NUMBERS AND ALGEBRA (SUPPLEMENTS)

MATH 1165: AUTUMN 2014
THIS DOCUMENT WAS TYPESET ON NOVEMBER 2, 2016.

Contents

A Supplemental Activities

A.54 Second Differences

In a previous activity, we developed strategies for finding the sum of arithmetic series. In this activity, we use arithmetic series to develop a formula for a sequence that has constant second differences. Then we demonstrate that all quadratic sequences have constant second differences.

A.54.1) Consider the sequence $f(n)$ given in the table below. In the rightmost column, Δ ("delta") means difference, computed by subtracting the current value of $f(n)$ from the next.

n	$f(n)$	Δ
0	4	3
1	7	3
2	10	3
3	13	3
4	16	3
5	19	

(a) Explain how $f(5)$ can be computed from the shaded cells in the table.

(b) Generalize your method to develop and explain a formula for $f(n)$.

(c) What was it about the differences that made this problem easy?

A.54.2) Consider the sequence $g(n)$ given in the table below.

n	$g(n)$	Δ	$\Delta\Delta$
0	1		
1	−2		
2	1		
3	10		
4	25		
5	46		
6	73		

(a) Compute Δ by subtracting the current value of $g(n)$ from the next.

(b) Explain the formula $\Delta(n) = g(n+1) - g(n)$.

(c) Check that the shaded cells sum to $g(5)$, and explain how that makes sense based upon how the Δ values were calculated.

(d) Because the Δ values ("first differences") are not constant, use the $\Delta\Delta$ column to compute the "differences of the differences" (also called "second differences").

(e) From the fact that the second differences are constant, develop an explicit formula for Δ in terms of n.

A.54.3) The same sequence $g(n)$ is given below, this time with a formula for Δ in terms of n.

n	$g(n)$	$\Delta(n) = 6n - 3$
0	1	−3
1	−2	3
2	1	9
3	10	15
4	25	21
5	46	27
6	73	

(a) Explain each of the following steps:

$$\begin{aligned} g(5) &= 1 + \Delta(0) + \Delta(1) + \Delta(2) + \Delta(3) + \Delta(4) \\ &= 1 + (6 \cdot 0 - 3) + (6 \cdot 1 - 3) + (6 \cdot 2 - 3) + (6 \cdot 3 - 3) + (6 \cdot 4 - 3) \\ &= 1 + 6 \cdot (0 + 1 + 2 + 3 + 4) + (-3 + -3 + -3 + -3 + -3) \\ &= 1 + 6 \cdot \frac{5 \cdot 4}{2} + 5 \cdot (-3) \end{aligned}$$

(b) Where do you see arithmetic series in the calculations you just explained?

(c) Generalize the above approach to yield an expression for $g(n)$.

(d) What kind of sequence is $g(n)$?

A.54.4) A general quadratic sequence $h(n)$ is given below.

n	$h(n) = an^2 + bn + c$	Δ	$\Delta\Delta$
0			
1			
2			
3			

(a) Compute the values of $h(n)$.

(b) Compute Δ by subtracting the next value of $h(n)$ from the current.

(c) Use the $\Delta\Delta$ column to compute the second differences.

(d) Generalize the result for first differences by computing $\Delta(n) = h(n+1) - h(n)$.

(e) Generalize the result for second differences by computing $\Delta\Delta(n) = \Delta(n+1) - \Delta(n)$.

(f) Explain how your work demonstrates that, for any quadratic sequence, the second differences must be constant.