NUMBERS AND ALGEBRA (SUPPLEMENTS)

MATH 1165: AUTUMN 2014
THIS DOCUMENT WAS TYPESET ON DECEMBER 20, 2014.

Contents

A Supplemental Activities

A.20 Poor Old Horatio (revised)

In this activity we are going to investigate ratios, which are the cornerstone of middle school mathematics.

We would like to avoid procedural approaches, such as, “set up a proportion and cross multiply.” Instead, we want to reason from the context and use pictures and tables to support the reasoning. In cases where you “set up an equation and solve,” explain the reasoning behind the equation and the steps in your solution.

A.20.1) A shade of orange is made by mixing 3 parts red paint with 5 parts yellow paint. Sam says we can add 4 cups of each color of paint and maintain the same color. Fred says we can quadruple both 3 and 5 and get the same color.

(a) Who (if either or both) is correct? Explain your reasoning.

(b) Use a table like the one below to show other paint mixtures that are the desired shade of orange.

Red	3								
Yellow	5								

A.20.2) If we wanted to make the same orange paint but could only use 73 cups of yellow paint, how many cups of red paint would we need? Give a detailed explanation of your solution. And explain what the units are for each value in your equation.

A.20.3) If we wanted to make the same orange paint but could only use 56 cups of red paint, how many cups of yellow paint would we need? Give a detailed explanation of your solution. And explain what the units are for each value in your equation.

A.20.4) To answer the question about 73 cups of yellow paint, some people use a table like the following:

Red	3			
Yellow	5	1	73	x

Fill out the remainder of the table and give a general formula for computing how much red paint is needed when x cups of yellow paint is used. Be very clear about how you are using the table to support your reasoning.

A.20.5) Use the following table to give a general formula for computing how much yellow paint is needed when y cups of red paint is used. Be very clear about how you are using the table to support your reasoning.

Red	3			
Yellow	5			

A.20.6) Suppose apples come in bags of 13 apples each.

(a) If Shel has 9 bags of apples, how many apples does Shel have total?

(b) Use a table like the one below to help Shel know how many apples are in any number of bags.

Bags									
Apples									

(c) What are the units in for each number in your calculation in part (a)? What are the units in the answer for part (a)? Where do you see these units in your table?

A.50 Integer Addition and Subtraction

In this activity, we explore various models and strategies for making sense of addition and subtraction of integers.

Useful language

Addition and subtraction problems arise in situations where we add to, take from, put together, take apart, or compare quantities.

Recall that addition and subtraction facts are related. For example, if we know that $8 + 5 = 13$, then we also know three related facts: $5 + 8 = 13$, $13 - 8 = 5$, and $13 - 5 = 8$. In school mathematics, these are often called *fact families*.

A.50.1) What are integers? Describe some situations in which both positive and negative integers arise. Use the word "opposite" in your descriptions.

Red and black chips

A.50.2) In a red-and-black-chip model of the integers, red and black chips each count for 1, but they are opposites, so that they cancel each other out. Using language from accounting, suppose black chips are assets and red chips are debts. We add by putting chips together. Use red and black chips (or draw the letters R and B) to model the following computations.

(a) $(-5) + (-3)$

(b) $6 + (-4)$.

(c) $(-7) + 9$

(d) $2 + (-5)$

A.50.3) In the previous problem, you saw different combinations of red and black chips that had the same numerical value.

(a) How many ways are there to represent -3? Draw two different representations.

(b) Use the phrase "zero pairs" to describe how your two representations are related.

A.50.4) To subtract in the red-and-black-chip model, we can "take away" chips, as you might expect. When we don't have enough chips of a particular color, we can always add "zero pairs." Use this idea to model the following subtraction problems:

(a) $6 - 8$

(b) $4 - (-3)$

(c) $(-6) - 5$

(d) $(-3) - (-7)$

Subtraction as missing addend

A.50.5) To evaluate a subtraction expression, we can solve a related addition equation. For example, $11 - 7$ is the solution to $7 + \_\_ = 11$. Use this idea to evaluate the subtraction expressions in the previous problem.

Subtraction as difference on the number line

A.50.6) Use a number line to reason about $b - a$ by asking how to get from a to b: How far? And in which direction? For example, to evaluate $11 - 7$, we can ask how to get from 7 to 11. We travel 4 units to the right. Use this idea to evaluate the subtraction expressions in the previous problems.

A.50.7) How is subtraction different from negation?

A.50.8) Use what you have learned to explain why $a - (-b) = a + b$.

Other Models

Use the following models for addition and subtraction of integers. Each model requires two decisions: (1) how positive and negative integers are 'opposite' in the situation, and (2) how addition and subtraction are 'opposite' in a different way.

- A postal carrier who brings checks and bills—and who also takes them away.
- Walking on an North-South number line, facing either North or South, and walking either forward or backward.

A.51 Integer Multiplication

In this activity, we explore various models and strategies for making sense of multiplication of integers.

Continuing patterns

A.51.1)

(a) Continue the following patterns, and explain why it makes sense to continue them in that way.

$4 \times 3 = 12$	$3 \times 6 = 18$	$(-7) \times 3 = -21$
$4 \times 2 =$	$2 \times 6 =$	$(-7) \times 2 =$
$4 \times 1 =$	$1 \times 6 =$	$(-7) \times 1 =$
$4 \times 0 =$	$0 \times 6 =$	$(-7) \times 0 =$
$4 \times (-1) =$	$(-1) \times 6 =$	$(-7) \times (-1) =$
$4 \times (-2) =$	$(-2) \times 6 =$	$(-7) \times (-2) =$
$4 \times (-3) =$	$(-3) \times 6 =$	$(-7) \times (-3) =$

(b) What rule of multiplication might a student infer from the first pattern?

(c) What rule of multiplication might a student infer from the second pattern?

(d) What rule of multiplication might a student infer from the third pattern?

Using properties of operations

A.51.2) Suppose we *do not know* how to multiply negative numbers but we do know that $4 \times 6 = 24$. We will use this fact and the properties of operations to reason about products involving negative numbers.

(a) What do we know about A and B if $A + B = 0$?

(b) Use the distributive property to show that the expression $4 \times 6 + 4 \times (-6)$ is equal to 0. Then use that fact to reason about what $4 \times (-6)$ should be.

(c) Use the distributive property to show that the expression $4 \times (-6) + (-4) \times (-6)$ is equal to 0. Then use that fact to reason about what $(-4) \times (-6)$ should be.

Walking on a number line

A.51.3) Matt is a member of the Ohio State University Marching Band. Being rather capable, Matt can take x steps of size y inches for all integer values of x and y. If x is positive it means *face North and take x steps*. If x is negative it means *face South and take* $|x|$ *steps*. If y is positive it means your step is a *forward step of* y *inches*. If y is negative it means your step *is a backward step of* $|y|$ *inches*.

(a) Discuss what the expressions $x \cdot y$ means in this context. In particular, what happens if $x = 1$? What if $y = 1$?

(b) If x and y are both positive, how does this fit with the "repeated addition" model of multiplication?

(c) Using the context above and specific numbers, demonstrate the general rule:

$$\text{negative} \cdot \text{positive} = \text{negative}$$

Clearly explain how your problem shows this.

(d) Using the context above and specific numbers, demonstrate the general rule:

$$\text{positive} \cdot \text{negative} = \text{negative}$$

Clearly explain how your problem shows this.

(e) Using the context above and specific numbers, demonstrate the general rule:

$$\text{negative} \cdot \text{negative} = \text{positive}$$

Clearly explain how your problem shows this.

A.52 Divisibility Statements (revised)

Let $a|b$ mean $b = aq$ for some integer q. (Read $a|b$ as "a divides b".)

A.52.1) Using the numbers 56 and 7, make some true statements using the notation above and one or more of the words factor, multiple, divisor, and divides.

A.52.2) Use the definition of *divides* to decide which of the following are true and which are false. If a statement is true, find q satisfying the definition of divides. If it is false, give an explanation. (Hint: Try to reason about multiplication rather than using your calculator.)

(a) $21|2121$

(b) $3|(9 \times 41)$

(c) $6|(2^4 \times 3^2 \times 7^3 \times 13^5)$

(d) $100000|(2^3 \times 3^9 \times 5^{11} \times 17^8)$

(e) $6000|(2^{21} \times 3^7 \times 5^{17} \times 29^5)$

(f) $p^3q^5r|(p^5q^{13}r^7s^2t^{27})$

(g) $7|(5 \times 21 + 14)$

A.52.3) If $a|b$ and $a|c$ does $a|(bc)$? Explain.

A.52.4) If $a|b$ and $a|c$ does $a|(b+c)$? Explain.

A.52.5) If $a|(b+c)$ and $a|c$ does $a|b$? Explain.

A.52.6) Suppose that

$$(3^5 \cdot 7^9 \cdot 11^x \cdot 13^y)|(3^a \cdot 7^b \cdot 11^{19} \cdot 13^7)$$

What values of a, b, x, and y make true statements?

A.53 Fraction Multiplication

A.53.1) Suppose x and y are counting numbers.

(a) What is our convention for the meaning of xy as repeated addition?

(b) In our convention for the meaning of the product xy, which letter describes *how many groups* and which letter describes *how many in one group*?

(c) In the product xy, the x is called the *multiplier* and y is called the *multiplicand*. Use these words to describe the meaning of xy as repeated addition.

A.53.2) In the Common Core State Standards, fractions and fraction operations are built from *unit fractions*, which are fractions with a 1 in the numerator. The meaning of a fraction $\frac{a}{b}$ involves three steps: (1) determining the whole; (2) describing the meaning of $\frac{1}{b}$; and (3) describe the meaning of the fraction $\frac{a}{b}$. Use pictures to illustrate these three steps for the fraction $\frac{3}{5}$.

A.53.3) Now we combine the ideas from the previous two problems to describe meanings for simple multiplication of fractions.

(a) Without computing the result, describe the meaning of the product $5 \times \frac{1}{3}$.

(b) Without computing the result, describe the meaning of the product $\frac{1}{3} \times 5$.

(c) Without using the commutativity of multiplication (which we have not established for fractions), use these meanings and pictures to explain what the products should be.

A.53.4) Beginning with a unit square, use an area model to illustrate the following:

(a) $\frac{1}{3} \times \frac{1}{4}$

(b) $\frac{7}{3} \times \frac{5}{4}$

A.53.5) When computing $2\frac{1}{3} \times 3\frac{2}{5}$, Byron says that the answer is $6\frac{2}{15}$.

(a) Explain Byron's method.

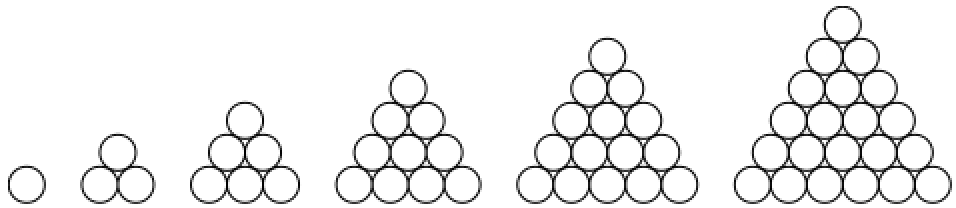
(b) How do you know that he is incorrect?

(c) Use what is right about his method to show what he is missing.

A.54 Arithmetic Series

In this activity, we explore *arithmetic series*, which are sums of consecutive terms from an arithmetic sequence.

Ms. Nguyen's math class has been looking at "triangular numbers." The first 6 triangular numbers are shown below.



A.54.1) Blair wanted to find the 551^{st} triangular number. She used a table and looked for a pattern in the *sequence of partial sums*: $1, 1+2, 1+2+3, \ldots$. Help her finish her idea.

A.54.2) Kaley realized the the 551^{st} triangular number would be the sum

$$1+2+3+4+\cdots+548+549+550+551$$

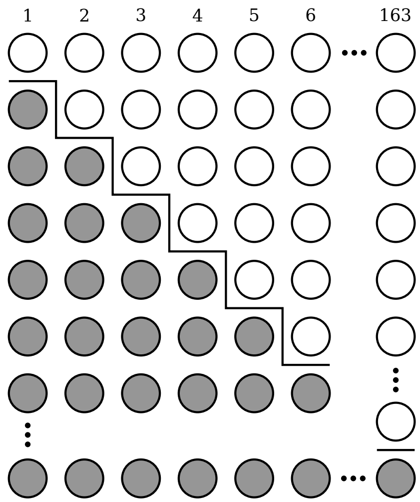
She started pairing the first with the last number; the second with the second-to-last; the third with the third-to-last; and so on. She saw that the averages are always the same. Help her finish her idea.

A.54.3) Ali begin by writing out the sum forward and backward and follows:

$$\begin{array}{c} 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 546 + 547 + 548 + 549 + 550 + 551 \\ 551 + 550 + 549 + 548 + 547 + 546 + \cdots + 6 + 5 + 4 + 3 + 2 + 1 \end{array}$$

Help her finish her idea. Be sure to explain clearly what happens "in the dots." Does it matter whether there are an even or an odd number of terms?

A.54.4) Cooper was interested in a different triangular number and drew the following picture:



Which triangular number was he finding? Help him finish his idea. Be sure to explain clearly what happens "in the dots."

A.54.5) Sum the numbers:

$$106 + 112 + 118 + \cdots + 514$$

A.54.6) Sum the numbers:

$$2.2 + 2.9 + 3.6 + 4.3 + \cdots + 81.3$$

A.54.7) Suppose you have an arithmetic sequence beginning with a, with a constant difference of d and with n terms.

(a) What is the n^{th} term of the sequence?

(b) Use dots to write the series consisting of the first n terms of this sequence.

(c) Find the sum of this series.

A.55 Geometic Series

In this activity, we explore *geometric series*, which are sums of consecutive terms from an geometric sequence.

Ms. Radigan's math class has been trying to compute the following sums:

$$1 + 2 + 4 + 8 + \cdots + 2^{19}$$

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{13}}$$

A.55.1) Kelsey used tables and looked for pattern in the *sequence of partial sums*: $1, 1 + 2, 1 + 2 + 4, \ldots$. Help her finish her idea for both sequences.

A.55.2) For the sum beginning with $\frac{2}{3}$, Erin started by drawing a large square (which she imagined as having area 1), and she shaded in $\frac{2}{3}$ of it. Then she shaded in $\frac{2}{9}$ more, and so on. Help her finish her idea.

A.55.3) Ryan wrote out all of the terms in the first sum, represented as powers of 2, beginning with $1 + 2 + 2^2 + 2^3$. Then he realized that because the terms formed a geometric sequence, he could multiply the sequence by the common ratio of 2, and the resulting sequence would be almost identical to the first, differing only at the beginning and the end. By subtracting the first sequence from the second, all of the middle terms would cancel. Help him finish his idea.

A.55.4) Ali said, "Here is a thought experiment. I take a sheet of paper, rip it perfectly into thirds, place one piece to start a pile that I will call A, another piece to start a pile I will call B, and I keep the third piece in my hands. I then rip that piece into thirds, place one piece on pile A, one piece on pile B, and keep the third. Notice that each of pile A and pile B have $\frac{1}{3} + \frac{1}{9}$ of a sheet of paper, and I still have $\frac{1}{9}$ of a sheet in my hands. I continue this process until I place $\frac{1}{3^{13}}$ of a sheet on each pile and still have $\frac{1}{3^{13}}$ of a sheet in my hands.

Help Ali finish her idea.

A.55.5) Sum the expression:

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \frac{2^n}{3^n}$$

What happens to this sum as n gets really large?

A.55.6) Consider the expression:

$$\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots + \frac{7}{10^n}$$

(a) Find the sum of the expression.

(b) What happens to this sum as n gets really large?

(c) How does this help you explain why a particular repeating decimal is a particular rational number? Be sure to indicate what repeating decimal and what rational number you are talking about.

A.55.7) Suppose you have an geometric sequence beginning with a, with a constant ratio of r and with n terms.

(a) What is the n^{th} term of the sequence?

(b) Use dots to write the series consisting of the first n terms of this sequence.

(c) Find the sum of this series.

A.56 Solving Quadratics

Here we explore various methods for solving quadratic equations in one variable.
Please read all instructions carefully.

A.56.1) Is $\sqrt{4} = \pm 2$? Explain.

A.56.2) Suppose that $\sqrt{4} = \pm 2$? Then evaluate $\sqrt{4} + \sqrt{9}$.

A.56.3) What does your calculator say about $\sqrt{4} + \sqrt{9}$?

A.56.4) In the following problems, you **may not use the quadratic formula**. But just for the record, write down the quadratic formula.

A.56.5) In the following list of equations, solve those that are **easy** to solve.

(a) $(x-3)(x+2)=0$

(b) $(x-3)(x+2)=1$

(c) $(2x-5)(3x+1)=0$

(d) $(x-a)(x-b)=0$

(e) $(x-1)(x-3)(x+2)(2x-3)=0$

A.56.6) Regarding the previous problem, state the property of numbers that made all but one of the equations easy to solve.

A.56.7) For each part below, write a quadratic equation with the stated solution(s) and no other solutions.

(a) $x=7$ and $x=-4$

(b) $x=p$ and $x=q$

(c) $x=3$

(d) $x=\dfrac{1\pm\sqrt{5}}{2}$

A.56.8) In the following list of equations, solve those that are **easy** to solve.

(a) $x^2=5$

(b) $x^2-4=2$

(c) $x^2-4x=2$

(d) $2x^2=1$

(e) $(x-2)^2=5$

A.56.9) Regarding the previous problem, state the property of numbers that made all but one of the equations easy to solve.

A.56.10) Although 160 is not a square in base ten, what could you add to 160 so that the result would be a square number?

A.56.11) Consider the polynomial expression $x^2 + 6x$ to be a number in base x. We want to add to this polynomial so that the result is a square in base x.

(a) Use "flats" and "longs" to draw a picture of this polynomial as a number in base x, adding enough "ones" so that you can arrange the polynomial into a square.

(b) What "feature" of the square does the new polynomial expression represent?

(c) Why does it make sense to call this technique "completing the square"?

(d) Use your picture to help you solve the equation $x^2 + 6x = 5$.

A.56.12) Complete the square to solve the following equations:

(a) $x^2 + 3x = 4$

(b) $x^2 + bx = q$

(c) $2x^2 + 8x = 12$

(d) $ax^2 + bx + c = 0$

A.56.13) Find all solutions to $x^3 - 3x^2 + x + 1 = 0$. Hint: One solution is $x = 1$.

A.57 Solving Cubic Equations

To solve the cubic equation $x^3 + px + q = 0$, we use methods that were discovered and advanced by various mathematicians, including Ferro, Tartaglia, and Cardano. The approach is organized in three steps. **Make notes in the margin as you follow along.**

A.57.1 Step 1: Replace x with $u + v$

In $x^3 + px + q = 0$, let $x = u + v$. Show that the result can be written as follows:

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

A.57.2 Step 2: Set uv to eliminate terms

If $3uv + p = 0$, then all of the terms are eliminated except for u^3, v^3, and constant terms. Explain why the equation simplifies nicely to:

$$u^3 + v^3 + q = 0.$$

Solve $3uv + p = 0$ for v, substitute, and show that we have:

$$u^3 + \left(\frac{-p}{3u}\right)^3 + q = 0.$$

A.57.3 Step 3: Recognize the equation as a quadratic in u^3 and solve

By multiplying by u^3, show that we get a quadratic in u^3:

$$u^6 + qu^3 + \left(\frac{-p}{3}\right)^3 = 0.$$

Show that this has solutions:

$$u^3 = \frac{-q \pm \sqrt{q^2 - 4\left(\frac{-p}{3}\right)^3}}{2}.$$

Now, use the facts $v = -p/(3u)$ and $x = u + v$ to write a formula for x:

$$x = \sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4\left(\frac{-p}{3}\right)^3}}{2}} + \frac{-p}{3\sqrt[3]{\frac{-q\pm\sqrt{q^2-4\left(\frac{-p}{3}\right)^3}}{2}}}.$$

A.57.1) How many values does this formula give for x? From the original equation $x^3 + px + q = 0$, how many solutions should we expect?

A.57.2) Use the above formula to solve the specific equation $x^3 - 15x - 4 = 0$. Show that

$$x = \sqrt[3]{2 \pm \sqrt{-121}} + \frac{5}{\sqrt[3]{2 \pm \sqrt{-121}}}.$$

Are these values of x real numbers?

A.57.3) Use technology to graph $y = x^3 - 15x - 4$. According to the graph, how many real roots does the polynomial have? What is going on?

A.57.4) Choose "plus" in the ±, and check that $2 + \sqrt{-1}$ is a cube root of $2 + \sqrt{-121}$. Use that fact to simplify the above expression for x. What do you notice?

A.57.5) Now choose "minus" in the ± above, and find the value of x. What do you notice?

In both cases, the formula requires computations with square roots of negative numbers, but the result is a real solution. These kinds of occurrences were the historical impetus behind the gradual acceptance of complex numbers.