Euclidean Constructions and the Geometry of Origami

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1. Introduction

Although the connection between geometry and origami is quite obvious, and has been researched in a limited way for centuries, very few people active in one field appear to be more than casually aware of the other. Occasionally an example in the problems section of some journal will concern itself with folding paper in some way, but origami is not generally considered a mathematical discipline. Some origamians (notably Kazuo Haga and Kunihiko Kashahara, see **[6]**, and Tomoko Fuse, see **[3]**) have done quite impressive work on the geometry of origami (especially on Platonic solids and related subjects), but most prefer to stick to the artistic side of the handicraft. In origami circles, excessive pondering of geometry is usually considered to detract in some way from the elegance and harmony of the art.

Nevertheless, the connection between geometry and origami is well established. Noted educators, such as the German, Friedrich Froebel, have suggested the use of origami as a tool for the teaching of elementary geometric forms. Some interesting work has been published on geometric aspects of origami, particularly as applied to specific models. Much is known about methods of folding regular polygons and polyhedra, for instance **[3, 5, 6, 9]**. It is also well known that the classic Delian problem (finding a cube twice the volume of a given cube) as well as the problem of angle trisection can be solved using methods of origami, despite being unsolvable by Euclidean methods. This article attempts to show the connections between origami folds and Euclidean constructions with straight-edge and compass. Somewhat surprisingly, we shall see that parabolas are elementary to origami construction, and play a role similar to that of circles in Euclidean constructions. There are also some aspects of projective geometry that sneak in unexpectedly.

Perhaps this paper will help in getting more mathematicians interested in origami from the scientific viewpoint. It is my firm belief that this can be fruitful for both the art of origami and the science of geometry.

2. Elementary Euclidean Procedures

When considering Euclidean constructions, it is assumed that specific points are known *a priori* in an infinite Euclidean plane. If needed, random points can be marked in addition to those already known. Using straight-edge and compass as tools, the following procedures are defined as being "allowed":

(E1) Given two non-identical points P and Q, one can draw the unique straight line $l = PQ$ containing both points, using the straight-edge.

(E2) Given a point M and the length of a line segment $r > 0$, one can draw the unique circle $c = \{M; r\}$ with M as center and r as radius, using the compass.

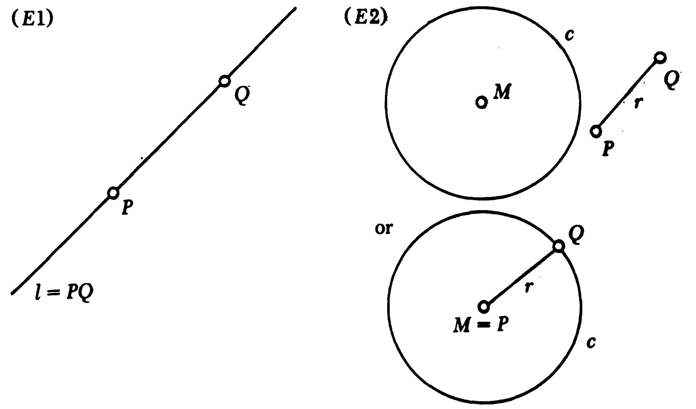

FIGURE 2.1

Specifically, the radius r must be given as the length of a line segment connecting two known points P and Q, one of which may be M, in which case the other is a point on the circle (FIGURE 2.1). Random lines or circles can be introduced, as their generation can always be understood as an application of (E1) and (E2) to random points.

Application of (E1) and (E2) to the points known *a priori* leads to specific straight lines and circles. Knowledge of these leads to further points by virtue of the following procedures of intersection, which are also defined as being "allowed":

(E3) Given two non-parallel straight lines l_1 and l_2, one can determine their unique point of intersection $P = l_1 \cap l_2$;
(E4) given a circle $c = \{M; r\}$ and a straight line l, such that the distance between M and l is not greater than r, one can determine the point(s) of intersection of c and l, and finally
(E5) given two circles $c_1 = \{M_1; r_1\}$ and $c_2 = \{M_2; r_2\}$, such that either
  i) neither contains the center of the other in its interior, and the distance between the centers is not greater than the sum of the radii, or
  ii) one contains the center of the other in its interior, and the distance between the centers is not less than the difference of the radii,
  one can determine the point(s) of intersection of c_1 and c_2.

(In practical applications, the locations of these points of intersection may not be known with sufficient precision. The angle of intersection may be very small, or the relative positions of intersecting straight lines or circles may be otherwise inconvenient, so that the points of intersection may not be practically accessible. They are nevertheless assumed to be known in theory.)

Iterated applications of (E1) to (E5) lead from *a priori* knowledge of specific points to specific straight lines and circles, then to further points, and so on. A geometric construction problem is said to be "solvable" by Euclidean methods, if it can be shown that iterated application exclusively of (E1) to (E5) leads from certain given points to those points and/or straight lines and/or circles that make up whatever geometric entity is sought.

It is the purpose of this paper to compare the solvability of problems in the Euclidean sense with that using elementary procedures of origami, which are defined in the next section.

3. Elementary Geometric Procedures of Origami

Origami is, of course, the art of paper folding. As anyone who has ever put a crease in a piece of paper knows, there are certain procedures in paper folding that seem natural and basic. It is natural to fold a straight line, for instance, whereas folding a curve is possible, but difficult to control. Although there are models in origami utilizing curved folds, for geometric purposes we shall exclude these as non-elementary.

As an origami model develops, its increasing complexity creates an increasingly complex analogous geometric pattern composed of straight lines (or line segments) on the paper used in producing the model. In most origami models, the result is in some way three-dimensional (although folding two-dimensional forms such as regular polygons or stars is also sometimes considered), but every model can be opened up, and what we will consider in this work is the geometry of the folds on the opened paper. Some abstraction will, of course, be necessary. For instance, although most origami is done with squares (or rectangles), and certainly all with finite pieces of paper, for theoretical purposes we will consider folding an infinite Euclidean plane. Also, folding multiple layers of paper leads to some interesting phenomena. For instance, the result of a multi-level fold may be a finite line segment. We will consider this line segment as automatically defining the infinite straight line of which it is a part. This is legitimized by the fact that every line segment has a unique pair of endpoints, which lead to a whole line by folding the infinite paper through these points (a procedure we will define as being allowed). Also, folding multiple layers allows points to be "marked" through the paper. That is, if a specific point comes to lie over another through folding, the multi-layered model can be folded (at least twice) along folds containing that specific point, thus "marking" the points on the other layers of paper immediately above or below that point. It therefore seems natural to assume that a known point, which is brought to lie over another through folding, defines the point on the other layer as equally well known. The same can be said to hold for a line that is brought into another position through folding; it too defines the overlying and underlying lines in the other layers of paper.

If we wish to compare Euclidean procedures to those used in origami, we must define "allowed" procedures, just as we did in the Euclidean case. The basic geometric entity of origami is the straight line. This differs from Euclidean geometry, where the basic entity is the point, knowledge of points then leading to straight lines and circles, and so on. It seems reasonable to assume that a straight line can be folded randomly anywhere on the plane, just as a point can be drawn randomly anywhere in the plane. (We shall use the short form verbs "draw" for Euclidean constructions and "fold" for origami constructions.) Given this, it seems reasonable to first define the following as "allowed":

(O1) Given two non-parallel straight lines l_1 and l_2, one can determine their unique point of intersection $P = l_1 \cap l_2$.

This is, of course, precisely the same as (E3), but in the origami case it defines how points can be considered to be known (which is not immediately obvious), whereas in the Euclidean sense it is *a posteriori* in the sense that points are considered well defined in the plane by simply marking them with a pencil or similar implement.

Two nonparallel straight lines are thus considered to define their point of intersection. This does not yet necessitate any folding other than that leading to the given lines. When two parallel straight lines are given, one can always fold one onto the other uniquely. The resulting fold is that straight line, which is parallel to both, and equidistant from them. When two intersecting straight lines are given, they can be folded onto each other in two ways, the resulting folds being the angle bisectors of the given lines. It therefore seems reasonable to assume the following as "allowed":

(O2) Given two parallel straight lines l_1 and l_2, one can fold the line m parallel to and equidistant from them ("mid-parallel"), and

(O3) Given two intersecting straight lines l_1 and l_2 one can fold their angle bisectors a and a'.

By virtue of the fact that folding a point or a line onto another spot implies knowledge of points and lines in the other layers immediately above or below, these two procedures include the transferring of a known angle to another line. (O2) is the transfer of a line to a parallel line and (O3) that to an intersecting line (FIGURE 3.1). Also, (O3) includes the rotation of a given line-segment, with one end-point in the point of intersection of the two lines, from one line to the other.

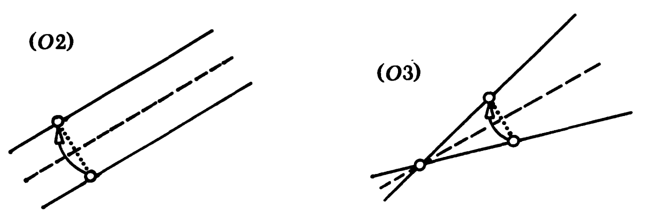

FIGURE 3.1

Given two points on a piece of paper, it is straightforward to fold the unique straight line joining the two points. Equally straightforward, however, is the folding of one given point onto the other. The resulting fold is obviously perpendicular to the line connecting the given points, and these points are equidistant from it. It is therefore the perpendicular bisector of the line segment defined by the two points. Further reasonable "allowed" procedures are therefore:

(O4) Given two non-identical points P and Q, one can fold the unique straight line $l = PQ$ connecting both points, and

(O5) Given two non-identical points P and Q, one can fold the unique perpendicular bisector b of the line segment PQ.

(O4) is, of course, identical to (E1).

As with the rotation of a line segment in (O3), (O5) includes transferring one end-point of a line segment of known length to another point (FIGURE 3.2). Together, (O5) and (O3) mean that one can transfer a line segment of known length to any spot on the paper, as any transfer can be achieved by combining reflections and rotations.

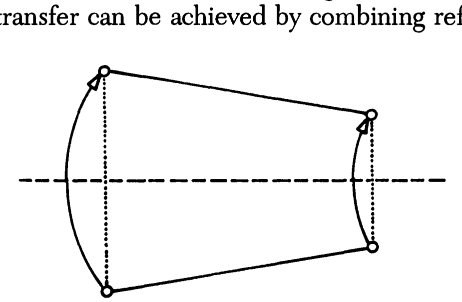

FIGURE 3.2

Given a point P and a straight line l, it is straightforward to fold the line onto itself, such that the given point lies on the fold. Such a fold is obviously unique. Since the given line is folded onto itself, the fold must be perpendicular to it. A further "allowed" procedure is thus:

(O6) Given a point P and a straight line l, one can fold the unique line l' perpendicular to l and containing P.

Finally, given a straight line l and a point P not on the line, it is straightforward to fold P onto any point on the line. The resulting (infinitely many) folds are the elements of the set of perpendicular bisectors of all line segments with the given point P at one end, and a point on the line l at the other. This is precisely the set of tangents of the parabola with P as its focus and l as its directrix (Figure 3.3). We see that parabolas, or rather the sets of tangents of parabolas, play an elementary role in the geometry of folding.

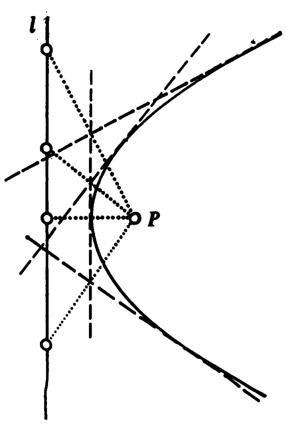

FIGURE 3.3

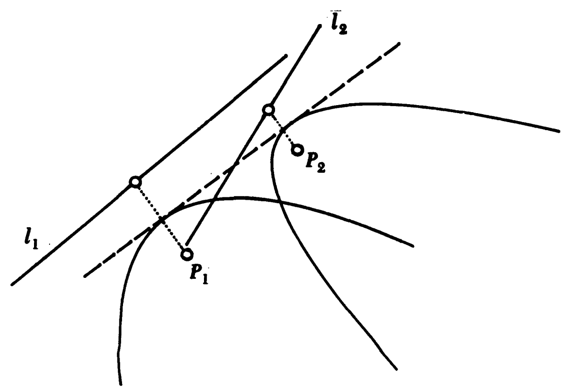

FIGURE 3.4

Given P and l and a farther point Q, it is also straightforward to fold P onto l such that Q lies on the fold. The fold is therefore a tangent of the parabola, containing Q. A further "allowed" procedure is therefore:

(O7) Given a point P and a straight line l, one can fold any tangent of the parabola with focus P and directrix l. Specifically, given a farther point Q, one can fold the tangents of the parabola containing Q.

As with (O3) and (O5), this includes the fact that one can find the points X and $\overline{X}$ on a given line l having the same distance from a given point Q as another given point P. This is slightly more specific than (O3). (O3) merely means that the points of a circle can be found by folding (see 4.2), but this implies that the points of intersection of a line and a circle can be found immediately (see 4.4 and Figure 4.2).

Given two points P_1 and P_2 and two lines l_1 and l_2, it is still straightforward to fold P_1 onto l_1 and P_2 onto l_2 with the same fold (Figure 3.4). The word "onto" must be understood in a wide sense here, as the points can come to lie above the lines, or the lines above the points, or one of each. In any case, the resulting fold is a common tangent of the two parabolas with foci P_1 and P_2 and directrices l_1 and l_2, respectively.

A final "allowed" procedure is thus:

(O7*) Given (possibly identical) points P_1 and P_2 and (possibly identical) lines l_1 and l_2, one can fold the common tangents of the parabolas p_1 and p_2 with foci P_1 and P_2 and directrices l_1 and l_2, respectively.

This procedure (O7*) is what makes the geometry of origami fundamentally different from Euclidean construction. As we shall see in the next two sections, Euclidean constructions are equivalent to that part of origami constructions utilizing (O1)–(O7). Procedure (O7*), however, allows constructions not accessible by Euclidean methods. The resulting constructions are similar (although not identical) to those utilizing a marked ruler, the theory of which is well established (see **[2]** pp. 74–78). It is not surprising that (O7*) goes beyond Euclidean constructions, if one considers what it means analytically. It is known that two conics in general have four common tangents. If both are parabolas, one of these common tangents is the line at infinity. It is thus a cubic problem to find the common tangents of two parabolas, and it is not to be expected that this cubic problem can be solved by Euclidean methods.

If the foci P_1 and P_2 are identical, the only common tangent is found by folding this point onto the point of intersection of l_1 and l_2. It is not surprising that there is only one further common tangent of p_1 and p_2 to be found, since it is known from projective geometry that the common focus is equivalent to a pair of common complex tangents.

If the directrices are identical, p_1 and p_2 not only have the line at infinity as a common tangent, but also a common point at infinity where the line at infinity is tangent. It therefore counts as a double common tangent, and there are only two further common tangents to be found. These are then, of course, the angle bisectors of the common directrix and the line joining the two foci. In these special cases, finding the common tangents can be solved by Euclidean methods, as the problem is reduced to a linear or quadratic one.

4. Reducing Euclidean Procedures to Origami

In this section, we shall show that each of the elementary Euclidean procedures (E1)–(E5) can be replaced by combinations of (O1)-(O7).

4.1. (E1) (E1) is identical to (O4).

4.2. (E2) A circle cannot be "drawn" by origami procedures. Nevertheless, a circle can still be considered to be well determined if one knows its center M and radius r, as one can determine any number of points and tangents of the circle. This can be achieved in the following manner (FIGURE 4.1):

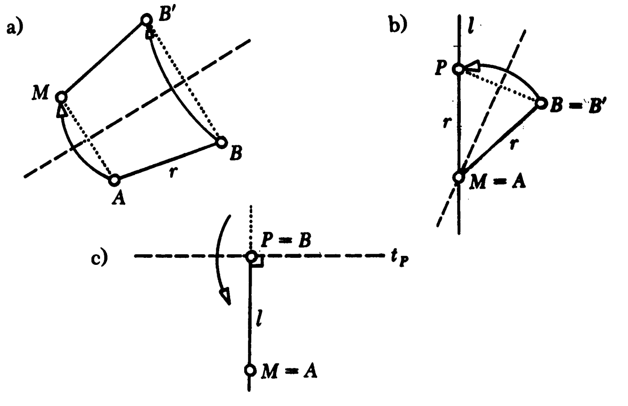

FIGURE 4.1

a) If the center M and radius $r = AB$ of a circle are known, it is possible to fold A to M by virtue of (O5) (folding the perpendicular bisector of MA). This brings B to a point B', and we have $r = MB'$.

b) If a specific line l through M is given, the radius $r = MB'$ can be folded onto it by virtue of (O3) (folding an angle bisector of $\angle MB', l$). This yields the point P as a point on the circle on the diameter l. (The other angle bisector yields the diametrically opposite point of the circle.)

c) Folding l onto itself through P by virtue of (O6) yields the line perpendicular to the diameter containing P, which is precisely the tangent of the circle in P.

4.3. (E3) (E3) is identical to (O1).

4.4. (E4) If a circle is known by its center M and a point P on its circumference, and a line l is given, the points of intersection of the circle and l can be found by folding P onto l such that the fold contains M. This is possible by virtue of (O7). In doing this, finding the points of intersection of a circle and a straight line is seen to be equivalent to finding the tangents of a specific parabola (with focus P and directrix l) containing a specific point (M).

4.5. (E5) Since circles are only accessible in origami through knowledge of specific points and tangents, it is not possible to find the common points of two circles

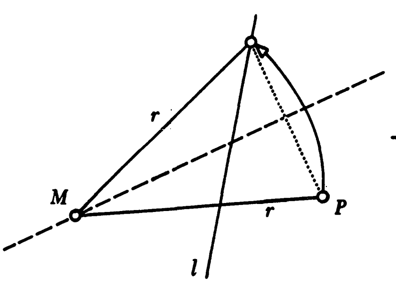

FIGURE 4.2

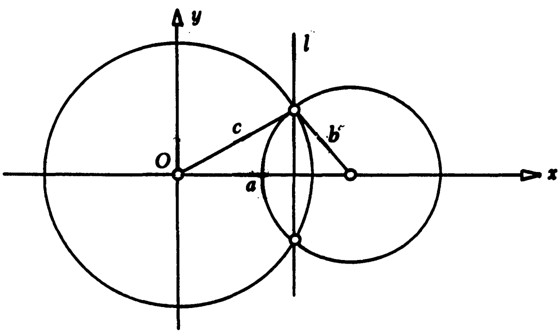

FIGURE 4.3

directly. It is, however, possible to find the common chord of intersecting circles, thus reducing (E5) to (E4). This can be achieved in the following manner.

We assume that two circles, whose points of intersection we wish to determine, are given. Let the distance between their centers be a, and let the radii be b and c, respectively. Assuming the center of one circle is the origin of a cartesian coordinate system, and the center of the other is on the x-axis, their equations are $x^2 + y^2 = c^2$ and $(x-a)^2 + y^2 = b^2$, respectively.

Their common chord is therefore the line represented by the equation

$$x^2 + y^2 - c^2 = x^2 - 2xa + a^2 + y^2 - b^2$$
$$\Leftrightarrow x = \frac{a^2 - b^2 + c^2}{2a}.$$

The common points of the two circles thus lie on the line that is perpendicular to the line connecting their centers, and whose distance to the center of the circle with radius c is $(a^2 - b^2 + c^2)/2a$ (or equivalently, whose distance to the center of the circle with radius b is $(a^2 + b^2 - c^2)/2a$). This line can be found by origami procedures. One way of doing this is described in the following four steps.

Step 1. Since the distances a and c are known, it is possible to fold a right triangle with sides a and c ((O3), (O5), (O6)). The length of the hypotenuse is then $\sqrt{a^2+c^2}$.

Step 2. The distance b and $\sqrt{a^2+c^2}$ are known. It is therefore possible to fold a right triangle with side b and hypotenuse $\sqrt{a^2+c^2}$ ((O3), (O5), (O6), (O7)). The length of the second side is then $\sqrt{a^2-b^2+c^2}$.

Step 3. A triangle can be folded with one side of unit length 1 and one side of length $\sqrt{a^2-b^2+c^2}$. A similar triangle can then be folded ((O2), (O3)) with a side of length $\sqrt{a^2-b^2+c^2}$ corresponding to that side of the first triangle with length 1. The side corresponding to that side of the first triangle with length $\sqrt{a^2-b^2+c^2}$ is then of length $a^2-b^2+c^2$.

Step 4. A triangle can be folded with one side of length $2a$ and another of length $a^2-b^2+c^2$ ((O2), (O3)). A similar triangle can be folded with a side of length 1 corresponding to the side of the first triangle with length $2a$ ((O2), (O3), (O5)). Then the side corresponding to the side of the first triangle of length $a^2-b^2+c^2$ has the length $a^2-b^2+c^2/(2a)$. This is precisely the length we had set out to produce.

(E5) is thus reduced to (E4), as we need only find the points of intersection of the common chord with either circle.

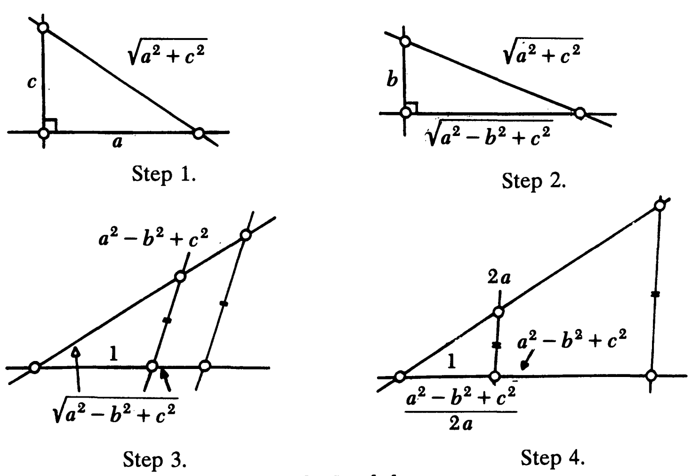

FIGURE 4.4

In summary, we have the following.

THEOREM 1. *Every construction that can be done by Euclidean methods can also be achieved by elementary methods of origami. Specifically, the Euclidean procedures* (E1)–(E5) *can all be replaced by combinations of the origami procedures* (O1)–(O7).

5. Reducing Origami Procedures to Euclidean Constructions

In this section we shall show that each of the elementary origami procedures (O1)–(O7) can be replaced by combinations of (E1)–(E5). That this is not to be expected for (O7*) has already been explained at the end of section 3.

5.1. Identical Procedures (O1) is identical to (E3), and (O4) is identical to (E1).

5.2. The Basic Procedures (O2), (O3), (O5), and (O6) The origami procedures (O2), (O3), (O5), and (O6) are the constructions of the mid-parallel of two parallel lines, the construction of the angle bisectors of two intersecting lines, the construction of the perpendicular bisector of a line segment, and the construction of a line through a given point and perpendicular to a given line, respectively. All of these constructions are known to be possible using Euclidean methods.

5.3. (O7) Determining the tangents of a parabola given by its focus F and directrix l through a given point P by Euclidean methods is not difficult, but the method is perhaps not as well known as those in 5.2. A review of some elementary properties of the parabola seems in order here (FIGURE 5.1). If a specific point T of a parabola with focus F and directrix l is known, we have $TF = Tl$ by definition of a parabola. The axis a of the parabola contains F and is perpendicular to l. The line parallel to a containing T intersects l in a point G. Obviously, $TG = Tl = TF$. If we determine the point X on a such that $FTGX$ is a rhombus, the diagonal TX of the rhombus is the tangent of the parabola in T.

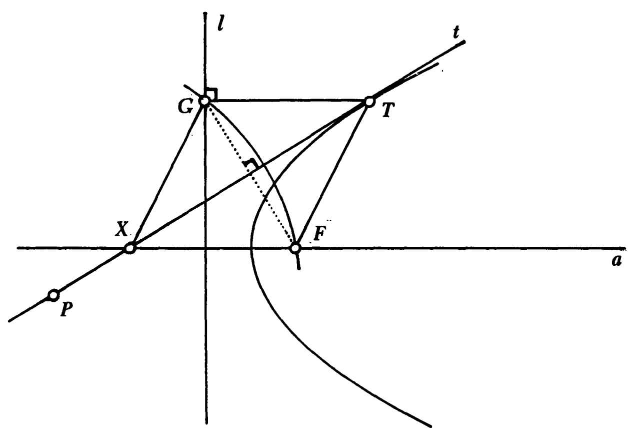

FIGURE 5.1

Knowing this, we see that we can find the tangents of a given parabola containing a given point P by the following method:

Since a tangent t containing P must be the perpendicular bisector of the line segment FG, where G is the point on l corresponding to that point T where t is tangent to the parabola, the distance from P to F must be equal to the distance from P to G. Using the compass, we can find the two possible points G, and can thus find the tangents t along with the points T by completing the rhombi defined by F and the two points G. The diagonals of these rhombi are then the tangents containing P.

We see that (O7) is also replaceable by Euclidean methods. In summary, we therefore have the following.

THEOREM 2. *Every construction that can be done by the origami methods* (O1)–(O7) *exclusively can also be achieved by Euclidean methods.*

Together with Theorem 1 this means that every geometric construction that can be achieved using the origami methods (O1)–(O7) can also be achieved by the Euclidean methods (E1)–(E5) and vice versa. The two sets of constructions are thus equivalent. As we have already seen, (O7*) adds additional geometric constructions to the set of possible constructions generated by these equivalent sets. We see that the set of constructions that can be generated by Euclidean methods is a true subset of the set *that can be generated by origami methods.*

6. Folding Cube Roots

As already mentioned in section 3, finding the common tangents of two parabolas is, in general, analytically a cubic problem. Since (O7*) makes it possible to fold the common tangents of two parabolas, it is to be expected to be possible to find cube roots utilizing (O7*), which is of course, not possible by Euclidean methods. In this section, we shall become acquainted with a simple method of folding the cube root of the quotient of the lengths of any two given line segments.

A method of folding $\sqrt[3]{2}$ is given in **[8]**. The more general method we shall use here is based on parabolas with a common vertex and perpendicular axes. That such parabolas intersect in points whose coordinates solve simple cubic equations was already known to Descartes, and that such parabolas have something to do with finding cube roots was even known in antiquity (see **[4]**, p. 12, or **[1]**, pp. 342–344). Since folding does not allow us to work with points of intersection, but rather with common tangents, we must deal with the dual problem, which works just as well, as we will see.

We consider the parabolas with the equations

$$p_1\colon\ y^2 = 2ax \quad \text{and} \quad p_2\colon\ x^2 = 2by.$$

Since these parabolas intersect in two points with real coordinates, they have only one real common tangent. We assume this tangent (which cannot be parallel to either axis) to be

$$t\colon\ y = cx + d.$$

We assume further that $P_1(x_1, y_1)$ is the point at which t is tangent to p_1. Then, t also has the equation

$$yy_1 = ax + ax_1 \quad \Leftrightarrow \quad y = \frac{a}{y_1}\cdot x + \frac{ax_1}{y_1}.$$

Therefore

$$\begin{aligned}
&c = \frac{a}{y_1} \quad \text{and} \quad d = \frac{ax_1}{y_1}\\
&\Rightarrow y_1 = \frac{a}{c} \quad \text{and} \quad x_1 = \frac{d}{c}\\
&\Rightarrow \frac{a^2}{c^2} = 2a\cdot\frac{d}{c}\\
&\Rightarrow a = 2cd.
\end{aligned}$$

We further assume that $P_2(x_2, y_2)$ is the point in which t is tangent to p_2. Then, t also has the equation

$$xx_2 = by + by_2 \quad \Leftrightarrow \quad y = \frac{x_2}{b}\cdot x - y_2.$$

Therefore

$$\begin{aligned}
&c = \frac{x_2}{b} \quad \text{and} \quad d = -y_2\\
&\Rightarrow x_2 = bc \quad \text{and} \quad y_2 = -d\\
&\Rightarrow b^2c^2 = -2bd\\
&\Rightarrow d = -\frac{bc^2}{2}.
\end{aligned}$$

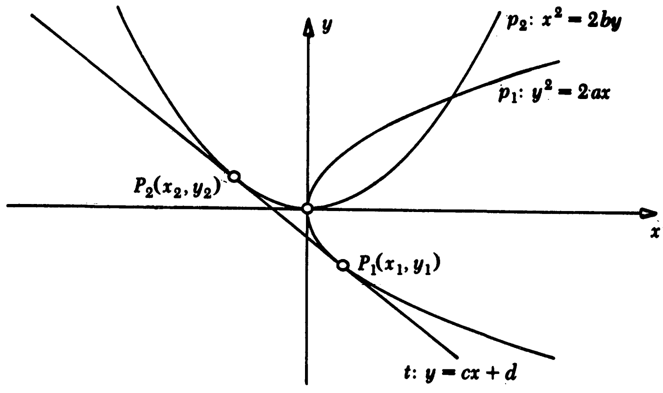

FIGURE 6.1

We therefore see that

$$a = 2cd \quad \text{and} \quad d = -\frac{bc^2}{2}$$

$$\Rightarrow a = -bc^3$$

$$\Rightarrow c = -\sqrt[3]{\frac{a}{b}}\,.$$

We see that the slope of the common tangent is the (negative) cube root of the quotient of the parameters of the parabolas.

If b is equal to the unit length, the slope of the common tangent is the cube root of the parameter of p_1.

A cube root can therefore be folded in the following manner. If a and b are given, we fold a right angle anywhere to represent the parabola axes. We fold $a/2$ to the left and right of the origin to obtain the directrix l_1 and the focus F_1 of p_1, respectively, and $b/2$ to the top and bottom to similarly obtain the directrix l_2 and the focus F_2 of p_2. (Since $\frac{a}{b} = \frac{2a}{2b}$, we can also use a and b rather than $a/2$ and $b/2$.) Folding F_1 onto l_1 and F_2 onto l_2 simultaneously by virtue of (O7*) gives us the common tangent t, whose slope is then $-\sqrt[3]{a/b}$. Folding the unit length from any point on t parallel to the x-axis and completing the right triangle with hypotenuse on t thus yields a line segment of length $\sqrt[3]{a/b}$ as the second side of the triangle.

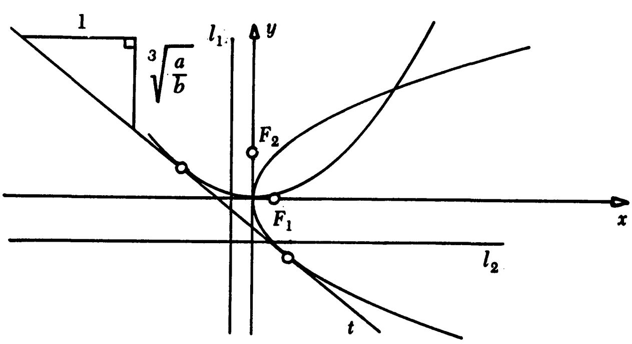

FIGURE 6.2

7. Solving General Cubic Equations

A slight generalization of the method presented in the preceding section allows us to solve general cubic equations. We can see this in the following manner.

We consider the parabolas with the equations

$$p_1\colon (y-n)^2 = 2a(x-m) \quad \text{and} \quad p_2\colon x^2 = 2by.$$

As before, we assume that the equation describing a common tangent of p_1 and p_2 (which need not be unique in this case), is

$$t\colon y = cx + d.$$

Again, such a common tangent cannot be parallel to either axis. We assume, as before, that $P_1(x_1, y_1)$ is the point in which t is tangent to p_1. Then, t is also represented by the equation

$$(y-n)(y_1-n) = a(x-m) + a(x_1-m)$$
$$\Leftrightarrow y = \frac{a}{y_1-n}\cdot x + n + \frac{ax_1 - 2am}{y_1-n}.$$

Therefore

$$c = \frac{a}{y_1-n} \quad \text{and} \quad d = n + \frac{ax_1-2am}{y_1-n}$$
$$\Rightarrow y_1 = \frac{a+nc}{c} \quad \text{and} \quad x_1 = \frac{d-n}{c} + 2m,$$

and

$$(y_1-n)^2 = 2a(x_1-m)$$
$$\Rightarrow \frac{a^2}{c^2} = 2a\left(\frac{d-n}{c} + m\right)$$
$$\Rightarrow a = 2c(d-n+cm).$$

As in the preceding section, assuming $P_2(x_2, y_2)$ to be the point in which t is tangent to p_2, we find that t is also represented by the equation

$$y = \frac{x_2}{b}\cdot x - y_2,$$

which again leads to

$$d = -\frac{bc^2}{2}.$$

Substituting for d, we obtain

$$a = 2c\left(-\frac{bc^2}{2} - n + cm\right)$$
$$\Leftrightarrow bc^3 - 2mc^2 + 2nc + a = 0$$
$$\Leftrightarrow c^3 - \frac{2m}{b}\cdot c^2 + \frac{2n}{b}\cdot c + \frac{a}{b} = 0.$$

The slope of the common tangent is therefore a solution c of the cubic equation

$$c^3 - \frac{2m}{b}\cdot c^2 + \frac{2n}{b}\cdot c + \frac{a}{b} = 0.$$

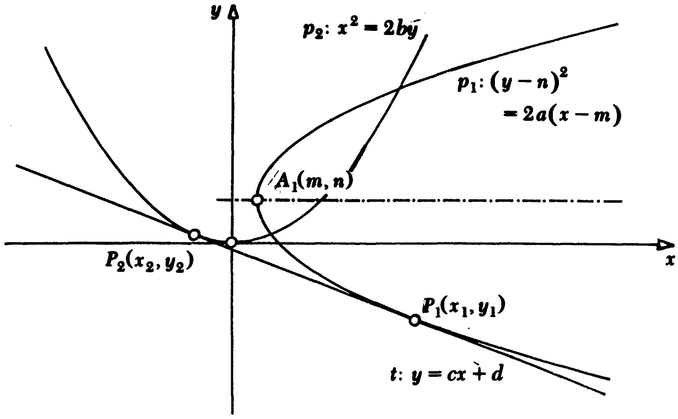

FIGURE 7.1

This equation can have either one real solution and a pair of complex solutions, or three real solutions, of which two or all three can be equal. This corresponds to parabolas that intersect, and parabolas that do not, respectively. Two solutions are equal if, and only if, two tangents are equal (that is, if the parabolas are tangent), and all three are equal if, and only if, the parabolas osculate (that is, if they have contact of third order).

Given a cubic equation, one can therefore fold the roots of the equation by the following method. Say the given equation is

$$x^3 + px^2 + qx + r = 0.$$

Assuming that the parameter b of p_2 is equal to the unit length 1, we have

$$p = -2m, q = 2n \quad \text{and} \quad r = a$$

or

$$m = -\frac{p}{2}, n = \frac{q}{2} \quad \text{and} \quad a = r.$$

We need therefore only find the point with coordinates

$$F_1\left(-\frac{p}{2} + \frac{r}{2}, \frac{q}{2}\right)$$

and the line l_1 represented by the equation

$$x = -\frac{p}{2} - \frac{r}{2}.$$

These are then focus and directrix of p_1, respectively. The focus F_2 of p_2 is $(0, \frac{1}{2})$, and its directrix l_2 is represented by the equation $y = -\frac{1}{2}$. Folding F_1 onto l_1 and F_2 onto l_2 simultaneously by virtue of (O7*) then yields the common tangent (or tangents) of p_1 and p_2, whose slope solves the given cubic equation.

8. Trisecting Angles

If we wish to trisect an angle using origami methods, we find that the preceding result gives us a straightforward method of doing so. It is known that the equation

$$\cos 3\alpha = 4\cos^3 \alpha - 3\cos \alpha$$

holds for any angle α. Assuming knowledge of $\cos 3\alpha$, finding $\cos\alpha$ is therefore merely a matter of solving the cubic equation

$$x^3 - \tfrac{3}{4}\cdot x - \tfrac{1}{4}\cdot\cos 3\alpha = 0.$$

As previously shown, this can be done by utilizing the parabola p_1 with the focus

$$F_1\left(-\tfrac{1}{8}\cos 3\alpha, -\tfrac{3}{8}\right)$$

and the directrix

$$l_1\colon\ x = \tfrac{1}{8}\cos 3\alpha$$

(since a is negative, the parabola is open to the left in this case), as well as the "unit parabola" p_2 with focus $F_2(0, \frac{1}{2})$ and directrix l_2: $y = -\frac{1}{2}$. The slope of the common tangent of these parabolas solves the cubic equation, yielding $\cos\alpha$, which immediately leads to α itself.

9. Conclusion

The method described in section 6 is by no means the only method of producing cube roots using origami methods, but it is effective, general, and acceptably easy to apply. Specifically, the Delian problem (doubling the volume of a cube) is reduced to a special case, and a fairly easy one at that. Perhaps this is what the oracle at Delos originally had in mind when it asked the Athenians to double the size of the cubic altar of Apollo in order to rid themselves of the plague. Maybe the oracle was really an origamian at heart.

Also, while the method of trisecting angles described in section 8 is not likely to be the most elegant for practical origami purposes, it is an immediate corollary of the solution of the general cubic equation using those origami methods defined as "allowed" in section 3, and therefore quite easy to grasp in this context. Perhaps angle trisectors should turn their attentions to origami in the future. In any case, we can hope that many more interesting results can be derived from the systematic study of the geometry of origami.

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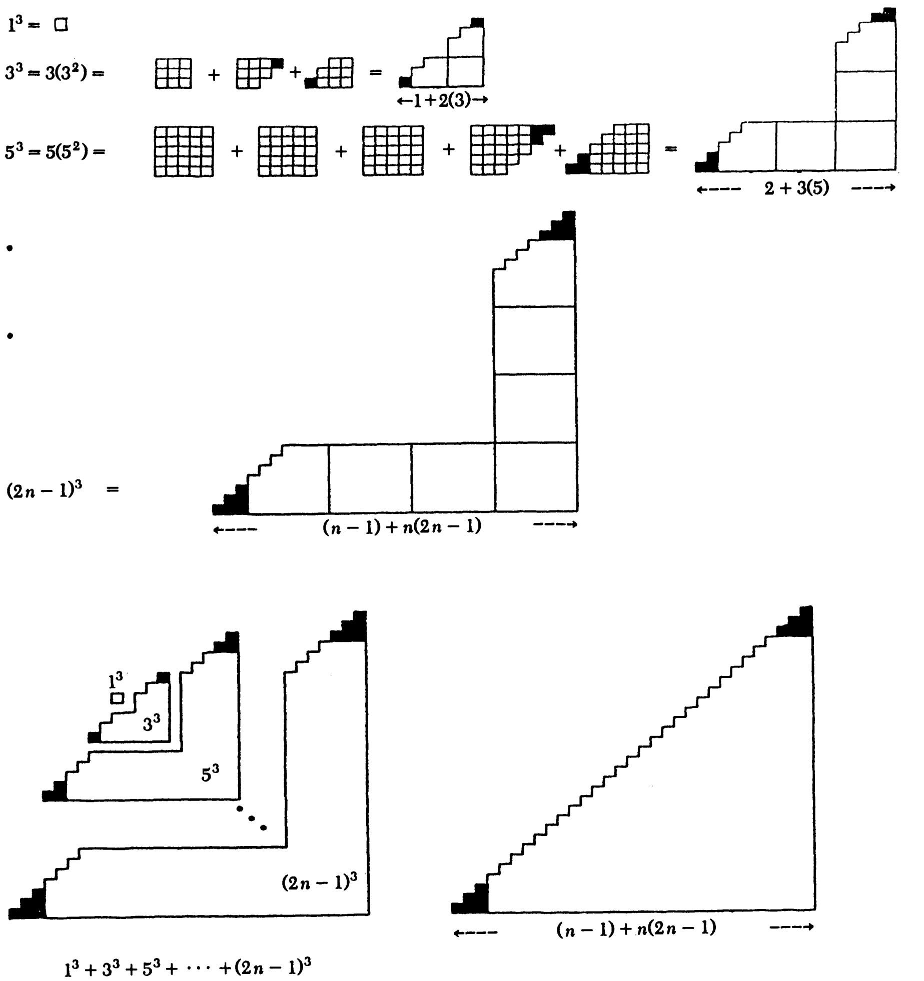
10. Postscript

After this paper was accepted for publication in this MAGAZINE, a paper titled "Reflections on a Mira" by John W. Emert, Kay I. Meeks, and Roger B. Nelsen appeared in the June-July 1994 edition of *The American Mathematical Monthly*. The

Mira is a semi-reflective geometric construction device, which allows constructions whose intrinsic geometry is essentially the same as that defined by origami methods. Some of the results of this paper are therefore quite similar to the results established there. Another interesting paper on the geometry of the mira is "Duplicating the Cube with a Mira" by George E. Martin, *Mathematics Teacher*, March 1979.

Proof without Words:
The Sum of Consecutive Odd Cubes is a Triangular Number



$$
\begin{aligned}
1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 &= 1 + 2 + 3 + \cdots + (2n^2 - 1) \\
&= \frac{(2n^2-1)(2n^2)}{2} \\
&= n^2(2n^2 - 1)
\end{aligned}
$$

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