# CP7: Arithmetic of complex numbers, complex conjugation

Throughout this quarter, we have developed number systems so that we can solve equations. We saw that equations like x - 7 = 0 had solutions in the counting numbers, but equations such as x + 7 = 0 did not. This gave rise to the integers (which contained the counting numbers and their negatives and 0). Then we saw equations like 3x + 4 = 0 did not have solutions in the integers, so the system of fractions, or rational numbers (which contain the integers) was developed to accommodate them. We then saw equations such as $x^2 - 2 = 0$ had no solutions in the rational numbers, so the irrational numbers were added to make the decimal numbers or real number system. The rationals are still inside (as terminating and repeating decimals) but there are other numbers, like

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^{2^2}} + \frac{1}{10^{2^{2^2}}} + \ldots.$$

The decimal numbers can be represented as the real number line.

From the quadratic formula

$$a_2x^2 + a_1x + a_0 = 0 \Leftrightarrow x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

we see that $a_1^2 - 4a_2a_0$ could be negative even if $a_2,\ a_1$ and $a_0$ are just counting numbers. (Examples are: $x^2 + 1 = 0$ and $3x^2 + 5x + 10 = 0$). If $a_1^2 - 4a_2a_0 < 0$, its square root can't be a real number $r$ because that would give

$$r^2 = a_1^2 - 4a_2a_0.$$

This is trouble since squares of real numbers are always $\geq 0$.

1. So we make our number system even bigger in order to solve quadratic equations. We consider the set of all polynomials

$$a_m i^m + a_{m-1} i^{m-1} + \ldots + a_1 i + a_0$$

where the $a_i$ are real numbers, BUT, every time we see an $i^2$ anywhere, we are allowed to replace it by $-1$. Show that any expression $a_m i^m + a_{m-1} i^{m-1} + \ldots + a_1 i + a_0$ can thereby be reduced to a unique expression

$$b_1 i + b_0$$

where $b_1$ and $b_0$ are real numbers. We call the set of such expressions the set of *complex numbers*.

2. We add, subtract and multiply complex numbers just like polynomials, BUT, every time we encounter an $i^2$, we replace it by $-1$. That is, we have constructed the square root of a negative number in our new number system, calling $\sqrt{-1} = i$, with every other square root of a negative number written in the form of *i*: e.g., $\sqrt{-7} = \sqrt{7} \cdot \sqrt{-1} = \sqrt{7} \cdot i$. Do the following multiplication problems:

$$\left(3 + \sqrt{7} \cdot i\right) \cdot \left(4 - 2 \cdot i\right) =$$

$$\left(3 + \sqrt{7} \cdot i\right) \cdot \left(3 - \sqrt{7} \cdot i\right) =$$

3. Show that
$$\left(b_0 + b_1 i\right)\cdot\left(b_0 - b_1 i\right)$$
is always a non-negative real number. $\left(b_0 - b_1 i\right)$ is called the *complex conguate* of the complex number $\left(b_0 + b_1 i\right)$ and written $\overline{\left(b_0 + b_1 i\right)}$. Show that, if
$$z = \left(b_0 + b_1 i\right) \text{ and } w = \left(c_0 + c_1 i\right)$$
are complex numbers, then
$$\begin{aligned}\overline{z+w} &= \overline{z} + \overline{w} \\ \overline{z\cdot w} &= \overline{z}\cdot\overline{w}\end{aligned}.$$

4. These laws for complex conjugation [and the fact that any number divided by itself should equal 1] are the key to defining the operation of division in the system of complex numbers.
For example:
$$\left(3+\sqrt{7}\cdot i\right) \div \left(4-2\cdot i\right) = \frac{3+\sqrt{7}\cdot i}{4-2\cdot i}$$
$$= \frac{3+\sqrt{7}\cdot i}{4-2\cdot i}\cdot\frac{4+2\cdot i}{4+2\cdot i} = \frac{\left(3+\sqrt{7}\cdot i\right)\cdot\left(4+2\cdot i\right)}{20}$$
$$= \frac{\left(12-2\sqrt{7}\right)+\left(6+4\sqrt{7}\right)\cdot i}{20}$$
$$= \left(\frac{12-2\sqrt{7}}{20}\right)+\left(\frac{6+4\sqrt{7}}{20}\right)\cdot i$$
Do the division problem
$$\left(1+2\cdot i\right) \div \left(1+7\cdot i\right) = \qquad .$$

5. *Extra credit*: You can represent any complex number $x + y\cdot i$ as a point $\left(x,y\right)$ in the Euclidean plane. Since points in the plane also have polar coordinates, you can write any complex number in the form
$$\left(r\cos\theta\right)+\left(r\sin\theta\right)\cdot i$$
where $r$ is a non-negative real number and $\theta$ is a real angle. Show geometrically in the $\left(x,y\right)$-plane what happens when you multiply complex numbers $x + y\cdot i$ by a fixed complex number $\left(r\cos\theta\right)+\left(r\sin\theta\right)\cdot i$. If you want, pick some fixed numerical value for $r$ and $\theta$ and multiply that complex number with lots of sample complex numbers $x + y\cdot i$.