CP5: Solving equations like $x^2 = 2$, real number system

Finding the square root of 2

1) Show that you can find a positive integer a_1 such that

$$\left(\frac{a_1}{10}\right)^2 \le 2 \le \left(\frac{a_1+1}{10}\right)^2.$$

Show that

$$\left(\frac{a_1+1}{10}\right)^2 - \left(\frac{a_1}{10}\right)^2 = \frac{2}{10}\left(\frac{a_1}{10}\right) + \frac{1}{10^2} \le \frac{2\cdot 2}{10} + \frac{1}{10^2} \le \frac{5}{10}.$$

2) Show that you can find a positive integer a_2 such that

$$\left(\frac{a_2}{100}\right)^2 \le 2 \le \left(\frac{a_2+1}{100}\right)^2.$$

Show that

$$\left(\frac{a_2+1}{10^2}\right)^2 - \left(\frac{a_2}{10^2}\right)^2 = \frac{2}{10^2}\left(\frac{a_2}{10^2}\right) + \frac{1}{10^4} \le \frac{2\cdot 2}{10^2} + \frac{1}{10^4} \le \frac{5}{10^2}.$$

3) Show that, for any positive integer n, no matter how large, you can find a positive integer a_n so that

$$\left(\frac{a_n}{10^n}\right)^2 \le 2 \le \left(\frac{a_n+1}{10^n}\right)^2$$

and

$$\left(\frac{a_n+1}{10^n}\right)^2 - \left(\frac{a_n}{10^n}\right)^2 \le \frac{5}{10^n}.$$

The real number system

The system of real numbers is the system of infinite decimals, that is the set of all (infinite) polynomial expressions of the form

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots + a_{-n}x^{-n} + \ldots$$

where $x = 10$ and all of the a_i are non-negative integers less than x. We add, subtract, multiply and divide them just like we do polynomials.

We call two real numbers equivalent (or equal) if the numerical value of their difference eventually gets smaller than $\frac{1}{10^n}$ no matter how large n is. (You have to start your subtraction of the two polynomial expressions *from the left* to make sense of this.)

4) Show that

$$0\overline{9} = 1.$$