Sequences and Functions

These pages serve as section 2.6 in the course notes for Math 1165: Math for Middle School Teachers, taught at Ohio State University. The notes are intended for current and future teachers.

Sequences. A *sequence* is an ordered set of numbers or other objects. Because "Sequences and Series" is a common topic in calculus and precalculus courses, the concept of sequence is often considered an advanced topic in high schools, but the idea of a sequence is much more elementary. In fact, many patterns explored in grades K-8 can be considered sequences. For example, the sequence $4, 7, 10, 13, 16, \ldots$ might be described as a "plus 3 pattern" because terms are computed by adding 3 to the previous term.

Functions. In the Common Core State Standards, students begin formal study of functions in grade 8.[8.F.1] In high school, the approach to functions becomes more formal, through the use of function notation and with explicit attention to the concepts of *domain* and *range*.[F-IF.1]

Sequences Are Functions. As students begin formal study of functions, it makes sense to use their patterning experience as a foundation for understanding functions.[F-IF.3] To show how the sequence above can be considered a function, we need an *index*, which indicates which term of the sequence we are talking about, and which serves as an input value to the function. Deciding that the 4 corresponds to an index value of 1, we can make a table showing the correspondence.•

Although sequences are sometimes notated with subscripts, function notation can help students remember that sequences are functions.• For example, the sequence can be described recursively by the rule $f(1) = 4$, $f(n+1) = f(n) + 3$ for $n \geq 2$. Notice that the recursive definition requires both a starting value and a rule for computing subsequent terms. The sequence can also be described with the closed (or explicit) formula $f(n) = 3n + 1$, for integers $n \geq 1$. Notice that the domain (i.e., integers $n \geq 1$) is included as part of the description. When a function is given without an explicit domain, the assumption is that the domain is all values for which the expression is valid. Thus, the function $g(x) = 3x + 1$ appears to be essentially

CCSS 8.F.1: Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. (Function notation is not required in Grade 8.)

CCSS F-IF.1: Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

CCSS F-IF.3: Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

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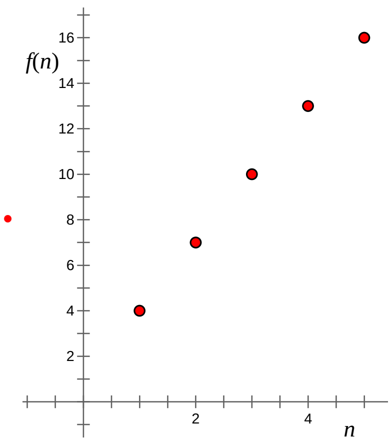
n	1	2	3	4	5	...
$f(n)$	4	7	10	13	16	...

• We recommend using subscript notation only in advanced high school courses.

the same as the function f because the formula is the same and because $f(n) = g(n)$ for all positive integers. But $g(2.5) = 8.5$, whereas $f(2.5)$ is not undefined.

A common habit in school mathematics is creating a table of (x, y) pairs, plotting those pairs (as dots), and then "connecting the dots." The above discussion demonstrates that this habit is sometimes not appropriate: A graph of the sequence consists of discrete dots, because the specification does not indicate what happens "between the dots." Connecting the dots requires the assumption that domain values between the dots make sense in some way.



Question In your own words, what does it mean to say that sequences are functions?

Question Given that $f(1) = f(2) = 1$, and $f(n+1) = f(n) + f(n-1)$ for integers $n > 2$, find $f(6)$.

Arithmetic and Geometric Sequences. An *arithmetic sequence* has is a constant difference between consecutive terms. A *geometric sequence* has a constant ratio between consecutive terms. Some sequences, of course, are neither arithmetic nor geometric.[F-BF.2]

CCSS F-BF.2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Question For each of the following sequences, decide whether it is arithmetic, geometric, or neither, and explain your reasoning:

- $1, 4, 9, 16, 25, \ldots$
- $4, 8, 16, 32, \ldots$
- $2, 4, 6, 8, 2, 4, 6, 8, \ldots$

- $-2, 5, 12, 19, \ldots$

Can you write both recursive and explicit formulas for each of these sequences?

Beginning in about grade 8, much of school mathematics is devoted to the study of linear, quadratic, and exponential functions.[F-LE.2] Here we provide only definitions and key questions about these types of functions.

CCSS F-LE.2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

- A linear function is of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. What do a and b tell you about the linear function?

- A quadratic function is of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. What do a, b, and c tell you about the function? Why is it important to specify that $a \neq 0$?

- An exponential function is of the form $f(x) = ab^x$, where a and b are real numbers and $b > 0$. What do a and b tell you about the function?

How can you identify these types of functions in tables, graphs, symbols, and contexts?[F-IF.4] [F-IF.7] For example, how can you recognize the slope in the graph of a linear function? What about in a table, in a symbolic expression, or in a context?

CCSS F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

CCSS F-IF.7: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

Question An arithmetic sequence is what kind of function? Explain.

Question A geometric sequence is what kind of function? Explain.

Question Sometimes quadratic functions are written in the form $f(x) = a(x - h)^2 + k$, where a, h, and k are real numbers and $a \neq 0$. What do a, k, and h tell you about the function? What are the advantages and disadvantages of this form of a quadratic, as compared to the alternative form given above?

Series. A *series* is a sum of consecutive terms from a sequence. A series with terms that form an arithmetic sequence is called an *arithmetic series*.

Question Find the sum: $1 + 3 + 5 + \cdots + 4999$. (First explain how you know this is an arithmetic series.)

In mathematics teaching and learning, it is useful to distinguish *problems* from *excercises*. *Problems* require that you formulate a solution strategy, whereas *exercises* are about using a procedure that you have been taught.• Whether a question is a problem or an exercise depends upon the learner.

• Problem solving is an essential part of mathematics.

Question Is the previous question a problem or an exercise for you?

When analyzing any series, it is often useful to consider the *sequence of partial sums*. For example, in response to the above question, the sequence of partial sums is as follows:

$$1, \qquad 1+3, \qquad 1+3+5, \qquad 1+3+5+7, \qquad \ldots$$

Sometimes you can see a pattern in the sequence of partial sums. Making a conjecture about a pattern is a type of inductive reasoning. Once you notice a pattern, an important next step is showing, deductively, that the pattern *must* continue.

For arithmetic series, there are several observations that can lead to a general deductive argument for the sum. For example, consider pairing the first term with the last term, the second term with the second-to-last term, and so on. What do you notice about the sum of each of these pairs? And how many such pairs are in the whole series? Alternatively, consider the average of each of the pairs and how those averages might help determine the sum of the whole series. An important general technique involves writing the series backward immediately below a forward version and then adding vertically.

Question Use one of these approaches to show that the sum is what it is. Can you use a picture to illustrate your reasoning?

Question When you consider the sequence of partial sums of an arithmetic series, what kind of function(s) can you get? Explain.

A series with terms that form a geometric sequence is called a *geometric series*.

Question Find the sum: $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^{10}}$. (First explain how you know the series is geometric.)

For the question above, it is not hard to see a pattern in the sequence of partial sums. In fact, it is reasonable to believe that the pattern holds for any (finite) partial sum of the infinite geometric series $\frac{2}{3}+\frac{2}{9}+\ldots$. But to show that the pattern always holds, we need a general argument.

For geometric series, the techniques for arithmetic series do not carry over. Instead, observe that if you multiply the series by the common ratio, the resulting series has most of the same terms as the original series. Thus, the difference between the two series (i.e., subtract the two) is a short expression that is not hard to work with.

Question Use these ideas to show that the sum is what it is. Can you use a picture to illustrate this sum?

Question Convert $0.\overline{42}$ to a fraction. What connections do you see with geometric series?

Concluding Remarks. When studying arithmetic and geometric sequences and series, it is easy to encapsulate common results into compact formulas. But formulas are easily confused with one another and otherwise misremembered. Furthermore, general formulas can obscure the ideas.

Question Find the missing terms in the following arithmetic sequence:

$$\_\_\_\_\_, \_\_\_\_\_, 2, \_\_\_\_\_, \_\_\_\_\_, 6, \ldots$$

Explain your reasoning.

Question Find exact values (not decimal approximations) for the missing terms in the following geometric sequence:

$$\_\_\_\_\_, \_\_\_\_\_, 2, \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_, 6, \ldots$$

Explain your reasoning. And describe how this problem and the rules of exponents might be used to explain the connection between radicals and exponents.

Question What key ideas behind arithmetic and geometric sequences did you use in the previous two problems?

Question Explain briefly the key ideas behind finding the sum of an arithmetic series. Then do the same for geometric series.

With the ideas, you can reconstruct the formulas you need. And without the ideas, formulas are empty.