CP4: Algebra with polynomials in integral powers of x

Polynomials in non-negative powers of an unknown

Suppose x is some unknown positive integer. A polynomial in x with integral coefficients is an expression of the form

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$$

where the a_i are integers. We will call the number you get when you substitute the numerical value for x into the polynomial and calculate out its *numerical value*.

1) For the polynomials above, show how to add and multiply these polynomials in x.

$$\begin{aligned}&\left(a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0\right) + \left(c_m x^m + c_{m-1}x^{m-1} + \ldots + c_1 x + c_0\right)\\ &= \left(a_m + c_m\right)x^m + \left(a_{m-1} + c_{m-1}\right)x^{m-1} + \ldots + \left(a_1 + c_1\right)x + \left(a_0 + c_0\right)\end{aligned}$$

$$\begin{aligned}&\left(a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0\right)\cdot\left(c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0\right) =\\ &\left(a_m c_n\right)x^{m+n} + \left(a_m c_{n-1} + a_{m-1}c_n\right)x^{m+n-1} + \left(a_m c_{n-2} + a_{m-1}c_{n-1} + a_{m-2}c_n\right)x^{m+n-2} + \ldots + \left(a_1 c_0 + a_0 c_1\right)x + \left(a_0 c_0\right)\end{aligned}$$

2) Suppose now that we make the rule that each a_i must be greater than or equal to zero. Which numbers can be represented by such polynomials?

> Any non-negatve integer N. Just set $a_0 = N$ and all the other $a_i = 0$.

3) Suppose now that we make the rule that each a_i must be greater than or equal to zero and less than 10. Now which numbers can be represented by such polynomials?

> It depends on what x is. If $x > 10$, then you will never be able to represent numbers between 9 and x or between $x + 9$ and $2x$, etc., etc.
> If $x \leq 10$, we can figure out the answer by doing Problem 4).

4) Suppose now that, instead, we make the rule that each a_i must be greater than or equal to zero and less than x. Now which numbers can be represented by such polynomials?

> Here the answer is again any non-negative integer N, but the reasoning is a bit more complicated. We start by finding the highest power of x that is less than or equal to N, say x^m. Then divide N by x^m to get
>
> $$N = a_m x^m + r_m.$$
>
> Then $r_m < x^m$. Why? Now find highest power of x that is less than or equal to r_m say $x^{m'}$. Then divide r_m by $x^{m'}$ to get
>
> $$r_m = a_{m'} x^{m'} + r_{m'}.$$
>
> Then $r_{m'} < x^{m'}$. Why? Also this lets us write
>
> $$N = a_m x^m + a_{m'} x^{m'} + r_{m'}.$$
>
> Now keep repeating the process until the remainder r is less that x and you will be done!

5) How can I rewrite a polynomial in 2) to get a polynomial in 4) without changing the numerical value of the polynomial?

> For $a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$, if any $a_i \geq x$, write
>
> $$a_i = b_{i+1}x + a'_i$$
>
> and substitute to get

$$\boxed{\begin{aligned}&a_m x^m + \ldots + a_{i+1}x^{i+1} + a_i x^i + \ldots + a_0 \\ &= a_m x^m + \ldots + a_{i+1}x^{i+1} + \left(b_{i+1}x + a'_i\right)x^i + \ldots + a_1 x + a_0 \\ &= a_m x^m + \ldots + \left(a_{i+1} + b_{i+1}\right)x^{i+1} + a'_i x^i + \ldots + a_1 x + a_0\end{aligned}}$$

Do this over and over any time there is still an $a_i \geq x$. Eventually you will get all the coefficients of all the powers of x to be less than x. (The final polynomial you get, though, may have degree higher than m.)

Polynomials in integral powers of an unknown (including negative powers)

x is an unknown positive integer greater than 1. A polynomial in integral powers of x with integer coefficients is an expression of the form

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots + a_{-n}x^{-n}$$

where the a_i are integers.

6) For the polynomials just above, show how to add and multiply these polynomials in x.

To add, just add the coefficients of like powers of x, just like in Problem 1). To multiply, write

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots + a_{-n}x^{-n}$$
$$= \frac{a_m x^{m+n} + a_{m-1}x^{m+n-1} + \ldots + a_1 x^{n+1} + a_0 x^n + a_{-1}x^{n-1} + a_{-2}x^{n-2} + \ldots + a_{-n}}{x^n}$$

and

$$c_{m'}x^{m'} + c_{m'-1}x^{m'-1} + \ldots + c_1 x + c_0 + c_{-1}x^{-1} + c_{-2}x^{-2} + \ldots + c_{-n'}x^{-n'}$$
$$= \frac{c_{m'}x^{m'+n'} + c_{m'-1}x^{m'+n'-1} + \ldots + c_1 x^{n'+1} + c_0 x^{n'} + c_{-1}x^{n'-1} + c_{-2}x^{n'-2} + \ldots + c_{-n'}}{x^{n'}}$$

Multiply the numerators like in Problem 1) and divide the answer by $x^{n+n'}$.

7) Suppose that each integer a_i must be greater than or equal to zero. Which numbers can be represented by such polynomials?

Again write:

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots + a_{-n}x^{-n}$$
$$= \frac{a_m x^{m+n} + a_{m-1}x^{m+n-1} + \ldots + a_1 x^{n+1} + a_0 x^n + a_{-1}x^{n-1} + a_{-2}x^{n-2} + \ldots + a_{-n}}{x^n}$$

We already saw that the numerator can take as value any positive integer N. The denominator can be x^n where n can be as large as you want. This means that the only fractions I can represent in this way are fractions whose denominator has only prime factors which are among the prime factors of x. That is the only restriction because I can cancel off any prime I don't want in the denominator by multiplying N by that prime.

8) Suppose now that we make the additional rule that each integer a_i must be greater than or equal to zero and less than x. Now which numbers can be represented by such polynomials?

Again write

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots + a_{-n}x^{-n}$$
$$= \frac{a_m x^{m+n} + a_{m-1}x^{m+n-1} + \ldots + a_1 x^{n+1} + a_0 x^n + a_{-1}x^{n-1} + a_{-2}x^{n-2} + \ldots + a_{-n}}{x^n}$$

The numerator can have value equal to any positive integer N. The denominator can be x^n where n can be as large as you want. This means that the only fractions I can represent in this way are fractions whose denominator has only prime factors which are among the prime factors of x. That is the only restriction because I can cancel off any prime I don't want in the denominator by multiplying N by that prime.

9) How can I rewrite a polynomial in 7) to get a polynomial in 8) without changing the numerical value of the polynomial?

Again write

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots + a_{-n}x^{-n}$$

$$= \frac{a_m x^{m+n} + a_{m-1}x^{m+n-1} + \ldots + a_1 x^{n+1} + a_0 x^n + a_{-1}x^{n-1} + a_{-2}x^{n-2} + \ldots + a_{-n}}{x^n}$$

Now apply what you did in Problems 4) and 5) to the numerator to get a new polynomial in x

$$c_{m'}x^{m'} + c_{m'-1}x^{m'-1} + \ldots + c_1 x + c_0$$

with the same numerical value but with all the c_i greater than or equal to 0 and less than x. Then

$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \ldots + a_{-n}x^{-n}$$

$$= \frac{a_m x^{m+n} + a_{m-1}x^{m+n-1} + \ldots + a_1 x^{n+1} + a_0 x^n + a_{-1}x^{n-1} + a_{-2}x^{n-2} + \ldots + a_{-n}}{x^n}$$

$$= \frac{c_{m'}x^{m'} + c_{m'-1}x^{m'-1} + \ldots + c_1 x + c_0}{x^n}$$

$$= c_{m'}x^{m'-n} + c_{m'-1}x^{m'-1-n} + \ldots + c_1 x^{1-n} + c_0{}^{-n}$$

and all the c_i are greater than or equal to 0 and less than x.