CP8: Fundamental theorem of algebra, factoring polynomials with real coefficients

Fundamental theorem of algebra: Every polynomial of the form
$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$$
where the a_i are complex numbers has a root r which is again a complex number. That is, there is a complex number r such that
$$a_m r^m + a_{m-1}r^{m-1} + \ldots + a_1 r + a_0 = 0.$$
That is, there is a complex number r such that
$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 = (x-r)\left(c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \ldots + c_1 x + c_0\right)$$

1. Show by induction that every polynomial
$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$$
where the a_i are complex numbers with $a_m \neq 0$ can be factored completely into the form
$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 = a_m(x-r_1)\cdot \ldots \cdot(x-r_m)$$
for some complex numbers r_i.

2. a) Factor the polynomial
$$3x^2 + 5x + 10$$
completely in the complex number system.
b) Factor the polynomial
$$x^3 - 1$$
completely in the complex number system.
c) Factor the polynomial
$$x^4 - 1$$
completely in the complex number system.

3. Suppose now that we are given a polynomial of the form
$$a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$$
where the a_i are *real* numbers, like the examples in Problem 2. Suppose that r is a *complex* number. Show that
$$\overline{a_m r^m + a_{m-1}r^{m-1} + \ldots + a_1 r + a_0} = a_m\bar{r}^m + a_{m-1}\bar{r}^{m-1} + \ldots + a_1\bar{r} + a_0.$$

4. a) Compute
$$\left(x-(2-3i)\right)\cdot\left(x-\overline{(2-3i)}\right) =$$
b)
$$\left(x-\left(1+\sqrt{3}i\right)\right)\cdot\left(x-\overline{\left(1+\sqrt{3}i\right)}\right) =$$
c)
$$\left(x-(b_0+b_1 i)\right)\cdot\left(x-\overline{((b_0+b_1 i))}\right) =$$

5. Suppose now that we are given a polynomial of the form

$$a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$

where the a_i are *real* numbers. Suppose that a *complex* number r is a root of this polynomial, that is,

$$a_m r^m + a_{m-1} r^{m-1} + \ldots + a_1 r + a_0 = 0.$$

Show that $\bar{r}$ is also a root. That is, show that

$$a_m \bar{r}^m + a_{m-1} \bar{r}^{m-1} + \ldots + a_1 \bar{r} + a_0 = 0.$$

6. Show that, if the a_i are *real* numbers, we can completely factor as follows:

$$a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0 =$$
$$a_m (x - r_1) \cdot (x - \bar{r}_1) \cdot \ldots \cdot (x - r_a) \cdot (x - \bar{r}_a) \cdot (x - r_{2a+1}) \cdot \ldots \cdot (x - r_m)$$

where $r_{2a+1}, \ldots, r_m$ are *real* numbers.

7. *Factorization of polynomials with real coefficients:* A polynomial of the form

$$a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$

where the a_i are *real* numbers can be factored into a product of polynomials with *real* coefficients such that the degree of each of the factor polynomials is of degree 1 or 2. Use Problems 4c) and 6 to say why this is true.