives a keen observer the ability to see and articulate fundamental structures in arithmetic and algebra.

The Associative Property An operation $\star$ is called **associative** if for all numbers a, b, and c:

$$a \star (b \star c) = (a \star b) \star c$$

The Commutative Property An operation $\star$ is called **commutative** if for all numbers a and b:

$$a \star b = b \star a$$

The Distributive Property An operation $\star$ is said to be **distributive** over another operation $\maltese$ if for all numbers a, b, and c:

$$a \star (b \maltese c) = (a \star b) \maltese (a \star c) \qquad \text{and} \qquad (b \maltese c) \star a = (b \star a) \maltese (c \star a)$$

You may find yourself a bit distressed over some of the notation used above. In particular you surly notice that we were using crazy symbols like $\star$ and $\maltese$. We did this for a reason. The properties above may hold for more than one operation. Let's explore this:

Question Can you give examples of operations that are associative? Can you give examples of operations that are not associative?

?

Question Can you give examples of operations that are commutative? Can you give examples of operations that are not commutative?

?

Question Can you give examples of two operations where one distributes over the other? Can you give examples of operations that do not distribute?

?

2.2.2 Algorithms

In elementary school you learned many algorithms. One of the first algorithms you learned was for adding numbers. Here we show you an example of the algorithm in action:

Basic Addition Algorithm Here is an example of the basic addition algorithm:

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

Question Can you give a description of how to perform this algorithm?

As a gesture of friendship, I'll take this one. All we are doing here is adding each column of digits at a time, starting with the right-most digit

$$\begin{array}{r} \\ 89\mathbf{2} \\ +39\mathbf{8} \\ \hline \mathbf{10} \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{1} \\ 89\mathbf{2} \\ +39\mathbf{8} \\ \hline \mathbf{0} \end{array}$$

If our column of digits sums to 10 or higher, then we must "carry" the tens-digit of our sum to the next column. This process repeats until we run out of digits on the left.

$$\begin{array}{r} \mathbf{1} \\ 8\mathbf{9}2 \\ +3\mathbf{9}8 \\ \hline \mathbf{19}0 \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{1}1 \\ \mathbf{8}92 \\ +\mathbf{3}98 \\ \hline \mathbf{12}90 \end{array}$$

We're done!

Question Can you show the "behind-the-scenes" algebra going on here?

I'll take this one too. Sure, you just write:

$$\begin{aligned} 892 + 398 &= (8 \cdot 10^2 + 9 \cdot 10 + 2) + (3 \cdot 10^2 + 9 \cdot 10 + 8) \\ &= 8 \cdot 10^2 + 9 \cdot 10 + 2 + 3 \cdot 10^2 + 9 \cdot 10 + 8 \\ &= 8 \cdot 10^2 + 3 \cdot 10^2 + 9 \cdot 10 + 9 \cdot 10 + 2 + 8 \\ &= (8+3) \cdot 10^2 + (9+9) \cdot 10 + (2+8) \\ &= (8+3) \cdot 10^2 + (9+9) \cdot 10 + 10 + 0 \\ &= (8+3) \cdot 10^2 + (9+9+1) \cdot 10 + 0 \\ &= (8+3) \cdot 10^2 + (10+9) \cdot 10 + 0 \\ &= (8+3+1) \cdot 10^2 + 9 \cdot 10 + 0 \\ &= 12 \cdot 10^2 + 9 \cdot 10 + 0 \\ &= 1290 \end{aligned}$$

Wow! That was a lot of algebra. At each step, you should be able to explain how to get to the next step, and state which algebraic properties are being used.

Basic Multiplication Algorithm Here is an example of the basic multiplication algorithm:

$$\begin{array}{r} 23 \\ 634 \\ \times \quad 8 \\ \hline 5072 \end{array}$$

Question Can you give a description of how to perform this algorithm?

Me me me me! All we are doing here is multiplying each digit of the multi-digit number by the single digit number.

$$\begin{array}{r} \\ 63\mathbf{4} \\ \times \quad \mathbf{8} \\ \hline \mathbf{32} \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{3} \\ 63\mathbf{4} \\ \times \quad \mathbf{8} \\ \hline \mathbf{2} \end{array}$$

If our product is 10 or higher, then we must "carry" the tens-digit of our product to the next column. This "carried" number is then added to our new product. This process repeats until we run out of digits on the left.

$$\begin{array}{r} \mathbf{3} \\ 6\mathbf{3}4 \\ \times \quad \mathbf{8} \\ \hline \mathbf{27}2 \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{2}3 \\ \mathbf{6}34 \\ \times \quad \mathbf{8} \\ \hline \mathbf{50}72 \end{array}$$

We're done!

Question Can you show the "behind-the-scenes" algebra going on here?

You betcha! Just write:

$$\begin{aligned} 634 \cdot 8 &= (6 \cdot 10^2 + 3 \cdot 10 + 4) \cdot 8 \\ &= 6 \cdot 8 \cdot 10^2 + 3 \cdot 8 \cdot 10 + 4 \cdot 8 \\ &= 6 \cdot 8 \cdot 10^2 + 3 \cdot 8 \cdot 10 + 32 && (✿) \\ &= 6 \cdot 8 \cdot 10^2 + (3 \cdot 8 + 3) \cdot 10 + 2 && (❀) \\ &= 6 \cdot 8 \cdot 10^2 + 270 + 2 && (❄) \\ &= (6 \cdot 8 + 2) \cdot 10^2 + 7 \cdot 10 + 2 && (✪) \\ &= 50 \cdot 10^2 + 7 \cdot 10 + 2 \\ &= 5 \cdot 10^3 + 0 \cdot 10^2 + 7 \cdot 10 + 2 \\ &= 5072 \end{aligned}$$

Ahhhhh! Algebra works. Remember just as before, at each step you should be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (✿) and (❀)? What about between lines (❄) and (❂)?

$$?$$

Basic Division Algorithm Once more we meet with this old foe—long division. Here is an example of the basic division algorithm:

$$\begin{array}{r} 97\ \text{R1} \\ 8\overline{)777} \\ \underline{72} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

Question Can you give a description of how to perform this algorithm?

Yes! I'm all about this sort of thing. All we are doing here is applying our division theorem for each digit of the multi-digit dividend (the number under the division symbol) by the single digit divisor (the left-most number). We start by noting that 8 won't go into 7, and so we see how many times 8 goes into 77.

$$\begin{array}{r} \mathbf{9} \\ \mathbf{8}\overline{)\mathbf{77}7} \\ \underline{\mathbf{72}} \\ \mathbf{5} \end{array} \qquad \leadsto \qquad \begin{aligned} n &= d\cdot q + r \\ 77 &= 8\cdot 9 + 5 \end{aligned}$$

Now we drop the other 7 down, and see how many times 8 goes into 57.

$$\begin{array}{r} 9\mathbf{7} \\ \mathbf{8}\overline{)777} \\ \underline{72} \\ \mathbf{57} \\ \underline{\mathbf{56}} \\ \mathbf{1} \end{array} \qquad \leadsto \qquad \begin{aligned} n &= d\cdot q + r \\ 57 &= 8\cdot 7 + 1 \end{aligned}$$

This process repeats until we run out of digits in the dividend.

Question Can you show the "behind-the-scenes" algebra going on here?

Of course—but this time things will be a bit different.

$$
\begin{aligned}
77 &= 8 \cdot 9 + 5 \\
77 \cdot 10 &= (8 \cdot 9 + 5) \cdot 10 \\
77 \cdot 10 &= 8 \cdot 9 \cdot 10 + 5 \cdot 10 \\
77 \cdot 10 + 7 &= 8 \cdot 9 \cdot 10 + 5 \cdot 10 + 7 \\
777 &= 8 \cdot (9 \cdot 10) + 57 && (\text{✿}) \\
777 &= 8 \cdot (9 \cdot 10) + (8 \cdot 7 + 1) && (\text{❀}) \\
777 &= 8 \cdot (9 \cdot 10) + 8 \cdot 7 + 1 && (\text{❄}) \\
777 &= 8 \cdot (9 \cdot 10 + 7) + 1 && (\text{❂}) \\
777 &= 8 \cdot 97 + 1
\end{aligned}
$$

Looks good to me, but remember: At each step you must be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (✿) and (❀)? What about between lines (❄) and (❂)?

?

Division Algorithm Without Remainder Do you remember that the division algorithm can be done in such a way that there is no remainder? Here is an example of the division algorithm without remainder:

$$
\begin{array}{r}
0.75 \\
4\overline{\big)3.00} \\
\underline{2\,8} \\
20 \\
\underline{\underline{20}}
\end{array}
$$

Question Can you give a description of how to perform this algorithm?

I'm getting a bit tired, but I think I can do this last one. Again, all we are doing here is applying our division theorem for each digit of the multi-digit dividend (the number under the division symbol) by the single digit divisor (the left-most number) adding zeros after the decimal point as needed. We start by noting that 4 won't go into 3, and so we see how many times 4 goes into 3.0. Mathematically this is the same question; however, by thinking of the 3.0 as 30, we put ourselves into familiar territory. Since

$$4 \cdot 7 = 30 \qquad \Rightarrow \qquad 4 \cdot 7 \cdot 10^{-1} = 30 \cdot 10^{-1} = 3$$

this will work as long as we put our 7 immediately to the right of the decimal point.

$$
\begin{array}{r}
\mathbf{0.7} \\
\mathbf{4}\overline{\big)\mathbf{3.0}} \\
\underline{\mathbf{2\,8}} \\
\mathbf{2}
\end{array}
\qquad \rightsquigarrow \qquad
\begin{aligned}
n &= d \cdot q + r \\
30 &= 4 \cdot 7 + 2
\end{aligned}
$$

Now we are left with a remainder of .2. To take care of this, we drop another 0 down and see how many times 4 goes into 20. Since

$$4 \cdot 5 = 20 \qquad \Rightarrow \qquad 4 \cdot 5 \cdot 10^{-2} = 5 \cdot 10^{-2} = 0.05$$

this will work as long as we put our 5 two spaces to the right of the decimal point.

$$\begin{array}{r} 0.7\mathbf{5} \\ \mathbf{4}\,\big)\overline{3.0\mathbf{0}} \\ \underline{2\,8} \\ \mathbf{20} \\ \underline{\underline{\mathbf{20}}} \end{array} \qquad \leftrightsquigarrow \qquad \begin{aligned} n &= d \cdot q + r \\ 20 &= 4 \cdot 5 + 0 \end{aligned}$$

This process repeats until we obtain a division with no remainder, or until we see repetition in the digits of the quotient.

Question Can you show the "behind-the-scenes" algebra going on here?

Let's do it:

$$\begin{aligned}
3 &= 4 \cdot 0 + 3 \\
3.0 &= (4 \cdot 7 + 2) \cdot 10^{-1} \\
3.0 &= 4 \cdot (7 \cdot 10^{-1}) + 2 \cdot 10^{-1} \\
3.00 &= 4 \cdot (7 \cdot 10^{-1}) + 20 \cdot 10^{-2} \\
3.00 &= 4 \cdot (7 \cdot 10^{-1}) + (4 \cdot 5) \cdot 10^{-2} && (\text{❄}) \\
3.00 &= 4 \cdot (7 \cdot 10^{-1}) + 4 \cdot (5 \cdot 10^{-2}) && (\text{✪}) \\
3.00 &= 4 \cdot (7 \cdot 10^{-1} + 5 \cdot 10^{-2}) \\
3.00 &= 4 \cdot 0.75
\end{aligned}$$

Looks good to me, but remember: At each step you must be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (❄) and (✪)?

?

Problems for Section 2.2

(1) Explain what it means for an operation $\star$ to be *associative*. Give some relevant and revealing examples.

(2) Explain what it means for an operation $\star$ to be *commutative*. Give some relevant and revealing examples.

(3) Explain what it means for an operation $\star$ to *distribute* over another operation $\maltese$. Give some relevant and revealing examples.

(4) Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.

(5) Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.

(6) Wacky Wally likes has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.

(7) Cheap Carl likes to give a tip of $13\frac{1}{3}\%$ when he is at restaurants. He does this by dividing his bill by 10 and then adding one-third more to this number. Explain why this works.

(8) Reasonable Rebbecca likes to give a tip of 18% when she is at restaurants. She does this by dividing her bill by 5 and then removing one-tenth of this number. Explain why this works.

(9) Can you think of and justify any other schemes for computing the tip?

(10) Here is an example of the basic addition algorithm:

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(11) Here is an example of the column addition algorithm:

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(12) Here is an example of the banker's addition algorithm:

$$\begin{array}{r} 892 \\ +398 \\ \hline 1\mathbf{0} \\ 1\mathbf{9} \\ \mathbf{12} \\ \hline 1290 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(13) Here is an example of the basic subtraction algorithm:

$$\begin{array}{rccc} & & 8 & \\ & 8 & \cancel{9} & {}^{1}2 \\ - & 3 & 7 & 8 \\ \hline & 5 & 1 & 4 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(14) Here is an example of the subtraction by addition algorithm:

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \quad \leadsto \quad \begin{array}{ll} 8 + \mathbf{4} = 12 & \text{add 1 to 7 to get 8} \\ 8 + \mathbf{1} = 9 & \\ 3 + \mathbf{5} = 8 & \end{array}$$

(a) Explain how to perform this algorithm.
(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
(c) Show the "behind-the-scenes" algebra that is going on here.

(15) Here is an example of the Austrian subtraction algorithm:

$$\begin{array}{r} 8\ \ 9\ {}^{1}2 \\ -3\ {}^{8}\cancel{7}\ \ 8 \\ \hline 5\ \ 1\ \ 4 \end{array}$$

(a) Explain how to perform this algorithm.
(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
(c) Show the "behind-the-scenes" algebra that is going on here.

(16) Here is an example of the basic multiplication algorithm:

$$\begin{array}{r} 23\ \\ 634 \\ \times\quad 8 \\ \hline 5072 \end{array}$$

(a) Explain how to perform this algorithm.
(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
(c) Show the "behind-the-scenes" algebra that is going on here.

(17) Here is an example of the partial-products multiplication algorithm:

$$\begin{array}{r} 634 \\ \times\quad 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

(a) Explain how to perform this algorithm.
(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.
(c) Show the "behind-the-scenes" algebra that is going on here.

(18) Here is an example of the basic division algorithm:

$$\begin{array}{r} 97 \text{ R1} \\ 8\overline{)777} \\ \underline{72}\ \\ 57 \\ \underline{56} \\ 1 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(19) Here is an example of the scaffolding division algorithm:

$$
\begin{array}{r}
7\\
90\\
8\,\overline{)777}\\
\underline{720}\\
57\\
\underline{56}\\
1
\end{array}
$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(20) Here is an example of the partial-quotients division algorithm:

$$
\begin{array}{r}
4\\
10\\
10\\
10\\
8\,\overline{)277}\\
\underline{80}\\
197\\
\underline{80}\\
117\\
\underline{80}\\
37\\
\underline{32}\\
5
\end{array}
$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(21) Here is an example of the multi-digit multiplication algorithm:

$$
\begin{array}{r}
634\\
\underline{\times 216}\\
3804\\
6340\\
\underline{126800}\\
136944
\end{array}
$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here—you may assume that you already know the algebra behind the basic multiplication algorithm.

(22) Here is an example of the addition algorithm with decimals:

$$\begin{array}{r} 1 \\ 37.2 \\ +8.74 \\ \hline 45.94 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(23) Here is an example of the multiplication algorithm with decimals:

$$\begin{array}{r} 3.40 \\ \times\ .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(24) Here is an example of the division algorithm without remainder:

$$\begin{array}{r} 0.75 \\ 4\,\overline{)3.00} \\ \underline{2\,8} \\ 20 \\ \underline{\underline{20}} \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(25) In the following addition problem, every number has been replaced with a letter.

```
   MOON
+  SUN
-------
  PLUTO
```

Recover the original problem and solution. Explain your reasoning. Hint: S = 6 and U = 5.

(26) In the following addition problem, every number has been replaced with a letter.

```
   SEND
+  MORE
-------
  MONEY
```

Recover the original problem and solution. Explain your reasoning.

(27) In the following subtraction problem, every number has been replaced with a letter.

```
  DEFER
-  DU7Y
-------
   N2G2
```

Recover the original problem and solution. Explain your reasoning.

(28) In the following two subtraction problems, every number has been replaced with a letter.

```
 NINE      NINE
- TEN     - ONE
-----     -----
  TWO       ALL
```

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

(29) In the following multiplication problem, every number has been replaced with a letter.

```
   LET
×   NO
------
   SOT
  NOT
------
  FRET
```

Recover the original problem and solution. Explain your reasoning.

(30) The following is a long division problem where every digit except 7 was replaced by X.

```
         X7X
    ________
XX ) XXXXX
     X77
     ---
      X7X
      X7X
      ---
        XX
        XX
        ==
```

Recover the numbers from this long division problem. Explain your reasoning.

(31) The following is a long division problem where the digits were replaced by X except in the quotient—where they were almost entirely removed.

```
            8
     ________
XXX )XXXXXXXX
     XXX
     ---
      XXXX
       XXX
       ---
        XXXX
        XXXX
        ====
```

One can see that the 8 is the third digit in a five digit answer. Can you recover what the numbers in this long division problem were? Explain your reasoning.

2.3 Algebra

Algebra is when you replace a number with a letter, usually x, right? OK—but you also do things with x, like make *polynomials* out of it.

2.3.1 Polynomial Basics

Question What's a polynomial?

I'll take this one:

Definition A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are all constants and n is a nonnegative integer.

Question Which of the following are polynomials?

$$3x^3 - 2x + 1 \qquad \frac{1}{3x^3 - 2x + 1} \qquad 3x^{-3} - 2x^{-1} + 1 \qquad 3x^{1/3} - 2x^{1/6} + 1$$

?

Given two polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

we treat these polynomials much the same way we treat numbers. Note, an easy fact is that polynomials are equal if and only if their coefficients are equal—this may come up again!

Question Are numbers equal if and only if their digits are equal?

?

Question Can you explain how to add two polynomials? Compare and contrast this procedure to the basic addition algorithm.

?

Question Can you explain how to multiply two polynomials? Compare and contrast this procedure to the basic multiplication algorithm.

?

Question Can you explain why someone might say that working with polynomials is like working in "base x?"

?

2.3.2 Division and Polynomials

For some reason you keep on signing up for classes with aloof old Professor Rufus. When he was asked to teach division of polynomials with remainders, he merely wrote

$$d(x)\,\overline{\big)\,n(x)}^{\,q(x)\ \ \mathrm{R}r(x)} \qquad \text{where} \qquad \begin{array}{l} d(x) \text{ is the divisor} \\ n(x) \text{ is the dividend} \\ q(x) \text{ is the quotient} \\ r(x) \text{ is the remainder} \end{array}$$

and walked out of the room, again! Do you have *déjà vu*?

Question Can you give 3 much needed examples of polynomial long division with remainders?

?

Question Given polynomials $d(x)$, $n(x)$, $q(x)$, and $r(x)$ how do you know if they leave us with a correct expression above?

?

Question Can you explain how to divide two polynomials? Compare and contrast how this procedure to the Division Theorem for integers.

?

Question Can you do the polynomial long division with remainder?

?

Again, this question can be turned into a theorem.

Theorem 4 (Division Theorem) *Given any polynomial $n(x)$ and a nonzero polynomial $d(x)$, there exist unique integers $q(x)$ and $r(x)$ such that*

The above space has intentionally been left blank for you to fill in.

Problems for Section 2.3

(1) Explain what is meant by a *polynomial* in a variable x.

(2) Given:

$$3x^7 - x^5 + x^4 - 16x^3 + 27 = a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0$$

Find a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7.

(3) Given:

$$6x^5 + a_4x^4 - x^2 + a_0 = a_5x^5 - 24x^4 + a_3x^3 + a_2x^2 - 5$$

Find a_0, a_1, a_2, a_3, a_4, a_5.

(4) Is it true that polynomials are equal if and only if their coefficients are equal? Explain your reasoning.

(5) Is it true that numbers are equal if and only if their digits are equal? Explain your reasoning.

(6) Explain how to add two polynomials.

(7) Explain how to multiply two polynomials.

(8) Here is an example of the polynomial division algorithm:

$$\begin{array}{r r} & x - 3 \quad \mathrm{R}\,9x+4 \\ \cline{2-2} x^2+3x+1 \,\big) & x^3 + 0x^2 + x + 1 \\ & \underline{x^3 + 3x^2 + x} \qquad \\ & -3x^2 + 0x + 1 \\ & \underline{-3x^2 - 9x - 3} \\ & 9x + 4 \end{array}$$

(a) Explain how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(9) State the *Division Theorem* for polynomials. Give some relevant and revealing examples of this theorem in action.

(10) Given a polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

can you find two numbers L and U such that $L \leqslant p(x) \leqslant U$ for all x? If so, explain why. If not, explain why not.

(11) Consider all all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are integers. If you substitute an integer for x will you always get an integer out? Explain your reasoning.

(12) Suppose I tell you that a certain polynomial $p(x)$ always returns an integer when an integer is substituted for x. Must the coefficients of $p(x)$ be integers? Big Hint: No. Find a counterexample.

(13) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0. Which numbers can be represented by such polynomials? Explain your reasoning.

(14) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than 10. Which numbers can be represented by such polynomials? Explain your reasoning.

(15) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 2 such that $p(2) = 35$. Discuss how your answer compares to the representation of 35 in base 2. Explain your reasoning.

(16) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 7 such that $p(7) = 200$. Discuss how your answer compares to the representation of 200 in base 7. Explain your reasoning.

(17) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 10 such that $p(10) = 18$. Discuss how your answer compares to the representation of 18 in base 10. Explain your reasoning.

(18) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 15 such that $p(15) = 25$. Discuss how your answer compares to the representation of 25 in base 15. Explain your reasoning.

(19) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than x. Which numbers can be represented by such polynomials? Explain your reasoning. Big Hint: Base x.

(20) Consider x^2+x+1. This can be thought of as a number in base x. Express this number in base $(x+1)$, that is, find b_0, b_1, b_2 such that

$$b_2(x+1)^2 + b_1(x+1) + b_0 = x^2 + x + 1.$$

Explain your reasoning.

(21) Consider x^2+2x+3. this can be thought of as a number in base x. Express this number in base $(x-1)$, that is, find b_0, b_1, b_2 such that

$$b_2(x-1)^2 + b_1(x-1) + b_0 = x^2 + 2x + 3.$$

Explain your reasoning.

(22) Consider x^3+2x+1. this can be thought of as a number in base x. Express this number in base $(x-1)$, that is, find b_0, b_1, b_2, b_3 such that

$$b_3(x-1)^3 + b_2(x-1)^2 + b_1(x-1) + b_0 = x^3 + 2x + 1.$$

Explain your reasoning.

(23) If the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is thought of as a number in base x, describe two different ways to find the base $(x-1)$ coefficients of $p(x)$.

Chapter 3

Solving Equations

> Politics is for the moment. An equation is for eternity.
>
> —Albert Einstein

3.1 Polynomial Equations

Solving equations is one of the fundamental activities in mathematics. Remember the definition of a *root* of a polynomial:

Definition A **root** of a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a number α where

$$a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 = 0.$$

OK—let's go!

3.1.1 Linear Equations

The simplest polynomials (besides constant polynomials) are linear polynomials. Solving equations of the form

$$ax + b = 0$$

poses no difficulty, we can write out the solution easily as

$$x = -b/a.$$

3.1.2 Quadratic Equations

Finding roots of quadratic polynomials is a bit more complex. We want to find x such that

$$ax^2 + bx + c = 0.$$

I know you already know how to do this. However, pretend for a moment that you don't. This would be a really hard problem. We have evidence that it took humans around 1000 years to solve this problem in generality, the first general solution appearing in Babylon and China around 2500 years ago. With this in mind, I think this topic warrants some attention. If you want to solve $ax^2 + bx + c = 0$, a good place to start would be with an easier problem. Let's make $a = 1$ and try to solve

$$x^2 + bx = c$$

Geometrically, you could visualize this as an $x \times x$ square along with a $b \times x$ rectangle.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Take your $b \times x$ rectangle and divide it into two $(b/2) \times x$ rectangles.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Now take both of your $(b/2)\times x$ rectangles and sunggie them next to your $x\times x$ square on adjacent sides. You should now have what looks like an $(x+\frac{b}{2})\times(x+\frac{b}{2})$ square with a corner cut out of it.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Finally—your big $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ has a piece missing, a $(b/2) \times (b/2)$ square, right? so if you add that piece in, the area of your picture had better be $c + (b/2)^2$. From your picture you will find that:

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

Question Can you find x at this point?

?

Question Explain how to solve $ax^2 + bx + c = 0$.

?

3.1.3 Cubic Equations

While the quadratic formula was discovered around 2500 years ago, cubic equations proved to be a tougher nut to crack. A general solution to a cubic equation was not found until the 1500's. At the time, mathematicians were a secretive and competitive bunch. Someone would solve a particular cubic equation, then challenge another mathematician to a sort of "mathematical duel." Each mathematician would give the other a list of problems to solve by a given date. The one who solved the most problems was the winner and glory everlasting[1] was theirs. One of the greatest duelists was Niccolò Fontana Tartaglia (pronounced *Tar-tah-lee-ya*). Why was he so great? He developed a general method for solving cubic equations! However, neither was he alone in this discovery nor was he the first. As sometimes happens, the method was discovered some years earlier by Scipione del Ferro—yet very few people knew of its existence. Since these discoveries were independent, we'll call the method the *Ferro-Tartaglia method*.

We'll show you the Ferro-Tartaglia method for finding at least one root of a cubic of the form:

$$x^3 + px + q$$

We'll illustrate with a specific example—you'll have to generalize yourself! Take

$$x^3 + 3x - 4 = 0$$

Step 1 Replace x with $u + v$.

$$\begin{aligned}(u+v)^3 + 3(u+v) - 4 &= u^3 + v^3 + 3uv(u+v) + 3(u+v) - 4\\ &= u^3 + v^3 - 4 + (3uv+3)(u+v).\end{aligned}$$

Step 2 Set uv so that all of the terms are eliminated except for u^3, v^3, and constant terms. Since we want

$$3uv + 3 = 0$$

we'll set $uv = -1$ and so

$$u^3 + v^3 - 4 = 0.$$

Since $uv = -1$, we see that $v = -1/u$ so

$$u^3 + \left(\frac{-1}{u}\right)^3 - 4 = u^3 - \frac{1}{u^3} - 4 = 0.$$

[1] This might be a slight exaggeration.

Step 3 Clear denominators and use the quadratic formula.

$$u^3 - \frac{1}{u^3} - 4 = 0 \qquad \Leftrightarrow \qquad u^6 - 4u^3 - 1 = 0$$

But now we may set $y = u^3$ and so we have

$$y^2 - 4y - 1 = 0$$

and by the quadratic formula

$$y = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}.$$

Putting this all together we find:

$$\begin{aligned} y &= 2 \pm \sqrt{5} \\ u &= \sqrt[3]{2 \pm \sqrt{5}} \\ v &= \frac{-1}{\sqrt[3]{2 \pm \sqrt{5}}} \end{aligned}$$

and finally (drum-roll please):

$$x = \sqrt[3]{2+\sqrt{5}} - \frac{1}{\sqrt[3]{2+\sqrt{5}}} \qquad \text{and} \qquad x = \sqrt[3]{2-\sqrt{5}} - \frac{1}{\sqrt[3]{2-\sqrt{5}}}$$

Question How many solutions are we supposed to have in total?

?

Question How do we do this procedure for other equations of the form

$$x^3 + px + q = 0?$$

?

3.1.4 Quartics, Quintics, and Beyond

While the Ferro-Tartaglia method may seem like it only solves the special case of $x^3 + px + q = 0$, it is a "wolf in sheep's clothing" and in fact is the key to giving a formula for solving any cubic equation

$$ax^3 + bx^2 + cx + d = 0.$$

The formula for solutions of the cubic equation is quite complex—we will spare you the details. Despite the fact that the key step of the formula is the Ferro-Tartaglia method, it is usually called *Cardano's formula* because Cardano was the first to publish.

It was wondered if there were formulas for solutions to polynomial equations of arbitrary degree. Cardano's student Ferrari, soon found the quartic formula, though it is too monstrous to write down in these notes. The search for the quintic equation began. Things started getting very difficult. The old tricks didn't work, and it wasn't until nearly 300 years later that this problem was settled.

Question Who was Niels Abel? Who was Évariste Galois?

?

Abel and Galois (pronounced *Gal-wah*), independently prove that there is no general formula for polynomial equations of degree 5 or higher. It is an amazing result and is only seen by students in advanced undergraduate or beginning graduated courses in pure mathematics. Nevertheless, in our studies we will not completely shy away from such demons. Read on!

Problems for Section 3.1

(1) Draw a rough timeline showing: The point when we realized we were interested in quadratic equations, the discovery of the quadratic formula, the discovery of the cubic formula, the discovery of the quartic formula, and the work of Abel and Galois proving the impossibility of a general formula for polynomial equations of degree 5 or higher.

(2) Given a polynomial, explain the connection between *linear factors* and *roots*. Are they the same thing or are they different things?

(3) In ancient and Medieval times the discussion of quadratic equations was often broken into three cases:

(a) $x^2 + bx = c$

(b) $x^2 = bx + c$

(c) $x^2 + c = bx$

where b and c are positive numbers. Create real-world word problems involving length and area for each case above.

(4) In ancient and Medieval times the discussion of quadratic equations was often broken into three cases:

(a) $x^2 + bx = c$

(b) $x^2 = bx + c$

(c) $x^2 + c = bx$

where b and c are positive numbers. Is this a complete list of cases? If not, what is missing and why is it (are they) missing? Explain your reasoning.

(5) Describe what happens geometrically when you complete the square of a quadratic equation of the form $x^2 + bx = c$ when b and c are positive. Explain your reasoning.

(6) Must a quadratic polynomial always have a real root? Explain your reasoning.

(7) Must a cubic polynomial always have a real root? Explain your reasoning.

(8) Must a quartic polynomial always have a real root? Explain your reasoning.

(9) Must a quintic polynomial always have a real root? Explain your reasoning.

(10) Derive the quadratic formula. Explain your reasoning.

(11) Solve $x^2 + 3x - 2 = 0$. Interlace an explanation with your work.

(12) Find two solutions to $x^4 + 3x^2 - 2 = 0$. Interlace an explanation with your work.

(13) Find two solutions to $x^6 + 3x^3 - 2 = 0$. Interlace an explanation with your work.

(14) Find two solutions to $x^{10} + 3x^5 - 2 = 0$. Interlace an explanation with your work.

(15) Find two solutions to $3x^{14} - 2x^7 + 6 = 0$. Interlace an explanation with your work.

(16) Find two solutions to $-4x^{22} + 13x^{11} + 1 = 0$. Interlace an explanation with your work.

(17) Give a general formula for finding two solutions to equations of the form: $ax^{2n} + bx^n + c = 0$ where n is an integer. Interlace an explanation with your work.

(18) Use the Ferro-Tartaglia method to find a solution to $x^3 + x + 1 = 0$. Interlace an explanation with your work.

(19) Use the Ferro-Tartaglia method to find a solution to $x^3 - x - 1 = 0$. Interlace an explanation with your work.

(20) Use the Ferro-Tartaglia method to find a solution to $x^3 + 3x - 4 = 0$. Interlace an explanation with your work.

(21) Use the Ferro-Tartaglia method to find a solution to $x^3 + 2x - 3 = 0$. Interlace an explanation with your work.

(22) Use the Ferro-Tartaglia method to find a solution to $x^3 + 5x - 1 = 0$. Interlace an explanation with your work.

(23) Solve $x^3 - 6x + 5 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning.

(24) Solve $x^3 - 6x + 4 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning.

(25) Solve $x^3 - 3x - 1 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning. Interlace an explanation with your work.

(26) Solve $x^3 - 12x - 8 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning. Interlace an explanation with your work.

(27) Solve $x^3 - 3x^2 + 5x - 3 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(28) Solve $x^3 + 4x^2 - 7x + 2 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(29) Find at least two solutions to $x^4 - x^3 - 3x^2 + 2x + 1 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(30) Explain what Abel and Galois proved to be impossible.

3.2 Time to get Real

We know what integers are right? We know what rational numbers are right?

Question Remind me what is $\mathbb{Z}$? What is $\mathbb{Q}$? What is the relationship between these two sets?

?

Are these all the numbers we need? Well, let's see. Consider the innocent equation:

$$x^2 - 2 = 0$$

Question Could $x^2 - 2$ have rational roots?

Stand back—I'll handle this. Remember, a root of $x^2 - 2$ is a number that solves the equation

$$x^2 - 2 = 0.$$

So suppose that there are integers a and b where a/b is a root of $x^2 - 2$ where a and b have no common factors. Then

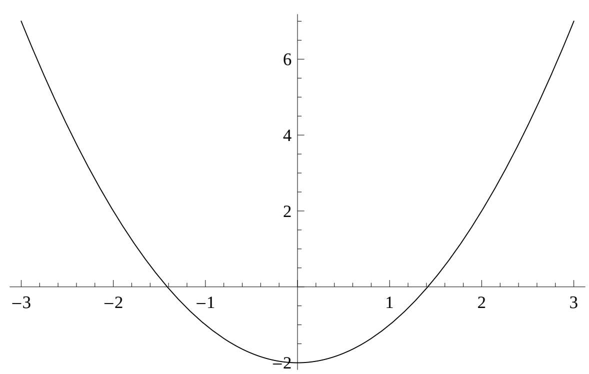
$$\left(\frac{a}{b}\right)^2 - 2 = 0.$$

So

$$a^2 - 2b^2 = 0 \qquad \text{thus} \qquad a^2 = 2b^2.$$

But a and b have no common factors—so by the Unique Factorization Theorem for the integers, $b^2 = 1$. This tells us that $a^2 = 2$ and that a is an integer—impossible!

Hmmm but now consider the plot of $y = x^2 - 2$:



The polynomial $x^2 - 2$ clearly has two roots! But we showed above that neither of them are rational—this means that there must be numbers that cannot be expressed as fractions of integers! In particular, this means:

The square-root of 2 is not rational!

Wow! But it still can be written as a decimal

$$\sqrt{2} = 1.4142135623\ldots$$

as the square-root of 2 is a *real number*.

Definition A **real number** is a number with a (possibly infinite) decimal representation. We use the symbol $\mathbb{R}$ to denote the real numbers.

For example:

$$-1.000\ldots \qquad 2.718281828459045\ldots \qquad 3.333\ldots \qquad 0.000\ldots$$

are all real numbers.

Question Another description of real numbers is that they are the numbers that can be approximated by rational numbers. Why does this follow from the definition above?

?

Famous examples of real numbers that are not rational are

$$\pi = 3.14159265358\ldots \qquad \text{and} \qquad e = 2.718281828459045\ldots$$

Question If a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b? What can you say about the decimal representation of an irrational number?

?

Problems for Section 3.2

(1) Describe the set of real numbers. Give some relevant and revealing examples/nonexamples.

(2) Explain what would happen if we "declared" the value of π to be 3?

(3) Explain why $x^2 - x - 1$ has no rational roots.

(4) Explain why $\sqrt{7}$ is irrational.

(5) Explain why $\sqrt[3]{5}$ is irrational.

(6) Explain why $\sqrt[5]{27}$ is irrational.

(7) Explain why if n is an integer and $\sqrt{n}$ is not an integer, then n is irrational.

(8) Solve $x^5 - 31x^4 + 310x^3 - 1240x^2 + 1984x - 1024 = 0$. Interlace an explanation with your work. Hint: Use reasoning from this section to find rational roots.

(9) Solve $x^5 - 28x^4 + 288x^3 - 1358x^2 + 2927x - 2310 = 0$. Interlace an explanation with your work. Hint: Use reasoning from this section to find rational roots.

(10) Knowing that π is irrational, explain why $\pi + 101$ is irrational.

(11) Knowing that π is irrational, explain why $101 \cdot \pi$ is irrational.

(12) Suppose we knew that α^2 was irrational. Could we conclude that α is also irrational? Explain your reasoning.

(13) Is $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$ rational or irrational? Explain your reasoning.

(14) In the discussion above, we give an argument showing that $\sqrt{2}$ is irrational. What happens if you try to use the exact same argument to try and show that $\sqrt{9}$ is irrational? Explain your reasoning.

3.3 Me, Myself, and a Complex Number

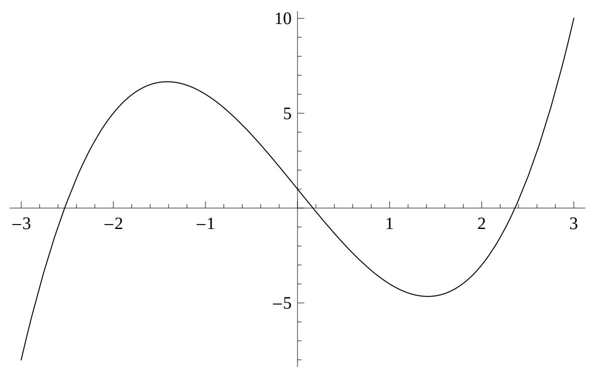
We'll start with the definition:

Definition A **complex number** is a number of the form

$$x + yi$$

where x and y are real numbers and i is the square-root of negative one. We use the symbol $\mathbb{C}$ to denote the complex numbers.

What that I hear? Yells of protest telling me that no such number exists? Well if it makes you feel any better, people denied the existence of such numbers for a long time. It wasn't until the 1800's until people finally changed their minds. Let's talk about some ideas that helped. Consider $x^3 - 6x + 1$. Here is its graph:



If you use the Ferro-Tartaglia method to find at least one solution to this cubic, then you find the following root:

$$\sqrt[3]{\frac{-1+\sqrt{-31}}{2}} + \frac{2}{\sqrt[3]{\frac{1}{2}(-1+\sqrt{-31})}}$$

This root looks like a complex number, since $\sqrt{-31}$ pops up twice. This might seem a bit redundant, but we should point out that $\sqrt{-31}$ is a complex number, it is:

$$0 + \sqrt{31}i$$

Even though our root has complex numbers in it, we know that it is real from the picture! Moral: If you want to give exact solutions to equations, then you'd better work with complex numbers, even if the roots are real!

Question If $u + vi$ is a nonzero complex number, is

$$\frac{1}{u + vi}$$

a complex number too?

You betcha! Let's do it. The first thing you must do is multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{u+vi} = \frac{1}{u+vi} \cdot \frac{u-vi}{u-vi} = \frac{u-vi}{u^2+v^2}$$

Now break up your fraction into two fractions:

$$\frac{u-vi}{u^2+v^2} = \frac{u}{u^2+v^2} + \frac{-v}{u^2+v^2} i$$

Ah! Since u and v are real numbers, so are

$$x = \frac{u}{u^2+v^2} \qquad \text{and} \qquad y = \frac{-v}{u^2+v^2}$$

Hence

$$\frac{1}{u+vi} = x + yi$$

and is definitely a complex number.

The real importance of the complex numbers came from Gauss, with the Fundamental Theorem of Algebra:

Theorem 5 (Fundamental Theorem of Algebra) *Every polynomial of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are complex numbers has a root r that is also also a complex number.

Said a different way, the Fundamental Theorem of Algebra says that every polynomial with complex coefficients

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

can be factored as

$$(x - r_1)(x - r_2) \cdots (x - r_n)$$

where each r_i is a complex number.

Question Why is that last statement true? That is, why is the Fundamental Theorem of Algebra the same as saying that every polynomial with complex coefficients can be factored into linear terms?

?

Problems for Section 3.3

(1) What is a real number?

(2) What is a complex number?

(3) Explain why every real number is a complex number.

(4) Explain why $\sqrt{-2}$ is a complex number.

(5) Is $\sqrt[3]{-2}$ a complex number? Explain your reasoning.

(6) Explain why $\sqrt[10]{-5}$ is a complex number.

(7) Explain why if $x + yi$ and $u + vi$ are complex numbers, then

$$(x + yi) + (u + vi)$$

is a complex number.

(8) Explain why if $x + yi$ and $u + vi$ are complex numbers, then

$$(x + yi)(u + vi)$$

is a complex number.

(9) Given a complex number $z = x + yi$, the **complex conjugate** of z is $x - yi$, we denote this as $\overline{z}$. Let $w = u + vi$ be another complex number.

- (a) Explain why $\overline{z} + \overline{w} = \overline{z + w}$.
- (b) Explain why $\overline{z} \cdot \overline{w} = \overline{z \cdot w}$.

(10) Explain why if $u + vi$ is a complex number, then

$$\frac{1}{u + vi}$$

is a complex number.

(11) Compute the following, expressing your answer in the form $x + yi$:

- (a) $(1 + 2i) + (1 + 7i)$
- (b) $(1 + 2i) \cdot (1 + 7i)$
- (c) $(1 + 2i)/(1 + 7i)$

Explain your reasoning.

(12) I'm going to show you something, see if you can see a connection to geometry:

- (a) Let $z = 3 + 4i$. Compute $\sqrt{z \cdot \overline{z}}$.
- (b) Let $z = 6 + 8i$. Compute $\sqrt{z \cdot \overline{z}}$.

(c) Let $z = 5 + 12i$. Compute $\sqrt{z \cdot \overline{z}}$.

What do you notice?

(13) Factor the polynomial $3x^2 + 5x + 10$ over the complex numbers. Explain your reasoning.

(14) Factor the polynomial $x^3 - 1$ over the complex numbers. Explain your reasoning.

(15) Factor the polynomial $x^4 - 1$ over the complex numbers. Explain your reasoning.

(16) Factor the polynomial $x^4 + 1$ over the complex numbers. Explain your reasoning.

(17) Factor the polynomial $x^4 + 4$ over the complex numbers. Can it be factored into polynomials with real coefficients of lower degree? Explain your reasoning.

(18) Suppose I tell you that $\sqrt{z \cdot \overline{z}} = 1$. What can you conclude about $z \cdot \overline{z}$? Explain your reasoning.

(19) Suppose I tell you that $\sqrt{z \cdot \overline{z}} = 1$ and that $\sqrt{w \cdot \overline{w}} = 1$. What can you conclude about $\sqrt{zw \cdot \overline{zw}}$? Explain your reasoning.

(20) The **complex plane** is obtained when one plots the complex number $x + yi$ as the point (x, y). Plot all complex numbers z such that $\sqrt{z \cdot \overline{z}} = 1$. Explain your reasoning.

(21) How many complex roots should $x^2 = 1$ have? What are they? Explain your reasoning.

(22) How many complex roots should $x^3 = 1$ have? What are they? Explain your reasoning.

(23) How many complex roots should $x^4 = 1$ have? What are they? Explain your reasoning.

(24) Suppose I told you that:

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots
\end{aligned}$$

Explain why we say:

$$e^{x \cdot i} = \cos(x) + i \sin(x)$$

(25) Use the problem above to explain Euler's famous formula:

$$e^{\pi \cdot i} + 1 = 0$$

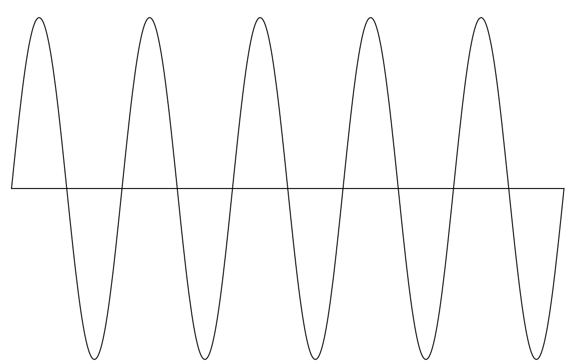
Chapter 4

Harmony of Numbers

> Let us despise the barbaric neighings [of war] which echo through these noble lands, and awaken our understanding and longing for the harmonies.
>
> —Johannes Kepler

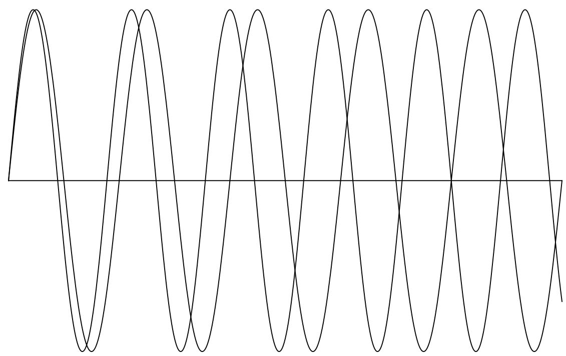
4.1 Tonic, Dominant, Octave

Someone once told me "Music is how math sounds." I'm not totally sure I believe them, but maybe there is some truth to what they are saying. Sound is made by compression waves in the air. Loosely speaking, the closer the compression waves come, the higher the pitch the sound is that we hear. We visualize these compression waves with a picture like this:

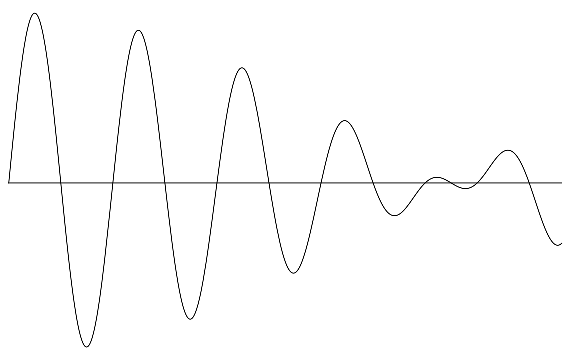


The peaks of the waves above represent high-pressure points where the air is very compressed, and the troughs of the waves above represent low-pressure points, where there are very few air molecules.

Let's call a sound produced by a single compression wave a *tone*. When two tones are played at the same time, their waves acting together act like the sum of the of the individual waves. Let's see this in action. If we play the following two tones at the same time,

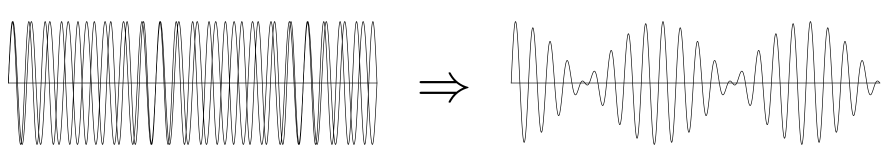


we end up with



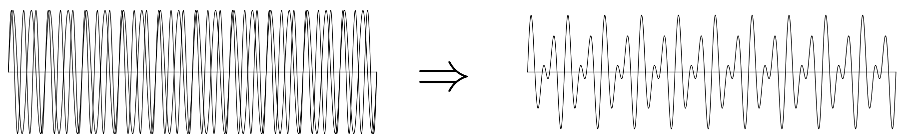
which is nothing more than every point on the first graph added to every point on the second graph. See how when the waves line up nicely, we get a nice big wave? See how when the waves disagree, our wave dwindles down to nothing?

Since the time when people started making sounds, we've noticed that some tones sound better together than others. There is an easy rule-of-thumb that will tell you when two tones will sound "right" together. If the graph of both waves combined has a lot of beats then it will probably sound "wrong," see the example below:



beats

If the beats are hard to see, then it will sound "right." You can see this in the following example:



no beats

Now let's define some words:

Definition The **wavelength** of a sinusoidal wave is length of a complete wave, the distance from peak to peak or the distance from trough to trough.

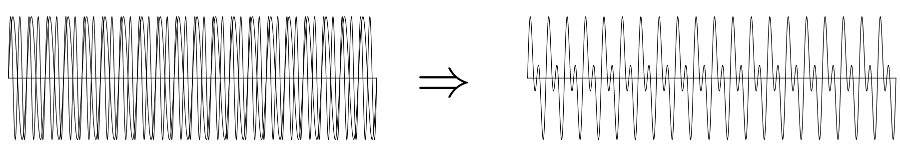
Definition The **frequency** of a sinusoidal wave is the number of complete waves per unit time.

If we're talking about sound waves, then wavelength and frequency are related by the following equation:

$$w \cdot f = s \qquad \text{where} \qquad \begin{array}{l} w \text{ is the wavelength} \\ f \text{ is the frequency} \\ s \text{ is the speed of sound} \end{array} \tag{4.1}$$

Definition The tone that you start with is called the **tonic**.

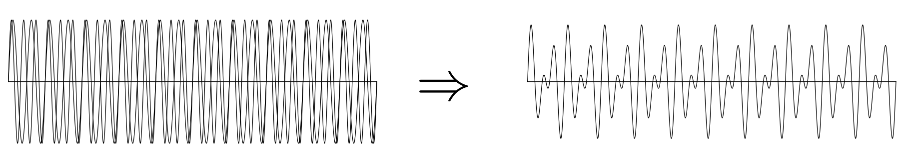
So let's start with the tonic, and add a tone whose wavelength is half that of the tonic:



That looks like it sounds really "right," as there are no beats to be seen.

Definition The tone whose wavelength is half that of the tonic is called the **octave**.

For some reason, the human ear and brain work together to identify the tonic and octave as *the same* tone, with the octave just being twice as high. Now let's get a little crazy, we'll start with the tonic, and add a tone whose wavelength is 2/3 that of the tonic:



That looks like it sounds "right" too, no beats again.

Definition The tone whose wavelength is 2/3 that of the tonic is called the **dominant**.

The dominant is central to all of western music. The tonic-dominant-octave trio of tones is sometimes called a *power chord* for its powerful sound.

Question Starting with just these three notions: Tonic, dominant, and octave what tones would we want an instrument to be able to play?

?

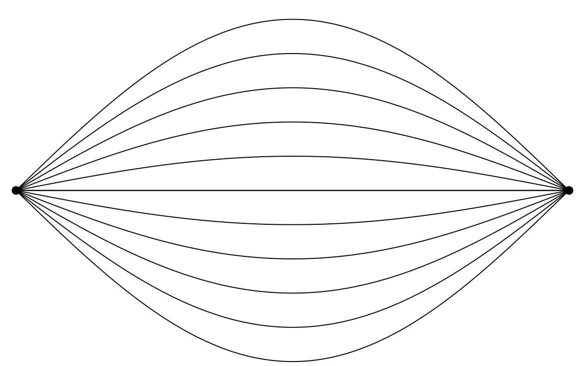
We seek the answer to this question.

4.1.1 Instrument Building

Close your eyes, and imagine you are on a beach in Brazil, imagine yourself swimming in the gentle waves. While you are swimming, you decide that when you return to your home town you are finally going to build that stringed instrument that you've always wanted—perhaps a lytherette. You'd have to pick some frequency for the open string. OK—done.

Question What do waves look like on your stringed instrument?

Let me explain a somewhat sticky point. The waves made by strings on instruments are not nice old sinusoidal waves, they are what we call *standing waves*. Standing waves look like a vibrating string:



Notice how the string is attached at the dots on the left and right? Points like those are called *nodes*.

Definition The **wavelength** of a standing wave is twice the length from node to node.

Definition The **frequency** of a standing wave is the number of complete vibrations (up and down) per unit time.

If we're talking about strings on instruments, then wavelength and frequency are related by the following equation:

$$w \cdot f = c \qquad \text{where} \qquad \begin{array}{l} w \text{ is the wavelength} \\ f \text{ is the frequency} \\ c \text{ is a constant based on the} \\ \text{mass and tension of the string} \end{array} \tag{4.2}$$

Question If you pluck a string, what will the wavelength of the sound wave be? Use equations (4.1) and (4.2) to express your answer in terms of w, f, c and s.

?

Question If you shorten a string's length by one-third, what will the wavelength of the new sound wave be? Use equations (4.1) and (4.2) to express your answer in terms of w, f, c and s.

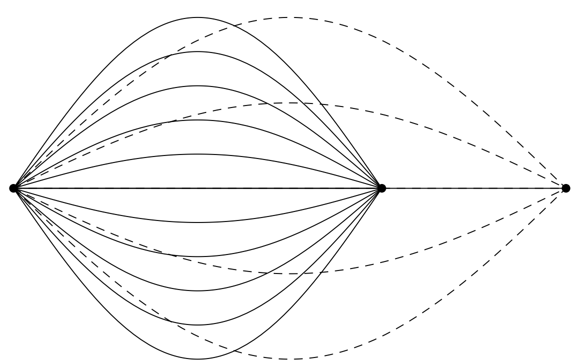
?

The upshot of the two questions above is that if you want to produce a tone whose wavelength is a fraction of another tone's wavelength, then your merely pluck a string whose length is the same fraction of the string that produced the original tone.

OK let's get back to instrument building.

Question When building a musical instrument, what tones do you want to be able to play?

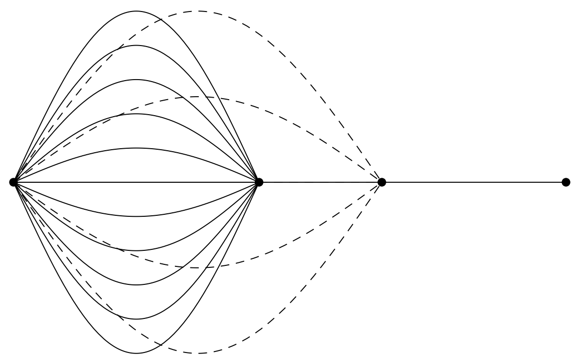
I'll take this one! You want to be able to play tonic (open string), the dominant, and the octave. How do we make this happen? Let's imagine we are building a stringed instrument. Many stringed instruments use *frets* (little metal things that help make new tones) to get the desired tones. To start, we'll want to put a fret 2/3 along the length of the string:



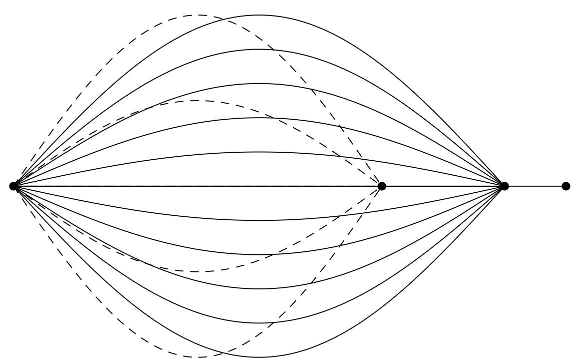
Additionally, we want to be able to play the dominant of this new tone as well. To do this, we'll might place a new fret

$$\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

of the length of the string.



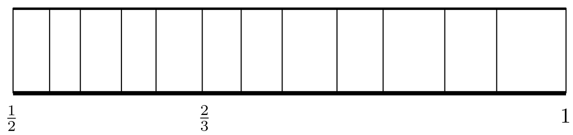
Hmmmm there is a slight problem though. This new fret would create a note *higher* than the octave (1/2 the length of the string). We want all of our frets to be placed between the original tonic and octave. So let's move this fret over by lowering our new tone by an octave. The new wavelength will be twice as long, so we'll put a fret 8/9 along the length of the string.



Now repeat this process, adding a new fret for the dominant of the previously added tone, at each step ensuring that the fret be placed between 1/2 and complete string length. Assuming you do this eleven times, you'll place frets at these locations

$$0.666\ldots, 0.888\ldots, 0.592\ldots, 0.790\ldots, 0.526\ldots, 0.702\ldots,$$
$$0.936\ldots, 0.624\ldots, 0.832\ldots, 0.554\ldots, 0.739\ldots,$$

with the last fret having been placed $0.739\ldots$ along the length of the string. If we compute the dominant of the tone obtained with this fret, we find we should place a twelfth fret $0.493\ldots$ along the length of the string. To the human ear, this tone will sound quite close to the octave of our starting tone (though slightly lower). Hence after twelve steps we *appear* to be at the octave. If we were to put frets on our instrument at all of these divisions, we would have something like this:



Remember though, the twelfth fret is close to an octave, but not perfect! Mathematically we might say

$$\frac{2^n}{3^m} \approx \frac{1}{2}$$

for some integers n and m. We note that in our case $m = 12$ and n is some other integer. Could we find integers n and m such that

$$\frac{2^n}{3^m} = \frac{1}{2}?$$

If so then we could write:

$$2^{n+1} = 3^m \qquad \Leftrightarrow \qquad 2^{n+1/m} = 3$$

Question What does the Unique Factorization Theorem for integers say about the above expressions? How do we proceed from here?

?

Problems for Section 4.1

(1) Explain what the octave of a given tone is.

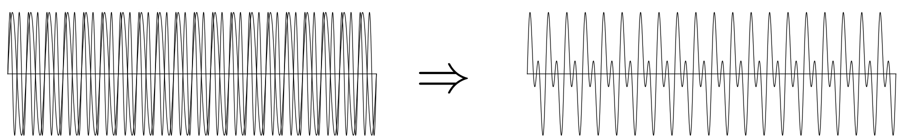
(2) Explain what the dominant of given tone is.

(3) Order the following wavelengths by which tone they produce from lowest to highest.

$$0.666\ldots, 0.888\ldots, 0.592\ldots, 0.790\ldots, 0.526\ldots, 0.702\ldots,$$
$$0.936\ldots, 0.624\ldots, 0.832\ldots, 0.554\ldots, 0.739\ldots, 0.493\ldots$$

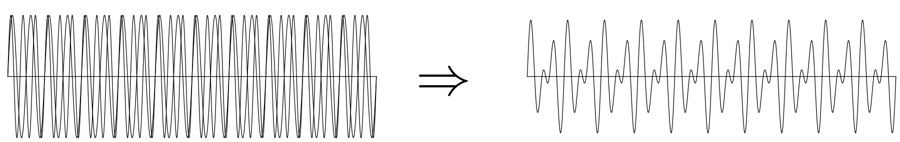
Explain your reasoning.

(4) Do the following tones sound "right" or "wrong" when played together?



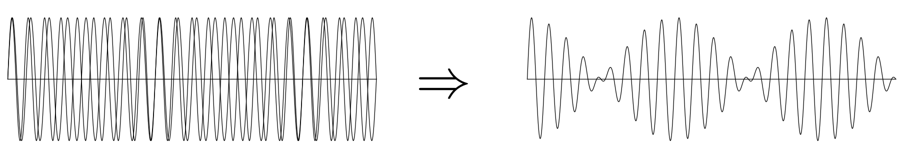
Explain your reasoning.

(5) Do the following tones sound "right" or "wrong" when played together?



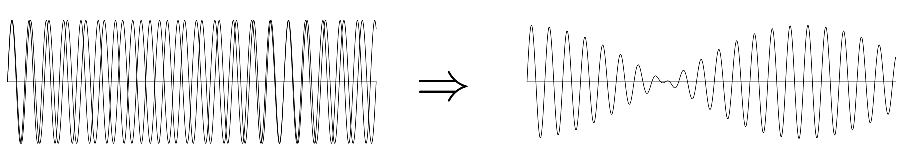
Explain your reasoning.

(6) Do the following tones sound "right" or "wrong" when played together?



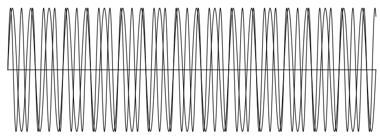
Explain your reasoning.

(7) Do the following tones sound "right" or "wrong" when played together?

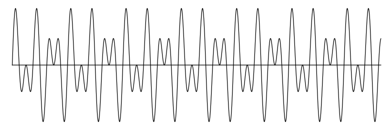


Explain your reasoning.

(8) Do the following tones sound "right" or "wrong" when played together?



$\Rightarrow$



Explain your reasoning.

(9) What is the wavelength of the dominant over the tone of wavelength 2/3? Explain your reasoning.

(10) What is the wavelength of the dominant over the tone of wavelength 3/4? Explain your reasoning.

(11) What is the wavelength of the dominant over the tone of wavelength 7/8? Explain your reasoning.

(12) What is the wavelength of the dominant over the tone of wavelength 6/13? Explain your reasoning.

(13) What is the wavelength of the dominant over the tone of wavelength 7/13? Explain your reasoning.

(14) What is the wavelength of the dominant over the tone of wavelength 2/3 if we insist that the resulting wavelength is between 1/2 and 1? Explain your reasoning.

(15) What is the wavelength of the dominant over the tone of wavelength 4/7 if we insist that the resulting wavelength is between 1/2 and 1? Explain your reasoning.

(16) What is the wavelength of the dominant over the tone of wavelength 5/8 if we insist that the resulting wavelength is between 1/2 and 1? Explain your reasoning.

(17) What is the wavelength of the dominant over the tone of wavelength 5/9 if we insist that the resulting wavelength is between 1/2 and 1? Explain your reasoning.

(18) What is the wavelength of the dominant over the tone of wavelength 6/11 if we insist that the resulting wavelength is between 1/2 and 1? Explain your reasoning.

(19) Give a precise derivation of how we obtained the fret positions

$$0.666\ldots, 0.888\ldots, 0.592\ldots, 0.790\ldots, 0.526\ldots, 0.702\ldots,$$
$$0.936\ldots, 0.624\ldots, 0.832\ldots, 0.554\ldots, 0.739\ldots, 0.493\ldots$$

using the ideas of the tonic and dominant.

4.2 Interlude of the Adders

If long division is a *forgotten foe*, then logarithms are a *supervillan*. When aloof old Professor Rufus was trying to explain logarithms to his class, he merely wrote

$$\log_b(a) = n \qquad \Leftrightarrow \qquad b^n = a$$

and walked out of the room.

Question Can you give 3 much needed examples of logarithms that are easily computed in one's head?

?

Question Can you give 3 much needed examples of logarithms that are more difficult to compute in one's head?

?

Question What conditions should be placed on a and b to make logarithms work nicely?

?

Question What is $\log_b(1)$?

?

Question What is $\log_b(0)$?

?

Question What is $\log_1(1)$?

?

Question Sketch the plot of $y = \log_b(x)$.

?

Question Why is this section named "interlude of the adders?!"

?

Problems for Section 4.2

(1) Explain what $\log_b(a) = n$ means.

(2) Sketch the plot of $y = \log_b(x)$ for some reasonable value of b. Explain your procedure.

(3) Sketch the plot of $y = b^x$ for some reasonable value of b. Explain your procedure. How does this plot compare to the one in the previous question?

(4) What is $\log_x(x^3)$? Explain your reasoning.

(5) Given that $\ln(x) = \log_e(x)$, explain why is it no big deal to say that $\ln(e^x) = x$.

(6) Compute $\log_5(125)$. Explain your reasoning.

(7) Compute $\log_{10}(10000)$. Explain your reasoning.

(8) Compute $\log_2(1024)$. Explain your reasoning.

(9) Compute $\log_{13}(169)$. Explain your reasoning.

(10) Compute $\log_7(2401)$. Explain your reasoning.

(11) Bound $\log_2(5)$ by two consecutive integers. Explain your reasoning.

(12) Bound $\log_3(43)$ by two consecutive integers. Explain your reasoning.

(13) Bound $\log_{11}(24)$ by two consecutive integers. Explain your reasoning.

(14) Bound $\log_{10}(999)$ by two consecutive integers. Explain your reasoning.

(15) Bound $\log_{10}(1032)$ by two consecutive integers. Explain your reasoning.

(16) What is the connection between the number of digits in some number n and $\log_{10}(n)$? Explain your reasoning.

(17) How many digits does the number 100 have in base 2? What does this have to do with $\log_2(100)$? Explain your reasoning.

(18) How many digits does the number 100 have in base 3? What does this have to do with $\log_3(100)$? Explain your reasoning.

(19) How many digits does the number 100 have in base 11? What does this have to do with $\log_{11}(100)$? Explain your reasoning.

(20) How many digits does the number 100 have in base 42? What does this have to do with $\log_{42}(100)$? Explain your reasoning.

(21) How many digits does the number 100 have in base 99? What does this have to do with $\log_{99}(100)$? Explain your reasoning.

(22) How many digits does the number 100 have in base 100? What does this have to do with $\log_{100}(100)$? Explain your reasoning.

(23) How many digits does the number 100 have in base 101? What does this have to do with $\log_{101}(100)$? Explain your reasoning.

(24) Explain why $\log_b(a) + \log_b(c) = \log_b(a \cdot c)$.

(25) Explain why $\log_b(a) - \log_b(c) = \log_b(a/c)$.

(26) Explain why $c \cdot \log_b(a) = \log_b(a^c)$.

(27) Explain why $\log_b(a) = \dfrac{1}{\log_a(b)}$.

(28) Explain why $\log_b(a) = \dfrac{\log_c(a)}{\log_c(b)}$.

(29) People have often told me something like "it is impossible to fold a piece of paper more than 7 times." What is meant by this statement and is it even true? Explain your reasoning. Note if you cannot solve this problem, no worries just say to yourself (aloud so all can hear) "I believe you can fold a piece of paper as much as you want" three times and then do the next problem and then come back to this one.

(30) Take a sheet of paper. If you fold it once, the resulting folded sheet of paper is twice as thick as the unfolded paper. If you fold it again, the resulting folded sheet is 4 times as thick as the unfolded piece of paper. How many times would you need to fold a sheet of paper to make the resulting sheet of paper as thick as you are tall? Explain your reasoning. Don't bother worrying about the physical limitations of this problem.

4.3 Rational and Irrational Temperament

Before our log interlude, we were thinking about how to build a stringed instrument with frets. With this in mind, we came up with the following equation

$$2^{\frac{n+1}{m}} = 3$$

where m is the number of divisions of the string that we would wish to make. Armed with the Unique Factorization Theorem for integers, we could (and you will!) explain that there is no rational solution of

$$2^x = 3,$$

meaning that finding appropriate values of n and m is actually impossible! It's a good thing that we are not the type of people to be deterred by the impossible. In light of our discussion above, we want to find a fraction:

$$\frac{n+1}{m} \approx \log_2(3) = 1.5849625007211561814537389439 4\dots$$

Question How do we find good fractional approximations of irrational numbers?

In two words: Continued fractions. Set:

$$x_1 = 1.58496250072115618145373894394\dots$$

Write x in terms of its whole-number part and its fractional part:

$$x_1 = \mathbf{1} + (x_1 - 1)$$

Now look at the reciprocal of $(x_1 - 1)$:

$$x_2 = \frac{1}{x_1 - 1} = 1.70951129135145477697619026217\dots$$
$$x_2 = \mathbf{1} + (x_2 - 1)$$

Continue on:

$$x_3 = \frac{1}{x_2 - 1} = 1.40942083965320900458240433081\dots$$
$$x_3 = \mathbf{1} + (x_3 - 1)$$

Again, again!

$$x_4 = \frac{1}{x_3 - 1} = 2.44247459618085927548717403238\dots$$
$$x_4 = \mathbf{2} + (x_4 - 2)$$

One last time:

$$\begin{aligned} x_5 &= \frac{1}{x_4 - 2} = 2.26001675267082453593127612260\ldots \\ x_5 &= \mathbf{2} + (x_5 - 2) \end{aligned}$$

Whew, now I'm tired, we could continue on but I think it is time to stop. Let's see what our continued fraction looks like:

$$\log_2(3) \approx 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}}$$

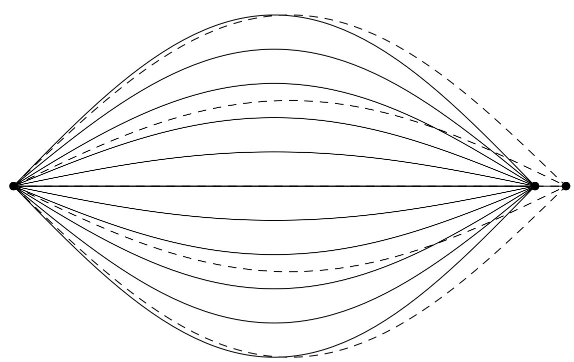
If we simplify this continued fraction into a regular old fraction we find:

$$\log_2(3) \approx \frac{19}{12} = 1.58333333333\ldots$$

Check this out:

$$2^{19/12} = 2.9966141537\ldots$$

This is really close to 3, thus $2^{19/12}$ is a good approximation of $\log_2(3)$. With this in mind, we will adopt the convention that the nth tone above the tonic will have a wavelength of exactly $1/2^{n/12}$ of the tonic:



In particular, since

$$\frac{1}{2^{1/12}} \cdot \frac{1}{2^{1/12}} = \frac{1}{2^{2/12}}$$

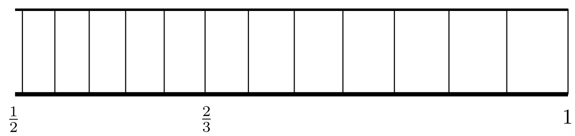
if we work with wavelengths of $1/2^{n/12}$ of the tonic, we will obtain a nice approximation of every tone we produced before. After 7 steps,

$$\frac{1}{2^{7/12}} = 0.66741992\cdots \approx \frac{2}{3}$$

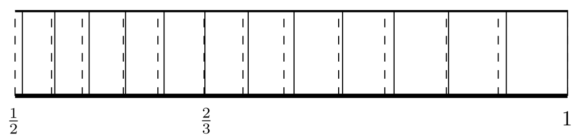
and additionally,

$$\frac{1}{2^{12/12}} = \frac{1}{2},$$

so after twelve steps we are exactly at the octave! If we put a frets at the points $\frac{1}{2^{n/12}}$ letting n run from 0 to 12, we'll obtain a picture like:



Let's compare this to our fret positions above:



Question Will this really work?

?

4.3.1 Equal Temperament

Spacing the division of the tones, by making use of wavelengths that are $\frac{1}{2^{1/12}}$ of the tonic, is called **equal temperament**. This is how modern guitars and pianos are tuned. People have also identified other ratios of the wavelength of the tonic that they thought sounded "right." Essentially, they have taken the idea that we should have as few "beats" as possible to the extreme. When they do this, and attempt to have 12 tones, they arrive at what is called **just intonation**. When *a cappella* groups sing and form chords, they usually sing using just intonation—the human brain seems to be somehow drawn to these sounds. Here are the ratios of the wavelength of the tonic used in just intonation:

$$1/2, \quad 8/15, \quad 5/9, \quad 3/5, \quad 5/8, \quad 2/3,$$
$$32/45, \quad 3/4, \quad 4/5, \quad 5/6, \quad 8/9, \quad 24/25$$

Let's compare these to the ratios used in equal temperament:

Tone	Equal Temperament	Just Intonation
12	$1/2^{12/12} = 0.5$	$1/2 = 0.5$
11	$1/2^{11/12} = 0.529\ldots$	$8/15 = 0.533\ldots$
10	$1/2^{10/12} = 0.561\ldots$	$5/9 = 0.555\ldots$
9	$1/2^{9/12} = 0.594\ldots$	$3/5 = 0.6$
8	$1/2^{8/12} = 0.629\ldots$	$5/8 = 0.625$
7	$1/2^{7/12} = 0.667\ldots$	$2/3 = 0.666\ldots$
6	$1/2^{6/12} = 0.707\ldots$	$32/45 = 0.711\ldots$
5	$1/2^{5/12} = 0.749\ldots$	$3/4 = 0.75$
4	$1/2^{4/12} = 0.793\ldots$	$4/5 = 0.8$
3	$1/2^{3/12} = 0.840\ldots$	$5/6 = 0.833\ldots$
2	$1/2^{2/12} = 0.890\ldots$	$8/9 = 0.888\ldots$
1	$1/2^{1/12} = 0.943\ldots$	$24/25 = 0.96\ldots$

The real issue with just intonation comes with we try to raise every note up by a given number of steps. Suppose a singer has trouble singing in the range of some song, yet has no problems if we raise every note of the song up by 1 half step. If our instrument is in equal temperament, then this shift of tones will have no adverse effects. However, if our instrument is in just intonation, then check out what happens:

- Let 24/25 of a wavelength be the new tonic—this is 1 half step.
- Now 7 steps up will be 5/8 of a wavelength.

Checking out the new ratio we find:

$$\frac{5/8}{24/25} = 0.651042$$

This is over a 2% difference from 2/3 of the tonic. Believe it or not, this will be noticeably "wrong" to the human ear. With equal temperament, the tone will still be spot-on.

These issues with musical instruments arise due to intrinsic differences between rational and irrational numbers. The tones that sound best to our ears are all defined by ratios of the wavelength of the tonic. However, moving up by these ratios is done via multiplication, and hence logarithms enter the scene. The Unique Factorization Theorem for integers tells us that the ratios that we are most interested in cannot be obtained simply from the other ratios we are interested in. From all this arises a fascinating problem that everyone experiences without even realizing it!

Problems for Section 4.3

(1) Explain why $\log_2(3)$ is an irrational number.

(2) Explain why $\log_3(5)$ is an irrational number.

(3) Explain why $\log_3(6)$ is an irrational number.

(4) Explain why $\log_4(6)$ is an irrational number.

(5) Explain why $\log_9(10)$ is an irrational number.

(6) Find the simple continued fraction expansion of 5/3. Explain your reasoning.

(7) Find the simple continued fraction expansion of 15/11. Explain your reasoning.

(8) Find the simple continued fraction expansion of 22/17. Explain your reasoning.

(9) Using a calculator, find the first five terms in the simple continued fraction expansion of e. What number do you get by only considering the first term? The first two terms? The first three terms? The first four? The first five? Explain your reasoning.

(10) Using a calculator, find the first five terms in the simple continued fraction expansion of π. What number do you get by only considering the first term? The first two terms? The first three terms? The first four? The first five? Explain your reasoning.

(11) Suppose you are building a stringed instrument. If the first octave of 12 tones has a length of 16 inches, how long is the next octave? What about the next octave? Explain your reasoning.

(12) A singer and a piano are playing a chord involving the sixth tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(13) A singer and a piano are playing a chord involving the seventh tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(14) A singer and a piano are playing a chord involving the fourth tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(15) A singer and a piano are playing a chord involving the twelfth tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(16) Some other cultures place 5 tones between octaves. Can you explain this if you know that they are trying to approximate $\log_2(3)$?

(17) Light also has wave-like properties. The wavelengths of the visible spectrum goes from around 380 nm to 750 nm. Sometimes colors are depicted as being in a line, other times they are depicted as being in a wheel. Can you use our discussion on music, thinking about tonics and octaves to give a plausible resolution to this paradox?

4.4 The Binomial Theorem

4.4.1 Varna-Sangita

In ancient Indian texts we find a description of a type of music called *varna-sangita*. This is music made from a variation of long and short syllables. When performing a varna-sangita, one starts off with a given number of short syllables and ends with the same number of long syllables. In between these verses, every possible combination of long and short syllables is supposed to occur. If s represents a short syllable and l represents a long syllable we might visualize this as:

$$ssss \xrightarrow{\text{every possible combination}} llll$$

To check their work, the people of ancient India counted how many of each combination appeared in a song. Suppose we started with sss and finished with lll. Our song should contain the following verses:

$$sss, \quad ssl, \quad sls, \quad lss, \quad sll, \quad lsl, \quad lls, \quad lll$$

We can construct the following table to summarize what we have found:

3 s's	2 s's and 1 l	1 s and 2 l's	3 l's
1	3	3	1

Question What would your table look like if you started with ss and finished with ll? What about if you started with $ssss$ and finished with $llll$?

?

The vedics of the time gave a rule for making tables like the one above. Their rule was based on the following diagram:

$$\begin{array}{lccccccccccccccl}
n=0: & & & & & & & 1 & & & & & & & & k=0 \\
n=1: & & & & & & 1 & & 1 & & & & & & & k=1 \\
n=2: & & & & & 1 & & 2 & & 1 & & & & & & k=2 \\
n=3: & & & & 1 & & 3 & & 3 & & 1 & & & & & k=3 \\
n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & k=4 \\
n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & k=5 \\
n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & k=6
\end{array}$$

Today people call this diagram **Pascal's triangle**.

Question How does Pascal's triangle relate to varna-sangitas? Is there an easy way to produce the above diagram?

?

And now for something completely different...

4.4.2 Expansions

Explain the following on a separate sheet of paper. Write the result of your work in the boxes below:

$$(a+b)^0 = \boxed{}$$

$$(a+b)^1 = \boxed{}$$

$$(a+b)^2 = \boxed{}$$

$$(a+b)^3 = \boxed{}$$

$$(a+b)^4 = \boxed{}$$

Question Is there a nice way to organize this data?

?

Question Can you explain the connection between expanding binomials and varna-sangitas?

?

4.4.3 Come Together

Let's see if we can bring these ideas together. Let's denote the following symbol:

$$\binom{n}{k} = \text{the number of ways we choose } k \text{ objects from } n \text{ objects.}$$

it is often said "n choose k" and is sometimes denoted as ${}_nC_k$.

Question What exactly does $\binom{n}{k}$ mean in terms of varna-sangitas? What does $\binom{n}{k}$ mean in terms of expansion of binomials?

?

Question How does $\binom{n}{k}$ relate to Pascal's triangle?

?

Question Pascal claims:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Explain how this single equation basically encapsulates the key to constructing Pascal's triangle.

?

Question Suppose that an oracle tells you that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

but we, being good skeptical people, are not convinced. How do we check this?

?

From the work above, we obtain a fabulous theorem:

Theorem 6 (Binomial Theorem) *If n is a nonnegative integer, then*

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

Question This looks like gibberish to me. Tell me what it is saying. Also, why is the Binomial Theorem true?

?

Problems for Section 4.4

(1) Write down the first 7 rows of Pascal's triangle.

(2) Explain how $\binom{n}{k}$ corresponds to the entries of Pascal's triangle. Feel free to draw diagrams and give examples.

(3) State the Binomial Theorem and give some examples of it in action.

(4) Explain the "physical" meaning of $\binom{n}{k}$. Give some examples illustrating this meaning.

(5) Explain how Pascal's triangle is formed. In your explanation, use the notation $\binom{n}{k}$. If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

(6) Explain why the formula you found in Problem (5) is true.

(7) State the formula for $\binom{n}{k}$.

(8) Expand $(a + b)^5$ using the Binomial Theorem.

(9) Expand $(a - b)^7$ using the Binomial Theorem.

(10) Expand $(-a - b)^8$ using the Binomial Theorem.

(11) Expand $(a + (b + c))^3$ using the Binomial Theorem.

(12) Expand $(a - b - c)^3$ using the Binomial Theorem.

(13) Let n be a positive integer.

- (a) Try some experiments to guess when $9^n + 1^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.
- (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 9 = 1$.

(14) Let n be a positive integer.

- (a) Try some experiments to guess when $6^n + 4^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.
- (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 6 = 4$.

(15) Let n be a positive integer.

- (a) Try some experiments to guess when $7^n - 3^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.
- (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 3 = 7$.

(16) Let n be a positive integer.

(a) Try some experiments to guess when $8^n - 2^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.

(b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 2 = 8$.

(17) Generalize Problems (13), (14), (15), and (16) above. Clearly articulate your new statement(s) and explain why they are true.

(18) Which is larger, $(1 + 1/2)^2$ or 2? Explain your reasoning.

(19) Which is larger, $(1 + 1/5)^5$ or 2? Explain your reasoning.

(20) Which is larger, $(1 + 1/27)^{27}$ or 2? Explain your reasoning.

(21) Which is larger, $(1 + 1/101)^{101}$ or 2? Explain your reasoning.

(22) Which is larger, $(1.0001)^{10000}$ or 2? Explain your reasoning.

(23) Generalize Problems (18), (19), (20), (21), and (22) above. Clearly articulate your new statement(s) and explain why it is true.

(24) Given a positive integer n, can you guess an upper bound for $(1 + 1/n)^n$?

(25) Use the Binomial Theorem to explain why:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

(26) Use the Binomial Theorem to explain why when n is a positive integer:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0$$

(27) Suppose I tell you:

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$$

Explain how to deduce:

$$(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} b^n$$

Appendix A

Activities

A.1 What Can Division Mean?

Here are some problems involving division. Someone once told me that most division problems could be broken into two types:

(a) Those that are asking "How many groups?"

(b) Those that are asking "How many in each group?"

Let's put this claim to the test. For each of the problems below:

(a) Numerically solve the problem.

(b) Draw a picture representing the situation or describe actions with objects a student could carry out to solve the problem.

(c) Identify whether the problem is asking "How many groups?" or "How many in each group?" or something else entirely.

1) There are a total of 35 hard candies. If there are 5 boxes with an equal number of candies in each box—and all the candy is accounted for, then how many candies are in each box?

2) There are a total of 28 hard candies. If there are 4 candies in each box, how many boxes are there?

3) A rectangle has a length of 6 inches and an area of 42 square inches. What is its width?

4) A chart has 72 cells and 8 rows. How many columns does it have?

5) Paolo has a total of 48 outfits (shirts and pants) he can wear. If he has 8 shirts, how many pants does he have?

6) Can you think of a division problem that is fundamentally different from the problems above?

7) In the context of the problems above, what might "division by zero" mean?

A.2 There's Always Another Prime

We'll start off with easy questions, then move to harder ones.

1) Use the Division Theorem to explain why 2 does not divide $2+1$.

2) Use the Division Theorem to explain why neither 2 nor 3 divides $2 \cdot 3 + 1$.

3) Use the Division Theorem to explain why neither 2 nor 3 nor 5 divides $2 \cdot 3 \cdot 5 + 1$.

4) Let $p_1, \dots, p_n$ be the first n primes. Do any of these primes divide

$$p_1 p_2 \cdots p_n + 1?$$

Explain your reasoning.

5) Let $p_1, \dots, p_n$ be the first n primes. Is $p_1 p_2 \cdots p_n + 1$ always a new prime? Explain your reasoning.

6) Suppose there were only a finite number of primes, say there were only n of them. Call them $p_1, \dots, p_n$. Could any of them divide

$$p_1 p_2 \cdots p_n + 1?$$

what does that mean? Can there really only be a finite number of primes?

A.3 Prome Factorization

Let's consider a crazy set of numbers—all multiples of 3. Let's use the symbol $3\mathbb{Z}$ to denote the set consisting of all multiples of 3. As a gesture of friendship, I have written down the first 100 nonnegative integers in $3\mathbb{Z}$:

0	3	6	9	12	15	18	21	24	27
30	33	36	39	42	45	48	51	54	57
60	63	66	69	72	75	78	81	84	87
90	93	96	99	102	105	108	111	114	117
120	123	126	129	132	135	138	141	144	147
150	153	156	159	162	165	168	171	174	177
180	183	186	189	192	195	198	201	204	207
210	213	216	219	222	225	228	231	234	237
240	243	246	249	252	255	258	261	264	267
270	273	276	279	282	285	288	291	294	297

1) Given any two integers in $3\mathbb{Z}$, will their sum be in $3\mathbb{Z}$? Explain your reasoning.

2) Given any two integers in $3\mathbb{Z}$, will their difference be in $3\mathbb{Z}$? Explain your reasoning.

3) Given any two integers in $3\mathbb{Z}$, will their product be in $3\mathbb{Z}$? Explain your reasoning.

4) Given any two integers in $3\mathbb{Z}$, will their quotient be in $3\mathbb{Z}$? Explain your reasoning.

Definition Call a number **prome** in $3\mathbb{Z}$ if it only has a single factor in $3\mathbb{Z}$. Equivalently, a number in $3\mathbb{Z}$ is prome if it cannot be expressed as the product of two integers both in $3\mathbb{Z}$.

As an example, I tell you that 6 is prome number in $3\mathbb{Z}$. You may object because $6 = 2 \cdot 3$, but remember—2 is not in $3\mathbb{Z}$!

5) List all the prome numbers less than 297.

6) Can you give some sort of algebraic characterization of prome numbers in $3\mathbb{Z}$?

7) Can you find numbers that factor completely into prome numbers in *two* different ways? How many can you find?

A.4 Whoops!

Niccolò Fontana Tartaglia was a great mathematician of the 1500's. Once he claimed claimed that the sums

$$1+2+4, \qquad 1+2+4+8, \qquad 1+2+4+8+16, \qquad \text{and so on,}$$

are alternately prime and composite.
1) Investigate Tartaglia's claim and report your findings.

One thing that would make analyzing this claim easier would be if we had a simple formula for his sums.

2) Using a table, see if you can guess a formula for Tartaglia's sums.

3) Can you explain why your formula holds? Can you generalize your formula?

4) Expand out the following:

(a) $(a^2-1)(1+a^2+a^{2\cdot 2})$

(b) $(a^2-1)(1+a^2+a^{2\cdot 2}+a^{3\cdot 2})$

(c) $(a^2-1)(1+a^2+a^{2\cdot 2}+a^{3\cdot 2}+a^{4\cdot 2})$

5) Expand out the following:

$$(2^a-1)(1+2^a+2^{2a}+2^{3a}+\cdots+2^{(b-1)a})$$

what does this say about Tartaglia's conjecture above?

A.5 Big Primes

All over the world, computers are churning away, trying to to find the next prime. This is what is known as the Great Internet Mersenne Prime Search—or (don't laugh) GIMPS. If you are willing, you too can play along and donate time on your computer. Every year or so a new prime is found, and glory-everlasting[1] goes to the person who finds it! I will not attempt to be up to date. Please use some inter-web-search-doo-hicky or whatever to find the largest known prime. Please write the actual number below:

There is little doubt in my mind that this largest known prime will be of the form:

$$M_n = 2^n - 1$$

where n is some number. Numbers of this form are called *Mersenne* numbers.
1) Make a table of the Mersenne numbers M_n for n from 1 to 10.

2) Which of the numbers in your table are prime? Make a conjecture.

3) Continue your table out to M_{14}. Check your conjecture up to this point.

4) Compare this activity to the activity *Whoops!* What do you notice?

By the way, we should add—M_n grows very fast as n grows. Let's do some comparisons:

5) Which is bigger, the largest known Mersenne prime or the number of atoms in *you*?

6) Which is bigger, the largest known Mersenne prime or the number of atoms in the planet Earth?

7) Which is bigger, the largest known Mersenne prime or the number of atoms in the Sun?

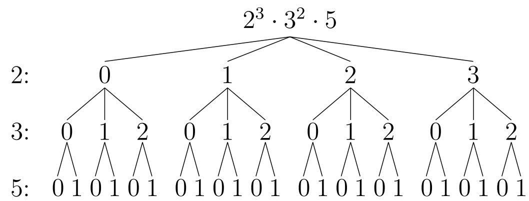
8) Which is bigger, the largest known Mersenne prime or the number of atoms in the visible Universe?

[1] This might be a slight exaggeration.

A.6 There are Many Factors to Consider

1) How many factors does the integer 12 have?

Consider the following diagram:



2) What is going on in this diagram? What do the numbers represent?

3) Make a similar diagram for 12.

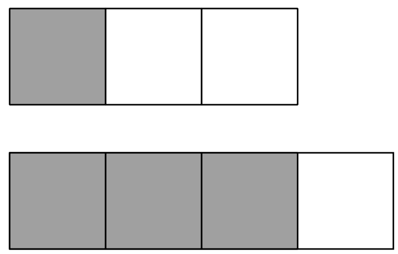
4) Can you devise a method for computing the number of factors that a number has? Explain why your method works.

5) How many factors does 735 have?

6) Which integers between 0 and 100 have the most factors?

A.7 Adders and Destiny (Density!)

1) Contrary Clark states that he knows a simpler method of adding fractions. He views 1/3 as “1 of 3” and 3/4 as “3 of 4.”



From this viewpoint Contrary Clark claims:

$$\frac{1}{3} + \frac{3}{4} = \frac{4}{7}$$

How do we set Contrary Clark straight?

Sometimes in mathematics, we go a little crazy—in a good way. Let me explain: Can we find value in Contrary Clark’s erroneous method of “adding” fractions? Before we can answer this question, we must gain a deeper understanding of the method.

2) Draw a number line, plot the points

$$\frac{1}{3}, \qquad \frac{3}{4}, \qquad \frac{1+3}{3+4}, \qquad \frac{1}{3} + \frac{3}{4}.$$

What do you observe? Play around with other numbers. Can you explain why the operation of “adding the numerators” and “adding the denominators” is sometimes called the *mediant* of two fractions?

3) I’m thinking of a number. Explain how to use the mediant to find other numbers that are arbitrarily close to my number.

4) In light of our discussion above, compare/contrast the two story problems below:

(a) Bart has been shooting free-throws. On Monday he made 18 out of 101. On Tuesday he made 18 out of 121. On Wednesday he made 17 out of 120. What is his average for the 3 days?

(b) Vic has been eating chocolate. On Monday he ate 1/3rd of a candy bar. On Tuesday he ate 3/5ths of a candy bar. On Wednesday he ate 1/4th of a candy bar. How much chocolate did Vic eat over the three days?

A.8 Cross Something-ing

1) What might you call the following statements:

(a) $\dfrac{a}{b} = \dfrac{c}{d} \Rightarrow ad = bc$

(b) $\dfrac{a}{b} \cdot \dfrac{b}{c} = \dfrac{a}{c}$

(c) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$

(d) $ad < bc \Rightarrow \dfrac{a}{b} < \dfrac{c}{d}$

(e) $ad < bc \Rightarrow \dfrac{c}{d} < \dfrac{a}{b}$

2) Which of the above statements are true? What name might you use to describe them?

3) Explain why the true statements above are true and give counterexamples showing that the false statements are false.

4) Can you think of other statements that should be grouped with those above?

5) If mathematics is a subject where you should strive to "say what you mean and mean what you say," what issue might arise with cross-multiplication?

A.9 Problem Solved!

Here's an old puzzler:

1) A man is riding a camel across a desert when he encounters a novel sight. Three young men are fiercely arguing surrounded by 17 camels. Dismounting, the stranger was told that their father had died, leaving (as their only real inheritance) these 17 camels. Now, the eldest son was to receive half of the camels, the second son one-third of the camels, the youngest son one-ninth of the camels. How could they divide the 17 camels amongst themselves? Explain your reasoning.

Uncharacteristically, I will solve this problem for you:

Solution The man should generously add his camel to the 17 being argued over. Now there are 18 camels to divide amongst the three brothers. With this being the case:

- The eldest son receives 9 camels.
- The middle son receives 6 camels.
- The youngest son receives 2 camels.

Ah! Since $9 + 6 + 2 = 17$, there is one left over, the man's original camel—he can now have it back. ■

2) What do you think of this solution?

3) Describe your thought process when addressing the above problem.

A.10 Decimals Aren't So Nice

We will investigate the following question: How is $0.999\ldots$ related to 1?

1) What symbol do you think you should use to fill in the box below?

$$.999\ldots \boxed{} 1$$

Should you use $<$, $>$, $=$ or something else entirely?

2) What is $1 - .999\ldots$?

3) How do you write $1/3$ in decimal notation? Express

$$1/3 + 1/3 + 1/3$$

in both fraction and decimal notation.

4) See what happens when you follow the directions below:

(a) Set $x = .999\ldots$.

(b) Compute $10x$.

(c) Compute $10x - x$.

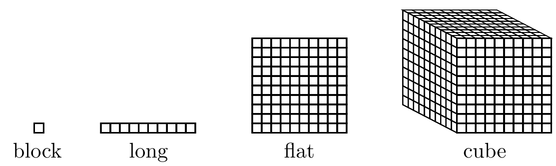
(d) From the step immediately above, what does $9x$ equal?

(e) From the step immediately above, what does x equal?

5) Are there other numbers with this weird property?

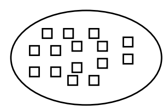
A.11 Playing with Blocks

I always enjoyed blocks quite a bit. Go find yourself some *base-ten blocks*. Just so that we are all on the same page, here are the basic blocks:



1) Model the number 247 with base-ten blocks.

2) Oscar modeled the number 15 in the following way:



What do you think of his model?

3) Here is an example of the basic addition algorithm:

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

Explain how to model this algorithm with base-ten blocks.

4) Here is an example of the basic multiplication algorithm:

$$\begin{array}{r} 23 \\ 634 \\ \times \quad 8 \\ \hline 5072 \end{array}$$

Explain how to model this algorithm with base-ten blocks.

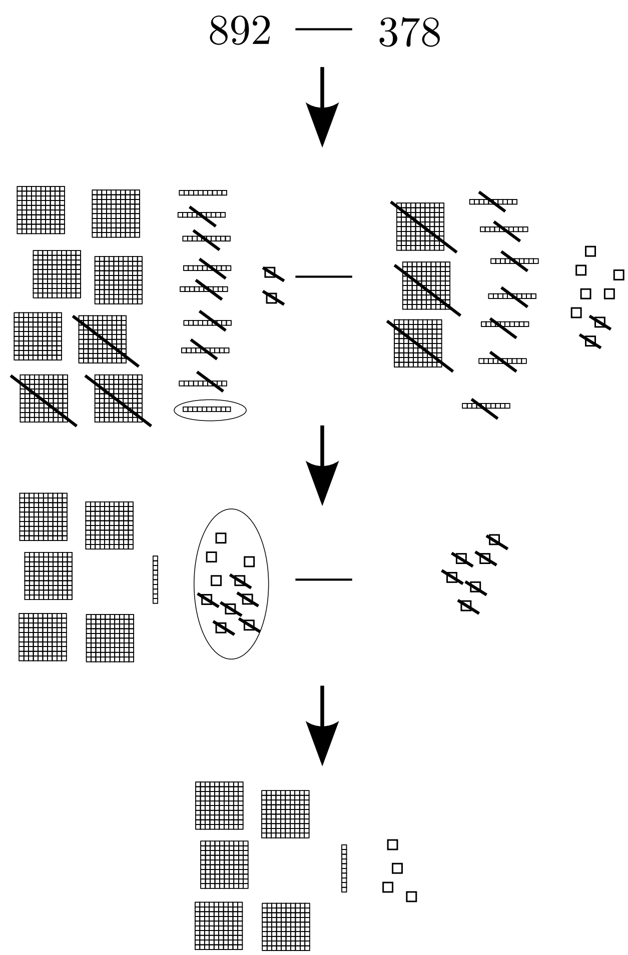
5) Here is an example of the basic division algorithm:

$$\begin{array}{r} 97 \text{ R1} \\ 8\overline{)777} \\ \underline{72} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

Explain how to model this algorithm with base-ten blocks.

6) Now Oscar is modeling the basic subtraction algorithm:

$$\begin{array}{r} \overset{8}{} \\ 8\cancel{9}\,{}^{1}2 \\ -3\,7\ \ 8 \\ \hline 5\,1\ \ 4 \end{array}$$



Can you explain what is going on? What do you think of his model?

7) Here is an example of the addition algorithm with decimals:

$$\begin{array}{r} 1 \\ 37.2 \\ +8.74 \\ \hline 45.94 \end{array}$$

Explain how to model this algorithm with base-ten blocks.

A.12 Shampoo, Rinse, ...

We're going to investigate the following question: If a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b? Let's see if we can get to the bottom of this.

1) Use long division to compute 5/6.

2) State the Division Theorem for integers.

3) Considering the solution of Problem 1, explicitly explain how the Division Theorem for integers appears in your work.

4) In each instance of the Division Theorem in Problem 3, what is the divisor? What does this say about the remainder?

5) Explain what can you say about the decimal representation of a/b when a and b are integers with $b \neq 0$.

6) Write the following fractions in decimal notation. Which have a "terminating" and which have a "repeating" decimal?

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{9}, \quad \frac{1}{10}, \quad \frac{1}{11}, \quad \frac{1}{12}, \quad \frac{1}{13},$$

$$\frac{1}{15}, \quad \frac{1}{20}, \quad \frac{1}{24}, \quad \frac{1}{25}, \quad \frac{1}{28}, \quad \frac{1}{32}, \quad \frac{1}{35}, \quad \frac{1}{40}, \quad \frac{1}{42}, \quad \frac{1}{48}.$$

7) Can you find a pattern from your results from Problem 6? Use your pattern to guess whether the following fractions "terminate" or "repeat."

$$\frac{1}{61} \qquad \frac{1}{625} \qquad \frac{1}{6251}$$

8) Can you explain why your conjecture from Problem 7 is true?

9) Compute $\frac{1}{9}$, $\frac{1}{99}$, and $\frac{1}{999}$. Can you find a pattern? Can you explain why your pattern holds?

10) Use your work from Problem 9 to give the fraction form of the following decimals:

(a) $0.\overline{357}$

(b) $0.23\overline{4598}$

(c) $23.\overline{459}$

(d) $76.34\overline{214}$

A.13 Geometry and Complex Numbers

Take a minute to recall the *Division Theorem*. Got it? OK we can do something similar with complex numbers. Check this out:
Definition A **Gaussian integer** is a number of the form

$$a + bi$$

where a and b are integers and i is the square-root of negative one.

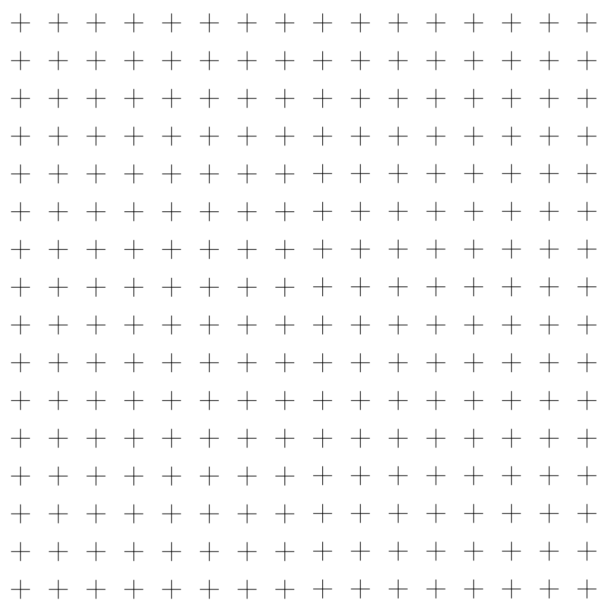
Just like with integers, we have a division theorem here too, check it out (this time I'll play nice):

Theorem 7 (Division Theorem) *Given any Gaussian integer α and a nonzero Gaussian integer β, there exist Gaussian integers θ and ρ such that*

$$\alpha = \beta \cdot \theta + \rho \qquad \text{with} \qquad \rho \cdot \overline{\rho} < \beta \cdot \overline{\beta}$$

where $\overline{a + bi} = a - bi$.

Suppose you want to divide $7 + 7i$ by $1 + 2i$ and end up with quotient and remainder that are both Gaussian integers. How do you do this? We'll use the complex plane to help us out.



1) Mark $1 + 2i$ and $7 + 7i$ on the complex plane. Use the grid above to help you and be sure to label your work.

2) Mark every Gaussian integer multiple of $1 + 2i$ on the plane above. Explain what happens and explain why it happens.

3) Find the nearest multiple of $1 + 2i$ to $7 + 7i$.

4) Use your work above to help find θ and ρ such that

$$7 + 7i = (1 + 2i) \cdot \theta + \rho \qquad \text{with} \qquad \rho \cdot \overline{\rho} < 5.$$

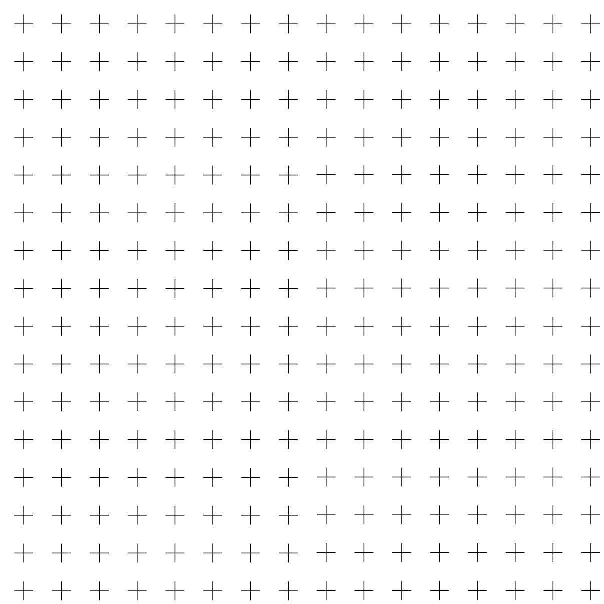
5) Are the θ and ρ you found above unique? Discuss.

6) Explain what is going on here in terms of geometry.

7) Find θ and ρ such that

$$9 + 8i = (5 + 2i) \cdot \theta + \rho \qquad \text{with} \qquad \rho \cdot \overline{\rho} < 29.$$

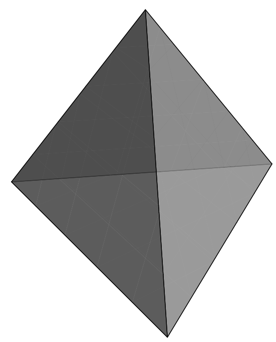
As a gesture of friendship, I have provided a fresh grid for your work.



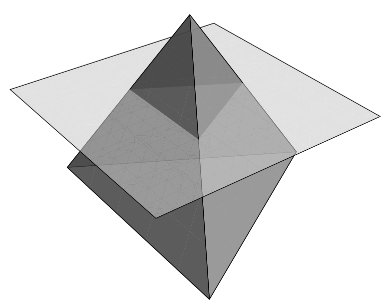
8) Are the θ and ρ you found above unique? Discuss.

A.14 Pascal's Pyramid

A *pyramid* is a three-dimensional object that has some polygon as its base and triangles that converge to a point as its sides. Fancy folks call a triangular-based pyramid a *tetrahedron*.



Since three-dimensional objects are hard to view on a flat sheet of paper, sometimes we think about them by taking cross sections:



We're going to build a triangular-based pyramid out of numbers. Here are the first four cross sections:

$$\begin{array}{c}1\end{array}\qquad \begin{array}{c}1\quad 1\\ 1\end{array}\qquad \begin{array}{c}1\quad 2\quad 1\\ 2\quad 2\\ 1\end{array}\qquad \begin{array}{c}1\quad 3\quad 3\quad 1\\ 3\quad 6\quad 3\\ 3\quad 3\\ 1\end{array}$$

This pyramid that we are building is called **Pascal's pyramid**.

1) Give the next two cross sections of Pascal's pyramid. Explain your reasoning.

2) Do you see any connections to Pascal's Triangle in Pascal's pyramid? Explain what you see.

3) Use the Binomial Theorem to expand:

$$(a + x)^3$$

4) Replace x above with $b + c$, and use the Binomial Theorem again along with your computation above to expand:

$$(a + b + c)^3$$

5) What do you notice about the coefficients in the expansion of $(a + b + c)^3$?

6) Explain how the trinomial coefficient

$$\binom{n}{j, k} = \frac{n!}{j!k!(n - j - k)!}$$

corresponds to entries of Pascal's pyramid. Feel free to draw diagrams and give examples.

7) The trinomial coefficient $\binom{n}{j,k}$ has the following "physical" meaning: It is the number of ways one can choose j objects and k objects from a set of n objects. Try a couple relevant and reveling examples to provide evidence for this claim.

8) Explain how Pascal's Triangle is formed. In your explanation, use the notation $\binom{n}{j,k}$. If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

9) Use Pascal's pyramid to expand:

$$(a + b + c)^4$$

Try to formulate a "Trinomial Theorem."

10) Use your Trinomial Theorem to explain why the numbers in the nth cross section of Pascal's pyramid sum to 3^n.

A.15 The Harmonic Triangle

Leibniz, one of the inventors of calculus, dreamed up this beast while trying to solve a problem posed to him by Huygens:

$$\begin{array}{ccccccccc}
 & & & & \frac{1}{1} & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5}
\end{array}$$

1) What relationships can you find between the entries of the triangle as we move from row to row?

2) What are the next two rows? Clearly articulate how to produce more rows of the Harmonic Triangle.

3) Explain how the following expression

$$\frac{1}{k\binom{n}{k}}$$

corresponds to entries of the Harmonic Triangle. Feel free to draw diagrams and give examples.

4) Explain how the Harmonic Triangle is formed. In your explanation, use the notation

$$\frac{1}{k\binom{n}{k}}$$

If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

5) Can you explain why the numerators of the fractions in the Harmonic Triangle must always be 1?

6) Explain how to use the Harmonic Triangle to go from:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

to

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$

Conclude by explaining why Leibniz said:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1$$

7) Explain how to use the Harmonic Triangle to go from:

$$\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\cdots$$

to

$$\left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{12}\right)+\left(\frac{1}{12}-\frac{1}{20}\right)+\left(\frac{1}{20}-\frac{1}{30}\right)+\cdots$$

Conclude by explaining why Leibniz said:

$$\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\cdots=\frac{1}{2}$$

8) Can you generalize the results above? Can you give a list of infinite sums and conjecture what they will converge to?

References and Further Reading

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