Numbers and Algebra

Math 1165: Fall 2012

With Teaching Notes

Distributing this Document

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Preface

These notes are designed with future middle grades mathematics teachers in mind. While most of the material in these notes would be accessible to an accelerated middle grades student, it is our hope that the reader will find these notes both interesting and challenging. In some sense we are simply taking the topics from a middle grades class and pushing "slightly beyond" what one might typically see in schools. In particular, there is an emphasis on the ability to communicate mathematical ideas. Three goals of these notes are:

- To enrich the reader's understanding of both numbers and algebra. From the basic algorithms of arithmetic—all of which have algebraic underpinnings, to the existence of irrational numbers, we hope to show the reader that numbers and algebra are deeply connected.

- To place an emphasis on problem solving. The reader will be exposed to problems that "fight-back." Worthy minds such as yours deserve worthy opponents. Too often mathematics problems fall after a single "trick." Some worthwhile problems take time to solve and cannot be done in a single sitting.

- To challenge the common view that mathematics is a body of knowledge to be memorized and repeated. The art and science of doing mathematics is a process of reasoning and personal discovery followed by justification and explanation. We wish to convey this to the reader, and sincerely hope that the reader will pass this on to others as well.

In summary—you, the reader, must become a doer of mathematics. To this end, many questions are asked in the text that follows. Sometimes these questions are answered, other times the questions are left for the reader to ponder. To let the reader know which questions are left for cogitation, a large question mark is displayed:

$$?$$

The instructor of the course will address some of these questions. If a question is not discussed to the reader's satisfaction, then I encourage the reader to put on a thinking-cap and think, think, think! If the question is still unresolved, go to the World Wide Web and search, search, search!

Please report corrections, suggestions, gripes, complaints, and criticisms to Bart Snapp at: `snapp@math.osu.edu`

Thanks and Acknowledgments

This document is based on a set of lectures originally given by Bart Snapp at the Ohio State University Fall 2009 and Fall 2010. In 2012, Bart Snapp and Vic Ferdinand worked on a major revision, incorporating many ideas from Vic's previous courses. Special thanks goes to Herb Clemens, Vic Ferdinand, and Betsy McNeal for many helpful comments which have greatly improved these notes.

Contents

Chapter 1

Arithmetic and Algebra

> As I made my way home, I thought Jem and I would get grown but there wasn't much else left for us to learn, except possibly algebra.
>
> —Harper Lee

Teaching Note: Here we outline a story with a series of puzzles. We suggest that the instructor simply present the puzzles (or similar puzzles) and have the students solve them rather than go through the entire story in class.

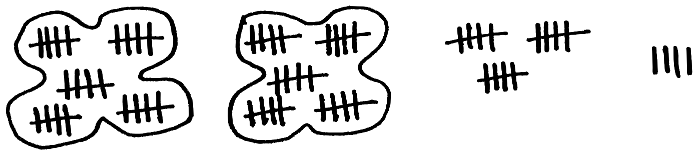
Teaching Note: Activity A.1 complements this section well.

1.1 Home Base

Imagine 600 generations past—that's on the order of 10000 years, the dawn of what we would call civilization. This is a long time ago, well before the *Epic of Gilgamesh*. Even then people already knew the need to keep track of numbers. However, they didn't use the numbers we know and love (that's right, *love*!), they used tally-marks. Now what if "a friend" of yours had a time machine? What if they traveled through time and space and they decided to take you back 500 generations? Perhaps you would meet a nice man named Lothar[1] who is

[1] *Lothar of the Hill People* is his full name

trying to keep track of his goats. He has the following written on a clay tablet:



From this picture you discern that Lothar has 69 goats. Lothar is studying the tablet intently when his wife, Gertrude, comes in. She tries in vain to get Lothar to keep track of his goats using another set of symbols:

0 1 2 3 4

A heated debate between Lothar and Gertrude ensues, the exact details of which are still a mystery. We do glean the following facts:

(1) Under Gertrude's scheme, five goats are denoted by:

10

(2) The total number of Lothar's goats is denoted by:

234

Question Can you explain Gertrude's counting scheme?

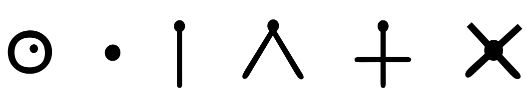
?

Did I mention that "your friend's" time machine is also a spaceship? Oh... Well it is. Now you both travel to the planet Omicron Persei 8. There are two things you should know about the inhabitants of Omicron Persei 8:

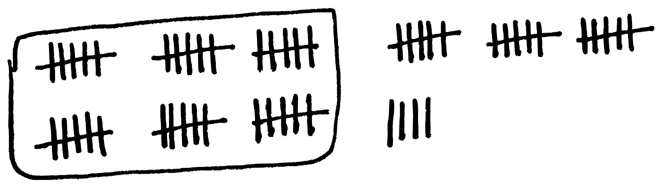
(1) They only have 3 fingers on each hand.

(2) They can eat a human in one bite.

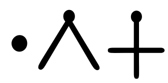
As you can see, there are serious issues that any human visitor to Omicron Persei 8 must deal with. For one thing, since the Omicronians only have 3 fingers on each hand, they've only written down the following symbols for counting:



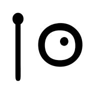
Emperor Lrrr of the Omicronians is tallying how many humans he ate last week



when his wife, Ndnd, comes in and reminds him that he can write this number using their fancy symbols as:



After reading some restaurant menus, you find out that twelve tally-marks are denoted by the symbols:



Question Can you explain the Omicronians' counting scheme?

?

At this point you hop back into "your friend's" space-time ship. "Your friend" kicks off their shoes. You notice that "your friend" has 6 toes on each foot. You strike up a conversation about the plethora of toes. Apparently this anomaly has enabled "your friend" to create their own counting scheme, which they say is based on:

- Toes
- Feets
- Feets of Feets
- and so on...

"Your friend" informs you that they would write the number you know as "twenty-six" as 22 or "two feets and two toes." What?! Though you find the conversation to be dull and stinky, you also find out that "your friend" uses two more symbols when they count. "Your friend" uses the letter A to mean what you call "ten," and the letter B to mean what you call "eleven!"

Question Can you explain "your friend's" counting scheme?

?

Problems for Section 1.1

(1) Explain why the following “joke” is “funny:” *There are* 10 *types of people in the world. Those who understand base* 2 *and those who don’t.*

(2) You meet some Tripod aliens, they tally by threes. Thankfully for everyone involved, they use the symbols 0, 1, and 2.

(a) Can you explain how a Tripod would count from 11 to 201? Be sure to carefully explain what number comes after 22.

(b) What number comes before 10? 210? 20110? Explain your reasoning.

(3) You meet some people who tally by sevens. They use the symbols O, A, B, C, D, E, and F.

(a) What do the individual symbols O, A, B, C, D, E, and F mean?

(b) Can you explain how they would count from DD to AOC? Be sure to carefully explain what number comes after FF.

(c) What number comes before AO? ABO? $EOFFO$? Explain your reasoning.

(4) Now, suppose that you meet a hermit who tallies by thirteens. Explain how he might count. Give some relevant and revealing examples.

(5) While visiting Mos Eisley spaceport, you stop by Chalmun’s Cantina. After you sit down, you notice that one of the other aliens is holding a discussion on fractions. Much to your surprise, they explain that 1/6 of 30 is 4. You are unhappy with this, knowing that 1/6 of 30 is in fact 5, yet their audience seems to agree with it, not you. Next the alien challenges its audience by asking, “what is 1/4 of 10?” What is the correct answer to this question and how many fingers do the aliens have? Explain your reasoning.

(6) When the first Venusian to visit Earth attended a 6th grade class, it watched the teacher show that

$$\frac{3}{12} = \frac{1}{4}.$$

“How strange,” thought the Venusian. “On Venus, $\frac{4}{12} = \frac{1}{4}$.” What base do Venusians use? Explain your reasoning.

(7) When the first Martian to visit Earth attended a high school algebra class, it watched the teacher show that the only solution of the equation

$$5x^2 - 50x + 125 = 0$$

is $x = 5$.

“How strange,” thought the Martian. “On Mars, $x = 5$ is a solution of this equation, but there also is another solution.” If Martians have more fingers than humans, how many fingers do Martians have? Explain your reasoning.

> ***Teaching Note: Here you cannot factor—you must first convert to base b.***

(8) In one of your many space-time adventures, you see the equation

$$\frac{3}{10} + \frac{4}{13} = \frac{21}{20}$$

written on a napkin. How many fingers did the beast who wrote this have? Explain your reasoning.

(9) What is the smallest number of weights needed to produce every integer-valued mass from 0 grams to say n grams? Explain your reasoning.

(10) Starting at zero, how high can you count using just your fingers?

(a) Explain how to count to 10.

(b) Explain how to count to 35.

(c) Explain how to count to 1023.

(d) Explain how to count to 59048.

(e) Can you count even higher?

Explain your reasoning.

1.2 Arithmetic

Consider this question:

Question Can you *think* about something if you lack the *vocabulary* required to discuss it?

?

Teaching Note: Activity A.2 complements this section well.

1.2.1 Nomenclature

The numbers and operations we work with have properties whose importance are so fundamental that we have given them names. Each of these properties is surely well known to you; however, the importance of the name is that it gives a keen observer the ability to see and articulate fundamental structures in arithmetic and algebra.

The Associative Property An operation $\star$ is called **associative** if for all numbers a, b, and c:

$$a \star (b \star c) = (a \star b) \star c$$

The Commutative Property An operation $\star$ is called **commutative** if for all numbers a and b:

$$a \star b = b \star a$$

The Distributive Property An operation $\star$ is said to be **distributive** over another operation $\maltese$ if for all numbers a, b, and c:

$$a \star (b \maltese c) = (a \star b) \maltese (a \star c) \qquad \text{and} \qquad (b \maltese c) \star a = (b \star a) \maltese (c \star a)$$

You may find yourself a bit distressed over some of the notation used above. In particular you surely notice that we were using crazy symbols like $\star$ and $\maltese$. We did this for a reason. The properties above may hold for more than one operation. Let's explore this:

Question Can you give examples of operations that are associative? Can you give examples of operations that are not associative?

?

Question Can you give examples of operations that are commutative? Can you give examples of operations that are not commutative?

?

Question Can you give examples of two operations where one distributes over the other? Can you give examples of operations that do not distribute?

?

1.2.2 Algorithms

Teaching Note: Here we seek to have the students acknowledge the algebra behind many algorithms. We have given a number of examples illustrating the sort of work we wish to see.

Teaching Note: Activities A.3 and A.4 complement this section well.

In elementary school you learned many algorithms. One of the first algorithms you learned was for adding numbers. Here we show you an example of the algorithm in action:

Basic Addition Algorithm Here is an example of the basic addition algorithm:

$$\begin{array}{r} 11\\ 892\\ +398\\ \hline 1290\end{array}$$

Question Can you describe how to perform this algorithm?

As a gesture of friendship, I'll take this one. All we are doing here is adding each column of digits at a time, starting with the right-most digit

$$\begin{array}{r} \\ 89\mathbf{2}\\ +39\mathbf{8}\\ \hline \mathbf{10}\end{array} \quad\rightsquigarrow\quad \begin{array}{r}\mathbf{1}\\ 89\mathbf{2}\\ +39\mathbf{8}\\ \hline \mathbf{0}\end{array}$$

If our column of digits sums to 10 or higher, then we must "carry" the tens-digit of our sum to the next column. This process repeats until we run out of digits on the left.

$$\begin{array}{r} \mathbf{1}\\ 8\mathbf{9}2\\ +3\mathbf{9}8\\ \hline \mathbf{19}0\end{array} \quad\rightsquigarrow\quad \begin{array}{r}\mathbf{1}1\\ \mathbf{8}92\\ +\mathbf{3}98\\ \hline \mathbf{12}90\end{array}$$

We're done!

Question Can you show the "behind-the-scenes" algebra going on here?

I'll take this one too. Sure, you just write:

$$\begin{aligned}
892 + 398 &= (8 \cdot 10^2 + 9 \cdot 10 + 2) + (3 \cdot 10^2 + 9 \cdot 10 + 8) \\
&= 8 \cdot 10^2 + 9 \cdot 10 + 2 + 3 \cdot 10^2 + 9 \cdot 10 + 8 \\
&= 8 \cdot 10^2 + 3 \cdot 10^2 + 9 \cdot 10 + 9 \cdot 10 + 2 + 8 \\
&= (8 + 3) \cdot 10^2 + (9 + 9) \cdot 10 + (2 + 8) \\
&= (8 + 3) \cdot 10^2 + (9 + 9) \cdot 10 + 10 + 0 \\
&= (8 + 3) \cdot 10^2 + (9 + 9 + 1) \cdot 10 + 0 \\
&= (8 + 3) \cdot 10^2 + (10 + 9) \cdot 10 + 0 \\
&= (8 + 3 + 1) \cdot 10^2 + 9 \cdot 10 + 0 \\
&= 12 \cdot 10^2 + 9 \cdot 10 + 0 \\
&= 1290
\end{aligned}$$

Wow! That was a lot of algebra. At each step, you should be able to explain how to get to the next step, and state which algebraic properties are being used.

Basic Multiplication Algorithm Here is an example of the basic multiplication algorithm:

$$\begin{array}{r}
23 \\
634 \\
\times \quad 8 \\
\hline
5072
\end{array}$$

Question Can you describe how to perform this algorithm?

Me me me me! All we are doing here is multiplying each digit of the multi-digit number by the single digit number.

$$\begin{array}{r}
\\
63\mathbf{4} \\
\times \quad \mathbf{8} \\
\hline
\mathbf{32}
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{r}
\mathbf{3} \\
63\mathbf{4} \\
\times \quad \mathbf{8} \\
\hline
\mathbf{2}
\end{array}$$

If our product is 10 or higher, then we must "carry" the tens-digit of our product to the next column. This "carried" number is then added to our new product. This process repeats until we run out of digits on the left.

$$\begin{array}{r}
\mathbf{3} \\
6\mathbf{3}4 \\
\times \quad \mathbf{8} \\
\hline
\mathbf{27}2
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{r}
\mathbf{2}3 \\
\mathbf{6}34 \\
\times \quad \mathbf{8} \\
\hline
\mathbf{50}72
\end{array}$$

We're done!

Question Can you show the "behind-the-scenes" algebra going on here?

You betcha! Just write:

$$
\begin{aligned}
634 \cdot 8 &= (6 \cdot 10^2 + 3 \cdot 10 + 4) \cdot 8 \\
&= 6 \cdot 8 \cdot 10^2 + 3 \cdot 8 \cdot 10 + 4 \cdot 8 \\
&= 6 \cdot 8 \cdot 10^2 + 3 \cdot 8 \cdot 10 + 32 && (✿) \\
&= 6 \cdot 8 \cdot 10^2 + (3 \cdot 8 + 3) \cdot 10 + 2 && (❀) \\
&= 6 \cdot 8 \cdot 10^2 + 270 + 2 && (❄) \\
&= (6 \cdot 8 + 2) \cdot 10^2 + 7 \cdot 10 + 2 && (✪) \\
&= 50 \cdot 10^2 + 7 \cdot 10 + 2 \\
&= 5 \cdot 10^3 + 0 \cdot 10^2 + 7 \cdot 10 + 2 \\
&= 5072
\end{aligned}
$$

Ahhhhh! Algebra works. Remember just as before, at each step you should be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (✿) and (❀)? What about between lines (❄) and (✪)?

?

Basic Division Algorithm Once more we meet with this old foe—long division. Here is an example of the basic division algorithm:

$$
\begin{array}{r@{\,}l}
 & \;\;97\,\mathrm{R}1 \\
8 & \overline{)777} \\
 & \;\underline{72} \\
 & \;\;\,57 \\
 & \;\;\,\underline{56} \\
 & \;\;\;\;1
\end{array}
$$

Question Can you describe how to perform this algorithm?

Yes! I'm all about this sort of thing. All we are doing here is single digit division for each digit of the multi-digit dividend (the number under the division symbol) by the single digit divisor (the left-most number). We start by noting that 8 won't go into 7, and so we see how many times 8 goes into 77.

$$
\begin{array}{r@{\,}l}
 & \;\;\mathbf{9} \\
\mathbf{8} & \overline{)\mathbf{77}7} \\
 & \underline{\mathbf{72}} \\
 & \;\;\mathbf{5}
\end{array}
\qquad \leftrightsquigarrow \qquad
\begin{aligned}
n &= d \cdot q + r \\
77 &= 8 \cdot 9 + 5
\end{aligned}
$$

Now we drop the other 7 down, and see how many times 8 goes into 57.

$$\begin{array}{r} 9\mathbf{7} \\ \mathbf{8}\,\overline{)777} \\ \underline{72} \\ \mathbf{57} \\ \underline{\mathbf{56}} \\ \mathbf{1} \end{array} \quad \leftrightsquigarrow \quad \begin{aligned} n &= d \cdot q + r \\ 57 &= 8 \cdot 7 + 1 \end{aligned}$$

This process repeats until we run out of digits in the dividend.

Question Can you show the "behind-the-scenes" algebra going on here?

Of course—but this time things will be a bit different.

$$\begin{aligned}
77 &= 8 \cdot 9 + 5 \\
77 \cdot 10 &= (8 \cdot 9 + 5) \cdot 10 \\
77 \cdot 10 &= 8 \cdot 9 \cdot 10 + 5 \cdot 10 \\
77 \cdot 10 + 7 &= 8 \cdot 9 \cdot 10 + 5 \cdot 10 + 7 \\
777 &= 8 \cdot (9 \cdot 10) + 57 && (\text{✿}) \\
777 &= 8 \cdot (9 \cdot 10) + (8 \cdot 7 + 1) && (\text{❀}) \\
777 &= 8 \cdot (9 \cdot 10) + 8 \cdot 7 + 1 && (\text{❄}) \\
777 &= 8 \cdot (9 \cdot 10 + 7) + 1 && (\text{✪}) \\
777 &= 8 \cdot 97 + 1
\end{aligned}$$

Looks good to me, but remember: At each step you must be able to explain how to get to the next step, and state which algebraic properties are being used.

Question Can you clearly explain what happened between lines (✿) and (❀)? What about between lines (❄) and (✪)?

?

Division Algorithm Without Remainder Do you remember that the division algorithm can be done in such a way that there is no remainder? Here is an example of the division algorithm without remainder:

$$\begin{array}{r} 0.75 \\ 4\,\overline{)3.00} \\ \underline{2\;8} \\ 20 \\ \underline{\underline{20}} \end{array}$$

Question Can you describe how to perform this algorithm?

I'm getting a bit tired, but I think I can do this last one. Again, all we are doing here is single digit division for each digit of the multi-digit dividend (the number under the division symbol) by the single digit divisor (the left-most number) adding zeros after the decimal point as needed. We start by noting that 4 won't go into 3, and so we see how many times 4 goes into 3.0. Mathematically this is the same question; however, by thinking of the 3.0 as 30, we put ourselves into familiar territory. Since

$$4 \cdot 7 = 30 \qquad \Rightarrow \qquad 4 \cdot 7 \cdot 10^{-1} = 30 \cdot 10^{-1} = 3$$

this will work as long as we put our 7 immediately to the right of the decimal point.

$$\begin{array}{r} \mathbf{0.7} \\ \mathbf{4}\overline{)\mathbf{3.0}} \\ \underline{\mathbf{2\,8}} \\ \mathbf{2} \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{r} n = d \cdot q + r \\ 30 = 4 \cdot 7 + 2 \end{array}$$

Now we are left with a remainder of .2. To take care of this, we drop another 0 down and see how many times 4 goes into 20. Since

$$4 \cdot 5 = 20 \qquad \Rightarrow \qquad 4 \cdot 5 \cdot 10^{-2} = 5 \cdot 10^{-2} = 0.05$$

this will work as long as we put our 5 two spaces to the right of the decimal point.

$$\begin{array}{r} 0.7\mathbf{5} \\ \mathbf{4}\overline{)3.0\mathbf{0}} \\ \underline{2\,8} \\ \mathbf{20} \\ \underline{\underline{\mathbf{20}}} \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{r} n = d \cdot q + r \\ 20 = 4 \cdot 5 + 0 \end{array}$$

This process repeats until we obtain a division with no remainder, or until we see repetition in the digits of the quotient.

Question Can you show the "behind-the-scenes" algebra going on here?

Let's do it:

$$\begin{aligned} 3 &= 4 \cdot 0 + 3 \\ 3.0 &= (4 \cdot 7 + 2) \cdot 10^{-1} \\ 3.0 &= 4 \cdot (7 \cdot 10^{-1}) + 2 \cdot 10^{-1} \\ 3.00 &= 4 \cdot (7 \cdot 10^{-1}) + 20 \cdot 10^{-2} \\ 3.00 &= 4 \cdot (7 \cdot 10^{-1}) + (4 \cdot 5) \cdot 10^{-2} && \text{(❄)} \\ 3.00 &= 4 \cdot (7 \cdot 10^{-1}) + 4 \cdot (5 \cdot 10^{-2}) && \text{(✪)} \\ 3.00 &= 4 \cdot (7 \cdot 10^{-1} + 5 \cdot 10^{-2}) \\ 3.00 &= 4 \cdot 0.75 \end{aligned}$$

Looks good to me, but remember: At each step you must be able to explain how to get to the next step, and state which algebraic properties are being used.

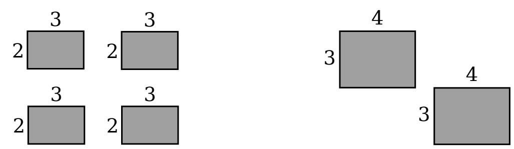
Question Can you clearly explain what happened between lines (❄) and (❂)?

?

Problems for Section 1.2

(1) Explain what it means for an operation $\star$ to be *associative*. Give some relevant and revealing examples.

(2) Consider the following pictures:



Jesse claims that these pictures represent $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$.

(a) Is Jesse's claim correct? Explain your reasoning.

(b) Do Jesse's pictures show the associativity of multiplication? If so, explain why. If not, draw new pictures representing $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ that do show the associativity of multiplication.

(3) Explain what it means for an operation $\star$ to be *commutative*. Give some relevant and revealing examples.

(4) Explain what it means for an operation $\star$ to *distribute* over another operation ✢. Give some relevant and revealing examples.

(5) Sometimes multiplication is described as *repeated addition*. Does this explain why multiplication is commutative? If so give the explanation. If not, give another description of multiplication that does explain why it is commutative.

(6) In a warehouse you obtain 20% discount but you must pay a 15% sales tax. Which would save you more money: To have the tax calculated first or the discount? Explain your reasoning—be sure to use relevant terminology.

(7) Money Bags Jon likes to give a tip of 20% when he is at restaurants. He does this by dividing his bill by 10 and then doubling it. Explain why this works.

(8) Regular Reggie likes to give a tip of 15% when he is at restaurants. He does this by dividing his bill by 10 and then adding half more to this number. Explain why this works.

(9) Wacky Wally has a strange way of giving tips when he is at restaurants. He does this by rounding his bill up to the nearest multiple of 7 and then taking the quotient (when that new number is divided by 7). Explain why this isn't as wacky as it might sound.

> ***Teaching Note: The problem above is fundamentally different than the other (related) problems involving tips.***

(10) Cheap Carl likes to give a tip of $13\frac{1}{3}\%$ when he is at restaurants. He does this by dividing his bill by 10 and then adding one-third more to this number. Explain why this works.

(11) Reasonable Rebbecca likes to give a tip of 18% when she is at restaurants. She does this by dividing her bill by 5 and then removing one-tenth of this number. Explain why this works.

(12) Can you think of and justify any other schemes for computing the tip?

(13) Here is an example of the basic addition algorithm:

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(14) Here is an example of the column addition algorithm:

$$\begin{array}{r} 892 \\ +398 \\ \hline 10 \\ 18 \\ 11 \\ \hline 1290 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(15) If you check out Problems (22) and (24), you will learn about "scaffolding" algorithms.

(a) Develop a scaffolding addition algorithm and describe how to perform this algorithm.

(b) Provide a relevant and revealing example demonstrating that you understand the algorithm.

   (c) Show the "behind-the-scenes" algebra that is going on here.

(16) Here is an example of the banker's addition algorithm:

$$\begin{array}{r} 892 \\ +398 \\ \hline 1\mathbf{0} \\ 1\mathbf{9} \\ \mathbf{12} \\ \hline 1290 \end{array}$$

   (a) Describe how to perform this algorithm.

   (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

   (c) Show the "behind-the-scenes" algebra that is going on here.

(17) Here is an example of the basic subtraction algorithm:

$$\begin{array}{rcc} & 8 & \\ 8 & \cancel{9} & {}^{1}2 \\ -3 & 7 & 8 \\ \hline 5 & 1 & 4 \end{array}$$

   (a) Describe how to perform this algorithm.

   (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

   (c) Show the "behind-the-scenes" algebra that is going on here.

(18) Here is an example of the subtraction by addition algorithm:

$$\begin{array}{r} 892 \\ -378 \\ \hline 514 \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ll} 8+\mathbf{4}=12 & \text{add 1 to 7 to get 8} \\ 8+\mathbf{1}=9 & \\ 3+\mathbf{5}=8 & \end{array}$$

   (a) Describe how to perform this algorithm.

   (b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

   (c) Show the "behind-the-scenes" algebra that is going on here.

(19) Here is an example of the Austrian subtraction algorithm:

$$\begin{array}{rcc} 8 & 9 & {}^{1}2 \\ -3 & {}^{8}\cancel{7} & 8 \\ \hline 5 & 1 & 4 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(20) If you check out Problems (22) and (24), you will learn about "scaffolding" algorithms.

(a) Develop a scaffolding subtraction algorithm and describe how to perform this algorithm.

(b) Provide a relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(21) Here is an example of the basic multiplication algorithm:

$$\begin{array}{r} 23 \\ 634 \\ \times\quad 8 \\ \hline 5072 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(22) Here is an example of the scaffolding multiplication algorithm:

$$\begin{array}{r} 634 \\ \times\quad 8 \\ \hline 4800 \\ 240 \\ 32 \\ \hline 5072 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(23) Here is an example of the basic division algorithm:

$$\begin{array}{r} 97\,\mathrm{R}1 \\ 8\,\overline{)777} \\ \underline{72} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(24) Here is an example of the scaffolding division algorithm:

$$\begin{array}{r}
7\\
90\\
8\,\overline{)777}\\
\underline{720}\\
57\\
\underline{56}\\
1
\end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(25) Here is an example of the partial-quotients division algorithm:

$$\begin{array}{r}
4\\
10\\
10\\
10\\
8\,\overline{)277}\\
\underline{80}\\
197\\
\underline{80}\\
117\\
\underline{80}\\
37\\
\underline{32}\\
5
\end{array}$$

(a) Describe how to perform this algorithm—be sure to explain how this is different from the scaffolding division algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(26) Here is an example of the multi-digit multiplication algorithm:

$$\begin{array}{r} 634 \\ \times 216 \\ \hline 3804 \\ 6340 \\ 126800 \\ \hline 136944 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here—you may assume that you already know the algebra behind the basic multiplication algorithm.

(27) Here is an example of the addition algorithm with decimals:

$$\begin{array}{l} \;1 \\ \;37.2 \\ +8.74 \\ \hline \;45.94 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(28) Here is an example of the multiplication algorithm with decimals:

$$\begin{array}{r} 3.40 \\ \times\ \ .21 \\ \hline 340 \\ 6800 \\ \hline .7140 \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(29) Here is an example of the division algorithm without remainder:

$$\begin{array}{r} 0.75 \\ 4\,\overline{)3.00} \\ \underline{2\;8} \\ 20 \\ \underline{\underline{20}} \end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(30) In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{MOON} \\ +\ \texttt{SUN} \\ \hline \texttt{PLUTO} \end{array}$$

Recover the original problem and solution. Explain your reasoning. Hint: $\texttt{S} = 6$ and $\texttt{U} = 5$.

(31) In the following addition problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{SEND} \\ +\texttt{MORE} \\ \hline \texttt{MONEY} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

(32) In the following subtraction problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{DEFER} \\ -\texttt{DU7Y} \\ \hline \texttt{N2G2} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

(33) In the following two subtraction problems, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{NINE} \\ -\texttt{TEN} \\ \hline \texttt{TWO} \end{array} \qquad \begin{array}{r} \texttt{NINE} \\ -\texttt{ONE} \\ \hline \texttt{ALL} \end{array}$$

Using both problems simultaneously, recover the original problems and solutions. Explain your reasoning.

(34) In the following multiplication problem, every digit has been replaced with a letter.

$$\begin{array}{r} \texttt{LET} \\ \times\ \texttt{NO} \\ \hline \texttt{SOT} \\ \texttt{NOT} \\ \hline \texttt{FRET} \end{array}$$

Recover the original problem and solution. Explain your reasoning.

> ***Teaching Note: The next two problems may seem tedious, but they are very rewarding for students when they are able to finally solve them. While the student should be encouraged to use a calculator, the solution is not pure "guess and check" and there is a lot of reasoning that goes into the solution.***

(35) The following is a long division problem where every digit except 7 was replaced by X.

$$
\begin{array}{r}
\mathrm{X7X} \\
\mathrm{XX}\,\overline{)\,\mathrm{XXXXX}} \\
\underline{\mathrm{X77}}\phantom{\mathrm{XX}} \\
\mathrm{X7X}\phantom{\mathrm{X}} \\
\underline{\mathrm{X7X}}\phantom{\mathrm{X}} \\
\mathrm{XX} \\
\underline{\underline{\mathrm{XX}}}
\end{array}
$$

Recover the digits from this long division problem. Explain your reasoning.

> ***Teaching Note: Remind students to use their calculator and that to start,*** $\mathrm{XX} \cdot \mathrm{X} = \mathrm{X77}$.

(36) The following is a long division problem where the digits were replaced by X except in the quotient—where they were almost entirely removed.

$$
\begin{array}{r}
8\phantom{\mathrm{XX}} \\
\mathrm{XXX}\,\overline{)\,\mathrm{XXXXXXXX}} \\
\underline{\mathrm{XXX}}\phantom{\mathrm{XXXX}} \\
\mathrm{XXXX}\phantom{\mathrm{XX}} \\
\underline{\mathrm{XXX}}\phantom{\mathrm{XX}} \\
\mathrm{XXXX} \\
\underline{\underline{\mathrm{XXXX}}}
\end{array}
$$

One can see that the 8 is the third digit in a five digit answer. Can you recover what the digits in this long division problem were? Explain your reasoning.

1.3 Algebra

Algebra is when you replace a number with a letter, usually x, right? OK—but you also do things with x, like make *polynomials* out of it.

1.3.1 Polynomial Basics

Teaching Note: Activity A.5 complements this section well.

Question What's a polynomial?

I'll take this one:

Definition A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are all constants and n is a nonnegative integer.

Question Which of the following are polynomials?

$$3x^3 - 2x + 1 \qquad \frac{1}{3x^3 - 2x + 1} \qquad 3x^{-3} - 2x^{-1} + 1 \qquad 3x^{1/3} - 2x^{1/6} + 1$$

?

Given two polynomials

$$\begin{gathered} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \end{gathered}$$

we treat these polynomials much the same way we treat numbers. Note, an easy fact is that polynomials are equal if and only if their coefficients are equal—this may come up again!

Question Are numbers equal if and only if their digits are equal?

?

Teaching Note: This question is foreshadowing a future discussion of real numbers. The students will probably suggest that it is true—this is OK. We will address this point later.

Question Can you explain how to add two polynomials? Compare and contrast this procedure to the basic addition algorithm.

?

Question Can you explain how to multiply two polynomials? Compare and contrast this procedure to the basic multiplication algorithm.

?

Question Can you explain why someone might say that working with polynomials is like working in "base x?"

?

1.3.2 Division and Polynomials

For some reason you keep on signing up for classes with aloof old Professor Rufus. When he was asked to teach division of polynomials with remainders, he merely wrote

$$\begin{array}{rl} & \;q(x)\ \ \mathrm{R}\, r(x) \\ d(x) & \overline{\smash{)}\,n(x)} \end{array} \qquad \text{where} \qquad \begin{array}{l} d(x) \text{ is the divisor} \\ n(x) \text{ is the dividend} \\ q(x) \text{ is the quotient} \\ r(x) \text{ is the remainder} \end{array}$$

and walked out of the room, again! Do you have *déjà vu*?

Question Can you give 3 much needed examples of polynomial long division with remainders?

?

Question Given polynomials $d(x)$, $n(x)$, $q(x)$, and $r(x)$ how do you know if they leave us with a correct expression above?

?

Question Can you explain how to divide two polynomials?

?

Question Can you do the polynomial long division with remainder?

?

Again, this question can be turned into a theorem.

Theorem 1 (Division Theorem) *Given any polynomial $n(x)$ and a nonzero polynomial $d(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that*

The above space has intentionally been left blank for you to fill in.

> ***Teaching Note: Here we want the students to realize that***
>
> $$n(x) = d(x)q(x) + r(x) \qquad \textbf{\textit{where}}\ 0 \leqslant \deg(r(x)) < \deg(d(x))$$

Problems for Section 1.3

(1) Explain what is meant by a *polynomial* in a variable x.

(2) Given:

$$3x^7-x^5+x^4-16x^3+27 = a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x^1+a_0$$

Find a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7.

(3) Given:

$$6x^5 + a_4x^4 - x^2 + a_0 = a_5x^5 - 24x^4 + a_3x^3 + a_2x^2 - 5$$

Find a_0, a_1, a_2, a_3, a_4, a_5.

(4) Is it true that polynomials are equal if and only if their coefficients are equal? Explain your reasoning.

(5) Is it true that numbers are equal if and only if their digits are equal? Explain your reasoning.

(6) Explain how to add two polynomials.

(7) Explain how to multiply two polynomials.

(8) Here is an example of the polynomial division algorithm:

$$\begin{array}{r|l}
 & \qquad\quad x - 3 \quad \mathrm{R}\, 9x+4 \\
\hline
x^2+3x+1 & x^3 + 0x^2 + x + 1 \\
 & \underline{x^3 + 3x^2 + x} \\
 & \quad -3x^2 + 0x + 1 \\
 & \quad \underline{-3x^2 - 9x - 3} \\
 & \qquad\qquad 9x + 4
\end{array}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

(c) Show the "behind-the-scenes" algebra that is going on here.

(9) State the *Division Theorem* for polynomials. Give some relevant and revealing examples of this theorem in action.

(10) Given a polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

can you find two numbers L and U such that $L \leqslant p(x) \leqslant U$ for all x? If so, explain why. If not, explain why not.

(11) Consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are integers. If you substitute an integer for x will you always get an integer out? Explain your reasoning.

(12) Consider the following polynomial:

$$p(x) = \frac{x^2}{2} + \frac{x}{2}$$

Will $p(x)$ always returns an integer when an integer is substituted for x? Explain your reasoning.

(13) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0. Which numbers can be represented by such polynomials? Explain your reasoning.

(14) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 2 such that $p(2) = 35$. Discuss how your answer compares to the representation of 35 in base 2. Explain your reasoning.

(15) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 7 such that $p(7) = 200$. Discuss how your answer compares to the representation of 200 in base 7. Explain your reasoning.

(16) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 10 such that $p(10) = 18$. Discuss how your answer compares to the representation of 18 in base 10. Explain your reasoning.

(17) Find a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that a_i's are integers greater than or equal to 0 and less than 15 such that $p(15) = 201$. Discuss how your answer compares to the representation of 201 in base 15. Explain your reasoning.

(18) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than x. Which numbers can be represented by such polynomials? Explain your reasoning. Big hint: Base x.

(19) Fix some integer value for x and consider all polynomials of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where the a_i's are integers greater than or equal to 0 and less than 10. Which numbers can be represented by such polynomials? Explain your reasoning.

(20) Consider $x^2 + x + 1$. This can be thought of as a "number" in base x. Express this number in base $(x+1)$, that is, find b_0, b_1, b_2 such that

$$b_2(x+1)^2 + b_1(x+1) + b_0 = x^2 + x + 1.$$

Explain your reasoning.

(21) Consider $x^2 + 2x + 3$. this can be thought of as a "number" in base x. Express this number in base $(x-1)$, that is, find b_0, b_1, b_2 such that

$$b_2(x-1)^2 + b_1(x-1) + b_0 = x^2 + 2x + 3.$$

Explain your reasoning.

(22) Consider $x^3 + 2x + 1$. this can be thought of as a "number" in base x. Express this number in base $(x-1)$, that is, find b_0, b_1, b_2, b_3 such that

$$b_3(x-1)^3 + b_2(x-1)^2 + b_1(x-1) + b_0 = x^3 + 2x + 1.$$

Explain your reasoning.

(23) If the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is thought of as a "number" in base x, describe two different ways to find the base $(x-1)$ coefficients of $p(x)$.

1.4 The Adders

If long division is a *forgotten foe*, then logarithms are a *supervillan*. When aloof old Professor Rufus was trying to explain logarithms to his class, he merely wrote

$$\log_b(a) = n \qquad \Leftrightarrow \qquad b^n = a$$

and walked out of the room.

Question Can you give 3 much needed examples of logarithms that are easily computed in one's head?

?

Question Can you give 3 much needed examples of logarithms that are more difficult to compute in one's head?

?

Question What conditions should be placed on a and b to make logarithms work nicely?

?

Question What is $\log_b(1)$?

?

Question What is $\log_b(0)$?

?

Question What is $\log_1(1)$?

?

Question Sketch the plot of $y = \log_b(x)$.

?

Question Why is this section named "interlude of the adders?!"

?

Problems for Section 1.4

(1) Explain what $\log_b(a) = n$ means.

(2) Sketch the plot of $y = \log_b(x)$ for some reasonable value of b. Explain your procedure.

(3) Sketch the plot of $y = b^x$ for some reasonable value of b. Explain your procedure. How does this plot compare to the one in the previous question?

(4) What is $\log_x(x^3)$? Explain your reasoning.

(5) Given that $\ln(x) = \log_e(x)$, explain why is it no big deal to say that $\ln(e^x) = x$.

(6) Compute $\log_5(125)$. Explain your reasoning.

(7) Compute $\log_{10}(10000)$. Explain your reasoning.

(8) Compute $\log_2(1024)$. Explain your reasoning.

(9) Compute $\log_{13}(169)$. Explain your reasoning.

(10) Compute $\log_7(2401)$. Explain your reasoning.

(11) Bound $\log_2(5)$ by two consecutive integers. Explain your reasoning.

(12) Bound $\log_3(43)$ by two consecutive integers. Explain your reasoning.

(13) Bound $\log_{11}(24)$ by two consecutive integers. Explain your reasoning.

(14) Bound $\log_{10}(999)$ by two consecutive integers. Explain your reasoning.

(15) Bound $\log_{10}(1032)$ by two consecutive integers. Explain your reasoning.

(16) What is the connection between the number of digits in some number n and $\log_{10}(n)$? Explain your reasoning.

(17) How many digits does the number 100 have in base 2? What does this have to do with $\log_2(100)$? Explain your reasoning.

(18) How many digits does the number 100 have in base 3? What does this have to do with $\log_3(100)$? Explain your reasoning.

(19) How many digits does the number 100 have in base 11? What does this have to do with $\log_{11}(100)$? Explain your reasoning.

(20) How many digits does the number 100 have in base 42? What does this have to do with $\log_{42}(100)$? Explain your reasoning.

(21) How many digits does the number 100 have in base 99? What does this have to do with $\log_{99}(100)$? Explain your reasoning.

(22) How many digits does the number 100 have in base 100? What does this have to do with $\log_{100}(100)$? Explain your reasoning.

(23) How many digits does the number 100 have in base 101? What does this have to do with $\log_{101}(100)$? Explain your reasoning.

(24) Explain why $\log_b(a) + \log_b(c) = \log_b(a \cdot c)$.

(25) Explain why $\log_b(a) - \log_b(c) = \log_b(a/c)$.

(26) Explain why $c \cdot \log_b(a) = \log_b(a^c)$.

(27) Explain why $\log_b(a) = \dfrac{1}{\log_a(b)}$.

(28) Explain why $\log_b(a) = \dfrac{\log_c(a)}{\log_c(b)}$.

(29) People have often told me something like "it is impossible to fold a piece of paper more than 7 times." What is meant by this statement and is it even true? Explain your reasoning. Note if you cannot solve this problem, no worries just say to yourself (aloud so all can hear) "I believe you can fold a piece of paper as much as you want" three times and then do the next problem and then come back to this one.

(30) Take a sheet of paper. If you fold it once, the resulting folded sheet of paper is twice as thick as the unfolded paper. If you fold it again, the resulting folded sheet is 4 times as thick as the unfolded piece of paper. How many times would you need to fold a sheet of paper to make the resulting sheet of paper as thick as you are tall? Explain your reasoning. Don't bother worrying about the physical limitations of this problem.

(31) Explain why the following "joke" is "funny:"

> The Flood is over and the ark has landed. Noah lets all the animals out and says, "Go forth and multiply."
>
> A few months later, Noah decides to take a stroll and see how the animals are doing. Everywhere he looks he finds baby animals. Everyone is doing fine except for one pair of little snakes. "What's the problem?" says Noah. "Cut down sssome treesss and let usss live there," say the snakes.
>
> Noah follows their advice. Several more weeks pass. Noah checks on the snakes again. Lots of little snakes, everybody is happy. Noah asks, "Want to tell me how the trees helped?"
>
> "SSSertainly," say the snakes. "We're addersss, ssso we need logsss to multiply."

Chapter 2

Numbers

> God created the integers, the rest is the work of man.
>
> —Leopold Kronecker

2.1 The Integers

In this course we will discuss several different sets of numbers. The first set we encounter is called the *integers*.

Definition The set of whole numbers, zero, and negative whole numbers is called the set of **integers**. We use the symbol $\mathbb{Z}$ to denote the integers:

$$\mathbb{Z} = \{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}$$

In case you're wondering, the symbol $\mathbb{Z}$ is used because *Zahlen* is the German word for "numbers."

2.1.1 Addition

Addition is probably the first operation we learn.

Question Write a story problem whose solution is given by the expression $19 + 17$. Let this context be a "working" model for addition.

?

Question Does your model show associativity of addition? If so, explain how. If not, can you come up with a new model (story problem) that does?

?

Question Does your model show commutativity of addition? If so, explain how. If not, can you come up with a new model that does?

?

Question Does your model work with negative integers, that is does it model say $19 + (-17)$? If so, explain how. If not, can you come up with a new model that does?

?

Question We know that

$$a - b = a + (-b),$$

but I insist that the left-hand side of the equation is conceptually different from the right-hand side of the equation. Write two story problems, one solved by $19 - 17$ and the other solved by $19 + (-17)$. What's the difference[1]?

?

> ***Teaching Note: Here we are trying to have the students develop the "take-away," along with the "missing addend," and a "comparison" model for subtraction.***

Question Can you use the two story problems above to model

$$(-19) - 17, \qquad 19 - (-17), \qquad (-19) - (-17)?$$

?

2.1.2 Multiplication

Multiplication is more multifaceted than addition.

Question Write a story problem whose solution is given by the expression $19 \cdot 17$. Let this context be a "working" model for multiplication.

?

> ***Teaching Note: We would like to point out that the units used in addition are generally the same for the different summands. However, with multiplication, the different factors often have different units.***

[1] Pun intended.

Teaching Note: The next question is with Problem* (2) *of Section 1.2 in mind. In particular to show associativity, we suggest appealing to the notion of volume.

Question Does your model show associativity of multiplication? If so, explain how. If not, can you come up with a new model that does?

?

Question Does your model show commutativity of multiplication? If so, explain how. If not, can you come up with a new model that does?

?

Question Does your model work with negative integers? In particular does your model show that

$$\text{positive} \cdot \text{negative} = \text{negative},$$

$$\text{negative} \cdot \text{positive} = \text{negative},$$

and

$$\text{negative} \cdot \text{negative} = \text{positive}?$$

If so, explain how. If not, can you come up with a new model that does?

?

***Teaching Note: This is difficult. The students may not be able to come up with a model that works. This is OK—as this issue is addressed in Problem* (32).**

2.1.3 Division

While addition and multiplication are good operations, the real "meat" of the situation comes with division.

Definition We say that an integer d **divides** an integer n if

$$n = dq$$

in this case we write $d|n$, which is said: "d divides n."

While this may seem easy, it is actually quite tricky. You must always remember the following synonyms for *divides*:

$$\text{“}d\ \textbf{divides}\ n\text{”} \leftrightsquigarrow \text{“}d \text{ is a } \textbf{factor} \text{ of } n\text{”} \leftrightsquigarrow \text{“}n \text{ is a } \textbf{multiple} \text{ of } d\text{”}$$

Teaching Note: Activity A.6 complements this section well.

Definition A **prime** number is a positive integer with exactly two positive divisors, namely 1 and itself.

Definition A **composite** number is a positive integer with more than two positive divisors.

I claim that every composite number is divisible by a prime number. Do you believe me? If not, consider this:

> Suppose there was a composite number that was *not* divisible by a prime. Then there would necessarily be a *smallest* composite number that is not divisible by a prime. Since this number is composite, this number is the product of two even smaller numbers, both of which have prime divisors. Hence our original number must have prime divisors.

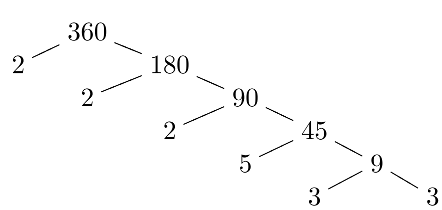
Question What the heck just happened?! Can you rewrite the above paragraph, drawing pictures and/or using symbols as necessary, making it more clear?

?

Teaching Note: Activity A.7 complements this section well.

Factoring

At this point we can factor any composite completely into primes. To do this, it is often convenient to make a *factor tree*:



From this tree[2] we see that

$$360 = 2^3 \cdot 3^2 \cdot 5.$$

[2]Why is this a tree? It looks more like roots to me!

At each step we simply divided by whichever prime number seemed most obvious, branched off the tree and kept on going. From our factor tree, we can see some of the divisors of the integer in question. However, there are many composite factors that can be built up from the prime divisors. One of the most important is the *greatest common divisor*.

Definition The **greatest common divisor** (GCD) of two numbers a and b is a number $g = \gcd(a, b)$ where:

(1) $g|a$ and $g|b$.

(2) If $d|a$ and $d|b$, then $d \leqslant g$.

Question How do you use a factor tree to compute the GCD of two integers?

?

So, to factor an integer or find the GCD, one could use a factor tree. However, when building the factor tree, we had to know what primes to divide by. What if no prime comes to mind? What if you want to factor the integer 391 or 397? This raises a new question:

Question How do you check to see if a given integer is prime? What possible divisors must you check? When can you stop checking?

?

Teaching Note: Activity A.8 complements this section well.

2.1.4 Division with Remainder

We all remember long division, or at least we remember *doing* long division. Sometimes, we need to be reminded of our *forgotten foes*. When aloof old Professor Rufus was trying to explain division to his class, he merely wrote

$$\begin{array}{r} q \;\;\mathrm{R}\, r \\ d\,\overline{)\,n\;\;\;\;} \end{array} \qquad \text{where} \qquad \begin{array}{l} d \text{ is the divisor} \\ n \text{ is the dividend} \\ q \text{ is the quotient} \\ r \text{ is the remainder} \end{array}$$

and walked out of the room.

Question Can you give 3 much needed examples of long division with remainders?

?

Question Given positive integers d, n, q, and r how do you know if they leave us with a correct expression above?

?

Question Given positive integers d and n, how many different sets of q and r can you find that will leave us with a correct expression above?

?

The innocuous questions above can be turned into a theorem. We'll start it for you, but you must finish it off yourself:

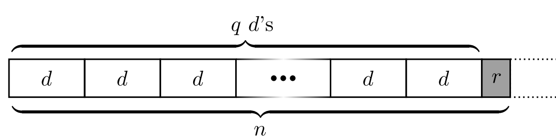
Theorem 2 (Division Theorem) *Given any integer n and a nonzero integer d, there exist unique integers q and r such that*

The above space has intentionally been left blank for you to fill in.

Teaching Note: Here we want the students to realize that

$$n = dq + r \qquad \textbf{\textit{where}}\ 0 \leqslant r < d$$

Now consider the following picture:



Question How does the picture above "prove" the Division Theorem for positive integers? How must we change the picture if we allow negative values for n and d?

?

Teaching Note: The second part of this question might be too hard for most students.

Teaching Note: Activity A.9 complements this section well.

Problems for Section 2.1

(1) Describe the set of integers. Give some relevant and revealing examples/nonexamples.

(2) Explain how to model integer addition with pictures or items. What relevant properties should your model show?

(3) Explain how to model integer multiplication with pictures or items. What relevant properties should your model show?

(4) Explain what it means for one integer to *divide* another integer. Give some relevant and revealing examples/nonexamples.

(5) Use the definition of *divides* to decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

(a) $5|30$

(b) $7|41$

(c) $6|(2^2 \cdot 3^4 \cdot 5 \cdot 7)$

(d) $1000|(2^7 \cdot 3^9 \cdot 5^{11} \cdot 17^8)$

(e) $6000|(2^{21} \cdot 3^{17} \cdot 5^{89} \cdot 29^{20})$

(6) *Incognito's Hall of Shoes* is a shoe store that just opened in Myrtle Beach, South Carolina. At the moment, they have 100 pairs of shoes in stock. At their grand opening 100 customers showed up. The first customer tried on every pair of shoes, the second customer tried on every 2nd pair, the third customer tried on every 3rd pair, and so on until the 100th customer, who only tried on the last pair of shoes.

(a) Which shoes were tried on by only 1 customer?

(b) Which shoes were tried on by exactly 2 customers?

(c) Which shoes were tried on by exactly 3 customers?

(d) Which shoes were tried on by the most number of customers?

Explain your reasoning.

(7) Factor the following integers:

(a) 111

(b) 1234

(c) 2345

(d) 4567

(e) 111111

In each case, how large of prime must you check before you can be sure of your answers? Explain your reasoning.

(8) Explain how to deduce that the following numbers are prime in as few calculations as possible:

$$29 \qquad 53 \qquad 101 \qquad 359 \qquad 5051$$

In each case, describe precisely which computations are needed and why those are the only computations needed.

(9) Suppose you were only allowed to perform at most 7 computations to see if a number is prime. How large a number could you check? Explain your reasoning.

(10) Find examples of integers a, b, and c such that $a \mid bc$ but $a \nmid b$ and $a \nmid c$. Explain your reasoning.

(11) Can you find at least 5 composite integers in a row? What about at least 6 composite integers? Can you find 7? What about n? Explain your reasoning. Hint: Consider something like $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

(12) Use the definition of the *greatest common divisor* to find the GCD of each of the pairs below. In each case, a detailed argument and explanation must be given justifying your claim.

(a) $\gcd(462, 1463)$
(b) $\gcd(541, 4669)$
(c) $\gcd(10000, 2^5 \cdot 3^{19} \cdot 5^7 \cdot 11^{13})$
(d) $\gcd(11111, 2^{14} \cdot 7^{21} \cdot 41^5 \cdot 101)$
(e) $\gcd(437^5, 8993^3)$

(13) Lisa wants to make a new quilt out of 2 of her favorite sheets. To do this, she is going to cut each sheet into as large of squares as possible while using the entire sheet and using whole inch measurements.

(a) If the first sheet is 72 inches by 60 inches what size squares should she cut?
(b) If the second sheet is 80 inches by 75 inches, what size squares should she cut?
(c) How she might sew these squares together?

Explain your reasoning.

(14) Deena and Doug like to feed birds. They want to put 16 cups of millet seed and 24 cups of sunflower seeds in their feeder.

(a) How many total scoops of seed (millet or sunflower) are required if their scoop holds 1 cup of seed?

(b) How many total scoops of seed (millet or sunflower) are required if their scoop holds 2 cups of seed?

(c) How large should the scoop be if we want to minimize the total number of scoops?

Explain your reasoning.

(15) Consider the expression:

$$\begin{array}{l} \;\;\; q \;\; \mathrm{R}\, r \\ d\,\overline{)\,n} \end{array} \qquad \text{where} \qquad \begin{array}{l} d \text{ is the divisor} \\ n \text{ is the dividend} \\ q \text{ is the quotient} \\ r \text{ is the remainder} \end{array}$$

(a) Give 3 relevant and revealing examples of long division with remainders.

(b) Given positive integers d, n, q, and r how do you know if they leave us with a correct expression above?

(c) Given positive integers d and n, how many different sets of q and r can you find that will leave us with a correct expression above?

(d) Give 3 relevant and revealing examples of long division with remainders where some of d, n, q, and r are negative.

(e) Still allowing some of d, n, q, and r to be negative, how do we know if they leave us with a correct expression above?

(16) State the *Division Theorem* for integers. Give some relevant and revealing examples of this theorem in action.

(17) Explain what it means for an integer to *not* divide another integer. That is, explain symbolically what it should mean to write:

$$a \nmid b$$

(18) Consider the following:

$$20 \div 8 = 2 \text{ remainder } 4,$$
$$28 \div 12 = 2 \text{ remainder } 4.$$

Is it correct to say that $20 \div 8 = 28 \div 12$? Explain your reasoning.

(19) Give a formula for the nth even number. Show-off your formula with some examples.

(20) Give a formula for the nth odd number. Show-off your formula with some examples.

(21) Give a formula for the nth multiple of 3. Show-off your formula with some examples.

(22) Give a formula for the nth multiple of -7. Show-off your formula with some examples.

(23) Give a formula for the nth number whose remainder when divided by 5 is 1. Show-off your formula with some examples.

(24) Explain the rule
$$\text{even} + \text{even} = \text{even}$$
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(25) Explain the rule
$$\text{odd} + \text{even} = \text{odd}$$
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(26) Explain the rule
$$\text{odd} + \text{odd} = \text{even}$$
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(27) Explain the rule
$$\text{even} \cdot \text{even} = \text{even}$$
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(28) Explain the rule
$$\text{odd} \cdot \text{odd} = \text{odd}$$
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(29) Explain the rule
$$\text{odd} \cdot \text{even} = \text{even}$$
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(30) Let $a \geqslant b$ be positive integers with $\gcd(a, b) = 1$. Compute $\gcd(a+b, a-b)$. Explain your reasoning. Hints:

   (a) Make a chart.

   (b) If $g|x$ and $g|y$ explain why $g|(x + y)$.

(31) Make a chart listing all pairs of positive integers whose product is 18. Do the same for 221, 462, and 924. Use this experience to help you explain why when factoring a number n, you only need to check factors less than or equal to $\sqrt{n}$.

(32) Matt is a member of the Ohio State University Marching Band. Being rather capable, Matt can take x steps of size y inches for all integer values of x and y. If x is positive it means *face North and take x steps.* If x is negative it means *face South and take $|x|$ steps.* If y is positive it means *take a forward step of y inches.* If y is negative it means *take a backward step of $|y|$ inches.*

(a) Discuss what the expressions $x \cdot y$ means in this context. In particular, what happens if $x = 1$? What if $y = 1$?

(b) Using the context above, write and solve a word problem that demonstrates the rule:

$$\text{negative} \cdot \text{positive} = \text{negative}$$

Clearly explain how your problem shows this.

(c) Using the context above, write and solve a word problem that demonstrates the rule:

$$\text{negative} \cdot \text{negative} = \text{positive}$$

Clearly explain how your problem shows this.

(33) Stewie decided to count the pennies he had in his piggy bank. He decided it would be quicker to count by fives. However, he ended with two uncounted pennies. So he tried counting by twos but ended up with one uncounted penny. Next he counted by threes and then by fours, each time there was one uncounted penny. Though he knew he had less than a dollars worth of pennies, and more than 50 cents, he still didn't have an exact count. Can you help Stewie out? Explain your reasoning.

2.2 The Euclidean Algorithm

Up to this point, computing the GCD of two integers required you to factor both numbers. This can be difficult to do. The following algorithm, called the *Euclidean algorithm*, makes finding GCD's quite easy. With that said, algorithms can be tricky to explain. Let's try this—study the following calculations, they are examples of the Euclidean algorithm in action:

$$\begin{aligned} 22 &= \mathbf{6}\cdot 3 + \mathbf{4} \\ \mathbf{6} &= \mathbf{4}\cdot 1 + \boxed{\mathbf{2}} \\ 4 &= 2\cdot 2 + 0 \qquad \boxed{\therefore \gcd(22,6) = 2} \end{aligned}$$

$$\begin{aligned} 33 &= \mathbf{24}\cdot 1 + \mathbf{9} \\ \mathbf{24} &= \mathbf{9}\cdot 2 + \mathbf{6} \\ \mathbf{9} &= \mathbf{6}\cdot 1 + \boxed{\mathbf{3}} \\ 6 &= 3\cdot 2 + 0 \qquad \boxed{\therefore \gcd(33,24) = 3} \end{aligned}$$

$$\begin{aligned} 42 &= \mathbf{16}\cdot 2 + \mathbf{10} \\ \mathbf{16} &= \mathbf{10}\cdot 1 + \mathbf{6} \\ \mathbf{10} &= \mathbf{6}\cdot 1 + \mathbf{4} \\ \mathbf{6} &= \mathbf{4}\cdot 1 + \boxed{\mathbf{2}} \\ 4 &= 2\cdot 2 + 0 \qquad \boxed{\therefore \gcd(42,16) = 2} \end{aligned}$$

Question Can you describe how to do the Euclidean algorithm?

?

Question Can you explain why the Euclidean algorithm will always stop? Hint: Division Theorem.

?

Teaching Note: Activity A.10 complements this section well.

The algorithm demonstrated above is called the *Euclidean algorithm* or *Euclid's algorithm* because Euclid uses it several times in Books VII and X of his book *The Elements*. Donald Knuth gives a description of the Euclidean algorithm in the first volume of his series of books *The Art of Computer Programming*. Given integers m and n, he describes it as follows:

(1) [Find remainder.] Divide m by n and let r be the remainder. (We will have $0 \leqslant r < n$.)

(2) [Is it zero?] If $r = 0$, the algorithm terminates; n is the answer.

(3) [Interchange.] Set $m \leftarrow n$, $n \leftarrow r$, and go back to step (1).

Question What do you think of this description? How does it compare to your description of the Euclidean algorithm?

?

While the Euclidean algorithm is handy and fun, its real power is that it helps us solve equations. Specifically it helps us solve *linear Diophantine equations*.

Definition A **Diophantine equation** is an equation where only integer solutions are deemed acceptable.

We're going to solve *linear Diophantine equations*, that is, equations of the form:

$$ax + by = c$$

where a, b, and c are integers and the only solutions we will accept are also integers. Let's study the following calculations:

$$\begin{aligned} 22 &= 6\cdot 3+4 &\Leftrightarrow\quad 22-6\cdot 3 &= \mathbf{4} \\ 6 &= 4\cdot 1+2 &\Leftrightarrow\quad 6-\mathbf{4}\cdot 1 &= 2 \\ 4 &= 2\cdot 2+0 & & \end{aligned} \qquad \begin{aligned} 6-4\cdot 1 &= 2 \\ 6-(22-6\cdot 3)\cdot 1 &= 2 \\ 6\cdot 4+22(-1) &= 2 \end{aligned}$$

$$\boxed{\therefore 22x+6y=2 \text{ where } x=-1 \text{ and } y=4}$$

$$\begin{aligned} 33 &= 24\cdot 1+9 &\Leftrightarrow\quad 33-24\cdot 1 &= \mathbf{9} \\ 24 &= 9\cdot 2+6 &\Leftrightarrow\quad 24-\mathbf{9}\cdot 2 &= \mathbf{6} \\ 9 &= 6\cdot 1+3 &\Leftrightarrow\quad 9-\mathbf{6}\cdot 1 &= 3 \\ 6 &= 3\cdot 2+0 & & \end{aligned} \qquad \begin{aligned} 9-6\cdot 1 &= 3 \\ 9-(24-9\cdot 2)\cdot 1 &= 3 \\ 9\cdot 3+24\cdot(-1) &= 3 \\ (33-24\cdot 1)\cdot 3+24\cdot(-1) &= 3 \\ 33\cdot 3+24\cdot(-4) &= 3 \end{aligned}$$

$$\boxed{\therefore 33x+24y=3 \text{ where } x=3 \text{ and } y=-4}$$

Question Can you explain how to solve Diophantine equations of the form

$$ax + by = g$$

where $g = \gcd(a, b)$?

?

2.2.1 Fundamental Theorems

Teaching Note: An important point of this section is to make the student think about the distributive property. One should try to point out each time

$$a(x+y) = ax + ay$$

occurs.

Teaching Note: Activity A.11 complements this section well.

The Euclidean algorithm is also useful for theoretical questions.

Question Given integers a and b, what is the smallest positive integer that can be expressed as

$$ax + by$$

where x and y are also integers?

I'm feeling chatty, so I'll take this one. I claim that $g = \gcd(a, b)$ is the smallest positive integer that can be expressed as

$$ax + by$$

where x and y are integers. How do I know? Well, suppose there was a smaller positive integer, say s where:

$$ax + by = s$$

Hmmm. . . but we know that $g|a$ and $g|b$. This means that g divides the left-hand-side of the equation. This means that g divides the right-hand-side of the equation. So $g|s$—but this is impossible, as $s < g$. Thus g is the smallest integer that can be expressed as $ax + by$.

Believe it or not, we're going somewhere with all this. The next *lemma* will help us out. What is a lemma, you ask? A lemma is nothing but a little theorem that helps us solve another problem. Note that a lemma should not be confused with the more sour *lemon*, as that is something different and unrelated to what we are discussing.

Lemma 3 *If a, b, and c are integers with* $\gcd(a, b) = 1$, *then*

$$a|bc \qquad \text{implies that} \qquad a|c.$$

Question Can you use the ideas above to explain why this lemma is true?

?

Now we have set the stage for our fundamental theorem—it is sometimes called the *Fundamental Theorem of Arithmetic*:

Theorem 4 (Unique Factorization) *Every positive integer can be factored uniquely (up to ordering) into primes.*

Proof Well, if an integer is prime, we are done. If an integer is composite, then it is divisible by a prime number. Divide and repeat with the quotient. If our original integer was n, we'll eventually get:

$$n = p_1 p_2 \cdots p_m$$

where some of the p_i's may be duplicates.

How do we know this factorization is unique? Well, suppose that

$$n = p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_l$$

where the p_i's are all prime and none of them equal any of the q_j's which are also prime. So, $\gcd(p_1, q_1) = 1$, and by the definition of "divides"

$$p_1 | q_1 (q_2 \cdots q_l).$$

So by our lemma above, p_1 must divide $(q_2 \cdots q_l)$. Repeat this enough times and you will find that $p_1 = q_j$ for one of the q_j's above. Repeat this process for the p_i's and you see that the factorization is unique. ■

Question Huh?! Can you explain what just happened drawing pictures and/or using symbols as necessary? Could you also give some examples?

?

Let's see the Unique Factorization Theorem for integers in action!

Question If $11|50a$, is it true that $11|a$?

I'll take this one. If $11|50a$, this means that

$$50a = 11 \cdot q \qquad \text{where } q \text{ is some integer.}$$

By the Unique Factorization Theorem for integers, we can factor both sides of the equation above in exactly one way. The upshot is that the primes that appear on the left-hand side of the equation must appear on the right-hand side of the equation. Since

$$2 \cdot 5^2 \cdot a = 11 \cdot q,$$

and I don't see 11 appearing as a factor on the left-hand side, but we know they must be there by the Unique Factorization Theorem! We conclude that 11 must be a factor of a, and hence $11|a$.

Problems for Section 2.2

(1) Explain what a *Diophantine equation* is. Give an example and explain why such a thing has real-world applications.

(2) Explain what the GCD of two integers is. Give some relevant and revealing examples/nonexamples.

(3) Explain what the LCM of two integers is. Give some relevant and revealing examples/nonexamples.

(4) Use the Euclidean algorithm to find: $\gcd(671, 715)$, $\gcd(667, 713)$, $\gcd(671, 713)$, $\gcd(682, 715)$, $\gcd(601, 735)$, and $\gcd(701, 835)$.

(5) Explain the advantages of using the Euclidean algorithm to find the GCD of two integers over factoring.

(6) Find integers x and y satisfying the following Diophantine equations:

(a) $671x + 715y = 11$
(b) $667x + 713y = 69$
(c) $671x + 713y = 1$
(d) $682x + 715y = 55$
(e) $601x + 735y = 4$
(f) $701x + 835y = 15$

(7) Given integers a, b, and c, explain how you know when a solution to a Diophantine equation of the form

$$ax + by = c$$

exists.

(8) Consider the Diophantine equation:

$$14x + 4y = 2$$

(a) Use the Euclidean Algorithm to find a solution to this equation. Explain your reasoning.
(b) Compute the slope of the line $14x + 4y = 2$ and write it in lowest terms. Show your work.
(c) Plot the line determined by $14x + 4y = 2$ on graph paper.
(d) Using your plot and the slope of the line, explain how to find 10 more solutions to the Diophantine equation above.

(9) Explain why a Diophantine equation

$$ax + by = c$$

has either an infinite number of solutions or zero solutions.

(10) Josh owns a box containing beetles and spiders. At the moment, there are 46 legs in the box. How may beetles and spiders are currently in the box? Explain your reasoning.

(11) How many different ways can thirty coins (nickles, dimes, and quarters) be worth five dollars? Explain your reasoning.

(12) Lisa collects lizards, beetles and worms. She has more worms than lizards and beetles together. Altogether in the collection there are twelve heads and twenty-six legs. How many lizards does Lisa have? Explain your reasoning.

(13) Can you make exactly \$5 with exactly 100 coins assuming you can only use pennies, dimes, and quarters? If so how, if not why not? Explain your reasoning.

(14) A merchant purchases a number of horses and bulls for the sum of 1770 talers. He pays 31 talers for each bull, and 21 talers for each horse. How many bulls and how many horses does the merchant buy? Solve this problem, explain what a *taler* is, and explain your reasoning—note this problem is an old problem by L. Euler, it was written in the 1700's.

(15) A certain person buys hogs, goats, and sheep, totaling 100 animals, for 100 crowns; the hogs cost him $3\frac{1}{2}$ crowns a piece, the goats $1\frac{1}{3}$ crowns, and the sheep go for $\frac{1}{2}$ crown a piece. How many did this person buy of each? Explain your reasoning—note this problem is an old problem from *Elements of Algebra* by L. Euler, it was written in the 1700's.

(16) How many zeros are at the end of the following numbers:

  (a) $2^4 \cdot 5^3 \cdot 7^3 \cdot 11^5$
  (b) $99!$
  (c) $1001!$
  (d) $28721!$

  In each case, explain your reasoning.

(17) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

  (a) $7|56$
  (b) $55|11$
  (c) $3|40$
  (d) $100|(2^4 \cdot 3^{17} \cdot 5^2 \cdot 7)$
  (e) $5555|(5^{20} \cdot 7^9 \cdot 11^{11} \cdot 13^{23})$
  (f) $3|(3 + 6 + 9 + \cdots 300 + 303)$

(18) Suppose that

$$(3^5 \cdot 7^9 \cdot 11^x \cdot 13^y) | (3^a \cdot 7^b \cdot 11^{19} \cdot 13^7)$$

What values of a, b, x and y, make true statements? Explain your reasoning.

(19) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

(a) If $7|13a$, then $7|a$.
(b) If $6|49a$, then $6|a$.
(c) If $10|65a$, then $10|a$.
(d) If $14|22a$, then $14|a$.
(e) $54|931^{21}$.
(f) $54|810^{33}$.

(20) Joanna thinks she can see if a number is divisible by 24 by checking to see if it's divisible by 4 and divisible by 6. She claims that if the number is divisible by 4 and by 6, then it must be divisible by 24.

Fred has a similar divisibility test for 24: He claims that if a number is divisible by 3 and by 8, then it must be divisible by 24.

Are either correct? Explain your reasoning.

(21) Generalize the problem above.

(22) Suppose that you have a huge bag of tickets. On each of the tickets is one of the following numbers.

$$\{6, 18, 21, 33, 45, 51, 57, 60, 69, 84\}$$

Could you ever choose some combination of tickets (you can use as many copies of the same ticket as needed) so that the numbers sum to 7429? If so, give the correct combination of tickets. If not explain why not.

(23) Decide whether the following statements are true or false. In each case, a detailed argument and explanation must be given justifying your claim.

(a) If $a^2|b^2$, then $a|b$.
(b) If $a|b^2$, then $a|b$.
(c) If $a|b$ and $\gcd(a, b) = 1$, then $a = 1$.

(24) Betsy is factoring the number 24949501. To do this, she divides by successively larger primes. She finds the smallest prime divisor to be 499 with quotient 49999. At this point she stops. Why doesn't she continue? Explain your reasoning.

(25) When Ann is half as old as Mary will be when Mary is three times as old as Mary is now, Mary will be five times as old as Ann is now. Neither Ann nor Mary may vote. How old is Ann? Explain your reasoning.

(26) If $x^2 = 11 \cdot y$, what can you say about y? Explain your reasoning.

(27) If $x^2 = 25 \cdot y$, what can you say about y? Explain your reasoning.

(28) When asked how many people were staying at the *Hotel Chevalier*, the clerk responded "The number you seek is the smallest positive integer such that dividing by 2 yields a perfect square, and dividing by 3 yields a perfect cube." How many people are staying at the hotel? Explain your reasoning.

2.3 Rational Numbers

Once you are familiar with integers, you start to notice something: Given an integer, it may or may not divide into another integer evenly. This property is at the heart of our notions of factoring and primality. Life would be very different if all nonzero integers divided evenly into one another. With this in mind, we introduce *rational numbers*.

Definition A **rational number** is a fraction of integers, where the denominator is nonzero.

The set of all rational numbers is denoted by the symbol $\mathbb{Q}$:

$$\mathbb{Q} = \left\{ \frac{a}{b} \text{ such that } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

The funny little "$\in$" symbol means "is in" or "is an element of." Fancy folks will replace the words *such that* with a colon ":" to get:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

We call this set the **rational numbers**. The letter $\mathbb{Q}$ stands for the word *quotient*, which should remind us of fractions.

> ***Teaching Note: Activities A.12 and A.13 complement this section well.***

Why do People Hate Fractions?

Why do so many people find fractions difficult? This is a question worth exploring. We'll guide you through some of the tough spots with some questions of our own.

Question Given a fraction a/b, come up with three other different fractions that are all equal to a/b. What features of fractions are we illustrating?

?

Question Given two fractions a/b and c/d, explain how to tell which fraction is larger. What features of fractions are we illustrating?

?

Question Given two fractions a/b and c/d with $a/b < c/d$, explain how one might find a fraction between them. What features of fractions are we illustrating?

?

Question Dream up numbers a, b, and c such that:

$$\frac{a/b}{c} = \frac{a}{b/c}$$

Can you dream up other numbers a', b', and c' such that:

$$\frac{a'/b'}{c'} \neq \frac{a'}{b'/c'}$$

What features of fractions are we illustrating?

?

Question Explain how to add two fractions a/b and c/d. What features of fractions are we illustrating?

?

Question Can you come up with any other reasons fractions are difficult?

?

> ***Teaching Note: Two key points of this dialog are:***
>
> (1) ***Equal fractions have different representations.***
>
> (2) ***It is difficult to compare fractions.***

2.3.1 Basic Meanings of Fractions

Like all numbers, fractions have meanings outside of their pure mathematical existence. Let's see if we can get to the heart of some of this meaning.

Question Draw a rectangle. Can you shade 3/8 of this rectangle? Explain the steps you took to do this.

?

Question Draw a rectangle. Given a fraction a/b where $0 < a \leqslant b$, explain how to shade a/b of this rectangle.

?

Question Draw a rectangle. How could you visualize 8/3 of this rectangle? Explain the steps you took to do this.

?

Question Draw a rectangle. Given a fraction a/b where $0 < b < a$, explain how to visualize a/b of this rectangle.

?

Question Draw a rectangle. Can you shade

$$\frac{3/8}{4}$$

of this rectangle? Explain the steps you took to do this.

?

Teaching Note: Activities A.14 through activity A.19 complement this section well.

Teaching Note: As a conclusion, we suggest doing Activity A.20.

Problems for Section 2.3

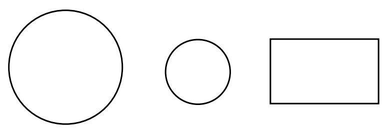
(1) Describe the set of rational numbers. Give some relevant and revealing examples/nonexamples.

(2) What algebraic properties do the rational numbers enjoy that the integers do not? Explain your reasoning.

(3) What number gives the same result when added to 1/2 as when multiplied by 1/2. Explain your reasoning.

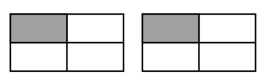
(4) Draw a rectangle to represent a garden. Shade in 3/5 of the garden. Without changing the shading, show why 3/5 of the garden is the same as 12/20 of the garden. Explain your reasoning.

(5) Shade in 2/5 of the entire picture below:



Explain your reasoning.

(6) What fractions could the following picture be illustrating?



Explain your reasoning.

(7) When Jesse was asked what the 7 in the fraction $\frac{3}{7}$ means, Jesse said that the "7" is the *whole*. Explain why this is not completely correct. What is a better description of what the "7" in the fraction $\frac{3}{7}$ means?

(8) Find yourself a sheet of paper. Now, suppose that this sheet of paper is actually 4/5 of some imaginary larger sheet of paper.

- Shade your sheet of paper so that 3/5 of the larger (imaginary) sheet of paper is shaded in. Explain why your shading is correct.
- Explain how this shows that

$$\frac{3/5}{4/5} = \frac{3}{4}.$$

(9) Try to find the largest rational number smaller than 3/7. Explain your solution or explain why this cannot be done.

(10) How many rational numbers are there between 3/4 and 4/7? Find 3 of them. Explain your reasoning.

(11) A youthful Bart loved to eat hamburgers. He ate 5/8 pounds of hamburger meat a day. After testing revealed that his blood consisted mostly of cholesterol, Bart decided to alter his eating habits by cutting his hamburger consumption by 3/4. How many pounds of hamburger a day did Bart eat on his new "low-cholesterol" diet? Explain your reasoning.

(12) A baseball coach once asked me the following question: If a pitcher can throw a 90 mph pitch during a game, but can only sustain a 60 mph pitch during practice, how close should the pitcher stand during practice to ensure that the amount of time it takes the ball to reach home plate is the same in practice as it is in the game? Explain your reasoning.

(13) Courtney and Paolo are eating popcorn. Unfortunately, 1/3rd of the popcorn is poisoned. If Courtney eats 5/16th of the bowl and Paolo eats 5/13ths of the bowl, did at least one of them eat poisoned kernel? Also, at least how many kernels of popcorn are in the bowl? Explain your reasoning.

(14) Three brothers and a sister won the lottery together and plan to share it equally. If the brothers alone had shared the money, then they would have increased the amount they each received by $20. How much was won in the lottery? Explain your reasoning.

(15) Chris is working on his Fiat. His car's cooling system holds 6 quarts of coolant, but it is currently 1 quart low. A car should be filled with a 50/50 mix of antifreeze and water. Chris accidentally added a 25/75 mix, 25 parts antifreeze, and 75 parts water. How much coolant does he have to remove from the cooling system to then add 100 percent antifreeze to restore his desired 50/50 mix? Explain your reasoning.

(16) Best of clocks, how much of the day is past if there remains twice two-thirds of what is gone? Explain what this strange question is asking and answer the question being sure to explain your reasoning—note this is an old problem from the *Greek Anthology* compiled by Metrodorus around the year 500.

(17) Monica, Tessa, and Jim are grading papers. If it would take Monica 2 hours to grade them all by herself, Tessa 3 hours to grade them all by herself, and Jim 4 hours to grade them all by himself how long would it take them to grade the exams if they all work together? Explain your reasoning.

(18) Say quickly, friend, in what portion of a day will four fountains, being let loose together, fill a container which would be filled by the individual fountains in one day, half a day, a third of a day, and a sixth of a day respectively? Explain your reasoning—note this is an old problem from the Indian text *Lilavati* written in the 1200s.

(19) John spent a fifth of his life as a boy growing up, another one-sixth of his life in college, one-half of his life as a bookie, and has spent the last six years in prison. How old is John now? Explain your reasoning

(20) Diophantus was a boy for 1/6th of his life, his beard grew after 1/12 more, he married after 1/7th more, and a son was born five years after his marriage. Alas! After attaining the measure of half his father's full life, chill fate took the child. Diophantus spent the last four years of his life consoling his grief through mathematics. How old was Diophantus when he died? Explain your reasoning—note this is an old problem from the *Greek Anthology* compiled by Metrodorus around the year 500.

(21) Wandering around my home town (perhaps trying to find my former self!), I suddenly realized that I had been in my job for one-quarter of my life. Perhaps the melancholia was getting the best of me, but I wondered: How long would it be until I had been in my job for one-third of my life? Explain your reasoning.

(22) In a certain adult condominium complex, 2/3 of the men are married to 3/5 of the women. Assuming that men are only married to women (and vice versa), and that married residents' spouses are also residents, what portion of the residents are married?

(a) Before any computations are done, use common sense to guess the solution to this problem.

(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your guess?

(c) Use algebra to solve the problem.

Explain your reasoning in each step above.

(23) Fred and Frank are two fitness fanatics on a run from A to B. Fred runs half the way and walks the other half. Frank runs for half the time and walks for the other half. They both run at the same speed and they both walk at the same speed. Who finishes first?

(a) Before any computations are done, use common sense to guess the solution to this problem.

(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your guess?

(c) Use algebra to solve the problem.

Explain your reasoning in each step above.

(24) Andy and Sandy run a race of a certain distance. Sandy finishes 1/10 of the distance ahead of Andy. After some discussion, Andy and Sandy decide to race the certain distance again, this time Sandy will start 1/10 of the distance behind Andy to "even-up" the competition. Who wins this time?

(a) Before any computations are done, use common sense to guess the solution to this problem.
(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your guess?
(c) Use algebra to solve the problem.

Explain your reasoning in each step above.

(25) You have two beakers, one that contains water and another that contains an equal amount of oil. A certain amount of water is transferred to the oil and thoroughly mixed. Immediately, the same amount of the mixture is transferred back to the water. Is there now more water in the oil or is there more oil in the water?

(a) Before any computations are done, use common sense to guess the solution to this problem.
(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your guess?
(c) Use algebra to solve the problem.

Explain your reasoning in each step above.

(26) While on a backpacking trip Lisa hiked five hours, first along a level path, then up a hill, then turned round and hiked back to her base camp along the same route. She walks 4 miles per hour on a level trail, 3 uphill, and 6 downhill. Find the total distance traveled. Explain your reasoning.

(27) Three drops of *Monica's XXX Hot Sauce* were mixed with five cups of chili mix to make a spicy treat—the hot sauce is much hotter than the chili. Later, two drops of *Monica's XXX Hot Sauce* were mixed with three cups of chili. Which mixture is hotter? Josh suggested the following method to compare the concentrations:

- Remove the second (recipe) from the first, that is: Start with 3 drops of hot sauce and 5 cups of chili, and remove 2 drops and 3 cups. So we are now comparing

  1 drop and 2 cups   with   2 drops and 3 cups.

- Now remove the first from the second, that is: Start with 2 drops and 3 cups, and remove 1 drop and 2 cups. So we are now comparing

  1 drop and 2 cups   with   1 drop and 1 cup.

Now you can see that the second is more concentrated (and hence hotter!) than the first. Is this correct? Will this strategy always/ever work? Explain your reasoning.

(28) Let a, b, c, and d be positive integers such that

$$a < b < c < d$$

Is it true that

$$\frac{a}{b} < \frac{c}{d}?$$

Explain your reasoning.

(29) Let a, b, c, and d be positive consecutive integers such that

$$a < b < c < d.$$

Is it true that

$$\frac{a}{b} < \frac{c}{d}?$$

Explain your reasoning.

(30) Let a, b, c, and d be positive consecutive integers such that

$$a < b < c < d.$$

Is it true that

$$\frac{a}{b} < \frac{b}{c} < \frac{c}{d}?$$

Explain your reasoning.

(31) Can you generalize Problem (29) and Problem (30) above? Explain your reasoning.

(32) Let a, b, c, and d be positive integers such that

$$\frac{a}{b} < \frac{c}{d}.$$

Is it true that

$$\frac{a}{a+b} < \frac{c}{c+d}?$$

Explain your reasoning.

Chapter 3

Solving Equations

Politics is for the moment. An equation is for eternity.

—Albert Einstein

Teaching Note: In this section, we are developing the idea that numbers are solutions to equations. Negative integers arise out of simple linear equations, as do rationals. However, these are not enough to solve all polynomial equations, and hence we need a "larger" number system.

3.1 Time to get Real

Remember the definition of a *root* of a polynomial:

Definition A **root** of a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a number α where

$$a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 = 0.$$

OK—let's go! We know what integers are right? We know what rational numbers are right?

Question Remind me, what is $\mathbb{Z}$? What is $\mathbb{Q}$? What is the relationship between these two sets of numbers?

While I do want **you** to think about this, I also want to tell you my answer: $\mathbb{Q}$ is the set of solutions to linear polynomial equations with coefficients in $\mathbb{Z}$.

Question What-with-the-who-in-the-where-now?

?

Are these all the numbers we need? Well, let's see. Consider the innocent equation:

$$x^2 - 2 = 0$$

Question Could $x^2 - 2$ have rational roots?

> ***Teaching Note: Here we essentially run through the proof of the rational roots test.***

Stand back—I'll handle this. Remember, a root of $x^2 - 2$ is a number that solves the equation

$$x^2 - 2 = 0.$$

So suppose that there are integers a and b where a/b is a root of $x^2 - 2$ where a and b have no common factors. Then

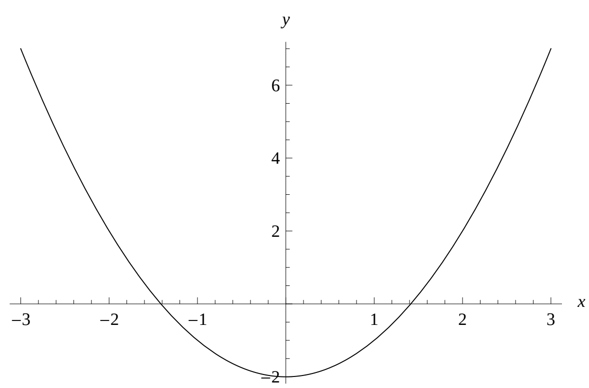
$$\left(\frac{a}{b}\right)^2 - 2 = 0.$$

So

$$a^2 - 2b^2 = 0 \qquad \text{thus} \qquad a^2 = 2b^2.$$

But a and b have no common factors—so by the Unique Factorization Theorem for the integers, $b^2 = 1$. If you find this step confusing, check out Problem (23) in Section 2.2. This tells us that $a^2 = 2$ and that a is an integer—impossible! So $x^2 - 2$ cannot have rational roots.

Hmmm but now consider the plot of $y = x^2 - 2$:



The polynomial $x^2 - 2$ clearly has two roots! But we showed above that neither of them are rational—this means that there must be numbers that cannot be expressed as fractions of integers! In particular, this means:

The square-root of 2 is not rational!

Wow! But it still can be written as a decimal

$$\sqrt{2} = 1.4142135623\ldots$$

as the square-root of 2 is a *real number*.

Definition A **real number** is a number with a (possibly infinite) decimal representation. We use the symbol $\mathbb{R}$ to denote the real numbers.

For example:

$$-1.000\ldots \qquad 2.718281828459045\ldots \qquad 3.333\ldots \qquad 0.000\ldots$$

are all real numbers.

Question Another description of real numbers is that they are the numbers that can be approximated by rational numbers. Why does this follow from the definition above?

?

Famous examples of real numbers that are not rational are

$$\pi = 3.14159265358\ldots \qquad \text{and} \qquad e = 2.718281828459045\ldots$$

Question If a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b? What can you say about the decimal representation of an irrational number?

?

Teaching Note: Activities A.21 and A.22 complement this section well.

Problems for Section 3.1

(1) Describe the set of real numbers. Give some relevant and revealing examples/nonexamples.

(2) Explain what would happen if we "declared" the value of π to be 3? What about if we declared it to have the value of 3.14?

(3) Explain why $x^2 - x - 1$ has no rational roots.

(4) Explain why $\sqrt{7}$ is irrational.

(5) Explain why $\sqrt[3]{5}$ is irrational.

(6) Explain why $\sqrt[5]{27}$ is irrational.

(7) Explain why if n is an integer and $\sqrt{n}$ is not an integer, then n is irrational.

(8) Solve $x^5 - 31x^4 + 310x^3 - 1240x^2 + 1984x - 1024 = 0$. Interlace an explanation with your work. Hint: Use reasoning from this section to find rational roots.

(9) Solve $x^5 - 28x^4 + 288x^3 - 1358x^2 + 2927x - 2310 = 0$. Interlace an explanation with your work. Hint: Use reasoning from this section to find rational roots.

(10) Knowing that π is irrational, explain why $101 \cdot \pi$ is irrational.

(11) Knowing that π is irrational, explain why $\pi + 101$ is irrational.

(12) Suppose we knew that α^2 was irrational. Could we conclude that α is also irrational? Explain your reasoning.

(13) Is $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$ rational or irrational? Explain your reasoning.

(14) In the discussion above, we give an argument showing that $\sqrt{2}$ is irrational. What happens if you try to use the exact same argument to try and show that $\sqrt{9}$ is irrational? Explain your reasoning.

3.2 Polynomial Equations

Teaching Note: Activity A.23 is a good warm-up to this section.

Solving equations is one of the fundamental activities in mathematics. We're going to separate our equations into sets:

(1) Linear Equations—polynomial equations of degree 1.

(2) Quadratic Equations—polynomial equations of degree 2.

(3) Cubic Equations—polynomial equations of degree 3.

(4) Quartic Equations—polynomial equations of degree 4.

(5) Quintic Equations—polynomial equations of degree 5.

We'll stop right there, for now…

3.2.1 Linear Equations

The simplest polynomials (besides constant polynomials) are linear polynomials. Solving equations of the form

$$ax + b = 0$$

poses no difficulty, we can write out the solution easily as

$$x = -b/a.$$

Teaching Note: Activity A.24 complements this section well.

3.2.2 Quadratic Equations

Finding roots of quadratic polynomials is a bit more complex. We want to find x such that

$$ax^2 + bx + c = 0.$$

I know you already know how to do this. However, pretend for a moment that you don't. This would be a really hard problem. We have evidence that it took humans around 1000 years to solve this problem in generality, the first general solution appearing in Babylon and China around 2500 years ago. With this in mind, I think this topic warrants some attention. If you want to solve $ax^2 + bx + c = 0$, a good place to start would be with an easier problem. Let's make $a = 1$ and try to solve

$$x^2 + bx = c$$

Geometrically, you could visualize this as an $x \times x$ square along with a $b \times x$ rectangle. Make a blob for c on the other side.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Take your $b \times x$ rectangle and divide it into two $(b/2) \times x$ rectangles.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Now take both of your $(b/2)\times x$ rectangles and snuggie them next to your $x\times x$ square on adjacent sides. You should now have what looks like an $(x+\frac{b}{2})\times(x+\frac{b}{2})$ square with a corner cut out of it.

Question What would a picture of this look like?

?

Question What is the total area of the shapes in your picture?

?

Finally, your big $(x + \frac{b}{2}) \times (x + \frac{b}{2})$ has a piece missing, a $(b/2) \times (b/2)$ square, right? So if you add that piece in on both sides, the area of both sides of your picture had better be $c + (b/2)^2$. From your picture you will find that:

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

Question Can you find x at this point?

?

Question Explain how to solve $ax^2 + bx + c = 0$.

?

Teaching Note: Activities A.25, A.26, and A.27, complement this section well.

Teaching Note: Activity A.28 could be done here too.

3.2.3 Cubic Equations

While the quadratic formula was discovered around 2500 years ago, cubic equations proved to be a tougher nut to crack. A general solution to a cubic equation was not found until the 1500's. At the time mathematicians were a secretive and competitive bunch. Someone would solve a particular cubic equation, then challenge another mathematician to a sort of "mathematical duel." Each mathematician would give the other a list of problems to solve by a given date. The one who solved the most problems was the winner and glory everlasting[1] was theirs. One of the greatest duelists was Niccolò Fontana Tartaglia (pronounced *Tar-tah-lee-ya*). Why was he so great? He developed a general method for solving cubic equations! However, neither was he alone in this discovery nor was he the first. As sometimes happens, the method was discovered some years earlier by another mathematician, Scipione del Ferro. However, due to the secrecy and competitiveness, very few people knew of Ferro's method. Since these discoveries were independent, we'll call the method the *Ferro-Tartaglia method.*

We'll show you the Ferro-Tartaglia method for finding at least one root of a cubic of the form:

$$x^3 + px + q$$

We'll illustrate with a specific example—you'll have to generalize yourself! Take

$$x^3 + 3x - 4 = 0$$

Step 1 Replace x with $u + v$.

$$\begin{aligned}(u+v)^3 + 3(u+v) - 4 &= u^3 + 3u^2v + 3uv^2 + v^3 + 3(u+v) - 4\\ &= u^3 + v^3 + 3uv(u+v) + 3(u+v) - 4\\ &= u^3 + v^3 - 4 + (3uv+3)(u+v).\end{aligned}$$

Step 2 Set uv so that all of the terms are eliminated except for u^3, v^3, and constant terms.

Since we want

$$3uv + 3 = 0$$

we'll set $uv = -1$ and so

$$u^3 + v^3 - 4 = 0.$$

[1]This might be a slight exaggeration.

Since $uv = -1$, we see that $v = -1/u$ so

$$u^3 + \left(\frac{-1}{u}\right)^3 - 4 = u^3 - \frac{1}{u^3} - 4 = 0.$$

Step 3 Clear denominators and use the quadratic formula.

$$u^3 - \frac{1}{u^3} - 4 = 0 \qquad \Leftrightarrow \qquad u^6 - 4u^3 - 1 = 0$$

But now we may set $y = u^3$ and so we have

$$y^2 - 4y - 1 = 0$$

and by the quadratic formula

$$y = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}.$$

Putting this all together we find:

$$\begin{aligned} y &= 2 \pm \sqrt{5} \\ u &= \sqrt[3]{2 \pm \sqrt{5}} \\ v &= \frac{-1}{\sqrt[3]{2 \pm \sqrt{5}}} \end{aligned}$$

and finally (drum-roll please):

$$x = \sqrt[3]{2+\sqrt{5}} - \frac{1}{\sqrt[3]{2+\sqrt{5}}} \qquad \text{and} \qquad x = \sqrt[3]{2-\sqrt{5}} - \frac{1}{\sqrt[3]{2-\sqrt{5}}}$$

Question How many solutions are we supposed to have in total?

?

Question How do we do this procedure for other equations of the form

$$x^3 + px + q = 0?$$

?

3.2.4 Quartics, Quintics, and Beyond

While the Ferro-Tartaglia method may seem like it only solves the special case of $x^3 + px + q = 0$, it is in fact a "wolf in sheep's clothing" and is the key to giving a formula for solving any cubic equation

$$ax^3 + bx^2 + cx + d = 0.$$

The formula for solutions of the cubic equation is quite complex—we will spare you the details. Despite the fact that the key step of the formula is the Ferro-Tartaglia method, it is usually called *Cardano's formula* because Cardano was the first to publish this method.

It was wondered if there were formulas for solutions to polynomial equations of arbitrary degree. When we say formulas, we mean formulas involving the coefficients of the polynomials and the symbols:

$$+ \quad - \quad \cdot \quad \div \quad \sqrt{}$$

Cardano's student Ferrari, (who incidentally went to the University of Bologna) soon found the quartic formula, though it is too monstrous to write down in these notes. The search for the quintic equation began. Things started getting very difficult. The old tricks didn't work, and it wasn't until nearly 300 years later that this problem was settled.

Question Who was Niels Abel? Who was Évariste Galois?

?

Abel and Galois (pronounced *Gal-wah*), independently prove that there is no general formula (using only the symbols above) for polynomial equations of degree 5 or higher. It is an amazing result and is only seen by students in advanced undergraduate or beginning graduated courses in pure mathematics. Nevertheless, in our studies we will not completely shy away from such demons. Read on!

Problems for Section 3.2

(1) Draw a rough timeline showing: The point when we realized we were interested in quadratic equations, the discovery of the quadratic formula, the discovery of the cubic formula, the discovery of the quartic formula, and the work of Abel and Galois proving the impossibility of a general formula for polynomial equations of degree 5 or higher.

(2) Given a polynomial, explain the connection between *linear factors* and *roots*. Are they the same thing or are they different things?

(3) In ancient and Medieval times the discussion of quadratic equations was often broken into three cases:

(a) $x^2 + bx = c$
(b) $x^2 = bx + c$
(c) $x^2 + c = bx$

where b and c are positive numbers. Create real-world word problems involving length and area for each case above.

(4) In ancient and Medieval times the discussion of quadratic equations was often broken into three cases:

(a) $x^2 + bx = c$
(b) $x^2 = bx + c$
(c) $x^2 + c = bx$

where b and c are positive numbers. Is this a complete list of cases? If not, what is missing and why is it (are they) missing? Explain your reasoning.

(5) Describe what happens geometrically when you complete the square of a quadratic equation of the form $x^2 + bx = c$ when b and c are positive. Explain your reasoning.

(6) Jim, Lydia, and Isabel are visiting China. Unfortunately they are stuck in a seemingly infinite traffic jam. The cars are moving at a very slow (but constant) rate. Jim and Lydia are 25 miles behind Isabel. Jim wants to send a sandwich to Isabel. So he hops on his motorcycle and rides through traffic to Isabel, gives her the sandwich, and rides back to Lydia at a constant speed. When he returns to Lydia, she has moved all the way to where Isabel was when Jim started. In total, how far did Jim travel on his motorcycle?

(a) Before any computations are done, use common sense to guess the solution to this problem.
(b) Try to get a feel for this problem by choosing numbers for the unknowns and doing some calculations. What do these calculations say about your guess?

   (c) Use algebra to solve the problem.

(7) Must a quadratic polynomial always have a real root? Explain your reasoning.

(8) Must a cubic polynomial always have a real root? Explain your reasoning.

(9) Must a quartic polynomial always have a real root? Explain your reasoning.

(10) Must a quintic polynomial always have a real root? Explain your reasoning.

(11) Derive the quadratic formula. Explain your reasoning.

(12) Solve $x^2 + 3x - 2 = 0$. Interlace an explanation with your work.

(13) Find two solutions to $x^4 + 3x^2 - 2 = 0$. Interlace an explanation with your work.

(14) Find two solutions to $x^6 + 3x^3 - 2 = 0$. Interlace an explanation with your work.

(15) Find two solutions to $x^{10} + 3x^5 - 2 = 0$. Interlace an explanation with your work.

(16) Find two solutions to $3x^{14} - 2x^7 + 6 = 0$. Interlace an explanation with your work.

(17) Find two solutions to $-4x^{22} + 13x^{11} + 1 = 0$. Interlace an explanation with your work.

(18) Give a general formula for finding two solutions to equations of the form: $ax^{2n} + bx^n + c = 0$ where n is an integer. Interlace an explanation with your work.

(19) Use the Ferro-Tartaglia method to find a solution to $x^3 + x + 1 = 0$. Interlace an explanation with your work.

(20) Use the Ferro-Tartaglia method to find a solution to $x^3 - x - 1 = 0$. Interlace an explanation with your work.

(21) Use the Ferro-Tartaglia method to find a solution to $x^3 + 3x - 4 = 0$. Interlace an explanation with your work.

(22) Use the Ferro-Tartaglia method to find a solution to $x^3 + 2x - 3 = 0$. Interlace an explanation with your work.

(23) Use the Ferro-Tartaglia method to find a solution to $x^3 + 6x - 20 = 0$. Interlace an explanation with your work.

(24) Find at least two solutions to $x^4 - x^3 - 3x^2 + 2x + 1 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(25) Explain what Abel and Galois proved to be impossible.

3.3 Me, Myself, and a Complex Number

Teaching Note: Activity A.29, complements this section well.

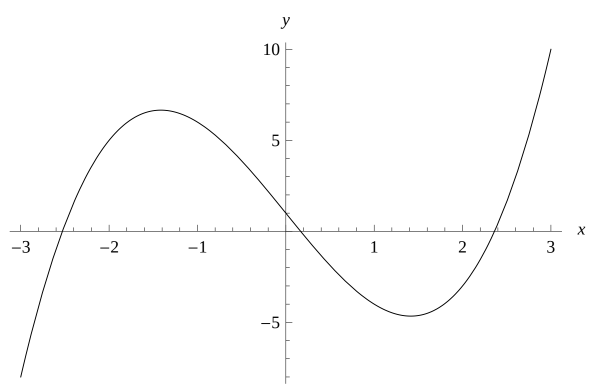
We'll start with the definition:

Definition A **complex number** is a number of the form

$$x + yi$$

where x and y are real numbers and i is the square-root of negative one. We use the symbol $\mathbb{C}$ to denote the complex numbers.

What's that I hear? Yells of protest telling me that no such number exists? Well if it makes you feel any better, people denied the existence of such numbers for a long time. It wasn't until the 1800's until people finally changed their minds. Let's talk about some ideas that helped. Consider the plot of $y = x^3 - 6x + 1$:



If you use the Ferro-Tartaglia method to find at least one solution to this cubic, then you find the following root:

$$\sqrt[3]{\frac{-1+\sqrt{-31}}{2}} + \frac{2}{\sqrt[3]{\frac{1}{2}(-1+\sqrt{-31})}}$$

This root looks like a complex number, since $\sqrt{-31}$ pops up twice. This might seem a bit redundant, but we should point out that $\sqrt{-31}$ is a complex number since it can be expressed as:

$$0 + \left(\sqrt{31}\right) i$$

Even though our root has complex numbers in it, we know that it is real from the picture! Moral: If you want to give exact solutions to equations, then you'd better work with complex numbers, even if the roots are real!

> ***Teaching Note: Here we are not ready to try to simplify the large expression above. We are leaving this as a mystery for a future course.***

Question If $u + vi$ is a nonzero complex number, is

$$\frac{1}{u+vi}$$

a complex number too?

You betcha! Let's do it. The first thing you must do is multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{u+vi} = \frac{1}{u+vi} \cdot \frac{u-vi}{u-vi} = \frac{u-vi}{u^2+v^2}$$

Now break up your fraction into two fractions:

$$\frac{u-vi}{u^2+v^2} = \frac{u}{u^2+v^2} + \frac{-v}{u^2+v^2} i$$

Ah! Since u and v are real numbers, so are

$$x = \frac{u}{u^2+v^2} \qquad \text{and} \qquad y = \frac{-v}{u^2+v^2}$$

Hence

$$\frac{1}{u+vi} = x + yi$$

and is definitely a complex number.

The real importance of the complex numbers came from Gauss, with the Fundamental Theorem of Algebra:

Theorem 5 (Fundamental Theorem of Algebra) *Every polynomial of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are complex numbers has exactly n (possibly repeated) complex roots.

Said a different way, the Fundamental Theorem of Algebra says that every polynomial with complex coefficients

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

can be factored as

$$a_n \cdot (x - r_1)(x - r_2) \cdots (x - r_n)$$

where each r_i is a complex number.

Question How many complex roots does $x^3 - 1$ have? What are they?

?

> ***Teaching Note: This problem should definitely be addressed. Again, we find an obvious root, $x = 1$ and use the division theorem.***

3.3.1 The Complex Plane

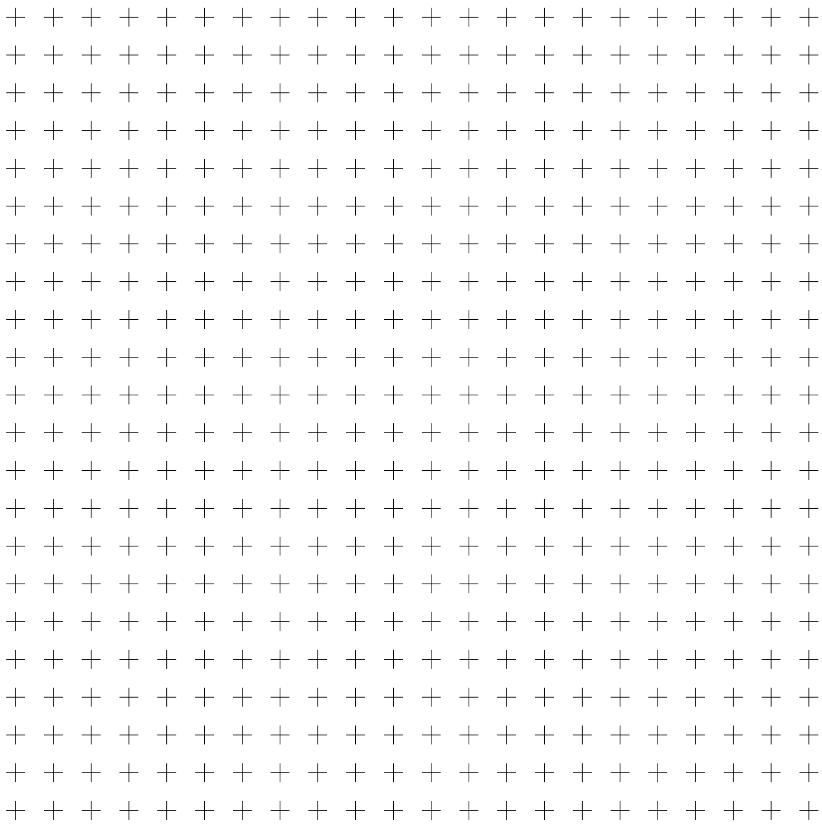
> ***Teaching Note: Activities A.30 and A.31 complement this section well.***

Complex numbers have a strong connection to geometry, we see this with the *complex plane*:

Definition The **complex plane** is obtained when one plots the complex number $x+yi$ as the point (x, y). When considering the complex plane, the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**.

Here is a grid. Draw the real and imaginary axes and plot the complex numbers:

$$3 \qquad -5i \qquad 4+6i \qquad -3+5i \qquad -6-i \qquad 6-6i$$



Be sure to label your plot.

Question Geometrically speaking, what does it mean to "add" complex numbers?

?

Question Geometrically speaking, what does it mean to "multiply" complex numbers?

?

Teaching Note: The activities A.32 and A.33 are both rather challenging, and can be skipped without real concern.

Problems for Section 3.3

(1) What is a real number?

(2) What is a complex number?

(3) Solve $x^3 - 6x + 5 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning.

(4) Solve $x^3 - 6x + 4 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning.

(5) Solve $x^3 - 2x - 1 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning. Interlace an explanation with your work.

(6) Solve $x^3 - 12x - 8 = 0$ two different ways. First, try to find an "obvious" root, call it r. Then divide your polynomial by $(x - r)$ and find the remaining roots. Second, use the Ferro-Tartaglia method to find (at least) one solution. Compare your answers. What do you notice—explain your reasoning. Interlace an explanation with your work.

(7) Solve $x^3 - 3x^2 + 5x - 3 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(8) Solve $x^3 + 4x^2 - 7x + 2 = 0$. Hint: Can you "guess" a solution to get you started? Interlace an explanation with your work.

(9) Draw a Venn diagram showing the relationship between $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Include relevant examples of numbers belonging to each set.

(10) Explain why the following "joke" is "funny:" *The number you have dialed is imaginary. Please, rotate your phone by 90 degrees and try again.*

(11) Explain why every real number is a complex number.

(12) Explain why $\sqrt{-2}$ is a complex number.

(13) Is $\sqrt[3]{-2}$ a complex number? Explain your reasoning.

(14) Explain why $\sqrt[10]{-5}$ is a complex number.

(15) Explain why if $x + yi$ and $u + vi$ are complex numbers, then

$$(x + yi) + (u + vi)$$

is a complex number.

(16) Explain why if $x + yi$ and $u + vi$ are complex numbers, then

$$(x + yi)(u + vi)$$

is a complex number.

(17) Given a complex number $z = x + yi$, the **complex conjugate** of z is $x - yi$, we denote this as $\overline{z}$. Let $w = u + vi$ be another complex number.

(a) Explain why $\overline{z} + \overline{w} = \overline{z + w}$.

(b) Explain why $\overline{z} \cdot \overline{w} = \overline{z \cdot w}$.

(18) Explain why if $u + vi$ is a complex number, then

$$\frac{1}{u + vi}$$

is a complex number.

(19) Compute the following, expressing your answer in the form $x + yi$:

(a) $(1 + 2i) + (1 + 7i)$

(b) $(1 + 2i) \cdot (1 + 7i)$

(c) $(1 + 2i)/(1 + 7i)$

Explain your reasoning.

(20) I'm going to show you something, see if you can see a connection to geometry:

(a) Let $z = 3 + 4i$. Compute $\sqrt{z \cdot \overline{z}}$.

(b) Let $z = 6 + 8i$. Compute $\sqrt{z \cdot \overline{z}}$.

(c) Let $z = 5 + 12i$. Compute $\sqrt{z \cdot \overline{z}}$.

What do you notice?

(21) Express $\sqrt{i}$ in the form $a + bi$. Hint: Solve the equation $z^2 = i$.

(22) Factor the polynomial $3x^2 + 5x + 10$ over the complex numbers. Explain your reasoning.

(23) Factor the polynomial $x^3 - 1$ over the complex numbers. Explain your reasoning.

(24) Factor the polynomial $x^4 - 1$ over the complex numbers. Explain your reasoning.

(25) Factor the polynomial $x^4 + 1$ over the complex numbers. Explain your reasoning. Hint: Factor as the difference of two squares and use Problem (21).

(26) Factor the polynomial x^4+4 over the complex numbers. Can it be factored into polynomials with real coefficients of lower degree? Explain your reasoning.

(27) Plot all complex numbers z in the complex plane such that $z \cdot \overline{z} = 1$. Explain your reasoning.

(28) Suppose I told you that:

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots
\end{aligned}$$

Explain why we say:

$$e^{x \cdot i} = \cos(x) + i \sin(x)$$

(29) This is Euler's famous formula:

$$e^{\pi \cdot i} + 1 = 0$$

Use Problem (28) to explain why it is true.

(30) How many complex roots should $x^2 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

(31) How many complex roots should $x^3 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

(32) How many complex roots should $x^4 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

(33) How many complex roots should $x^5 = 1$ have? What are they? Plot them in the complex plane. Explain your reasoning.

Chapter 4

Harmony of Numbers

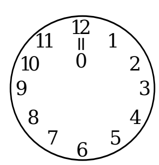
> Let us despise the barbaric neighings [of war] which echo through these noble lands, and awaken our understanding and longing for the harmonies.
>
> —Johannes Kepler

Teaching Note: This section is a hodge-podge of applications and modeling.

Teaching Note: Activity A.34 complements this section well.

4.1 Clocks

It is now time to think about clocks. Consider the usual run-of-the-mill clock:



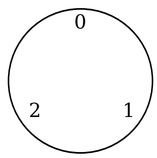
Question Suppose you start grading papers at 3 o'clock and then 5 hours pass. What time is it? Now suppose that you find more papers to grade, and 5 more hours pass—now what time is it? How do you do these problems? Why are there so many papers to grade?

?

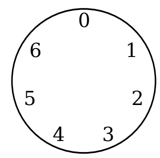
We have a mathematical way of writing these questions:

$$3 + 5 \equiv 8 \pmod{12}$$
$$8 + 5 \equiv 1 \pmod{12}$$

We call arithmetic on clocks **modular arithmetic**. Being rather fearless in our quest for knowledge, we aren't content to stick with 12 hour clocks:

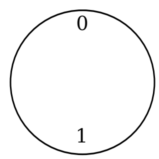


(mod 3)



(mod 7)

Question Suppose you are working on a 2 hour clock:



(mod 2)

Suppose you started at time zero, and finished after 10245 hours.

(1) Where is the hand of the clock pointing?

(2) How does the answer change if you are working on a 5 hour clock?

(3) What if you are working on a 7 hour clock?

?

OK—clocks are great. Here is something slightly different: Denote the set of all integers that are r greater than a multiple of 5 by $[r]_5$. So for example:

$$[0]_5 = \{\dots, -15, -10, -5, 0, 5, 10, 15, \dots\}$$

Write down the following sets:

$[1]_5 =$

$[2]_5 =$

$[3]_5 =$

$[4]_5 =$

$[5]_5 =$

Question With our work above, see if you can answer the following:

(1) Explain why one could say that $[4]_5 = [9]_5$.

(2) Explain why one could say that $[2]_5 = [-3]_5$.

(3) Explain what you think is meant by the expression:

$$[1]_5 + [2]_5 = [3]_5$$

(4) Explain what you think is meant by the expression:

$$[1]_5 + [4]_5 = [0]_5$$

?

Question How many different descriptions of modular arithmetic can you give? To aid you in this quest, I suggest you start your descriptions off with the words:

The number a is congruent to b modulo m when ...

?

OK—I know I was supposed to leave that question for you, but there is one description that I just gotta tell you about—check this out:

$$a \equiv b \pmod{m} \qquad \Leftrightarrow \qquad a - b = m \cdot q$$

Question What is the deal with the junk above? What is q? How does it help you solve congruences like

$$3x \equiv 1 \pmod{11}?$$

?

Question Is it the case that

$$5 + x \equiv 2 + x \pmod{3}$$

for all integers x? Why or why not? Use each of the descriptions of modular arithmetic above to answer this question.

?

Problems for Section 4.1

(1) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $3 + x = 10$
(b) $3 + x \equiv 10 \pmod{12}$
(c) $3 + x \equiv 10 \pmod{7}$
(d) $3 + x \equiv 10 \pmod{6}$
(e) $3 + x \equiv 10 \pmod{5}$
(f) $3 + x \equiv 10 \pmod{3}$
(g) $3 + x \equiv 10 \pmod{2}$

In each case explain your reasoning.

(2) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $10 + x = 1$
(b) $10 + x \equiv 1 \pmod{12}$
(c) $10 + x \equiv 1 \pmod{11}$
(d) $10 + x \equiv 1 \pmod{9}$
(e) $10 + x \equiv 1 \pmod{5}$
(f) $10 + x \equiv 1 \pmod{3}$
(g) $10 + x \equiv 1 \pmod{2}$

In each case explain your reasoning.

(3) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $217 + x = 1022$
(b) $217 + x \equiv 1022 \pmod{100}$
(c) $217 + x \equiv 1022 \pmod{20}$
(d) $217 + x \equiv 1022 \pmod{12}$
(e) $217 + x \equiv 1022 \pmod{5}$
(f) $217 + x \equiv 1022 \pmod{3}$
(g) $217 + x \equiv 1022 \pmod{2}$

In each case explain your reasoning.

(4) Solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $11 + x \equiv 7 \pmod 2$
(b) $11 + x \equiv 7 \pmod 3$
(c) $11 + x \equiv 7 \pmod 5$
(d) $11 + x \equiv 7 \pmod 8$
(e) $11 + x \equiv 7 \pmod{10}$

In each case explain your reasoning.

(5) List out 6 elements of $[3]_4$, including 3 positive and 3 negative elements. Explain your reasoning.

(6) List out 6 elements of $[6]_7$, including 3 positive and 3 negative elements. Explain your reasoning.

(7) List out 6 elements of $[7]_6$, including 3 positive and 3 negative elements. Explain your reasoning.

(8) One day you walk into a mathematics classroom and you see the following written on the board:

$$[4]_6 = \{\dots, -14, -8, -2, 4, 10, 16, 22, \dots\}$$
$$\left[\frac{1}{2}\right] = \left\{\dots, \frac{-3}{-6}, \frac{-2}{-4}, \frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \dots\right\}$$

What is going on here? Can you figure out what $\left[\frac{3}{4}\right]$ would be? Explain your reasoning.

(9) If possible, solve the following equations/congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $3x = 1$
(b) $3x \equiv 1 \pmod{11}$
(c) $3x \equiv 1 \pmod 9$
(d) $3x \equiv 1 \pmod 8$
(e) $3x \equiv 1 \pmod 7$
(f) $3x \equiv 1 \pmod 3$
(g) $3x \equiv 1 \pmod 2$

In each case explain your reasoning.

(10) Solve the following congruences, expressing your answer as a number between 0 and the relevant modulus:

(a) $11x \equiv 7 \pmod 2$
(b) $11x \equiv 7 \pmod 3$

(c) $11x \equiv 7 \pmod 5$
(d) $11x \equiv 7 \pmod 8$
(e) $11x \equiv 7 \pmod{10}$

In each case explain your reasoning.

(11) Solve the following congruences or explain why there is no solution, expressing your answer as a number between 0 and the relevant modulus:

(a) $15x \equiv 7 \pmod 2$
(b) $15x \equiv 7 \pmod 3$
(c) $15x \equiv 7 \pmod 5$
(d) $15x \equiv 7 \pmod 9$
(e) $15x \equiv 7 \pmod{10}$

In each case explain your reasoning.

(12) Make an “addition table” for arithmetic modulo 6.

(13) Make an “addition table” for arithmetic modulo 7.

(14) Make a “multiplication table” for arithmetic modulo 6.

(15) Make a “multiplication table” for arithmetic modulo 7.

(16) Explain the connection between writing an integer in base b and reducing an integer modulo b.

(17) Is
$$5 + x \equiv 12 + x \pmod 3$$
ever/always true? Explain your reasoning.

(18) Is
$$20 + x \equiv 32 + x \pmod 3$$
ever/always true? Explain your reasoning.

(19) Recalling that $i^2 = -1$, can you find “i” in the integers modulo 5? Explain your reasoning.

(20) Recalling that $i^2 = -1$, can you find “i” in the integers modulo 17? Explain your reasoning.

(21) Recalling that $i^2 = -1$, can you find “i” in the integers modulo 13? Explain your reasoning.

(22) Recalling that $i^2 = -1$, can you find “i” in the integers modulo 11? Explain your reasoning.

(23) Today is Saturday. What day will it be in 3281 days? Explain your reasoning.

(24) It is now December. What month will it be in 219 months? What about 111 months ago? Explain your reasoning.

(25) What is the remainder when 2^{999} is divided by 3? Explain your reasoning.

(26) What is the remainder when 3^{26} is divided by 7? Explain your reasoning.

(27) What is the remainder when 14^{30} is divided by 11? Explain your reasoning.

(28) What is the remainder when 5^{28} is divided by 11? Explain your reasoning.

(29) What is the units digit of 123^{456}? Explain your reasoning.

(30) Factor $x^2 + 1$ over the integers modulo 2. Explain your reasoning.

(31) Factor $x^3 + x^2 + x + 1$ over the integers modulo 2. Explain your reasoning.

(32) Factor $x^5 + x^4 + x + 1$ over the integers modulo 2. Explain your reasoning.

4.2 In the Real World

Perhaps the coolest thing about mathematics is that you can actually solve "real world" problems. Let's stroll through some of these "real world" problems.

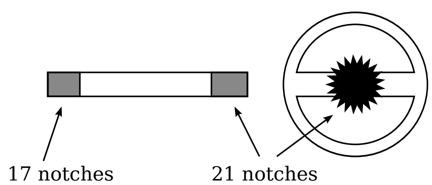
4.2.1 Automotive Repair

A Geometry Problem

One Thanksgiving Day I had a neat conversation with my cousin Chris at the dinner table. You see he works on cars—specifically vintage Italian sports cars. He had been doing some routine maintenance on one of his cars and needed to remove the steering wheel and the steering column. All was fine until it came time to put the parts back together. The steering wheel was no longer centered! The car could drive down the street just fine, but when the car drove straight ahead the steering wheel was off by a rotation of 5 degrees to the right. This would not do! This sounds like a geometry problem.

An Algebra Problem

How did this happen you ask? Well the *steering wheel* attaches to the car via the *steering column*:



there were 21 notches on the back of the wheel, which connects to the column. There were also 17 notches on the other end of the column that then connected to the car itself.

Chris had noticed that moving the wheel 1 notch changed its position by

$$\frac{360}{21} \approx 17 \text{ degrees},$$

and that adjusting the columns by 1 notch changed its position by

$$\frac{360}{17} \approx 21 \text{ degrees}.$$

Hmmm so if we want to center the wheel, we want to solve the following equation:

$$17w + 21c = -5$$

where w represents how many notches we turn the wheel and c represents how many notches we turn the column. Ah! This sounds like an algebra problem! There is only one issue: We have two unknowns and a single variable.

Question How do we proceed from here? Can you solve the problem? Where does modular arithmetic factor in to the solution?

?

4.2.2 Check Digits

Our world is full of numbers. Sometimes if you are in a large organization—say a large university—you feel a bit like a number. How do you know if you are the right number? Allow me to clarify. Most items you buy have some sort of UPC (Universal Product Code) on them. This allows them to be put into a computer in an organized fashion. When you buy items in a grocery story, you want the item you scanned to come up—and not some other (potentially embarrassing!) item. To ensure you get what is coming to you, we have *check digits*. These are digits that "check" to make sure that the code has scanned correctly. Typically, what you see are either UPC-A codes or UPC-E codes. Here is an example of a UPC-A code:

0 42572 05001 4

The check digit is the right most digit (in this case 4). The check digit is not used in identifying the item, instead it is used purely to check if the other digits are correct. Here is how you check to see if a UPC-A code is valid:

(1) Working modulo 10, add the digits in the odd positions and multiply by 3:

$$\begin{aligned} 0+2+7+0+0+1 &= 10 \\ 10 \cdot 3 &= 30 \\ 30 &\equiv 0 \pmod{10}. \end{aligned}$$

(2) Working modulo 10, add the digits in the even positions (including the check digit):

$$\begin{aligned} 4+5+2+5+0+4 &= 20 \\ 20 &\equiv 0 \pmod{10} \end{aligned}$$

(3) Add the outcomes from the previous steps together and take the result modulo 10:

$$0+0 \equiv 0 \pmod{10}$$

If the result is congruent to 0 modulo 10, as it is in this case, then you have a correct UPC-A number and you are good to go!

We should note, sometimes at stores you see UPC-E codes:

0 123456 5

These are compressed UPC-A codes where 5 zeros have been removed. The rules for transforming UPC-A codes to UPC-E codes are a bit tedious, so we'll skip them for now—though they are easy to look up on the internet.

Question Can you find a UPC-E code and verify that it is valid?

?

Problems for Section 4.2

(1) Which of the following is a correct UPC-A number?

8 12556 01041 0

8 12565 01091 0

8 12556 01091 0

Explain your reasoning.

(2) Which of the following is a correct UPC-A number?

7 17664 13387 0

7 17669 13387 0

7 17669 73387 0

Explain your reasoning.

(3) Find the missing digit in the following UPC-A number:

8 14371 0■354 2

Explain your reasoning.

(4) Find the missing digit in the following UPC-A number:

0 76484 86■97 3

Explain your reasoning.

(5) How similar can two different UPC numbers be? Explain your reasoning.

(6) In the United States some bank check codes are nine digit numbers

$$a_1a_2a_3a_4a_5a_6a_7a_8a_9$$

where

$$7a_1 + 3a_2 + 9a_3 + 7a_4 + 3a_5 + 9a_6 + 7a_7 + 3a_8 \equiv a_9 \pmod{10}.$$

(a) Give three examples of valid bank check codes.
(b) If adjacent digits were accidentally switched, could a machine detect the error? Explain your reasoning.

(7) ISBN-10 numbers are ten digit numbers

$$a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}$$

where

$$10a_1 + 9a_2 + 8a_3 + 7a_4 + 6a_5 + 5a_6 + 4a_7 + 3a_8 + 2a_9 + a_{10} \equiv 0 \pmod{11}.$$

(a) Give three examples of ISBN-10 numbers.
(b) If adjacent digits were accidentally switched, could a machine detect the error? Explain your reasoning.

4.3 The Binomial Theorem

> ***Teaching Note: This is a good point to do Activities A.35 through activity A.37. Though they are somewhat independent of the rest of the chapter.***

4.3.1 Varna-Sangita

In ancient Indian texts we find a description of a type of music called *varna-sangita*. This is music made from a variation of long and short syllables. When performing a varna-sangita, one starts off with a given number of short syllables and ends with the same number of long syllables. In between these verses, every possible combination of long and short syllables is supposed to occur. If s represents a short syllable and l represents a long syllable we might visualize this as:

$$ssss \xrightarrow{\text{every possible combination}} llll$$

To check their work, the people of ancient India counted how many of each combination appeared in a song. Suppose we started with sss and finished with lll. Our song should contain the following verses:

$$sss, \quad ssl, \quad sls, \quad lss, \quad sll, \quad lsl, \quad lls, \quad lll$$

We can construct the following table to summarize what we have found:

3 s's	2 s's and 1 l	1 s and 2 l's	3 l's
1	3	3	1

Question What would your table look like if you started with ss and finished with ll? What about if you started with $ssss$ and finished with $llll$?

?

The vedics of the time gave a rule for making tables like the one above. Their rule was based on the following diagram:

$$\begin{array}{lccccccccccccccr}
n=0: & & & & & & & 1 & & & & & & & & k=0 \\
n=1: & & & & & & 1 & & 1 & & & & & & & k=1 \\
n=2: & & & & & 1 & & 2 & & 1 & & & & & & k=2 \\
n=3: & & & & 1 & & 3 & & 3 & & 1 & & & & & k=3 \\
n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & k=4 \\
n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & k=5 \\
n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & k=6
\end{array}$$

Today people call this diagram **Pascal's triangle**.

Question How does Pascal's triangle relate to varna-sangitas? Is there an easy way to produce the above diagram?

?

And now for something completely different...

4.3.2 Expansions

Expand the following on a separate sheet of paper. Write the result of your work in the boxes below:

$$(a+b)^0 =$$

$$(a+b)^1 =$$

$$(a+b)^2 =$$

$$(a+b)^3 =$$

$$(a+b)^4 =$$

Question Is there a nice way to organize this data?

?

Question Can you explain the connection between expanding binomials and varna-sangitas?

?

Teaching Note: Activity A.38 complements this section well.

4.3.3 Come Together

Let's see if we can bring these ideas together. Let's denote the following symbol:

$$\binom{n}{k} = \text{the number of ways we choose } k \text{ objects from } n \text{ objects.}$$

it is often said "n choose k" and is sometimes denoted as ${}_nC_k$.

Question What exactly does $\binom{n}{k}$ mean in terms of varna-sangitas? What does $\binom{n}{k}$ mean in terms of expansion of binomials?

?

Question How does $\binom{n}{k}$ relate to Pascal's triangle?

?

Question Pascal claims:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Explain how this single equation basically encapsulates the key to constructing Pascal's triangle.

?

Question Suppose that an oracle tells you that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

but we, being good skeptical people, are not convinced. How do we check this?

?

From the work above, we obtain a fabulous theorem:

Theorem 6 (Binomial Theorem) *If n is a nonnegative integer, then*

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

Question This looks like gibberish to me. Tell me what it is saying. Also, why is the Binomial Theorem true?

?

Teaching Note: Activity A.39 complements this section well.

Teaching Note: Activity A.40 is difficult and can be skipped.

Teaching Note: The counting/probability activities, A.41 through A.45 can now be done.

Problems for Section 4.3

(1) Write down the first 7 rows of Pascal's triangle.

(2) Explain how $\binom{n}{k}$ corresponds to the entries of Pascal's triangle. Feel free to draw diagrams and give examples.

(3) State the Binomial Theorem and give some examples of it in action.

(4) Explain the "physical" meaning of $\binom{n}{k}$. Give some examples illustrating this meaning.

(5) Explain how Pascal's triangle is formed. In your explanation, use the notation $\binom{n}{k}$. If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

(6) Explain why the formula you found in Problem (5) is true.

(7) State the formula for $\binom{n}{k}$.

(8) Expand $(a+b)^5$ using the Binomial Theorem.

(9) Expand $(a-b)^7$ using the Binomial Theorem.

(10) Expand $(-a-b)^8$ using the Binomial Theorem.

(11) Expand $(a+(b+c))^3$ using the Binomial Theorem.

(12) Expand $(a-b-c)^3$ using the Binomial Theorem.

(13) Let n be a positive integer.

  (a) Try some experiments to guess when $9^n + 1^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.

  (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 9 = 1$.

(14) Let n be a positive integer.

  (a) Try some experiments to guess when $6^n + 4^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.

  (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 6 = 4$.

(15) Let n be a positive integer.

  (a) Try some experiments to guess when $7^n - 3^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.

  (b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 3 = 7$.

(16) Let n be a positive integer.

(a) Try some experiments to guess when $8^n - 2^n$ is divisible by 10. What do you find? Clearly articulate your conjecture.

(b) Use the Binomial Theorem to explain why your conjecture is true. Hint: $10 - 2 = 8$.

(17) Generalize Problems (13), (14), (15), and (16) above. Clearly articulate your new statement(s) and explain why they are true.

(18) Which is larger, $(1 + 1/2)^2$ or 2? Explain your reasoning.

(19) Which is larger, $(1 + 1/5)^5$ or 2? Explain your reasoning.

(20) Which is larger, $(1 + 1/27)^{27}$ or 2? Explain your reasoning.

(21) Which is larger, $(1 + 1/101)^{101}$ or 2? Explain your reasoning.

(22) Which is larger, $(1.0001)^{10000}$ or 2? Explain your reasoning.

(23) Generalize Problems (18), (19), (20), (21), and (22) above. Clearly articulate your new statement(s) and explain why it is true.

(24) Given a positive integer n, can you guess an upper bound for $(1 + 1/n)^n$?

(25) Use the Binomial Theorem to explain why:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

What does this mean in terms of Pascal's Triangle?

(26) Use the Binomial Theorem to explain why when n is a positive integer:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n\binom{n}{n} = 0$$

What does this mean in terms of Pascal's Triangle?

(27) Suppose I tell you:

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Explain how to deduce:

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

Appendix A

Activities

A.1 Shelby and Scotty

Shelby and Scotty want to express the number 27 in base 4. However, they used very different methods to do this. Let's check them out.

1) Consider Shelby's work:

$$\begin{array}{r} 6 \text{ R}\,\mathbf{3} \\ 4\overline{)27} \end{array} \qquad \begin{array}{r} 1 \text{ R}\,\mathbf{2} \\ 4\overline{)6} \end{array} \qquad \begin{array}{r} 0 \text{ R}\,\mathbf{1} \\ 4\overline{)1} \end{array} \qquad \Rightarrow \qquad \boxed{123}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

2) Consider Scotty's work:

$$\begin{array}{r} 0 \text{ R}\,27 \\ 4^3\overline{)27} \end{array} \qquad \begin{array}{r} \mathbf{1} \text{ R}\,11 \\ 4^2\overline{)27} \end{array} \qquad \begin{array}{r} \mathbf{2} \text{ R}\,\mathbf{3} \\ 4\overline{)11} \end{array} \qquad \Rightarrow \qquad \boxed{123}$$

(a) Describe how to perform this algorithm.

(b) Provide an additional relevant and revealing example demonstrating that you understand the algorithm.

3) Create an illustration (or series of illustrations) based on the 27 marks below that models Shelby's method for changing bases.

| | | | | | | | | | | | | | | | | | | | | | | | | | |

Further, explain why Shelby's method works.

4) Create an illustration (or series of illustrations) based on the 27 marks below that models Scotty's method for changing bases.

| | | | | | | | | | | | | | | | | | | | | | | | | | |

Further, explain why Scotty's method works.

A.2 Hieroglyphical Arithmetic

Consider the following addition and multiplication tables:

+	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
fish	fork	balloon	fish	eyeball	cinder-block	skull	lolly-pop	man	DNA
lolly-pop	balloon	eyeball	lolly-pop	fork	fish	man	cinder-block	DNA	skull
skull	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
cinder-block	eyeball	fork	cinder-block	balloon	lolly-pop	DNA	fish	skull	man
DNA	cinder-block	fish	DNA	lolly-pop	man	eyeball	skull	fork	balloon
fork	skull	man	fork	DNA	eyeball	fish	balloon	lolly-pop	cinder-block
man	lolly-pop	cinder-block	man	fish	skull	balloon	DNA	eyeball	fork
balloon	man	DNA	balloon	skull	fork	lolly-pop	eyeball	cinder-block	fish
eyeball	DNA	skull	eyeball	man	balloon	cinder-block	fork	fish	lolly-pop

= fish

= lolly-pop

= skull

= cinder-block

= DNA

= fork

= man

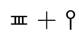
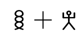
= balloon

= eyeball

·	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish
fork	fish	lolly-pop	cinder-block	DNA	skull	eyeball	balloon	man	fork
eyeball	lolly-pop	balloon	man	fork	skull	cinder-block	DNA	fish	eyeball
balloon	cinder-block	man	fork	lolly-pop	skull	DNA	fish	eyeball	balloon
man	DNA	fork	lolly-pop	cinder-block	skull	fish	eyeball	balloon	man
skull	skull	skull	skull	skull	skull	skull	skull	skull	skull
lolly-pop	eyeball	cinder-block	DNA	fish	skull	balloon	man	fork	lolly-pop
cinder-block	balloon	DNA	fish	eyeball	skull	man	fork	lolly-pop	cinder-block
DNA	man	fish	eyeball	balloon	skull	fork	lolly-pop	cinder-block	DNA
fish	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish

This activity is based on an activity originally designed by Lee Wayand.

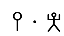
1) Use the addition table to compute the following:

 and 

2) Do you notice any patterns in the addition table? Tell us about them.

3) Can you tell me which glyph represents 0? How did you arrive at this conclusion?

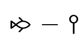
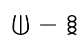
4) Use the multiplication table to compute the following:

 and 

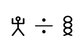
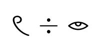
5) Do you notice any patterns in the multiplication table? Tell us about them.

6) Can you tell me which glyph represents 1? How did you arrive at this conclusion?

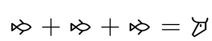
7) Compute:

 and 

8) Compute:

 and 

9) Keen Kelley was working with our tables above. All of a sudden, she writes

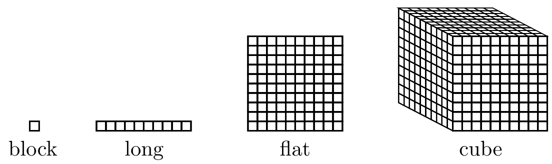


and shouts "Weird!" Why is she so surprised? Try repeated addition with other glyphs. What do you find? Can you explain this?

10) Can you find any other oddities of the arithmetic above? Hint: Try repeated multiplication!

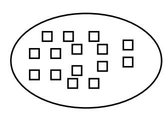
A.3 Playing with Blocks

I always enjoyed blocks quite a bit. Go find yourself some *base-ten blocks*. Just so that we are all on the same page, here are the basic blocks:



1) Model the number 247 with base-ten blocks.

2) Oscar modeled the number 15 in the following way:

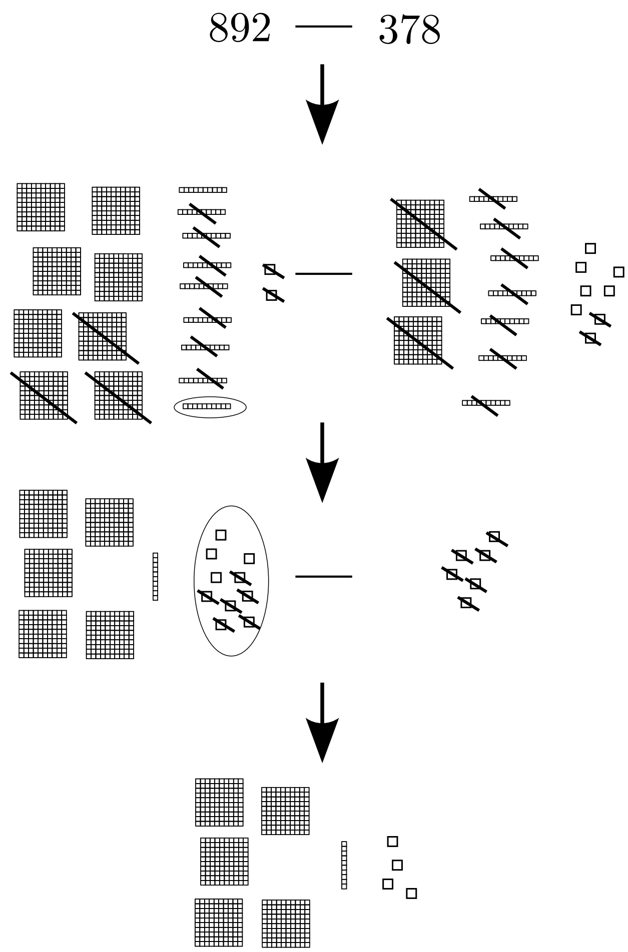


What do you think of his model?

> ***Teaching Note: The issue here is that the place-value system is not modeled. When working with base-10 blocks, we will demand that the place value system is always modeled.***

3) Now Oscar is modeling the basic subtraction algorithm:

$$\begin{array}{r} \;\;8 \\ 8\cancel{9}^1 2 \\ -3\,7\;\;8 \\ \hline 5\,1\;\;4 \end{array}$$



Can you explain what is going on? What do you think of his model? Can you give a better model?

> ***Teaching Note: Here the issue is that the actual operations of the algorithm are not modeled.***

4) Create a "new" subtraction algorithm based on Oscar's model.

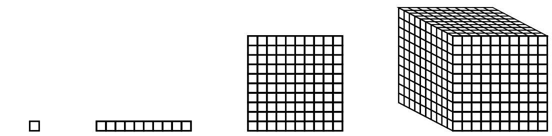
5) Here is an example of the basic addition algorithm:

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

Explain how to model this algorithm with base-ten blocks.

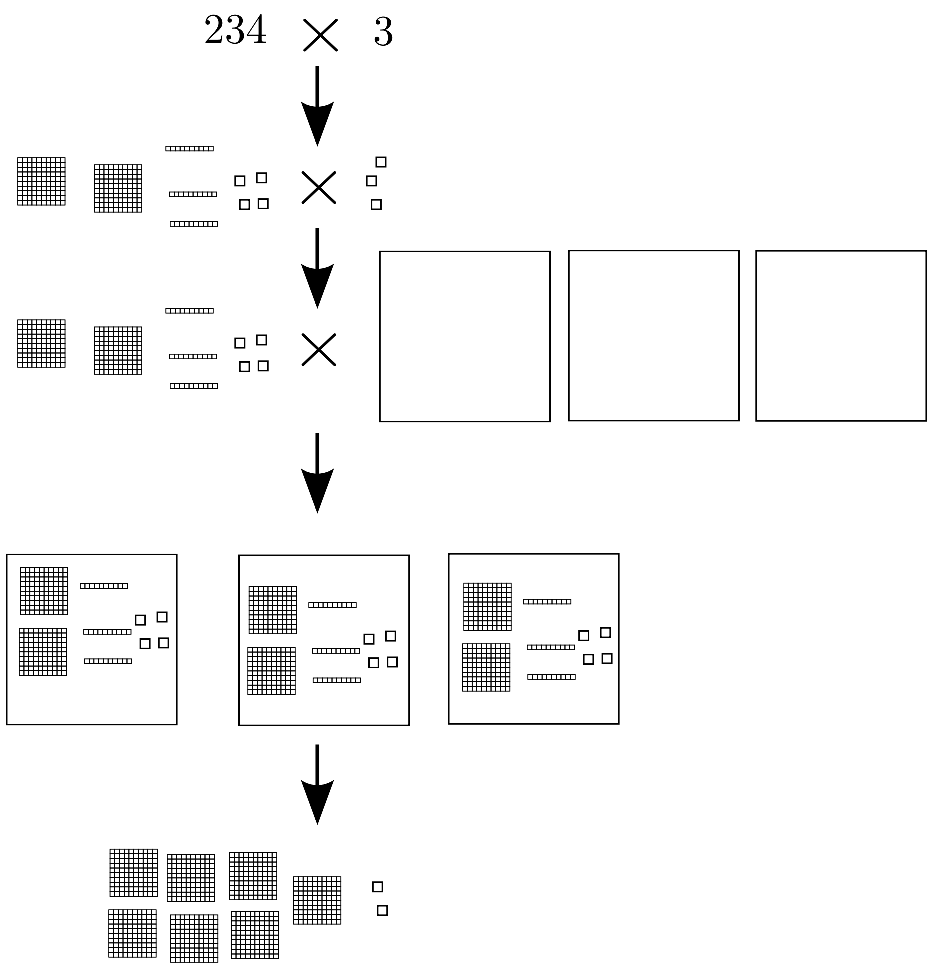
A.4 More Playing with Blocks

Did you put your blocks away? Darn—there is still more to be done!. Just so that we are all on the same page, here are the basic blocks:



1) Now Oscar is modeling the basic multiplication algorithm:

$$\begin{array}{r} {}^{1\,1} \\ 234 \\ \times \quad 3 \\ \hline 702 \end{array}$$



Can you explain what is going on? What do you think of his model?

2) Here is an example of the basic division algorithm:

$$
\begin{array}{r@{}r}
 & 67 \;\; \text{R1} \\
3\,\big) & \overline{202}\phantom{\;\; \text{R1}} \\
 & \underline{18}\phantom{\;\; \text{R1}} \\
 & 22\phantom{\;\; \text{R1}} \\
 & \underline{21}\phantom{\;\; \text{R1}} \\
 & 1\phantom{\;\; \text{R1}}
\end{array}
$$

Explain how to model this algorithm with base-ten blocks.

A.5 Comparative Arithmetic

1) Compute:

$$\begin{array}{r} 131 \\ \underline{+122} \end{array} \qquad \text{and} \qquad \begin{array}{r} x^2+3x+1 \\ \underline{+x^2+2x+2} \end{array}$$

Compare, contrast, and describe your experiences.

2) Compute:

$$\begin{array}{r} 139 \\ \underline{+122} \end{array} \qquad \text{and} \qquad \begin{array}{r} x^2+3x+9 \\ \underline{+x^2+2x+2} \end{array}$$

Compare, contrast, and describe your experiences. In particular, discuss how this is different from the first problem.

3) Compute:

$$\begin{array}{r} 121 \\ \underline{\times 32} \end{array} \qquad \text{and} \qquad \begin{array}{r} x^2+2x+1 \\ \underline{\times \quad 3x+2} \end{array}$$

Compare, contrast, and describe your experiences.

4) Expand:

$$(x^2+2x+1)(3x+2)$$

Compare, contrast, and describe your experiences. In particular, discuss how this problem relates to the one above.

A.6 What Can Division Mean?

Here are some problems involving division. Someone once told me that most division problems could be broken into two types:

(a) Those that are asking "How many groups?"

(b) Those that are asking "How many in each group?"

Let's put this claim to the test. For each of the problems below:

(a) Numerically solve the problem. Should our answer be a decimal, or a quotient and reminder?

(b) Draw a picture representing the situation and describe actions with objects a student could carry out to solve the problem.

(c) Identify whether the problem is asking "How many groups?" or "How many in each group?" or something else entirely.

1) There are a total of 35 hard candies. If there are 5 boxes with an equal number of candies in each box—and all the candy is accounted for, then how many candies are in each box? What if you had 39 candies?

2) There are a total of 28 hard candies. If there are 4 candies in each box, how many boxes are there? What if you had 34 candies?

3) There is a total of 29 gallons of milk to be put in 6 containers. If each container holds the same amount of milk and all the milk is accounted for, how much milk will each container hold?

4) There is a total of 29 gallons of milk to be put in containers holding 6 gallons each. If all the milk is used, how many containers were used?

5) If there were 29 kids and each van holds 5 kids, how many vans would we need for the field trip?

A.7 There's Always Another Prime

We'll start off with easy questions, then move to harder ones.

1) Use the Division Theorem to explain why 2 does not divide $2 + 1$.

2) Use the Division Theorem to explain why neither 2 nor 3 divides $2 \cdot 3 + 1$.

3) Use the Division Theorem to explain why neither 2 nor 3 nor 5 divides $2 \cdot 3 \cdot 5 + 1$.

4) Let $p_1, \ldots, p_n$ be the first n primes. Do any of these primes divide

$$p_1 p_2 \cdots p_n + 1?$$

Explain your reasoning.

5) Suppose there were only a finite number of primes, say there were only n of them. Call them $p_1, \ldots, p_n$. Could any of them divide

$$p_1 p_2 \cdots p_n + 1?$$

what does that mean? Can there really only be a finite number of primes?

6) Consider the following:

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509$$

Does this contradict our work above? If so, explain why. If not, explain why not.

A.8 Sieving it All Out

1) *Incognito's Hall of Shoes* is a shoe store that just opened in Myrtle Beach, South Carolina. At the moment, they have 100 pairs of shoes in stock. At their grand opening 100 customers showed up. The first customer tried on every pair of shoes, the second customer tried on every 2nd pair, the third customer tried on every 3rd pair, and so on until the 100th customer, who only tried on the last pair of shoes.

(a) Which shoes were tried on by only 1 customer?

(b) Which shoes were tried on by exactly 2 customers?

(c) Which shoes were tried on by exactly 3 customers?

(d) Which shoes were tried on by the most number of customers?

Explain your reasoning.

2) Can you use the techniques from the problem above to find a way to systematically find all the primes from 1 to 120 *without* doing any division? As a jesture of friendship, we've listed out the numbers from 1 to 120.

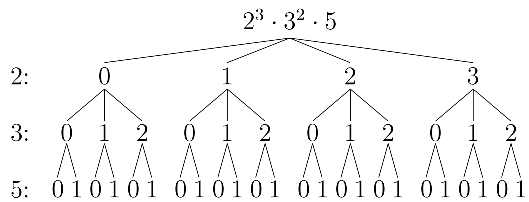
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

3) Find all of the prime factors of 1008. How can you be sure you've found them all?

A.9 There are Many Factors to Consider

1) How many factors does the integer 60 have?

2) Consider the following diagram:



What is going on in this diagram? What do the numbers represent? How does it help you count the number of factors of $2^3 \cdot 3^2 \cdot 5$?

3) Make a similar diagram for 60.

4) Can you devise a method for computing the number of factors that a number has? Explain why your method works.

5) How many factors does 735 have?

6) If p is a prime number, how many factors does p^n have?

7) If p and q are both prime numbers, how many factors does $p^n q^m$ have?

8) Which integers between 0 and 100 have the most factors?

A.10 Why Does It Work?

The Euclidean Algorithm is pretty neat. Let's see if we can figure out **why** it works. As a gesture of friendship, I'll compute $\gcd(351, 153)$:

$$\begin{aligned} 351 &= \mathbf{153} \cdot 2 + \mathbf{45} \\ \mathbf{153} &= \mathbf{45} \cdot 3 + \mathbf{18} \\ \mathbf{45} &= \mathbf{18} \cdot 2 + \boxed{\mathbf{9}} \\ 18 &= 9 \cdot 2 + 0 \qquad \boxed{\therefore \gcd(351, 153) = 9} \end{aligned}$$

Let's look at this line-by-line.

The First Line

1) Since $351 = 153 \cdot 2 + 45$, explain why $\gcd(153, 45)$ divides 351.

2) Since $351 = 153 \cdot 2 + 45$, explain why $\gcd(351, 153)$ divides 45.

3) Since $351 = 153 \cdot 2 + 45$, explain why $\gcd(351, 153) = \gcd(153, 45)$.

The Second Line

4) Since $153 = 45 \cdot 3 + 18$, explain why $\gcd(45, 18)$ divides 153.

5) Since $153 = 45 \cdot 3 + 18$, explain why $\gcd(153, 45)$ divides 18.

6) Since $153 = 45 \cdot 3 + 18$, explain why $\gcd(153, 45) = \gcd(45, 18)$.

The Third Line

7) Since $45 = 18 \cdot 2 + 9$, explain why $\gcd(18, 9)$ divides 45.

8) Since $45 = 18 \cdot 2 + 9$, explain why $\gcd(45, 18)$ divides 9.

9) Since $45 = 18 \cdot 2 + 9$, explain why $\gcd(45, 18) = \gcd(18, 9)$.

The Final Line

10) Why are we done? How do you know that the Euclidean Algorithm will **always** terminate?

A.11 Prome Factorization

Let's consider a crazy set of numbers—all multiples of 3. Let's use the symbol $3\mathbb{Z}$ to denote the set consisting of all multiples of 3. As a gesture of friendship, I have written down the first 100 nonnegative integers in $3\mathbb{Z}$:

0	3	6	9	12	15	18	21	24	27
30	33	36	39	42	45	48	51	54	57
60	63	66	69	72	75	78	81	84	87
90	93	96	99	102	105	108	111	114	117
120	123	126	129	132	135	138	141	144	147
150	153	156	159	162	165	168	171	174	177
180	183	186	189	192	195	198	201	204	207
210	213	216	219	222	225	228	231	234	237
240	243	246	249	252	255	258	261	264	267
270	273	276	279	282	285	288	291	294	297

1) Given any two integers in $3\mathbb{Z}$, will their sum be in $3\mathbb{Z}$? Explain your reasoning.

2) Given any two integers in $3\mathbb{Z}$, will their difference be in $3\mathbb{Z}$? Explain your reasoning.

3) Given any two integers in $3\mathbb{Z}$, will their product be in $3\mathbb{Z}$? Explain your reasoning.

4) Given any two integers in $3\mathbb{Z}$, will their quotient be in $3\mathbb{Z}$? Explain your reasoning.

Definition Call a positive integer **prome** in $3\mathbb{Z}$ if it cannot be expressed as the product of two integers *both* in $3\mathbb{Z}$.

As an example, I tell you that 6 is prome number in $3\mathbb{Z}$. You may object because $6 = 2 \cdot 3$, but remember—2 is not in $3\mathbb{Z}$!

5) List all the prome numbers less than 297.

6) Can you give some sort of algebraic characterization of prome numbers in $3\mathbb{Z}$?

7) Can you find numbers that factor completely into prome numbers in *two* different ways? How many can you find?

A.12 Picture Models for Equivalent Fractions

1) Draw pictures to explain why:

$$\frac{2}{3} = \frac{4}{6}$$

Explain how your pictures show this.

2) Draw pictures to explain why:

$$\frac{3}{6} = \frac{2}{4}$$

Explain how your pictures show this.

3) Given equivalent fractions with $0 < a \leqslant b$ and $0 < c \leqslant d$:

$$\frac{a}{b} = \frac{c}{d}$$

Give a procedure for representing this equation with pictures.

4) Explain why if $0 < a \leqslant b$ and $0 < c \leqslant d$:

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc$$

Feel free to use pictures as part of your explanation.

A.13 Picture Models for Fraction Operations

1) Draw pictures that model:

$$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

Explain how your pictures show this. Write a story problem whose solution is given by the expression above.

2) Draw pictures that model:

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$

Explain how your pictures model this equation. Be sure to carefully explain how common denominators are represented in your pictures. Write a story problem whose solution is given by the expression above.

3) Given $0 < a \leqslant b$ and $0 < c \leqslant d$, explain how to draw pictures that model the sum:

$$\frac{a}{b} + \frac{c}{d}$$

Use pictures to find this sum and carefully explain how common denominators are represented in your pictures.

4) Given positive integers a and b, explain how to draw pictures that model the product $a \cdot \mathrm{b}$—give an example of your process.

5) Draw pictures that model:

$$\frac{5}{7} \cdot \frac{2}{3} = \frac{10}{21}$$

Explain how your pictures model this equation. Write a story problem whose solution is given by the expression above. Does your story work with

$$\frac{11}{7} \cdot \frac{2}{3} = \frac{22}{21}?$$

6) Given $0 < a \leqslant b$ and $0 < c \leqslant d$, explain how to draw pictures that model the product:

$$\frac{a}{b} \cdot \frac{c}{d}$$

Use pictures to find this product and explain how this product is shown in your pictures—give an example of your process.

A.14 Flour Power

1) Suppose a cookie recipe calls for 2 cups of flour. If you have 6 cups of flour total, how many batches of cookies can you make?

(a) Numerically solve the problem.

(b) Draw a picture representing the situation or describe actions with objects a student could carry out to solve the problem.

(c) Identify whether the problem is asking "How many groups?" or "How many in each group?" or something else entirely.

2) You decide that 2 cups of flour per batch is too much for your taste—you think you'll try $1\frac{1}{2}$ cups per batch. If you have 6 cups of flour, how many batches of cookies can you make?

(a) Numerically solve the problem.

(b) Draw a picture representing the situation or describe actions with objects a student could carry out to solve the problem.

(c) Identify whether the problem is asking "How many groups?" or "How many in each group?" or something else entirely.

3) Somebody once told you that "to divide fractions, you invert and multiply." Discuss how this rule is manifested in part (b) of the problem above.

4) You have 2 snazzy stainless steel containers, which hold a total of 6 cups of flour. How many cups of flour does 1 container hold?

(a) Numerically solve the problem.

(b) Draw a picture representing the situation or describe actions with objects a student could carry out to solve the problem.

(c) Identify whether the problem is asking "How many groups?" or "How many in each group?" or something else entirely.

5) Now you have 3 beautiful decorative bowls, which hold a total of 1/2 cup of flour. How many cups of flour does 1 decorative bowl hold?

(a) Numerically solve the problem.

(b) Draw a picture representing the situation or describe actions with objects a student could carry out to solve the problem.

(c) Identify whether the problem is asking "How many groups?" or "How many in each group?" or something else entirely.

6) Somebody once told you that "to divide fractions, you invert and multiply." Discuss how this rule is manifested in part (b) of the problem above.

A.15 Picture Yourself Dividing

We want to understand how to visualize

$$\frac{a}{b} \div \frac{c}{d}$$

Let's see if we can ease into this like a cold swimming pool.

1) Draw a picture that shows how to compute:

$$6 \div 3$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in each group."

2) Try to use a similar process to the one you used in the first problem to draw a picture that shows how to compute:

$$\frac{1}{4} \div 3$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in each group."

3) Try to use a similar process to the one you used in the first two problems to draw a picture that shows how to compute:

$$3 \div \frac{1}{4}$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in each group."

4) Try to use a similar process to the one you used in the first three problems to draw a picture that shows how to compute:

$$\frac{7}{5} \div \frac{3}{4}$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in each group."

5) Explain how to draw pictures to visualize:

$$\frac{a}{b} \div \frac{c}{d}$$

6) Use pictures to explain why:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

A.16 Cross Something-ing

1) What might someone call the following statements:

(a) $\dfrac{a}{b} = \dfrac{c}{d} \Rightarrow ad = bc$

(b) $\dfrac{a}{b} \cdot \dfrac{b}{c} = \dfrac{a}{c}$

(c) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$

(d) $ad < bc \Rightarrow \dfrac{a}{b} < \dfrac{c}{d}$

(e) $ad < bc \Rightarrow \dfrac{c}{d} < \dfrac{a}{b}$

2) Which of the above statements are true? What specific name might you use to describe them?

3) Use pictures to help explain why the true statements above are true and give counterexamples showing that the false statements are false.

4) Can you think of other statements that should be grouped with those above?

5) If mathematics is a subject where you should strive to "say what you mean and mean what you say," what issue might arise with cross-multiplication?

A.17 Poor Old Horatio (Ratios: The Cornerstone of Middle Childhood Math)

In this activity we are going to investigate ratios.

1) A shade of orange is made by mixing 3 parts red paint with 5 parts yellow paint. Sam says we can add 4 cups of each color of paint and maintain the same color. Fred says we can quadruple both 3 and 5 and get the same color. Who (if either or both) is correct? Explain your reasoning.

2) In keeping with the orange paint described above, if we wanted to make the same orange paint but could only use 73 cups of yellow paint, how many cups of red paint would we need? Give a very detailed explanation of your solution. In particular, if you write an equation, you must justify why the equation holds, and explain what the units are for each value in your equation.

3) Consider the following table:

Red	5			
Yellow	3	1	73	x

Fill out the remainder of the table. Give a general formula for computing how much red paint is needed, and explain why this makes sense within the context of the problem.

4) Is "cross-multiplication" a legitimate way to solve the equations arising from this sort of problem—be sure to think of the weird units that are generated by doing so. What good is this kooky method? What exactly is one doing when they "cross-multiply" and what type of problem does it solve?

5) Consider the following question:

> If Shel has 9 bags each with 13 apples in them, how many apples does Shel have total?

What is somewhat unusual about the units assigned the values? Why does this have the potential to confuse people? Where do we see/use this sort of quantity in real life and throughout the standard school curriculum?

A.18 Ratio Oddities

In this activity we are going to investigate thinking about and adding ratios.

1) There are 3 boys for every 4 girls in Mrs. Sanders' class.

(a) What fraction of the class are girls?

(b) List ratios that can describe this situation.

(c) If each of the number of boys and number of girls quadruples, what is the new ratio of girls to boys?

(d) Write an equation relating the number of boys in the class to the number of girls in the class.

(e) If the number of boys and number of girls each increase by 6, what can you say about the new ratio of boys to girls?

2) Suppose the ratio of girls to boys in Smith's class is 7:3 while the ratio of girls to boys in Jones' class is 6:5.

(a) If there are 50 students in Smith's class and 55 students in Jones' class, and both classes get together for an assembly, what is the ratio of girls to boys? Explain your reasoning.

(b) If there are 500 students in Smith's class and 55 students in Jones' class, and both classes get together for an assembly, what is the ratio of girls to boys? Explain your reasoning.

(c) If there are 500 students in Smith's class and 55 students in Jones' class, and both classes get together for an assembly, what is the ratio of girls to boys? Explain your reasoning.

(d) Now suppose you don't know how many student are in Smith's class and there are 55 students in Jones' class. What is the best answer you can give for the ratio of girls to boys at the assembly.

3) Suppose you are teaching a class, and a student writes

$$\frac{1}{4} + \frac{3}{5} = \frac{4}{9}$$

(a) How would you respond to this?

(b) This student is most contrary, and presents you with the following problem:

> Suppose you have two cars, a 4 seater and a 5 seater. If the first car is 1/4 full and the second car is 3/5 full, how full are they together?

The student then proceeds to answer their question with "The answer is 4/9." How do you address this?

> ***Teaching Note: You might want to ask what happens if the first car is** 1/4 **full and** 6/10 **full.***

4) Again, suppose the ratio of girls to boys in Smith's class is 7:3 while the ratio of girls to boys in Jones' class is 6:5. If there are 40 students in Smith's class and 55 students in Jones' class, and both classes get together for an assembly, what is the ratio of girls to boys at the assembly? Explain your reasoning.

5) Let's use $\oplus$ for this new form of "addition." So

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

I claim that in Problem 2 we could solve using

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

However, in Problem 4 we could not. What is going on here?

6) Let's think a bit more about $\oplus$. If you were going to plot $\frac{a}{b}$ and $\frac{c}{d}$ on a number line, where is $\frac{a}{b} \oplus \frac{c}{d}$? Is this always the case, or does it depend on the values of a, b, c, and d? You should give an explanation based on context, and an explanation based on algebra.

> ***Teaching Note: Here you will probably not only want to have the students realize that** $\frac{a}{b} \oplus \frac{c}{d}$ **is between both** $\frac{a}{b}$ **and** $\frac{c}{d}$**, but that the location varies by which denominator is largest.***

A.19 Problem Solved!

Here's an old puzzler:

1) A man is riding a camel across a desert when he encounters a novel sight. Three young men are fiercely arguing surrounded by 17 camels. Dismounting, the stranger was told that their father had died, leaving (as their only real inheritance) these 17 camels. Now, the eldest son was to receive half of the camels, the second son one-third of the camels, the youngest son one-ninth of the camels. How could they divide the 17 camels amongst themselves? Explain your reasoning.

Uncharacteristically, I will solve this problem for you:

Solution The man should generously add his camel to the 17 being argued over. Now there are 18 camels to divide amongst the three brothers. With this being the case:

- The eldest son receives 9 camels.
- The middle son receives 6 camels.
- The youngest son receives 2 camels.

Ah! Since $9 + 6 + 2 = 17$, there is one left over, the man's original camel—he can now have it back. ■

2) What do you think of this solution?

3) Describe your thought process when addressing the above problem.

A.20 The Dreaded Story Problem

Let's try our hand at a problem involving ratios.

> ***Teaching Note: This problem is challenging. Students must try out actually numbers and do experiments. There are many false "solutions" that can be obtained.***

1) On orders from his doctor, every day, Marathon Marty must run from his house to a statue of Millard Fillmore and run back home along the same path. So Marty doesn't lollygag, the doctor orders him to average 8 miles per hour for the round trip or endure painful electrical shock therapy to his big toes. Today, Marty ran into Gabby Gilly on his way to the statue and averaged only 6 miles per hour for the trip out to the statue.

(a) Name some quantities that might be associated with this problem. Which of the quantities are constant and which can change? Which quantities affect Marty's average rate for the round trip and what are those effects?

(b) What must Marty do to ensure he's obeyed his doctor's orders? Once your model is set up, solve the problem in as many ways that you can (guess and check, algebraically, graphically, with tables, pure reasoning, etc.).

(c) Assume the doctor orders him to average 8.346 miles per hour for the round trip. Today, Marty ran into Gabby Gilly on his way to the statue and averaged only 6.597 miles per hour for the trip out to the statue. What must Marty do to ensure he's obeyed his doctor's orders?

(d) Now assume the doctor orders him to average 6 miles per hour for the round trip. Being fired up, Marty ignores Gabby Gilly and averages 8 miles per hour on the way out. What must Marty do to ensure he's obeyed his doctor's orders?

(e) The doctor now goes back and orders Marty to average 8 miles per hour for the round trip again. This time, Gabby Gilly tackles Marty and starts yakking at him so much that Marty only averages 4 miles per hour on the way out. What must Marty do to ensure he's obeyed his doctor's orders?

(f) Now the doctor orders him to average n miles per hour for the round trip. Assuming that Marty, for whatever reason, averages m miles per hour on the trip out to the statue. What must Marty do to ensure he's obeyed his doctor's orders?

2) What changes can you make to this problem to make it different? Easier? Harder?

A.21 Decimals Aren't So Nice

We will investigate the following question: How is $0.999\ldots$ related to 1?

1) What symbol do you think you should use to fill in the box below?

$$.999\ldots \boxed{}\, 1$$

Should you use $<$, $>$, $=$ or something else entirely?

2) What is $1 - .999\ldots$?

3) How do you write $1/3$ in decimal notation? Express

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

in both fraction and decimal notation.

4) See what happens when you follow the directions below:

(a) Set $x = .999\ldots$.

(b) Compute $10x$.

(c) Compute $10x - x$.

(d) From the step immediately above, what does $9x$ equal?

(e) From the step immediately above, what does x equal?

5) Are there other numbers with this weird property?

A.22 Shampoo, Rinse, ...

We're going to investigate the following question: If a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b? Let's see if we can get to the bottom of this.

1) Use long division to compute $1/7$.

2) State the Division Theorem for integers.

3) Considering the solution of Problem 1, explicitly explain how the Division Theorem for integers appears in your work.

4) In each instance of the Division Theorem in Problem 3, what is the divisor? What does this say about the remainder?

5) What can you say about the decimal representation of a/b when a and b are integers with $b \neq 0$?

6) Assuming that the pattern holds, is the number

$$.123456789101112131415161718192021\dots$$

a rational number? Explain your reasoning.

7) Write the following fractions in decimal notation. Which have a "terminating" and which have a "repeating" decimal?

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{9}, \quad \frac{1}{10}, \quad \frac{1}{11}, \quad \frac{1}{12}, \quad \frac{1}{13},$$

$$\frac{1}{15}, \quad \frac{1}{20}, \quad \frac{1}{24}, \quad \frac{1}{25}, \quad \frac{1}{28}, \quad \frac{1}{32}, \quad \frac{1}{35}, \quad \frac{1}{40}, \quad \frac{1}{42}, \quad \frac{1}{48}.$$

8) Can you find a pattern from your results from Problem 7? Use your pattern to guess whether the following fractions "terminate" or "repeat."

$$\frac{1}{61} \qquad \frac{1}{625} \qquad \frac{1}{6251}$$

9) Can you explain why your conjecture from Problem 8 is true?

10) Compute $\frac{1}{9}$, $\frac{1}{99}$, and $\frac{1}{999}$. Can you find a pattern? Can you explain why your pattern holds?

11) Use your work from Problem 10 to give the fraction form of the following decimals:

(a) $0.\overline{357}$

(b) $0.23\overline{4598}$

(c) $23.\overline{459}$

(d) $76.34\overline{214}$

A.23 Hieroglyphical Algebra

Consider the following addition and multiplication tables:

+	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
fish	fork	balloon	fish	eyeball	cinder-block	skull	lolly-pop	man	DNA
lolly-pop	balloon	eyeball	lolly-pop	fork	fish	man	cinder-block	DNA	skull
skull	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
cinder-block	eyeball	fork	cinder-block	balloon	lolly-pop	DNA	fish	skull	man
DNA	cinder-block	fish	DNA	lolly-pop	man	eyeball	skull	fork	balloon
fork	skull	man	fork	DNA	eyeball	fish	balloon	lolly-pop	cinder-block
man	lolly-pop	cinder-block	man	fish	skull	balloon	DNA	eyeball	fork
balloon	man	DNA	balloon	skull	fork	lolly-pop	eyeball	cinder-block	fish
eyeball	DNA	skull	eyeball	man	balloon	cinder-block	fork	fish	lolly-pop

fish = fish

lolly-pop = lolly-pop

skull = skull

cinder-block = cinder-block

DNA = DNA

fork = fork

man = man

balloon = balloon

eyeball = eyeball

·	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish
fork	fish	lolly-pop	cinder-block	DNA	skull	eyeball	balloon	man	fork
eyeball	lolly-pop	balloon	man	fork	skull	cinder-block	DNA	fish	eyeball
balloon	cinder-block	man	fork	lolly-pop	skull	DNA	fish	eyeball	balloon
man	DNA	fork	lolly-pop	cinder-block	skull	fish	eyeball	balloon	man
skull	skull	skull	skull	skull	skull	skull	skull	skull	skull
lolly-pop	eyeball	cinder-block	DNA	fish	skull	balloon	man	fork	lolly-pop
cinder-block	balloon	DNA	fish	eyeball	skull	man	fork	lolly-pop	cinder-block
DNA	man	fish	eyeball	balloon	skull	fork	lolly-pop	cinder-block	DNA
fish	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish

This activity is based on an activity originally designed by Lee Wayand.

1) Can you tell me which glyph represents 0? How did you arrive at this conclusion?

2) Can you tell me which glyph represents 1? How did you arrive at this conclusion?

3) A number x has an *additive inverse* if you can find another number y with

$$x + y = 0.$$

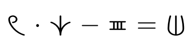
and we say that "y is the additive inverse for x." If possible, find the additive inverse of every number in the table above.

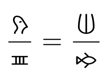
4) A number x has a *multiplicative inverse* if you can find another number y with

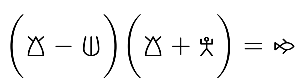
$$x \cdot y = 1.$$

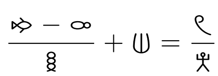
and we say that "y is the multiplicative inverse for x." If possible, find the multiplicative inverse of every number in the table above.

5) If possible, solve the following equations:

(a) 

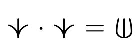
(b) 

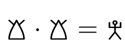
(c) 

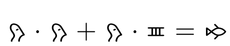
(d) 

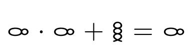
In each case explain your reasoning.

6) If possible, solve the following equations:

(a) 

(b) 

(c) 

(d) 

In each case explain your reasoning.

A.24 I Walk the Line

1) Slimy Sam is on the lam from the law. Being not-too-smart, he drives the clunker of a car he stole east on I-70 across Ohio. Because the car can only go a maximum of 52 miles per hour, he floors it all the way from where he stole the car (just now at the Rest Area 5 miles west of the Indiana line) and goes as far as he can before running out of gas 3.78 hours from now.

(a) At what mile marker will he be 3 hours after stealing the car?

(b) At what mile marker will he be when he runs out of gas and is arrested?

(c) At what mile marker will he be x hours after stealing the car?

(d) At what time will he be at mile marker 99 (east of Indiana)?

(e) At what time will he be at mile marker 71.84?

(f) At what time will he be at mile marker y?

(g) Do parts (c) and (f) supposing that the car goes m miles per hour and Sam started b miles east of the Ohio-Indiana border.

(h) What "form" of an equation for a line does this problem motivate?

2) Free-Lance Freddy works for varying hourly rates, depending on the job. He also carries some spare cash for lunch. To make his customers sweat, Freddy keeps a meter on his belt telling how much money they currently owe (with his lunch money added in).

(a) On Monday, 3 hours into his work as a gourmet burger flipper, Freddy's meter reads \$42. 7 hours into his work, his meter reads \$86. If he works for 12 hours, how much money will he have? When will he have \$196? Solve this problem **without** finding his lunch money.

(b) On Tuesday, Freddy is CEO of the of *We Say So* Company. After 2.53 hours of work, his meter reads \$863.15 and after 5.71 hours of work, his meter reads \$1349.78. If he works for 10.34 hours, how much money will he have? How much time will he be in office to have \$1759.21?

(c) On Wednesday, Freddy is starting goalie for the *Columbus Blue Jackets*. After x_1 hours of work, his meter reads y_1 dollars and after x_2 hours of work, his meter reads y_2 dollars. Without finding his amount of lunch money, if he works for x hours, how much money will he have? How much time will he be in front of the net to have y dollars?

(d) What "form" of an equation for a line does this problem motivate?

3) Counterfeit Cathy sells two kinds of fake cereal: Square Cheerios for \$4 per pound and Sugarless Sugar Pops for \$5 per pound.

(a) If Cathy's goal for today is to sell \$1000 of cereal, how much of each kind could she sell? Give five possible answers.

(b) Plot your answers. What does the slope represent in this situation? What do the points where your curve intercepts the axes represent?

(c) If she sells Square Cheerios for a dollars per pound and Sugarless Sugar Pops for b dollars per pound and she wants to sell c dollars of cereal, write an equation that relates the amount of Sugar Pops Cathy sells to the amount of Cheerios she sells. What "form" of the equation of a line does this problem motivate?

(d) Write a function in the form

$$\#\text{Sugar Pops Sold} = f(\#\text{pounds of Cheerios sold}).$$

4) Given points $p = (3, 7)$ and $q = (4, 9)$, find the formula for the line that connects these points.

5) In each of the situations above, write an equation relating the two variables (hours and position, hours and current financial status, pounds of Square Cheerios and pounds of Sugarless Sugar Pops) and answer the following questions:

(a) How did (or could) the equations help you solve the problems above? What about a table or a graph?

(b) Organize the information in each problem into a table and then into a graph. What patterns do you see, if any?

(c) What do the different features of your graph represent for each situation?

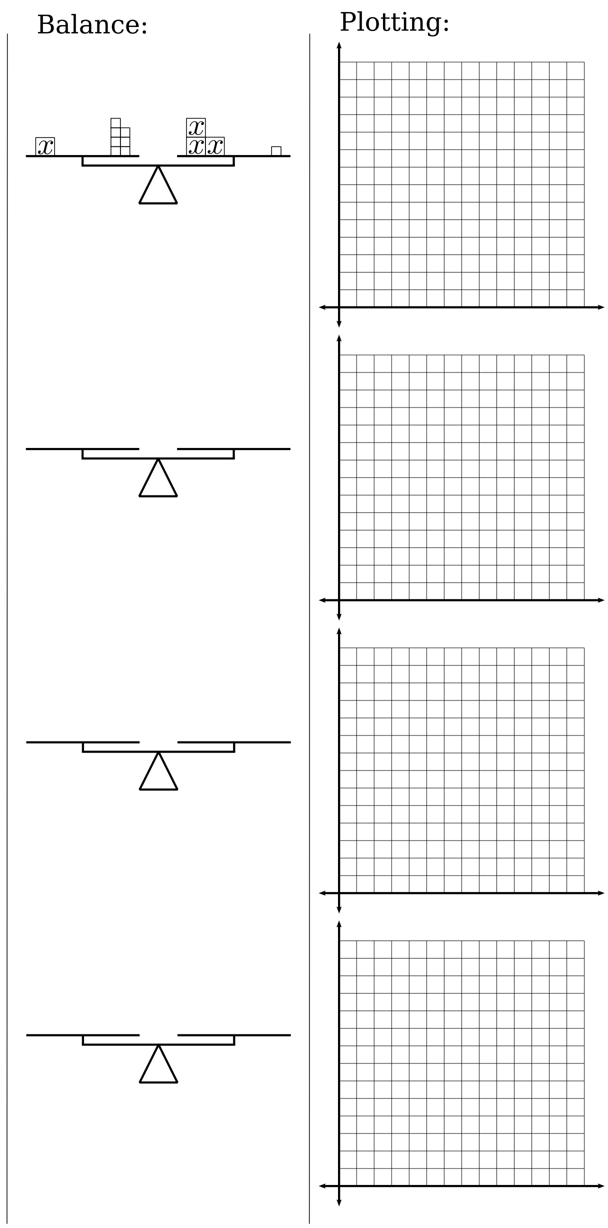
A.25 The Other Side—Solving Equations

In this activity, we will explore ideas related to solving equations.

1) Solve the following equation three ways: Using algebra, using the balance, and with the graph. At each step, each model should be in complete alignment.

Equations:

$$x + 7 = 3x + 1$$



2) Critically analyze the three "different" methods of solving equations. Can you solve quadratic equations using the methods above? If so give an example. If not, explain why not.

> ***Teaching Note: The key point here is that it is difficult to make "balances" work for anything but linear equations.***

3) Can you think of an example when the undoing via algebraic manipulation would fail?

> ***Teaching Note: Here we are looking for something where an inverse function must be applied, as in*** $.6 = \sin(x)$***.***

While sometimes we solve equations via a process of algebraic manipulation, other times we have a formula.

4) Give a formula for solving linear equations of the form $ax + b = 0$.

5) Complete the square to give a formula for solving equations of the form

$$x^2 + bx + c.$$

Of course these formulas can only take us so far. The key to solving polynomial equations is that finding any root will allow you to divide, and lower the degree.

6) Solve the following equation

$$x^5 - 4x^4 - 18x^3 + 64x^2 + 17x - 60 = 0$$

assuming you know that 1, -1, and 3 are roots.

A.26 Maximums and Minimums

Teaching Note: This activity will be necessary for computing least squares approximation.

While you might have encountered completing the square first when solving quadratic equations, its real power is in transforming the form of an expression. In this activity, we'll see it in action.

1) Consider the curve $f(x) = x^2 - 3$. Find the x and y values for the maximum/minimum value(s) of this curve. Explain how you know you are correct.

2) Consider the curve $f(x) = 3(x - 5)^2 + 7$. Find the x and y values for the maximum/minimum value(s) of this curve. Explain how you know you are correct.

3) Consider the curve $f(x) = -2(x + 3)^2 + 7$. Find the x and y values for the maximum/minimum value(s) of this curve. Explain how you know you are correct.

4) What type of curve is drawn by $f(x) = a(x - h)^2 + k$? Find the x and y values for the maximum/minimum value(s) of this curve. Explain how you know you are correct.

Teaching Note: This is the vertex form of a parabola.

5) Consider the parabola $f(x) = x^2 + 4x + 2$. Complete the square to put this expression in the form above and identify the maximum/minimum value(s) of this curve.

6) Consider the parabola $f(x) = 2x^2 - 8x + 6$. Complete the square to put this expression in the form above and identify the maximum/minimum value(s) of this curve.

7) Consider the parabola $f(x) = 3x^2 + 7x - 1$. Complete the square to put this expression in the form above and identify the maximum/minimum value(s) of this curve.

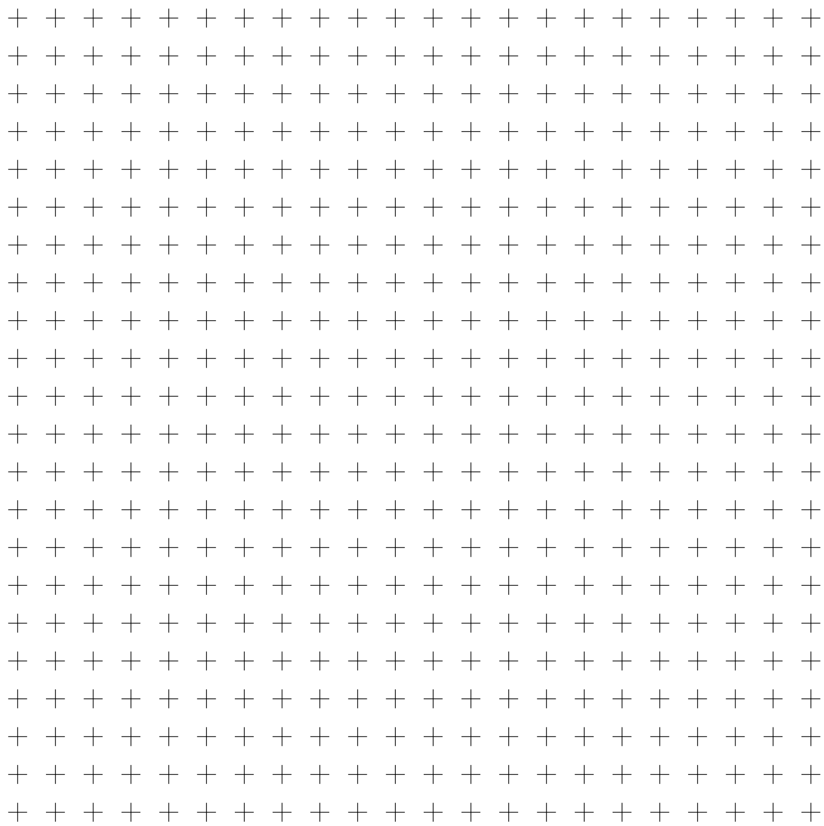
8) Given a parabola $f(x) = ax^2 + bx + c$. Complete the square to put this expression in the form above and identify the maximum/minimum value(s) of this curve.

9) Could you find the same formula found in the previous question by appealing to the symmetry of the roots?

A.27 Least Squares Approximation

In this activity, we are going to investigate *least squares approximation*.

1) Consider the following data: {(2,3), (4,5), (6,11)}



Plot the data and use a ruler to sketch a "best fit" line.

2) Now we are going to record some more data in the chart below:

(a) For each data point, use a ruler to measure the vertical distance between the point and the line. Record this in the first empty row of the table below.

(b) For each data point, square the vertical distance. Record this in the second empty row of the table below.

Point	$(2,3)$	$(4,5)$	$(6,11)$
Vertical Distance			
Squares			

3) Add up the squares of the vertical distances. You want this to be as small as possible. Compare your sum with that of a friend, or enemy. Whoever got the smallest value has the best approximation of the given data.

4) Now find the equation of the line you drew. Write it down and don't forget it!

So far we've just been "eye-balling" our data. Let's roll up our sleaves and do some real math.

5) Suppose that your line is $\ell(x) = ax + b$. Give an expression representing the sum of squares you get with your data above.

> ***Teaching Note: Here we're looking for something like:***
>
> $$(a \cdot 2 + b - 3)^2 + (a \cdot 4 + b - 5)^2 + (a \cdot 6 + b - 11)^2$$

6) Simplfy the expression above. You should now have a quadratic in two variables a and b. Find the minimum, thinking of this as quadratic equation in a and then thinking of this as a quadratic equation in b.

7) You should now have two equations, and two unknowns—solve!

8) Compare your computed formula with the line you guessed—how did you do?

A.28 It Takes All Kinds. . .

Data can come in all shapes and sizes. While a line is the simplest approximation, it might not be the best.

1) Consider the data below:

x	0	1	2	3
y	8.1	22.1	60.1	165

What type of data is this? To get the "brain juices" flowing here are some choices. It could be:

(a) A parabola.

(b) An exponential.

(c) A quartic.

(d) Something else.

Hint: Think about the most famous graph of all, the one you know most about. And see if you can somehow convert the above data to get that type of graph. You will probably need to make some plots.

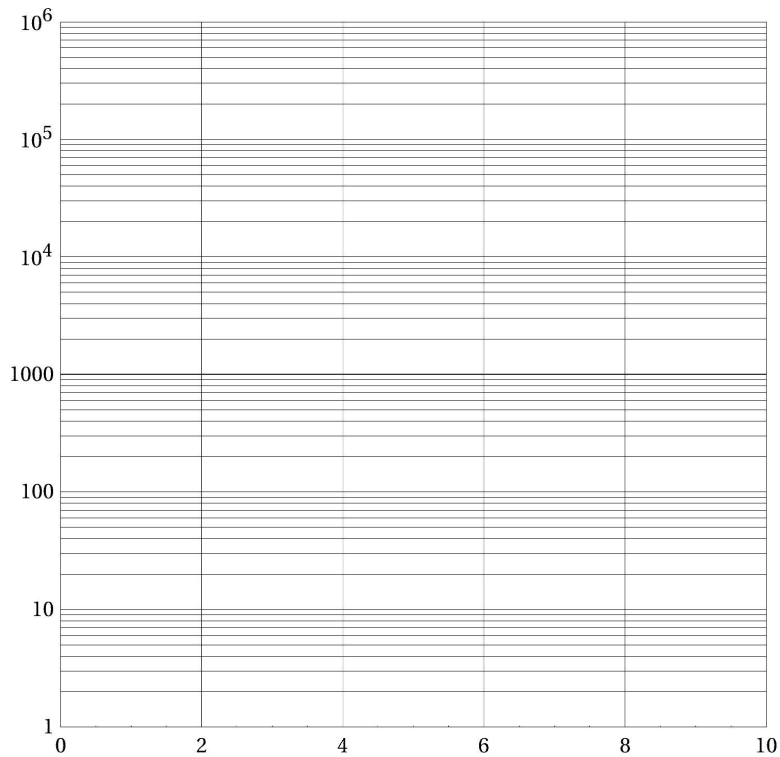
2) Now do the same with this data:

x	1	2	3	4
y	8.3	443.6	24420.8	1364278.6

3) Now do the same with this data:

x	1	2	3	4	5
y	7	62	220	506	1012

4) Here is a sample of semi-log paper. What's going on here?



A.29 Sketching Roots

In this activity we seek to better understand the connection between roots and the plots of polynomials.

1) Sketch the plot of a quadratic polynomial with real coefficients that has:

(a) Two real roots.

(b) One repeated real root.

(c) No real roots.

In each case, give an example of such a polynomial.

2) Can you have a quadratic polynomial with exactly one real root and 1 complex root? Explain why or why not.

3) Sketch the plot of a cubic polynomial with real coefficients that has:

(a) Three distinct real roots.

(b) One real root and two complex roots.

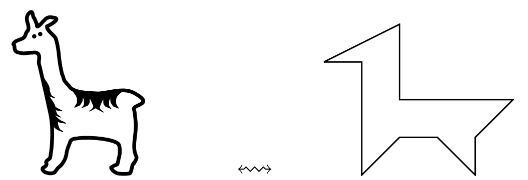
In each case, give an example of such a polynomial.

4) Can you have a cubic polynomial with no real roots? Explain why or why not. What about two distinct real roots and one complex root?

5) For polynomials with real coefficients of degree 1 to 5, classify exactly which types of roots can be found. For example, in our work above, we classified polynomials of degree 2 and 3.

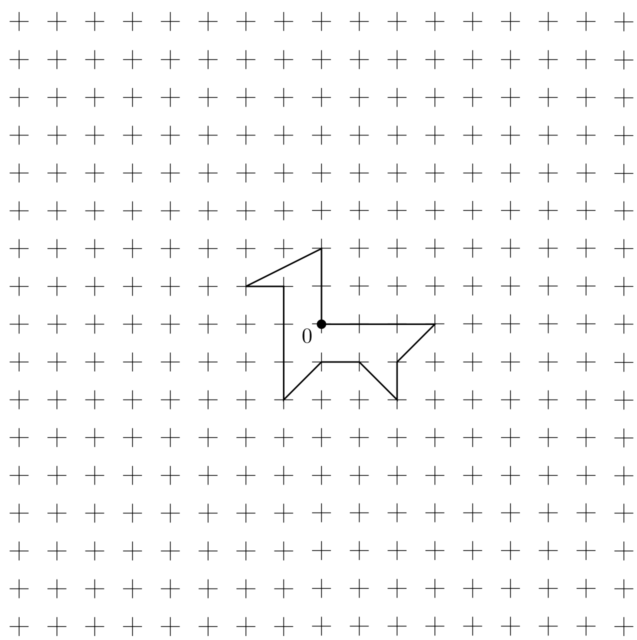
A.30 Geometry and Adding Complex Numbers

Let's think about the geometry of adding complex numbers. We won't be alone on our journey—Louie Llama is here to help us out:



Louie Llama how we'll draw him

1) Here's Louie Llama hanging out near the point 0 in the complex plane. Add $4+4i$ to him. Make a table and show in the plane below what happens.



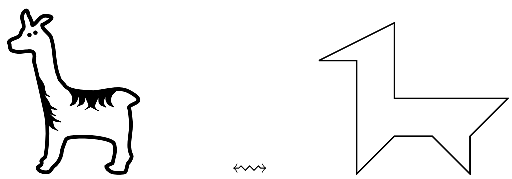
2) Explain what it means to "add" a complex number to Louie Llama. Describe the process(es) used when doing this.

3) Put Louie Llama back where he started, now add $1-5i$ to him. Make a table and show what happens in the plane.

4) Geometrically speaking, what does it mean to "add" complex numbers?

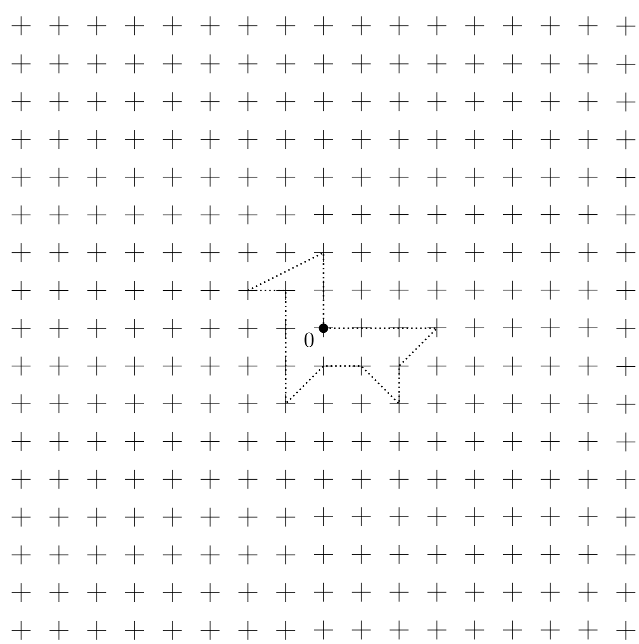
A.31 Geometry and Multiplying Complex Numbers

Now we'll investigate the geometry of multiplying complex numbers. Louie Llama is here to help us out:

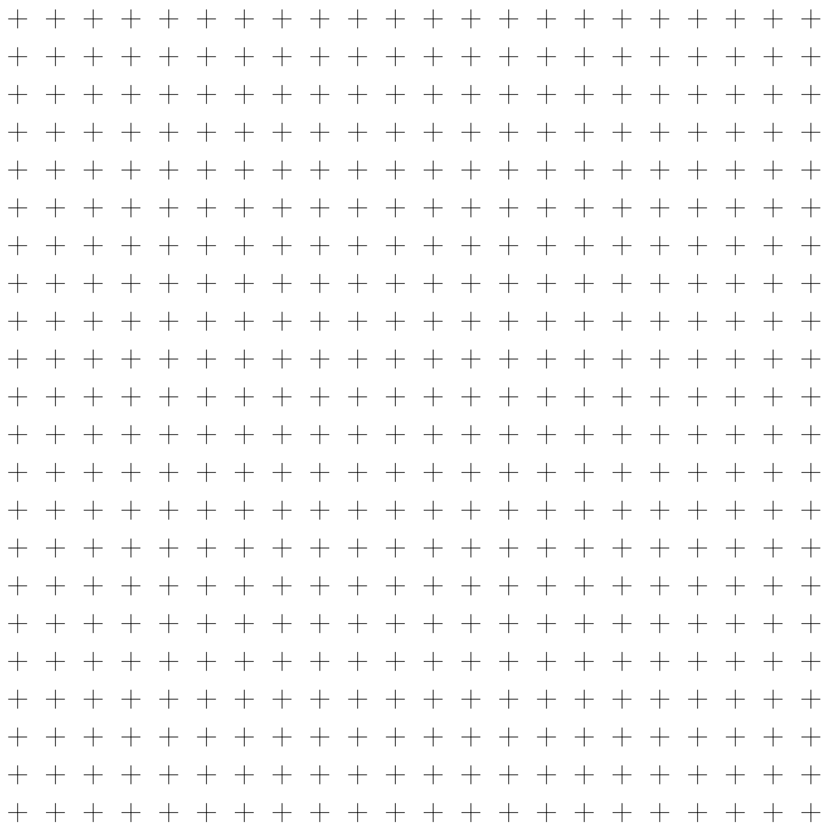


Louie Llama ⇝ how we'll draw him

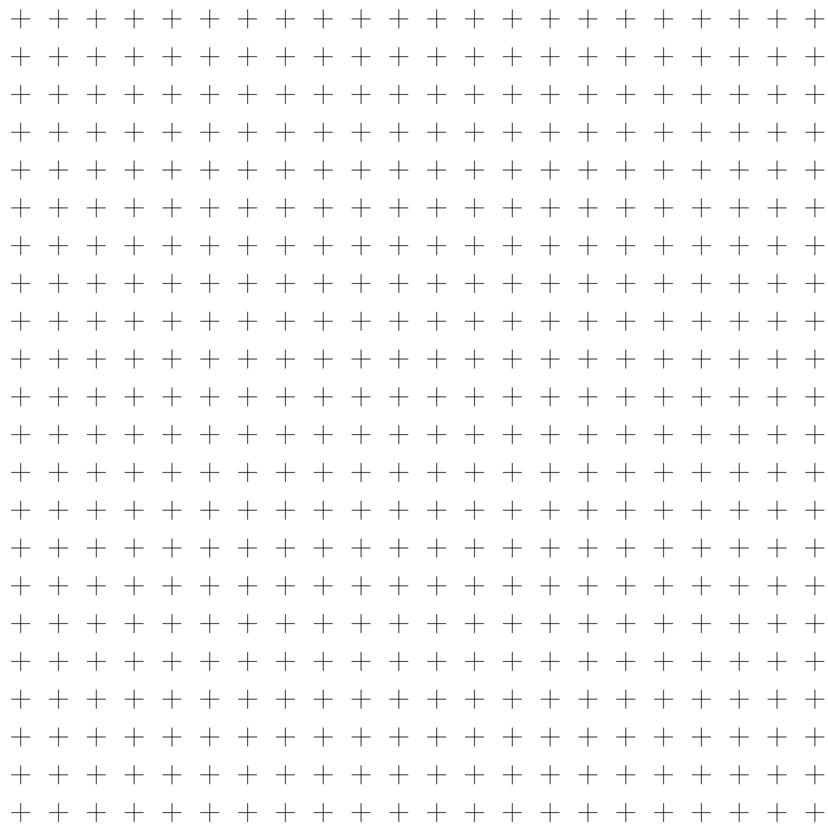
1) Here's Louie Llama hanging out near the point 0 in the complex plane. Multiply him by 2. Make a table and show in the plane below what happens.



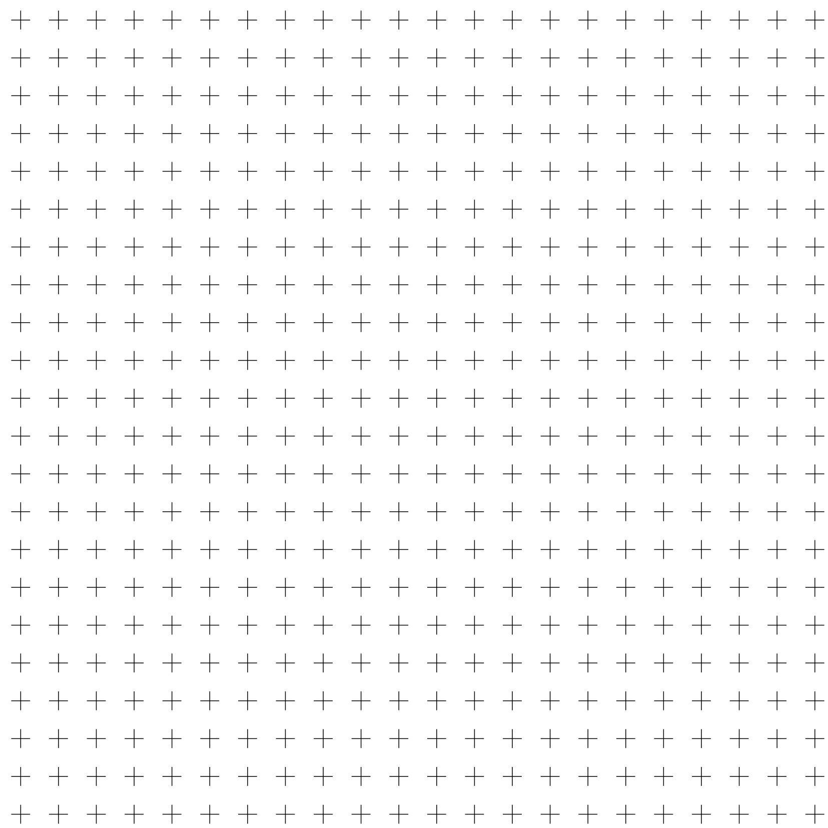
2) Now multiply him by i. Make a table and show in the plane below what happens.



3) Now multiply Louie Llama by $2 + i$. Make a table and show in the plane below what happens.



4) Now multiply Louie Llama by $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Make a table and show in the plane below what happens.



5) Geometrically speaking, what does it mean to "multiply" complex numbers?

6) Explain what it means to "multiply" Louie Llama by a complex number. Considering the different cases above, describe the process(es) used when doing this.

A.32 To the Second Degree

In this activity, we seek to understand why roots of polynomials with real coefficients must always come in conjugate pairs.

1) Consider your favorite (non-real) complex number, I'll call it ξ. Find a polynomial with real coefficients whose degree is as small as possible having your number as a root. What is the degree of your polynomial?

2) I'll call the polynomial found in the first problem $s(x)$. Let $f(x)$ be some other polynomial with

$$f(\xi) = 0.$$

I claim $s(x)|f(x)$. Explain why if $s(x) \nmid f(x)$ then there exist $q(x)$ and $r(x)$ with

$$f(x) = s(x) \cdot q(x) + r(x) \qquad \text{with } \deg(r) < \deg(s).$$

3) Plug in ξ for x in the equation above. What does this tell you about $r(\xi)$? Is this possible?

4) Explain why complex roots must always come in conjugate pairs. Also plot some conjugate pairs in the complex plane and explain what "conjugation" means geometrically.

A.33 Yet Another Division Theorem

Take a minute to recall the *Division Theorem*. Got it? OK we can do something similar with complex numbers. Check this out:

Definition A **Gaussian integer** is a number of the form

$$a + bi$$

where a and b are integers and i is the square-root of negative one.

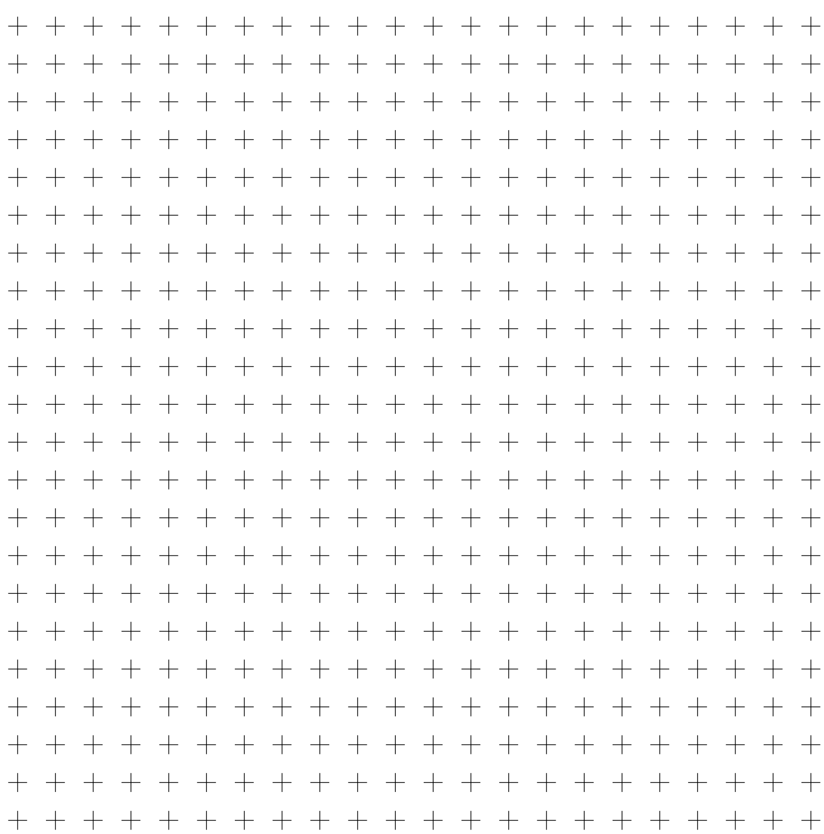
Just like with integers, we have a division theorem here too, check it out (this time I'll play nice):

Theorem 7 (Division Theorem) *Given any Gaussian integer α and a nonzero Gaussian integer β, there exist Gaussian integers θ and ρ such that*

$$\alpha = \beta \cdot \theta + \rho \qquad \textit{with} \qquad \rho \cdot \overline{\rho} < \beta \cdot \overline{\beta}$$

where $\overline{a + bi} = a - bi$.

Suppose you want to divide $7 + 7i$ by $1 + 2i$ and end up with quotient and remainder that are both Gaussian integers. How do you do this? We'll use the complex plane to help us out.



1) Mark $1+2i$ and $7+7i$ on the complex plane. Use the grid above to help you and be sure to label your work.

2) Mark every Gaussian integer multiple of $1+2i$ on the plane above. Explain what happens and explain why it happens.

3) Find the nearest multiple of $1+2i$ to $7+7i$.

4) Use your work above to help find θ and ρ such that

$$7+7i = (1+2i)\cdot\theta+\rho \qquad \text{with} \qquad \rho\cdot\overline{\rho} < 5.$$

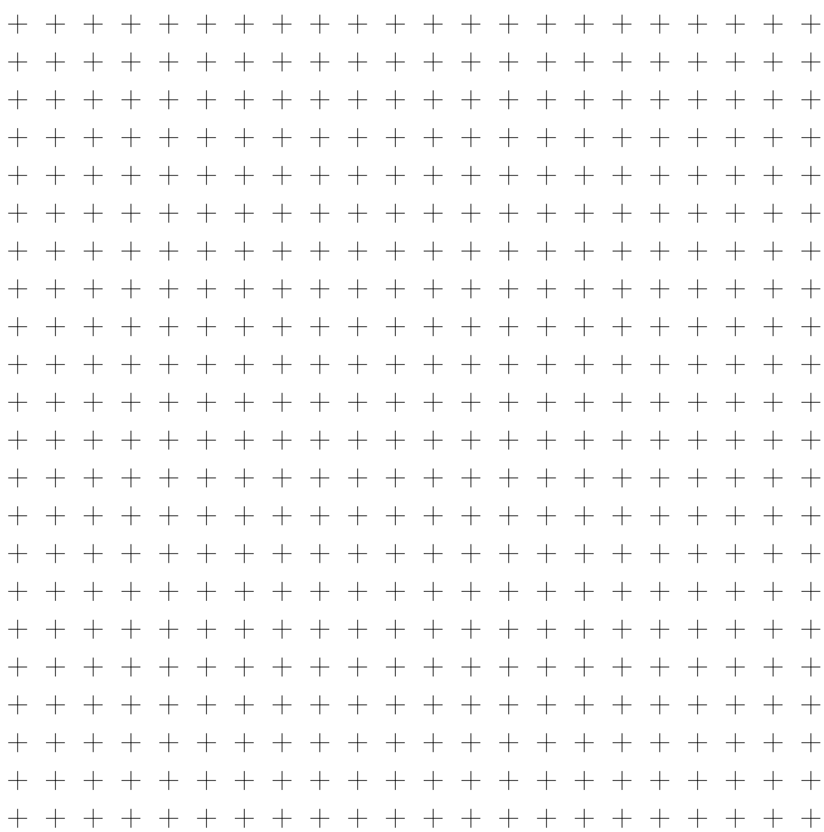
5) Are the θ and ρ you found above unique? Discuss.

6) Explain what is going on here in terms of geometry.

7) Find θ and ρ such that

$$9+8i = (5+2i)\cdot\theta+\rho \qquad \text{with} \qquad \rho\cdot\overline{\rho} < 29.$$

As a gesture of friendship, I have provided a fresh grid for your work.



8) Are the θ and ρ you found above unique? Discuss.

A.34 Broken Records

Fill in the following table:

modulus:	2	3	4	5	6	7	8	9	10	11
$2 \cdot 1 \equiv$										
$2 \cdot 2 \equiv$										
$2 \cdot 3 \equiv$										
$2 \cdot 4 \equiv$										
$2 \cdot 5 \equiv$										
$2 \cdot 6 \equiv$										
$2 \cdot 7 \equiv$										
$2 \cdot 8 \equiv$										
$2 \cdot 9 \equiv$										
$2 \cdot 10 \equiv$										
$2 \cdot 11 \equiv$										

1) Find patterns in your table above, clearly describe the patterns you find.

2) Consider the patterns you found. Can you explain why they happen?

3) When does a column have a 0? When does a column have a 1?

4) Describe what would happen if you extend the table for bigger moduli and bigger multiplicands.

modulus:	2	3	4	5	6	7	8	9	10	11
$3 \cdot 1 \equiv$										
$3 \cdot 2 \equiv$										
$3 \cdot 3 \equiv$										
$3 \cdot 4 \equiv$										
$3 \cdot 5 \equiv$										
$3 \cdot 6 \equiv$										
$3 \cdot 7 \equiv$										
$3 \cdot 8 \equiv$										
$3 \cdot 9 \equiv$										
$3 \cdot 10 \equiv$										
$3 \cdot 11 \equiv$										

5) Find patterns in your table above, clearly describe the patterns you find.

6) Consider the patterns you found. Can you explain why they happen?

7) When does a column have a 0? When does a column have a 1?

8) Can you describe what would happen if you extend the table for bigger moduli and bigger multiplicands?

9) Describe precisely when a column of the table will contain representatives for each integer modulo n. Explain why your description is true.

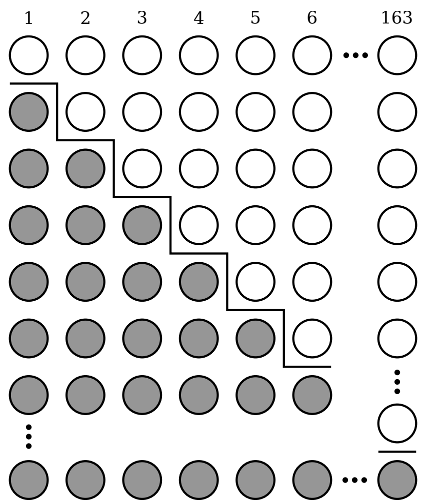
A.35 Something Doesn't Add Up...

1) Sum all the numbers starting with 551 and ending at 5051. Use the rows of numbers below to help you.

$$\begin{array}{c} 551 + 552 + 553 + 554 + 555 + 556 + \cdots + 5046 + 5047 + 5048 + 5049 + 5050 + 5051 \\ 5051 + 5050 + 5049 + 5048 + 5047 + 5046 + \cdots + 556 + 555 + 554 + 553 + 552 + 551 \end{array}$$

Explain your reasoning—be sure to clearly explain what happens in the "dots."

2) How many unshaded circles are in the diagram below?



Explain your reasoning—be sure to clearly explain what happens in the "dots." Compare this with question 1.

3) Sum the numbers:

$$106 + 112 + 118 + \cdots + 514$$

Compare this with questions 2 and 1 above.

4) Sum the numbers:

$$2.2 + 2.9 + 3.6 + 4.3 + \cdots + 81.3$$

A.36 Gertrude the Gumchewer

1) Gertrude the Gumchewer has an addiction to *Xtra Sugarloaded Gum*, and it's getting worse. Each day, she goes to her always stocked storage vault and grabs gum to chew. At the beginning of her habit, she chewed three pieces and then, each day, she chews 8 more pieces than she chewed the day before to satisfy her ever-increasing cravings.

(a) How many pieces will she chew on the 10th day of her habit?

(b) How many pieces will she chew on the kth day of her habit?

(c) How many pieces will she chew on the 793rd day of her habit? How do you know you are right?

(d) How many pieces will she chew over the course of the first 793 days of her habit?

2) Assume now that Gertrude, at the beginning of her habit, chewed m pieces of gum and then, each day, she chews n more pieces than she chewed the day before to satisfy her ever-increasing cravings. How many pieces will she chew over the course of the first k days of her habit? Explain your formula and how you know it will work for any m, n and k.

3) Use the method you developed in questions 1 and 2 to find the sum:

$$19 + 26 + 33 + \cdots + 1720$$

Give a story problem that is represented by this sum.

A.37 Billy the Bouncing Ball

1) Sum the numbers:

$$1 + 2 + 4 + 8 + 16 + \cdots + 8388608$$

2) Billy the Bouncing Ball is dropped from a height of 13.5 feet. After each bounce, Billy only goes up by 60% of what he did on the previous bounce.

(a) How high will Billy go after the 38th bounce?

(b) How much distance will Billy travel over the course of 38 bounces (not including the height he went up after the 38th bounce)?

3) Assume now that Billy the Bouncing Ball is dropped from a height of h feet. After each bounce, Billy goes up a distance equal to r times the distance of the previous bounce. (For example, $r = .60$ in part 1.)

(a) If $r < 1$, what can you say about Billy's bounces? What if $r = 1$? What if $r > 1$?

(b) How high will Billy go after the kth bounce?

(c) How much distance will Billy travel over the course of k bounces (not including the height he went up after the kth bounce)?

A.38 On The Road

1) Steve likes to drive the city roads. Suppose he is driving down a road with three traffic lights. Note, Steve is a very cautious driver and if he sees a yellow light, he waits for it to turn red.

(a) How many ways could he see one red light and two green lights?

(b) How many ways could he see one green light and two red lights?

(c) How many ways could he see all red lights?

2) Now suppose Steve is driving down a road with four traffic lights.

(a) How many ways could he see two red light and two green lights?

(b) How many ways could he see one green light and three red lights?

(c) How many ways could he see all green lights?

3) In the following chart let n be the number of traffic lights and k be the number of green lights seen. In each square, write the number of ways this number of green lights could be seen while Steve drives down the street.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$n = 0$							
$n = 1$							
$n = 2$							
$n = 3$							
$n = 4$							
$n = 5$							
$n = 6$							

Describe any patterns you see in your table and try to explain them in terms of traffic lights.

A.39 Pascal's Triangle: Fact or Fiction?

Consider the numbers $\binom{n}{k}$. These numbers can be arranged into a "triangle" form that is popularly called "Pascal's Triangle". Assuming that the "top" entry is $\binom{0}{0} = 1$, we write the numbers row by row, with n fixed for each row. Write out the first 7 rows of Pascal's Triangle.

Note that there are many patterns to be found. Your job is to justify the following patterns in the context of relevant models. Here are three patterns, can you explain them?

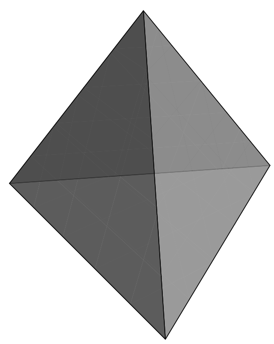
(a) $\binom{n}{k} = \binom{n}{n-k}$.

(b) The sum of the entries in each row is 2^n.

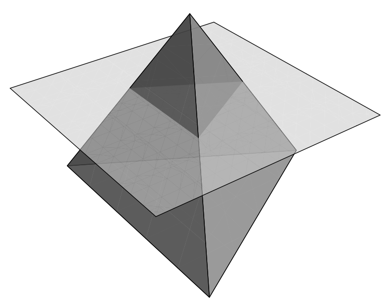
(c) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

A.40 Pascal's Pyramid

A *pyramid* is a three-dimensional object that has some polygon as its base and triangles that converge to a point as its sides. Fancy folks call a triangular-based pyramid a *tetrahedron*.



Since three-dimensional objects are hard to view on a flat sheet of paper, sometimes we think about them by taking cross sections:



We're going to build a triangular-based pyramid out of numbers. Here are the first four cross sections:

$$
\begin{array}{c} 1 \end{array}
\qquad
\begin{array}{c} 1 \quad 1 \\ 1 \end{array}
\qquad
\begin{array}{c} 1 \quad 2 \quad 1 \\ 2 \quad 2 \\ 1 \end{array}
\qquad
\begin{array}{c} 1 \quad 3 \quad 3 \quad 1 \\ 3 \quad 6 \quad 3 \\ 3 \quad 3 \\ 1 \end{array}
$$

This pyramid that we are building is called **Pascal's pyramid**.

1) Give the next two cross sections of Pascal's pyramid. Explain your reasoning.

2) Do you see any connections to Pascal's Triangle in Pascal's pyramid? Explain what you see.

3) Use the Binomial Theorem to expand:

$$(a+x)^3$$

4) Replace x above with $b+c$, and use the Binomial Theorem again along with your computation above to expand:

$$(a+b+c)^3$$

5) What do you notice about the coefficients in the expansion of $(a+b+c)^3$?

6) Explain how the trinomial coefficient

$$\binom{n}{j,k} = \frac{n!}{j!k!(n-j-k)!}$$

corresponds to entries of Pascal's pyramid. Feel free to draw diagrams and give examples.

7) The trinomial coefficient $\binom{n}{j,k}$ has the following "physical" meaning: It is the number of ways one can choose j objects and k objects from a set of n objects. Try a couple of relevant and revealing examples to provide evidence for this claim.

8) Explain how Pascal's Triangle is formed. In your explanation, use the notation $\binom{n}{j,k}$. If you were so inclined to do so, could you state a single equation that basically encapsulates your explanation above?

9) Use Pascal's pyramid to expand:

$$(a+b+c)^4$$

Try to formulate a "Trinomial Theorem."

10) Use your Trinomial Theorem to explain why the numbers in the nth cross section of Pascal's pyramid sum to 3^n.

A.41 You Can Count on It!

1) The Diet-Lite restaurant offers 5 entrees, 8 side dishes, 12 desserts, and 6 kinds of drinks. If you were going to select a dinner with one entrée, one side dish, one dessert, and one drink, how many different dinners could you order?

2) A standard Ohio license plate consists of two letters followed by two digits followed by two letters. How many different standard Ohio license plates can be made if:

(a) There are no more restrictions on the numbers or letters.

(b) There are no repeats of numbers or letters.

3) Seven separate coins are flipped. How many different results are possible (e.g., HTHHTHT is different than THHHTTH)?

4) A pizza shop always puts cheese on their pizzas. If the shop offers n additional toppings, how many different pizzas can be ordered (Note: A plain cheese pizza is an option)?

5) There are 10 students in the auto mechanics club. Elections are coming up and the members are holding nominations for President, Vice President, Secreatary, and Treasurer. If all members are eligible, how many possible tickets are there?

6) Same as the previous question, but now there are n members of the club and k offices.

7) Now the club (with n members) is not electing officers anymore, but instead deciding to send k delegates to the state auto mechanics club convention. How many possible groups of delegates can be made?

8) The Pig-Out restaurant offers 5 entrees, 8 side dishes, 12 desserts, and 6 kinds of drinks. If you were going to select a dinner with 3 entrées, 4 side dishes, 7 desserts, and one drink, how many different dinners could you order?

A.42 Which Road Should We Take?

1) Consider a six-sided die. Without actually rolling a die, guess the number of 1's, 2's, 3's, 4's, 5's, and 6's you would obtain in 50 rolls. Record your predictions in the chart below:

# of 1's	# of 2's	# of 3's	# of 4's	# of 5's	# of 6's	Total

Now roll a die 50 times and record the number of 1's, 2's, 3's, 4's, 5's, and 6's you obtain.

# of 1's	# of 2's	# of 3's	# of 4's	# of 5's	# of 6's	Total

How did you come up with your predictions? How do your predictions compare with your actual results? Now make a chart to combine your data with that of the rest of the class.

Experiment 1 We investigated the results of throwing one die and recording what we saw (a 1, a 2, ..., or a 6). We said that the probability of an event (for example, getting a "3" in this experiment) predicts the frequency with which we expect to see that event occur in a large number of trials. You argued the $P(\text{seeing } 3) = 1/6$ (meaning we expect to get a 3 in about 1/6 of our trials) because there were six different outcomes, only one of them is a 3, and you expected each outcome to occur about the same number of times.

Experiment 2 We are now investigating the results of throwing two dice and recording the sum of the faces. We are trying to analyze the probabilities associated with these sums. Let's focus first on $P(sum = 2) =?$. We might have some different theories, such as the following:

Theory 1 $P(\text{sum} = 2) = 1/11$.
It was proposed that a sum of 2 was 1 out of the 11 possible sums $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Theory 2 $P(\text{sum} = 2) = 1/21$.
It was proposed that a sum of 2 was 1 of 21 possible results, counting $1 + 3$ as the same as $3 + 1$:

$$\begin{array}{cccccc}
1+1 & - & - & - & - & - \\
2+1 & 2+2 & - & - & - & - \\
3+1 & 3+2 & 3+3 & - & - & - \\
4+1 & 4+2 & 4+3 & 4+4 & - & - \\
5+1 & 5+2 & 5+3 & 5+4 & 5+5 & - \\
6+1 & 6+2 & 6+3 & 6+4 & 6+5 & 6+6
\end{array}$$

2) Is there another theory?

3) Test all theories by computing the following probabilities using each theory:

$$P(\text{sum} = 5) =? \qquad P(\text{sum} = 6) =? \qquad P(\text{sum} = 7) =? \qquad P(\text{sum} = 8) =?$$

4) Which theory do you like best? Why?

5) How does your theory compare to the dice rolls we recorded in our class?

6) How could we test our theory further?

A.43 Lumpy and Eddie

Two ancient philosophers, Lumpy and Eddie, were sitting on rocks flipping coins.

1) Lumpy and Eddie wondered about the probability of obtaining both a head and a tail. Here is how it went:

> Eddie argued the following: “Look Lumpy, it’s clear to me that when we flip two coins, we should get one of each about half the time because there are two possibilities: They’re either the same or different.” Lumpy, on the other hand, argued this way: “Eddie, stop being a wise guy! If we flipped two coins, we should expect both a head and tail to come up about a third of the time because there are only three possibilities: two heads, two tails, and one of each.”

Which, if any, of these two guys is right? Is there another answer?

2) Next Lumpy and Eddie threw a third coin in the mix and wondered about the probability of obtaining 2 heads and a tail or 2 tails and a head.

(a) What would Lumpy say in this case?

(b) What would Eddie say in this case?

Be sure to clearly explain why you think they would answer in the way you suggest.

A.44 Go Climb a Tree!

In this activity, we'll evaluate the probabilities of complex events using tree diagrams, fraction arithmetic, and counting.

1) Give a story problem that is modeled by the expression:

$$\frac{3}{7} \times \frac{2}{5}$$

Let the start of the story be: "2/5 of the class are girls." Once you have the story, solve it using pictures (use rectangles for the wholes) and explain why it makes sense that multiplying fractions is the same as multiplying the numerators and multiplying the denominators.

2) The Weather Channel has predicted that there is a 70% chance of rain today, a 20% chance of rain tomorrow, and a 40% chance of rain the day after tomorrow. Use a tree diagram to help answer the following:

(a) What is the probability that it will rain today and not rain tomorrow?

(b) What is the probability it will rain on exactly one of the first two days?

(c) What is the probability that it will rain today, not rain tomorrow, and rain the following day?

(d) What is the probability that it will rain on exactly two of the three days?

(e) What is the probability it will rain on all three days?

(f) What is the probability it won't rain at all?

(g) What is the probability it will rain on at least one of the days?

3) The Indians and the Yankees are to face each other in a best-of-seven series. The probability that the Indians will win any game is 30%.

(a) What is the probability that the Indians win games 1, 3, 4, and 6 to win the series?

(b) What is the probability that the Indians win the series in exactly 6 games?

(c) What is the probability that the Indians win the series?

4) Fred the Slob has an unreliable car that starts only 65% of the days. If the car doesn't start, poor Fred must walk the one block to work. This week, he is slated to work 6 days (Monday through Saturday).

(a) What is the probability that Fred will walk on Monday and Wednesday and drive the other days?

(b) What is the probability that Fred will drive on exactly 4 of the days?

(c) What is the probability that poor Fred will have to walk on at least two of the days?

5) Use the techniques of this activity (i.e., using a special case and fraction arithmetic to help investigate a more general case) to find the probability of passing a 10-question multiple choice test if you must get 70% or more correct to pass.

A.45 They'll Fall for Anything!

What is incorrect about the following reasoning? Be specific!

1) Herman says that if you pick a United States citizen at random, the probability of selecting a citizen from Indiana is because Indiana is one of 50 equally likely states to be selected.

2) Jerry has set up a game in which one wins a prize if he/she selects an orange chip from a bag. There are two bags to choose from. One has 2 orange and 4 green chips. The other bag has 7 orange and 7 green chips. Jerry argues that you have a better chance of winning by drawing from the second bag because there are more orange chips in it.

3) Gil the Gambler says that it is just as likely to flip 5 coins and get exactly 3 heads as it is to flip 10 coins and get exactly 6 heads because

$$\frac{3}{5} = \frac{6}{10}$$

4) We draw 4 cards without replacement from a deck of 52. Know-it-all Ned says the probability of obtaining all four 7's is $\frac{4}{\binom{52}{4}}$ because there are ways to select the $\binom{52}{4}$ 4 cards and there are four 7's in the deck.

5) At a festival, Stealin' Stan gives Crazy Chris the choice of one of three prizes—each of which was hidden behind a door. One of the doors has a fabulous prize behind it while the other two doors each have a "zonk" (a free used tube of toothpaste, etc.). Crazy Chris chooses Door #1. Before opening that door, Stealin' Stan shows Chris that hidden behind Door #3 is a zonk and gives Chris the option to keep Door #1 or switch to Door #2. Chris says, "Big deal. It doesn't help my chances of winning to switch or not switch."

Appendix B

Enrichment Topics

B.1 Continued Fractions

We're going to use some tricks involving fractions to study numbers that have a nasty form. As an example, consider

$$\sqrt{2} = 1.4142135623\ldots$$

Yuck! That's just some crazy decimal. It would be nice if we could somehow see some order in this chaos! To do this, we'll need some definitions:

Definition A fraction of the form

$$a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}$$

is called a **continued fraction**. If $a_1, a_2, a_3, \ldots$ are all 1, we will call this a **simple continued fraction**.

Definition The **whole-number part** of a number is the largest whole number which is less than or equal to the given number.

Example *The whole-number part of* 2 *is* 2*, while the whole-number part of* 5.32 *is* 5.

Definition The **fractional part** of a number is the number minus its whole-number part.

Example *The fractional part of* 2 *is* 0*, while the fractional part of* 5.32 *is* 0.32*.*

Question Why don't we just describe the fractional part of a number as the part that is to the right of the decimal point? Hint: Think about $0.99999\ldots$.

?

Given any number, we can write it as a simple continued fraction. Consider 13/5. To start note that

$$3 > \frac{13}{5} > 2.$$

So this means that

$$\frac{13}{5} = 2 + \frac{3}{5}.$$

Here 2 is the whole-number part and 3/5 is the fractional part of 13/5. But in the simple continued fraction, our numerator is 1, not 3. How do we deal with this? Well,

$$\frac{13}{5} = 2 + \frac{3}{5} = 2 + \frac{1}{\frac{5}{3}}.$$

This is an improvement but we only want whole numbers in our simple continued fractions and not 5/3. So we write

$$\frac{5}{3} = 1 + \frac{2}{3}$$

which gives us

$$\frac{13}{5} = 2 + \cfrac{1}{1 + \cfrac{2}{3}}.$$

Again, we want our numerator to be 1, not 2 so we will repeat the steps above to get

$$\frac{13}{5} = 2 + \cfrac{1}{1 + \cfrac{2}{3}} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

and this last expression is the simple continued fraction for 13/5. We could also list our steps as:

$$\begin{aligned} \frac{13}{5} &= \mathbf{2} + \frac{3}{5} \\ \frac{1}{\frac{3}{5}} = \frac{5}{3} &= \mathbf{1} + \frac{2}{3} \\ \frac{1}{\frac{2}{3}} = \frac{3}{2} &= \mathbf{1} + \frac{1}{2} \\ \frac{1}{\frac{1}{2}} &= \mathbf{2} + 0 \end{aligned}$$

These boldface numbers tell us our continued fraction expansion.

We can also find the simple continued fraction of numbers which are not already fractions (otherwise this would all be a bit silly). Consider $\sqrt{2}$, remember how yucky it was?

$$\sqrt{2} = 1.4142135623\ldots$$

To beautify this number, note that $2 > \sqrt{2} > 1$. So this means that

$$\sqrt{2} = 1 + (\sqrt{2} - 1).$$

Where 1 is the whole-number part and $(\sqrt{2} - 1)$ is the fractional part of $\sqrt{2}$. Alright, now look at $1/(\sqrt{2} - 1)$. Again we want to separate the whole-number part and the fractional part. With a little algebra we see that

$$\frac{1}{\sqrt{2} - 1} = \frac{\sqrt{2} + 1}{2 - 1} = \sqrt{2} + 1 = 2 + (\sqrt{2} + 1 - 2) = 2 + (\sqrt{2} - 1).$$

Now don't you get bogged down in the steps. Here it is in fast forward:

$$\begin{aligned}
\sqrt{2} &= \mathbf{1} + (\sqrt{2}-1)\\
\frac{1}{(\sqrt{2}-1)} &= \mathbf{2} + (\sqrt{2}-1)\\
\frac{1}{(\sqrt{2}-1)} &= \mathbf{2} + (\sqrt{2}-1)\\
\frac{1}{(\sqrt{2}-1)} &= \mathbf{2} + (\sqrt{2}-1),\\
&\vdots
\end{aligned}$$

At each step we want:

$$\text{number} = \text{whole-number part} + \text{fractional part}$$

Now from the bold-faced numbers above we will make our continued fraction:

$$\sqrt{2} = \mathbf{1} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{2} + \cdots}}}}$$

Beautiful!

Question Can you explain why this works?

?

Question Do you think you could find a regular fraction equal to $\sqrt{2}$?

?

Some Hidden Beauties

Continued fractions allow us to see patterns that are otherwise totally hidden.

Check out $e = 2.718281828459045\ldots$. It turns out that

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cdots}}}}}}}}}$$

Also check out $\pi = 3.14159265358\ldots$. In 1999 L.J. Lange found this amazing continued fraction for π:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cfrac{9^2}{6 + \cfrac{11^2}{6 + \cdots}}}}}}$$

Wow!

Problems for Section B.1

(a) Explain what the **whole-number part** and what the **fractional part** of a number are. Give examples.

(b) Find the simple continued fraction expansion of $1/2$. Explain your work.

(c) Find the simple continued fraction expansion of 11. Explain your work.

(d) Find the simple continued fraction expansion of $5/3$. Explain your work.

(e) Find the simple continued fraction expansion of $15/11$. Explain your work.

(f) Find the simple continued fraction expansion of $22/17$. Explain your work.

(g) Using a calculator, find the first five terms in the simple continued fraction expansion of π. What number do you get by only considering the first term? The first four?

(h) Find the simple continued fraction expansion of $\sqrt{5}$. Explain your work.

(i) Find the simple continued fraction expansion of $\sqrt{10}$. Explain your work.

(j) Find the simple continued fraction expansion of $\sqrt{17}$. Explain your work.

(k) Find the simple continued fraction expansion of $\sqrt{26}$. Explain your work.

(l) Find the simple continued fraction expansion of

$$\frac{1+\sqrt{5}}{2}$$

Explain your work. Note—this is a special number, it is called the *golden ratio*. More on this later.

(m) Courtney Gibbons is someone who has a rather unusual tattoo. She was kind enough to let an unusual person like me take a picture of it. What does her tattoo represent? Explain your reasoning.

(n) What is it about the numbers $2, 5, 10, 17, 26$ that makes it easy to compute the continued fraction expansion of the square-roots of these numbers? Explain your answer.

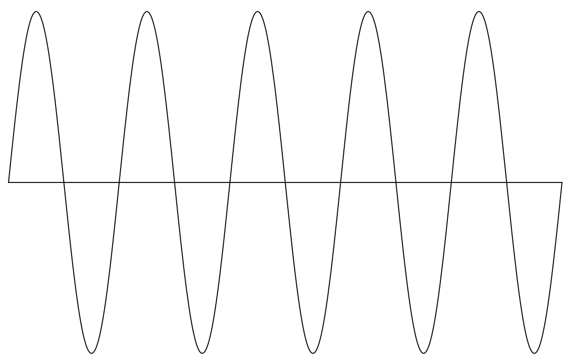
(o) What is the best rational approximation of $\sqrt{2}$ where the denominator is less than 10? Less than 20? Less than 30? Less than 100?

(p) What is the best rational approximation of $\sqrt{5}$ where the denominator is less than 10? Less than 20? Less than 30? Less than 100?

(q) What is the best rational approximation of $\sqrt{3}$ where the denominator is less than 10? Less than 20? Less than 30? Less than 100?

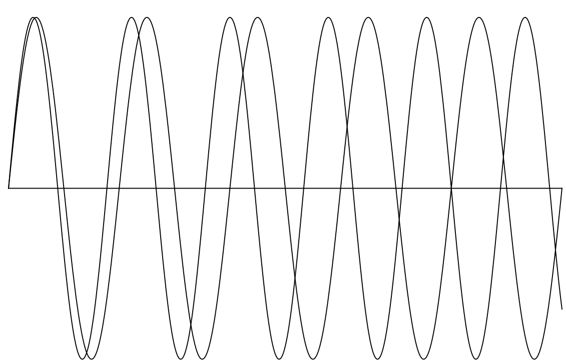
B.2 Tonic, Dominant, Octave

Someone once told me "Music is how math sounds." I'm not totally sure I believe them, but maybe there is some truth to what they are saying. Sound is made by compression waves in the air. Loosely speaking, the closer the compression waves come, the higher the pitch the sound is that we hear. We visualize these compression waves with a picture like this:

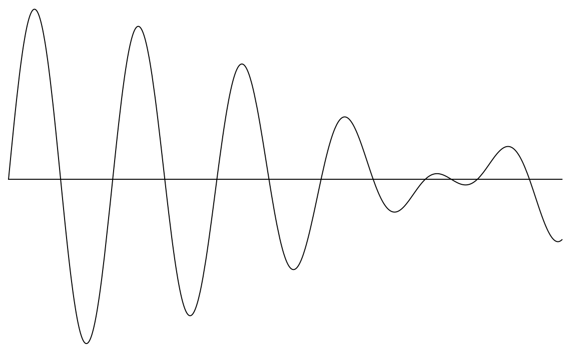


The peaks of the waves above represent high-pressure points where the air is very compressed, and the troughs of the waves above represent low-pressure points, where there are very few air molecules.

Let's call a sound produced by a single compression wave a *tone*. When two tones are played at the same time, their waves act together like the sum of the individual waves. Let's see this in action. If we play the following two tones at the same time,

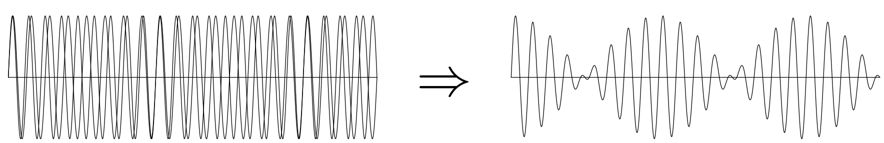


we end up with



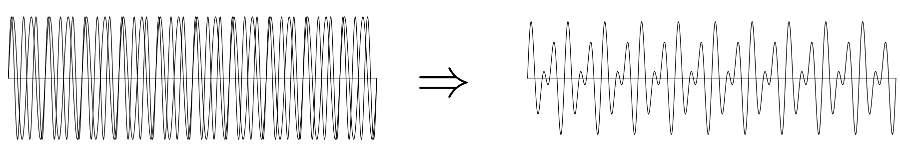
which is nothing more than every point on the first graph added to every point on the second graph. See how when the waves line up nicely, we get a nice big wave? See how when the waves disagree, our wave dwindles down to nothing?

Since the time when people started making sounds, we've noticed that some tones sound better together than others. There is an easy rule-of-thumb that will tell you when two tones will sound "right" together. If the graph of both waves combined has a lot of beats then it will probably sound "wrong," see the example below:



beats

If the beats are hard to see, then it will sound "right." You can see this in the following example:



no beats

Now let's define some words:

Definition The **wavelength** of a sinusoidal wave is length of a complete wave, the distance from peak to peak or the distance from trough to trough.

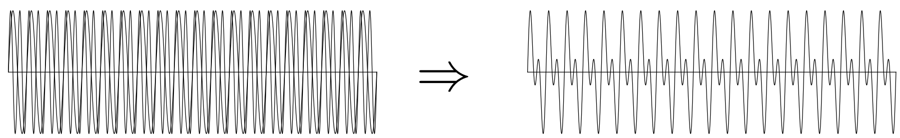
Definition The **frequency** of a sinusoidal wave is the number of complete waves per unit time.

If we're talking about sound waves, then wavelength and frequency are related by the following equation:

$$w_s \cdot f = s \qquad \text{where} \qquad \begin{array}{l} w_s \text{ is the wavelength of the sound wave} \\ f \text{ is the frequency} \\ s \text{ is the speed of sound} \end{array} \tag{B.1}$$

Definition The tone that you start with is called the **tonic**.

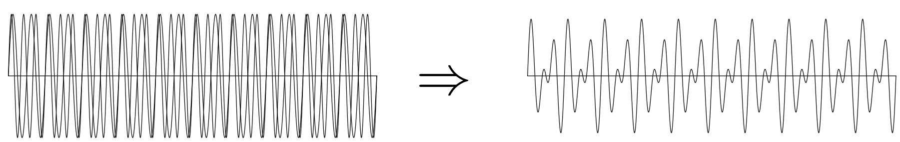
So let's start with the tonic, and add a tone whose wavelength is half that of the tonic:



That looks like it sounds really "right," as there are no beats to be seen.

Definition The tone whose wavelength is half that of the tonic is called the **octave**.

For some reason, the human ear and brain work together to identify the tonic and octave as *the same* tone, with the octave just being twice as high. Now let's get a little crazy, we'll start with the tonic, and add a tone whose wavelength is 2/3 that of the tonic:



That looks like it sounds "right" too, no beats again.

Definition The tone whose wavelength is 2/3 that of the tonic is called the **dominant**.

The dominant is central to all of western music. The tonic-dominant-octave trio of tones is sometimes called a *power chord* for its powerful sound.

Question Starting with just these three notions: Tonic, dominant, and octave what tones would we want an instrument to be able to play?

?

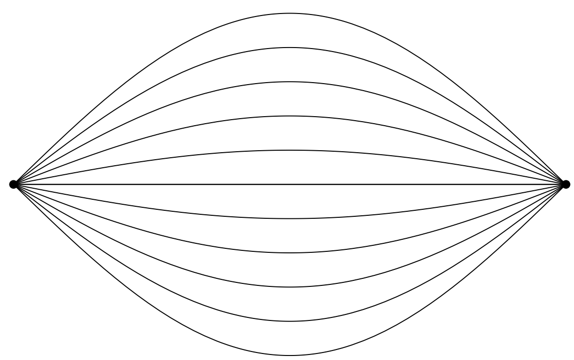
We seek the answer to this question.

B.2.1 Instrument Building

Close your eyes, and imagine you are on a beach in Brazil, imagine yourself swimming in the gentle waves. While you are swimming, you decide that when you return to your home town you are finally going to build that stringed instrument that you've always wanted—perhaps a lytherette. You'd have to pick some frequency for the open string. OK—done.

Question What do waves look like on your stringed instrument?

Let me explain a somewhat sticky point. The waves made by strings on instruments are not nice old sinusoidal waves, they are what we call *standing waves*. Standing waves look like a vibrating string:



Notice how the string is attached at the dots on the left and right? Points like those are called *nodes*. While there are many places that nodes can be, for us nodes will always be at the endpoints of the string.

Definition The **wavelength** of a standing wave is twice the length from node to node.

Definition The **frequency** of a standing wave is the number of complete vibrations (up and down) per unit time.

If we're talking about strings on instruments, then wavelength and frequency are related by the following equation:

$$w_t \cdot f = c \qquad \text{where} \qquad \begin{array}{l} w_t \text{ is the wavelength of the standing wave} \\ f \text{ is the frequency} \\ c \text{ is a constant based on the} \\ \text{mass and tension of the string} \end{array} \tag{B.2}$$

Question If you pluck a string, what will the wavelength of the sound wave be? Use equations (B.1) and (B.2) to express your answer in terms of w_t, f, c and s.

?

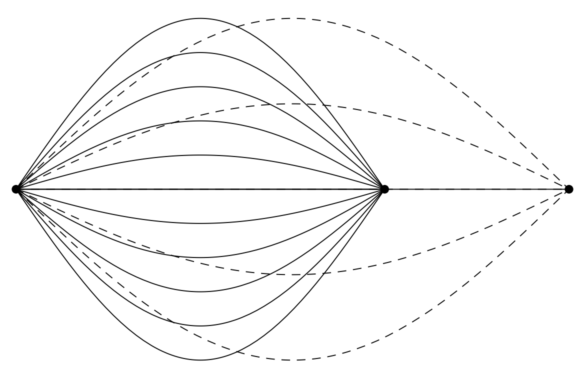
Question If you shorten a string's length by one-third, what will the wavelength of the new sound wave be? Use equations (B.1) and (B.2) to express your answer in terms of w_t, f, c and s.

?

The upshot of the two questions above is that if you want to produce a tone whose wavelength is a fraction of another tone's wavelength, then your merely pluck a string whose length is the same fraction of the string that produced the original tone. Let's get back to instrument building.

Question When building a musical instrument, what tones do you want to be able to play?

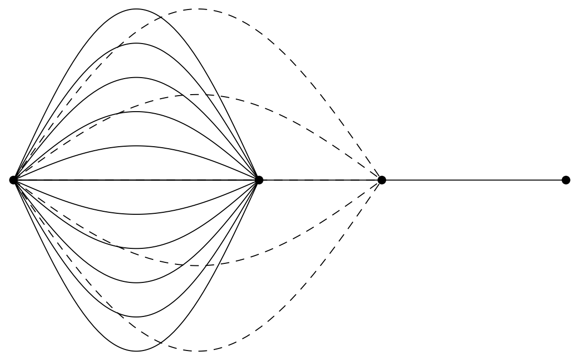
I'll take this one! You want to be able to play tonic (open string), the dominant, and the octave. How do we make this happen? Let's imagine we are building a stringed instrument. Many stringed instruments use *frets* (little metal things that help make new tones) to get the desired tones. To start, we'll want to put a fret 2/3 along the length of the string:



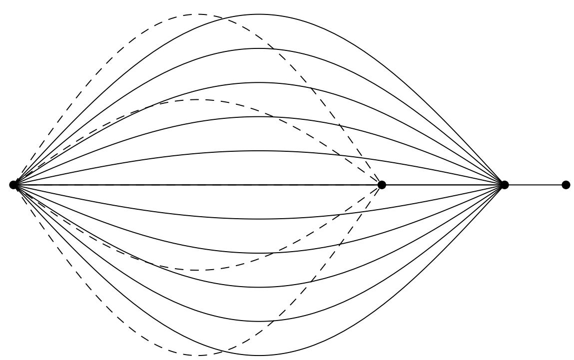
Additionally, we want to be able to play the dominant of this new tone as well. To do this, we'll might place a new fret

$$\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

of the length of the string.



Hmmmm there is a slight problem though. This new fret would create a note *higher* than the octave (1/2 the length of the string). We want all of our frets to be placed between the original tonic and octave. So let's move this fret over by lowering our new tone by an octave. The new wavelength will be twice as long, so we'll put a fret 8/9 along the length of the string.

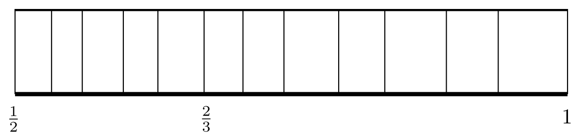


Now repeat this process, adding a new fret for the dominant of the previously added tone, at each step ensuring that the fret be placed between 1/2 and complete string length. Do this 12 times, let's see what happens. Fill in the fractions in the boxes below—as a gesture of friendship, I've given you the correct decimal approximation for each fraction:

fraction						
decimal	0.666...	0.888...	0.592...	0.790...	0.526...	0.702...

fraction						
decimal	0.936...	0.624...	0.832...	0.554...	0.739...	0.493...

From this we find we should place a twelfth fret $0.493\ldots$ along the length of the string. To the human ear, this tone will sound quite close to the octave of our starting tone (though slightly lower). Hence after twelve steps we *appear* to be at the octave. If we were to put frets on our instrument at all of these divisions, we would have something like this:



(B.3)

Remember though, the twelfth fret is close to an octave, but not perfect! Mathematically we might say

$$\frac{2^n}{3^m} \approx \frac{1}{2}$$

for some integers n and m. We note that in our case $m = 12$ and n is some other integer. Could we find integers n and m such that

$$\frac{2^n}{3^m} = \frac{1}{2}?$$

If so then we could write:

$$2^{n+1} = 3^m \qquad \Leftrightarrow \qquad 2^{(n+1)/m} = 3$$

Question What does the Unique Factorization Theorem for integers say about the above expressions? How do we proceed from here?

$$\textbf{?}$$

Problems for Section B.2

(a) Explain what the octave of a given tone is.

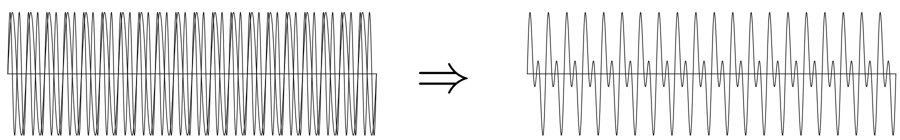
(b) Explain what the dominant of given tone is.

(c) Order the following wavelengths by which tone they produce from lowest to highest.

$$0.666\ldots, 0.888\ldots, 0.592\ldots, 0.790\ldots, 0.526\ldots, 0.702\ldots,$$
$$0.936\ldots, 0.624\ldots, 0.832\ldots, 0.554\ldots, 0.739\ldots, 0.493\ldots$$

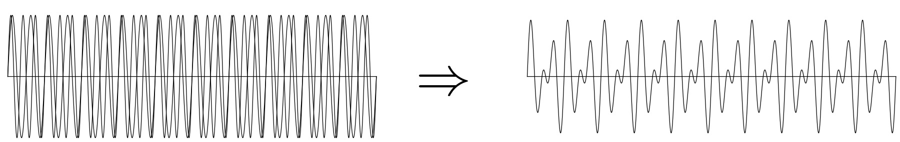
Explain your reasoning.

(d) Do the following tones sound "right" or "wrong" when played together?



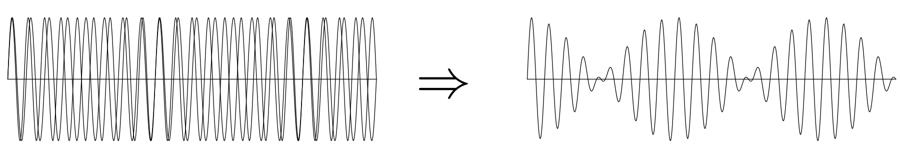
Explain your reasoning.

(e) Do the following tones sound "right" or "wrong" when played together?



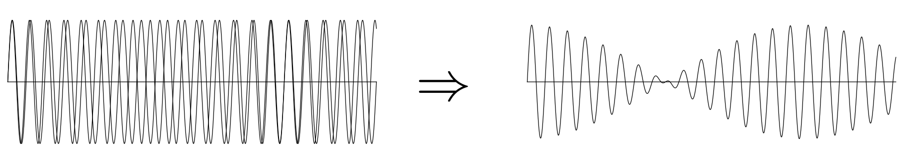
Explain your reasoning.

(f) Do the following tones sound "right" or "wrong" when played together?



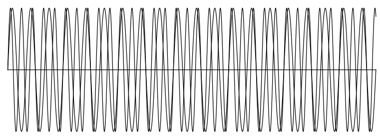
Explain your reasoning.

(g) Do the following tones sound "right" or "wrong" when played together?

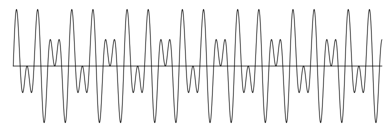


Explain your reasoning.

(h) Do the following tones sound "right" or "wrong" when played together?



$\Rightarrow$



Explain your reasoning.

(i) What is the wavelength of the dominant over the tone of wavelength $2/3$? Explain your reasoning.

(j) What is the wavelength of the dominant over the tone of wavelength $3/4$? Explain your reasoning.

(k) What is the wavelength of the dominant over the tone of wavelength $7/8$? Explain your reasoning.

(l) What is the wavelength of the dominant over the tone of wavelength $6/13$? Explain your reasoning.

(m) What is the wavelength of the dominant over the tone of wavelength $7/13$? Explain your reasoning.

(n) What is the wavelength of the dominant over the tone of wavelength $2/3$ if we insist that the resulting wavelength is between $1/2$ and 1? Explain your reasoning.

(o) What is the wavelength of the dominant over the tone of wavelength $4/7$ if we insist that the resulting wavelength is between $1/2$ and 1? Explain your reasoning.

(p) What is the wavelength of the dominant over the tone of wavelength $5/8$ if we insist that the resulting wavelength is between $1/2$ and 1? Explain your reasoning.

(q) What is the wavelength of the dominant over the tone of wavelength $5/9$ if we insist that the resulting wavelength is between $1/2$ and 1? Explain your reasoning.

(r) What is the wavelength of the dominant over the tone of wavelength $6/11$ if we insist that the resulting wavelength is between $1/2$ and 1? Explain your reasoning.

(s) Give a precise derivation of how we obtained the fret positions

$$\begin{gathered}0.666\ldots, 0.888\ldots, 0.592\ldots, 0.790\ldots, 0.526\ldots, 0.702\ldots,\\ 0.936\ldots, 0.624\ldots, 0.832\ldots, 0.554\ldots, 0.739\ldots, 0.493\ldots\end{gathered}$$

using the ideas of the tonic and dominant.

B.3 Rational and Irrational Temperament

In the last section, we were thinking about how to build a stringed instrument with frets. With this in mind, we came up with the following equation

$$2^{\frac{n+1}{m}} = 3$$

where m is the number of divisions of the string that we would wish to make. Armed with the Unique Factorization Theorem for integers, we could (and you will!) explain that there is no rational solution of

$$2^x = 3,$$

meaning that finding appropriate values of n and m is actually impossible! It's a good thing that we are not the type of people to be deterred by the impossible. In light of our discussion above, we want to find a fraction:

$$\frac{n+1}{m} \approx \log_2(3) = 1.58496250072115618145373894394\ldots$$

Question How do we find good fractional approximations of irrational numbers?

In two words: Continued fractions. Set:

$$x_1 = 1.58496250072115618145373894394\ldots$$

Write x in terms of its whole-number part and its fractional part:

$$x_1 = \mathbf{1} + (x_1 - 1)$$

Now look at the reciprocal of $(x_1 - 1)$:

$$\begin{aligned} x_2 &= \frac{1}{x_1 - 1} = 1.70951129135145477697619026217\ldots \\ x_2 &= \mathbf{1} + (x_2 - 1) \end{aligned}$$

Continue on:

$$\begin{aligned} x_3 &= \frac{1}{x_2 - 1} = 1.40942083965320900458240433081\ldots \\ x_3 &= \mathbf{1} + (x_3 - 1) \end{aligned}$$

Again, again!

$$\begin{aligned} x_4 &= \frac{1}{x_3 - 1} = 2.44247459618085927548717403238\ldots \\ x_4 &= \mathbf{2} + (x_4 - 2) \end{aligned}$$

One last time:

$$
\begin{aligned}
x_5 &= \frac{1}{x_4 - 2} = 2.260016752670824535931276122 60\ldots \\
x_5 &= \mathbf{2} + (x_5 - 2)
\end{aligned}
$$

Whew, now I'm tired, we could continue on but I think it is time to stop. Let's see what our continued fraction looks like:

$$
\log_2(3) \approx 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}}
$$

If we simplify this continued fraction into a regular old fraction we find:

$$
\log_2(3) \approx \frac{19}{12} = 1.58333333333\ldots
$$

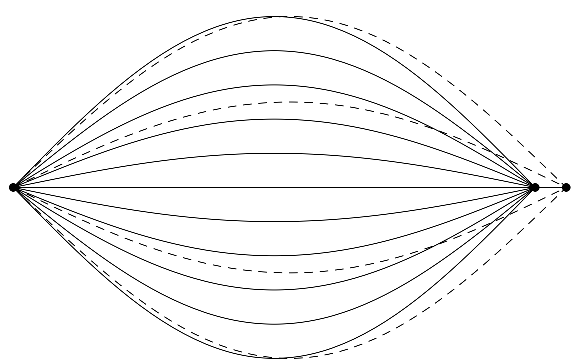
Check this out:

$$
2^{19/12} = 2.9966141537\ldots
$$

This is really close to 3, thus $2^{19/12}$ is a good approximation of $\log_2(3)$. Write

$$
\begin{aligned}
3 &\approx 2^{19/12} \\
\frac{2}{3} &\approx \frac{2}{2^{19/12}} \\
&\approx \frac{1}{2^{7/12}}
\end{aligned}
$$

With this in mind, we will adopt the convention that the nth tone above the tonic will have a wavelength of exactly $1/2^{n/12}$ of the tonic:



In particular, since

$$
\frac{1}{2^{1/12}} \cdot \frac{1}{2^{1/12}} = \frac{1}{2^{2/12}}
$$

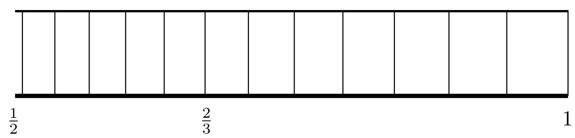
if we work with wavelengths of $1/2^{n/12}$ of the tonic, we will obtain a nice approximation of every tone we produced before. After 7 steps,

$$\frac{1}{2^{7/12}} = 0.66741992\cdots \approx \frac{2}{3}$$

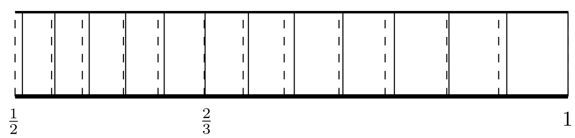
and additionally,

$$\frac{1}{2^{12/12}} = \frac{1}{2},$$

so after twelve steps we are exactly at the octave! If we put a frets at the points $\frac{1}{2^{n/12}}$ letting n run from 0 to 12, we'll obtain a picture like:



Let's compare this to our fret positions in diagram (B.3):



Question Will this really work?

?

B.3.1 Equal Temperament

Spacing the division of the tones by making use of wavelengths that are $\frac{1}{2^{1/12}}$ of the tonic is called **equal temperament**. This is how modern guitars and pianos are tuned. People have also identified other ratios of the wavelength of the tonic that they thought sounded "right." Essentially, they have taken the idea that we should have as few "beats" as possible to the extreme. When they do this, and attempt to have 12 tones, they arrive at what is called **just intonation**. When *a cappella* groups sing and form chords, they usually sing using just intonation—the human brain seems to be somehow drawn to these sounds. Here are the ratios of the wavelength of the tonic used in just intonation:

$$\begin{array}{cccccc} 1/2, & 8/15, & 5/9, & 3/5, & 5/8, & 2/3, \\ 32/45, & 3/4, & 4/5, & 5/6, & 8/9, & 24/25 \end{array}$$

Let's compare these to the ratios used in equal temperament:

Tone	Equal Temperament	Just Intonation
12	$1/2^{12/12} = 0.5$	$1/2 = 0.5$
11	$1/2^{11/12} = 0.529\ldots$	$8/15 = 0.533\ldots$
10	$1/2^{10/12} = 0.561\ldots$	$5/9 = 0.555\ldots$
9	$1/2^{9/12} = 0.594\ldots$	$3/5 = 0.6$
8	$1/2^{8/12} = 0.629\ldots$	$5/8 = 0.625$
7	$1/2^{7/12} = 0.667\ldots$	$2/3 = 0.666\ldots$
6	$1/2^{6/12} = 0.707\ldots$	$32/45 = 0.711\ldots$
5	$1/2^{5/12} = 0.749\ldots$	$3/4 = 0.75$
4	$1/2^{4/12} = 0.793\ldots$	$4/5 = 0.8$
3	$1/2^{3/12} = 0.840\ldots$	$5/6 = 0.833\ldots$
2	$1/2^{2/12} = 0.890\ldots$	$8/9 = 0.888\ldots$
1	$1/2^{1/12} = 0.943\ldots$	$24/25 = 0.96\ldots$

The real issue with just intonation comes with we try to raise every note up by a given number of steps. Suppose a singer has trouble singing in the range of some song, yet has no problems if we raise every note of the song up by 1 half step. If our instrument is in equal temperament, then this shift of tones will have no adverse effects. However, if our instrument is in just intonation, then check out what happens:

- Let 24/25 of a wavelength be the new tonic—this is 1 half step.
- Now 7 steps up will be 5/8 of a wavelength.

Checking out the new ratio we find:

$$\frac{5/8}{24/25} = 0.651042$$

This is over a 2% difference from 2/3 of the tonic. Believe it or not, this will be noticeably "wrong" to the human ear. With equal temperament, the tone will still be spot-on.

These issues with musical instruments arise due to intrinsic differences between rational and irrational numbers. The tones that sound best to our ears are all defined by ratios of the wavelength of the tonic. However, moving up by these ratios is done via multiplication, and hence logarithms enter the scene. The Unique Factorization Theorem for integers tells us that the ratios that we are most interested in cannot be obtained simply from the other ratios we are interested in. From all this arises a fascinating problem that everyone experiences without even realizing it!

Problems for Section B.3

(a) Explain why $\log_2(3)$ is an irrational number.

(b) Explain why $\log_3(5)$ is an irrational number.

(c) Explain why $\log_3(6)$ is an irrational number.

(d) Explain why $\log_4(6)$ is an irrational number.

(e) Explain why $\log_9(10)$ is an irrational number.

(f) Find the simple continued fraction expansion of $5/3$. Explain your reasoning.

(g) Find the simple continued fraction expansion of $15/11$. Explain your reasoning.

(h) Find the simple continued fraction expansion of $22/17$. Explain your reasoning.

(i) Using a calculator, find the first five terms in the simple continued fraction expansion of e. What number do you get by only considering the first term? The first two terms? The first three terms? The first four? The first five? Explain your reasoning.

(j) Using a calculator, find the first five terms in the simple continued fraction expansion of π. What number do you get by only considering the first term? The first two terms? The first three terms? The first four? The first five? Explain your reasoning.

(k) Suppose you are building a stringed instrument. If the first octave of 12 tones has a length of 16 inches, how long is the next octave? What about the next octave? Explain your reasoning.

(l) A singer and a piano are playing a chord involving the sixth tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(m) A singer and a piano are playing a chord involving the seventh tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(n) A singer and a piano are playing a chord involving the fourth tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(o) A singer and a piano are playing a chord involving the twelfth tone. If the singer is singing in just intonation, and the piano is in equal temperament, does the singer believe that the piano is playing too high or too low? Explain your reasoning.

(p) Some other cultures place 5 tones between octaves. Can you explain this if you know that they are trying to approximate $\log_2(3)$?

(q) Light also has wave-like properties. The wavelengths of the visible spectrum goes from around 380 nm to 750 nm. Sometimes colors are depicted as being in a line, other times they are depicted as being in a wheel. Can you use our discussion on music, thinking about tonics and octaves to give a plausible resolution to this paradox?

References and Further Reading

[1] B. Averbach and O. Chein. *Problem Solving Through Recreational Mathematics*. Dover, 2000.

[2] A.K. Bag. *Binomial Theorem in Ancient India*. Indian Journal of History of Science, vol. 1, 1966 68–74.

[3] M. Baron. *The Origins of the Infinitesimal Calculus*. Dover, 1987.

[4] S. Beckmann. *Mathematics for Elementary Teachers*. Addison Wesley, 2007.

[5] A. Bogomolny. Leibniz and Pascal Triangles. *Interactive Mathematics Miscellany and Puzzles*.
`http://www.cut-the-knot.org/`

[6] C.B. Boyer and U.C. Merzbach. *A History of Mathematics*. Wiley, 1991.

[7] H. Colebrooke. *Classics of Indian Mathematics*. Sharada Publishing House, 2005.

[8] U. Dudley. *Elementary Number Theory*. Dover, 2008.

[9] R.W. Dull. *Mathematics for Engineers*. Second Edition. McGraw-Hill, 1941.

[10] E. Dunne and M. McConnell. *Pianos and Continued Fractions*. Mathematics Magazine, vol. 72, no. 2 (1999), 104-115.

[11] Euclid and T.L. Heath. *The Thirteen Books of Euclid's Elements: Volumes 1, 2, and 3*. Dover, 1956.

[12] H. Eves. *An Introduction to the History of Mathematics*. Saunders College Publishing, 1990.

[13] M. Gardner. *My Best Mathematical and Logic Puzzles*. Dover, 1994.

[14] L.J. Lange. *An Elegant Continued Fraction for π*. The American Mathematical Monthly, vol. 106, no. 5, (1999), pp. 456–458.

[15] M. Livio. *The Equation that Couldn't be Solved.* Simon&Schuster, 2005.

[16] R.L. Keeney and J.F. Ramaley. *On the Trinomial Coefficients.* Mathematics Magazine, vol. 42, no. 4 (Sep., 1969), pp. 210-213.

[17] D.E. Knuth. *The Art of Computer Programming. Volume 1: Fundamental Algorithms.* Addison-Wesley, 1975.

[18] J.A. Mira. *Mathematical Teasers.* Barnes&Noble, 1970.

[19] NRICH website.
`http://nrich.maths.org/`

[20] I. Papick. *Algebra Connections: Mathematics for Middle School Teachers.* Pearson, 2007.

[21] M. Petković. *Famous Puzzles of Great Mathematicians.* American Mathematical Society, 2009.

[22] G. Polya. *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving.* Wiley, 1981.

[23] A. Posamentier. *Teaching Secondary Mathematics.* Prentice-Hall, 2002.

[24] M. Tahan. *The Man Who Counted: A Collection of Mathematical Adventures.* W.W. Norton&Company, 1993.

[25] R. Solomon. *Abstract Algebra.* Brooks/Cole, 2003.

[26] C.R. Wylie Jr. *101 Puzzles in Thought and Logic.* Dover, 1957.

[27] B.K. Youse. *Arithmetic: A Modern Approach.* Prentice-Hall, 1963.

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