NUMBERS AND ALGEBRA

WITH TEACHING NOTES

MATH 1165: AUTUMN 2016

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Preface

These notes are designed with future middle grades mathematics teachers in mind. While most of the material in these notes would be accessible to an accelerated middle grades student, it is our hope that the reader will find these notes both interesting and challenging. In some sense we are simply taking the topics from a middle grades class and pushing "slightly beyond" what one might typically see in schools. In particular, there is an emphasis on the ability to communicate mathematical ideas. Three goals of these notes are:

- To enrich the reader's understanding of both numbers and algebra. From the basic algorithms of arithmetic—all of which have algebraic underpinnings—to the existence of irrational numbers, we hope to show the reader that numbers and algebra are deeply connected.

- To place an emphasis on problem solving. The reader will be exposed to problems that "fight-back." Worthy minds such as yours deserve worthy opponents. Too often mathematics problems fall after a single "trick." Some worthwhile problems take time to solve and cannot be done in a single sitting.

- To challenge the common view that mathematics is a body of knowledge to be memorized and repeated. The art and science of doing mathematics is a process of reasoning and personal discovery followed by justification and explanation. We wish to convey this to the reader, and sincerely hope that the reader will pass this on to others as well.

In summary—you, the reader, must become a doer of mathematics. To this end, many questions are asked in the text that follows. Sometimes these questions are answered; other times the questions are left for the reader to ponder. To let the reader know which questions are left for cogitation, a large question mark is displayed:

?

The instructor of the course will address some of these questions. If a question is not discussed to the reader's satisfaction, then we encourage the reader to put on a thinking-cap and think, think, think! If the question is still unresolved, go to the World Wide Web and search, search, search!

Much of the mathematics content in this course is strongly tied to the mathematics that you may be teaching in grades 4 through 9. To emphasize these connections, you will sometimes see margin notes that begin "CCSS." These are drawn from the *Common Core State Standards*, which describe goals for mathematics learning in grades K–12 in Ohio and many other states. For more information, see `http://www.corestandards.org`.

Please report corrections, suggestions, gripes, complaints, and criticisms to Bart Snapp at snapp@math.osu.edu or Brad Findell at findell.2@osu.edu.

Thanks and Acknowledgments

This document is based on a set of lectures originally given by Bart Snapp at the Ohio State University Fall 2009 and Fall 2010. Since then, additional text and many activities have been added by Vic Ferdinand, Brad Findell, and Betsy McNeal as part of our ongoing revision process to better serve our audience of future middle grades teachers. Special thanks goes to Herb Clemens for many helpful comments that have greatly improved these notes.

Contents

1 Arithmetic and Algebra

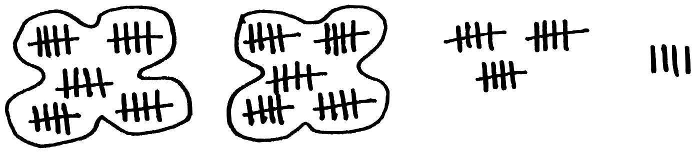
> As I made my way home, I thought Jem and I would get grown but there wasn't much else left for us to learn, except possibly algebra.
>
> —Harper Lee

Teaching Note: Here we outline a story with a series of puzzles. We suggest that the instructor simply present the puzzles (or similar puzzles) and have the students solve them rather than go through the entire story in class.

Activity A.1 complements this section well.

1.1 Home Base

Imagine 600 generations past—that's on the order of 10000 years, the dawn of what we would call civilization. This is a long time ago, well before the *Epic of Gilgamesh*. Even then people already knew the need to keep track of numbers. However, they didn't use the numbers we know and love (that's right, *love*!), they used tally-marks. Now what if "a friend" of yours had a time machine? What if they traveled through time and space and they decided to take you back 500 generations? Perhaps you would meet a nice man named Lothar (*Lothar of the Hill People* is his full name) who is trying to keep track of his goats. He has the following written on a clay tablet:



From this picture you discern that Lothar has 69 goats. Lothar is studying the tablet intently when his wife, Gertrude, comes in. She tries in vain to get Lothar to keep track of his goats using another set of symbols:

0 1 2 3 4

A heated debate between Lothar and Gertrude ensues, the exact details of which are still a mystery. We do glean the following facts:

(1) Under Gertrude's scheme, five goats are denoted by:

10

(2) The total number of Lothar's goats is denoted by:

234

Question Can you explain Gertrude's counting scheme?

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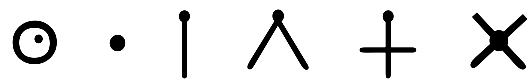
Did I mention that "your friend's" time machine is also a spaceship? Oh. . . Well it is. Now you both travel to the planet Omicron Persei 8. There are two things you should know about the inhabitants of Omicron Persei 8:

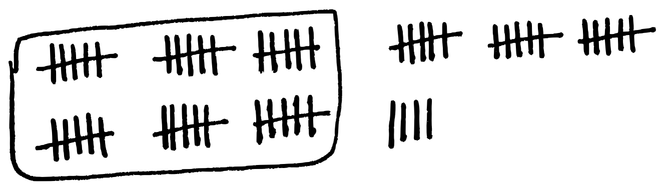
(1) They only have 3 fingers on each hand.

(2) They can eat a human in one bite.

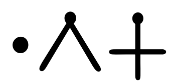
As you can see, there are serious issues that any human visitor to Omicron Persei 8 must deal with. For one thing, since the Omicronians only have 3 fingers on each hand, they've only written down the following symbols for counting:



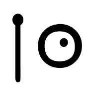
Emperor Lrrr of the Omicronians is tallying how many humans he ate last week



when his wife, Ndnd, comes in and reminds him that he can write this number using their fancy symbols as:



After reading some restaurant menus, you find out that twelve tally-marks are denoted by the symbols:



Question Can you explain the Omicronians' counting scheme?

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At this point you hop back into "your friend's" space-time ship. "Your friend" kicks off their shoes. You notice that "your friend" has 6 toes on each foot. You strike up a conversation about the plethora of toes. Apparently this anomaly has enabled “your friend” to create their own counting scheme, which they say is based on:

- Toes
- Feets
- Feets of Feets
- and so on. . .

“Your friend” informs you that they would write the number you know as “twenty-six” as 22 or “two feets and two toes.” What?! Though you find the conversation to be dull and stinky, you also find out that “your friend” uses two more symbols when they count. “Your friend” uses the letter *A* to mean what you call “ten,” and the letter *B* to mean what you call “eleven!”

Question Can you explain “your friend’s” counting scheme?

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Problems for Section 1.1

(1) Explain why the following "joke" is "funny:" *There are* 10 *types of people in the world. Those who understand base* 2 *and those who don't.*

(2) You meet some Tripod aliens, they tally by threes. Thankfully for everyone involved, they use the symbols 0, 1, and 2.

(a) Can you explain how a Tripod would count from 11 to 201? Be sure to carefully explain what number comes after 22.

(b) What number comes immediately before 10? 210? 20110? Explain your reasoning.

(3) You meet some people who tally by sevens. They use the symbols O, A, B, C, D, E, and F.

(a) What do the individual symbols O, A, B, C, D, E, and F mean?

(b) Can you explain how they would count from DD to AOC? Be sure to carefully explain what number comes after FF.

(c) What number comes immediately before AO? ABO? $EOFFA$? Explain your reasoning.

(4) Now, suppose that you meet a hermit who tallies by thirteens. Explain how he might count. Give some relevant and revealing examples.

(5) While visiting Mos Eisley spaceport, you stop by Chalmun's Cantina. After you sit down, you notice that one of the other aliens is holding a discussion on fractions. Much to your surprise, they explain that 1/6 of 36 is 7. You are unhappy with this, knowing that 1/6 of 36 is in fact 6, yet their audience seems to agree with it, not you. Next the alien challenges its audience by asking, "What is 1/4 of 10?" What is the correct answer to this question, and how many fingers do the aliens have? Explain your reasoning.

(6) When the first Venusian to visit Earth attended a 6Te grade class, it watched the teacher show that

$$\frac{3}{12} = \frac{1}{4}.$$

"How strange," thought the Venusian. "On Venus, $\frac{4}{12} = \frac{1}{4}$." What base do Venusians use? Explain your reasoning.

(7) When the first Martian to visit Earth attended a high school algebra class, it watched the teacher show that the only solution of the equation

$$5x^2 - 50x + 125 = 0$$

is $x = 5$.

"How strange," thought the Martian. "On Mars, $x = 5$ is a solution of this equation, but there also is another solution." If Martians have more fingers than humans, how many fingers do Martians have on both hands? Explain your reasoning.

(8) In one of your many space-time adventures, you see the equation

$$\frac{3}{10} + \frac{4}{13} = \frac{21}{20}$$

written on a napkin. How many fingers did the beast who wrote this have? Explain your reasoning.

(9) What is the smallest number of weights needed to produce every integer-valued mass from 0 grams to say n grams? Explain your reasoning.

(10) Starting at zero, how high can you count using just your fingers?

(a) Explain how to count to 10.

(b) Explain how to count to 35.

(c) Explain how to count to 1023.

(d) Explain how to count to 59048.

(e) Can you count even higher?

Explain your reasoning.

1.2 Arithmetic

Consider this question:

Activity A.2 complements this section well.

Question Can you *think* about something if you lack the *vocabulary* required to discuss it?

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1.2.1 Nomenclature

The numbers and operations we work with have properties whose importance are so fundamental that we have given them names. Each of these properties is surely well known to you; however, the importance of the name is that it gives a keen observer the ability to see and articulate fundamental structures in arithmetic and algebra.

The Associative Property An operation ★ is called **associative** if for all numbers a, b, and c:

$$a \bigstar (b \bigstar c) = (a \bigstar b) \bigstar c$$

The Commutative Property An operation ★ is called **commutative** if for all numbers a and b:

$$a \bigstar b = b \bigstar a$$

The Distributive Property An operation ★ is said to be **distributive** over another operation ✢ if for all numbers a, b, and c:

$$a \bigstar (b \maltese c) = (a \bigstar b) \maltese (a \bigstar c) \qquad \text{and} \qquad (b \maltese c) \bigstar a = (b \bigstar a) \maltese (c \bigstar a)$$

The Closure Property An operation ★ is called **closed** on a set of numbers if for all numbers a and b in the set:

$$a \bigstar b \qquad \text{is another number in the set.}$$

Fixnote: Possibly add identities and inverses. (See also Activity A.2.)

You may find yourself a bit distressed over some of the notation used above. In particular you surely notice that we were using crazy symbols like $\star$ and ✣. We did this for a reason. The properties above may hold for more than one operation. Let's explore this:

Question Can you give examples of operations that are associative? Can you give examples of operations that are not associative?

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Question Can you give examples of operations that are commutative? Can you give examples of operations that are not commutative?

?

Question Can you give examples of two operations where one distributes over the other? Can you give examples of operations that do not distribute?

?

Question Can you give examples of an operation and a set of numbers where the operation is closed on the set of numbers? Can you give examples of an operation and a set of numbers where the operation is not closed on the set of numbers?

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1.2.2 Algorithms

In elementary school you learned many strategies for addition and subtraction.[2.NBT.9] Some of these strategies can be developed into *algorithms*, which are general step-by-step procedures for computation. In this section, we aim to explain various strategies and algorithms for addition and subtraction, and our tools are place value and the properties of operations.[3.NBT.2]

CCSS 2.NBT.9: Explain why addition and subtraction strategies work, using place value and the properties of operations. (Explanations may be supported by drawings or objects.)

CCSS 3.NBT.2: Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

Teaching Note: Here we seek to have the students acknowledge the algebra behind many algorithms. We have given a number of examples illustrating the sort of work we wish to see.

Activities A.3 and A.4 complement this section well.

Standard Addition Algorithm Here is an example of a standard addition algorithm:

$$\begin{array}{r} 1\,1 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

Question Can you describe how to perform this algorithm?

As a gesture of friendship, I'll take this one. All we are doing here is adding each column of digits at a time, starting with the right-most digit

$$\begin{array}{r} \\ 89\mathbf{2} \\ +39\mathbf{8} \\ \hline \mathbf{10} \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{1} \\ 89\mathbf{2} \\ +39\mathbf{8} \\ \hline \mathbf{0} \end{array}$$

If our column of digits sums to 10 or higher, then we must "carry" the tens-digit of our sum to the next column. This process repeats until we run out of digits on the left.

$$\begin{array}{r} \mathbf{1} \\ 8\mathbf{9}2 \\ +3\mathbf{9}8 \\ \hline \mathbf{19}0 \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{1}1 \\ \mathbf{8}92 \\ +\mathbf{3}98 \\ \hline \mathbf{12}90 \end{array}$$

We're done!

Question Can you show the "behind-the-scenes" algebra going on here?

I'll take this one too. Sure, you just write:

$$\begin{aligned}
892 + 398 &= (8 \cdot 10^2 + 9 \cdot 10 + 2) + (3 \cdot 10^2 + 9 \cdot 10 + 8) \\
&= 8 \cdot 10^2 + 9 \cdot 10 + 2 + 3 \cdot 10^2 + 9 \cdot 10 + 8 \\
&= 8 \cdot 10^2 + 3 \cdot 10^2 + 9 \cdot 10 + 9 \cdot 10 + 2 + 8 \\
&= (8 + 3) \cdot 10^2 + (9 + 9) \cdot 10 + (2 + 8) \\
&= (8 + 3) \cdot 10^2 + (9 + 9) \cdot 10 + 10 + 0 \\
&= (8 + 3) \cdot 10^2 + (9 + 9 + 1) \cdot 10 + 0 \\
&= (8 + 3) \cdot 10^2 + (10 + 9) \cdot 10 + 0 \\
&= (8 + 3 + 1) \cdot 10^2 + 9 \cdot 10 + 0 \\
&= 12 \cdot 10^2 + 9 \cdot 10 + 0 \\
&= 1290
\end{aligned}$$

Wow! That was a lot of algebra. At each step, you should be able to explain how to get to the next step, and state which algebraic properties are being used.

Standard Multiplication Algorithm Here is an example of a standard multiplication algorithm:

$$\begin{array}{r} 23 \\ 634 \\ \times \quad 8 \\ \hline 5072 \end{array}$$

Question Can you describe how to perform this algorithm?

Me me me me! All we are doing here is multiplying each digit of the multi-digit number by the single digit number.

$$\begin{array}{r} 63\mathbf{4} \\ \times \quad \mathbf{8} \\ \hline \mathbf{32} \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{3} \\ 63\mathbf{4} \\ \times \quad \mathbf{8} \\ \hline \mathbf{2} \end{array}$$

If our product is 10 or higher, then we must "carry" the tens-digit of our product to the next column. This "carried" number is then added to our new product. This process repeats until we run out of digits on the left.

$$\begin{array}{r} \mathbf{3} \\ 6\mathbf{3}4 \\ \times \quad \mathbf{8} \\ \hline \mathbf{27}2 \end{array} \quad \rightsquigarrow \quad \begin{array}{r} \mathbf{2}3 \\ \mathbf{6}34 \\ \times \quad \mathbf{8} \\ \hline \mathbf{50}72 \end{array}$$

We're done!

Question Can you show the "behind-the-scenes" algebra going on here?

You betcha! Just write:

$$\begin{aligned}
634 \cdot 8 &= (6 \cdot 10^2 + 3 \cdot 10 + 4) \cdot 8 \\
&= 6 \cdot 8 \cdot 10^2 + 3 \cdot 8 \cdot 10 + 4 \cdot 8 \\
&= 6 \cdot 8 \cdot 10^2 + 3 \cdot 8 \cdot 10 + 32 && \text{(✿)} \\
&= 6 \cdot 8 \cdot 10^2 + (3 \cdot 8 + 3) \cdot 10 + 2 && \text{(❀)} \\
&= 6 \cdot 8 \cdot 10^2 + 270 + 2 && \text{(❄)} \\
&= (6 \cdot 8 + 2) \cdot 10^2 + 7 \cdot 10 + 2 && \text{(❂)} \\
&= 50 \cdot 10^2 + 7 \cdot 10 + 2 \\
&= 5 \cdot 10^3 + 0 \cdot 10^2 + 7 \cdot 10 + 2 \\
&= 5072
\end{aligned}$$

Ahhhhh! Algebra works. Remember just as before, at each step you should be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (✿) and (❀)? What about between lines (❄) and (✪)?

?

Long-Division Algorithm With Remainder Once more we meet with this old foe—long division. Here is an example:

$$\begin{array}{r@{\,}l} & 97\,\mathrm{R}1 \\ 8 & \overline{)777} \\ & \underline{72} \\ & 57 \\ & \underline{56} \\ & 1 \end{array}$$

Question Can you describe how to perform this algorithm?

Yes! I'm all about this sort of thing. All we are doing here is single digit division for each digit of the multi-digit dividend (the number under the division symbol) by the single digit divisor (the left-most number). We start by noting that 8 won't go into 7, and so we see how many times 8 goes into 77.

$$\begin{array}{r@{\,}l} & \mathbf{9} \\ \mathbf{8} & \overline{)\mathbf{77}7} \\ & \underline{\mathbf{72}} \\ & \mathbf{5} \end{array} \quad \rightsquigarrow \quad \begin{array}{r} n = d \cdot q + r \\ 77 = 8 \cdot 9 + 5 \end{array}$$

Now we drop the other 7 down, and see how many times 8 goes into 57.

$$\begin{array}{r} 9\mathbf{7} \\ \mathbf{8}\,\overline{)777} \\ \underline{72} \\ \mathbf{57} \\ \underline{\mathbf{56}} \\ \mathbf{1} \end{array} \quad \leftrightsquigarrow \quad \begin{aligned} n &= d \cdot q + r \\ 57 &= 8 \cdot 7 + 1 \end{aligned}$$

This process repeats until we run out of digits in the dividend.

Question Can you show the "behind-the-scenes" algebra going on here?

Of course—but this time things will be a bit different.

$$\begin{aligned} 77 &= 8 \cdot 9 + 5 \\ 77 \cdot 10 &= (8 \cdot 9 + 5) \cdot 10 \\ 77 \cdot 10 &= 8 \cdot 9 \cdot 10 + 5 \cdot 10 \\ 77 \cdot 10 + 7 &= 8 \cdot 9 \cdot 10 + 5 \cdot 10 + 7 \\ 777 &= 8 \cdot (9 \cdot 10) + 57 && (✿) \\ 777 &= 8 \cdot (9 \cdot 10) + (8 \cdot 7 + 1) && (❀) \\ 777 &= 8 \cdot (9 \cdot 10) + 8 \cdot 7 + 1 && (❄) \\ 777 &= 8 \cdot (9 \cdot 10 + 7) + 1 && (❂) \\ 777 &= 8 \cdot 97 + 1 \end{aligned}$$

Looks good to me, but remember: At each step you must be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (✿) and (❀)? What about between lines (❄) and (❂)?

?

Long-Division Algorithm Without Remainder Do you remember that the division algorithm can be done in such a way that there is no remainder? Here is an example of the division algorithm without remainder:

$$\begin{array}{r} 0.75 \\ 4\,\overline{)\,3.00} \\ \underline{2\,8} \\ 20 \\ \underline{\underline{20}} \end{array}$$

Question Can you describe how to perform this algorithm?

I'm getting a bit tired, but I think I can do this last one. Again, all we are doing here is single digit division for each digit of the multi-digit dividend (the number under the division symbol) by the single digit divisor (the left-most number) adding zeros after the decimal point as needed. We start by noting that 4 won't go into 3, and so we see how many times 4 goes into 3.0. Mathematically this is the same question; however, by thinking of the 3.0 as 30, we put ourselves into familiar territory. Since

$$4 \cdot 7 = 30 \qquad \Rightarrow \qquad 4 \cdot 7 \cdot 10^{-1} = 30 \cdot 10^{-1} = 3$$

this will work as long as we put our 7 immediately to the right of the decimal point.

$$\begin{array}{r} \mathbf{0.7} \\ \mathbf{4}\,\overline{)\,\mathbf{3.0}} \\ \underline{\mathbf{2\,8}} \\ \mathbf{2} \end{array} \qquad \leftrightsquigarrow \qquad \begin{aligned} n &= d \cdot q + r \\ 30 &= 4 \cdot 7 + 2 \end{aligned}$$

Now we are left with a remainder of .2. To take care of this, we drop another 0 down and see how many times 4 goes into 20. Since

$$4 \cdot 5 = 20 \qquad \Rightarrow \qquad 4 \cdot 5 \cdot 10^{-2} = 5 \cdot 10^{-2} = 0.05$$

this will work as long as we put our 5 two spaces to the right of the decimal point.

$$
\begin{array}{r}
0.7\mathbf{5} \\
\mathbf{4}\,\overline{)3.0\mathbf{0}} \\
\underline{2\,8} \\
\mathbf{20} \\
\underline{\underline{\mathbf{20}}}
\end{array}
\qquad \leftrightsquigarrow \qquad
\begin{aligned}
n &= d \cdot q + r \\
20 &= 4 \cdot 5 + 0
\end{aligned}
$$

This process repeats until we obtain a division with no remainder, or until we see repetition in the digits of the quotient.

Question Can you show the "behind-the-scenes" algebra going on here?

Let's do it:

$$
\begin{aligned}
3 &= 4 \cdot 0 + 3 \\
3.0 &= (4 \cdot 7 + 2) \cdot 10^{-1} \\
3.0 &= 4 \cdot (7 \cdot 10^{-1}) + 2 \cdot 10^{-1} \\
3.00 &= 4 \cdot (7 \cdot 10^{-1}) + 20 \cdot 10^{-2} \\
3.00 &= 4 \cdot (7 \cdot 10^{-1}) + (4 \cdot 5) \cdot 10^{-2} && (❄) \\
3.00 &= 4 \cdot (7 \cdot 10^{-1}) + 4 \cdot (5 \cdot 10^{-2}) && (❂) \\
3.00 &= 4 \cdot (7 \cdot 10^{-1} + 5 \cdot 10^{-2}) \\
3.00 &= 4 \cdot 0.75
\end{aligned}
$$

Looks good to me, but remember: At each step you must be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (❄) and (❂)?

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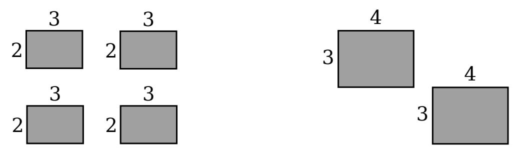
Problems for Section 1.2

Fixnote: Add a problem about the doubling/halving base-two multiplication algorithm.... Perhaps also include the Mysterious base-two game, or make it an activity.

(1) Explain what it means for an operation ★ to be *associative*. Give some relevant and revealing examples and non-examples.

(2) Consider the following pictures:



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$.

(a) Is Jesse's claim correct? Explain your reasoning.

(b) Do Jesse's pictures show the associativity of multiplication? If so, explain why. If not, draw new pictures representing $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ that do show the associativity of multiplication.

(3) Explain what it means for an operation ★ to be *commutative*. Give some relevant and revealing examples and non-examples.

(4) Explain what it means for an operation ★ to *distribute* over another operation ✢. Give some relevant and revealing examples and non-examples.

(5) Explain what it means for an operation ★ to be *closed* on a set of numbers. Give some relevant and revealing examples and non-examples.

(6) Sometimes multiplication is described as *repeated addition*. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.

(7) In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology. In particular, which property of which operation(s) do you use?

(8) Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.

(9) Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.

(10) Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.

(11) Cheap Carl likes to give a tip of $13\frac{1}{3}\%$ when he is at restaurants. He does this by dividing his bill by 10 and then adding one-third more to this number. Explain why this works.

(12) Reasonable Rebbecca likes to give a tip of 18% when she is at restaurants. She does this by dividing her bill by 5 and then removing one-tenth of this number. Explain why this works.

(13) Can you think of and justify any other schemes for computing the tip?

(14) Here is an example of a standard addition algorithm:

$$\begin{array}{r} {\scriptstyle 1\,1\ } \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(15) Here is an example of the column addition algorithm:

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(16) If you check out Problems (23) and (25), you will learn about "partial" algorithms.

(a) Develop a "partial" algorithm for addition, give it a name, and describe how to perform this algorithm.

(b) Provide a relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(17) Here is an example of the banker's addition algorithm:

$$\begin{array}{r} 892 \\ +398 \\ \hline 1\mathbf{0} \\ 1\mathbf{9} \\ \mathbf{12} \\ \hline 1290 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(18) Here is an example of a standard subtraction algorithm:

$$\begin{array}{rrr} & 8 & \\ 8 & \cancel{9} & {}^{1}2 \\ -3 & 7 & 8 \\ \hline 5 & 1 & 4 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(19) Here is an example of the subtraction by addition algorithm:

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ll} 8+\mathbf{4}=12 & \text{add 1 to 7 to get 8} \\ 8+\mathbf{1}=9 & \\ 3+\mathbf{5}=8 & \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(20) Here is an example of the Austrian subtraction algorithm:

$$\begin{array}{rrr} 8 & 9 & {}^{1}2 \\ -3 & {}^{8}\cancel{7} & 8 \\ \hline 5 & 1 & 4 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(21) If you check out Problems (23) and (25), you will learn about "partial" algorithms.

(a) Develop a "partial" algorithm for subtraction, give it a name, and describe how to perform this algorithm.

(b) Provide a relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(22) Here is an example of a standard multiplication algorithm:

$$\begin{array}{r} 23 \\ 634 \\ \times\quad 8 \\ \hline 5072 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the “behind-the-scenes” algebra that is going on here.

(23) Here is an example of the partial-products algorithm:

$$\begin{array}{r} 634 \\ \times\quad 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the “behind-the-scenes” algebra that is going on here.

(24) Here is an example of a standard division algorithm:

$$\begin{array}{r} 97\,\mathrm{R}1 \\ 8\,\overline{)777} \\ \underline{72} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the “behind-the-scenes” algebra that is going on here.

(25) Here is an example of the partial quotients algorithm:

$$\begin{array}{r} 7 \\ 90 \\ 8\,\overline{)777} \\ \underline{720} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the “behind-the-scenes” algebra that is going on here.

(26) Here is another example of the partial-quotients division algorithm:

$$\begin{array}{r} 4 \\ 10 \\ 10 \\ 10 \\ 8\,\overline{)277} \\ \underline{80} \\ 197 \\ \underline{80} \\ 117 \\ \underline{80} \\ 37 \\ \underline{32} \\ 5 \end{array}$$

(a) Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the “behind-the-scenes” algebra that is going on here.

(27) Here is an example of a standard multiplication algorithm:

$$\begin{array}{r} 634 \\ \times 216 \\ \hline 3804 \\ 6340 \\ 126800 \\ \hline 136944 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the “behind-the-scenes” algebra that is going on here—you may assume that you already know the algebra behind the standard multiplication algorithm.

(28) Here is an example of the addition algorithm with decimals:

$$\begin{array}{r} 1 \\ 37.2 \\ +8.74 \\ \hline 45.94 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(29) Here is an example of the multiplication algorithm with decimals:

$$\begin{array}{r} 3.40 \\ \times\ .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(30) Here is an example of the division algorithm without remainder:

$$\begin{array}{r} 0.75 \\ 4\,\overline{)3.00} \\ \underline{2\ 8} \\ 20 \\ \underline{\underline{20}} \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(31) In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{MOON} \\ \texttt{+ SUN} \\ \hline \texttt{PLUTO} \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint: S = 6 and U = 5.

(32) In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{SEND} \\ \texttt{+MORE} \\ \hline \texttt{MONEY} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

(33) In the following subtraction problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{DEFER} \\ \texttt{-DU7Y} \\ \hline \texttt{N2G2} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

(34) In the following two subtraction problems, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{NINE} \\ \texttt{-TEN} \\ \hline \texttt{TWO} \end{array} \qquad \begin{array}{r} \texttt{NINE} \\ \texttt{-ONE} \\ \hline \texttt{ALL} \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

(35) In the following multiplication problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{LET} \\ \times\ \texttt{NO} \\ \hline \texttt{SOT} \\ \texttt{NOT}\phantom{\texttt{T}} \\ \hline \texttt{FRET} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

(36) The following is a long division problem where every digit except 7 was replaced by X.

$$\begin{array}{r} X7X \\ XX\,\overline{)XXXXX} \\ \underline{X77} \\ X7X \\ \underline{X7X} \\ XX \\ \underline{\underline{XX}} \end{array}$$

Recover the digits from this long division problem. Explain your reasoning.

(37) The following is a long division problem where the various digits were replaced by X except for a single 8. The double bar indicates that the remainder is 0.

```
          XX8XX
    ___________
XXX ) XXXXXXXX
      XXX
      ---
       XXXX
        XXX
        ---
         XXXX
         XXXX
         ====
```

Recover the digits from this long division problem. Explain your reasoning.

1.3 Algebra

Algebra is when you replace a number with a letter, usually x, right? OK—but you also do things with x, like make *polynomials* out of it.

1.3.1 Polynomial Basics

Activity A.5 complements this section well.

Question What's a polynomial?

?

I'll take this one:

Definition An n^{th}-degree **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are all constants, n is a nonnegative integer, and $a_n \neq 0$.

Question Which of the following are polynomials?

$$3x^3 - 2x + 1 \qquad \frac{1}{3x^3 - 2x + 1} \qquad 3x^{-3} - 2x^{-1} + 1 \qquad 3x^{1/3} - 2x^{1/6} + 1$$

?

Given two polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

we treat these polynomials much the same way we treat numbers. Note, an easy fact is that polynomials are equal if and only if their coefficients are equal—this may come up again!

Question Are numbers equal if and only if their digits are equal?

?

Teaching Note: This question is foreshadowing a future discussion of real numbers. The students will probably suggest that it is true—this is OK. We will address this point later.

Question Can you explain how to add two polynomials? Compare and contrast this procedure to the standard addition algorithm for counting numbers.

?

Question Can you explain how to multiply two polynomials? Compare and contrast this procedure to the standard multiplication algorithm for counting numbers.

?

Question Can you explain why someone might say that working with polynomials is like working in "base x?"

?

1.3.2 Division and Polynomials

For some reason you keep on signing up for classes with aloof old Professor Rufus. When he was asked to teach division of polynomials with remainders, he merely wrote

$$d(x)\overline{\smash{)}n(x)}^{\;q(x)\ \mathrm{R}\, r(x)} \qquad \text{where} \qquad \begin{array}{l} d(x) \text{ is the divisor} \\ n(x) \text{ is the dividend} \\ q(x) \text{ is the quotient} \\ r(x) \text{ is the remainder} \end{array}$$

and walked out of the room, again! Do you have *déjà vu*?

Question Can you give 3 much needed examples of polynomial long division with remainders?

?

Question Given polynomials $d(x)$, $n(x)$, $q(x)$, and $r(x)$ how do you know if they leave us with a correct expression above?

?

Question Can you explain how to divide two polynomials?

?

Question Can you do the polynomial long division with remainder?

?

Again, this question can be turned into a theorem.

Theorem 1.3.1 (Division Theorem) *Given any polynomial $n(x)$ and a non-constant polynomial $d(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that*

The above space has intentionally been left blank for you to fill in.

Teaching Note: Here we want the students to realize that

$$n(x) = d(x)q(x) + r(x) \qquad \text{where } 0 \leqslant \deg(r(x)) < \deg(d(x))$$

Highlight uniqueness, using the requirement on the degree of the remainder.

Problems for Section 1.3

(1) Explain what is meant by a *polynomial* in a variable x.

(2) Given:

$$3x^7-x^5+x^4-16x^3+27 = a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x^1+a_0$$

Find a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7.

(3) Given:

$$6x^5 + a_4x^4 - x^2 + a_0 = a_5x^5 - 24x^4 + a_3x^3 + a_2x^2 - 5$$

Find a_0, a_1, a_2, a_3, a_4, a_5.

(4) Is it true that polynomials are equal if and only if their coefficients are equal? Explain your reasoning.

(5) Is it true that numbers are equal if and only if their digits are equal? Explain your reasoning.

(6) Explain how to add two polynomials. Explain, in particular, how "collecting like terms" is an application of the properties of arithmetic.

(7) Explain how to multiply two polynomials.

(8) Here is an example of the polynomial division algorithm:

$$\begin{array}{r@{}l}
 & \phantom{x^3 + 0x^2 + {}} x - 3 \quad \mathrm{R}\, 9x+4 \\
\cline{2-2}
x^2+3x+1 \big) & x^3 + 0x^2 + x + 1 \\
 & \underline{x^3 + 3x^2 + x} \\
 & \quad -3x^2 + 0x + 1 \\
 & \quad \underline{-3x^2 - 9x - 3} \\
 & \qquad\qquad 9x + 4
\end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(9) State the *Division Theorem* for polynomials. Give some relevant and revealing examples of this theorem in action.

(10) Given a polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

can you find two numbers L and U such that $L \leqslant p(x) \leqslant U$ for all x? If so, explain why. If not, explain why not.

(11) Consider all polynomials of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where the a_i's are integers. If you substitute an integer for x will you always get an integer out? Explain your reasoning.

(12) Consider the following polynomial:

$$p(x) = \frac{x^2}{2} + \frac{x}{2}$$

Will $p(x)$ always returns an integer when an integer is substituted for x? Explain your reasoning.

(13) Fix some integer value for x and consider all polynomials of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

Where the a_i's are integers greater than or equal to 0. Which numbers can be represented by such polynomials? Explain your reasoning.

(14) Find a polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 2 such that $p(2) = 35$. Discuss how your answer compares to the representation of 35 in base 2. Explain your reasoning.

(15) Find a polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 7 such that $p(7) = 234$. Discuss how your answer compares to the representation of 234 in base 7. Explain your reasoning.

(16) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 10 such that $p(10) = 18$. Discuss how your answer compares to the representation of 18 in base 10. Explain your reasoning.

(17) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 15 such that $p(15) = 201$. Discuss how your answer compares to the representation of 201 in base 15. Explain your reasoning.

(18) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than x. Which numbers can be represented by such polynomials? Explain your reasoning. Big hint: Base x.

(19) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than 10. Which numbers can be represented by such polynomials? Explain your reasoning.

(20) Consider $x^2 + x + 1$. This can be thought of as a "number" in base x. Express this number in base $(x + 1)$, that is, find b_0, b_1, b_2 such that

$$b_2(x + 1)^2 + b_1(x + 1) + b_0 = x^2 + x + 1.$$

Explain your reasoning.

(21) Consider $x^2 + 2x + 3$. This can be thought of as a "number" in base x. Express this number in base $(x - 1)$, that is, find b_0, b_1, b_2 such that

$$b_2(x - 1)^2 + b_1(x - 1) + b_0 = x^2 + 2x + 3.$$

Explain your reasoning.

(22) Consider $x^3 + 2x + 1$. This can be thought of as a "number" in base x. Express this number in base $(x - 1)$, that is, find b_0, b_1, b_2, b_3 such that

$$b_3(x - 1)^3 + b_2(x - 1)^2 + b_1(x - 1) + b_0 = x^3 + 2x + 1.$$

Explain your reasoning.

(23) If the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is thought of as a "number" in base x, describe two different ways to find the base $(x - 1)$ coefficients of $p(x)$.

2 Numbers

> God created the integers, the rest is the work of man.
>
> —Leopold Kronecker

2.1 The Integers

An important theme in this course is distinguishing among the various number systems of school mathematics. A *number system* is a set of numbers together with arithmetic operations, such as addition and multiplication, on those numbers. We have already been using *counting numbers*. Now we need to be more precise.

Teaching Note: A theme in this chapter and throughout the course is "general reasoning with specific numbers."

Definition The **counting numbers**, often called the **natural numbers**, are (naturally) those use for counting. We use the symbol $\mathbb{N}$ to denote the counting numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

When 0 is included with the counting numbers, we have the set of **whole numbers**, denoted $\mathbb{W}$:

$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

The set of counting numbers, zero, and negative counting numbers is called the set of **integers**, denoted $\mathbb{Z}$:

$$\mathbb{Z} = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

In case you're wondering, the symbol $\mathbb{Z}$ is used because *Zahlen* is the German word for "numbers."

2.1.1 Addition

Teaching Note: Key meanings of addition are "adding to" and "putting together."

Addition is probably the first operation we learn.

Question Write a story problem whose solution is given by the expression $19 + 17$. Let this context be a "working model" for addition.

?

Question Does your model show associativity of addition? If so, explain how. If not, can you come up with a new model (story problem) that does?

?

Question Does your model show commutativity of addition? If so, explain how. If not, can you come up with a new model that does?

?

An observant reader might notice that we have thus far given no reason to have negative integers.

Question Describe some contexts (story problems) in which negative numbers are useful. It will help to think of contexts in which there are “opposite” numbers in some sense.

?

Question Does your addition model work with negative integers? In other words, does it model $19 + (-17)$ and $8 + (-13)$? If so, explain how. If not, can you modify your model or come up with a new model that does work?

?

2.1.2 Subtraction

Question Write a story problem whose solution is given by the expression $19 - 17$. Let this context be a “working” model for subtraction.

?

Question We know that

$$a - b = a + (-b),$$

but the left-hand side of the equation is conceptually different from the right-hand side of the equation. Write two story problems, one solved by $19 - 17$ and the other solved by $19 + (-17)$. What’s the difference? (Pun intended!)

?

Teaching Note: Here we are trying to have the students develop the “take-away,” along with the “missing addend,” and a “comparison” model for subtraction.

Question Can you use the two story problems above to model

$$(-19) - 17, \qquad 19 - (-17), \qquad (-19) - (-17)?$$

?

Question How is *subtraction* different from *negation*?

?

2.1.3 Multiplication

Multiplication is more multifaceted than addition.

Question Write a story problem whose solution is given by the expression $19 \cdot 17$. Let this context be a “working” model for multiplication.

?

Teaching Note: We would like to point out that the units used in addition are generally the same for the different summands. However, with multiplication, the different factors often have different units.

Question Does your model show commutativity of multiplication? If so, explain how. If not, can you come up with a new model that does?

?

Question Does your model show associativity of multiplication? If so, explain how. If not, can you come up with a new model that does?

?

Teaching Note: This question is with Problem (2) of Section 1.2 in mind. In particular to show associativity, we suggest appealing to the notion of volume.

Question Does your model work with negative integers? In particular does your model show that

$$\text{positive} \cdot \text{negative} = \text{negative},$$

$$\text{negative} \cdot \text{positive} = \text{negative},$$

and

$$\text{negative} \cdot \text{negative} = \text{positive?}$$

If so, explain how. If not, can you come up with a new model that does?

?

Teaching Note: This is difficult. The students may not be able to come up with a model that works. This is OK—as this issue is addressed in Problem (32).

2.1.4 Division

While addition and multiplication are good operations, the real "meat" of the situation comes with division.

Definition We say that a non-zero integer d **divides** an integer n if there is an integer q such that

$$n = dq.$$

In this case we write $d \mid n$, which is said: "d divides n." If d does not divide n, we sometimes write $d \nmid n$.

While this may seem easy, it is actually quite tricky. You must always remember the following synonyms for *divides*:

"d **divides** n" $\leftrightsquigarrow$ "d is a **divisor** of n" $\leftrightsquigarrow$ "d is a **factor** of n" $\leftrightsquigarrow$ "n is a **multiple** of d"

Activity A.8 complements this section well.

Definition A **prime** number is a positive integer with exactly two positive divisors, namely 1 and itself.

Definition A **composite** number is a positive integer with more than two positive divisors.

I claim that every composite number is divisible by a prime number. Do you believe me? If not, consider this:

> Suppose there was a composite number that was *not* divisible by a prime. Then there would necessarily be a *smallest* composite number that is not divisible by a prime. Since this number is composite, this number is the product of two even smaller numbers, both of which have prime divisors. Hence our original number must have prime divisors.

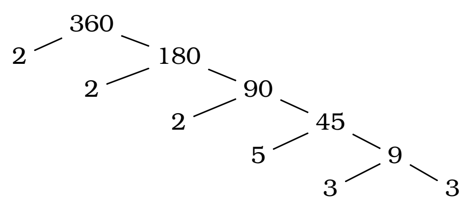
Question What the heck just happened?! Can you rewrite the above paragraph, drawing pictures and/or using symbols as necessary, making it more clear?

?

Activity A.12 complements this section well.

2.1.5 Factoring

At this point we can factor any composite completely into primes. To do this, it is often convenient to make a *factor tree*:



From this tree (Why is this a tree? It looks more like roots to me!) we see that

$$360 = 2^3 \cdot 3^2 \cdot 5.$$

At each step we simply divided by whichever prime number seemed most obvious, branched off the tree and kept on going. From our factor tree, we can see some of the divisors of the integer in question. However, there are many composite factors that can be built up from the prime divisors. One of the most important is the *greatest common divisor*.

Definition The **greatest common divisor** (GCD) of two integers a and b (not both 0) is a positive integer $g = \gcd(a, b)$ where:

(1) $g|a$ and $g|b$.

(2) If $d|a$ and $d|b$, then $0 < d \leqslant g$.

Question Describe informally what the greatest common divisor of two numbers means. Explain how the two conditions in the formal definition appear in your informal description.

?

Question What can you conclude when $\gcd(a, b) = 1$? Explain.

?

Question How can you use a factor tree to compute the GCD of two integers?

?

Question Describe informally what the least common multiple (LCM) of two numbers means. Write a formal definition of LCM. Explain how the two conditions in the formal definition appear in your informal description.

?

Question How can you use a factor tree to compute the LCM of two integers?

?

So, to factor an integer or find the GCD, one could use a factor tree. However, when building the factor tree, we had to know what primes to divide by. What if no prime comes to mind? What if you want to factor the integer 391 or 397? This raises a new question:

Question How do you check to see if a given integer is prime? What possible divisors must you check? When can you stop checking?

?

Activities A.10 and A.11 complements this section well.

2.1.6 Division with Remainder

We all remember long division, or at least we remember *doing* long division. Sometimes, we need to be reminded of our *forgotten foes*. When aloof old Professor Rufus was trying to explain division to his class, he merely wrote

$$d\,\overline{)\,n}^{\;q\ \mathrm{R}\,r} \qquad \text{where} \qquad \begin{array}{l} d \text{ is the divisor} \\ n \text{ is the dividend} \\ q \text{ is the quotient} \\ r \text{ is the remainder} \end{array}$$

and walked out of the room.

Question Can you give 3 much needed examples of long division with remainders?

?

Question Given positive integers d, n, q, and r how do you know if they leave us with a correct expression above?

?

Question Given positive integers d and n, how many different sets of q and r can you find that will leave us with a correct expression above?

?

The innocuous questions above can be turned into a theorem. We'll start it for you, but you must finish it off yourself:

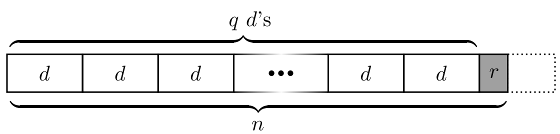
Theorem 2.1.1 (Division Theorem) *Given any integer n and a nonzero integer d, there exist unique integers q and r such that*

The above space has intentionally been left blank for you to fill in.

Teaching Note: Here we want the students to realize that

$$n = dq + r \qquad \text{where } 0 \leqslant r < d$$

Now consider the following picture:



Question How does the picture above "prove" the Division Theorem for positive integers? How must we change the picture if we allow negative values for n and d?

?

Teaching Note: Highlight uniqueness: The requirement that $0 \leqslant r < d$ makes both d and r unique. Without that requirement, many d and r pairs will work.

The second part of this question is quite challenging. Some specific examples can help.

Activity A.13 complements this section well.

Problems for Section 2.1

(1) Describe the set of integers. Give some relevant and revealing examples/nonexamples.

(2) Explain how to model integer addition with pictures or items. What relevant properties should your model show?

(3) Explain how to model integer multiplication with pictures or items. What relevant properties should your model show?

(4) Explain what it means for one integer to *divide* another integer. Give some relevant and revealing examples/nonexamples.

(5) Use the definition of *divides* to decide whether the following statements are true or false. In each case, an explanation must be given justifying your claim.

(a) $5|30$
(b) $7|41$
(c) $0|3$
(d) $3|0$
(e) $6|(2^2 \cdot 3^4 \cdot 5 \cdot 7)$
(f) $1000|(2^7 \cdot 3^9 \cdot 5^{11} \cdot 17^8)$
(g) $6000|(2^{21} \cdot 3^{17} \cdot 5^{89} \cdot 29^{20})$

(6) *Incognito's Hall of Shoes* is a shoe store that just opened in Myrtle Beach, South Carolina. At the moment, they have 100 pairs of shoes in stock. At their grand opening 100 customers showed up. The first customer tried on every pair of shoes, the second customer tried on every 2nd pair, the third customer tried on every 3rd pair, and so on until the 100th customer, who only tried on the last pair of shoes.

(a) Which shoes were tried on by only 1 customer?
(b) Which shoes were tried on by exactly 2 customers?
(c) Which shoes were tried on by exactly 3 customers?
(d) Which shoes were tried on by the most number of customers?

Explain your reasoning.

(7) Factor the following integers:

(a) 111
(b) 1234
(c) 2345
(d) 4567
(e) 111111

In each case, how large a prime must you check before you can be sure of your answers? Explain your reasoning.

(8) Which of the following numbers are prime? Explain how could deduce whether the numbers are prime in as few calculations as possible:

$$29 \qquad 53 \qquad 101 \qquad 359 \qquad 779 \qquad 839 \qquad 841$$

In each case, describe precisely which computations are needed and why those are the only computations needed.

(9) Suppose you were only allowed to perform at most 7 computations to see if a number is prime. How large a number could you check? Explain your reasoning.

(10) Find examples of integers a, b, and c such that $a \mid bc$ but $a \nmid b$ and $a \nmid c$. Explain your reasoning.

(11) Can you find at least 5 composite integers in a row? What about at least 6 composite integers? Can you find 7? What about n? Explain your reasoning. Hint: Consider something like $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

(12) Use the definition of the *greatest common divisor* to find the GCD of each of the pairs below. In each case, a detailed argument and explanation must be given justifying your claim.

(a) $\gcd(462, 1463)$
(b) $\gcd(541, 4669)$
(c) $\gcd(10000, 2^5 \cdot 3^{19} \cdot 5^7 \cdot 11^{13})$
(d) $\gcd(11111, 2^{14} \cdot 7^{21} \cdot 41^5 \cdot 101)$
(e) $\gcd(437^5, 8993^3)$

(13) Lisa wants to make a new quilt out of 2 of her favorite sheets. To do this, she is going to cut each sheet into as large squares as possible while using the entire sheet and using whole inch measurements.

(a) If the first sheet is 72 inches by 60 inches what size squares should she cut?

(b) If the second sheet is 80 inches by 75 inches, what size squares should she cut?

(c) How she might sew these squares together?

Explain your reasoning.

(14) Deena and Doug like to feed birds. They want to put 16 cups of millet seed and 24 cups of sunflower seeds in their feeder.

(a) How many total scoops of seed (millet or sunflower) are required if their scoop holds 1 cup of seed?

(b) How many total scoops of seed (millet or sunflower) are required if their scoop holds 2 cups of seed?

(c) How large should the scoop be if we want to minimize the total number of scoops?

Explain your reasoning.

(15) Consider the expression:

$$d\overline{\smash{)}n}^{\,q\ \mathrm{R}\,r} \qquad \text{where} \qquad \begin{array}{l} d \text{ is the divisor} \\ n \text{ is the dividend} \\ q \text{ is the quotient} \\ r \text{ is the remainder} \end{array}$$

(a) Give 3 relevant and revealing examples of long division with remainders.

(b) Given positive integers d, n, q, and r how do you know if they leave us with a correct expression above?

(c) Given positive integers d and n, how many different sets of q and r can you find that will leave us with a correct expression above?

(d) Give 3 relevant and revealing examples of long division with remainders where some of d, n, q, and r are negative.

(e) Still allowing some of d, n, q, and r to be negative, how do we know if they leave us with a correct expression above?

(16) State the *Division Theorem* for integers. Give some relevant and revealing examples of this theorem in action.

(17) Explain what it means for an integer to *not* divide another integer. That is, explain symbolically what it should mean to write:

$$a \nmid b$$

(18) Consider the following:

$$20 \div 8 = 2 \text{ remainder } 4,$$
$$28 \div 12 = 2 \text{ remainder } 4.$$

Is it correct to say that $20 \div 8 = 28 \div 12$? Explain your reasoning.

(19) Give a formula for the nth even number. Show-off your formula with some examples.

(20) Give a formula for the nth odd number. Show-off your formula with some examples.

(21) Give a formula for the nth multiple of 3. Show-off your formula with some examples.

(22) Give a formula for the nth multiple of -7. Show-off your formula with some examples.

(23) Give a formula for the nth number whose remainder when divided by 5 is 1. Show-off your formula with some examples.

(24) Explain the rule

$$\text{even} + \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra. Your explanations must be general, not based on specific examples.

(25) Explain the rule

$$\text{odd} + \text{even} = \text{odd}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra. Your explanations must be general, not based on specific examples.

(26) Explain the rule

$$\text{odd} + \text{odd} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra. Your explanations must be general, not based on specific examples.

(27) Explain the rule

$$\text{even} \cdot \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra. Your explanations must be general, not based on specific examples.

(28) Explain the rule

$$\text{odd} \cdot \text{odd} = \text{odd}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra. Your explanations must be general, not based on specific examples.

(29) Explain the rule

$$\text{odd} \cdot \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra. Your explanations must be general, not based on specific examples.

(30) Let $a \geqslant b$ be positive integers with $\gcd(a, b) = 1$. Compute $\gcd(a + b, a - b)$. Explain your reasoning. Hints:

(a) Make a chart.

(b) If $g|x$ and $g|y$ explain why $g|(x + y)$.

(31) Make a chart listing all pairs of positive integers whose product is 18. Do the same for 221, 462, and 924. Use this experience to help you explain why when factoring a number n, you only need to check factors less than or equal to $\sqrt{n}$.

(32) Matt is a member of the Ohio State University Marching Band. Being rather capable, Matt can take x steps of size y inches for all integer values of x and y. If x is positive it means *face North and take x steps*. If x is negative it means *face South and take $|x|$ steps*. If y is positive it means your step is a *forward step of y inches*. If y is negative it means your step *is a backward step of $|y|$ inches*.

Fixnote: We need additional models, e.g., checks and bills, red and black chips. Some of these are incorporated into Activities A.6 and A.7

(a) Discuss what the expressions $x \cdot y$ means in this context. In particular, what happens if $x = 1$? What if $y = 1$?

(b) Using the context above, write and solve a word problem that demonstrates the rule:

$$\text{negative} \cdot \text{positive} = \text{negative}$$

Clearly explain how your problem shows this.

(c) Using the context above, write and solve a word problem that demonstrates the rule:

$$\text{negative} \cdot \text{negative} = \text{positive}$$

Clearly explain how your problem shows this.

(33) Stewie decided to count the pennies he had in his piggy bank. He decided it would be quicker to count by fives. However, he ended with two uncounted pennies. So he tried counting by twos but ended up with one uncounted penny. Next he counted by threes and then by fours, each time there was one uncounted penny. Though he knew he had less than a dollars worth of pennies, and more than 50 cents, he still didn't have an exact count. Can you help Stewie out? Explain your reasoning.

2.2 The Fundamental Theorem of Arithmetic

In the previous section, we found divisors, greatest common divisors, and prime factors of positive integers. And when we found prime factorizations of integers, we used factor trees to organize our work.

Question Jake and Jenna use factor trees to find prime factorizations of the same large number. Assuming that they don't make any mistakes will their prime factorizations be the same or could they be different? Explain.

?

Let's try a simpler question.

Question If $11|50a$, is it true that $11|a$? Explain and generalize.

?

The following *lemma* will help us tie these ideas together. What is a lemma, you ask? A lemma is nothing but a little theorem that helps us solve another problem.

Lemma 2.2.1 (Euclid's Lemma) *If p is a prime number and a and b are integers*

$$p|ab \qquad \text{implies that} \qquad p|a \text{ or } p|b.$$

We are going to assume Euclid's Lemma without proof (at least for now) because we want to use it to prove our fundamental theorem—sometimes called the *Fundamental Theorem of Arithmetic*:

Activity A.15 complements this section well.

Theorem 2.2.1 (Unique Factorization) *Every integer greater than 1 can be factored uniquely (up to ordering) into primes.*

Proof *Well, if an integer is prime, we are done. If an integer is composite, then it is divisible by a prime number. Divide and repeat with the quotient. If our original integer was n, we'll eventually get:*

$$n = p_1 p_2 \cdots p_m$$

where some of the p_i's may be duplicates.

How do we know this factorization is unique? Well, suppose that

$$n = p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_l$$

where the p_i's and the q_j's are all prime. By the definition of "divides"

$$p_1 | q_1 (q_2 \cdots q_l).$$

So by Euclid's Lemma, p_1 must divide either q_1 or $(q_2 \cdots q_l)$. If $p_1 \nmid q_1$, then

$$p_1 | q_2 (q_3 \cdots q_l).$$

Repeat this enough times and you will find that $p_1 | q_j$ for one of the q_j's above, which implies that $p_1 = q_j$. Repeat this process for all the p_i's and you see that the factorization is unique.

Question Huh?! Can you explain what just happened drawing pictures and/or using symbols as necessary? How do we know the process will terminate? Once we see that $p_i | q_j$ for some j, how do we know that $p_i = q_j$? Could you also give some examples?

?

Question Thinking about Unique Factorization of the Integers, explain why it makes sense to exclude 1 from the prime numbers.

?

Question Thinking about Unique Factorization of the Integers, what must be the case when a number in base ten has units digit of 0? What about in other bases?

?

From high school algebra, you have lots of tools for solving equations. But in some situations, we are interested only in whole number or integer solutions to these equations. These kinds of equations have a special name:

Definition A **Diophantine equation** is an equation where only integer solutions are deemed acceptable.

In this section, we are particularly interested solve *linear Diophantine equations*, that is, equations of the form:

$$ax + by = c$$

where a, b, and c are integers and the only solutions we will accept are pairs of integers x and y.

Problems for Section 2.2

(1) Explain what the GCD of two integers is. Give some relevant and revealing examples/nonexamples.

(2) Explain what the LCM of two integers is. Give some relevant and revealing examples/nonexamples.

(3) Consider the Diophantine equation:

$$15x + 4y = 1$$

(a) Find a solution to this equation. Explain your reasoning.

(b) Compute the slope of the line $15x + 4y = 1$ and write it in lowest terms. Show your work.

(c) Plot the line determined by $15x + 4y = 1$ on graph paper.

(d) Using your plot and the slope of the line, explain how to find 10 more solutions to the Diophantine equation above.

(4) Explain why a Diophantine equation

$$ax + by = c$$

has either an infinite number of solutions or zero solutions.

(5) Josh owns a box containing beetles and spiders. At the moment, there are 46 legs in the box. How may beetles and spiders are currently in the box? Explain your reasoning.

(6) How many different ways can thirty coins (nickles, dimes, and quarters) be worth five dollars? Explain your reasoning.

(7) Lisa collects lizards, beetles and worms. She has more worms than lizards and beetles together. Altogether in the collection there are twelve heads and twenty-six legs. How many lizards does Lisa have? Explain your reasoning.

(8) Can you make exactly \$5 with exactly 100 coins assuming you can only use pennies, dimes, and quarters? If so how, if not why not? Explain your reasoning.

(9) A merchant purchases a number of horses and bulls for the sum of 1770 talers. He pays 31 talers for each bull, and 21 talers for each horse. How many bulls and how many horses does the merchant buy? Solve this problem, explain what a *taler* is, and explain your reasoning—note this problem is an old problem by L. Euler, it was written in the 1700's.

(10) A certain person buys hogs, goats, and sheep, totaling 100 animals, for 100 crowns; the hogs cost him $3\frac{1}{2}$ crowns a piece, the goats $1\frac{1}{3}$ crowns, and the sheep go for $\frac{1}{2}$ crown a piece. How many did this person buy of each? Explain your reasoning—note this problem is an old problem from *Elements of Algebra* by L. Euler, it was written in the 1700's.

(11) How many zeros are at the end of the following numbers:

(a) $2^2 \cdot 5^8 \cdot 7^3 \cdot 11^5$

(b) $11!$

(c) $27!$

(d) $99!$

(e) $1001!$

In each case, explain your reasoning.

(12) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

(a) $7|56$

(b) $55|11$

(c) $3|40$

(d) $100|(2^4 \cdot 3^{17} \cdot 5^2 \cdot 7)$

(e) $5555|(5^{20} \cdot 7^9 \cdot 11^{11} \cdot 13^{23})$

(f) $3|(3 + 6 + 9 + \cdots + 300 + 303)$

(13) Suppose that

$$(3^5 \cdot 7^9 \cdot 11^x \cdot 13^y)|(3^a \cdot 7^b \cdot 11^{19} \cdot 13^7)$$

What values of a, b, x and y, make true statements? Explain your reasoning.

(14) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

(a) If $7|13a$, then $7|a$.

(b) If $6|49a$, then $6|a$.

(c) If $10|65a$, then $10|a$.

(d) If $14|22a$, then $14|a$.

(e) $54|931^{21}$.

(f) $54|810^{33}$.

(15) Joanna thinks she can see if a number is divisible by 24 by checking to see if it's divisible by 4 and divisible by 6. She claims that if the number is divisible by 4 and by 6, then it must be divisible by 24.

Lindsay has a similar divisibility test for 24: She claims that if a number is divisible by 3 and by 8, then it must be divisible by 24.

Are either correct? Explain your reasoning.

(16) Generalize the problem above.

(17) Suppose that you have a huge bag of tickets. On each of the tickets is one of the following numbers.

$$\{6, 18, 21, 33, 45, 51, 57, 60, 69, 84\}$$

Could you ever choose some combination of tickets (you can use as many copies of the same ticket as needed) so that the numbers sum to 7429? If so, give the correct combination of tickets. If not explain why not.

(18) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

(a) If $a^2|b^2$, then $a|b$.

(b) If $a|b^2$, then $a|b$.

(c) If $a|b$ and $\gcd(a, b) = 1$, then $a = 1$.

(19) Betsy is factoring the number 24949501. To do this, she divides by successively larger primes. She finds the smallest prime divisor to be 499 with quotient 49999. At this point she stops. Why doesn't she continue? Explain your reasoning.

(20) When Ann is half as old as Mary will be when Mary is three times as old as Mary is now, Mary will be five times as old as Ann is now. Neither Ann nor Mary may vote. How old is Ann? Explain your reasoning.

(21) If $x^2 = 11 \cdot y$, what can you say about y? Explain your reasoning.

(22) If $x^2 = 25 \cdot y$, what can you say about y? Explain your reasoning.

(23) When asked how many people were staying at the *Hotel Chevalier*, the clerk responded "The number you seek is the smallest positive integer such that dividing by 2 yields a perfect square, and dividing by 3 yields a perfect cube." How many people are staying at the hotel? Explain your reasoning.

2.3 The Euclidean Algorithm

In section 2.2, we assumed Euclid's Lemma and used it to prove the Fundamental Theorem of Arithmetic (aka Unique Factorization). In this section, we backtrack to prove Euclid's Lemma.

Question What was Euclid's Lemma?

?

Teaching Note: An important point of this section is to make the student think about the distributive property. One should try to point out each time

$$a(x + y) = ax + ay$$

occurs.

Up to this point, computing the GCD of two integers required you to factor both numbers. This can be difficult to do. The following algorithm, called the *Euclidean algorithm*, makes finding GCD's quite easy. With that said, algorithms can be tricky to explain. Let's try this—study the following calculations, they are examples of the Euclidean algorithm in action:

$$\begin{aligned} 22 &= 6 \cdot 3 + 4 \\ 6 &= 4 \cdot 1 + \boxed{2} \\ 4 &= 2 \cdot 2 + 0 \qquad \boxed{\therefore \gcd(22, 6) = 2} \end{aligned}$$

$$\begin{aligned} 33 &= 24 \cdot 1 + 9 \\ 24 &= 9 \cdot 2 + 6 \\ 9 &= 6 \cdot 1 + \boxed{3} \\ 6 &= 3 \cdot 2 + 0 \qquad \boxed{\therefore \gcd(33, 24) = 3} \end{aligned}$$

$$
\begin{aligned}
42 &= 16 \cdot 2 + 10 \\
16 &= 10 \cdot 1 + 6 \\
10 &= 6 \cdot 1 + 4 \\
6 &= 4 \cdot 1 + \boxed{2} \\
4 &= 2 \cdot 2 + 0 \qquad \boxed{\therefore \gcd(42, 16) = 2}
\end{aligned}
$$

Question Can you describe how to do the Euclidean algorithm?

?

Question Can you explain why the Euclidean algorithm will always stop? Hint: Division Theorem.

?

Activity A.14 complements this section well.

The algorithm demonstrated above is called the *Euclidean algorithm* or *Euclid's algorithm* because Euclid uses it several times in Books VII and X of his book *The Elements*. Donald Knuth gives a description of the Euclidean algorithm in the first volume of his series of books *The Art of Computer Programming*. Given integers m and n, he describes it as follows:

(1) [Find remainder.] Divide m by n and let r be the remainder. (We will have $0 \leqslant r < n$.)

(2) [Is it zero?] If $r = 0$, the algorithm terminates; n is the answer.

(3) [Interchange.] Set $m \leftarrow n$, $n \leftarrow r$, and go back to step (1).

Question What do you think of this description? How does it compare to your description of the Euclidean algorithm?

?

While the Euclidean algorithm is handy and fun, its real power is that it helps us solve equations. Specifically it helps us solve linear Diophantine equations.

Let's study the following calculations:

$$\begin{aligned} 22 &= 6\cdot 3+4 &\Leftrightarrow\quad 22-6\cdot 3&=4 \\ 6 &= 4\cdot 1+2 &\Leftrightarrow\quad 6-4\cdot 1&=2 \\ 4&=2\cdot 2+0 \end{aligned} \qquad \begin{aligned} 6-4\cdot 1&=2\\ 6-(22-6\cdot 3)\cdot 1&=2\\ 6\cdot 4+22(-1)&=2 \end{aligned}$$

$$\boxed{\therefore 22x+6y=2 \text{ where } x=-1 \text{ and } y=4}$$

$$\begin{aligned} 33 &= 24\cdot 1+9 &\Leftrightarrow\quad 33-24\cdot 1&=9 \\ 24 &= 9\cdot 2+6 &\Leftrightarrow\quad 24-9\cdot 2&=6 \\ 9 &= 6\cdot 1+3 &\Leftrightarrow\quad 9-6\cdot 1&=3 \\ 6&=3\cdot 2+0 \end{aligned} \qquad \begin{aligned} 9-6\cdot 1&=3\\ 9-(24-9\cdot 2)\cdot 1&=3\\ 9\cdot 3+24\cdot(-1)&=3\\ (33-24\cdot 1)\cdot 3+24\cdot(-1)&=3\\ 33\cdot 3+24\cdot(-4)&=3 \end{aligned}$$

$$\boxed{\therefore 33x+24y=3 \text{ where } x=3 \text{ and } y=-4}$$

Question Can you explain how to solve Diophantine equations of the form

$$ax+by=g$$

where $g=\gcd(a,b)$?

?

The Euclidean algorithm is also useful for theoretical questions.

Question Given integers a and b, what is the smallest positive integer that can be expressed as

$$ax+by$$

where x and y are also integers?

Fixnote: This argument could be improved, as could the corresponding activity, A.14.

I'm feeling chatty, so I'll take this one. I claim that $g = \gcd(a, b)$ is the smallest positive integer that can be expressed as

$$ax + by$$

where x and y are integers. How do I know? Well first, the Euclidean algorithm shows that g can be expressed as a sum $ax + by$. (Why?)

Second, suppose there was a smaller positive integer, say s where:

$$ax + by = s$$

Hmmm. . . but we know that $g|a$ and $g|b$. This means that g divides the left-hand-side of the equation. This means that g divides the right-hand-side of the equation. So $g|s$—but this is impossible, as $s < g$. Thus g is the smallest integer that can be expressed as $ax + by$.

Question Can you now use the Euclidean Algorithm to prove Euclid's Lemma?

?

Problems for Section 2.3

(1) Explain what a *Diophantine equation* is. Give an example and explain why such a thing has real-world applications.

(2) Use the Euclidean algorithm to find: $\gcd(671, 715)$, $\gcd(667, 713)$, $\gcd(671, 713)$, $\gcd(682, 715)$, $\gcd(601, 735)$, and $\gcd(701, 835)$.

(3) Explain the advantages of using the Euclidean algorithm to find the GCD of two integers over factoring.

(4) Find integers x and y satisfying the following Diophantine equations:

(a) $671x + 715y = 11$

(b) $667x + 713y = 69$

(c) $671x + 713y = 1$

(d) $682x + 715y = 55$

(e) $601x + 735y = 4$

(f) $701x + 835y = 15$

(5) Given integers a, b, and c, explain how you know when a solution to a Diophantine equation of the form

$$ax + by = c$$

exists.

(6) Consider the Diophantine equation:

$$15x + 4y = 1$$

(a) Use the Euclidean Algorithm to find a solution to this equation. Explain your reasoning.

(b) Compute the slope of the line $15x + 4y = 1$ and write it in lowest terms. Show your work.

(c) Plot the line determined by $15x + 4y = 1$ on graph paper.

(d) Using your plot and the slope of the line, explain how to find 10 more solutions to the Diophantine equation above.

(7) Explain why a Diophantine equation

$$ax + by = c$$

has either an infinite number of solutions or zero solutions.

2.4 Rational Numbers

Once you are familiar with integers, you start to notice something: Given an integer, it may or may not divide into another integer evenly. This property is at the heart of our notions of factoring and primality. Life would be very different if all nonzero integers divided evenly into one another. With this in mind, we introduce *rational numbers*.

Activities A.16 and A.17 complement this section well.

Definition A **rational number** can be written as $\frac{a}{b}$ where $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $b \neq 0$.

In other words, rational numbers can be written as a fraction of integers, where the denominator is nonzero.

Warning Note the words "can be" in the definition. Rational numbers do not have to be represented as fractions. And fractions are not necessarily rational numbers.

Question Which of the following numbers are rational?

$$\frac{5}{4}, \quad 718, \quad \sqrt{2}, \quad 2.718, \quad \frac{22}{7}, \quad \frac{12}{4}, \quad \frac{\pi}{3}, \quad \frac{\sqrt{2}}{\sqrt{8}}, \quad \frac{1}{43}$$

The set of all **rational numbers** is denoted by the symbol $\mathbb{Q}$:

$$\mathbb{Q} = \left\{ \frac{a}{b} \text{ such that } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

The letter $\mathbb{Q}$ stands for the word *quotient*, which should remind us of fractions. The funny little "$\in$" symbol means "is in" or "is an element of." Fancy folks will replace the words *such that* with a colon ":" to get:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

2.4.1 Why do People Hate Fractions?

Why do so many people find fractions difficult? This is a question worth exploring. We'll guide you through some of the tough spots with some questions of our own.

Question Given a rational number $\dfrac{a}{b}$, come up with three other different rational numbers that are all equal to $\dfrac{a}{b}$. What features of fractions are we illustrating?

?

Question Given two positive rational numbers $\dfrac{a}{b}$ and $\dfrac{c}{d}$, explain how to tell which is greater. What features of fractions are we illustrating?

?

Question Given two rational numbers $\dfrac{a}{b}$ and $\dfrac{c}{d}$ with $\dfrac{a}{b} < \dfrac{c}{d}$, explain how one might find a rational number between them. What features of fractions are we illustrating?

?

Question Dream up counting numbers a, b, and c such that:

$$\frac{a/b}{c} = \frac{a}{b/c}$$

Can you dream up other counting numbers a', b', and c' such that:

$$\frac{a'/b'}{c'} \neq \frac{a'}{b'/c'}$$

What features of fractions are we illustrating?

?

Question Explain how to add two fractions $\frac{a}{b}$ and $\frac{c}{d}$. What features of fractions are we illustrating?

?

Question Can you come up with any other reasons fractions are difficult?

?

Teaching Note: Two key points of this dialog are:

(1) Equal fractions have different representations.

(2) It is difficult to compare fractions.

2.4.2 Basic Meanings of Fractions

Like all numbers, fractions have meanings outside of their pure mathematical existence. Let's see if we can get to the heart of some of this meaning.[3.NF.1]

Activities A.19 through A.21 complement this section well.

CCSS 3.NF.1: Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

Question Draw a rectangle. Can you shade 3/8 of this rectangle? Explain the steps you took to do this.

?

Question Draw a rectangle. Given a fraction a/b where $0 < a \leqslant b$, explain how to shade a/b of this rectangle.

?

Question Draw a rectangle. How could you visualize $8/3$ of this rectangle? Explain the steps you took to do this.

?

Question Draw a rectangle. Given a fraction a/b where $0 < b < a$, explain how to visualize a/b of this rectangle.

?

Question Draw a rectangle. Can you shade

$$\frac{3/8}{4}$$

of this rectangle? Explain the steps you took to do this.

?

Problems for Section 2.4

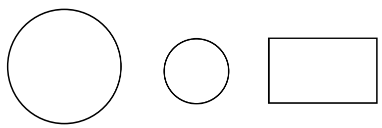
(1) Describe the set of rational numbers. Give some relevant and revealing examples/nonexamples.

(2) What algebraic properties do the rational numbers enjoy that the integers do not? Explain your reasoning.

(3) What number gives the same result when added to $1/2$ as when multiplied by $1/2$. Explain your reasoning.

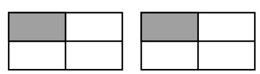
(4) Draw a rectangle to represent a garden. Shade in $3/5$ of the garden. Without changing the shading, show why $3/5$ of the garden is the same as $12/20$ of the garden. Explain your reasoning.

(5) Shade in $2/3$ of the entire picture below:



Explain your reasoning.

(6) What fractions could the following picture be illustrating?



Explain your reasoning.

(7) When Jesse was asked what the 7 in the fraction $\frac{3}{7}$ means, Jesse said that the "7" is the *whole*. Explain why this is not completely correct. What is a better description of what the "7" in the fraction $\frac{3}{7}$ means?

(8) Find yourself a sheet of paper. Now, suppose that this sheet of paper is actually $4/5$ of some imaginary larger sheet of paper.

- Shade your sheet of paper so that $3/5$ of the larger (imaginary) sheet of paper is shaded in. Explain why your shading is correct.
- Explain how this shows that

$$\frac{3/5}{4/5} = \frac{3}{4}.$$

(9) Try to find the largest rational number smaller than $3/7$. Explain your solution or explain why this cannot be done.

(10) How many rational numbers are there between $3/4$ and $4/7$? Find 3 of them. Explain your reasoning.

(11) A youthful Bart loved to eat hamburgers. He ate $5/8$ pounds of hamburger meat a day. After testing revealed that his blood consisted mostly of cholesterol, Bart decided to alter his eating habits by cutting his hamburger consumption by $3/4$. How many pounds of hamburger a day did Bart eat on his new "low-cholesterol" diet? Explain your reasoning.

(12) Courtney and Paolo are eating popcorn. Unfortunately, 1/3rd of the popcorn kernels are poisoned. If Courtney eats exactly 5/16th of the kernels and Paolo eats exactly 5/13ths of the kernels, did at least one of them eat a poisoned kernel? Explain your reasoning. Also, at least how many kernels of popcorn are in the bowl? Again, explain your reasoning.

(13) Best of clocks, how much of the day is past if there remains twice two-thirds of what is gone? Explain what this strange question is asking and answer the question being sure to explain your reasoning—note this is an old problem from the *Greek Anthology* compiled by Metrodorus around the year 500.

(14) John spent a fifth of his life as a boy growing up, another one-sixth of his life in college, one-half of his life as a bookie, and has spent the last six years in prison. How old is John now? Explain your reasoning

(15) Diophantus was a boy for 1/6th of his life, his beard grew after 1/12 more, he married after 1/7th more, and a son was born five years after his marriage. Alas! After attaining the measure of half his father's full life, chill fate took the child. Diophantus spent the last four years of his life consoling his grief through mathematics. How old was Diophantus when he died? Explain your reasoning—note this is an old problem from the *Greek Anthology* compiled by Metrodorus around the year 500.

(16) Wandering around my home town (perhaps trying to find my former self!), I suddenly realized that I had been in my job for one-quarter

of my life. Perhaps the melancholia was getting the best of me, but I wondered: How long would it be until I had been in my job for one-third of my life? Explain your reasoning.

(17) In a certain adult condominium complex, 2/3 of the men are married to 3/5 of the women. Assuming that men are only married to women (and vice versa), and that married residents' spouses are also residents, what portion of the residents are married?

(a) Before any computations are done, use common sense to guess the solution to this problem.

(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your guess?

(c) Use algebra to solve the problem.

Explain your reasoning in each step above.

(18) Let a, b, c, and d be positive integers such that

$$a < b < c < d$$

Is it true that

$$\frac{a}{b} < \frac{c}{d}?$$

Explain your reasoning.

(19) Let a, b, c, and d be positive consecutive integers such that

$$a < b < c < d.$$

Is it true that

$$\frac{a}{b} < \frac{c}{d}?$$

Explain your reasoning.

(20) Let a, b, c, and d be positive consecutive integers such that

$$a < b < c < d.$$

Is it true that

$$\frac{a}{b} < \frac{b}{c} < \frac{c}{d}?$$

Explain your reasoning.

(21) Can you generalize Problem (19) and Problem (20) above? Explain your reasoning.

(22) Let a, b, c, and d be positive integers such that

$$\frac{a}{b} < \frac{c}{d}.$$

Is it true that

$$\frac{a}{a+b} < \frac{c}{c+d}?$$

Explain your reasoning.

2.5 Decimal Representations

There are two ways that we usually write real numbers that aren't whole numbers: as fractions and as decimals. Let's explore the relationship between these two representations of numbers.

Activities A.22, A.23, and A.24 are intended for this section.

Question How is a "fraction" different from a rational number?

?

First, let's work on translating fraction representations into decimal representations. You probably already know from school that some numbers have decimal representations that end (these are called "terminating" decimals) and the rest of them have decimal representations that never end (these are "non-terminating"). Try to figure out what it would take for a fraction to have a terminating decimal representation.

Question Write .465, 0.72895, 0.00673, and 34.062 as fractions of integers. What do you notice about terminating decimals when they are written as fractions?

?

Question Write 4/5, 7/16, 43/20, and 3/6250 as decimals. What about these fractions makes the decimals terminate?

?

Question Your calculator is not trustworthy for determining whether a number's decimal representation terminates or repeats. Why? How can you use your calculator carefully to judge whether a decimal terminates or repeats?

?

In Activity A.23, you separate a bunch of fractions according to whether they appear to have a terminating or non-terminating decimals. The rational numbers that have a terminating decimals are straighforward to describe, once you see the idea. The real action (and the intrigue) lies with the non-terminating ones.

Let's investigate with a fraction that has a non-terminating representation: 4/7. As you know, 4/7 is the same as "4 divided by 7." So, use long division to find the decimal representation.[7.NS.2d]Bring a pillow, because you already know that it will take an infinite number of steps to complete the work!

CCSS 7.NS.2d: Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Now that you've spent your life doing long division, can you carefully explain why the fraction's non-terminating representation will "repeat"? (Hint: Think about remainders.) Try a few others, like 2/13, 3/11, or 4/17. Will the same sort of thing happen with, say, 3457/213678940753? What can you say about how soon the process will repeat?

Here are some cool things you can investigate on the side:

- Some non-terminating decimals have a "delay" before they start repeating. (The most famous one is probably 1/6.) I happen to know 1/123750000 will have a delay of 7 places before the repeating starts. Can you look at a fraction and predict whether it will or will not have a delay (and how long that delay will be)?

- What are the restrictions to the sizes of the "blocks" for the repeating decimal representation of a rational number? For example, any fraction with denominator 37 can only possibly repeat in blocks of 1, 2, 3, 4, 6, 9, 12, 18, or 36.

Based on the ideas you have explored, you can prove that a non-terminating decimal representation of a rational number must repeat. Is the converse true? Can any repeating decimal be written as a fraction? It turns out indeed to be the case, as can be found by taking advantage of a nice pattern involving the decimals representations of 1/9, 1/99, 1/999, etc., or by noting that each repeating decimal is a "geometric series," as we will explore later.

Thus, we have it that every rational number can be written as either a terminating or repeating decimal.[8.NS.1] Can every decimal be written as a fraction? That is, we have that all fractions are decimals, but are all decimals fractions? Have we let any decimals out in the cold here?

CCSS 8.NS.1: Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Question Describe the decimal representations of 3 “homemade” decimals that could never be written as fractions of integers. Explain your thinking. Warning: Do *not* say $\sqrt{2}$, π, e, or the like, unless you are ready to convince the class that these are not rational numbers.

?

2.5.1 A Note on Infinite Processes

Mathematical reasoning often involves “infinite processes” in which direct calculation is impossible.• Infinite processes become central in calculus, where both differentiation and integration are defined via limits. These approaches are made rigorous in advanced undergraduate courses, such as Real Analysis. But infinite processes arise from time to time even in middle grades mathematics, and so it is important that teachers are able to talk about them sensibly and accurately. Here we explain some key ideas for reasoning about infinite processes.

• This discussion draws heavily on ideas described in *Where Mathematics Comes From: How the embodied mind brings mathematics into being* by Lakoff and Núñez (2000).

First, there is the idea of a process that continues, over and over, without end. Here are some examples:

- Perhaps the earliest of these is counting: 1, 2, 3, 4, ... We do not imagine completing the process of counting. Nonetheless, for any large positive number you name, we can imagine exceeding that number, eventually, if we have enough time.
- We can approximate $1/3$ with a sequence, 0.3, 0.33, 0.333, and so on. We can get as close to $1/3$ as we like by including enough digits. Note, on the other hand, that it is false to say that $0.3333 = 1/3$ or even $0.3333333333 = 1/3$, because any finite number of digits will miss $1/3$ by an amount that can be calculated precisely.
- If we look at a sequence of regular n-gons of the same diameter, as n gets large, we can get as close to a circle as we might like. But for any finite number of sides, the regular n-gon will not actually be a circle.

The above examples use what is sometimes called *potential infinity*, for in none of the cases do we actually complete the process, and we do not need to. We imagine these things as going on “forever,” and a process that goes on forever never ends.

But the interesting uses of infinity in mathematics involve *actual infinity*.

Question In each of the above examples, what would happen if the process could end?

?

In order to conceptualize actual infinity, we imagine, metaphorically, that the process *does* end. In a literal sense, an infinite process cannot end, but through the use of metaphor, we consider what would happen if the process were to end. And with the help of intuitions about completed processes, we then infer the "ultimate result" of the completed infinite process.

With the metaphor of actual infinity, counting yields the infinite set of counting numbers, $\mathbb{N}$. All of them. In the repeating decimal for $1/3$, we get an exact decimal representation, so that $0.33333\ldots = 1/3$. Exactly. And in the case of the regular n-gon with an infinite number of sides, we get a circle. Perfectly.

Warning With the metaphor of actual infinity, it is *false* to say that $0.3333\ldots$ never gets to $1/3$ because the dots imply that the infinite process has been completed. Although any finite number of digits fails reach $1/3$, an infinite number of digits reaches $1/3$ exactly: The error has gone to 0.

In summary, reasoning about infinite processes involves the following steps:

(1) Describing the finite process carefully and accurately;

(2) Considering the process to go on forever, and describing how the result can get arbitrarily close to some goal;

(3) Imagining that the infinite process has been completed; and

(4) Reasoning about the "ultimate result" of the infinite process.

For some infinite processes, it is quite helpful in the second and fourth steps to talk about the "error," which is to say how much the finite process falls short of the ultimate goal, and then to argue that the error becomes arbitrarily small (i.e., it goes to 0).

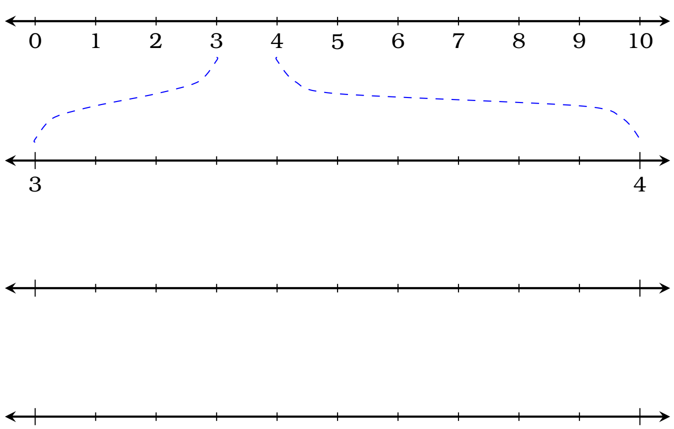
Happy infinite reasoning!

Problems for Section 2.5

Exercises

(1) What does 3.417 mean in the base-ten place-value system? Using the rectangle below as 1, draw a picture the illustrates the place-value meaning of 3.417. Draw as accurately as you can, indicating how the picture would be drawn perfectly (if you could). Indicate whether your model is primarily about length, area, or something else.

(2) Plot 3.417 on each of the following number lines, zooming in to show how to make the placement more accurate at each step. Draw dotted curves (as shown) to indicate where the zooming takes place, and label the large tick marks on each number line.



(3) How would your plotted points in the four number lines have been different if the number had been 341.7? What about 0.003417? Or 34,170,000? What does your answer say about the consistent structure of the base-ten place value system? (Hint: In each number, how does the meaning of the 4 compare to the meaning of the 1 to its right? How does the meaning of the 4 compare to the meaning of the 3 to its left?)

(4) How would your plotted points in the four number lines have been different if the number had been 3.41708? What about 3.41708667999? Explain.

(5) You should know or be able to figure out (in your head) decimal equivalents of fractions with many small or “nice” denominators (i.e., 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 16, 20, 25, 30, 40, and 50). Describe how to figure out quickly any that you might forget.

(6) Here is a nice relationship between twelfths and eighths: $1/8 \approx 0.12$ and $1/12 \approx 0.08$. Find other such pairs, and explain why the pairs “work” this way.

(7) Compare the decimal representations of $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$, and $\frac{6}{7}$.

(a) Notice that the repeating digits always appear in the same order. Explain why this is the case.

(b) Suppose you are able to remember the decimal representation of $\frac{1}{7}$. Explain how to use that to write quickly the decimal representation of any of the other sevenths.

(8) Compare the decimal representations of $\frac{1}{13}, \frac{2}{13}, \ldots, \frac{12}{13}$.

(a) Describe carefully how the order of the digits is somewhat like and also different from what you noticed for sevenths.

(b) Explain why the decimal representations of thirteenths work as you described.

(9) Without a calculator, predict whether the decimal representations of the following numbers will terminate or not. For those that terminate, predict the number of decimal places.

(a) $\frac{13}{400}$

(b) $\frac{11}{70}$

(c) $\frac{21}{70}$

(d) $\frac{27}{6250}$

(e) $\dfrac{23}{2^7 \cdot 5^2}$

(f) $\dfrac{23}{2^7 \cdot 5^2 \cdot 11}$

(g) $\dfrac{22}{2^7 \cdot 5^2 \cdot 11}$

(10) The clearest way to demonstrate that a number is rational is to show that it satisfies the definition. (What is the definition of a rational number?) Show that the following numbers are rational:

(a) $0.\overline{324}$

(b) $15.\overline{324}$

(c) $0.15\overline{324}$

(d) $0.2\overline{5643}$

Generalizations

(11) Use long division to explain why the decimal representation of a rational number must either terminate or repeat.

(12) Suppose $\dfrac{m}{n}$ is a rational number in lowest terms. If the number's decimal representation terminates, what can you conclude about m and about n? Explain.

(13) Suppose $\dfrac{m}{n}$ is a rational number in lowest terms. If you know the number's decimal representation repeats, what can you conclude about the number of repeating digits? Explain.

(14) You have seen three types of decimal representations for rational numbers between 0 and 1: terminating, repeating, and delayed-repeating. Suppose that m and n are counting numbers with no common factors and $m < n$. Explain why the type of decimal representation of $\dfrac{m}{n}$ depends only on n and not on m. Hint: Consider the three types separately.

Explorations

(15) The rational number $\dfrac{1}{19}$ has decimal representation $0.\overline{052631578947368421}$. To verify this, your calculator is unlikely to display enough digits, and long division would be quite tedious. Devise a method for "piecing together" this decimal representation in "chunks," using your calculator. Then use the method to compute the decimal representation of $\dfrac{7}{23}$. Be sure to indicate how you know that it repeats as you claim.

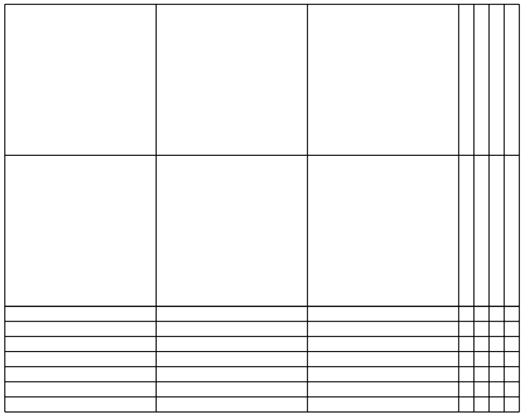
(16) Given a prime number p, find the smallest positive integer n so that p divides $10^n - 1$, or explain why there is no such integer n.

(a) Do this for all primes less than 15, and also for the primes 37, 41, 73, and 101.

(b) For each prime, compare the n you found with the number of repeating digits in the decimal representation of $\dfrac{1}{p}$. Make a conjecture about what you notice. Provide a brief explanation of why your conjecture ought to be true.

(17) Explain 2.7×3.4 in **two different ways**. Be sure your explanations address two key questions: (i) Why can you almost ignore the decimal point and multiply as though the digits described whole numbers? And (ii) How do you know where to place the decimal point in the result? Here are some ideas:

- Use behind the scenes algebra to explain why the digits in the 27×34 should be the same as the digits in the desired product 2.7×3.4.
- Convert the decimals to fractions, compute the product of the fractions, and then convert the result to a decimal.
- Use the picture below to compute 2.7×3.4 with neither an algorithm nor a calculator. Explain your reasoning.



- Explain why the above picture can also represent 27×34. Explain the lengths and areas for both calculations.

(18) Explain $3.96 \div 2.4$ in **two different ways**. Be sure your explanations address two key questions: (i) Why can you almost ignore the decimal point and divide as though the digits described whole numbers? And (ii) How do you know where to place the decimal point in the result? Here are some ideas:

- Use the measurement model of division to reason how many groups of size 2.4 are in 3.96.
- Use bundles or base ten blocks where the single stick or unit block represents a quantity other than 1.
- Multiply both the dividend and the divisor by a suitable power of 10 and then divide.
- Convert both decimals to fractions, divide the fractions, then convert the result back to a decimal.
- Divide 396 by 24 and then use estimation to place the decimal point.

3 Ratios, Functions, and Beyond

3.1 Ratios and Proportional Relationships

As a topic in school mathematics, ratios and proportions are often isolated entities with their own special vocabulary, habits, and procedures. When studying ratios and proportional situations, students often learn, "Set up a proportion and cross multiply." But what is a proportion? When does cross multiplication work? Why does it work?

For the problems in this section, try to take a more general approach: "Write an equation and solve." More precisely, "Write an equation relating the quantities, and solve the equation for the desired quantity (usually an unknown)." These are skills that serve students well throughout school mathematics and beyond.

As you work through the problems and activities for this section, you will find it useful to make use of reasoning tools such as the following:

- Equivalent fractions
- Equivalent ratios
- Ratio tables
- Unit rates
- Double number lines
- Graphs

During the process, be on the lookout for a wide variety of strategies, including part:part comparisons, part:whole comparisons, common denominators, and common numerators. And note how the problems simultaneously build on understandings of fractions and pave the way for functions.

3.1.1 Ratios

Fractions, ratios, and rates are three connected ideas with differing histories and differing usage:

- *Fractions* are numbers, often used to express results of sharing, cutting, or measuring.
- *Ratios* have historically been used to compare quantities of the same kind, such as two lengths or two volumes. Ratios are often expressed as pairs of counting numbers, without units, e.g., 3 : 2.
- *Rates* are typically used to compare different quantities (e.g., meters and seconds). Rates are often expressed as quotients with units (e.g., 1.5 m/sec).

In high school and beyond, these rough historical distinctions become blurred, and the uses of these terms are varied, sometimes conflicting, and often muddled. Thus, we will not attempt to write precise definitions that distinguish these terms from one another. Instead, we aim toward the end-goal that students see all of these as quotients that provide differing perspectives on closely related ideas.

To this end, we invest our energy in solving problems. We will see that it is sometimes useful to attend only to the numbers in a situation, so that we can notice that two apparently different problems are abstractly the same if we "decontextualize" the problem by removing the units. At other times, we will see the importance of using the units to interpret answers in context. This interplay is the essence of modeling.

Teaching Note: You will find insightful discussion and pictures in the draft 6-7 Progression on Ratios and Proportional Relationships, available at
`http://math.arizona.edu/~ime/progressions/`

3.1.2 Proportional Relationships

For situations that involve two varying quantities, perhaps the most fundamental are those in which the quantities are proportional to one another.

Definition Quantities x and y are in a *proportional relationship* if there is a constant k such that $y = kx$.

When solving problems, a critical skill is the ability to distinguish propotional situations from situations in which quantities are not proportional.

Question Give a table of data for two quantities that are in a proportional relationship.

?

Question Give a table of data for two quantities that are **not** in a proportional relationship.

?

Activities A.25, A.26, and A.27 complement this section well. As a conclusion, we suggest doing Activity A.29.

Problems for Section 3.1

Fixnote: Need easy problems, strip diagrams, double number lines. See Beckmann.

(1) A baseball coach once asked me the following question: If a pitcher can throw a 90 mph pitch during a game, but can only sustain a 60 mph pitch during practice, how close should the pitcher stand during practice to ensure that the amount of time it takes the ball to reach home plate is the same in practice as it is in the game? Explain your reasoning.

(2) Three brothers and a sister won the lottery together and plan to share it equally. If the brothers alone had shared the money, then they would have increased the amount they each received by \$20. How much was won in the lottery? Explain your reasoning.

(3) Chris is working on his Fiat. His car's cooling system holds 6 quarts of coolant, and should be filled with a 50/50 mix of antifreeze and water. Chris noticed that the car was 1 quart low with the correct 50/50 mix. But then he added a 25/75 mix, 25 parts antifreeze, and 75 parts water. How much coolant does he have to remove from the cooling system to then add 100 percent antifreeze to restore his desired 50/50 mix? Explain your reasoning.

(4) If a hen and a half lays an egg and a half in a day and a half, how many eggs will 6 hens lay in 4 days? How many days will it take for 8 hens to lay 16 eggs? How many hens would it take to lay 12 eggs in three days? How many hens would it take to lay a dozen eggs per week? In each case, explain your reasoning.

(5) Fred and Frank are two fitness fanatics on a run from A to B. Fred runs half the way and walks the other half. Frank runs for half the time and walks for the other half. They both run at the same speed and they both walk at the same speed. Who finishes first?

(a) Before any computations are done, guess the solution to this problem and record your guess.

(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your initial guess?

(c) Use algebra to solve the problem. What does your solution say about your initial guess?

(6) Andy and Sandy run a race of a certain distance. When Sandy finishes, she is 1/10 of the distance ahead of Andy, who then finishes the race. After some discussion, Andy and Sandy decide to race the distance again, but this time Sandy will start 1/10 of the distance behind Andy (at the starting line) to "even-up" the competition. Who wins this time? Explain your reasoning.

(a) Before any computations are done, guess the solution to this problem and record your guess.

(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your initial guess?

(c) Use algebra to solve the problem. What does your solution say about your initial guess?

(7) You have two beakers, one that contains water and another that contains an equal amount of oil. A certain amount of water is transferred to the oil and thoroughly mixed. Immediately, the same amount of the mixture is transferred back to the water. Is there now more water in the oil or is there more oil in the water?

(a) Before any computations are done, guess the solution to this problem and record your guess.

(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your initial guess?

(c) Use algebra to solve the problem. What does your solution say about your initial guess?

(8) While on a backpacking trip Lisa hiked five hours, first along a level path, then up a hill, then turned around and hiked back to her base camp along the same route. She walks 4 miles per hour on a level trail, 3 uphill, and 6 downhill. Find the total distance traveled. Explain your reasoning.

(9) Monica, Tessa, and Jim are grading papers. If it would take Monica 2 hours to grade them all by herself, Tessa 3 hours to grade them

all by herself, and Jim 4 hours to grade them all by himself how long would it take them to grade the exams if they all work together? Explain your reasoning.

(10) Say quickly, friend, in what portion of a day will four fountains, being let loose together, fill a container which would be filled by the individual fountains in one day, half a day, a third of a day, and a sixth of a day respectively? Explain your reasoning—note this is an old problem from the Indian text *Lilavati* written in the 1200s.

(11) Three drops of *Monica's XXX Hot Sauce* were mixed with five cups of chili mix to make a spicy treat—the hot sauce is much hotter than the chili. Later, two drops of *Monica's XXX Hot Sauce* were mixed with three cups of chili. Which mixture is hotter?

Josh made the following observation: "If two different recipes are added together, the result will be a chili with hotness between the two." Explain why this makes sense.

To compare the given recipes, Josh suggested using this reasoning backwards, as follows:

- Remove the second (recipe) from the first, that is: Start with 3 drops of hot sauce and 5 cups of chili, and remove 2 drops and 3 cups. So we are now comparing

  1 drop and 2 cups with 2 drops and 3 cups.

- Now remove the first from the second, that is: Start with 2 drops and 3 cups, and remove 1 drop and 2 cups. So we are now comparing

  1 drop and 2 cups with 1 drop and 1 cup.

Now you can see that the second is more concentrated (and hence hotter!) than the first. Is this correct? Will this strategy always/ever work? Explain your reasoning.

3.2 Sequences and Functions

Sequences. Because "Sequences and Series" is a common topic in calculus and precalculus courses, the concept of sequence is often considered an advanced topic in high schools, but the idea of a sequence is much more elementary. In fact, many patterns explored in grades K-8 can be considered sequences. For example, the sequence $4, 7, 10, 13, 16, \ldots$ might be described as a “plus 3 pattern” because terms are computed by adding 3 to the previous term.

Activities A.30 through **??** are intended for this section.

Definition A **sequence** is an ordered set of numbers or other objects. The numbers or objects are called the **terms** of the sequence.

Functions. In the Common Core State Standards, students begin formal study of functions in grade 8.[8.F.1] In high school, the approach to functions becomes more formal, through the use of function notation and with explicit attention to the concepts of domain and range.[F-IF.1]

CCSS 8.F.1: Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. (Function notation is not required in Grade 8.)

CCSS F-IF.1: Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

Definition A **function** is a rule in which each input value determines a corresponding output value. The **domain** of a function is the set of input values. The **range** of the function is the set of output values.

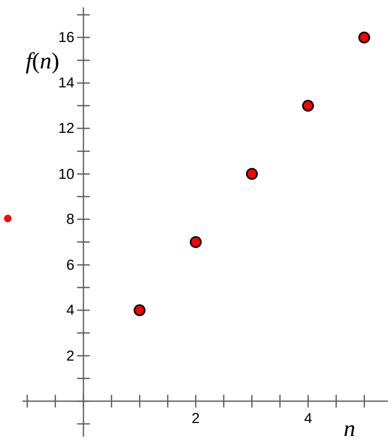
Sequences Are Functions. As students begin formal study of functions, it makes sense to use their patterning experience as a foundation for understanding functions.[F-IF.3] To show how the sequence above can be considered a function, we need an *index*, which indicates which term of the sequence we are talking about, and which serves as an input value to the function. After deciding that the 4 corresponds to an index value of 1, we can make a table showing the correspondence.•

CCSS F-IF.3: Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

n	1	2	3	4	5	...
$f(n)$	4	7	10	13	16	...

Although sequences are sometimes notated with subscripts, function notation can help students remember that sequences are functions. For example, the sequence can be described recursively by the rule $f(1) = 4$, $f(n + 1) = f(n) + 3$ for $n \geq 2$. Notice that the recursive definition requires both a starting value and a rule for computing subsequent terms. The sequence can also be described with the closed (or explicit) formula $f(n) = 3n + 1$, for integers $n \geq 1$. Notice that the domain (i.e., integers $n \geq 1$) is included as part of the description. When a function is given without an explicit domain, the assumption is that the domain is all values for which the expression is valid. Thus, the function $g(x) = 3x + 1$ appears to be essentially the same as the function f because the formula is the same and because $f(n) = g(n)$ for all positive integers. To see that the functions are different, observe that $g(2.5) = 8.5$, but $f(2.5)$ is undefined.

A common habit in school mathematics is creating a table of (x, y) pairs, plotting those pairs (as dots), and then "connecting the dots." The above discussion demonstrates that this habit is sometimes not appropriate: A graph of the sequence consists of discrete dots, because the specification does not indicate what happens "between the dots." Connecting the dots requires the assumption that domain values between the dots make sense in some way.



Question In your own words, what does it mean to say that sequences are functions?

Question Given that $f(1) = f(2) = 1$, and $f(n+1) = f(n) + f(n-1)$ for integers $n > 2$, find $f(6)$.

Arithmetic and Geometric Sequences. F-BF.2

CCSS F-BF.2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Definition An **arithmetic sequence** has a constant difference between consecutive terms. A **geometric sequence** has a constant ratio between consecutive terms. Some sequences, of course, are neither arithmetic nor geometric.

Question For each of the following sequences, decide whether it is arithmetic, geometric, or neither, and explain your reasoning:

- $1, 4, 9, 16, 25, \ldots$

- $4, 8, 16, 32, \ldots$
- $2, 4, 6, 8, 2, 4, 6, 8, \ldots$
- $-2, 5, 12, 19, \ldots$

Can you write both recursive and explicit formulas for each of these sequences?

Beginning in about grade 8, much of school mathematics is devoted to the study of linear, quadratic, and exponential functions.[F-LE.2] Here we provide only definitions and key questions about these types of functions.

CCSS F-LE.2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

- A linear function is of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. What do a and b tell you about the linear function?
- A quadratic function is of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. What do a, b, and c tell you about the function? Why is it important to specify that $a \neq 0$?
- An exponential function is of the form $f(x) = ab^x$, where a and b are real numbers, $a \neq 0$, and $b > 0$. What do a and b tell you about the function?

Question Why do these definitions require that $a \neq 0$?

?

Question Why does the definition of exponential function require that $b > 0$? What happens if $b = 0$? What happens if $b < 0$?

?

How can you identify these types of functions in tables, graphs, symbols, and contexts?[F-IF.4 F-IF.7] For example, how can you recognize the slope in the graph of a linear function? What about in a table, in a symbolic expression, or in a context?

CCSS F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

CCSS F-IF.7: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

Question An arithmetic sequence is what kind of function? Explain.

?

Question A geometric sequence is what kind of function? Explain.

?

Question Sometimes quadratic functions are written in the form $f(x) = a(x - h)^2 + k$, where a, h, and k are real numbers and $a \neq 0$. What do a, k, and h tell you about the function? What are the advantages and disadvantages of this form of a quadratic, as compared to the alternative form given above?

?

Concluding Remarks. When studying arithmetic and geometric sequences, it is tempting to encapsulate common results into compact formulas. But formulas are easily confused with one another and otherwise misremembered. Furthermore, general formulas often obscure the ideas.

Question Find the missing terms in the following arithmetic sequence:

$$\_\_\_\_, \_\_\_\_, 2, \_\_\_\_, \_\_\_\_, 6, \ldots$$

Explain your reasoning.

Question Find exact values (not decimal approximations) for the missing terms in the following geometric sequence:

$$\_\_\_\_, \_\_\_\_, 2, \_\_\_\_, \_\_\_\_, \_\_\_\_, 6, \ldots$$

Explain your reasoning. And describe how this problem and the rules of exponents might be used to explain the connection between radicals and exponents.

Question What key ideas behind arithmetic and geometric sequences did you use in the previous two problems?

With the ideas, you can reconstruct the formulas you need. And without the ideas, formulas are empty.

Problems for Section 3.2

(1) A park consists of a row of circular gardens. "Garden #0" has radius 3 feet, and each successive garden after that has a radius 2 feet greater than the previous garden.

(a) Using tables as a guide, write both explicit and recursive representations that will allow us to predict the area of the n^{th} garden.

(b) Make a graph that shows the areas of the gardens in the park. Which variable do you plot on the horizontal axis? Explain.

(c) Does it make sense to connect the dots on your graph? Explain your reasoning.

(d) Using your table, compute the area differences between the successive gardens. What do you notice? Why does this happen?

(2) An oil spill starts out as a circle with radius 3 feet and is expanding outward in all directions at a rate of 2 feet per minute.

(a) Use tables, graphs, and formulas to describe the area of the oil region x minutes after the spill.

(b) How is this question fundamentally different than that of the gardens?

(c) Dumb Question: At any one time, how many different areas are possible for the oil region?

Fixnote: Need more problems. Doubling time? Half life? Use the ones from HW 8, in comments above.

3.3 Series

Definition A **series** is a sum of consecutive terms from a sequence. A series with terms that form an arithmetic sequence is called an **arithmetic series**.

Activities A.34 and A.35 are intended for this section.

Question Find the sum: $1 + 3 + 5 + \cdots + 4999$. (First explain how you know this is an arithmetic series.)

?

In mathematics teaching and learning, it is useful to distinguish problems from excercises. *Problems* require that you formulate a solution strategy, whereas *exercises* are about using a procedure that you have been taught.• Whether a question is a problem or an exercise depends upon the learner.

• Problem solving is an essential part of mathematics.

Question Is the previous question a problem or an exercise for you?

When analyzing any series, it is often useful to consider the *sequence of partial sums*. For example, in response to the above question, the sequence of partial sums is as follows:

$$1, \qquad 1+3, \qquad 1+3+5, \qquad 1+3+5+7, \qquad \ldots$$

Sometimes you can see a pattern in the sequence of partial sums. Making a conjecture about a pattern is a type of inductive reasoning. Once you notice a pattern, an important next step is showing, deductively, that the pattern *must* continue.

For arithmetic series, here are some approaches that can lead to general deductive arguments for the sum:

- Consider pairing the first term with the last term, the second term with the second-to-last term, and so on. What do you notice about the sum of each of these pairs? And how many such pairs are in the whole series?

- Consider the average of each of the same pairs. How might those averages help determine the sum of the whole series?
- Consider writing the series backward immediately below a forward version, line up the terms, and then add vertically.

Question Use one of these approaches to show that the sum is what it is. Can you use a picture to illustrate your reasoning?

?

Question When you consider the sequence of partial sums of an arithmetic series, what kind of function(s) can you get? Explain.

?

Definition A series with terms that form a geometric sequence is called a **geometric series**.

Question Find the sum: $\frac{2}{3} + \frac{2}{9} + \cdots + \frac{2}{3^{10}}$. (First explain how you know the series is geometric.)

?

For the question above, it is not hard to see a pattern in the sequence of partial sums. In fact, it is reasonable to believe that the pattern holds for any (finite) partial sum of the infinite geometric series $\frac{2}{3} + \frac{2}{9} + \ldots$. But to show that the pattern always holds, we need a general argument.

For geometric series, the techniques for arithmetic series do not carry over. Instead, observe that if you multiply the series by the common ratio, the resulting series has most of the same terms as the original series. Thus, the difference between the two series (i.e., subtract the two) is a short expression that is not hard to work with.

Question Use these ideas to show that the sum is what it is. Can you use a picture to illustrate this sum?

?

Question Convert $0.\overline{42}$ to a fraction. What connections do you see with geometric series?

?

Question Explain briefly the key ideas behind finding the sum of an arithmetic series. Then do the same for geometric series.

?

Problems for Section 3.3

(1) Recall the story of Gertrude the Gumchewer, who has an addiction to Xtra Sugarloaded Gum. Each day, she goes to her always stocked storage vault and grabs gum to chew. At the beginning of her habit, she chewed three pieces and then, each day, she chews eight more pieces than she chewed the day before to satisfy her ever-increasing cravings. We want to find out how many pieces of gum did Gertrude chew over the course of the first 973 days of her habit?

(2) Assume now that Gertrude, at the beginning of her habit, chewed m pieces of gum and then, each day, she chews n more pieces than she chewed the day before to satisfy her ever-increasing cravings. How many pieces will she chew over the course of the first k days of her habit? Explain your formula and how you know it will work for any m, n and k.

(3) Find the sum:

$$19 + 26 + 33 + \cdots + 1720$$

Give a story problem that is represented by this sum.

(4) Now remember the story of Billy the bouncing ball. He is dropped from a height of 13 feet and each bounce goes up 92% of the bounce before it. Assume that the first time Billy hits the ground is bounce #1. How far did Billy travel over the course of 38 bounces (up to when he hits the ground on his 38th bounce)?

(5) Assume now that Billy the Bouncing Ball is dropped from a height of h feet. After each bounce, Billy goes up a distance equal to r times the distance of the previous bounce. (For example, $r = .92$ above.)

(a) How high will Billy go after the kth bounce?

(b) How much distance will Billy travel over the course of k bounces (not including the height he went up after the kth bounce)?

(c) If $r < 1$, what can you say about Billy's bounces? What if $r = 1$? What if $r > 1$?

(6) Joey starts out with \$456. He plays one hand of poker each day with the same stakes of \$10. Because he doesn't know anything about poker, he is on an extended losing streak. Write explicit and recursive representations for the amount of money Joey has after n days.

(7) Suppose Buzz Aldrin could fold a piece of paper in half as many times as he wanted—for rectangular paper of any size. How many folds would Buzz need for the thickness of the paper to reach or exceed the distance from the earth to the moon? How many folds would it take to reach or exceed halfway to the moon?

(8) A ball is thrown up in the air from a 200 foot cliff with an initial velocity of 15 feet per second. What is the height of the ball at any time t? Write explicit and recursive representations of the relationship between the time after the ball is thrown and its height above the base of the cliff.

(9) While waiting for Mark Pi to arrive to address the Mathematics Party, the members all shake hands with one another. When Mark Pi walks in, he shakes hands with only the big-wigs of the Party. A total of 6357 handshakes took place that day, and no one shook hands with the same person more than once. How many members were there? How many big-wigs were there?

(10) If you deposit \$347 at the beginning of each year (starting on Jan. 1, 2014), into an account that compounds interest annually at a rate of 6.7%, how much will be in the account on January 2, 2056?

(11) You take out a \$100,000 loan on Jan. 1, 2014 to buy one Michigan ticket. The loan shark charges 13% annual interest. You agree to pay back the loan through equal annual payments beginning on Jan. 1, 2015 and ending with the final payment on Jan. 1, 2028. How much should each annual payment be? How much interest will you pay over the course of the payment plan? If you hit the lottery on Dec. 31, 2020 and decide to pay off the rest of the loan the next day, how much will you owe?

(12) 5 mg of a drug are administered to a patient at the start of a treatment regimen. Each day at the same time, 3 more mg of the drug is administered. Assuming that the drug still dissipates by 21% each day, how much of the drug will be in the body immediately following the 34^{th} 3 mg dose?

(13) Find the fractional representation of the number $0.78\overline{4396}$.

(14) A park consists of a row of circular gardens. The "Garden 0" has radius 3 feet, and each successive garden after that has a radius 2

feet longer than the previous garden. If there are 37 gardens, how much total area would the gardens comprise?

(15) During their hour play time, the two oldest Brady kids (Greg and Marcia) went to the park to play with their walkie-talkies. They used them for an hour. The next day, the next-oldest kid came along. Because there were only two walkie-talkies, they needed to share so that each possible pair got equal time. The next day, the next-oldest also came along. This continued until on the ninth day, all ten kids were there wanting to use the walkie-talkies. How much time did Greg and Marcia spend with each other on the walkie-talkies over the course of the nine days?

(16) Find the total number of gifts given in the song "The 365 Days of Christmas".

4 Solving Equations

> Politics is for the moment. An equation is for eternity.
>
> —Albert Einstein

Teaching Note: In this section, we are developing the idea that numbers are solutions to equations. Negative integers arise out of simple linear equations, as do rationals. However, these are not enough to solve all polynomial equations, and hence we need a "larger" number system.

Fixnote: Somewhere we need attention to uses of the equals sign.

4.1 Time to Get Real

Remember the definition of a *root* of a polynomial:

Definition A **root** of a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a number r where

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

OK—let's go! We know what integers are right? We know what rational numbers are right?

Question Remind me, what is $\mathbb{Z}$? What is $\mathbb{Q}$? What is the relationship between these two sets of numbers?

While I do want **you** to think about this, I also want to tell you my answer: $\mathbb{Q}$ is the set of solutions to linear polynomial equations with coefficients in $\mathbb{Z}$.

Question What-with-the-who-in-the-where-now?

?

Are these all the numbers we need? Well, let's see. Consider the innocent equation:

$$x^2 - 2 = 0$$

Question Could $x^2 - 2$ have rational roots?

Teaching Note: Here we essentially run through the proof of the rational roots test.

Stand back—I'll handle this. Remember, a root of $x^2 - 2$ is a number that solves the equation

$$x^2 - 2 = 0.$$

So suppose that there are integers a and b where a/b is a root of $x^2 - 2$ where a and b have no common factors. Then

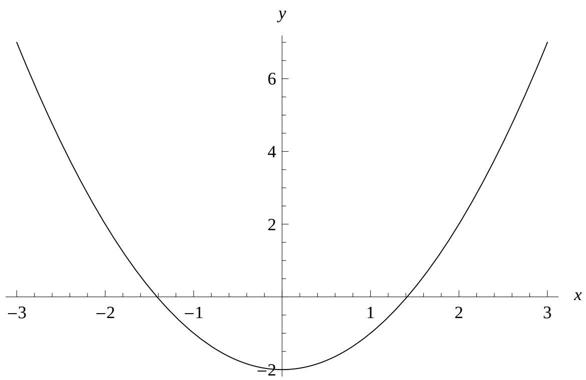
$$\left(\frac{a}{b}\right)^2 - 2 = 0.$$

So

$$a^2 - 2b^2 = 0 \qquad \text{thus} \qquad a^2 = 2b^2.$$

But a and b have no common factors—so by the Unique Factorization Theorem for the integers, $b^2 = 1$. If you find this step confusing, check out Problem (18) in Section 2.3. This tells us that $a^2 = 2$ and that a is an integer—impossible! So $x^2 - 2$ cannot have rational roots.

Hmmm but now consider the plot of $y = x^2 - 2$:



The polynomial $x^2 - 2$ clearly has two roots! But we showed above that neither of them are rational—this means that there must be numbers that cannot be expressed as fractions of integers! In particular, this means:

The square-root of 2 is not rational!

Wow! But it still can be written as a decimal

$$\sqrt{2} = 1.4142135623\ldots$$

as the square-root of 2 is a *real number*.

Definition A **real number** is a number with a (possibly infinite) decimal representation. We use the symbol $\mathbb{R}$ to denote the real numbers.

For example:

$$-1.000\ldots \qquad 2.718281828459045\ldots \qquad 3.333\ldots \qquad 0.000\ldots$$

are all real numbers.

Question Another description of real numbers is that they are the numbers that can be approximated by rational numbers. Why does this follow from the definition above?

?

Famous examples of real numbers that are not rational are

$$\pi = 3.14159265358\ldots \qquad \text{and} \qquad e = 2.718281828459045\ldots$$

Question If a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b? What can you say about the decimal representation of an irrational number?

?

Teaching Note: This is an opportunity to revisit the ideas from activities A.24 and A.23.

Problems for Section 4.1

(1) Describe the set of real numbers. Give some relevant and revealing examples/nonexamples.

(2) Explain what would happen if we "declared" the value of π to be 3? What about if we declared it to have the value of 3.14?

(3) Explain why $x^2 - x - 1$ has no rational roots.

(4) Explain why $\sqrt{7}$ is irrational.

(5) Explain why $\sqrt[3]{5}$ is irrational.

(6) Explain why $\sqrt[5]{27}$ is irrational.

(7) Explain why if n is an integer and $\sqrt{n}$ is not an integer, then n is irrational.

(8) Consider the following numbers:

$$\frac{1}{47} \quad \text{and} \quad \frac{1}{78125}$$

For each, determine whether the decimal representation terminates or repeates, **without** actually computing the decimal representation. Explain your reasoning. If the decimal repeats, indicate and explain what the maximum possible number of digits in the repeating pattern is.

(9) Solve $x^5 - 31x^4 + 310x^3 - 1240x^2 + 1984x - 1024 = 0$. Interlace an explanation with your work. Hint: Use reasoning from this section to find rational roots.

(10) Solve $x^5 - 28x^4 + 288x^3 - 1358x^2 + 2927x - 2310 = 0$. Interlace an explanation with your work. Hint: Use reasoning from this section to find rational roots.

(11) Knowing that π is irrational, explain why $101 \cdot \pi$ is irrational.

(12) Knowing that π is irrational, explain why $\pi + 101$ is irrational.

(13) Knowing that π is irrational, explain why $77.2835 \cdot \pi$ is irrational.

(14) Knowing that π is irrational, explain why $\pi + 77.2835$ is irrational.

(15) Suppose we knew that a^2 was irrational. Could we conclude that a is also irrational? Explain your reasoning.

(16) Is $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$ rational or irrational? Explain your reasoning.

(17) In the discussion above, we give an argument showing that $\sqrt{2}$ is irrational. What happens if you try to use the exact same argument to try and show that $\sqrt{9}$ is irrational? Explain your reasoning.

(18) For each of the following statements, indicate whether the expression is a "rational number," an "irrational number," or whether it could be either. Note: For parts (d)–(f) assume that neither of the numbers is zero.

(a) rational + rational =?

(b) rational + irrational =?

(c) irrational + irrational =?

(d) rational · rational =?

(e) rational · irrational =?

(f) irrational · irrational =?

Give careful explanations for parts (a), (e), and (f).

4.2 Polynomial Equations

Activity A.36 is a good warm-up to this section.

Solving equations is one of the fundamental activities in mathematics. We're going to separate our equations into sets:

(1) Linear Equations—polynomial equations of degree 1.

(2) Quadratic Equations—polynomial equations of degree 2.

(3) Cubic Equations—polynomial equations of degree 3.

(4) Quartic Equations—polynomial equations of degree 4.

(5) Quintic Equations—polynomial equations of degree 5.

We'll stop right there, for now. . .

4.2.1 Linear Equations

The simplest polynomials (besides constant polynomials) are linear polynomials. Solving equations of the form

$$ax + b = 0$$

poses no difficulty, we can write out the solution easily as

$$x = -b/a.$$

4.2.2 Quadratic Equations

Finding roots of quadratic polynomials is a bit more complex. We want to find x such that

$$ax^2 + bx + c = 0.$$

I know you already know how to do this. However, pretend for a moment that you don't. This would be a really hard problem. We have evidence that it took humans around 1000 years to solve this problem in generality, the first general solution appearing in Babylon and China around 2500 years ago. With this in mind, I think this topic warrants some attention. If you want to solve $ax^2 + bx + c = 0$, a good place to start would be with an easier problem. Let's make $a = 1$ and try to solve

$$x^2 + bx = c$$

Geometrically, you could visualize this as an $x \times x$ square along with a $b \times x$ rectangle. Make a blob for c on the other side.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Take your $b \times x$ rectangle and divide it into two $(b/2) \times x$ rectangles.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Now take both of your $(b/2) \times x$ rectangles and snuggie them next to your $x \times x$ square on adjacent sides. You should now have what looks like an $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ square with a corner cut out of it.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Finally, your big $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ has a piece missing, a $(b/2) \times (b/2)$ square, right? So if you add that piece in on both sides, the area of both sides of your picture had better be $c + (b/2)^2$. From your picture you will find that:

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

Question Can you find x at this point?

?

Question Explain how to solve $ax^2 + bx + c = 0$.

?

Activities A.37, A.40, and **??**, complement this section well. Activity A.41 could be done here too.

4.2.3 Cubic Equations

While the quadratic formula was discovered around 2500 years ago, cubic equations proved to be a tougher nut to crack. A general solution to a cubic equation was not found until the 1500's. At the time mathematicians were a secretive and competitive bunch. Someone would solve a particular cubic equation, then challenge another mathematician to a sort of "mathematical duel." Each mathematician would give the other a list of problems to solve by a given date. The one who solved the most problems was the winner and glory everlasting (This might be a slight exaggeration.) was theirs. One of the greatest duelists was Niccolò Fontana Tartaglia (pronounced *Tar-tah-lee-ya*). Why was he so great? He developed a general method for solving cubic equations! However, neither was he alone in this discovery nor was he the first. As sometimes happens, the method was discovered some years earlier by another mathematician, Scipione del Ferro. However, due to the secrecy and competitiveness, very few people knew of Ferro's method. Since these discoveries were independent, we'll call the method the *Ferro-Tartaglia method*.

We'll show you the Ferro-Tartaglia method for finding at least one root of a cubic of the form:

$$x^3 + px + q$$

We'll illustrate with a specific example—you'll have to generalize yourself! Take

$$x^3 + 3x - 2 = 0$$

Step 1 Replace x with $u + v$.

$$\begin{aligned}(u+v)^3 + 3(u+v) - 2 &= u^3 + 3u^2v + 3uv^2 + v^3 + 3(u+v) - 2\\ &= u^3 + v^3 + 3uv(u+v) + 3(u+v) - 2\\ &= u^3 + v^3 - 2 + (3uv+3)(u+v).\end{aligned}$$

Step 2 Set uv so that all of the terms are eliminated except for u^3, v^3, and constant terms.

Since we want

$$3uv + 3 = 0$$

we'll set $uv = -1$ and so

$$u^3 + v^3 - 2 = 0.$$

Since $uv = -1$, we see that $v = -1/u$ so

$$u^3 + \left(\frac{-1}{u}\right)^3 - 2 = u^3 - \frac{1}{u^3} - 2 = 0.$$

Step 3 Clear denominators and use the quadratic formula.

$$u^3 - \frac{1}{u^3} - 2 = 0 \qquad \Leftrightarrow \qquad u^6 - 2u^3 - 1 = 0$$

But now we may set $y = u^3$ and so we have

$$y^2 - 2y - 1 = 0$$

and by the quadratic formula

$$y = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

Putting this all together we find:

$$\begin{aligned} y &= 1 \pm \sqrt{2} \\ u &= \sqrt[3]{1 \pm \sqrt{2}} \\ v &= \frac{-1}{\sqrt[3]{1 \pm \sqrt{2}}} \end{aligned}$$

and finally (drum-roll please):

$$x = \sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}} \qquad \text{and} \qquad x = \sqrt[3]{1-\sqrt{2}} - \frac{1}{\sqrt[3]{1-\sqrt{2}}}$$

Question How many solutions are we supposed to have in total?

?

Question How do we do this procedure for other equations of the form

$$x^3 + px + q = 0?$$

?

4.2.4 Quartics, Quintics, and Beyond

While the Ferro-Tartaglia method may seem like it only solves the special case of $x^3 + px + q = 0$, it is in fact a "wolf in sheep's clothing" and is the key to giving a formula for solving any cubic equation

$$ax^3 + bx^2 + cx + d = 0.$$

The formula for solutions of the cubic equation is quite complex—we will spare you the details. Despite the fact that the key step of the formula is the Ferro-Tartaglia method, it is usually called *Cardano's formula* because Cardano was the first to publish this method.

It was wondered if there were formulas for solutions to polynomial equations of arbitrary degree. When we say formulas, we mean formulas involving the coefficients of the polynomials and the symbols:

$$+ \quad - \quad \cdot \quad \div \quad \sqrt{}$$

Cardano's student Ferrari, (who incidentally went to the University of Bologna) soon found the quartic formula, though it is too monstrous to write down in these notes. The search for the quintic equation began. Things started getting very difficult. The old tricks didn't work, and it wasn't until nearly 300 years later that this problem was settled.

Question Who was Niels Abel? Who was Évariste Galois?

?

Abel and Galois (pronounced *Gal-wah*), independently prove that there is no general formula (using only the symbols above) for polynomial equations of degree 5 or higher. It is an amazing result and is only seen by students in advanced undergraduate or beginning graduated courses in pure mathematics. Nevertheless, in our studies we will not completely shy away from such demons. Read on!

Problems for Section 4.2

(1) Draw a rough timeline showing: The point when we realized we were interested in quadratic equations, the discovery of the quadratic formula, the discovery of the cubic formula, the discovery of the quartic formula, and the work of Abel and Galois proving the impossibility of a general formula for polynomial equations of degree 5 or higher.

(2) Given a polynomial, explain the connection between *linear factors* and *roots*. Are they the same thing or are they different things?

(3) In ancient and Medieval times the discussion of quadratic equations was often broken into three cases:

(a) $x^2 + bx = c$

(b) $x^2 = bx + c$

(c) $x^2 + c = bx$

where b and c are positive numbers. Create real-world word problems involving length and area for each case above.

(4) In ancient and Medieval times the discussion of quadratic equations was often broken into three cases:

(a) $x^2 + bx = c$

(b) $x^2 = bx + c$

(c) $x^2 + c = bx$

where b and c are positive numbers. Is this a complete list of cases? If not, what is missing and why is it (are they) missing? Explain your reasoning.

(5) Describe what happens geometrically when you complete the square of a quadratic equation of the form $x^2 + bx = c$ when b and c are positive. Explain your reasoning.

(6) Jim, Lydia, and Isabel are visiting China. Unfortunately they are stuck in a seemingly infinite traffic jam. The cars are moving at a very slow (but constant) rate. Jim and Lydia are 25 miles behind Isabel. Jim wants to send a sandwich to Isabel. So he hops on his motorcycle and rides through traffic to Isabel, gives her the sandwich, and rides back to Lydia at a constant speed. When he returns to Lydia, she has moved all the way to where Isabel was when Jim started. In total, how far did Jim travel on his motorcycle?

(a) Before any computations are done, use common sense to guess the solution to this problem.

(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your guess?

(c) Use algebra to solve the problem.

(7) Must a quadratic polynomial always have a real root? Explain your reasoning.

(8) Must a cubic polynomial always have a real root? Explain your reasoning.

(9) Must a quartic polynomial always have a real root? Explain your reasoning.

(10) Must a quintic polynomial always have a real root? Explain your reasoning.

(11) Derive the quadratic formula. Explain your reasoning.

(12) Solve $x^2 + 3x - 2 = 0$. Interlace an explanation with your work.

(13) Find two solutions to $x^4 + 3x^2 - 2 = 0$. Interlace an explanation with your work.

(14) Find two solutions to $x^6 + 3x^3 - 2 = 0$. Interlace an explanation with your work.

(15) Find two solutions to $x^{10} + 3x^5 - 2 = 0$. Interlace an explanation with your work.

(16) Find two solutions to $3x^{14} - 2x^7 + 6 = 0$. Interlace an explanation with your work.

(17) Find two solutions to $-4x^{22} + 13x^{11} + 1 = 0$. Interlace an explanation with your work.

(18) Give a general formula for finding two solutions to equations of the form: $ax^{2n} + bx^n + c = 0$ where n is an integer. Interlace an explanation with your work.

(19) Use the Ferro-Tartaglia method to find a solution to $x^3 + x + 1 = 0$. Interlace an explanation with your work.

(20) Use the Ferro-Tartaglia method to find a solution to $x^3 - x - 1 = 0$. Interlace an explanation with your work.

(21) Use the Ferro-Tartaglia method to find a solution to $x^3 + 3x - 4 = 0$. Interlace an explanation with your work.

(22) Use the Ferro-Tartaglia method to find a solution to $x^3 + 2x - 3 = 0$. Interlace an explanation with your work.

(23) Use the Ferro-Tartaglia method to find a solution to $x^3 + 6x - 20 = 0$. Interlace an explanation with your work.

(24) Find at least two solutions to $x^4 - x^3 - 3x^2 + 2x + 1 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(25) Explain what Abel and Galois proved to be impossible.

4.3 Me, Myself, and a Complex Number

Activity A.42 complements this section well.

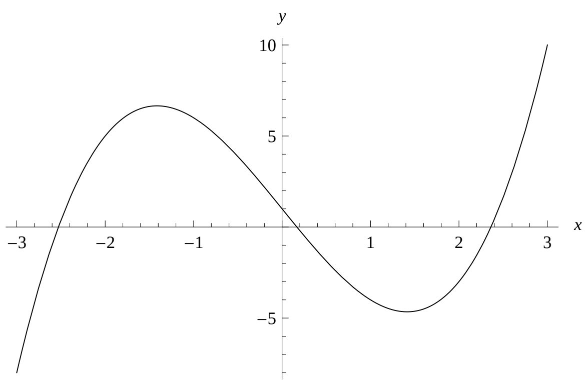
We'll start with the definition:

Definition A **complex number** is a number of the form

$$x + yi$$

where x and y are real numbers and $i^2 = -1$. We use the symbol $\mathbb{C}$ to denote the complex numbers.

What's that I hear? Yells of protest telling me that no such number exists? Well if it makes you feel any better, people denied the existence of such numbers for a long time. It wasn't until the 1800's until people finally changed their minds. Let's talk about some ideas that helped. Consider the plot of $y = x^3 - 6x + 1$:



If you use the Ferro-Tartaglia method to find at least one solution to this cubic, then you find the following root:

$$\sqrt[3]{\frac{-1+\sqrt{-31}}{2}} + \frac{2}{\sqrt[3]{\frac{1}{2}(-1+\sqrt{-31})}}$$

This root looks like a complex number, since $\sqrt{-31}$ pops up twice. This might seem a bit redundant, but we should point out that $\sqrt{-31}$ is a complex number since it can be expressed as:

$$0 + \left(\sqrt{31}\right) i$$

Even though our root has complex numbers in it, we know that it is real from the picture! Moral: If you want to give exact solutions to equations, then you'd better work with complex numbers, even if the roots are real!

Teaching Note: Here we are not ready to try to simplify the large expression above. We are leaving this as a mystery for a future course.

Question If $u + vi$ is a nonzero complex number, is

$$\frac{1}{u + vi}$$

a complex number too?

You betcha! Let's do it. The first thing you must do is multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{u + vi} = \frac{1}{u + vi} \cdot \frac{u - vi}{u - vi} = \frac{u - vi}{u^2 + v^2}$$

Now break up your fraction into two fractions:

$$\frac{u - vi}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2} i$$

Ah! Since u and v are real numbers, so are

$$x = \frac{u}{u^2 + v^2} \qquad \text{and} \qquad y = \frac{-v}{u^2 + v^2}$$

Hence

$$\frac{1}{u + vi} = x + yi$$

and is definitely a complex number.

The real importance of the complex numbers came from Gauss, with the Fundamental Theorem of Algebra:

Theorem 4.3.1 (Fundamental Theorem of Algebra) *Every polynomial of the form*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are complex numbers has exactly n (possibly repeated) complex roots.

Said a different way, the Fundamental Theorem of Algebra says that every polynomial with complex coefficients

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

can be factored as

$$a_n \cdot (x - r_1)(x - r_2) \cdots (x - r_n)$$

where each r_i is a complex number.

Question How many complex roots does $x^3 - 1$ have? What are they?

?

Teaching Note: This problem should definitely be addressed. Again, we find an obvious root, $x = 1$ and use the division theorem.

4.3.1 The Complex Plane

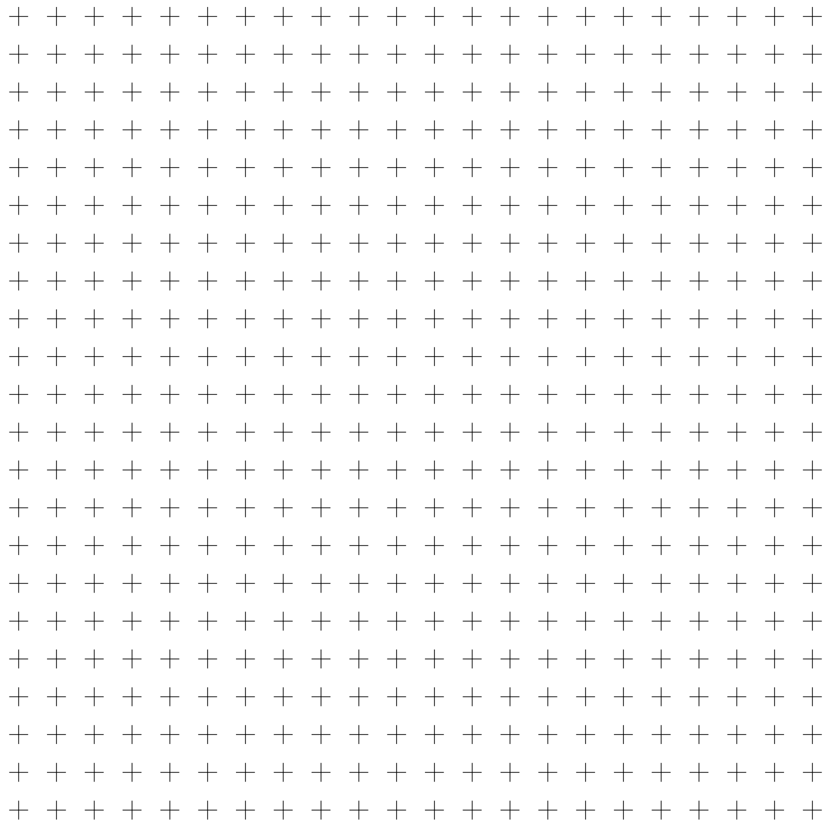
Complex numbers have a strong connection to geometry, we see this with the *complex plane*:

Activities A.43 and A.44 complement this section well.

Definition The **complex plane** is obtained when one plots the complex number $x + yi$ as the point (x, y). When considering the complex plane, the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**.

Here is a grid. Draw the real and imaginary axes and plot the complex numbers:

$$3 \qquad -5i \qquad 4 + 6i \qquad -3 + 5i \qquad -6 - i \qquad 6 - 6i$$



Be sure to label your plot.

Question Geometrically speaking, what does it mean to "add" complex numbers?

?

Question Geometrically speaking, what does it mean to "multiply" complex numbers?

?

Activity A.45 is worth considering here.

Problems for Section 4.3

(1) What is a real number?

(2) What is a complex number?

(3) Solve $x^3 - 6x + 5 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning.

(4) Solve $x^3 - 6x + 4 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning.

(5) Solve $x^3 - 2x - 1 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning. Interlace an explanation with your work.

(6) Solve $x^3 - 12x - 8 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning. Interlace an explanation with your work.

(7) Solve $x^3 - 3x^2 + 5x - 3 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(8) Solve $x^3 + 4x^2 - 7x + 2 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(9) Draw a Venn diagram showing the relationship between $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Include relevant examples of numbers belonging to each set.

(10) Explain why the following "joke" is "funny:" *The number you have dialed is imaginary. Please rotate your phone by 90 degrees and try again.*

(11) Explain why every real number is a complex number.

(12) Explain why $\sqrt{-2}$ is a complex number.

(13) Is $\sqrt[3]{-2}$ a complex number? Explain your reasoning.

(14) Explain why $\sqrt[10]{-5}$ is a complex number.

(15) Explain why if $x + yi$ and $u + vi$ are complex numbers, then

$$(x + yi) + (u + vi)$$

is a complex number.

(16) Explain why if $x + yi$ and $u + vi$ are complex numbers, then

$$(x + yi)(u + vi)$$

is a complex number.

(17) Given a complex number $z = x + yi$, the **complex conjugate** of z is $x - yi$, we denote this as $\overline{z}$. Let $w = u + vi$ be another complex number.

(a) Explain why $\overline{z} + \overline{w} = \overline{z + w}$.

(b) Explain why $\overline{z} \cdot \overline{w} = \overline{z \cdot w}$.

(18) Explain why if $u + vi$ is a complex number, then

$$\frac{1}{u + vi}$$

is a complex number.

(19) Compute the following, expressing your answer in the form $x + yi$:

(a) $(1 + 2i) + (1 + 7i)$

(b) $(1 + 2i) \cdot (1 + 7i)$

(c) $(1 + 2i)/(1 + 7i)$

Explain your reasoning.

(20) I'm going to show you something, see if you can see a connection to geometry:

(a) Let $z = 3 + 4i$. Compute $\sqrt{z \cdot \overline{z}}$.

(b) Let $z = 6 + 8i$. Compute $\sqrt{z \cdot \overline{z}}$.

(c) Let $z = 5 + 12i$. Compute $\sqrt{z \cdot \overline{z}}$.

What do you notice?

(21) Express $\sqrt{i}$ in the form $a + bi$. Hint: Solve the equation $z^2 = i$.

(22) Factor the polynomial $3x^2 + 5x + 10$ over the complex numbers. Explain your reasoning.

(23) Factor the polynomial $x^3 - 1$ over the complex numbers. Explain your reasoning.

(24) Factor the polynomial $x^4 - 1$ over the complex numbers. Explain your reasoning.

(25) Factor the polynomial $x^4 + 1$ over the complex numbers. Explain your reasoning. Hint: Factor as the difference of two squares and use Problem (21).

(26) Factor the polynomial $x^4 + 4$ over the complex numbers. Can it be factored into polynomials with real coefficients of lower degree? Explain your reasoning.

(27) Plot all complex numbers z in the complex plane such that $z \cdot \overline{z} = 1$. Explain your reasoning.

(28) Suppose I told you that:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

Explain why we say:

$$e^{x \cdot i} = \cos(x) + i\sin(x)$$

(29) This is Euler's famous formula:

$$e^{\pi \cdot i} + 1 = 0$$

Use Problem (28) to explain why it is true.

(30) How many complex roots should $x^2 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

(31) How many complex roots should $x^3 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

(32) How many complex roots should $x^4 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

(33) How many complex roots should $x^5 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

5 Harmony of Numbers

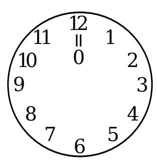
> Let us despise the barbaric neighings [of war] which echo through these noble lands, and awaken our understanding and longing for the harmonies.
>
> —Johannes Kepler

Teaching Note: This section is a hodge-podge of applications and modeling.

Activity A.46 complements this section well.

5.1 Clocks

It is now time to think about clocks. Consider the usual run-of-the-mill clock:



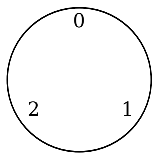
Question Suppose you start grading papers at 3 o'clock and then 5 hours pass. What time is it? Now suppose that you find more papers to grade, and 5 more hours pass—now what time is it? How do you do these problems? Why are there so many papers to grade?

?

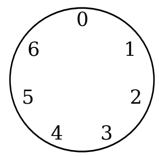
We have a mathematical way of writing these questions:

$$3 + 5 \equiv 8 \pmod{12}$$
$$8 + 5 \equiv 1 \pmod{12}$$

We call arithmetic on clocks **modular arithmetic**. Being rather fearless in our quest for knowledge, we aren't content to stick with 12 hour clocks:

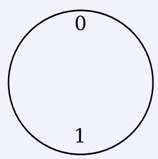


(mod 3)



(mod 7)

Question Suppose you are working on a 2 hour clock:



(mod 2)

Suppose you started at time zero, and finished after 10245 hours.

(1) Where is the hand of the clock pointing?

(2) How does the answer change if you are working on a 5 hour clock?

(3) What if you are working on a 7 hour clock?

?

OK—clocks are great. Here is something slightly different: Denote the set of all integers that are r greater than a multiple of 5 by $[r]_5$. So for example:

$$[0]_5 = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$$

Write down the following sets:

$[1]_5 =$ ☐

$[2]_5 =$ ☐

$[3]_5 =$ ☐

$[4]_5 =$ ☐

$[5]_5 =$ ☐

Question With our work above, see if you can answer the following:

(1) Explain why one could say that $[4]_5 = [9]_5$.

(2) Explain why one could say that $[2]_5 = [-3]_5$.

(3) Explain what you think is meant by the expression:

$$[1]_5 + [2]_5 = [3]_5$$

(4) Explain what you think is meant by the expression:

$$[1]_5 + [4]_5 = [0]_5$$

?

Question How many different descriptions of modular arithmetic can you give? To aid you in this quest, I suggest you start your descriptions off with the words:

The number a is congruent to b modulo m when . . .

?

OK—I know I was supposed to leave that question for you, but there is one description that I just gotta tell you about—check this out:

$$a \equiv b \pmod{m} \qquad \Leftrightarrow \qquad a - b = m \cdot q$$

Question What is the deal with the junk above? What is q? How does it help you solve congruences like

$$3x \equiv 1 \pmod{11}?$$

?

Question Is it the case that

$$5 + x \equiv 2 + x \pmod{3}$$

for all integers x? Why or why not? Use each of the descriptions of modular arithmetic above to answer this question.

?

Problems for Section 5.1

(1) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $3 + x = 10$
(b) $3 + x \equiv 10 \pmod{12}$
(c) $3 + x \equiv 10 \pmod{7}$
(d) $3 + x \equiv 10 \pmod{6}$
(e) $3 + x \equiv 10 \pmod{5}$
(f) $3 + x \equiv 10 \pmod{3}$
(g) $3 + x \equiv 10 \pmod{2}$

In each case explain your reasoning.

(2) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $10 + x = 1$
(b) $10 + x \equiv 1 \pmod{12}$
(c) $10 + x \equiv 1 \pmod{11}$
(d) $10 + x \equiv 1 \pmod{9}$
(e) $10 + x \equiv 1 \pmod{5}$
(f) $10 + x \equiv 1 \pmod{3}$
(g) $10 + x \equiv 1 \pmod{2}$

In each case explain your reasoning.

(3) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $217 + x = 1022$
(b) $217 + x \equiv 1022 \pmod{100}$
(c) $217 + x \equiv 1022 \pmod{20}$
(d) $217 + x \equiv 1022 \pmod{12}$
(e) $217 + x \equiv 1022 \pmod{5}$
(f) $217 + x \equiv 1022 \pmod{3}$
(g) $217 + x \equiv 1022 \pmod{2}$

In each case explain your reasoning.

(4) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $11 + x \equiv 7 \pmod{2}$
(b) $11 + x \equiv 7 \pmod{3}$
(c) $11 + x \equiv 7 \pmod{5}$
(d) $11 + x \equiv 7 \pmod{8}$
(e) $11 + x \equiv 7 \pmod{10}$

In each case explain your reasoning.

(5) List out 6 elements of $[3]_4$, including 3 positive and 3 negative elements. Explain your reasoning.

(6) List out 6 elements of $[6]_7$, including 3 positive and 3 negative elements. Explain your reasoning.

(7) List out 6 elements of $[7]_6$, including 3 positive and 3 negative elements. Explain your reasoning.

(8) One day you walk into a mathematics classroom and you see the following written on the board:

$$[4]_6 = \{\dots, -14, -8, -2, 4, 10, 16, 22, \dots\}$$
$$\left[\frac{1}{2}\right] = \left\{\dots, \frac{-3}{-6}, \frac{-2}{-4}, \frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \dots\right\}$$

What is going on here? Can you figure out what $\left[\frac{3}{4}\right]$ would be? Explain your reasoning.

(9) If possible, solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $3x = 1$
(b) $3x \equiv 1 \pmod{11}$
(c) $3x \equiv 1 \pmod{9}$
(d) $3x \equiv 1 \pmod{8}$
(e) $3x \equiv 1 \pmod{7}$
(f) $3x \equiv 1 \pmod{3}$
(g) $3x \equiv 1 \pmod{2}$

In each case explain your reasoning.

(10) Solve the following congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $11x \equiv 7 \pmod 2$
(b) $11x \equiv 7 \pmod 3$
(c) $11x \equiv 7 \pmod 5$
(d) $11x \equiv 7 \pmod 8$
(e) $11x \equiv 7 \pmod{10}$

In each case explain your reasoning.

(11) Solve the following congruences or explain why there is no solution, expressing your answer as a number between 0 and the relevant modulus:

(a) $15x \equiv 7 \pmod 2$
(b) $15x \equiv 7 \pmod 3$
(c) $15x \equiv 7 \pmod 5$
(d) $15x \equiv 7 \pmod 9$
(e) $15x \equiv 7 \pmod{10}$

In each case explain your reasoning.

(12) Make an "addition table" for arithmetic modulo 6.

(13) Make an "addition table" for arithmetic modulo 7.

(14) Make a "multiplication table" for arithmetic modulo 6.

(15) Make a "multiplication table" for arithmetic modulo 7.

(16) Explain the connection between writing an integer in base b and reducing an integer modulo b.

(17) Is

$$5 + x \equiv 12 + x \pmod 3$$

ever/always true? Explain your reasoning.

(18) Is

$$20 + x \equiv 32 + x \pmod 3$$

ever/always true? Explain your reasoning.

(19) Recalling that $i^2 = -1$, can you find "i" in the integers modulo 5? Explain your reasoning.

(20) Recalling that $i^2 = -1$, can you find "i" in the integers modulo 17? Explain your reasoning.

(21) Recalling that $i^2 = -1$, can you find "i" in the integers modulo 13? Explain your reasoning.

(22) Recalling that $i^2 = -1$, can you find "i" in the integers modulo 11? Explain your reasoning.

(23) Today is Saturday. What day will it be in 3281 days? Explain your reasoning.

(24) It is now December. What month will it be in 219 months? What about 111 months ago? Explain your reasoning.

(25) What is the remainder when 2^{999} is divided by 3? Explain your reasoning.

(26) What is the remainder when 3^{26} is divided by 7? Explain your reasoning.

(27) What is the remainder when 14^{30} is divided by 11? Explain your reasoning.

(28) What is the remainder when 5^{28} is divided by 11? Explain your reasoning.

(29) What is the units digit of 123^{456}? Explain your reasoning.

(30) Factor $x^2 + 1$ over the integers modulo 2. Explain your reasoning.

(31) Factor $x^3 + x^2 + x + 1$ over the integers modulo 2. Explain your reasoning.

(32) Factor $x^5 + x^4 + x + 1$ over the integers modulo 2. Explain your reasoning.

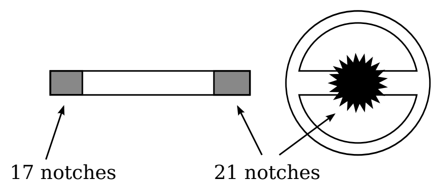
5.2 In the Real World

Perhaps the coolest thing about mathematics is that you can actually solve “real world” problems. Let’s stroll through some of these “real world” problems.

5.2.1 Automotive Repair

A Geometry Problem One Thanksgiving Day I had a neat conversation with my cousin Chris at the dinner table. You see he works on cars—specifically vintage Italian sports cars. He had been doing some routine maintenance on one of his cars and needed to remove the steering wheel and the steering column. All was fine until it came time to put the parts back together. The steering wheel was no longer centered! The car could drive down the street just fine, but when the car drove straight ahead the steering wheel was off by a rotation of 5 degrees to the right. This would not do! This sounds like a geometry problem.

An Algebra Problem How did this happen you ask? Well the *steering wheel* attaches to the car via the *steering column*:



there were 21 notches on the back of the wheel, which connects to the column. There were also 17 notches on the other end of the column that then connected to the car itself.

Chris had noticed that moving the wheel 1 notch changed its position by

$$\frac{360}{21} \approx 17 \text{ degrees},$$

and that adjusting the columns by 1 notch changed its position by

$$\frac{360}{17} \approx 21 \text{ degrees}.$$

Hmmm so if we want to center the wheel, we want to solve the following equation:

$$17w + 21c = -5$$

where w represents how many notches we turn the wheel and c represents how many notches we turn the column. Ah! This sounds like an algebra problem! There is only one issue: We have two unknowns and a single variable.

Question How do we proceed from here? Can you solve the problem? Where does modular arithmetic factor in to the solution?

?

5.2.2 Check Digits

Our world is full of numbers. Sometimes if you are in a large organization—say a large university—you feel a bit like a number. How do you know if you are the right number? Allow me to clarify. Most items you buy have some sort of UPC (Universal Product Code) on them. This allows them to be put into a computer in an organized fashion. When you buy items in a grocery story, you want the item you scanned to come up—and not some other (potentially embarrassing!) item. To ensure you get what is coming to you, we have *check digits*. These are digits that "check" to make sure that the code has scanned correctly. Typically, what you see are either UPC-A codes or UPC-E codes. Here is an example of a UPC-A code:

0 42572 05001 4

The check digit is the right most digit (in this case 4). The check digit is not used in identifying the item, instead it is used purely to check if the other digits are correct. Here is how you check to see if a UPC-A code is valid:

(1) Working modulo 10, add the digits in the odd positions and multiply by 3:

$$\begin{aligned} 0+2+7+0+0+1 &= 10 \\ 10 \cdot 3 &= 30 \\ 30 &\equiv 0 \pmod{10}. \end{aligned}$$

(2) Working modulo 10, add the digits in the even positions (including the check digit):

$$4 + 5 + 2 + 5 + 0 + 4 = 20$$
$$20 \equiv 0 \pmod{10}$$

(3) Add the outcomes from the previous steps together and take the result modulo 10:

$$0 + 0 \equiv 0 \pmod{10}$$

If the result is congruent to 0 modulo 10, as it is in this case, then you have a correct UPC-A number and you are good to go!

We should note, sometimes at stores you see UPC-E codes:

0 123456 5

These are compressed UPC-A codes where 5 zeros have been removed. The rules for transforming UPC-A codes to UPC-E codes are a bit tedious, so we'll skip them for now—though they are easy to look up on the internet.

Question Can you find a UPC-E code and verify that it is valid?

?

Problems for Section 5.2

(1) Which of the following is a correct UPC-A number?

8 12556 01041 0

8 12565 01091 0

8 12556 01091 0

Explain your reasoning.

(2) Which of the following is a correct UPC-A number?

7 17664 13387 0

7 17669 13387 0

7 17669 73387 0

Explain your reasoning.

(3) Find the missing digit in the following UPC-A number:

8 14371 0■354 2

Explain your reasoning.

(4) Find the missing digit in the following UPC-A number:

0 76484 86■97 3

Explain your reasoning.

(5) How similar can two different UPC numbers be? Explain your reasoning.

(6) In the United States some bank check codes are nine digit numbers

$$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9$$

where

$$7a_1 + 3a_2 + 9a_3 + 7a_4 + 3a_5 + 9a_6 + 7a_7 + 3a_8 \equiv a_9 \pmod{10}.$$

(a) Give three examples of valid bank check codes.

(b) If adjacent digits were accidentally switched, could a machine detect the error? Explain your reasoning.

(7) ISBN-10 numbers are ten digit numbers

$$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}$$

where

$$10a_1+9a_2+8a_3+7a_4+6a_5+5a_6+4a_7+3a_8+2a_9+a_{10} \equiv 0 \pmod{11}.$$

(a) Give three examples of ISBN-10 numbers.

(b) If adjacent digits were accidentally switched, could a machine detect the error? Explain your reasoning.

5.3 The Binomial Theorem

5.3.1 Varna-Sangita

In ancient Indian texts we find a description of a type of music called *varna-sangita*. This is music made from a variation of long and short syllables. When performing a varna-sangita, one starts off with a given number of short syllables and ends with the same number of long syllables. In between these verses, every possible combination of long and short syllables is supposed to occur. If s represents a short syllable and l represents a long syllable we might visualize this as:

$$ssss \xrightarrow{\text{every possible combination}} llll$$

To check their work, the people of ancient India counted how many of each combination appeared in a song. Suppose we started with sss and finished with lll. Our song should contain the following verses:

$$sss, \quad ssl, \quad sls, \quad lss, \quad sll, \quad lsl, \quad lls, \quad lll$$

We can construct the following table to summarize what we have found:

3 s's	2 s's and 1 l	1 s and 2 l's	3 l's
1	3	3	1

Question What would your table look like if you started with ss and finished with ll? What about if you started with $ssss$ and finished with $llll$?

?

The vedics of the time gave a rule for making tables like the one above. Their rule was based on the following diagram:

$n = 0$:							1							$k = 0$
$n = 1$:						1		1						$k = 1$
$n = 2$:					1		2		1					$k = 2$
$n = 3$:				1		3		3		1				$k = 3$
$n = 4$:			1		4		6		4		1			$k = 4$
$n = 5$:		1		5		10		10		5		1		$k = 5$
$n = 6$:	1		6		15		20		15		6		1	$k = 6$

Today people call this diagram **Pascal's triangle**.

Question How does Pascal's triangle relate to varna-sangitas? Is there an easy way to produce the above diagram?

?

And now for something completely different. . .

5.3.2 Expansions

Expand the following on a separate sheet of paper. Write the result of your work in the boxes below:

$$(a + b)^0 = \boxed{\qquad\qquad\qquad\qquad}$$

$$(a + b)^1 = \boxed{\qquad\qquad\qquad\qquad}$$

$$(a + b)^2 = \boxed{\qquad\qquad\qquad\qquad}$$

$$(a + b)^3 = \boxed{\qquad\qquad\qquad\qquad}$$

$$(a + b)^4 = \boxed{\qquad\qquad\qquad\qquad}$$

Question Is there a nice way to organize this data?

?

Question Can you explain the connection between expanding binomials and varna-sangitas?

?

Activity A.47 complements this section well.

5.3.3 Come Together

Let's see if we can bring these ideas together. Let's denote the following symbol:

$$\binom{n}{k} = \text{the number of ways we choose } k \text{ objects from } n \text{ objects.}$$

it is often said "n choose k" and is sometimes denoted as ${}_nC_k$.

Question What exactly does $\binom{n}{k}$ mean in terms of varna-sangitas? What does $\binom{n}{k}$ mean in terms of expansion of binomials?

?

Question How does $\binom{n}{k}$ relate to Pascal's triangle?

?

Question Pascal claims:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Explain how this single equation basically encapsulates the key to constructing Pascal's triangle.

?

Question Suppose that an oracle tells you that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

but we, being good skeptical people, are not convinced. How do we check this?

?

From the work above, we obtain a fabulous theorem:

Theorem 5.3.1 (Binomial Theorem) *If n is a nonnegative integer, then*

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n.$$

Question This looks like gibberish to me. Tell me what it is saying. Also, why is the Binomial Theorem true?

?

Activity A.48 complements this section well.

Activity **??** is worth considering here, too.

The counting/probability activities, A.49 through A.53 can now be done.

Problems for Section 5.3

(1) Write down the first 7 rows of Pascal's triangle.

(2) Explain how $\binom{n}{k}$ corresponds to the entries of Pascal's triangle. Feel free to draw diagrams and give examples.

(3) State the Binomial Theorem and give some examples of it in action.

(4) Explain the "physical" meaning of $\binom{n}{k}$. Give some examples illustrating this meaning.

(5) Explain how Pascal's triangle is formed. In your explanation, use the notation $\binom{n}{k}$. If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

(6) Explain why the formula you found in Problem (5) is true.

(7) State the formula for $\binom{n}{k}$.

(8) Expand $(a+b)^5$ using the Binomial Theorem.

(9) Expand $(a-b)^7$ using the Binomial Theorem.

(10) Expand $(-a-b)^8$ using the Binomial Theorem.

(11) Expand $(a+(b+c))^3$ using the Binomial Theorem.

(12) Expand $(a-b-c)^3$ using the Binomial Theorem.

(13) Let n be a positive integer.

- (a) Try some experiments to guess when $9^n + 1^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.
- (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 9 = 1$.

(14) Let n be a positive integer.

- (a) Try some experiments to guess when $6^n + 4^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.
- (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 6 = 4$.

(15) Let n be a positive integer.

- (a) Try some experiments to guess when $7^n - 3^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.
- (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 3 = 7$.

(16) Let n be a positive integer.

- (a) Try some experiments to guess when $8^n - 2^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.
- (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 2 = 8$.

(17) Generalize Problems (13), (14), (15), and (16) above. Clearly articulate your new statement(s) and explain why they are true.

(18) Which is larger, $(1 + 1/2)^2$ or 2? Explain your reasoning.

(19) Which is larger, $(1 + 1/5)^5$ or 2? Explain your reasoning.

(20) Which is larger, $(1 + 1/27)^{27}$ or 2? Explain your reasoning.

(21) Which is larger, $(1 + 1/101)^{101}$ or 2? Explain your reasoning.

(22) Which is larger, $(1.0001)^{10000}$ or 2? Explain your reasoning.

(23) Generalize Problems (18), (19), (20), (21), and (22) above. Clearly articulate your new statement(s) and explain why it is true.

(24) Given a positive integer n, can you guess an upper bound for $(1 + 1/n)^n$?

(25) Let n be a positive integer. Use the Binomial Theorem to explain why:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

What does this mean in terms of Pascal's Triangle?

(26) Let n be a positive integer. Use the Binomial Theorem to explain why:

$$(-1)^0\binom{n}{0} + (-1)^1\binom{n}{1} + (-1)^2\binom{n}{2} + \cdots + (-1)^n\binom{n}{n} = 0$$

What does this mean in terms of Pascal's Triangle?

(27) Suppose I tell you:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Explain how to deduce:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

A Activities

A.1 Shelby and Scotty

Note: In this activity, we use words (rather than numerals) to indicate bases. And we use use a subscript after a numeral to specify its base.

Shelby and Scotty want to express the (base ten) number 27 in base four. However, they used very different methods to do this. Let's check them out.

> **Teaching Note:** Consider first asking students to do this problem themselves. Then they are more likely ready to interpret the methods below, and chances are some students will do it like Shelby and others like Scotty.

A.1.1) Consider Shelby's work:

$$\begin{array}{r} 6\ \text{R}3 \\ 4\overline{)27} \end{array} \qquad \begin{array}{r} 1\ \text{R}2 \\ 4\overline{)6} \end{array} \qquad \begin{array}{r} 0\ \text{R}1 \\ 4\overline{)1} \end{array} \qquad \Rightarrow \qquad \boxed{123_{\text{four}}}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

A.1.2) Using the 27 marks below, create an illustration (or series of illustrations) that models Shelby's method for changing bases.

$$| \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ | \ |$$

Further, explain why Shelby's method works.

A.1.3) Consider Scotty's work:

$$\begin{array}{r} 0\ \text{R}27 \\ 4^3\overline{)27} \end{array} \qquad \begin{array}{r} 1\ \text{R}11 \\ 4^2\overline{)27} \end{array} \qquad \begin{array}{r} 2\ \text{R}3 \\ 4\overline{)11} \end{array} \qquad \Rightarrow \qquad \boxed{123_{\text{four}}}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

A.1.4) Using the 27 marks below, create an illustration (or series of illustrations) that models Scotty's method for changing bases.

| | | | | | | | | | | | | | | | | | | | | | | | | | |

Further, explain why Scotty's method works.

A.1.5) Use both methods to write 1644_{ten} in base seven.

A.1.6) Now let's try to be more efficient.

(a) Convert 8630_{ten} to base thirteen. Use *A* for ten, *B* for eleven, and *C* for twelve.

(b) Quickly convert 2102_{three} to base nine.

(c) Without using base ten, convert 341_{six} to base four.

(d) Without using base ten, convert 341_{six} to base eleven.

A.2 Hieroglyphical Arithmetic

Note: This activity is based on an activity originally designed by Lee Wayand.

Fixnote: Perhaps include something about closure.

Consider the following addition and multiplication tables:

+	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
fish	fork	balloon	fish	eyeball	cinder-block	skull	lolly-pop	man	DNA
lolly-pop	balloon	eyeball	lolly-pop	fork	fish	man	cinder-block	DNA	skull
skull	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
cinder-block	eyeball	fork	cinder-block	balloon	lolly-pop	DNA	fish	skull	man
DNA	cinder-block	fish	DNA	lolly-pop	man	eyeball	skull	fork	balloon
fork	skull	man	fork	DNA	eyeball	fish	balloon	lolly-pop	cinder-block
man	lolly-pop	cinder-block	man	fish	skull	balloon	DNA	eyeball	fork
balloon	man	DNA	balloon	skull	fork	lolly-pop	eyeball	cinder-block	fish
eyeball	DNA	skull	eyeball	man	balloon	cinder-block	fork	fish	lolly-pop

fish = fish

lolly-pop = lolly-pop

skull = skull

cinder-block = cinder-block

DNA = DNA

fork = fork

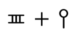
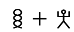
man = man

balloon = balloon

eyeball = eyeball

·	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish
fork	fish	lolly-pop	cinder-block	DNA	skull	eyeball	balloon	man	fork
eyeball	lolly-pop	balloon	man	fork	skull	cinder-block	DNA	fish	eyeball
balloon	cinder-block	man	fork	lolly-pop	skull	DNA	fish	eyeball	balloon
man	DNA	fork	lolly-pop	cinder-block	skull	fish	eyeball	balloon	man
skull	skull	skull	skull	skull	skull	skull	skull	skull	skull
lolly-pop	eyeball	cinder-block	DNA	fish	skull	balloon	man	fork	lolly-pop
cinder-block	balloon	DNA	fish	eyeball	skull	man	fork	lolly-pop	cinder-block
DNA	man	fish	eyeball	balloon	skull	fork	lolly-pop	cinder-block	DNA
fish	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish

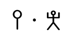
A.2.1) Use the addition table to compute the following:

 and 

A.2.2) Do you notice any patterns in the addition table? Tell us about them.

A.2.3) Can you tell me which glyph represents 0? How did you arrive at this conclusion?

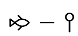
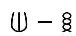
A.2.4) Use the multiplication table to compute the following:

 and 

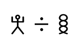
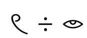
A.2.5) Do you notice any patterns in the multiplication table? Tell us about them.

A.2.6) Can you tell me which glyph represents 1? How did you arrive at this conclusion?

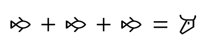
A.2.7) Compute:

 and 

A.2.8) Compute:

 and 

A.2.9) Keen Kelley was working with our tables above. All of a sudden, she writes

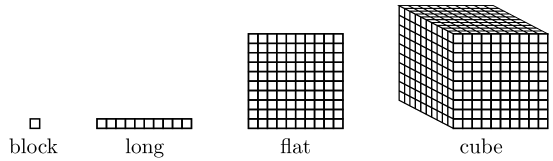


and shouts “Weird!” Why is she so surprised? Try repeated addition with other glyphs. What do you find? Can you explain this?

A.2.10) Can you find any other oddities of the arithmetic above? Hint: Try repeated multiplication!

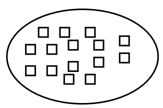
A.3 Playing with Blocks

I always enjoyed blocks quite a bit. Go find yourself some *base-ten blocks*. Just so that we are all on the same page, here are the basic blocks:



A.3.1) Sketch a model of the number 247 with base-ten blocks.

A.3.2) Oscar modeled the number 15 in the following way:



What do you think of his model? Can you improve upon it?

Teaching Note: The issue here is that the place-value system is not modeled. When working with base-10 blocks, we will demand that the place value system is always modeled. We want to do this with all algorithms.

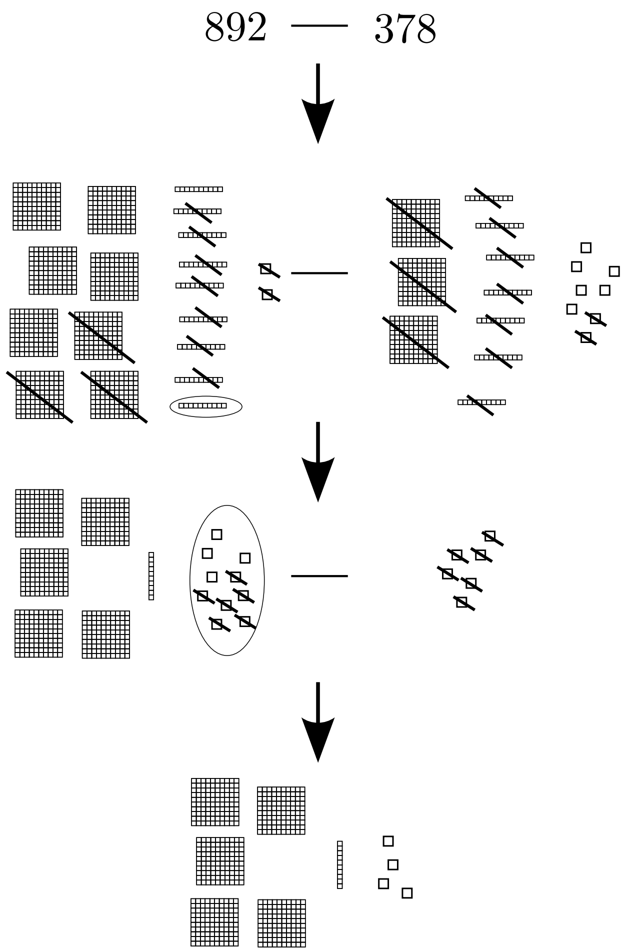
A.3.3) Many problems involving subtraction can be considered one of the following types: take-away, comparison, and missing addend. Describe what might be meant by each of those types.

A.3.4) Here is a standard subtraction algorithm:

$$\begin{array}{rrr} & 8 & \\ 8 & \cancel{9} & {}^{1}2 \\ -3 & 7 & 8 \\ \hline 5 & 1 & 4 \end{array}$$

Use base-ten blocks to model this algorithm. Which type of subtraction are you using?

A.3.5) Oscar uses base-ten blocks to model subtraction.



Can you explain what is going on? Which type of subtraction is Oscar using?

A.3.6) Create a “new” subtraction algorithm based on Oscar’s model.

A.3.7) Here is an example of a standard addition algorithm:

$$\begin{array}{r} {}^{1\,1} \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

Model this algorithm with base-ten blocks.

A.4 More Playing with Blocks

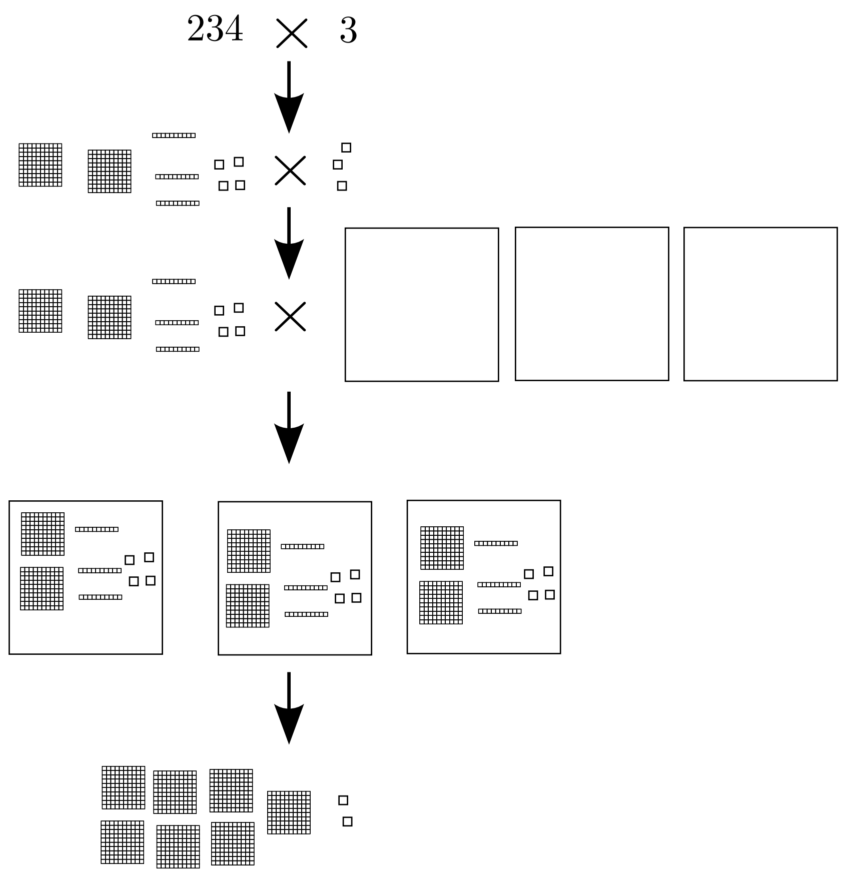
Teaching Note: In each of the following problems, use the blocks to explain each step in the algorithm. Then show the "behind the scenes algebra."

- Three copies of four blocks makes 12 longs. Regroup as 1 long and 2 blocks. Write 2 below in blocks column and 1 above in longs column.
- Three copies of three longs makes 9 longs. Plus 1 long (from regrouping) makes 10 longs. Regroup as 1 flat and 0 longs. Write 0 below in the longs column and 1 above in the flats column.
- Three copies of 2 flats makes 6 flats. Plus 1 flat (from regrouping) makes 7 flats. Write 7 below in the flats column.

$$
\begin{aligned}
234 \times 3 &= (2 \cdot 10^2 + 3 \cdot 10 + 4) \times 3 \\
&= 3 \cdot 2 \cdot 10^2 + 3 \cdot 3 \cdot 10 + 3 \cdot 4 \\
&= 3 \cdot 2 \cdot 10^2 + 3 \cdot 3 \cdot 10 + 12 \\
&= 3 \cdot 2 \cdot 10^2 + 3 \cdot 3 \cdot 10 + 10 + 2 \\
&= 3 \cdot 2 \cdot 10^2 + (3 \cdot 3 + 1) \cdot 10 + 2 \\
&= 3 \cdot 2 \cdot 10^2 + 10 \cdot 10 + 2 \\
&= (3 \cdot 2 + 1) \cdot 10^2 + 0 \cdot 10 + 2 \\
&= 7 \cdot 10^2 + 0 \cdot 10 + 2 \\
&= 702
\end{aligned}
$$

A.4.1) Now Oscar is modeling the basic multiplication algorithm:

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 234 \\ \times \quad 3 \\ \hline 702 \end{array}$$



Can you explain what is going on? What do you think of his model?

Fixnote: Maybe fix the pictures to show missing steps.

A.4.2) Here is an example of the basic division algorithm:

$$\begin{array}{r@{}l} & \;\;67 \text{ R1} \\ 3 & \overline{)202} \\ & \underline{18} \\ & 22 \\ & \underline{21} \\ & 1 \end{array}$$

Explain how to model this algorithm with base-ten blocks, assuming that you start with 202 as two flats and two blocks and that you intend to organize them into three equal piles.

Teaching Note:

- For $202 \div 3$, 3 doesn't go into 2 flats. Regroup the 2 flats as 20 longs. Then 3 groups gives 6 longs with for a total of 18 longs. After subtraction, 2 longs are remaining.
- Regroup the 2 remaining longs as 20 blocks and combine with the 2 blocks to give 22 blocks.
- Organize the blocks into 3 groups give 7 blocks in each group for a total of 21 blocks. After subtraction 1 block remains.

A.5 Comparative Arithmetic

Teaching Note: The point of the activity is that the properties that govern base-ten algorithms carry over to polynomials, but there is no carrying or borrowing. Ultimately, we want students to see polynomials as numbers in base x and to see base-ten numbers as polynomials in 10.

A.5.1) Compute:

$$\begin{array}{r} 131 \\ \underline{+122} \end{array} \quad \text{and} \quad \begin{array}{r} x^2+3x+1 \\ \underline{+x^2+2x+2} \end{array}$$

Compare, contrast, and describe your experiences.

A.5.2) Compute:

$$\begin{array}{r} 139 \\ \underline{+122} \end{array} \quad \text{and} \quad \begin{array}{r} x^2+3x+9 \\ \underline{+x^2+2x+2} \end{array}$$

Compare, contrast, and describe your experiences. In particular, discuss how this is different from the first problem.

A.5.3) Compute:

$$\begin{array}{r} 121 \\ \underline{\times 32} \end{array} \quad \text{and} \quad \begin{array}{r} x^2+2x+1 \\ \underline{\times \quad 3x+2} \end{array}$$

Compare, contrast, and describe your experiences.

A.5.4) Expand:

$$(x^2+2x+1)(3x+2)$$

Compare, contrast, and describe your experiences. In particular, discuss how this problem relates to the one above.

Fixnote: Need division examples.

A.6 Integer Addition and Subtraction

In this activity, we explore various models and strategies for making sense of addition and subtraction of integers.

Useful language

Addition and subtraction problems arise in situations where we add to, take from, put together, take apart, or compare quantities.

Recall that addition and subtraction facts are related. For example, if we know that $8 + 5 = 13$, then we also know three related facts: $5 + 8 = 13$, $13 - 8 = 5$, and $13 - 5 = 8$. In school mathematics, these are often called *fact families*.

A.6.1) What are integers? Describe some situations in which both positive and negative integers arise. Use the word “opposite” in your descriptions.

Red and black chips

A.6.2) In a red-and-black-chip model of the integers, red and black chips each count for 1, but they are opposites, so that they cancel each other out. Using language from accounting, suppose black chips are assets and red chips are debts. We add by putting chips together. Use red and black chips (or draw the letters R and B) to model the following computations.

(a) $(-5) + (-3)$

(b) $6 + (-4)$.

(c) $(-7) + 9$

(d) $2 + (-5)$

A.6.3) In the previous problem, you saw different combinations of red and black chips that had the same numerical value.

(a) How many ways are there to represent -3? Draw two different representations.

(b) Use the phrase “zero pairs” to describe how your two representations are related.

A.6.4) To subtract in the red-and-black-chip model, we can "take away" chips, as you might expect. When we don't have enough chips of a particular color, we can always add "zero pairs." Use this idea to model the following subtraction problems:

(a) $6 - 8$

(b) $4 - (-3)$

(c) $(-6) - 5$

(d) $(-3) - (-7)$

Subtraction as missing addend

A.6.5) To evaluate a subtraction expression, we can solve a related addition equation. For example, $11 - 7$ is the solution to $7 + \_\_ = 11$. Use this idea to evaluate the subtraction expressions in the previous problem.

Subtraction as difference on the number line

A.6.6) Use a number line to reason about $b - a$ by asking how to get from a to b: How far? And in which direction? For example, to evaluate $11 - 7$, we can ask how to get from 7 to 11. We travel 4 units to the right. Use this idea to evaluate the subtraction expressions in the previous problems.

A.6.7) How is subtraction different from negation?

A.6.8) Use what you have learned to explain why $a - (-b) = a + b$.

Other Models

Use the following models for addition and subtraction of integers. Each model requires two decisions: (1) how positive and negative integers are 'opposite' in the situation, and (2) how addition and subtraction are 'opposite' in a different way.

- A postal carrier who brings checks and bills—and who also takes them away.
- Walking on an North-South number line, facing either North or South, and walking either forward or backward.

A.7 Integer Multiplication

In this activity, we explore various models and strategies for making sense of multiplication of integers.

Continuing patterns

A.7.1)

(a) Continue the following patterns, and explain why it makes sense to continue them in that way.

$4 \times 3 = 12$	$3 \times 6 = 18$	$(-7) \times 3 = -21$
$4 \times 2 =$	$2 \times 6 =$	$(-7) \times 2 =$
$4 \times 1 =$	$1 \times 6 =$	$(-7) \times 1 =$
$4 \times 0 =$	$0 \times 6 =$	$(-7) \times 0 =$
$4 \times (-1) =$	$(-1) \times 6 =$	$(-7) \times (-1) =$
$4 \times (-2) =$	$(-2) \times 6 =$	$(-7) \times (-2) =$
$4 \times (-3) =$	$(-3) \times 6 =$	$(-7) \times (-3) =$

(b) What rule of multiplication might a student infer from the first pattern?

(c) What rule of multiplication might a student infer from the second pattern?

(d) What rule of multiplication might a student infer from the third pattern?

Using properties of operations

A.7.2) Suppose we *do not know* how to multiply negative numbers but we do know that $4 \times 6 = 24$. We will use this fact and the properties of operations to reason about products involving negative numbers.

(a) What do we know about A and B if $A + B = 0$?

(b) Use the distributive property to show that the expression $4 \times 6 + 4 \times (-6)$ is equal to 0. Then use that fact to reason about what $4 \times (-6)$ should be.

(c) Use the distributive property to show that the expression $4 \times (-6) + (-4) \times (-6)$ is equal to 0. Then use that fact to reason about what $(-4) \times (-6)$ should be.

Walking on a number line

A.7.3) Matt is a member of the Ohio State University Marching Band. Being rather capable, Matt can take x steps of size y inches for all integer values of x and y. If x is positive it means *face North and take x steps*. If x is negative it means *face South and take $|x|$ steps*. If y is positive it means your step is a *forward step of y inches*. If y is negative it means your step *is a backward step of $|y|$ inches*.

(a) Discuss what the expressions $x \cdot y$ means in this context. In particular, what happens if $x = 1$? What if $y = 1$?

(b) If x and y are both positive, how does this fit with the "repeated addition" model of multiplication?

(c) Using the context above and specific numbers, demonstrate the general rule:

$$\text{negative} \cdot \text{positive} = \text{negative}$$

Clearly explain how your problem shows this.

(d) Using the context above and specific numbers, demonstrate the general rule:

$$\text{positive} \cdot \text{negative} = \text{negative}$$

Clearly explain how your problem shows this.

(e) Using the context above and specific numbers, demonstrate the general rule:

$$\text{negative} \cdot \text{negative} = \text{positive}$$

Clearly explain how your problem shows this.

A.8 What Can Division Mean?

Teaching Note: Two models of division:

- "How many in one group?" questions use the "sharing" model of division.
- "How many groups?" questions use what is sometimes called the "measurement" model of division.

Division problems can have many types of answers, depending on the numbers and the context:

(a) exact division (i.e., no remainder)

(b) complete division (e.g., with a fractional part)

(c) quotient and remainder (i.e., two numbers with different meanings)

(d) just the quotient (i.e.,ignoring a non-zero remainder, rounding down)

(e) quotient rounded up

(f) just the remainder

Solve each of the problems below, explain your reasoning, and indicate whether the problem is asking "**How many in one group?**" or "**How many groups?**" or something else entirely.

A.8.1) There are a total of 35 hard candies. If there are 5 boxes with an equal number of candies in each box—and all the candy is accounted for, then how many candies are in each box? What if you had 39 candies?

A.8.2) There are a total of 28 hard candies. If there are 4 candies in each box, how many boxes are there? What if you had 34 candies?

A.8.3) There is a total of 29 gallons of milk to be put in 6 containers. If each container holds the same amount of milk and all the milk is accounted for, how much milk will each container hold?

A.8.4) There is a total of 29 gallons of milk to be sold in containers holding 6 gallons each. If all the milk is used, how many containers can be sold?

A.8.5) There is a total of 29 gallons of milk to be sold in containers holding 6 gallons each. If all the milk is used, how much milk cannot be sold?

A.8.6) If there are 29 kids and each van holds 6 kids, how many vans do we need for the field trip?

A.9 Divisibility Statements

Let $a|b$ mean $b = aq$ for some integer q. (Read $a|b$ as "a divides b".)

A.9.1) Using the numbers 56 and 7, make some true statements using the notation above and one or more of the words factor, multiple, divisor, and divides.

A.9.2) Use the definition of *divides* to decide which of the following are true and which are false. If a statement is true, find q satisfying the definition of divides. If it is false, give an explanation. (Hint: Try to reason about multiplication rather than using your calculator.)

(a) $21|2121$

(b) $3|(9 \times 41)$

(c) $6|(2^4 \times 3^2 \times 7^3 \times 13^5)$

(d) $100000|(2^3 \times 3^9 \times 5^{11} \times 17^8)$

(e) $6000|(2^{21} \times 3^7 \times 5^{17} \times 29^5)$

(f) $p^3q^5r|(p^5q^{13}r^7s^2t^{27})$

(g) $7|(5 \times 21 + 14)$

> Fixnote: Need some easier examples above. Below, we need at least one that isn't true. Use converses of some of these?

A.9.3) If $a|b$ and $a|c$ does $a|(bc)$? Explain.

A.9.4) If $a|b$ and $a|c$ does $a|(b+c)$? Explain.

A.9.5) If $a|(b+c)$ and $a|c$ does $a|b$? Explain.

A.9.6) Suppose that

$$(3^5 \cdot 7^9 \cdot 11^x \cdot 13^y)|(3^a \cdot 7^b \cdot 11^{19} \cdot 13^7)$$

What values of a, b, x, and y make true statements?

A.10 Hall of Shoes

A.10.1) *Incognito's Hall of Shoes* is a shoe store that just opened in Myrtle Beach, South Carolina. At the moment, they have 100 pairs of shoes in stock. At their grand opening 100 customers showed up. The first customer tried on every pair of shoes, the second customer tried on every 2nd pair, the third customer tried on every 3rd pair, and so on until the 100th customer, who only tried on the last pair of shoes.

(a) Which shoes were tried on by only 1 customer?

(b) Which shoes were tried on by exactly 2 customers?

(c) Which shoes were tried on by exactly 3 customers?

(d) Which shoes were tried on by exactly 4 customers?

(e) How many customers tried on the 45th pair?

(f) How many customers tried on the 81st pair?

(g) Challenge: Which shoes were tried on by the most customers?

In each case, explain your reasoning.

A.10.2) Which pairs of shoes were tried on by both

(a) customers 3 and 5?

(b) customers 6 and 8?

(c) customers 12 and 30?

(d) customers 7 and 13?

(e) customers a and b?

A.10.3) Which customers tried on both

(a) pairs 24 and 36?

(b) pairs 30 and 60?

(c) pairs 42 and 12?

(d) pairs 28 and 15?

(e) pairs a and b?

A.11 Sieving It All Out

A.11.1) Try to find all the primes from 1 to 120 *without* doing any division. Try to circle numbers that are prime and cross out numbers that are not prime. As a gesture of friendship, here are the numbers from 1 to 120.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Describe your method.

A.11.2) Now let's be systematic. Ignore 1 (we'll talk about why later). As you identify a prime, first circle it, then cross out its multiples that are not already crossed out. Keep track of your work so that you can answer the following questions:

(a) After circling a new prime, note the first number crossed out with that prime. Record your results in a table.

prime	first # crossed out
2	

(b) What was the biggest prime for which you crossed out at least one multiple?

Teaching Note: When being systematic, the first number crossed out should be the square of the circled prime. (All earlier multiples should have been crossed out because of smaller primes.)

A quick check for the carefulness of the process is to look at $119 = 7 \times 17$.

Side note: When the sieve is done in six columns, we can observe that any prime greater than 3 must be one more or one less than a multiple of 6.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

A.12 There's Always Another Prime

We'll start off with easy questions, then move to harder ones.

A.12.1) Use the Division Theorem to explain why neither 2 nor 3 divides $2 \cdot 3 + 1$. (Hint: Do not multiply and add. Use the expression as written to reason what the quotient and remainder must be.)

A.12.2) Use the Division Theorem to explain why neither 2 nor 3 nor 5 divides $2 \cdot 3 \cdot 5 + 1$.

A.12.3) Let $p_1, \ldots, p_n$ be the first n primes. Do any of these primes divide

$$p_1 p_2 \cdots p_n + 1?$$

Explain your reasoning.

A.12.4) Suppose there were only a finite number of primes, say there were only n of them. Call them $p_1, \ldots, p_n$. Could any of them divide

$$p_1 p_2 \cdots p_n + 1?$$

what does that mean? Can there really only be a finite number of primes?

A.12.5) Consider the following:

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509$$

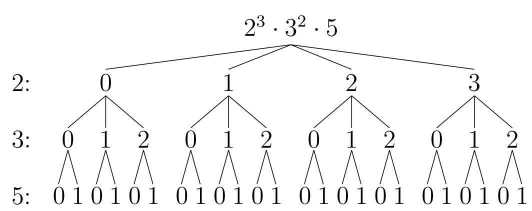
Does this contradict our work above? If so, explain why. If not, explain why not.

A.13 There Are Many Factors to Consider

Fixnote: Revise. List all factors of several numbers. Need an example where being systematic by consecutive numbers (rather than by using prime factorization) is inefficient. List all the factors of 60 with their prime factorization.

A.13.1) How many factors does the integer 60 have?

A.13.2) Consider the following diagram:



What is going on in this diagram? What do the numbers represent? How does it help you count the number of factors of $2^3 \cdot 3^2 \cdot 5$?

A.13.3) Make a similar diagram for 60.

A.13.4) Can you devise a method for computing the number of factors that a number has? Explain why your method works.

A.13.5) How many factors does 735 have?

A.13.6) If p is a prime number, how many factors does p^n have?

A.13.7) If p and q are both prime numbers, how many factors does $p^n q^m$ have?

A.13.8) Which integers between 0 and 100 have the most factors?

A.14 Why Does It Work?

The Euclidean Algorithm is pretty neat. Let's see if we can figure out **why** it works. As a gesture of friendship, I'll compute gcd(351, 153):

$$\begin{aligned} 351 &= 153 \cdot 2 + 45 \\ 153 &= 45 \cdot 3 + 18 \\ 45 &= 18 \cdot 2 + \boxed{9} \\ 18 &= 9 \cdot 2 + 0 \qquad \boxed{\therefore \gcd(351, 153) = 9} \end{aligned}$$

Let's look at this line-by-line.

The First Line

A.14.1) Since $351 = 153 \cdot 2 + 45$, explain why $\gcd(153, 45)$ divides 351.

A.14.2) Since $351 = 153 \cdot 2 + 45$, explain why $\gcd(351, 153)$ divides 45.

A.14.3) Since $351 = 153 \cdot 2 + 45$, explain why $\gcd(351, 153) = \gcd(153, 45)$.

The Second Line

A.14.4) Since $153 = 45 \cdot 3 + 18$, explain why $\gcd(45, 18)$ divides 153.

A.14.5) Since $153 = 45 \cdot 3 + 18$, explain why $\gcd(153, 45)$ divides 18.

A.14.6) Since $153 = 45 \cdot 3 + 18$, explain why $\gcd(153, 45) = \gcd(45, 18)$.

The Third Line

A.14.7) Since $45 = 18 \cdot 2 + 9$, explain why $\gcd(18, 9)$ divides 45.

A.14.8) Since $45 = 18 \cdot 2 + 9$, explain why $\gcd(45, 18)$ divides 9.

A.14.9) Since $45 = 18 \cdot 2 + 9$, explain why $\gcd(45, 18) = \gcd(18, 9)$.

The Final Line

A.14.10) Why are we done? How do you know that the Euclidean Algorithm will **always** terminate?

Fixnote: New question: What does the final line look like when the GCD is 1?

A.15 Prome Factorization

Teaching Note: In the course, we first assume Euclid's Lemma and use it to prove the Fundamental Theorem of Arithmetic (FTA). In this activity, both Euclid's Lemma and the FTA fail.

It might help to begin the class with the following questions:

- If $7|ab$ (where a and b integers), does it follow that 7 must divide either a or b?
- If $14|ab$ (where a and b integers), does it follow that 14 must divide either a or b?

Through discussion, students should decide that the answers are "yes" and "no," respectively, and the reason is that 7 is prime but 14 is not. Some students should realize that, in the second case, the factors of 2 and 7 (of 14) might be "split" between a and b. This realization can be stated as Euclid's Lemma:

Suppose a and b are integers and p is prime. If $p|ab$, then $p|a$ or $p|b$.

Students are not be responsible for its name. At this point, we accept it without proof.

The purpose of questions 1-4 is to see that this "new" number system is very much like the integers: You can always add, subtract, and multiply, but you cannot necessarily divide. Students can use their reasoning about integers to explain these facts about the system $3\mathbb{Z}$.

The purpose of questions 5-7 is to notice that both Euclid's Lemma and Unique Factorization fail in this number system. Some examples:

$$36 = 3 \times 13 = 6 \times 6$$

$$72 = 3 \times 24 = 6 \times 12$$

Let's consider a crazy set of numbers—all multiples of 3. Let's use the symbol $3\mathbb{Z}$ to denote the set consisting of all multiples of 3. As a gesture of friendship, I have written down the first 100 nonnegative integers in $3\mathbb{Z}$:

0	3	6	9	12	15	18	21	24	27
30	33	36	39	42	45	48	51	54	57
60	63	66	69	72	75	78	81	84	87
90	93	96	99	102	105	108	111	114	117
120	123	126	129	132	135	138	141	144	147
150	153	156	159	162	165	168	171	174	177
180	183	186	189	192	195	198	201	204	207
210	213	216	219	222	225	228	231	234	237
240	243	246	249	252	255	258	261	264	267
270	273	276	279	282	285	288	291	294	297

A.15.1) Given any two integers in $3\mathbb{Z}$, will their sum be in $3\mathbb{Z}$? Explain your reasoning.

A.15.2) Given any two integers in $3\mathbb{Z}$, will their difference be in $3\mathbb{Z}$? Explain your reasoning.

Teaching Note: Yes, but students might need to be reminded that $3\mathbb{Z}$ includes negative integers.

A.15.3) Given any two integers in $3\mathbb{Z}$, will their product be in $3\mathbb{Z}$? Explain your reasoning.

A.15.4) Given any two integers in $3\mathbb{Z}$, will their quotient be in $3\mathbb{Z}$? Explain your reasoning.

Definition Call a positive integer **prome** in $3\mathbb{Z}$ if it cannot be expressed as the product of two integers *both* in $3\mathbb{Z}$.

As an example, I tell you that 6 is prome number in $3\mathbb{Z}$. You may object because $6 = 2 \cdot 3$, but remember—2 is not in $3\mathbb{Z}$!

A.15.5) List some of the prome numbers less than 297. Hint: What numbers in $3\mathbb{Z}$ *can* be expressed as a product of two integers *both* in $3\mathbb{Z}$?

A.15.6) Can you give some sort of algebraic characterization of prome numbers in $3\mathbb{Z}$?

A.15.7) Can you find numbers that factor completely into prome numbers in *two* different ways? How many can you find?

A.16 Picture Models for Equivalent Fractions

Teaching Note:

Step 1. Use the first problem (use paper to show 3/8) to generate the meaning of fraction from the Common Core State Standards:

> 3.NF.1. Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.
>
> Source: http://www.corestandards.org/Math/Content/3/NF/A/1/

The code 3.NF.1 means "third grade, number and operations–fractions, standard 1." These standards are written to be read by teachers, not students.

Step 2. Introduce the formal definition of rational number and the set of rational numbers, as in the beginning of section 2.4.

A rational number can be represented as a/b with integers a and b, where b is not 0.

Distinguish fraction (a representation) from rational number, highlighting the phrase "can be" in the definition. Have students generate fractions that are not rational numbers as well as rational numbers not represented as fractions.

Introduce the letter $\mathbb{Q}$ (usually in "black-board bold" font) to denote the set of all rational numbers.

Step 3. Complete Activity A.16. The point is to use the meaning of fractions above to explain why fractions are equivalent. And the approach is "reasoning generally with specific numbers."

A.16.2. For 2/3 = 4/6, some students will be tempted to draw 2/3, draw 4/6 and then say, "See!" With this method, it is not clear why the pieces should line up. Much better to use 2/3 to create 4/6 by cutting each of the thirds into two equal pieces.

A.16.3. For 3/6 = 2/4, some students will be tempted to say "Because they both equal 1/2." To explain why the pieces will have to line up, it is clearer (and more general) to go through a common denominator, such as 12ths or 24ths.

A.16.4. To show that $a/b = c/d$, generalize the approach from the previous problem: Thinking of the common denominator bd, cut the a/b parts each into d parts. Then we have ad parts of size $1/(bd)$. Cut the c/d parts each into b parts. Then we have cb parts of size $1/(db)$. For the two fractions to be equal, the ad parts of size $1/(bd)$ must be equal to the cb parts of size $1/(db)$.

A.16.5. In the picture from A.16.4, because the parts are the same size (i.e., $1/(bd)$), it must follow that $ad = bc$. (Argue both directions: if the fractions are equal, then $ad = bc$; if $ad = bc$, then the fractions must be equal.)

A.16.1) Get out a piece of paper and show $\frac{3}{8}$. Explain how you know.

A.16.2) Draw pictures to explain why:

$$\frac{2}{3} = \frac{4}{6}$$

Explain how your pictures show this.

A.16.3) Draw pictures to explain why:

$$\frac{3}{6} = \frac{2}{4}$$

Explain how your pictures show this.

A.16.4) Given equivalent fractions with $0 < a \leqslant b$ and $0 < c \leqslant d$:

$$\frac{a}{b} = \frac{c}{d}$$

Give a procedure for representing this equation with pictures.

A.16.5) Explain, without cross-multiplication, why if $0 < a \leqslant b$ and $0 < c \leqslant d$:

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc$$

Feel free to use pictures as part of your explanation.

A.17 Picture Models for Fraction Operations

A.17.1) Draw pictures that model:

$$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

Explain how your pictures show this. Write a story problem whose solution is given by the expression above.

A.17.2) Draw pictures that model:

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$

Explain how your pictures model this equation. Be sure to carefully explain how common denominators are represented in your pictures. Write a story problem whose solution is given by the expression above.

A.17.3) Given $0 < a \leqslant b$ and $0 < c \leqslant d$, explain how to draw pictures that model the sum:

$$\frac{a}{b} + \frac{c}{d}$$

Use pictures to find this sum and carefully explain how common denominators are represented in your pictures.

A.17.4) Given positive integers a and b, explain how to draw pictures that model the product $a \cdot b$—give an example of your process.

A.17.5) Draw pictures that model:

$$\frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$$

Explain how your pictures model this equation. Write a story problem whose solution is given by the expression above. Does your story work with

$$\frac{7}{5} \cdot \frac{2}{3} = \frac{14}{15}?$$

A.17.6) Given $0 < a \leqslant b$ and $0 < c \leqslant d$, explain how to draw pictures that model the product:

$$\frac{a}{b} \cdot \frac{c}{d}$$

Use pictures to find this product and explain how this product is shown in your pictures—give an example of your process.

A.18 Fraction Multiplication

A.18.1) Suppose x and y are counting numbers.

(a) What is our convention for the meaning of xy as repeated addition?

(b) In our convention for the meaning of the product xy, which letter describes *how many groups* and which letter describes *how many in one group*?

(c) In the product xy, the x is called the *multiplier* and y is called the *multiplicand*. Use these words to describe the meaning of xy as repeated addition.

A.18.2) In the Common Core State Standards, fractions and fraction operations are built from *unit fractions*, which are fractions with a 1 in the numerator. The meaning of a fraction $\frac{a}{b}$ involves three steps: (1) determining the whole; (2) describing the meaning of $\frac{1}{b}$; and (3) describe the meaning of the fraction $\frac{a}{b}$. Use pictures to illustrate these three steps for the fraction $\frac{3}{5}$.

A.18.3) Now we combine the ideas from the previous two problems to describe meanings for simple multiplication of fractions.

(a) Without computing the result, describe the meaning of the product $5 \times \frac{1}{3}$.

(b) Without computing the result, describe the meaning of the product $\frac{1}{3} \times 5$.

(c) Without using the commutativity of multiplication (which we have not established for fractions), use these meanings and pictures to explain what the products should be.

A.18.1 Area Models

Fixnote: Incorporate story problems. Make use of units.

A.18.4) Beginning with a unit square, use an area model to illustrate the following:

(a) $\frac{1}{3} \times \frac{1}{4}$

(b) $\frac{7}{3} \times \frac{5}{4}$

A.18.5) When computing $2\frac{1}{3} \times 3\frac{2}{5}$, Byron says that the answer is $6\frac{2}{15}$.

(a) Explain Byron's method.

(b) How do you know that he is incorrect?

(c) Use what is right about his method to show what he is missing.

A.19 Flour Power

A.19.1) Suppose a cookie recipe calls for 2 cups of flour. If you have 6 cups of flour total, how many batches of cookies can you make?

(a) Draw a picture representing the situation, and use pictures to solve the problem.

(b) Identify whether the problem is asking "How many groups?" or "How many in one group?" or something else entirely.

(c) You find another recipe that calls for $1\frac{1}{2}$ cups per batch. If you have 6 cups of flour, how many batches of these cookies can you make? Again use pictures to solve the problem.

(d) Somebody once told you that "to divide fractions, you invert and multiply." Discuss how this rule is manifested in this problem.

A.19.2) You have 2 snazzy stainless steel containers (both the same size), which hold a total of 6 cups of flour. How many cups of flour does 1 container hold?

(a) Draw a picture representing the situation, and use pictures to solve the problem.

(b) Identify whether the problem is asking "How many groups?" or "How many in one group?" or something else entirely.

(c) It turned out that the 6 cups of flour filled exactly $1\frac{1}{2}$ of your containers. How many cups of flour does 1 container hold? Again use pictures to solve the problem.

(d) Somebody once told you that "to divide fractions, you invert and multiply." Discuss how this rule is manifested in this problem.

A.20 Picture Yourself Dividing

We want to understand how to visualize

$$\frac{a}{b} \div \frac{c}{d}$$

Let's see if we can ease into this like a cold swimming pool.

A.20.1) Draw a picture that shows how to compute:

$$10 \div 5$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in one group."

A.20.2) Try to use a similar process to the one you used in the first problem to draw a picture that shows how to compute:

$$\frac{1}{4} \div 3$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in one group."

A.20.3) Try to use a similar process to the one you used in the first two problems to draw a picture that shows how to compute:

$$3 \div \frac{1}{4}$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in one group."

Fixnote: Also incorporate $1\frac{3}{4} \div \frac{1}{2}$

A.20.4) Try to use a similar process to the one you used in the first three problems to draw a picture that shows how to compute:

$$\frac{7}{5} \div \frac{3}{4}$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in each group."

A.20.5) Explain how to draw pictures to visualize:

$$\frac{a}{b} \div \frac{c}{d}$$

A.20.6) Use pictures to explain why:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

A.21 Cross Something-ing

A.21.1) What might someone call the following statements:

(a) $\dfrac{a}{b} = \dfrac{c}{d} \Rightarrow ad = bc$

(b) $\dfrac{a}{b} \cdot \dfrac{b}{c} = \dfrac{a}{c}$

(c) $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{ad}{bc}$

(d) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$

(e) $ad < bc \Rightarrow \dfrac{a}{b} < \dfrac{c}{d}$

(f) $ad < bc \Rightarrow \dfrac{c}{d} < \dfrac{a}{b}$

A.21.2) Which of the above statements are true? What specific name might you use to describe them?

A.21.3) Use pictures to help explain why the true statements above are true and give counterexamples showing that the false statements are false.

A.21.4) Can you think of other statements that should be grouped with those above?

A.21.5) If mathematics is a subject where you should strive to "say what you mean and mean what you say," what issue might arise with cross-multiplication?

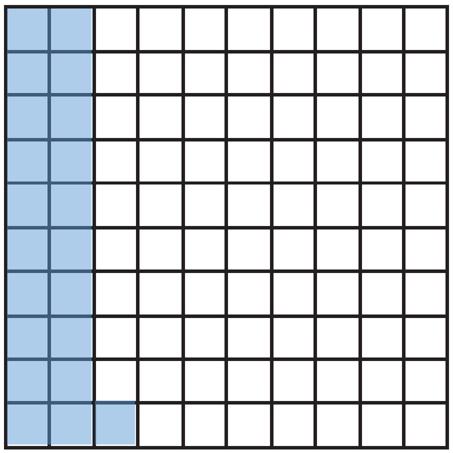
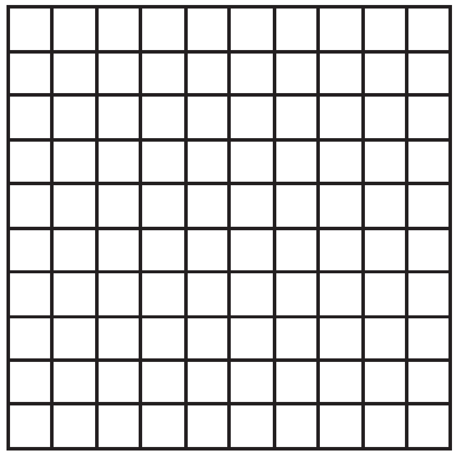
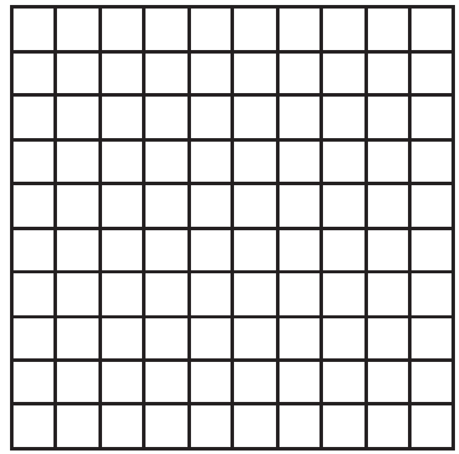
A.22 Hundredths Grids for Rational Numbers

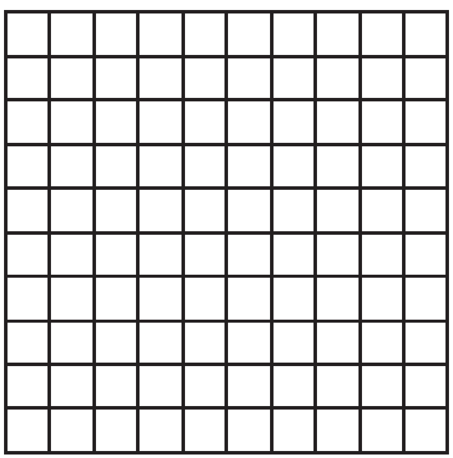
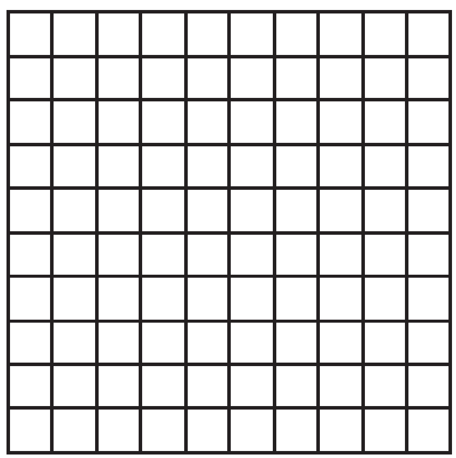
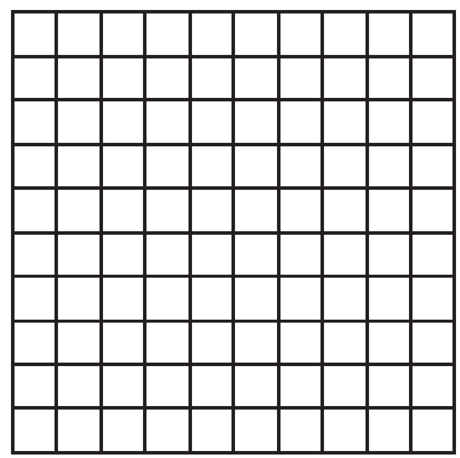
When a 10×10 square is taken to be 1 whole, it can be used as a "hundredths grid" to represent fractions and decimals between 0 and 1.•

• For example, one of the grids below is shaded to represent $\frac{21}{100}$.

A.22.1) Shade the hundredths grids to show each of the following fractions. Then use your shading to determine a decimal equivalent for each fraction.

(a) $\frac{3}{20}$ (b) $\frac{1}{8}$ (c) $\frac{1}{6}$ (d) $\frac{7}{12}$

A.23 Shampoo, Rinse, . . .

We're going to investigate the following question: If a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b?

As a middle school teacher, you should know from memory the decimal equivalents of many fractions, and you should be able to compute others quickly in your head. Use this activity to hone this skill, and use your calculator as backup support.

A.23.1) Complete the following table. For type, write "T" for "Terminating," and use other letters for other types you observe.

Fraction	Decimal	Type
$\frac{1}{2}$		
$\frac{1}{3}$		
$\frac{1}{4}$		
$\frac{1}{5}$		
$\frac{1}{6}$		
$\frac{1}{7}$		
$\frac{1}{8}$		
$\frac{1}{9}$		
$\frac{1}{10}$		
$\frac{1}{11}$		
$\frac{1}{12}$		
$\frac{1}{13}$		
$\frac{1}{14}$		
$\frac{1}{15}$		
$\frac{1}{16}$		
$\frac{1}{20}$		
$\frac{1}{24}$		
$\frac{1}{25}$		
$\frac{1}{28}$		
$\frac{1}{32}$		
$\frac{1}{35}$		
$\frac{1}{40}$		
$\frac{1}{42}$		
$\frac{1}{48}$		
$\frac{1}{64}$		
$\frac{1}{80}$		

A.23.2) Can you find a pattern from your results from Problem A.23.1? Use your pattern to guess whether the following fractions “terminate”?

$$\frac{1}{61} \qquad \frac{1}{625} \qquad \frac{1}{6251}$$

A.23.3) Can you explain why your conjecture from Problem A.23.2 is true?

A.23.4) Now let’s consider fractions with decimal representations that do not terminate.

(a) Use long division to compute $1/7$.

(b) State the Division Theorem for integers.

(c) How does the Division Theorem for integers appears in your computation for $1/7$?

(d) In each instance of the Division Theorem, what is the divisor? And what does this imply about the remainder?

(e) Generalize: When a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b, assuming it does not terminate? Explain your reasoning.

A.23.5) Compute $\frac{1}{9}$, $\frac{1}{99}$, and $\frac{1}{999}$. Can you find a pattern? Can you explain why your pattern holds?

Teaching Note: From long division, $\frac{1}{9} = 0.\overline{1}$, $\frac{1}{99} = 0.\overline{01}$, and $\frac{1}{999} = 0.\overline{001}$.

A.23.6) Use your work from Problem A.23.5 to give the fraction form of the following decimals:

(a) $0.\overline{357}$

(b) $23.\overline{459}$

(c) $0.23\overline{4598}$

(d) $76.3\overline{421}$

Teaching Note: The technique is to reason from decimal multiplication as follows:

$$0.\overline{357} = 357 \times 0.\overline{001} = 357 \times \frac{1}{999}$$

No need to simplify these anticipated solutions:

(a) $0.\overline{357} = \dfrac{357}{999}$

(b) $23.\overline{459} = 23 + \dfrac{459}{999}$

(c) $0.23\overline{4598} = \dfrac{23}{100} + \dfrac{1}{100} \times \dfrac{4598}{9999}$

(d) $76.3\overline{421} = 76 + \dfrac{3}{10} + \dfrac{1}{10} \times \dfrac{421}{999}$

A.23.7) Assuming that the pattern holds, is the number

$$.1234567891011121314151617181920 21\ldots$$

a rational number? Explain your reasoning.

Teaching Note: Reasoning from the finite list of remainders, the decimal representation of any rational number either terminates or (eventually) repeats. In this problem, the number shows an interesting and predictiable pattern, but it does not show a sequence of digits that appears in exactly the same way again and again.

A.24 Decimals Aren't So Nice

We will investigate the following question: How is $0.999\ldots$ related to 1?

A.24.1) What symbol do you think you should use to fill in the box below?

$$.999\ldots \boxed{}\, 1$$

Should you use $<$, $>$, $=$ or something else entirely?

A.24.2) What is $1 - .999\ldots$?

A.24.3) How do you write $1/3$ in decimal notation? Express

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

in both fraction and decimal notation.

A.24.4) See what happens when you follow the directions below:

(a) Set $x = .999\ldots$.

(b) Compute $10x$.

(c) Compute $10x - x$.

(d) From the step immediately above, what does $9x$ equal?

(e) From the step immediately above, what does x equal?

A.24.5) Are there other numbers with this weird property?

A.25 Ratios and Proportional Relationships

Here begins our work with ratios and proportional reasoning, which are the cornerstone of middle school mathematics. Try to avoid procedural approaches, such as, "set up a proportion and cross multiply." Instead, try to reason from the context and **use pictures and tables to support your reasoning**.

As you solve these problems, note how the problems simultaneously build on understandings of fractions and pave the way for functions.

Stacking Paper

A.25.1) Suppose you want to know how many sheets are in a particular stack of paper, but don't want to count the pages directly. You have the following information:

- The given stack has height 4.50 cm.
- A ream of 500 sheets has height 6.25 cm.

How many sheets of paper do you think are in the given stack?

> **Teaching Note:** Points to make:
>
> - To solve this problem, many people write a proportion and cross multiply, which might be fine if the only goal is the answer. But writing a proportion and cross multiplying misses opportunities for proportional reasoning.
> - Draw out unit rates (80 sheets/cm and 0.0125 cm/sheet) and scale factors.
> - Draw out quantities that are in a proportional relationship and write equations relating them.
> - What is proportional to what?

A.25.2) In your solution to the previous problem, what did you assume was proportional to what other quantity? Be precise.

Mixing Punch

A.25.3) Jenny is mixing punch and is considering two recipes:

- Recipe A: 3 parts orange juice for every 5 parts ginger ale
- Recipe B: 2 parts orange juice for every 3 parts ginger ale

(a) Which recipe will give juice that is the most "orangey"? Explain your reasoning.

(b) Use a table to show various ways to make recipe B.

(c) To make 12 gallons of recipe B, how much of each will you need?

> **Teaching Note:** Draw out part:part versus part:whole comparisons, using quotients, common numerators, and common denominators. In all cases, interpret the fractions and quotients.
>
> Use ratio tables, graphs, etc. Note that graphs could be made relating any two of the three quantities (orange juice, ginger ale, punch).
>
> Draw out various unit rates: 3/5 parts orange juice for every 1 part of ginger ale...

Racing Snails

A.25.4) Mike is racing snails that move at a constant speed:

- Snail A travels 3 inches in 5 minutes.
- Snail B travels 2 inches in 3 minutes.

(a) Which snail moves faster? Explain your reasoning.

(b) Use a table to show other distances and times for snail B.

Teaching Note: These problems are very much the same as the previous problems. But this time, the units are of different types, and they don't combine to make a new whole.

Draw out both 2/3 in/min and 1.5 min./in as meaningful unit rates.

Ratios are sometimes represented by fractions, but there is an important distinction: A fraction is a single number, whereas a ratio is often conceived as a relationship between two quantities.

A.26 Poor Old Horatio

A.26.1) A shade of orange is made by mixing 3 parts red paint with 5 parts yellow paint. Sam says we can add 4 cups of each color of paint and maintain the same color. Fred says we can quadruple both 3 and 5 and get the same color.

(a) Who (if either or both) is correct? Explain your reasoning.

(b) Use a table like the one below to show other paint mixtures that are the desired shade of orange.

Red	3							
Yellow	5							

A.26.2) If we wanted to make the same orange paint but were required to use 73 cups of yellow paint, how many cups of red paint would we need? Explain your reasoning.

Red	3				
Yellow	5				

A.26.3) If we wanted to make the same orange paint but were required to use 56 cups of red paint, how many cups of yellow paint would we need? Explain your reasoning.

Red	3				
Yellow	5				

A.26.4) Generalize your approaches to the previous problems.

(a) Give a general formula for computing how much red paint is needed when y cups of yellow paint is used.

(b) Give a general formula for computing how much yellow paint is needed when r cups of red paint is used.

Red	3				
Yellow	5				

A.26.5) Now suppose we want to make a **different shade** of orange, this time made with $\frac{3}{4}$ cup of red paint and $\frac{2}{3}$ cup of yellow paint. How many cups of each color do you need in order to make 15 cups of the mixture? Use the table below.

Red	$\frac{3}{4}$				
Yellow	$\frac{2}{3}$				
		17	1	15	

Fixnote: Need to distinguish ratios and rates. Maybe earlier distinguish part/part to part/whole.

A.26.6) In proportional reasoning problems, a *unit rate* describes the amount of one quantity for 1 unit of another quantity.

(a) What are the units for the various numbers in these problems?

(b) Identify some unit rates in this activity.

(c) In solving the above problems, it is likely that you or your classmates use strategies that made use of unit rates on the way to your solution. Explain why this strategy is sometimes called *going through one*.

Fixnote: Revise these problems drawn from Beckmann. Use dollars/pound, or meters/second, etc.

A.26.7) If $2\frac{1}{2}$ pints of jelly fills $3\frac{1}{2}$ jars, then how many jars will you need for 12 pints of jelly? (Assume the jars are all the same size.) If the last jar is not totally full, indicate how full it will be.

Jelly								
Jars								

A.27 Ratio Oddities

In this activity we are going to investigate thinking about and adding ratios.

> **Teaching Note:** Distinguish part/part from part/whole ratios.
>
> When writing equations, be sure to work through typical wrong answers: (1) using letters as units versus the number of boys or girls; and (2) saying $3x = 4x$ to indicate batches.
>
> There is no need to reduce the ratios in the answers. Yet where appropriate, encourage answers with variables in them.

A.27.1) There are 3 boys for every 4 girls in Mrs. Sanders' class.

(a) What fraction of the class are girls?

(b) List ratios that can describe this situation.

(c) If each of the number of boys and number of girls quadruples, what is the new ratio of girls to boys?

(d) Write an equation relating the number of boys in the class to the number of girls in the class.

(e) If the number of boys and number of girls each increase by 6, what can you say about the new ratio of boys to girls?

A.27.2) Suppose the ratio of girls to boys in Smith's class is 7:3 while the ratio of girls to boys in Jones' class is 6:5.

(a) If there are 50 students in Smith's class and 55 students in Jones' class, and both classes get together for an assembly, what is the ratio of girls to boys? Explain your reasoning.

(b) What if there are 500 students in Smith's class and still 55 students in Jones' class?

(c) What if there are 5000 students in Smith's class and still 55 students in Jones' class?

(d) How do the ratios of girls to boys in the combined assembly compare to the ratios of girls to boys in the original classes?

(e) Now suppose you don't know how many students are in Smith's class and there are 55 students in Jones' class. What can you say about the ratio of girls to boys at the assembly?

A.27.3) Suppose you are teaching a class, and a student writes

$$\frac{1}{4} + \frac{3}{5} = \frac{4}{9}$$

(a) How would you respond to this?

(b) This student is most contrary, and presents you with the following problem:

> Suppose you have two cars, a 4 seater and a 5 seater. If the first car is 1/4 full and the second car is 3/5 full, how full are they together?

The student then proceeds to answer their question with "The answer is 4/9." How do you address this?

Teaching Note: You might want to ask what happens if the first car is 1/4 full and 6/10 full; or suggest the first car is 2/8 full, because 2/8 = 1/4.

(c) This student's reasoning suggest a new kind of "addition" of ratios. Let's use $\oplus$ for this new form of "addition." So

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

For which of the previous problems is does this "addition" give the correct answer? What is going on?

(d) Use the student's context of seats and cars to reason about how $\frac{a}{b} \oplus \frac{c}{d}$ compares with $\frac{a}{b}$ and $\frac{c}{d}$.

A.27.4) Let's think a bit more about $\oplus$. If you were going to plot $\frac{a}{b}$ and $\frac{c}{d}$ on a number line, where iswhat can you say about the location of $\frac{a}{b} \oplus \frac{c}{d}$? Is this always the case, or does it depend on the values of a, b, c, and d? *Hint*: Assume that all of the letters are positive. Use specific numbers and a context; then try to reason generally.

Teaching Note: Here you will probably not only want to have the students realize that $\frac{a}{b} \oplus \frac{c}{d}$ is between both $\frac{a}{b}$ and $\frac{c}{d}$, but that the location varies by which denominator is largest.

Another approach is to compare slopes of vectors (b, a), (d, c), and $(b+d, a+c)$, all originating at the origin. Through specific examples, students can reason that the vector sum (and therefore its slope) is between the others.

A.28 The Triathlete

A.28.1) On Friday afternoon, just as Laine got off the bus, she realized that she had left her bicycle at school. In order to have her bicycle at home for the weekend, she decided to run to school and then ride her bike back home. If she averaged 6 mph running and 12 mph on her bike, what was her average speed for the round trip? Explain your reasoning.

Teaching Note: A key idea here is that the average speed is independent of the distance. Here are several ways that students can solve it:

- Pick a distance, say 12 miles. Then running to school will take 2 hours, and biking back home will take 1 hour. That's a total of 24 miles in 3 hours, or an average of 8 mph.
- Let the distance be d. Then running to school will take $\frac{d}{6}$ hours, and biking back home will take $\frac{d}{12}$ hours. The total distance is $2d$. So the average rate is

  $$\frac{2d}{\frac{d}{6}+\frac{d}{12}} = \frac{2}{\frac{1}{6}+\frac{1}{12}} = \frac{2}{\frac{3}{12}} = \frac{2}{\frac{1}{4}} = 8 \text{ mph}$$

  Notice that the d factors out of both the numerator and the denominator (and hence cancels), which shows that the average speed is independent of the distance.

  Notice also that this calculation can be expressed as a different kind of average:

  $$\frac{\frac{1}{6}+\frac{1}{12}}{2} = \frac{1}{8}$$

  This average is called the *harmonic mean*. Specifically, 8 is the harmonic mean of 6 and 12 because it is the reciprocal of the average of their reciprocals. (Math 1165 students do not need to know this language.)
- Sometimes is helpful to think of speed as "time per distance," which is the reciprocal of "distance per time." In this problem, we can reason that

Lanie runs a “ten-minute mile" as follows: Her speed of 6 mph would be $\frac{1}{6}$ hour/mile, which is the same as 10 min/mile. Similarly, she bikes at 5 min/mile. With this insight, we can cut the distance between home and school into 1-mile pieces and reason that she will take 10 minutes to run each mile and 5 minutes to bike the same mile on the way home. That would be 15 minutes for both directions (2 miles), for an average speed of 7.5 min/mile, which is the same as 8 mph.

A.28.2) On Saturday, Laine completed a workout in which she split the time evenly between running and cycling. If she again averaged 6 mph running and 12 mph on her bike, what was her average speed for the workout? Explain your reasoning.

Teaching Note: Here the naive calculations works: The average speed is the average of the two speeds, so the answer is $\frac{6+12}{2} = 9$ mph. But we should be clear why this works. Here are two approaches:

- Pick a time, say 1 hour, to spend on each activity. Lanie will run 6 miles in 1 hour and will bike 12 miles in 1 hour. That will be 18 miles in 2 hours, or an average of 9 mph. This will work, of course, for every hour of activity, which suggests that the result is independent of time.
- Algebra: Let the t represent the time spent on each activity. The distance running will be $6t$, the distance biking will be $12t$, and the total time will be $2t$. So the average speed will be

$$\frac{6t + 12t}{2t} = \frac{18t}{2t} = 9 \text{ mph.}$$

  Notice the common factor of t cancels, which shows that the average speed is independent of the time.

A.28.3) Why was her average speed on Saturday different from her average speed on Friday? Can you reason, without computation, which average speed should be faster?

> **Teaching Note:** One approach: When the times are the same, the average will be midway between the two speeds. When the distances are the same, in contrast, she will spend more time traveling at the slower speed than at the faster speed, so the average will be closer to the slower speed, which implies that the same-distance average is slower than the same-time average.

A.28.4) On Sunday, Laine's workout included swimming. Assuming that she can swim at an average speed of 2 mph, describe two running-cycling-swimming workouts, one similar to Friday's scenario (same distance) and a second similar to Saturday's (same time). Compute the average speed for each and explain your reasoning.

> **Teaching Note:** Same distance (a la Friday):
>
> $$\text{Average speed } = \frac{3d}{\frac{d}{2}+\frac{d}{6}+\frac{d}{12}} = \frac{3}{\frac{1}{2}+\frac{1}{6}+\frac{1}{12}} = \frac{3}{\frac{9}{12}} = \frac{3}{\frac{3}{4}} = 4 \text{ mph.}$$
>
> Same time (a la Saturday):
>
> $$\text{Average speed } = \frac{2t+6t+12t}{3t} = \frac{20t}{3t} = 6\frac{1}{3} \text{ mph.}$$

A.28.5) Which of the workout scenarios (same distance or same time) most closely resembles an actual triathlon? Why do you think that is the case?

> **Teaching Note:** In actual triathlons, the running distances are much shorter than the biking distances, and the swimming distances are much shorter still. It would not be reasonable to swim any reasonable biking distance. So the "same time" scenario is closer. But in reality, the swimming times are quite a bit shorter than the running and biking times.

A.28.6) After two months of intense training, Laine is able to average s mph swimming, r mph running, and c mph cycling. Again describe two running-cycling-swimming workouts, one similar to each of the two original scenarios, and compute her average speeds.

Teaching Note: Same distance (a la Friday):

$$\text{Average speed} = \frac{3d}{\frac{d}{s}+\frac{d}{r}+\frac{d}{c}} = \frac{3}{\frac{1}{s}+\frac{1}{r}+\frac{1}{c}}$$

Same time (a la Saturday):

$$\text{Average speed} = \frac{st+rt+ct}{3t} = \frac{(s+r+c)t}{3t} = \frac{s+r+c}{3}$$

A.29 The Dreaded Story Problem

Let's try our hand at a problem involving ratios.

> **Teaching Note:** This problem is challenging. Naive "solutions" are likely to be wrong.

A.29.1) On orders from his doctor, every day, Marathon Marty must run from his house to a statue of Millard Fillmore and run back home along the same path. So Marty doesn't lollygag, the doctor orders him to average 8 miles per hour for the round trip.

(a) On Monday, Marty ran into Gabby Gilly on his way to the statue and averaged only 6 miles per hour for the trip out to the statue.What must Marty do to ensure he's obeyed his doctor's orders?

> **Teaching Note:** Here is a sketch of an algebraic solution.
>
> Identify the following constants and variables:
>
> - 6 mph = the rate traveling from home to the statue
> - 8 mph = the average rate traveling for the round trip
> - x = the rate traveling from the statue back home
> - d = the distance to from home to the statue
> - t_1 = the time traveling from home to the statue
> - t_2 = the time traveling from the statue back home
>
> There are several relationships among distance, rate, and time. Write the following equations:
>
> - Going to the statue: $d = 6t_1$
> - Returning from the statue: $d = xt_2$

- For the round trip: $2d = 8(t_1 + t_2)$

1. Explain each of the above equations.
2. Explain how to use these equations to yield the following equation:

$$8 = \frac{2d}{\frac{d}{6} + \frac{d}{x}}$$

3. Now solve this equation for x.
4. Explain how the solution shows that the answer is "independent of the distance." (Hint: What does the phrase in quotes mean?)

(b) On Tuesday, Marty did not see Gilly on his way to the statue and averaged 9.23 miles per hour for the trip out to the statue. What must Marty do to ensure he's obeyed his doctor's orders?

(c) On Wednesday, Gilly talks so much that Marty only averages 4 miles per hour on the way out. What must Marty do to ensure he's obeyed his doctor's orders?

Teaching Note: Encourage students also to try numbers near 4, such as 4.1, and 4.01.

(d) Assuming that Marty, for whatever reason, averages r miles per hour on the trip out to the statue. What must Marty do to ensure he's obeyed his doctor's orders?

Teaching Note: Assuming that the previous result is $x = \dfrac{8r}{2r-8}$, analyze this to explain the previous results, including the vertical and horizontal asymptotes.

A.30 I Walk the Line

Fixnote: Maybe reverse the first two problems.

Solve the problems below initially without using letters and without algebraic procedures. Rely on numerical reasoning only, and then generalize your numerical approaches.

A.30.1) Slimy Sam is on the lam from the law. Being not-too-smart, he drives the clunker of a car he stole east on I-70 across Ohio. Because the car can only go a maximum of 52 miles per hour, he floors it all the way from where he stole the car (just now at the Rest Area 5 miles west of the Indiana line) and goes as far as he can before running out of gas 3.78 hours from now.

(a) At what mile marker will he be 3 hours after stealing the car?

(b) At what mile marker will he be when he runs out of gas and is arrested?

(c) At what mile marker will he be x hours after stealing the car?

(d) At what time will he be at mile marker 99 (east of Indiana)?

(e) At what time will he be at mile marker 71.84?

(f) At what time will he be at mile marker y?

(g) Do parts (c) and (f) supposing that the car goes m miles per hour and Sam started b miles east of the Ohio-Indiana border.

(h) What "form" of an equation for a line does this problem motivate?

A.30.2) Free-Lance Freddy works for varying hourly rates, depending on the job. He also carries some spare cash for lunch. To make his customers sweat, Freddy keeps a meter on his belt telling how much money they currently owe (with his lunch money added in).

(a) On Monday, 3 hours into his work as a gourmet burger flipper, Freddy's meter reads \$42. 7 hours into his work, his meter reads \$86. If he works for 12 hours, how much money will he have? When will he have \$196? Solve this problem **without** finding his lunch money.

(b) On Tuesday, Freddy is CEO of the of *We Say So* Company. After 2.53 hours of work, his meter reads \$863.15 and after 5.71 hours of work, his meter reads \$1349.78. If he works for 10.34 hours, how much money will he have? How much time will he be in office to have \$1759.21?

(c) On Wednesday, Freddy is starting goalie for the *Columbus Blue Jackets*. After x_1 hours of work, his meter reads y_1 dollars and after x_2 hours of work, his meter reads y_2 dollars. Without finding his amount of lunch money, if he works for x hours, how much money will he have? How much time will he be in front of the net to have y dollars?

(d) What "form" of an equation for a line does this problem motivate?

A.30.3) Counterfeit Cathy buys two kinds of fake cereal: Square Cheerios for \$4 per pound and Sugarless Sugar Pops for \$5 per pound.

(a) If Cathy's goal for today is to buy \$1000 of cereal, how much of each kind could she purchase? Give five possible answers.

(b) Plot your answers. What does the slope represent in this situation? What do the points where your curve intercepts the axes represent?

(c) If she buys Square Cheerios for a dollars per pound and Sugarless Sugar Pops for b dollars per pound and she wants to buy c dollars of cereal, write an equation that relates the amount of Sugar Pops Cathy buys to the amount of Cheerios she buys. What "form" of the equation of a line does this problem motivate?

(d) Write a function in the form

$$\text{pounds of Sugar Pops} = f(\text{pounds of Cheerios}).$$

A.30.4) Given points $p = (3, 7)$ and $q = (4, 9)$, find the formula for the line that connects these points.

A.30.5) In each of the situations above, write an equation relating the two variables (hours and position, hours and current financial status, pounds of Square Cheerios and pounds of Sugarless Sugar Pops) and answer the following questions:

(a) How did (or could) the equations help you solve the problems above? What about a table or a graph?

(b) Organize the information in each problem into a table and then into a graph. What patterns do you see, if any?

(c) What do the different features of your graph represent for each situation?

A.31 Constant Amount Changes

In this section, we explore sequences and functions and the fact that sequences are functions.

Sometimes you compute the output value of a function from a previous output value. This is called a *recursive* representation of the function. Other times, you compute the output value directly from the input value. This is called a *closed form* representation of the function. Both approaches are important, as they provide different insights.

A.31.1) We can use function notation for sequences, with $f(n)$ representing the n^{th} term of a sequence. Here is an example of a sequence specified recursively:

$$f(0) = 1, f(1) = 1, \text{ and } f(n) = f(n-1) + f(n-2) \text{ for } n \geqslant 2.$$

(a) Find $f(6)$ and explain your reasoning.

(b) Why was it important to give the values $f(0) = 1$ and $f(1) = 1$?

A.31.2) Gertrude the Gumchewer has an addiction to Xtra Sugarloaded Gum, and it's getting worse. At the beginning of her habit, on day 0, she chews 3 pieces and then, each day afterward, she chews 8 more pieces than she chewed the day before.

(a) Gertrude's friend Wanda notices that Gertrude chewed 35 pieces on day 4. Wanda claims that, because Gertrude is increasing the number of pieces she chews at a constant rate, we can just use proportions with the given piece of information to find out how many pieces Gertrude chewed on any other day. Is Wanda correct or not? Explain.

(b) Make a table of how many pieces of gum Gertrude chewed on each of the first 10 days of her addiction.

(c) Think of what a 4^{th} grader would do to predict the next day's number of pieces given the previous day's number of pieces. Use the variables *Next* and *Now* to write an equation that describes the thinking.

(d) Write a recursive specification for a function $g(n)$ that gives the number of pieces of gum Gertrude chewed on the n^{th} day.

(e) How many pieces of gum Gertrude did chew on the 793rd day of her habit? Explain your reasoning.

(f) How would the 4th grader answer the previous question? How does this differ from how you solved it?

(g) Write a closed formula for computing $g(n)$ directly from n.

(h) Make a graph of your data about Gertrude's gum chewing. Which variable do you plot on the horizontal axis? Explain.

(i) Does it make sense to connect the dots on your graph? Explain.

(j) Locate the values $g(6)$ and $g(5)$ in your table from above, compute $g(6) - g(5)$, and interpret your result. How might you have known the answer without doing any calculation?

A.31.3) Slimy Sam steals a car from a rest area 3 miles east of the Indiana-Ohio state line and starts heading east along the side of I-70. Because the car is a real clunker, it can only go 8 miles per hour.

(a) Assuming the police are laughing too hard to arrest Sam, describe Sam's position on I-70 (via mile markers) t hours after stealing the car.

(b) Make a graph of your data about Sam's travel. Which variable do you plot on the horizontal axis? Explain.

(c) Does it make sense to connect the dots on your graph? Explain your reasoning.

(d) Write a recursive specification for a function $s(t)$ that gives Sam's position on I-70 at hour t.

(e) Write closed formula for $s(t)$.

(f) How is this problem the same and how is it different from the Gertrude problem?

(g) Dumb Question: At any specific time, how many positions could Sam be in?

A.32 Constant Percentage (Ratio) Changes

Fixnote: Give them a table and a graph. Or put them in the margin.

A.32.1) Billy is a bouncing ball. He is dropped from a height of 13 feet and each bounce goes up 92% of the bounce before it. Assume that the first time Billy hits the ground is bounce 1.

(a) Make a table of how high Billy bounced after each of the first 10 times he hit the ground. Be sure to indicate the arithmetic process you go through for each bounce (i.e., not just the final height). Find a pattern that will predict an answer.

(b) Think of what a 4^{th} grader would do to predict the next bounce's height given the previous bounce's height. How would the 4^{th} grader answer the previous question? How does this differ from how you solved it?

(c) Make a graph of your data about Billy. Which variable do you plot on the horizontal axis? Explain.

(d) Does it make sense to connect the dots on your graph? Explain your reasoning.

(e) How high will Billy bounce after the 38th bounce? How high will Billy bounce after the n^{th} bounce? Explain your reasoning.

(f) Use function notation, $f(n)$, and a recursive formula to specify the height of Billy's bounces, including the initial condition and general term.

(g) Use function notation, $f(n)$, and an explicit formula to specify the height of Billy's bounces. Indicate the domain of the function.

(h) Using your table from above, compute the differences between the heights on successive bounces (e.g., $f(1) - f(0)$, $f(2) - f(1)$, etc.). What do you notice? Why does this happen?

(i) Compare and contrast the explicit and recursive representations from Billy and from Gertrude. How do the role(s) of the operations and initial values differ, remain the same, or relate?

A.32.2) Supppose 13 mg of a drug is administered to a patient once, and the amount of the drug in the patient's body decreases by 8% each hour.

(a) Describe the amount of the drug in the patient's body x hours after it was administered.

(b) Make a graph of your data about the amount of drug in the body over time. Which variable do you plot on the horizontal axis? Explain.

(c) Does it make sense to connect the dots on your graph? Explain your reasoning.

(d) Use function notation, $g(t)$, and an explicit formula to specify the the amount of drug remaining in the body after t hours. Indicate the domain of the function.

(e) How is this problem fundamentally different from the Billy problem? What is the same and different about the functions f and g?

(f) Dumb Question: At any one time, how many different amounts of the drug are possible in the patient's body?

A.33 Meanings of Exponents

Students in grades 3-7 can use their understanding of counting number arithmetic to build understandings of the arithmetic of negative integers and rational numbers. Here are the key ideas:

- The properties of operations (commutative, associative, and distributive properties) are established for counting numbers based on meanings of operations.
- As we extend arithmetic to negative integers and rational numbers, we want the properties of operations to continue to hold.

This activity follows an analogous process for exponents: Students use their understanding of counting number exponents to build an understanding of negative integer and rational exponents. Here are the key ideas:

- The rules of exponents are established for counting number exponents based on the meaning of an exponent.
- As we extend to negative and rational exponents, we want the rules of exponents to continue to hold.

A.33.1) Students sometimes say that a^n means "a multiplied by itself n times." But for counting number exponents, this is not correct. For example, how many multiplications are there in 3^5? Write a better definition for a^n, where n is a counting number.

A.33.2) Why is x^3 not the same function as 3^x? We often think of multiplication as "repeated addition," and we find that adding a copies of b gives the same result as adding b copies of a. Does this idea work for thinking of exponentiation as "repeated multiplication"? Explain.

A.33.3) If you do not know (or do not remember) the rules for exponents, you can still use your definition of a^n to figure out other ways of writing expressions with exponents. Use **specific values** for letters in expressions of the form $a^n a^m$, a^n/a^m, $(a^n)^m$, and $(ab)^n$ for counting-number exponents, to explain what the rules must be. Choose specific values that help you explain generally.

A.33.4) Patterns. One way to reason about the meanings of zero and negative exponents is to use patterns. As you complete the following table, **imagine that you**

know nothing about zero and negative exponents. Instead, use the patterns in the values for positive exponents to reason about what the values should be for zero and negative exponents. Then reason generally about the meaning of a^0 and a^{-n}, where n is a counting number and a is a real number. Are there any values of a for which your reasoning is not valid? Explain.

$2^3 =$	$3^3 =$	$(-2)^3 =$	$\left(\frac{1}{2}\right)^3 =$
$2^2 =$	$3^2 =$	$(-2)^2 =$	$\left(\frac{1}{2}\right)^2 =$
$2^1 =$	$3^1 =$	$(-2)^1 =$	$\left(\frac{1}{2}\right)^1 =$
$2^0 =$	$3^0 =$	$(-2)^0 =$	$\left(\frac{1}{2}\right)^0 =$
$2^{-1} =$	$3^{-1} =$	$(-2)^{-1} =$	$\left(\frac{1}{2}\right)^{-1} =$
$2^{-2} =$	$3^{-2} =$	$(-2)^{-2} =$	$\left(\frac{1}{2}\right)^{-2} =$
$2^{-3} =$	$3^{-3} =$	$(-2)^{-3} =$	$\left(\frac{1}{2}\right)^{-3} =$

A.33.5) Extending the rules. A careful way to approach zero and negative integer exponents is to use the rules of exponents (which you established above for counting-number exponents) to determine what 0 and negative integer exponents must mean if the exponent rules continue to hold in this extended domain.

(a) Use the exponent rules to provide two explanations for a sensible definition of a^0, being clear about why your definition makes sense. Note any restrictions on a.

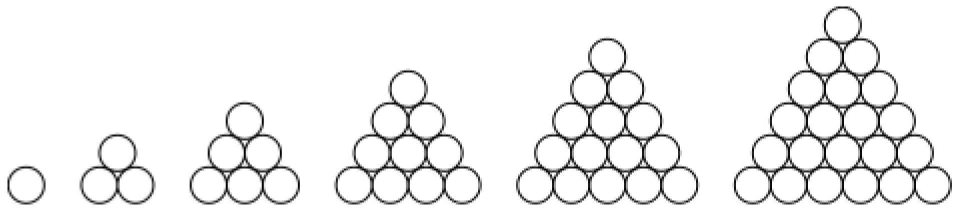
(b) Use the exponent rules to provide two explanations for a sensible definition of a^{-n}, where n is a counting number. Again, note any restrictions on a.

A.33.6) While trying to decide what $3^{\frac{2}{5}}$ should mean, Katie wondered about the expression $\left(3^{\frac{2}{5}}\right)^5$. What should Katie's expression be equal to? Explain, using rules of exponents. Then use Katie's idea to determine a value for $3^{\frac{2}{5}}$.

A.34 Arithmetic Series

In this activity, we explore *arithmetic series*, which are sums of consecutive terms from an arithmetic sequence.

Ms. Nguyen's math class has been looking at "triangular numbers." The first 6 triangular numbers are shown below.



Fixnote: Need an original graphic here.

Fixnote: Start this problem without any help.

A.34.1) Blair wanted to find the 551^{st} triangular number. She used a table and looked for a pattern in the *sequence of partial sums*: $1, 1+2, 1+2+3, \ldots$. Help her finish her idea.

A.34.2) Kaley realized the the 551^{st} triangular number would be the sum

$$1+2+3+4+\cdots+548+549+550+551$$

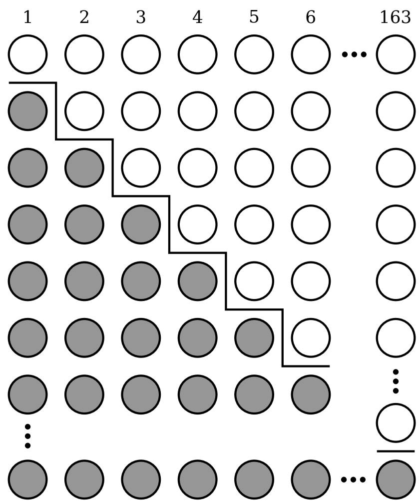
She started pairing the first with the last number; the second with the second-to-last; the third with the third-to-last; and so on. She saw that the averages are always the same. Help her finish her idea.

A.34.3) Ali begin by writing out the sum forward and backward and follows:

$$\begin{array}{c} 1\ +\ 2\ +\ 3\ +\ 4\ +\ 5\ +\ 6\ +\cdots+546+547+548+549+550+551 \\ 551+550+549+548+547+546+\cdots+\ 6\ +\ 5\ +\ 4\ +\ 3\ +\ 2\ +\ 1 \end{array}$$

Help her finish her idea. Be sure to explain clearly what happens "in the dots." Does it matter whether there are an even or an odd number of terms?

A.34.4) Cooper was interested in a different triangular number and drew the following picture:



Which triangular number was he finding? Help him finish his idea. Be sure to explain clearly what happens "in the dots."

A.34.5) Sum the numbers:

$$106 + 112 + 118 + \cdots + 514$$

A.34.6) Sum the numbers:

$$2.2 + 2.9 + 3.6 + 4.3 + \cdots + 81.3$$

A.34.7) Suppose you have an arithmetic sequence beginning with a, with a constant difference of d and with n terms.

(a) What is the n^{th} term of the sequence?

(b) Use dots to write the series consisting of the first n terms of this sequence.

(c) Find the sum of this series.

A.35 Geometic Series

In this activity, we explore *geometric series*, which are sums of consecutive terms from an geometric sequence.

Ms. Radigan's math class has been trying to compute the following sums:

$$1 + 2 + 4 + 8 + \cdots + 2^{19}$$

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{13}}$$

A.35.1) Kelsey used tables and looked for pattern in the *sequence of partial sums*: $1, 1+2, 1+2+4, \ldots$. Help her finish her idea for both sequences.

A.35.2) For the sum beginning with $\frac{2}{3}$, Erin started by drawing a large square (which she imagined as having area 1), and she shaded in $\frac{2}{3}$ of it. Then she shaded in $\frac{2}{9}$ more, and so on. Help her finish her idea.

A.35.3) Ryan wrote out all of the terms in the first sum, represented as powers of 2, beginning with $1 + 2 + 2^2 + 2^3$. Then he realized that because the terms formed a geometric sequence, he could multiply the series by the common ratio of 2, and the resulting series would be almost identical to the first, differing only at the beginning and the end. By subtracting the first series from the second, all of the middle terms would cancel. Help him finish his idea.

A.35.4) Ali said, "Here is a thought experiment. I take a sheet of paper, rip it perfectly into thirds, place one piece to start a pile that I will call A, another piece to start a pile I will call B, and I keep the third piece in my hands. I then rip that piece into thirds, place one piece on pile A, one piece on pile B, and keep the third. Notice that each of pile A and pile B have $\frac{1}{3} + \frac{1}{9}$ of a sheet of paper, and I still have $\frac{1}{9}$ of a sheet in my hands. I continue this process until I place $\frac{1}{3^{13}}$ of a sheet on each pile and still have $\frac{1}{3^{13}}$ of a sheet in my hands.

Help Ali finish her idea.

A.35.5) Sum the expression:

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \frac{2^n}{3^n}$$

What happens to this sum as n gets really large?

A.35.6) Consider the expression:

$$\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots + \frac{7}{10^n}$$

(a) Find the sum of the expression.

(b) What happens to this sum as n gets really large?

(c) How does this help you explain why a particular repeating decimal is a particular rational number? Be sure to indicate what repeating decimal and what rational number you are talking about.

A.35.7) Suppose you have an geometric sequence beginning with a, with a constant ratio of r and with n terms.

(a) What is the n^{th} term of the sequence?

(b) Use dots to write the series consisting of the first n terms of this sequence.

(c) Find the sum of this series.

A.36 Hieroglyphical Algebra

([)0]This activity is based on an activity originally designed by Lee Wayand.

Consider the following addition and multiplication tables:

+	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
fish	fork	balloon	fish	eyeball	cinder-block	skull	lolly-pop	man	DNA
lolly-pop	balloon	eyeball	lolly-pop	fork	fish	man	cinder-block	DNA	skull
skull	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
cinder-block	eyeball	fork	cinder-block	balloon	lolly-pop	DNA	fish	skull	man
DNA	cinder-block	fish	DNA	lolly-pop	man	eyeball	skull	fork	balloon
fork	skull	man	fork	DNA	eyeball	fish	balloon	lolly-pop	cinder-block
man	lolly-pop	cinder-block	man	fish	skull	balloon	DNA	eyeball	fork
balloon	man	DNA	balloon	skull	fork	lolly-pop	eyeball	cinder-block	fish
eyeball	DNA	skull	eyeball	man	balloon	cinder-block	fork	fish	lolly-pop

fish = fish

lolly-pop = lolly-pop

skull = skull

cinder-block = cinder-block

DNA = DNA

fork = fork

man = man

balloon = balloon

eyeball = eyeball

·	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish
fork	fish	lolly-pop	cinder-block	DNA	skull	eyeball	balloon	man	fork
eyeball	lolly-pop	balloon	man	fork	skull	cinder-block	DNA	fish	eyeball
balloon	cinder-block	man	fork	lolly-pop	skull	DNA	fish	eyeball	balloon
man	DNA	fork	lolly-pop	cinder-block	skull	fish	eyeball	balloon	man
skull	skull	skull	skull	skull	skull	skull	skull	skull	skull
lolly-pop	eyeball	cinder-block	DNA	fish	skull	balloon	man	fork	lolly-pop
cinder-block	balloon	DNA	fish	eyeball	skull	man	fork	lolly-pop	cinder-block
DNA	man	fish	eyeball	balloon	skull	fork	lolly-pop	cinder-block	DNA
fish	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish

A.36.1) Can you tell me which glyph represents 0? How did you arrive at this conclusion?

A.36.2) Can you tell me which glyph represents 1? How did you arrive at this conclusion?

A.36.3) A number x has an *additive inverse* if you can find another number y with

$$x + y = 0.$$

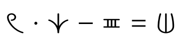
and we say that "y is the additive inverse for x." If possible, find the additive inverse of every number in the table above.

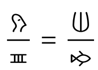
A.36.4) A number x has a *multiplicative inverse* if you can find another number y with

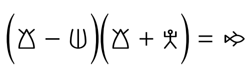
$$x \cdot y = 1.$$

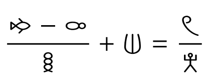
and we say that "y is the multiplicative inverse for x." If possible, find the multiplicative inverse of every number in the table above.

A.36.5) If possible, solve the following equations:

(a) 

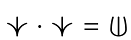
(b) 

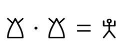
(c) 

(d) 

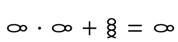
In each case explain your reasoning.

A.36.6) If possible, solve the following equations:

(a) 

(b) 

(c) 

(d) 

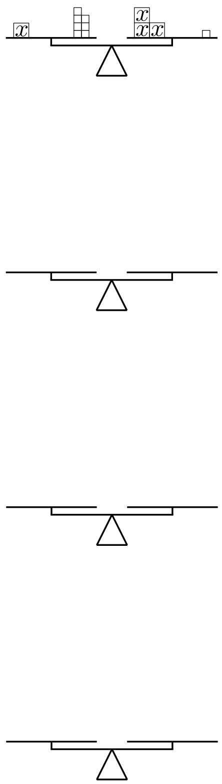
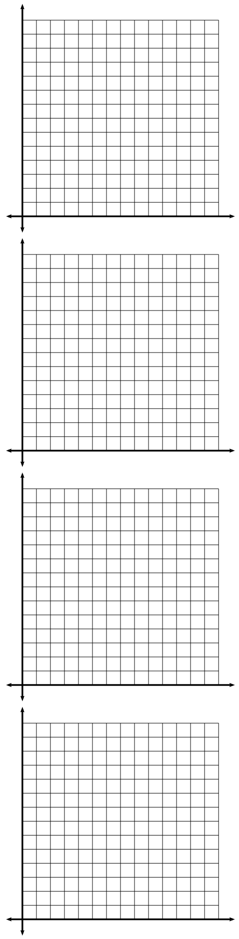
In each case explain your reasoning.

A.37 The Other Side---Solving Equations

In this activity, we will explore ideas related to solving equations.

> Fixnote: Need some help here for using graphs, noting that we are looking for where $f(x) = g(x)$ for the two sides of the equation.

A.37.1) Solve the following equation three ways: Using algebra, using the balance, and with the graph. At each step, the three models should be in complete alignment.

Equations:	Balance:	Plotting:
$x + 7 = 3x + 1$		

A.37.2) Critically analyze the three “different” methods of solving equations, noting the advantages and disadvantages of each.

A.37.3) Can you solve quadratic equations using the methods above? If so give an example. If not, explain why not.

Teaching Note: The key point here is that it is difficult to make "balances" work for anything but linear equations.

A.37.4) Can you think of an example when the undoing via algebraic manipulation would fail?

Teaching Note: Here we are looking for something where an inverse function must be applied, as in $.6 = \sin(x)$.

While sometimes we solve equations via a process of algebraic manipulation, other times we have a formula.

A.37.5) Give a formula for solving linear equations of the form $ax + b = 0$.

A.38 Solving Quadratics

Here we explore various methods for solving quadratic equations in one variable. **Please read all instructions carefully.**

A.38.1) Is $\sqrt{4} = \pm 2$? Explain.

> **Teaching Note:** Both 2 and -2 are "square roots" of 4 because $2^2 = 4$ and $(-2)^2 = 4$, and both of them are solutions to the equation $x^2 = 4$. The question is whether the radical symbol refers to both of them, either of them (you choose?), or a specific one of them.

A.38.2) Suppose that $\sqrt{4} = \pm 2$? Then evaluate $\sqrt{4} + \sqrt{9}$.

> **Teaching Note:**
> $$\sqrt{4} + \sqrt{9} = \pm 2 + \pm 3 = 5, -1, 1, or - 5$$

A.38.3) What does your calculator say about $\sqrt{4} + \sqrt{9}$?

> **Teaching Note:** Now emphasize the conventional meaning of the radical symbol: For $a > 0$ then $\sqrt{a}$ means the positive square root of a.

A.38.4) In the following problems, you **may not use the quadratic formula**. But just for the record, write down the quadratic formula.

> **Teaching Note:** Many students will write only $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, but we want them to write the following:
> $$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note that if the radical symbol were to refer to both a positive and negative square root, then there would be no reason to write ± outside the radical symbol.

A.38.5) In the following list of equations, solve those that are **easy** to solve.

(a) $(x-3)(x+2)=0$

(b) $(x-3)(x+2)=1$

(c) $(2x-5)(3x+1)=0$

(d) $(x-a)(x-b)=0$

(e) $(x-1)(x-3)(x+2)(2x-3)=0$

A.38.6) Regarding the previous problem, state the property of numbers that made all but one of the equations easy to solve.

Teaching Note: Zero product property: If $ab=0$ then $a=0$ or $b=0$. Note that this doesn't work when the right side is not 0.

A.38.7) For each part below, write a quadratic equation with the stated solution(s) and no other solutions.

(a) $x=7$ or $x=-4$

(b) $x=p$ or $x=q$

(c) $x=3$

(d) $x=\dfrac{1\pm\sqrt{5}}{2}$

Teaching Note: In the following problem, discuss ways of explaining that

$$\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$$

A.38.8) In the following list of equations, solve those that are **easy** to solve.

(a) $x^2 = 5$

(b) $x^2 - 4 = 2$

(c) $x^2 - 4x = 2$

(d) $2x^2 = 1$

(e) $(x-2)^2 = 5$

A.38.9) Regarding the previous problem, state the property of numbers that made all but one of the equations easy to solve.

> **Teaching Note:** If $u^2 = a$ then $u = \pm\sqrt{a}$.

A.38.10) Although 160 is not a square in base ten, what could you add to 160 so that the result would be a square number?

A.38.11) Consider the polynomial expression $x^2 + 6x$ to be a number in base x. We want to add to this polynomial so that the result is a square in base x.

(a) Use "flats" and "longs" to draw a picture of this polynomial as a number in base x, adding enough "ones" so that you can arrange the polynomial into a square.

(b) What "feature" of the square does the new polynomial expression represent?

(c) Why does it make sense to call this technique "completing the square"?

(d) Use your picture to help you solve the equation $x^2 + 6x = 5$.

A.38.12) Complete the square to solve the following equations:

(a) $x^2 + 3x = 4$

(b) $x^2 + bx = q$

(c) $2x^2 + 8x = 12$

(d) $ax^2 + bx + c = 0$

A.38.13) Find all solutions to $x^3 - 3x^2 + x + 1 = 0$. Hint: One solution is $x = 1$.

A.38.14) Solve the following equation

$$x^5 - 4x^4 - 18x^3 + 64x^2 + 17x - 60 = 0$$

assuming you know that 1, −1, and 3 are roots.

A.39 Solving Cubic Equations

To solve the cubic equation $x^3 + px + q = 0$, we use methods that were discovered and advanced by various mathematicians, including Ferro, Tartaglia, and Cardano. The approach is organized in three steps. **Make notes in the margin as you follow along.**

A.39.1 Step 1: Replace x with $u + v$

In $x^3 + px + q = 0$, let $x = u + v$. Show that the result can be written as follows:

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

A.39.2 Step 2: Set uv to eliminate terms

If $3uv + p = 0$, then all of the terms are eliminated except for u^3, v^3, and constant terms. Explain why the equation simplifies nicely to:

$$u^3 + v^3 + q = 0.$$

Solve $3uv + p = 0$ for v, substitute, and show that we have:

$$u^3 + \left(\frac{-p}{3u}\right)^3 + q = 0.$$

A.39.3 Step 3: Recognize the equation as a quadratic in u^3 and solve

By multiplying by u^3, show that we get a quadratic in u^3:

$$u^6 + qu^3 + \left(\frac{-p}{3}\right)^3 = 0.$$

Show that this has solutions:

$$u^3 = \frac{-q \pm \sqrt{q^2 - 4\left(\frac{-p}{3}\right)^3}}{2}.$$

Now, use the facts $v = -p/(3u)$ and $x = u + v$ to write a formula for x:

$$x = \sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4\left(\frac{-p}{3}\right)^3}}{2}} + \frac{-p}{3\sqrt[3]{\frac{-q\pm\sqrt{q^2-4\left(\frac{-p}{3}\right)^3}}{2}}}.$$

A.39.1) How many values does this formula give for x? From the original equation $x^3 + px + q = 0$, how many solutions should we expect?

A.39.2) Use the above formula to solve the specific equation $x^3 - 15x - 4 = 0$. Show that

$$x = \sqrt[3]{2 \pm \sqrt{-121}} + \frac{5}{\sqrt[3]{2 \pm \sqrt{-121}}}.$$

Are these values of x real numbers?

A.39.3) Use technology to graph $y = x^3 - 15x - 4$. According to the graph, how many real roots does the polynomial have? What is going on?

A.39.4) Choose "plus" in the $\pm$, and check that $2 + \sqrt{-1}$ is a cube root of $2 + \sqrt{-121}$. Use that fact to simplify the above expression for x. What do you notice?

A.39.5) Now choose "minus" in the $\pm$ above, and find the value of x. What do you notice?

In both cases, the formula requires computations with square roots of negative numbers, but the result is a real solution. These kinds of occurrences were the historical impetus behind the gradual acceptance of complex numbers.

A.40 Maximums and Minimums (Revised)

In high school mathematics, you saw three different forms for quadratic functions. In this activity, we explore the advantages and disadvantages of each.

Note: We use only real numbers for x. And we begin by agreeing that the shape of the graph of a quadratic function is a parabola.

A.40.1) Consider the function $f(x) = x^2 - 3$. What are the maximum/minimum value(s) of $f(x)$, and for what x values do they occur? Explain your reasoning. Use this information to sketch a graph.

A.40.2) Consider the function $f(x) = 3(x-5)^2+7$. What are the maximum/minimum value(s) of $f(x)$, and for what x values do they occur? Explain your reasoning. Use this information to sketch a graph.

A.40.3) Consider the function $f(x) = -2(x+3)^2+7$. What are the maximum/minimum value(s) of $f(x)$, and for what x values do they occur? Explain your reasoning. Use this information to sketch a graph.

A.40.4) What are the advantages of the form $f(x) = a(x-h)^2 + k$ for a quadratic function? Why is it called vertex form? What do the values of a, h, and k tell you about the graph?

A.40.5) Consider the function $f(x) = x^2 + 4x + 2$. Complete the square to put this function into vertex form, and sketch a graph.

A.40.6) Consider the function $f(x) = 2x^2 - 8x + 6$. Complete the square to put this function into vertex form, and sketch a graph.

A.40.7) Consider the function $f(x) = (x + 1)(x + 5)$.

(a) What points on the graph are easy to locate?

(b) How can you use those points to find the vertex?

(c) Sketch the graph.

A.40.8) Consider the function $f(x) = -2(x - 2)(x + 3)$.

(a) What points on the graph are easy to locate?

(b) How can you use those points to find the vertex?

(c) Sketch the graph.

A.40.9) Consider the function $f(x) = a(x - r_1)(x - r_2)$.

(a) What do the values of a, r_1, and r_2 tell you about the graph?

(b) How can you use that information to find the vertex?

(c) What is this form called and why?

A.40.10) Consider the function $f(x) = ax^2 + bx + c$.

(a) What do the values of a, b, and c tell you about the graph?

(b) What are the advantages and disadvantages of this form?

A.41 It Takes All Kinds. . .

Fixnote: Maybe move these to problems. Replace with linear, quadratic, exponential sheet.

Data can come in all shapes and sizes. While a line is the simplest approximation, it might not be the best.

A.41.1) Consider the data below:

x	0	1	2	3
y	8.1	22.1	60.1	165

What type of data is this? To get the "brain juices" flowing here are some choices. It could be:

(a) A parabola.

(b) An exponential.

(c) A quartic.

(d) Something else.

Hint: Think about the most famous graph of all, the one you know most about. And see if you can somehow convert the above data to get that type of graph. You will probably need to make some plots.

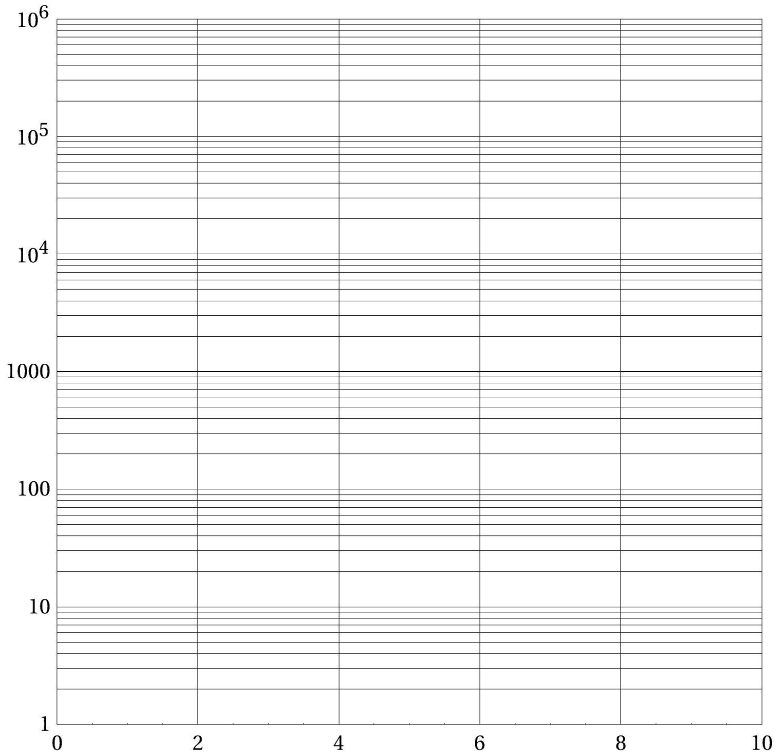
A.41.2) Now do the same with this data:

x	1	2	3	4
y	8.3	443.6	24420.8	1364278.6

A.41.3) Now do the same with this data:

x	1	2	3	4	5
y	7	62	220	506	1012

A.41.4) Here is a sample of semi-log paper. What's going on here?



A.42 Sketching Roots

In this activity we seek to better understand the connection between roots and the plots of polynomials. We will restrict our attention to polynomials with real coefficients.

First, we need to be precise about the correct usage of some important language:

- Expressions have *values*.
- Equations have *solutions*: values of the variables that make the equation true.
- Functions have *zeros*: input values that give output values of 0.
- Polynomials (i.e., polynomial expressions) have *roots*.

These ideas are related, of course, as follows: A zero of a polynomial function, $p(x)$, is a root of the polynomial $p(x)$ and a solution to the equation $p(x) = 0$.

Please try to use this language correctly: Equations do not have zeros, and functions do not have solutions.

A.42.1) Give an example of a polynomial, and write a true sentence about related equations, functions, zeros, equations, and roots.

A.42.2) Sketch the plot of a quadratic polynomial with real coefficients that has:

(a) Two real roots.

(b) One repeated real root.

(c) No real roots.

In each case, give an example of such a polynomial.

A.42.3) Can you have a quadratic polynomial with exactly one real root and 1 complex root? Explain why or why not.

A.42.4) Sketch the plot of a cubic polynomial with real coefficients that has:

(a) Three distinct real roots.

(b) One real root and two complex roots.

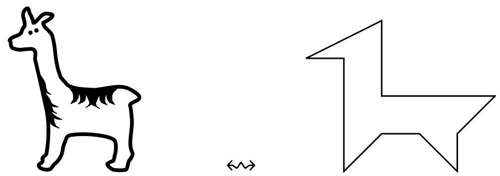
In each case, give an example of such a polynomial.

A.42.5) Can you have a cubic polynomial with no real roots? Explain why or why not. What about two distinct real roots and one complex root?

A.42.6) For polynomials with real coefficients of degree 1 to 5, classify exactly which types of roots can be found. For example, in our work above, we classified polynomials of degree 2 and 3.

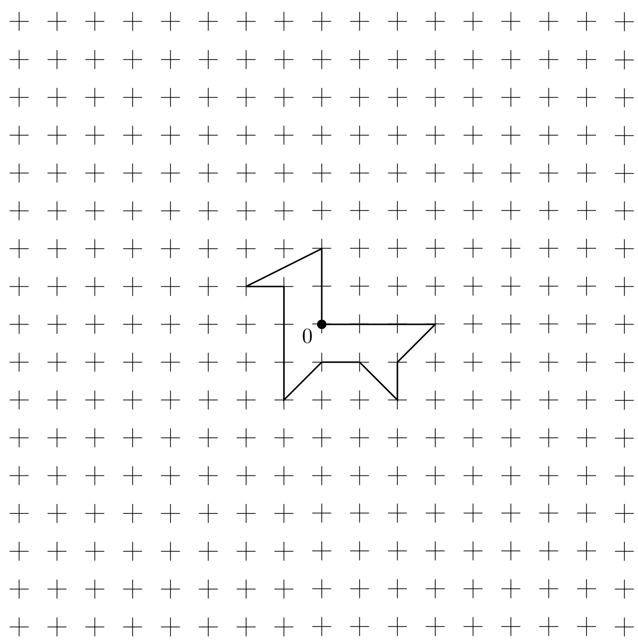
A.43 Geometry and Adding Complex Numbers

Let's think about the geometry of adding complex numbers. We won't be alone on our journey—Louie Llama is here to help us out:



Louie Llama $\quad\leftrightsquigarrow\quad$ how we'll draw him

A.43.1) Here's Louie Llama hanging out near the point 0 in the complex plane. Add $4 + 4i$ to him. Make a table and show in the plane below what happens.



A.43.2) Explain what it means to "add" a complex number to Louie Llama. Describe the process(es) used when doing this.

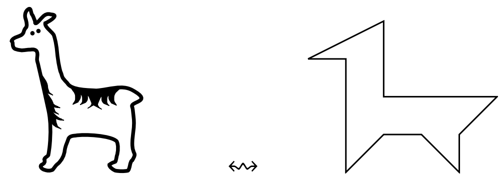
A.43.3) Put Louie Llama back where he started, now add $1 - 5i$ to him. Make a table and show what happens in the plane.

A.43.4) Geometrically speaking, what does it mean to "add" complex numbers?

A.44 Geometry and Multiplying Complex Numbers

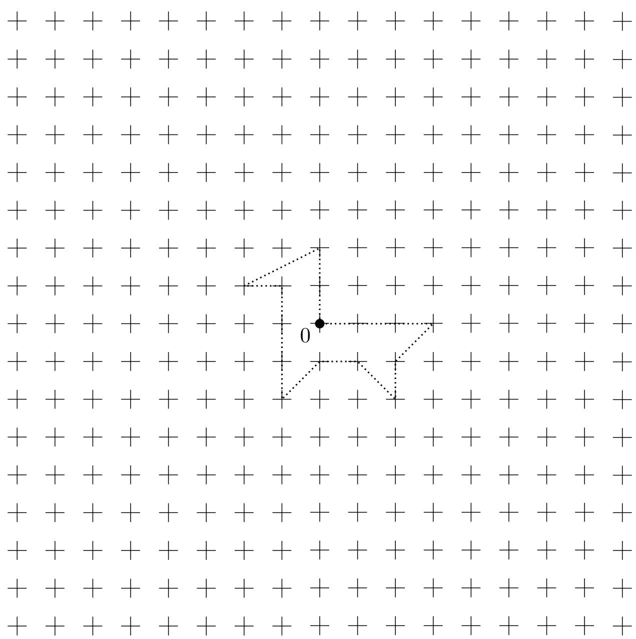
Now we'll investigate the geometry of multiplying complex numbers. In each case, specify the transformation. For example, if you see a rotation, specify the angle and the center of rotation.

Louie Llama is here to help us out:

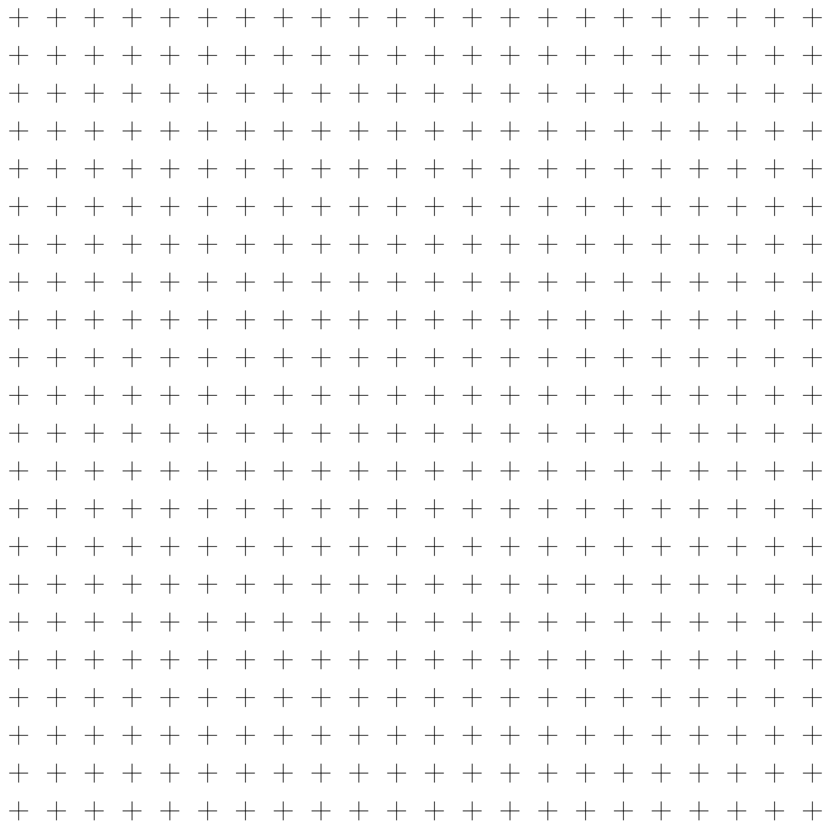


Louie Llama how we'll draw him

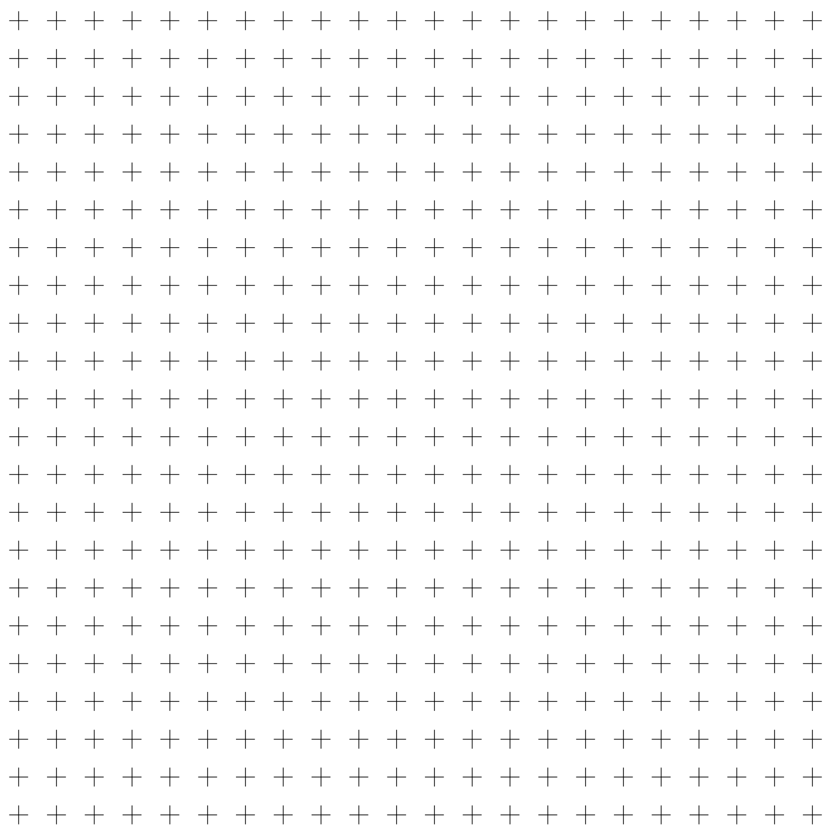
A.44.1) Here's Louie Llama hanging out near the point 0 in the complex plane. Multiply him by 2. Make a table and show in the plane below what happens. What transformation do you see?



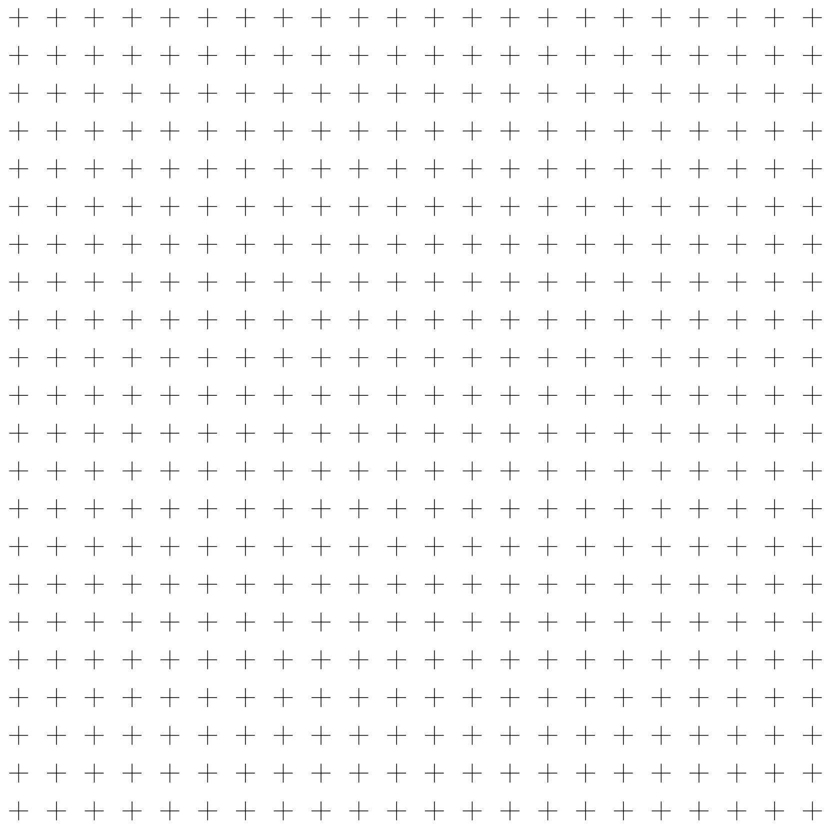
A.44.2) Now multiply him by i. Make a table and show in the plane below what happens. What transformation do you see?



A.44.3) Now multiply Louie Llama by $1 + i$. Make a table and show in the plane below what happens. What transformation do you see?



A.44.4) Now multiply Louie Llama by $1 - 2i$. Make a table and show in the plane below what happens. What transformation do you see?



A.44.5) Make a table to summarize your results from the previous problems. Then describe what happens geometrically when we "multiply" by a complex number.

A.45 To the Second Degree

In this activity, we seek to understand why roots of polynomials with real coefficients must always come in conjugate pairs.

A.45.1) Consider your favorite (non-real) complex number, I'll call it ξ. Find a polynomial with real coefficients whose degree is as small as possible having your number as a root. What is the degree of your polynomial?

A.45.2) I'll call the polynomial found in the first problem $s(x)$. Let $f(x)$ be some other polynomial with

$$f(\xi) = 0.$$

I claim $s(x)|f(x)$. Explain why if $s(x) \nmid f(x)$ then there exist $q(x)$ and $r(x)$ with

$$f(x) = s(x) \cdot q(x) + r(x) \qquad \text{with } \deg(r) < \deg(s).$$

A.45.3) Plug in ξ for x in the equation above. What does this tell you about $r(\xi)$? Is this possible?

A.45.4) Explain why complex roots must always come in conjugate pairs. Also plot some conjugate pairs in the complex plane and explain what "conjugation" means geometrically.

A.46 Broken Records

Fill in the following table:

modulus:	2	3	4	5	6	7	8	9	10	11
$2 \cdot 1 \equiv$										
$2 \cdot 2 \equiv$										
$2 \cdot 3 \equiv$										
$2 \cdot 4 \equiv$										
$2 \cdot 5 \equiv$										
$2 \cdot 6 \equiv$										
$2 \cdot 7 \equiv$										
$2 \cdot 8 \equiv$										
$2 \cdot 9 \equiv$										
$2 \cdot 10 \equiv$										
$2 \cdot 11 \equiv$										

A.46.1) Find patterns in your table above, clearly describe the patterns you find.

A.46.2) Consider the patterns you found. Can you explain why they happen?

A.46.3) When does a column have a 0? When does a column have a 1?

A.46.4) Describe what would happen if you extend the table for bigger moduli and bigger multiplicands.

modulus:	2	3	4	5	6	7	8	9	10	11
$3 \cdot 1 \equiv$										
$3 \cdot 2 \equiv$										
$3 \cdot 3 \equiv$										
$3 \cdot 4 \equiv$										
$3 \cdot 5 \equiv$										
$3 \cdot 6 \equiv$										
$3 \cdot 7 \equiv$										
$3 \cdot 8 \equiv$										
$3 \cdot 9 \equiv$										
$3 \cdot 10 \equiv$										
$3 \cdot 11 \equiv$										

A.46.5) Find patterns in your table above, clearly describe the patterns you find.

A.46.6) Consider the patterns you found. Can you explain why they happen?

A.46.7) When does a column have a 0? When does a column have a 1?

A.46.8) Can you describe what would happen if you extend the table for bigger moduli and bigger multiplicands?

A.46.9) Describe precisely when a column of the table will contain representatives for each integer modulo n. Explain why your description is true.

A.47 On the Road

A.47.1) Steve likes to drive the city roads. Suppose he is driving down a road with three traffic lights. For this activity, we will ignore yellow lights, and pretend that lights are either red or green.

(a) How many ways could he see one red light and two green lights?

(b) How many ways could he see one green light and two red lights?

(c) How many ways could he see all red lights?

A.47.2) Now suppose Steve is driving down a road with four traffic lights.

(a) How many ways could he see two red light and two green lights?

(b) How many ways could he see one green light and three red lights?

(c) How many ways could he see all green lights?

A.47.3) In the following chart let n be the number of traffic lights and k be the number of green lights seen. In each square, write the number of ways this number of green lights could be seen while Steve drives down the street.

Fixnote: Hard for students to see the patterns. Maybe sheer the table.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$n = 0$							
$n = 1$							
$n = 2$							
$n = 3$							
$n = 4$							
$n = 5$							
$n = 6$							

Describe any patterns you see in your table and try to explain them in terms of traffic lights.

A.48 Pascal's Triangle: Fact or Fiction?

Fixnote: Connect to previous activity.

Consider the numbers $\binom{n}{k}$. These numbers can be arranged into a "triangle" form that is popularly called "Pascal's Triangle". Assuming that the "top" entry is $\binom{0}{0} = 1$, we write the numbers row by row, with n fixed for each row. Write out the first 7 rows of Pascal's Triangle.

Note that there are many patterns to be found. Your job is to justify the following patterns in the context of relevant models. Here are three patterns. Can you explain them?

(a) $\binom{n}{k} = \binom{n}{n-k}$.

(b) The sum of the entries in each row is 2^n.

(c) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

A.49 You Can Count on It!

Teaching Note: The words permutation and combination tend to promote formulaic thinking rather than careful reasoning. And students tend to ask "Does order matter?" in ways that don't help them with the reasoning.

The major purposes of this activity: (1) the multiplication principle of counting, supported by a drawn or imagined tree diagram; and (2) strong explanatory thinking around "n choose k," connecting to both traffic lights and Pascal's triangle.

Although students will find formulas for "n choose k" to be useful, we need not emphasize formulas for permutations because the multiplication principle provides it for free. (See also the final exam review document from 2014.)

A.49.1) The Diet-Lite restaurant offers 5 entrées and 8 side dishes. If you were going to select a dinner with one entrée and one side dish, how many different dinners could you order?

A.49.2) In addition to the 5 entrées and 8 side dishes, The Diet-Lite restaurant offers 12 desserts and 6 kinds of drinks. If you were going to select a dinner with one entrée, one side dish, one dessert, and one drink, how many different dinners could you order?

A.49.3) A standard Ohio license plate consists of two letters followed by two digits followed by two letters. How many different standard Ohio license plates can be made if:

(a) There are no more restrictions on the numbers or letters.

(b) There are no repeats of numbers or letters.

A.49.4) Seven separate coins are flipped. How many different results are possible (e.g., HTHHTHT is different from THHHTTH)?

A.49.5) There are 10 students in the auto mechanics club. Elections are coming up and the members are holding nominations for President, Vice President, Secretary, and Treasurer. If all members are eligible, how many possible slates of nominees are there?

A.49.6) Now the club is not electing officers anymore, but instead deciding to send 3 delegates to the state auto mechanics club convention. How many possible groups of delegates can be made?

A.49.7) Describe how to generalize the previous two questions to n members of the club and k offices or delegates.

Teaching Note: The following problems are optional.

A.49.8) A pizza shop always puts cheese on their pizzas. If the shop offers n additional toppings, how many different pizzas can be ordered (Note: A plain cheese pizza is an option)?

A.49.9) The Pig-Out restaurant offers 5 entrees, 8 side dishes, 12 desserts, and 6 kinds of drinks. If you were going to select a dinner with 3 entrées, 4 side dishes, 7 desserts, and one drink, how many different dinners could you order?

A.50 Which Road Should We Take?

A.50.1) Consider a six-sided die. Without actually rolling a die, guess the number of 1's, 2's, 3's, 4's, 5's, and 6's you would obtain in 50 rolls. Record your predictions in the chart below:

Predictions

# of 1's	# of 2's	# of 3's	# of 4's	# of 5's	# of 6's	Total

Now roll a die 50 times and record the number of 1's, 2's, 3's, 4's, 5's, and 6's you obtain.

Experimental Results

# of 1's	# of 2's	# of 3's	# of 4's	# of 5's	# of 6's	Total

How did you come up with your predictions? How do your predictions compare with your actual results? Now make a chart to combine your data with that of the rest of the class.

Experiment 1 We investigated the results of throwing one die and recording what we saw (a 1, a 2, ..., or a 6). We said that the probability of an event (for example, getting a "3" in this experiment) predicts the frequency with which we expect to see that event occur in a large number of trials. You argued the $P(\text{seeing } 3) = 1/6$ (meaning we expect to get a 3 in about $1/6$ of our trials) because there were six different outcomes, only one of them is a 3, and you expected each outcome to occur about the same number of times.

Experiment 2 We are now investigating the results of throwing two dice and recording the sum of the faces. We are trying to analyze the probabilities associated with these sums. Let's focus first on $P(sum = 2) =?$. We might have some different theories, such as the following:

Theory 1 $P(\text{sum} = 2) = 1/11$.

It is proposed that a sum of 2 was 1 out of the 11 possible sums $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Theory 2 $P(\text{sum} = 2) = 1/21$.

It is proposed that a sum of 2 was 1 of 21 possible results, counting $1 + 3$ as the same as $3 + 1$:

$1+1$	–	–	–	–	–
$2+1$	$2+2$	–	–	–	–
$3+1$	$3+2$	$3+3$	–	–	–
$4+1$	$4+2$	$4+3$	$4+4$	–	–
$5+1$	$5+2$	$5+3$	$5+4$	$5+5$	–
$6+1$	$6+2$	$6+3$	$6+4$	$6+5$	$6+6$

Teaching Note: Some might suggest that the sample space is better thought of as ordered pairs rather than sums. In any case, the upshot is that for anything we do with the two dice, the sample space has 36 elements. Then the following activity, Lumpy and Eddy, might be redundant.

A.50.2) Propose your own Theory 3.

A.50.3) Test all theories by computing $P(2)$, $P(3)$, $\ldots$, $P(12)$ for each theory and comparing to the dice rolls recorded by the class. What do you notice?

A.50.4) Which theory do you like best? Why?

A.50.5) How could we test our theory further?

A.51 Lumpy and Eddie

Two ancient philosophers, Lumpy and Eddie, were sitting on rocks flipping coins.

A.51.1) Lumpy and Eddie wondered about the probability of obtaining both a head and a tail. Here is how it went:

> Eddie argued the following: “Look Lumpy, it’s clear to me that when we flip two coins, we should get one of each about half the time because there are two possibilities: They’re either the same or different.” Lumpy, on the other hand, argued this way: “Eddie, stop being a wise guy! If we flipped two coins, we should expect both a head and tail to come up about a third of the time because there are only three possibilities: two heads, two tails, and one of each.”

Which, if any, of these two guys is right? Is there another answer?

A.51.2) Next Lumpy and Eddie threw a third coin in the mix and wondered about the probability of obtaining 2 heads and a tail or 2 tails and a head.

(a) What would Lumpy say in this case?

(b) What would Eddie say in this case?

Be sure to clearly explain why you think they would answer in the way you suggest.

A.52 Go Climb a Tree!

Fixnote: Make this more careful and focused. Perhaps begin with one and one freethrows in basketball as a reason to draw a tree diagram. In all problems, we need to assume independence. In problem 3, best of five games would be enoug. In problem 5, we mean by guessing.

In this activity, we'll evaluate the probabilities of complex events using tree diagrams, fraction arithmetic, and counting.

A.52.1) Give a story problem that is modeled by the expression:

$$\frac{3}{7} \times \frac{2}{5}$$

Let the start of the story be: "2/5 of the class are girls." Once you have the story, solve it using pictures (use rectangles for the wholes) and explain why it makes sense that multiplying fractions is the same as multiplying the numerators and multiplying the denominators.

A.52.2) The Weather Channel has predicted that there is a 70% chance of rain today, a 20% chance of rain tomorrow, and a 40% chance of rain the day after tomorrow. Use a tree diagram to help answer the following:

(a) What is the probability that it will rain today and not rain tomorrow?

(b) What is the probability it will rain on exactly one of the first two days?

(c) What is the probability that it will rain today, not rain tomorrow, and rain the following day?

(d) What is the probability that it will rain on exactly two of the three days?

(e) What is the probability it will rain on all three days?

(f) What is the probability it won't rain at all?

(g) What is the probability it will rain on at least one of the days?

A.52.3) The Indians and the Yankees are to face each other in a best-of-seven series. The probability that the Indians will win any game is 30%.

(a) What is the probability that the Indians win games 1, 3, 4, and 6 to win the series?

(b) What is the probability that the Indians win the series in exactly 6 games?

(c) What is the probability that the Indians win the series?

A.52.4) Fred the Slob has an unreliable car that starts only 65% of the days. If the car doesn't start, poor Fred must walk the one block to work. This week, he is slated to work 6 days (Monday through Saturday).

(a) What is the probability that Fred will walk on Monday and Wednesday and drive the other days?

(b) What is the probability that Fred will drive on exactly 4 of the days?

(c) What is the probability that poor Fred will have to walk on at least two of the days?

A.52.5) Use the techniques of this activity (i.e., using a special case and fraction arithmetic to help investigate a more general case) to find the probability of passing a 10-question multiple choice test by guessing if you must get 70% or more correct to pass.

A.53 They'll Fall for Anything!

What is incorrect about the following reasoning? Be specific!

A.53.1) Herman says that if you pick a United States citizen at random, the probability of selecting a citizen from Indiana is because Indiana is one of 50 equally likely states to be selected.

A.53.2) Jerry has set up a game in which one wins a prize if he/she selects an orange chip from a bag. There are two bags to choose from. One has 2 orange and 4 green chips. The other bag has 7 orange and 7 green chips. Jerry argues that you have a better chance of winning by drawing from the second bag because there are more orange chips in it.

A.53.3) Gil the Gambler says that it is just as likely to flip 5 coins and get exactly 3 heads as it is to flip 10 coins and get exactly 6 heads because

$$\frac{3}{5} = \frac{6}{10}$$

A.53.4) We draw 4 cards without replacement from a deck of 52. Know-it-all Ned says the probability of obtaining all four 7's is $\frac{4}{\binom{52}{4}}$ because there are ways to select the $\binom{52}{4}$ 4 cards and there are four 7's in the deck.

A.53.5) At a festival, Stealin' Stan gives Crazy Chris the choice of one of three prizes—each of which was hidden behind a door. One of the doors has a fabulous prize behind it while the other two doors each have a "zonk" (a free used tube of toothpaste, etc.). Crazy Chris chooses Door #1. Before opening that door, Stealin' Stan shows Chris that hidden behind Door #3 is a zonk and gives Chris the option to keep Door #1 or switch to Door #2. Chris says, "Big deal. It doesn't help my chances of winning to switch or not switch."

B References and Further Reading

[1] B. Averbach and O. Chein. *Problem Solving Through Recreational Mathematics*. Dover, 2000.

[2] A.K. Bag. *Binomial Theorem in Ancient India*. Indian Journal of History of Science, vol. 1, 1966 68–74.

[3] M. Baron. *The Origins of the Infinitesimal Calculus*. Dover, 1987.

[4] S. Beckmann. *Mathematics for Elementary Teachers*. Addison Wesley, 2007.

[5] A. Bogomolny. Leibniz and Pascal Triangles. *Interactive Mathematics Miscellany and Puzzles*. http://www.cut-the-knot.org/

[6] C.B. Boyer and U.C. Merzbach. *A History of Mathematics*. Wiley, 1991.

[7] H. Colebrooke. *Classics of Indian Mathematics*. Sharada Publishing House, 2005.

[8] U. Dudley. *Elementary Number Theory*. Dover, 2008.

[9] R.W. Dull. *Mathematics for Engineers*. Second Edition. McGraw-Hill, 1941.

[10] E. Dunne and M. McConnell. *Pianos and Continued Fractions*. Mathematics Magazine, vol. 72, no. 2 (1999), 104-115.

[11] Euclid and T.L. Heath. *The Thirteen Books of Euclid's Elements: Volumes 1, 2, and 3*. Dover, 1956.

[12] H. Eves. *An Introduction to the History of Mathematics*. Saunders College Publishing, 1990.

[13] M. Gardner. *My Best Mathematical and Logic Puzzles*. Dover, 1994.

[14] L.J. Lange. *An Elegant Continued Fraction for π*. The American Mathematical Monthly, vol. 106, no. 5, (1999), pp. 456–458.

[15] M. Livio. *The Equation that Couldn't be Solved*. Simon&Schuster, 2005.

[16] R.L. Keeney and J.F. Ramaley. *On the Trinomial Coefficients*. Mathematics Magazine, vol. 42, no. 4 (Sep., 1969), pp. 210-213.

[17] D.E. Knuth. *The Art of Computer Programming. Volume 1: Fundamental Algorithms*. Addison-Wesley, 1975.

[18] J.A. Mira. *Mathematical Teasers*. Barnes&Noble, 1970.

[19] NRICH website.
http://nrich.maths.org/

[20] I. Papick. *Algebra Connections: Mathematics for Middle School Teachers*. Pearson, 2007.

[21] M. Petković. *Famous Puzzles of Great Mathematicians*. American Mathematical Society, 2009.

[22] G. Polya. *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving*. Wiley, 1981.

[23] A. Posamentier. *Teaching Secondary Mathematics*. Prentice-Hall, 2002.

[24] M. Tahan. *The Man Who Counted: A Collection of Mathematical Adven- tures*. W.W. Norton&Company, 1993.

[25] R. Solomon. *Abstract Algebra*. Brooks/Cole, 2003.

[26] C.R. Wylie Jr. *101 Puzzles in Thought and Logic*. Dover, 1957.

[27] B.K. Youse. *Arithmetic: A Modern Approach*. Prentice-Hall, 1963.