Workbook of two-dimensional geometries:

Exploring the complete, simply connected two-dimensional geometries of constant curvature

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1 Introduction

As a young mathematician I was introduced to the classic *Leçons sur la Géométrie des Espaces de Riemann*, written by the great French geometer Élie Cartan. Early in his treatise on geometries in all dimensions, the author presents the case of two-dimensional geometries, in particular, those two-dimensional geometries that look the same at all points and in all directions. (For example, a cylinder looks the same at each of its points but not in all directions emanating from any one of its points, whereas a sphere looks the same at all points and in all directions.) It turns out that there is one and only one such geometry for each real number K, called the *curvature* of the geometry. The case $K = 0$ is the (flat) Euclidean geometry that you learned in high school .

These twenty-five pages of Cartan's book (Chapter VI, §i-v) so captivated me that I have returned to them regularly throughout my career and have adapted and taught them many times at the advanced undergraduate level. They form the basis for this little book. To me they tell one of the most beautiful and satisying stories in all of geometry, one which exemplifies a fundamental principle of all great mathematics, namely that, using the tools at hand but in a slightly novel way, the clouds part and one sees that objects and relationships that seemed so different are in fact parts of a single elegant story!

When $K > 0$ it turns out that the K-geometry is the geometry of the sphere of radius $R = 1/K^{1/2}$ that we can see as a subset of Euclidean three-space $\mathbb{R}^3$. But the geometries with $K < 0$ are not so easy to visualize. They are the so-called 'hyperbolic' geometries. In fact it took mathematicians a couple thousand years to realize that the existed at all! It turns out that the secret to understanding all the two-dimensional geometries, including the ones with $K < 0$, in a unified way is to simply rescale the third coordinate in $\mathbb{R}^3$ and, in these 'unusual' coordinates (x, y, z), to look at each two-dimensional geometry as the solution set to the equation

$$K\left(x^2 + y^2\right) + z^2 = 1.$$

However the idea of changing coordinates without changing the underlying geometry described by those coordinates is a challenging one that did not come into mathematics until a couple of centuries ago. It will require that, before we get into the beautiful uniform study of all two-dimensional geometries, we practice the coordinate change we are going to use, namely the rescaling of the third coordinate in Euclidean 3-space. That practice, together with a review of some concepts from several variable calculus and linear algebra, will comprise Part IV of this book.

It has often been said that "mathematics is not a spectator sport." This truism is very much in evidence in the writing of this book. It is written so as to guide you through the entire story, yet permit you, when possible, to construct the mathematical story for yourself, that is, to do some mathematics yourself rather than just observe it done by others. This 'doing mathematics oneself' takes the form of Exercises with enough help (Hints) provided so that the 'doing' is not so onerous as to get in the way of the story itself.

Strong evidence has been provided by students of mathematics over many centuries that such guided 'doing' is indispensible for understanding and retention. In fact the very form of this book, as a loose-leaf or electronic notebook, is intended to encourage you to write out (in correctable form) solutions to the Exercises that can be inserted at the appropriate places into the text.

This book supposes familiarity with several variable calculus and the linear algebra of matrices. In particular you will need to remember and apply the chain rule for differentiable functions of several variables, written in matrix notation. That is:

Theorem 1 ***(Chain Rule)*** *Given differentiable mappings*

$$\begin{array}{l} (y_1(x_1,\ldots,x_m),\ldots,y_n(x_1,\ldots,x_m)) \\ (z_1(y_1,\ldots,y_n),\ldots,z_p(y_1,\ldots,y_n)) \end{array}$$

then

$$\left(\frac{\partial z_k}{\partial x_i}\right) = \left(\frac{\partial z_k}{\partial y_j}\right) \cdot \left(\frac{\partial y_j}{\partial x_i}\right).$$

As a help, at some points in the text and in some of the Exercises, a more complete treatment of a particular topic can be found in one of the following two texts:

[MJG]: Greenberg, Marvin Jay. *Euclidean and Non-Euclidean Geometry: Development and History.* W.H. Freeman & Co. 3rd Ed., 1994.

[DS]: Davis, H. and Snider, A.D. *Introduction to Vector Analysis.* Wm. C. Brown Publishers, 7th Ed. 1994.

The corresponding topics in these texts are referenced. For example, [MJG,311] refers to page 311 in the Greenberg book and [DS,59ff] refers to page 59 and those pages just following page 59 in the Davis-Snider book.

Some final remarks about notation in this book. The letters '**EG**' will always mean Euclidean (usually plane but occasionally 3-dimensional) Geometry, the letters '**SG**' will always mean Spherical Geometry, and the letters '**HG**' will always mean Hyperbolic Geometry. One further kind of geometry, which we call Neutral Geometry, will be explained in the book and denoted by '**NG**.'

Also, it will often be useful to consider a vector, for example, $V = (a, b, c)$, as a 1×3 matrix and we will write

$$(V) = \begin{pmatrix} a & b & c \end{pmatrix}$$

or as a 3×1 matrix and we will write

$$(V)^t = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This will allow us, for example, to write the scalar product of two vectors

$$\begin{aligned} V \bullet V' &= (a, b, c) \bullet (a', b', c') \\ &= aa' + bb' + cc' \end{aligned}$$

as a product of matrices

$$V \cdot V' = \begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}.$$

It is my hope and intention in writing this little book that you engage with and enjoy this uniform way of understanding all two-dimensional geometries as much as I did!

Remark 2 *Special message to current or future teachers of high school geometry: Parts II and III of this book are especially relevant to your teaching of the subject. Look especially closely at the treatment of congruence (rigid motion), similarity (dilation), circles, expressing geometric properties with equations, and geometric measurement and dimension, and compare them with the high school geometry sections of the Common Core State Standards in Mathematics. The latter can be found at:*

http://www.corestandards.org/Math/Content/HSG/introduction.

A useful companion course to one based on this book, one that might be called *Geometry for Teaching*, would explicitly make the connections between the material covered as in this book and what you do (or will do) in your high school geometry classroom. The idea is *not* that the material we will cover in Parts II and III will tell you how to teach that material but rather that the treatment given here will give you the depth and breadth of geometric understanding that will allow you to design what you teach and bring it into your classroom in ways that those who lack that understanding cannot.

Remark 3 *This book can also be used as a bridge to a first course in Riemannian geometry. It treats the case of two-dimensional geometries that are homogeneous, that is, that look the same at all their points. But to treat these geometries efficiently, we introduce the notion of changing coordinates for the geometry without changing the geometry itself. It is that notion that allowed geometers to treat surfaces and higher-dimensional smooth spaces that look different at different points, ones that can often not be treated at all their points using a single set of coordinates.*

Part I
Neutral geometry

2 Euclid's postulates for plane geometry

2.1 Neutral geometry

We first turn our attention to plane (or 'flat') two-dimensional geometry.

In Western civilization, the primary source of our understanding of this geometry comes from Euclid's *Elements*. The treatise is of transcendant importance well beyond geometry itself, because it is among the first, and perhaps the most influential single example of organized, formal logical deductive reasoning. Certain fundamentals, that are called *axioms*, are postulated or 'given,' providing the platform on which a 'geometry' is built, that is, a mathematical entity modeling a physical 'reality'. Its properties are arrived at by applying the laws of logic to the given fundamentals. Euclid gives five axioms for plane geometry, the first four of which seem to be 'obvious' reflections of physical reality. In paraphrased form, they are:

Axiom 4 *(E1) Through any point P and any other point Q, there lies a unique line.*

Axiom 5 *(E2) Given any two segments $\overline{AB}$ and $\overline{CD}$, there is a segment $\overline{AE}$ such that B lies on $\overline{AE}$ and $|CD| = |BE|$*

(NB: In plane geometry we often use the notation $|CD|$ to denote the distance between two points A and B rather than the notation $d(A, B)$ used previously.

Axiom 6 *(E3) Given and point P and any positive real number r, there exists a (unique) circle of radius r and center P. (Said another way, if you move away from P along a line in any direction, you will encounter a unique point at distance r from P.)*

Axiom 7 *(E4) All right angles are congruent. (A right angle is defined as follows. Let C be the midpoint on the segment $\overline{AB}$. Let E be any point not equal to C. The angle $\angle ACE$ is called a right angle if $\angle ACE$ is congruent to $\angle BCE$.) [MJG,17-18]*

Definition 8 *If we are only given E1-E4, we will call our geometry **Neutral Geometry (NG)**.*

Definition 9 *In **NG**, two distinct lines are called parallel if and only if they don't intersect.*

One implicit assumption of two-dimensional Neutral (and Euclidean) geometry is the existence of (a group of) rigid motions or congruences. That is, it is assumed that given any point $\hat{A}$ and any tangent vector $\hat{V}$ emanating from

$\hat{A}$ and given any second point $\hat{B}$ in the geometry and any tangent vector $\hat{W}$ emanating from $\hat{B}$, then there is a transformation $\hat{M}$ of the geometry such that

1) $\hat{M}$ takes $\hat{A}$ to $\hat{B}$,

2) $\hat{M}$ takes $\hat{V}$ and to a positive scalar multiple times $\hat{W}$ to $\hat{M}\left(\hat{V}\right)$,

3) for all points $\hat{A}', \hat{A}''$ in the geometry, $\hat{M}$ leaves the distance between them unchanged, that is,

$$\left|\hat{M}\left(\hat{A}'\right)\hat{M}\left(\hat{A}''\right)\right| = \left|\hat{A}'\hat{A}''\right|,$$

4) for any two tangent vectors $\hat{V}'$ and $\hat{V}''$ emanating from $\hat{A}$, the angle between $\hat{M}\left(\hat{V}'\right)$ and $\hat{M}\left(\hat{V}''\right)$ is the same as the angle between $\hat{V}'$ and $\hat{V}''$.

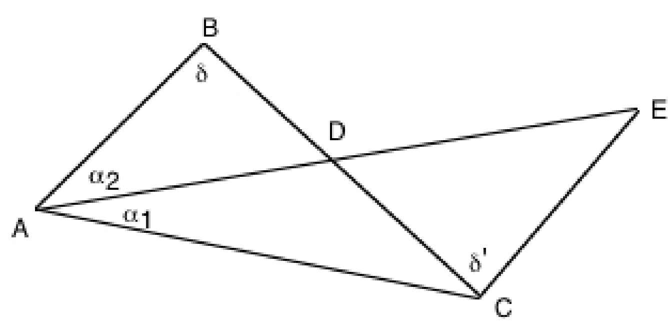
Exercise 10 *Using a sketch on grid paper or an algebraic formultation in the Euclidean plane, give a concrete example of a rigid motion that takes* $(1,2)$ *to* $(3,5)$ *and the tangent vector* $(1,0)$ *emanating from* $(1,2)$ *to a positive multiple of the tangent vector* $(0,2)$ *emanating from* $(3,5)$.

Exercise 11 *(**NG**) Think back to high school days and write the congruence rules SSS, SAS, and ASA. Be very careful with your wording–it had better be that triangles can be moved onto each other by a rigid motion if and only if they satisfy any one (and hence all) of the three properties(SSS, SAS, ASA).*

Exercise 12 *Give a counterexample to show that there is no universal SSA law.*

Although it is a bit tedious to show (and we will not ask you to do it here), using only E1-E4 you can derive the usual rules for congruent triangles (SSS, SAS, ASA). Thus these laws hold in any neutral geometry, that is, in any geometry satisfying E1-E4.

Exercise 13 *(**NG**) Suppose, in the diagram below that* $|BD| = |CD|$ *and* $|AD| = |ED|$.



Show that triangle $\triangle BDA$ *and triangle* $\triangle CDE$ *are congruent. [MJG,138]*

Exercise 14 *a) Show in Neutral Geometry that, for $\triangle ABC$ as in Exercise 13, the exterior angle of the triangle at C is greater than either remote interior angle. [MJG,119]*

b) Use a) to show that the sum of any two angles of a triangle is less than $180°$.

Exercise 15 *(**NG**) Show in **NG** that, if two lines cut by a transversal line have a pair of congruent alternate interior angles, then they are parallel. [MJG,117]*

Hint: Suppose the assertion is false for some pair of lines. Find a triangle that violates the conclusion of Exercise 14a).

2.2 Sum of angles in a triangle in NG

We will now reason to one of the fundamental results about Neutral Geometry, one that puzzled mathematicians for many centuries, in fact, until the discovery of hyperbolic geometry about two centuries ago. That discovery showed that there was more than one geometry that satisfied the axioms of Neutral Geometry, and that attempts to show that Euclid's fifth axiom (below) was a consequence of the four axioms of Neutral Geometry were futile. There was another geometry, namely Hyperbolic Geometry, that satisfied E1-E4. The thing that separates Hyperbolic Geometry from Euclidean (plane) Geometry is the sum of the angles in a triangle. (If you had a good geometry course in high school, you may remember that you had to use Euclid's fifth axiom in order to show that the sum of the angles in a triagle was $180°$. But more on that later.)

There is one important fact about the sum of the angles in a triangle that you *can* prove in **NG**, that is, without invoking Euclid's fifth Axiom. We will in fact accomplish that in this section.

Exercise 16 *(**NG**) For the diagram in Exercise 13, show that the sum of the angles in $\triangle ACE$ is the same as the sum of the angles in $\triangle ACB$*

Exercise 17 *(**NG**) Suppose that there is a triangle $\triangle ABC$ in **NG** for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)^o$ with $x > 0$. For the $\triangle ABC$ in Exercise 13, show that one of the angles of $\triangle ACE$ is no more than half the size of $\angle BAC$. Yet by Exercise 16 the sum of the angles in a triangle $\triangle ABC$ is still $(180 + x)^o$.*

Hint: In the diagram in Exercise 13, this new 'smaller' angle may or may not have vertex A. [MJG,125-127]

Exercise 18 *(**NG**) Suppose that there is a triangle $\triangle ABC$ in **NG** for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)^o$ with $x > 0$. Let α denote the measure of $\angle BAC$. Repeat the construction in Exercise 17 over and over again n-times to construct a triangle with the sum of its angles still equal to $(180 + x)^o$ but such that one of its angles has size less than*

$$\frac{1}{2^n}\alpha.$$

Exercise 19 *(**NG**) Suppose that there is a triangle $\triangle ABC$ in **NG** for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)^o$ with $x > 0$. Show that there is a positive integer n so that, if you repeat the construction in Exercise 17 over and over again n-times, the result will be a triangle with the sum of its angles still equal to $(180 + x)^o$ but with one of its angles having measure less than x. [MJG,125-127]*

Exercise 20 *(**NG**) On the other hand, by Exercise 14b), you cannot have a triangle with two angles summing to more than 180^o. [MJG,124] Use this fact to conclude the following theorem:*

Theorem 21 *In **NG**, the sum of the interior angles in any triangle is no greater than* 180^o*.*

Proof. We argue by contradiction. Start with $\triangle ABC$ as in Exercise 17 for which the sum of the angles in a triangle $\triangle ABC$ is $(180 + x)^{\,o}$ with $x > 0$. Suppose the measure of the angle at A is denoted by α. By Exercise 17 there exists a triangle $\triangle A^{(1)}B^{(1)}C^{(1)}$ such that the sum of the angles in a triangle $\triangle A^{(1)}B^{(1)}C^{(1)}$ is $(180 + x)^{\,o}$ and the measure of the angle at the vertex $A^{(1)}$ is less than or equal to $\frac{\alpha}{2}$. By Exercise 18 there is a triangle $\triangle A^{(n)}B^{(n)}C^{(n)}$ such that the sum of the angles in a triangle $\triangle A^{(n)}B^{(n)}C^{(n)}$ is $(180 + x)^{\,o}$ and the measure α_n of the angle at the vertex $A^{(n)}$ is less than or equal to $\frac{\alpha}{2^n}$. If n is sufficiently big,

$$\frac{\alpha}{2^n} < x.$$

So, for that value of n, if β_n is the measure of the angle of $\triangle A^{(n)}B^{(n)}C^{(n)}$ at the vertex $B^{(n)}$ and γ_n is the measure of the angle of $\triangle A^{(n)}B^{(n)}C^{(n)}$ at the vertex $C^{(n)}$, then we have the two relations

$$\alpha_n < \frac{\alpha}{2^n} < x$$
$$\alpha_n + \beta_n + \gamma_n = 180 + x.$$

So

$$\beta_n + \gamma_n = 180 + (x - \alpha_n) > 180.$$

But this is impossible by Exercise 14b). ■

Exercise 22 *(**NG**) Show the following:*

a) The sum of the interior angles in any quadrilateral is no greater than $360°$*.*

b) The sum of the interior angles of an n*-gon is no greater than* $(n-2)\cdot 180^o$*.*

Part II
Euclidean (plane) geometry

3 Rectangles and cartesian coordinates

3.1 Euclid's Axiom 5, the Parallel Postulate

We are finally ready to introduce Euclid's fifth and final axiom, the so-called *Parallel Postulate*.

Axiom 23 *(E5) Through a point not on a line there passes a unique parallel line.*

NG together with E5 is called Euclidean geometry (**EG**). As mentioned above, we will see later that there is another geometry (**HG**) that satisfies all the postulates of **NG** but not E5. In it, the sum of the interior angles of a triangle will *always* be less than 180°! [MJG,134]

Exercise 24 ***(EG)*** *a) Show that, if two parallel lines are cut by a transversal line, opposite interior angles are equal.*

We will call the set of points on a line which lie on one side of a given point a *ray*. We call the given point the *origin* of the ray.

We call two rays in the plane parallel if they lie on parallel lines and they both lie on the same side of the transversal line passing through their origins.

Strictly speaking, an angle in the plane is the union of two ordered rays with common origin and choice of one of the two connected regions into which the union of the rays divides the plane. We often denote angles by $\angle BAC$ where A is the common origin and B a point along one of the rays, called the initial ray, and C is a point along the other ray, called the final ray. The choice of the region is either clear from the context or explicitly given.

Exercise 25 *a) Given an angle $\angle BAC$ show by drawings the two regions into which it divides the plane. Show how the (signed) measure of the angle depends on which region you pick and on which is the initial ray and which is the final ray of the angle.*

b) Show that angles $\angle BAC$ and $\angle B'A'C'$ in the Euclidean plane are either equal (i.e. have the same measure) if corresponding rays are parallel.

c) Show that $\angle BAC$ and $\angle B'A'C'$ are equal (i.e. have the same measure) if $\angle B'A'C'$ can be rotated around A' to obtain an angle $\angle B''A'C''$ with corresponding rays parallel to those of $\angle BAC$.

Exercise 26 ***(EG)*** *Use the 'uniqueness' assertion in E5 together with what we have established about Neutral Geometry to show that in **EG** the sum of the interior angles of any triangle is 180^o.*

Exercise 27 *(**EG**) Show that in **EG** the sum of the interior angles of a quadrilateral is* 360^o.

Exercise 28 *(**EG**) Show in **EG** that, given any positive real numbers* a *and* b*, there exist rectangles with adjacent side of lengths* a *and* b.

Hint: To show that opposite sides are of equal length, suppose not. For example, suppose the top of the rectangle has length a' *and the bottom has length* a *and, for example,* $a' > a$*. Mark the point at length* a *along the top, starting at the left-hand vertex. Connect tht point to the right-hand bottom vertex. Find a triangle that contradicts Exercise 14a).*

Exercise 29 *(**EG**) Show that there is a cartesian coordinate system on **EG**, that is, the set of points of **EG** are in* $1-1$ *correspondence with the set of pairs of real numbers.*

3.2 The distance formula in EG

It is the existence of a cartesian coordinate system in **EG** that allows us to define distance between points

$$d\left(\left(a_1,b_1\right),\left(a_2,b_2\right)\right)=\sqrt{\left(a_2-a_1\right)^2+\left(b_2-b_1\right)^2} \tag{1}$$

and so gives rigorous mathematical meaning to a concept that the ancient Greeks were never able to describe precisely, namely the similarity of figures in **EG**. For that we will require the notion of a dilation or magnification in **EG**. And we need a cartesian coordinate system to describe dilation precisely, a reality backed up by the fact that similarities do not exist in **HG** or **SG**. (Try drawing two triangles that are similar but not congruent on a perfectly spherical balloon!)

Exercise 30 *(**EG**) State the Pythagorean theorem in **EG** and use Exercise 28 to prove it.*

Hint: In the cartesian plane, construct a square with vertices $(0,0)$, $(a+b,0)$, $(0,a+b)$ *and* $(a+b,a+b)$. *Inside that square, construct the square with vertices* $(a,0)$, $(a+b,a)$, $(b,a+b)$ *and* $(0,b)$. *Show that the area of the big square is the area of the little square plus the area of* 4 *right triangles, each of area* $\frac{ab}{2}$.

Exercise 31 *Use the Pythagorean theorem to justify the definition* (1).

3.3 Dilations in EG

Definition 32 *(**EG**) A **dilation** is a one-to-one onto transformation of the cartesian plane to itself that*

a) fixes one point called the center of the dilation,

b) takes each line through the fixpoint to itself,

c) multiples all distances by a fixed positive real number called the magnification factor of the dilation.

Definition 33 *Given a point (x_0, y_0) in the (cartesian) plane and a positive real number r, we define a mapping D with center (x_0, y_0) and magnification factor r by the formula*

$$D(x, y) = (x_0, y_0) + r(x - x_0, y - y_0). \tag{2}$$

We will also denote the output $D(x, y)$ of the dilation as $\left(\underline{x}, \underline{y}\right)$.

Exercise 34 *(**EG**) Using cartesian coordinates for the plane, show that the mapping D defined in (2) is a dilation with magnification factor r and center (x_0, y_0).*

Exercise 35 *(**EG**) Show (using several-variable calculus if you wish) that a dilation with magnification factor r multiplies all areas by a factor of r^2.*

Exercise 36 *(**EG**) a) Show that the inverse mapping of a dilation is again a dilation with the same center but with magnification factor r^{-1}.*

b) Show that a dilation takes lines to lines.

Hint: For a) solve for (x, y) in terms of $\left(\underline{x}, \underline{y}\right)$. For b) write the equation

$$ax + by = c$$

for the given line. Then substitute for (x, y) its expression in terms of $\left(\underline{x}, \underline{y}\right)$.

Exercise 37 *(**EG**) Show that a dilation takes any line to a line parallel (or equal) to itself.*

Hint: Compute slopes.

Exercise 38 *(**EG**) Show that a dilation by a factor of r takes any vector to r times itself.*

Hint: Realize the vector as the difference of two points.

Exercise 39 *(**EG**) Show that a dilation of the plane preserves angles.*

Hint: Use the dot product of vectors emanating from the same point to measure angles

3.4 Similarity in EG

Definition 40 *(**EG**) Two triangles are **similar** if there is a dilation of the plane that takes one to a triangle which is congruent to the other. We write*

$$\triangle ABC \sim \triangle A'B'C'$$

to denote that these two triangles are similar (where the order of the vertices tells us which vertices correspond).

Exercise 41 *(**EG**) a) Show that, if two triangles are similar, then corresponding sides are proportional.*

Hint: You have to start from the supposition that the two triangles satisfy the definition of similar triangles.

b) Show that, if corresponding sides of two triangles are proportional, then the two triangles are similar.

Hint: You have to start from the supposition that corresponding sides of the two triangles are proportional and use SSS to show that there is a dilation of $\triangle ABC$ *is congruent to* $\triangle A'B'C'$.

Exercise 42 *(**EG**) a) Show that, if two triangles are similar, then corresponding angles are equal.*

Hint: You have to start from the supposition that the two triangles satisfy the definition of similar triangles.

b) Show that, if corresponding angles of two triangles are equal, then the two triangles are similar.

Hint: You have to start from the supposition that corresponding angles of the two triangles are equal, then use a dilation with $r = |A'B'|/|AB|$ *and ASA to show that the dilation of one triangle that is congruent to the other.*

Exercise 43 *(**EG**) Show that two triangles are similar if corresponding sides are parallel.*

Hint: Use Exercise 24b).

Exercise 44 *Show that two triangles are similar if corresponding sides are perpendicular.*

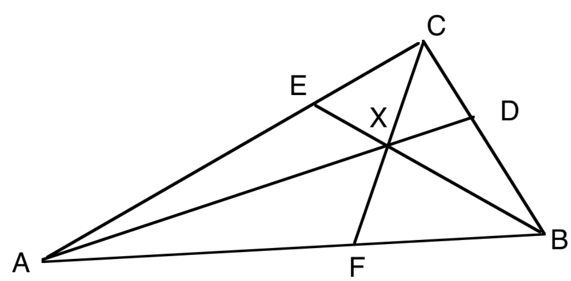
Hint: Extend one of the rays of the first angle until it crosses the corresponding ray of the second angle.

3.5 Concurrence theorems in EG, Ceva's theorem

Before leaving (plane) Euclidean Geometry, we will visit two more of its many sets of memorable properties, one's that you may or may not have seen in high school. The first of these comes under the name of concurrence theorems–these theorems relate the measures of the three sides (or angles) of a triangle to the measure of quantities constructed from those sides by some uniform rule.

Exercise 45 *(**EG**) Denote the measure or area of a triangle $\triangle ABC$ as $|\triangle AFC|$. Show that, in the diagram below,*

$$\frac{|AF|}{|FB|} = \frac{|\triangle AFC|}{|\triangle CFB|} = \frac{|\triangle AFX|}{|\triangle XFB|}.$$



Exercise 46 *(**EG**) Use Exercise 45 to show by pure algebra that*

$$\frac{|AF|}{|FB|} = \frac{|\triangle AXC|}{|\triangle CXB|}. \tag{3}$$

Exercise 47 *(**EG**) For three concurrent segments $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ as given in Exercise 45, use Exercise 46 to show that*

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Hint: Use (3), (3) *with side $\overline{BC}$ replacing $\overline{AB}$,and* (3) *with side $\overline{CA}$ replacing $\overline{AB}$.*

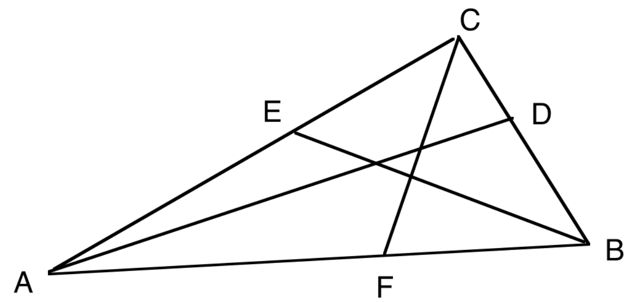
This last result with its converse, which you will show in the next Exercise, is called *Ceva's Theorem.* [MJG,287-288]

Exercise 48 *(**EG**) Show the converse of the result in Exercise 47, namely that, if*

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1$$

then the lines AD, BE, and CF pass through a common point.

Hint: Suppose that they do not pass through a common point.



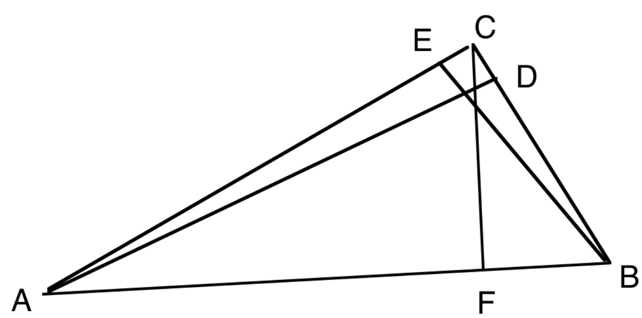
Notice that if, for example, F moves along the segment $\overline{AB}$, then $\frac{|AF|}{|FB|}$ is a strictly increasing function of $|AF|$. Now use Exercise 47 to determine a position F' for F along the segment $\overline{AB}$ at which

$$\frac{|AF'|}{|F'B|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1.$$

Exercise 49 ***(EG)*** *A* ***median*** *of a triangle is a line segment from a vertex to the midpoint of the opposite side. Show that the medians of any triangle meet in a common point.*

Hint: Use Ceva's Theorem.

Exercise 50 ***(EG)*** *Use Ceva's theorem to show that the three altitudes of a triangle are concurrent.*



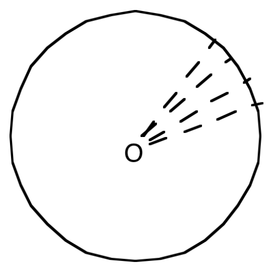
Hint: Use all three similarities of the form $\triangle CEB \sim \triangle CDA$ and then apply Ceva's theorem.

4 Properties of circles in EG

4.1 Basics

Our final topic before leaving Euclidean Geometry is circles. We include this partly for its own interest, and partly because the properites we visit here will be useful later on. Again we explore the topic through a sequence of Exercises (with Hints to their solutions to ease the way). We begin with perhaps the most basic fact of all about circles in **EG**.

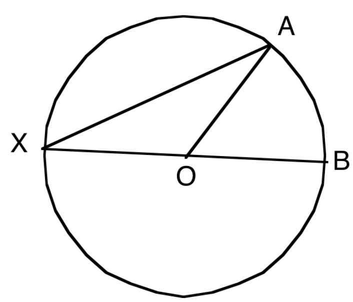
Exercise 51 *(**EG**) The circle of radius* 1 *has (interior) area* π*. Use this to reason to the fact that the circle of radius* 1 *has circumference* 2π*.*



Hint: Approximate a rectangle by rearranging the slices in the picture. Compute the area of the "rectangle."

Next we turn to some facts about chords in circles and angels inscribed in circles.

Exercise 52 *(**EG**) On the circle with center O below,*

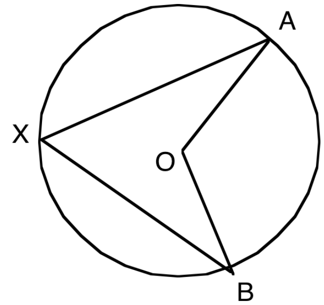


show that

$$\angle AXB = (1/2)(\angle AOB).$$

Hint: $\triangle OAX$ *is isosceles.*

Exercise 53 *(**EG**) On the circle with center O below,*

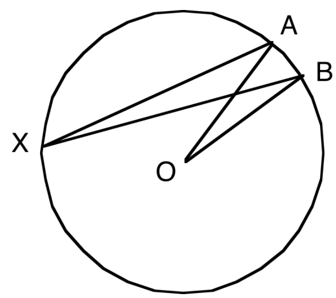


show that

$$\angle AXB = (1/2)(\angle AOB).$$

Hint: Draw the diameter through O and X and add.

Exercise 54 *(**EG**) On the circle with center O below,*



show that

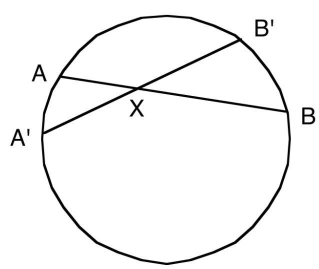
$$\angle AXB = (1/2)(\angle AOB).$$

Hint: Draw the diameter through O and X and subtract.

We can summarize the results of the last three exercises into the following Theorem.

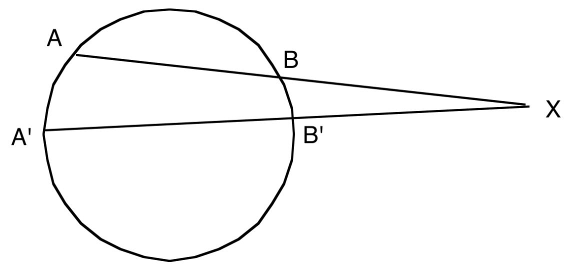
Theorem 55 *The measure of any angle inscribed in a circle is one-half of the measure of the corresponding central angle.*

Exercise 56 *(**EG**) Use similar triangles and the previous Exercises to show that $|AX|\cdot|XB| = |A'X|\cdot|XB'|$ in the figure below.*



Hint: Draw $\overline{AB'}$ and $\overline{A'B}$.

Exercise 57 *(**EG**) Use similar triangles and the previous Exercises to show that $|AX|{\cdot}|XB| = |A'X|{\cdot}|XB'|$ in the figure below.*



Hint: Draw $\overline{AB'}$ and $\overline{A'B}$.

Exercise 58 *Show that, given any three non-collinear points in the Euclidean plane, there is a unique circle passing through the three points.*

Hint: Show that the center of the circle must be the intersection of the perpendicular bisectors of any two of the sides of the triangle whose vertices are the three given points.

But how about four points in the plane, no three of which are collinear?

Exercise 59 *a) Draw four points in the Euclidean plane, no* 3 *of which are collinear, that cannot lie on a single circle.*

b) Draw four points in the Euclidean plane that do lie on a single circle.

The issue we will explore in the next two sections is the question of finding a numerical condition about the four points that tells us exactly when they all lie on a single circle. For that, we will need a very famous mathematical relationship, one very closely related to the notion of perspective in painting. That is, how do you faithfully render depth on a flat canvas? This relationship is called the *cross-ratio* of the four points.

4.2 Cross-ratio of points on the number line

We begin by studying the cross-ratio of four points on a line. Start with the set of points on the real number line with coordinate t and add one extra point called $t = \infty$. Call the resulting set $\overline{\mathbb{R}}$. You could think of the resulting set as the set of all lines through the origin in $\mathbb{R}^2$ by assigning to each line the real number that is its slope and to the y-axis the slope ∞.

Exercise 60 *a) Show that the transformation*

$$(x, y) \mapsto (\underline{x}, \underline{y}) = (x, y) \cdot \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

is a $1-1$*, onto (linear) transformation of* $\mathbb{R}^2$ *as long as*

$$\begin{vmatrix} d & b \\ c & a \end{vmatrix} \neq 0. \tag{4}$$

b) For the transformation in the previous Exercise, show that every line through the origin in (x, y)*-space is sent to a line through the origin in* $(\underline{x}, \underline{y})$*-space. The slope* t *of the line through* $(0, 0)$ *and* (x, y) *is of course* $t = \frac{y}{x}$*. What is the slope* $\underline{t}$ *of the line through* $(\underline{0}, \underline{0})$ *and* $(\underline{x}, \underline{y})$*? Show that*

$$\underline{t} = \frac{at + b}{ct + d} \tag{5}$$

Definition 61 *Functions* (5) *for which the condition* (4) *holds are called **linear fractional transformations**.*

Exercise 62 *Show that a linear fractional transformation*

$$\overline{\mathbb{R}} \to \overline{\mathbb{R}}$$
$$t \mapsto \underline{t} = \frac{at + b}{ct + d}$$

is $1 - 1$ *and onto. What is its inverse function? (Your answer should show that the inverse function is also a linear fractional transformation.)*

Hint: By algebra solve for t *in terms of* $\underline{t}$*. Then graph*

$$\underline{t} = \frac{at + b}{ct + d}$$

in the $(t, \underline{t})$*-plane. If* $c = 0$ *show that the graph is a straight line with non-zero slope and*

$$\infty \mapsto \infty.$$

If $c \neq 0$*, show that the graph has exactly one horizontal asymptote where* $t \mapsto \infty$ *and one vertical asymptote where* $\underline{t} \mapsto \infty$*.*

Exercise 63 *Show that the set of linear fractional transformations form a group under the operation of composition of functions. That is, check associativity, identity element and existence of inverses.*

Exercise 64 *Show that, for any three distinct points t_2, t_3 and t_4, the function of t given by the formula*

$$\underline{t} = \frac{t_3 - t_4}{t_3 - t_2}\frac{t - t_2}{t - t_4} = \frac{t - t_2}{t_3 - t_2} \div \frac{t - t_4}{t_3 - t_4}$$

takes t_2 to 0, takes t_3 to 1 and takes t_4 to ∞. Show that this function is a linear fractional transformation, that is, a function of the form (5) for which the condition (4) holds.

Exercise 65 *Show that any linear fractional transformation (5) that leaves 0, 1, and ∞ fixed is the identity map.*

Exercise 66 *Suppose that you are given a function (5) and four points t_1, t_2, t_3 and t_4. Let*

$$\underline{t_i} = \frac{at_i + b}{ct_i + d}$$

for $i = 1, 2, 3, 4$. Show that

$$\frac{\underline{t_1} - \underline{t_2}}{\underline{t_3} - \underline{t_2}} \div \frac{\underline{t_1} - \underline{t_4}}{\underline{t_3} - \underline{t_4}} = \frac{t_1 - t_2}{t_3 - t_2} \div \frac{t_1 - t_4}{t_3 - t_4}.$$

[MJG,288]

Hint: Just write out the formula for each side and do the high school algebra. There is a fancier way that uses that the set of linear fractional transformations form a group whose operation is composition. It goes like this. Use Exercise 64 to show that the inverse of the linear fractional transformation

$$t \mapsto \frac{t - t_2}{t_3 - t_2} \div \frac{t - t_4}{t_3 - t_4}$$

followed by

$$t \mapsto \underline{t}$$

and then followed by

$$t \mapsto \frac{t - \underline{t_2}}{\underline{t_3} - \underline{t_2}} \div \frac{t - \underline{t_4}}{\underline{t_3} - \underline{t_4}}$$

fixes 0, 1, and ∞ and so is the identity transformation by Exercise 65. So

$$t \mapsto \frac{t - t_2}{t_3 - t_2} \div \frac{t - t_4}{t_3 - t_4}$$

is the same transformation as

$$t \mapsto \frac{\underline{t} - \underline{t_2}}{\underline{t_3} - \underline{t_2}} \div \frac{\underline{t} - \underline{t_4}}{\underline{t_3} - \underline{t_4}}.$$

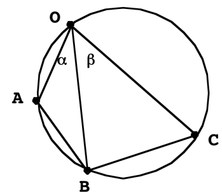
Definition 67 *The cross-ratio* $(t_1 : t_2 : t_3 : t_4)$ *of four (ordered) points* t_1, t_2, t_3 *and* t_4 *is defined by*

$$(t_1 : t_2 : t_3 : t_4) = \frac{t_1 - t_2}{t_3 - t_2} \div \frac{t_1 - t_4}{t_3 - t_4}.$$

Exercise 66 shows that if four points are moved by any function (5) the cross-ratio $\left(\underline{t_1} : \underline{t_2} : \underline{t_3} : \underline{t_4}\right)$ of the output four points is the same as the cross-ratio $(t_1 : t_2 : t_3 : t_4)$ of the original four points.

4.3 Cross-ratio of points on a circle

Exercise 68 *(**EG**) a) In the diagram*



show that

$$\frac{|AB|}{|CB|} = \frac{\sin\alpha}{\sin\beta} = \frac{\sin(\angle AOB)}{\sin(\angle COB)}.$$

Hint: Notice that by Theorem 55

$$m(\angle BAO) + m(OCB) = 180°$$

so that

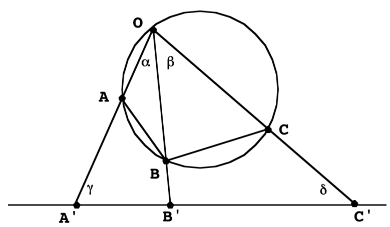
$$\sin(\angle BAO) = \sin(OCB).$$

Now use the Law of Sines.

Exercise 69 *(**EG**) Show that if, in the above figure, B moves along the circle to the other side of C, it is still true that*

$$\frac{|AB|}{|CB|} = \frac{\sin(\angle AOB)}{\sin(\angle COB)}$$

Exercise 70 *(**EG**) In the diagram*



(6)

show that

$$\frac{|A'B'|}{|C'B'|} = \frac{\sin\alpha}{\sin\beta} \div \frac{\sin\gamma}{\sin\delta} = \frac{\sin(\angle A'OB')}{\sin(\angle C'OB')} \div \frac{\sin(\angle B'A'O)}{\sin(\angle B'C'O)}.$$

[MJG,266-267]

Exercise 71 *(**EG**) Show that if, in the above figure, B' moves along the line to the other side of C', it is still true that*

$$\frac{|A'B'|}{|C'B'|} = \frac{\sin(\angle A'OB')}{\sin(\angle C'OB')} \div \frac{\sin(\angle B'A'O)}{\sin(\angle B'C'O)}.$$

These last two Exercises allow us to define the cross-ratio of four points on a circle.

Definition 72 *(**EG**) For a sequence of four (ordered) points A, B, C, and D on a circle, we define*

$$(A : B : C : D) = \frac{|AB|}{|CB|} \div \frac{|AD|}{|CD|}$$

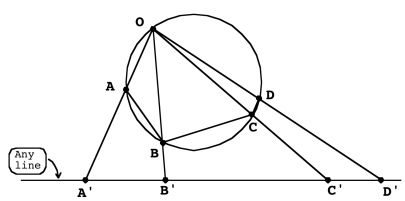
which we call the cross-ratio of the ordered sequence of the four points. Similarly for a sequence of four (ordered) points A', B', C', and D' on a line, we define

$$(A' : B' : C' : D') = \frac{|A'B'|}{|C'B'|} \div \frac{|A'D'|}{|C'D'|}$$

which we call the cross-ratio of the ordered sequence of the four points.

Notice that Definition 67 is just a refinement of the definition of $(A' : B' : C' : D')$ just above. In Definition 67 we are keeping track of the signs of the terms in the quotients whereas $(A' : B' : C' : D')$ is always non-negative.

Exercise 73 *a) Show that, in the figure*



we have the equality

$$(A : B : C : D) = (A' : B' : C' : D').$$

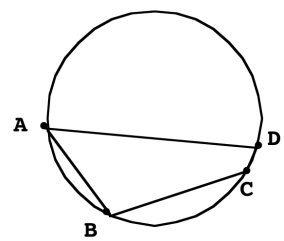
Hint: Use Exercises 68-71.
b) What happens in a) if we move B to the other side of C?

We say that "Cross-ratio is invariant under stereographic projection."

4.4 Ptolemy's Theorem

Given any three non-collinear points in the Euclidean plane, there is one and only one circle that passes through the three points. (How do you construct it?) You can easily convince yourself with a few examples that, given four non-collinear points A, B, C and D in the plane, it is not always true that there is a circle that passes through all four. A famous theorem of classical Euclidean geometry gives the condition that there is a circle that passes through all four.

Theorem 74 *(Ptolemy) If the ordered sequence of points A, B, C and D lies on a circle,*



then

$$|AC| \cdot |BD| = |AD| \cdot |BC| + |AB| \cdot |CD| \,.$$

That is, the product of the diagonals of the quadrilateral $ABCD$ is the sum of the products of pairs of opposite sides.

Proof. We need to check that

$$|AC| \cdot |BD| = |AD| \cdot |BC| + |AB| \cdot |CD|$$

or, what is the same, we need to check that

$$\frac{|AC| \cdot |BD|}{|AD| \cdot |BC|} = 1 + \frac{|AB| \cdot |CD|}{|AD| \cdot |BC|}.$$

That is, we need to check that

$$(A : C : B : D) = 1 + (A : B : C : D)\,.$$

But by Exercise 73 this is the same as checking that

$$(A' : C' : B' : D') = 1 + (A' : B' : C' : D')$$

for the projection of the four points onto a line from a point O on the circle. But that is the same thing as showing that

$$\frac{|A'C'| \cdot |B'D'|}{|A'D'| \cdot |B'C'|} = 1 + \frac{|A'B'| \cdot |C'D'|}{|A'D'| \cdot |B'C'|}$$

which is the same thing as showing that

$$|A'C'| \cdot |B'D'| = |A'D'| \cdot |B'C'| + |A'B'| \cdot |C'D'| \,.$$

Now check this last equality by high-school algebra. ■

Part III

Spherical Geometry I

5 Surface area and volume of the R-sphere in Euclidean 3-space

5.1 Volumes of pyramids

We are now going to study the geometry of the R-sphere in Euclidean 3-space. We will first study this geometry in the usual way, namely using $(\hat{x}, \hat{y}, \hat{z})$-coordinates. We start with a relationship that shows why there is a factor of 1/3 in many formulas for volumes in 3-dimensional Euclidean geometry.just like there is a factor of 1/2 in many formulas for areas in 2-dimensional Euclidean geometry.

Exercise 75 ***(EG)*** *Show that an $r \times r \times r$ cube can be constructed from three equal pyramids with an $r \times r$ square base. Conclude that the volume of each pyramid is* 1/3 *the volume of the cube, namely*

$$\frac{r^3}{3}.$$

Hint: Suppose the cube had a hollow interior and infinitely thin faces. Put your (infinitely tiny) eye at one vertex of the cube and look inside. How many faces of the cube can you see?

We next want to show why any pyramid with $r \times r$ square base and vertical altitude r has the same volume. That is, if we put the vertex of the pyramid anywhere in a plane parallel to the base and at distance r, the volume is unchanged.

This fact is an example of *Cavalieri's Principle*: Shearing a figure parallel to a fixed direction does not change the n-dimensional measure of an object in Euclidean n-space. (Think of a stack of (very thin) books.)

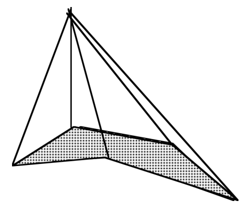
Exercise 76 *Show that Cavalieri's Principle is true for the pyramid using several variable calculus.*

Hint: Put the base of the pyramid P so that its vertices are $(0,0)$, $(r,0)$, $(0,r)$ and (r,r) in 3-dimensional Euclidean space. Consider the transformation

$$\left(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\right) = (\hat{x}, \hat{y}, \hat{z}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

and notice that

$$\int_{\underline{P}} d\underline{\hat{x}} d\underline{\hat{y}} d\underline{\hat{z}} = \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \right| \int_P d\hat{x} d\hat{y} d\hat{z}.$$



5.2 Magnification principle

Magnification principle: If an object in Euclidean n-space is magnified by factors of r_1,$\ldots, r_n$, its n-dimensional measure is multiplied by $r_1 \cdot \ldots . r_n$.

Exercise 77 ***(EG)*** *Use this magnification principle to justify the volume formula*

$$(1/3)B \cdot h$$

for any pyramid with rectangular base of area B and vertical altitude h.

Exercise 78 *Prove the magnification principle using several variable calculus.*
Hint: Consider the transformation

$$(\underline{\hat{x}_1}, \ldots, \underline{\hat{x}_n}) = (\hat{x}_1, \ldots, \hat{x}_n) \begin{pmatrix} r_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & r_n \end{pmatrix}$$

and notice that

$$\int_{\underline{P}} d\underline{\hat{x}_1} \ldots d\underline{\hat{x}_n} = r_1 \cdot \ldots \cdot r_n \int_P d\hat{x}_1 \ldots d\hat{x}_n.$$

Now suppose we have any pyramid with any shaped base of area B and any verticle altitude h. Approximate the base as close as you want (i.e ε-close) by tiling its interior with rectangles. Let t denote the sum of the areas of these rectangles. Approximate the base as close as you want (i.e ε-close) by covering it entirely with rectangles. Let T denote the sum of the areas of these rectangles. Why is the area B of the base of the pyramid caught between $B - \varepsilon$ and $B + \varepsilon$?

Exercise 79 ***(EG)*** *Show that the volume V of the pyramid is caught between $(1/3) \cdot t \cdot h$ and $(1/3) \cdot T \cdot h$.*

Exercise 80 ***(EG)*** *Argue that, given any positive real number ε, however small, the volume V of the pyramid is caught between $(1/3) \cdot (B - \varepsilon) \cdot h$ and $(1/3) \cdot (B + \varepsilon) \cdot h$.*

Exercise 81 ***(EG)*** *Show that*

$$V = (1/3) \cdot B \cdot h.$$

5.3 Relation between volume and surface area of a sphere

Think of a disco-ball. Its surface is approximately a sphere, but that surface is made up of tiny flat mirrors.

Exercise 82 ***(SG)*** *a) Why can you think of the disco-ball as being made up of pyramids, with each pyramid having base one of the tiny mirrors and vertex at the interior point O at the center of the disco-ball.*

b) Argue that the volume of the disco-ball is $(1/3)$ *times the distance h from a mirror to O times the sum of the areas of all the mirrors.*

Exercise 83 ***(SG)*** *Argue that, as the mirrors are made to be smaller and smaller,*

1) the sum of the areas of the mirrors approaches the surface area of a sphere,

2) the distance h approaches the radius R of that sphere,

3) the volume of the disco-ball approaches the volume of the sphere.

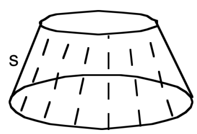
Conclude that, for a sphere of radius R in Euclidean 3*-space, the relation betwee the volume V of the sphere and the surface area S of the sphere is given by the formula*

$$V = \frac{R \cdot S}{3}.$$

Our goal in the next Subsection is to compute the surface area of the sphere of radius R in Euclidean 3-space. The formula just above will then let us compute the volume of the same sphere.

5.4 Surface area

To compute the surface area of the sphere of radius R in 3-dimensional Euclidean space, we will show that its surface area is equal to the surface area of something we can lay out flat. The argument for this goes way back to the great physicist and mathematician, Archimedes of Alexandria, in the 2nd century B.C. To follow his argument, we have to begin by computing the area of a 'lamp shade' or 'collar.' We think of a circular collar as in the figure below



as approximated by an arrangement of trapezoids. To achieve this, we approximate the bottom circle of the collar by an inscribed regular n-gon whose vertices are the points of intersection with the slant lines in the figure. Similarly approximate the top circle by an inscribed regular n-gon positioned directly above the bottom one, again with vertices given by the points of intersection with the slant lines. Complete a side of the bottom n-gon and the side of the top n-gon directly above it to a trapezoid by adjoining the two slant lines in the figure that connect endpoints. Let b_n denote the length of a side of the bottom regular n-gon and let t_n denote the length of a side of the top n-gon. Then the trapezoid has area

$$\left(\frac{b_n + t_n}{2}\right) \cdot h_n$$

where h_n is the vertical height of the trapeziod. The collar is approximated by the union of these n trapezoids, so the area of the collar is approximated by the sum of the areas of the n congruent trapezoids, namely

$$n \cdot \left(\frac{b_n + t_n}{2}\right) \cdot h_n = \left(\frac{n \cdot b_n + n \cdot t_n}{2}\right) \cdot h_n.$$

As n goes to infinity, the area of the approximation approaches the area of the collar. But

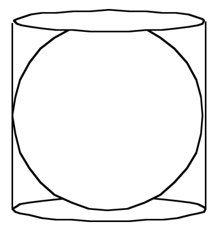
$$\lim_{n\to\infty} b_n = c_b$$
$$\lim_{n\to\infty} t_n = c_t$$
$$\lim_{n\to\infty} h_n = s$$

where c_b is the circumference of the bottom circle and c_t is the circumference of the top circle and s is the slant height of the collar as shown in the above figure. We conclude that the area of the collar is

$$\begin{aligned} \frac{c_b + c_t}{2} \cdot s &= \pi \cdot (r_b + r_t) \cdot s \qquad (7) \\ &= 2\pi \cdot r_a \cdot s \end{aligned}$$

where r_b and r_t are the radii of the respective circles and r_a is the average of the two.

Theorem 84 ***(SG)*** *The surface area of the sphere of radius R is the same as the surface area of the label of the smallest can into which the sphere will fit.*



Namely the surface area of the sphere of radius R is

$$2\pi R \cdot 2R = 4\pi R^2.$$

Exercise 85 ***(SG)*** *Show why the above Theorem is true.*

Hint: Slice the picture above into n horizontal slices. Approximate the piece of the surface of the sphere between the i-th pair of successive slices by a collar C_i. Let $a(C_i)$ denote the area of C_i, let r_i denote its average radius and let s_i denote its slant height. Then the surface area of the sphere is approximately

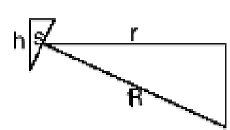
$$\sum_{i=1}^{n} 2\pi \cdot r_i \cdot s_i,$$

at least if the slices are pretty thin. (Why?) Also the approximate area $\sum_{i=1}^{n} a(C_i)$ approaches the exact surface area of the sphere as the slices get thinner and thinner.

Next let h_i denote the vertical height of the label on the can between the i-th pair of successive slices. The area of the label is exactly

$$\sum_{i=1}^{n} 2\pi \cdot h_i.$$

(Why?) Explain why the relationship between each r_i, s_i and h_i is given by the picture below.



Now use Exercise 43b) to explain why

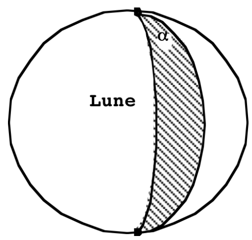
$$r_i \cdot s_i = h_i \cdot R.$$

Finally explain why this completes the proof of the Theorem.

6 Areas on spheres in Euclidean 3-space

6.1 Lunes

In the picture we have shaded in an ' α-lune' on the R-sphere in Euclidean 3-space.

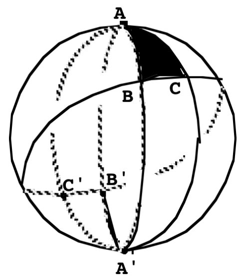


The lune has two vertices. They are at opposite (antipodal) points on the R-sphere, that is, the line in Euclidean 3-space that joins the two vertices runs through the center of the sphere. The angle at a vertex of the lune is α radians.

Exercise 86 *(**SG**) Explain why the area of the α-lune is $2\alpha{\cdot}R^2$.*

6.2 Spherical triangles

If a triangle on the sphere of radius R has interior angles with radian measures α, β, and γ, it can be covered three times by lunes as shown in the figure below.



Notice that each lune has one vertex at a vertex of the triangle and angle equal to that interior angle of the triangle. The other vertices of each lune are vertices of an 'opposite' triangle that has the same area as the given one since it is just the image of the given one under the rigid motion

$$\left(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\right) = (\hat{x}, \hat{y}, \hat{z}) \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(See, for example, the formula (38).) The three lunes cover the triangle three times. The three opposite lunes cover the opposite triangle three times. If you take all six lunes together, they cover each of the two triangles three times and everything else exactly once.

Exercise 87 *(**SG**) a) Show that the area of the spherical triangle is given by the formula*

$$R^2\left((\alpha+\beta+\gamma)-\pi\right),$$

that is,

$$|K|^{-1}\left((\alpha+\beta+\gamma)-\pi\right).$$

Hint: Use Exercise 86 to turn the sentence just preceding the Exercise into an equation.

b) Give a formula for the area of any spherical n-gon.

Hint: Divide the spherical n-gon into spherical triangles.

Part IV
Usual and unusual coordinates for three-dimensional Euclidean space

7 Euclidean three-space as a metric space

7.1 Points and vectors in Euclidean 3-space

In this book, we will study all the *two*-dimensional geometries (spheres, the plane and hyperbolic spaces). Each one of these geometries looks the same at each of its points and it also looks the same in every direction emanating from any of its points. But to study them all at the same time and in a uniform way we will need to visualize them all as different surfaces lying in some common *three*-dimensional space. We start with the most familiar cases, namely the spherical geometries.

For those we begin with three-dimensional Euclidean space

$$\mathbb{R}^3 = \{(\hat{x}, \hat{y}, \hat{z}) : \hat{x}, \hat{y}, \hat{z} \in \mathbb{R}\},$$

where there is a standard way to measure distance between two points

$$\begin{aligned} \hat{X}_1 &= (\hat{x}_1, \hat{y}_1, \hat{z}_1) \\ \hat{X}_2 &= (\hat{x}_2, \hat{y}_2, \hat{z}_2), \end{aligned}$$

namely

$$d\left(\hat{X}_1, \hat{X}_2\right) = \sqrt{(\hat{x}_2 - \hat{x}_1)^2 + (\hat{y}_2 - \hat{y}_1)^2 + (\hat{z}_2 - \hat{z}_1)^2}. \tag{8}$$

As we will see, the formula (8) is compatible with distances on such objects as spheres

$$\left\{(\hat{x}, \hat{y}, \hat{z}) \in \mathbb{R}^3 : \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2\right\}$$

of a fixed radius R, since these can be faithfully represented in ordinary Euclidean three-space. However, there is one disconcerting fact about studying the geometry of spheres in this way. Namely, as R approaches infinity, the geometry of the R-sphere at any point looks more and more like plane geometry, but on the other hand, that ‘limit’ plane geometry is ‘out at infinity.’ So in order to study all the two-dimensional geometries, including plane geometry and the hyperbolic geometries, in a uniform way we will have to *change* the coordinate system we use, or, what will turn out to be the same thing, we will have to change the distance formula slightly for each geometry. We will do that in later sections, but first we want to review some of the basic properties of ordinary Euclidean three-space you learned about it in calculus.

We write $(\hat{x}, \hat{y}, \hat{z})$ for our ordinary Euclidean coordinates. When you see $(\hat{x}, \hat{y}, \hat{z})$ in what follows, that means that distance between points is measured by the formula (8). One more thing–in Euclidean three-space it will be important thoughout to make the distinction between **points** and **vectors**: Although each will be represented by a triple of real numbers we will use

$$\hat{X} = (\hat{x}, \hat{y}, \hat{z})$$

to denote **points**, that is, **position** in Euclidean 3-space, and

$$\hat{V} = \left(\hat{a}, \hat{b}, \hat{c}\right)$$

to denote **vectors**, that is, **displacement** by which we mean the amount and direction a given point is being moved. So vectors always indicate *motion* from an explicit (or implicit) *point* of reference.

7.2 Dot product of vectors emanating from a point

There are various operations we can perform on one or more vectors when we think of them as emanating from the same point in Euclidean 3-space. The first is the dot product of two vectors.

Definition 88 *The dot product of two vectors*

$$\hat{V}_1 = \left(\hat{a}_1, \hat{b}_1, \hat{c}_1\right)$$
$$\hat{V}_2 = \left(\hat{a}_2, \hat{b}_2, \hat{c}_2\right)$$

emanating from the same point in 3-dimensional Euclidean space is defined as the real number given by the formula

$$\hat{a}_1\hat{a}_2 + \hat{b}_1\hat{b}_2 + \hat{c}_1\hat{c}_2$$

or in matrix notation as

$$\begin{pmatrix} \hat{a}_1 & \hat{b}_1 & \hat{c}_1 \end{pmatrix} \cdot \begin{pmatrix} \hat{a}_2 \\ \hat{b}_2 \\ \hat{c}_2 \end{pmatrix}.$$

It is also denoted as

$$\hat{V}_1 \bullet \hat{V}_2$$

or in matrix notation as

$$\hat{V}_1 \cdot \left(\hat{V}_2\right)^t.$$

Exercise 89 *Give the formula for the length* $\left|\hat{V}\right|$ *of a vector* $\hat{V} = \left(\hat{a}, \hat{b}, \hat{c}\right)$ *in 3-dimensional Euclidean space in terms of dot product.*

Exercise 90 *As you work through the proof of the Law of Cosines in the following Lemma, construct a diagram or picture for each step.*

Lemma 91 *(Law of Cosines) The (smaller) angle* ϑ *between two vectors* $\hat{V}_1$ *and* $\hat{V}_2$ *emanating from* $O = (0,0,0)$ *satisfies the relation*

$$\left|\hat{V}_2 - \hat{V}_1\right|^2 = \left|\hat{V}_1\right|^2 + \left|\hat{V}_2\right|^2 - 2\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \cos\vartheta.$$

Proof. Without loss of generality we can assume that $\left|\hat{V}_1\right| \leq \left|\hat{V}_2\right|$. Consider the triangle with one side given by the interval from $O = (0,0,0)$ to the endpoint P_1 of $\hat{V}_1$, with a second side S_2 given by the interval from O to the endpoint P_2 of $\hat{V}_2$ and with the third side given by the interval joining P_1 and P_2. Let P be the point on S_2 so that the interval between P_1 and P is perpendicular to S_2. By the Pythagorean theorem

$$\begin{aligned}
|P_1P_2|^2 - |P_2P|^2 &= |PP_1|^2 \\
&= |OP_1|^2 - |OP|^2 \\
|P_1P_2|^2 &= |OP_1|^2 + \left(|P_2P|^2 - |OP|^2\right) \\
&= |OP_1|^2 + (|P_2P| + |OP|)(|P_2P| - |OP|) \\
&= |OP_1|^2 + |OP_2|(|P_2P| - |OP|) \\
&= |OP_1|^2 + |OP_2|(|OP_2| - 2|OP|)
\end{aligned}$$

But

$$|OP| = |OP_1| \cdot \cos\vartheta.$$

■

Exercise 92 *What can you say about the cosine of the larger of the two angles between two vectors $\hat{V}_1$ and $\hat{V}_2$, that is about $(360° - \vartheta)$?*

Lemma 93 *The angle ϑ between two vectors $\hat{V}_1$ and $\hat{V}_2$ emanating from the same point in Euclidean 3-space satisfies the relation*

$$\hat{V}_1 \bullet \hat{V}_2 = \left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \cos\vartheta. \tag{9}$$

[DS,30ff]

Proof. Multipying out using the definition and algebraic properties of dot product,

$$\begin{aligned}
\left|\hat{V}_2 - \hat{V}_1\right|^2 &= \left(\hat{V}_2 - \hat{V}_1\right) \bullet \left(\hat{V}_2 - \hat{V}_1\right) \\
&= \left|\hat{V}_1\right|^2 + \left|\hat{V}_2\right|^2 - 2\left(\hat{V}_1 \bullet \hat{V}_2\right).
\end{aligned}$$

Now apply Lemma 91. ■

The significance of Lemma 93 is that the measure of angles between vectors depends only on the definition of the dot product.

Corollary 94 *The formula for the angle ϑ between two vectors $\hat{V}_1 = \left(\hat{a}_1, \hat{b}_1, \hat{c}_1\right)$ and $\hat{V}_2 = \left(\hat{a}_2, \hat{b}_2, \hat{c}_2\right)$ in 3-dimensional Euclidean space depends only on the dot products of the two vectors with themselves and with each other. Namely*

$$\vartheta = \arccos\left(\frac{\hat{V}_1 \bullet \hat{V}_2}{\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right|}\right).$$

In fact it is also true that the formula for the area of the parallelogram determined by two vectors $\hat{V}_1$ and $\hat{V}_2$ depends only on the dot products of the two vectors with themselves and with each other. You will see this by answering the following Exercises.

Exercise 95 *Show that the area of the parallelogram determined by $\hat{V}_1$ and $\hat{V}_2$ emanating from the same point in Euclidean 3-space is given by*

$$\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \sin\vartheta. \tag{10}$$

Exercise 96 *Show that the area of the parallelogram determined by $\hat{V}_1$ and $\hat{V}_2$ emanating from the same point in Euclidean 3-space is also given by*

$$\sqrt{\left|\begin{array}{cc} \hat{V}_1 \bullet \hat{V}_1 & \hat{V}_2 \bullet \hat{V}_1 \\ \hat{V}_1 \bullet \hat{V}_2 & \hat{V}_2 \bullet \hat{V}_2 \end{array}\right|}.$$

Hint: Start from the square of (10)*, substitute* $\left(1-\cos^2\vartheta\right)$ *for* $\sin^2\vartheta$*, and use* (9)*. Alternatively start from* (10) *and show that*

$$\sin\left(\arccos\left(\frac{\hat{V}_1 \bullet \hat{V}_2}{\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right|}\right)\right) = \frac{\sqrt{\left|\begin{array}{cc} \hat{V}_1 \bullet \hat{V}_1 & \hat{V}_2 \bullet \hat{V}_1 \\ \hat{V}_1 \bullet \hat{V}_2 & \hat{V}_2 \bullet \hat{V}_2 \end{array}\right|}}{\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right|}.$$

Exercise 97 *Show that we have the following equality of matrices*

$$\left(\begin{array}{cc} \hat{V}_1 \bullet \hat{V}_1 & \hat{V}_2 \bullet \hat{V}_1 \\ \hat{V}_1 \bullet \hat{V}_2 & \hat{V}_2 \bullet \hat{V}_2 \end{array}\right) = \left(\begin{array}{c} \left(\hat{V}_1\right) \\ \left(\hat{V}_2\right) \end{array}\right) \cdot \left(\begin{array}{cc} \left(\hat{V}_1\right)^t & \left(\hat{V}_2\right)^t \end{array}\right).$$

Again, the significance of Exercise 96 is that, to compute areas, we only need to know how to compute dot-products–the definition of the dot-product of the vectors completely determines the calculation of the area of the parallelogram they generate.

We finish this section with one other related fact.

Lemma 98 *The area of the parallelogram determined by two vectors* $\hat{V}_1 = \left(\hat{a}_1, \hat{b}_1, \hat{c}_1\right)$ *and* $\hat{V}_2 = \left(\hat{a}_2, \hat{b}_2, \hat{c}_2\right)$ *emanating from the same point in Euclidean 3-space is given by the length of the cross-product*

$$\hat{V}_1 \times \hat{V}_2 = \left(\left|\begin{array}{cc} \hat{b}_1 & \hat{c}_1 \\ \hat{b}_2 & \hat{c}_2 \end{array}\right|, \left|\begin{array}{cc} \hat{c}_1 & \hat{a}_1 \\ \hat{c}_2 & \hat{a}_2 \end{array}\right|, \left|\begin{array}{cc} \hat{a}_1 & \hat{b}_1 \\ \hat{a}_2 & \hat{b}_2 \end{array}\right|\right).$$

Proof. We use some facts from linear algebra. First of all, develop the determinant

$$\left|\begin{array}{c} \left(\hat{V}_1\right) \\ \left(\hat{V}_2\right) \\ \left(\hat{V}_1 \times \hat{V}_2\right) \end{array}\right|$$

along the third row. Then writing out both sides of the equation

$$\begin{vmatrix} \left(\hat{V}_1\right) \\ \left(\hat{V}_2\right) \\ \left(\hat{V}_1 \times \hat{V}_2\right) \end{vmatrix} = \left|\hat{V}_1 \times \hat{V}_2\right|^2 \tag{11}$$

we conclude that they are equal. On the other hand,

$$\begin{vmatrix} \left(\hat{V}_1\right) \\ \left(\hat{V}_2\right) \\ \left(\hat{V}_1\right) \end{vmatrix} = 0$$

and developing the left-hand determinant along the third row, we conclude that $\hat{V}_1$ is perpendicular to $\hat{V}_1 \times \hat{V}_2$. Similarly $\hat{V}_2$ is perpendicular to $\hat{V}_1 \times \hat{V}_2$. Finally, the absolute value of the determinant of a 3×3 matrix

$$\begin{vmatrix} \left(\hat{V}_1\right) \\ \left(\hat{V}_2\right) \\ \left(\hat{V}_1 \times \hat{V}_2\right) \end{vmatrix}$$

is the volume of the parallelepiped determined by the row vectors of the matrix. But that volume is

$$\left(\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \sin\vartheta\right) \cdot \left|\hat{V}_1 \times \hat{V}_2\right|$$

since $\left(\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \sin\vartheta\right)$ is the area of the base of the parallelepiped and $\hat{V}_1 \times \hat{V}_2$ is perpendicular to both V_1 and V_2. So using (11)

$$\left(\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \sin\vartheta\right) \cdot \left|\hat{V}_1 \times \hat{V}_2\right| = \begin{vmatrix} \left(\hat{V}_1\right) \\ \left(\hat{V}_2\right) \\ \left(\hat{V}_1 \times \hat{V}_2\right) \end{vmatrix} = \left|\hat{V}_1 \times \hat{V}_2\right|^2.$$

So

$$\left(\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \sin\vartheta\right) = \left|\hat{V}_1 \times \hat{V}_2\right|.$$

[DS,42-47] ■

7.3 Curves in Euclidean 3-space and vectors tangent to them

Definition 99 *A* ***smooth curve in 3-dimensional Euclidean space*** *is given by a differentiable mapping*

$$\begin{aligned} \hat{X} &: [b,e] \to \mathbb{R}^3 \\ t &\mapsto (\hat{x}(t), \hat{y}(t), \hat{z}(t)) \end{aligned}$$

from an interval $[b,e]$ on the real line. We shall sometimes use the notation

$$(\hat{x}(t), \hat{y}(t), \hat{z}(t)) = \hat{X}(t).$$

The mapping $\hat{X}(t)$ must have the additional property that the tangent vector

$$\left(\hat{a}(t), \hat{b}(t), \hat{c}(t)\right) = \left(\frac{d\hat{x}}{dt}, \frac{d\hat{y}}{dt}, \frac{d\hat{z}}{dt}\right) = \frac{d\hat{X}}{dt}$$

is not the zero vector for any t in $[b,e]$.

Exercise 100 *a) Give two examples of smooth curves,*

$$\begin{aligned} \hat{X}_1(s) &= (\hat{x}_1(s), \hat{y}_1(s), \hat{z}_1(s)) \\ \hat{X}_2(t) &= (\hat{x}_2(t), \hat{y}_2(t), \hat{z}_2(t)) \end{aligned}$$

neither of which is a straight line, in 3-dimensional Euclidean space. Do this so that the two curves pass through a common point and go in distinct tangent directions at that point. Please choose curves so that none of the coordinate functions of s or t is a constant function. [DS, 71ff]

b) Compute the tangent vectors of each of the two curves at each of their points.

c) For the two curves you defined in a), what are the coordinates of the point in Euclidean 3-space at which the two curves intersect?

d) Use the dot product formula to compute the angle ϑ between (the tangent vectors to) your two example curves in a) at the point at which the curves intersect. [DS,20-21]

Sometimes displacement is measured by showing how a given point is displaced, as in

$$\hat{V} = \hat{X}_2 - \hat{X}_1 = (\hat{x}_2 - \hat{x}_1, \hat{y}_2 - \hat{y}_1, \hat{z}_2 - \hat{z}_1),$$

and sometimes displacement is expressed as the instantaneous velocity of a point moving along a curve as in

$$\hat{V} = \frac{d\hat{X}(t)}{dt} = \left(\frac{d\hat{x}(t)}{dt}, \frac{d\hat{y}(t)}{dt}, \frac{d\hat{z}(t)}{dt}\right).$$

[DS,30ff].

In matrix notation we can think of

$$\hat{X}_2 - \hat{X}_1 = (\hat{x}_2 - \hat{x}_1, \hat{y}_2 - \hat{y}_1, \hat{z}_2 - \hat{z}_1)$$

as a 1×3 matrix $\left(\hat{X}_2 - \hat{X}_1\right)$. Then we can write the formula for the distance between two points $\hat{X}_1$ and $\hat{X}_2$ in Euclidean 3-space in terms of the dot-product

$$d\left(\hat{X}_1, \hat{X}_2\right) = \sqrt{\left(\hat{X}_2 - \hat{X}_1\right) \bullet \left(\hat{X}_2 - \hat{X}_1\right)} \tag{12}$$

or in terms of the matrix product

$$d\left(\hat{X}_1, \hat{X}_2\right) = \sqrt{\left(\left(\hat{X}_2 - \hat{X}_1\right)\right) \cdot \left(\left(\hat{X}_2 - \hat{X}_1\right)\right)^t}.$$

7.4 Length of a smooth curve in Euclidean 3-space

Exercise 101 *Compute the length of the tangent vector*

$$l(t) = \sqrt{\frac{d\hat{X}}{dt} \bullet \frac{d\hat{X}}{dt}}$$

to each of your two example curves in Exercise 100 at each of their points.

Definition 102 *The length L of the curve $\hat{X}(t)$, $t \in [b, e]$, in Euclidean 3-space is obtained by integrating the length of the tangent vector to the curve, that is,*

$$L = \int_b^e l(t)\, dt.$$

[DS,82] Notice that the length of any curve only depends on the definition of the dot-product. That is, if we know the formula for the dot-product, we know (the formula for) the length of any curve.

Our first example is the path

$$\begin{aligned} (\hat{x}(t), \hat{y}(t), \hat{z}(t)) &= (R \cdot sin(t), 0, R \cdot cos(t)) \\ 0 &\leq t \leq \pi. \end{aligned} \tag{13}$$

Notice that this path lies on the sphere of radius R.

Exercise 103 *Write the formula for the tangent vector to the path (13) at each point using $(\hat{x}(t), \hat{y}(t), \hat{z}(t))$-coordinates. Show that the length of this path is $R\pi$.*

Exercise 104 *Compute the length of each of your two example curves in Exercise 100.*

Remark 105 *In this last Exercise, you may easily be confronted with an integral that you cannot compute. For example, if your curve $\hat{X}_1(t)$ happens to describe an ellipse that is not circular, it was proved in the 19 th century that no formula involving only the standard functions from calculus will give you the length of your path from a fixed beginning point to a variable ending point on the ellipse. If that kind of thing occurs, go back and change the definitions of your curves in Exercise 100 until you get two curves for which you can compute length of your path from a fixed beginning point to a fixed ending point. [DS,81-82]*

We will want to reserve the notation (x, y, z) for some new coordinates that we will put on the 'same' objects in the next section. These new coordinates will be chosen to keep the north and south poles from going to infinity as the radius R of a sphere increases without bound. This change of viewpoint will eventually let us go non-Euclidean or, in the language of Buzz Lightyear "to infinity and beyond." The idea will be like the change from rectangular to polar coordinates for the plane that you encountered in calculus, only easier.

8 Changing coordinates

8.1 Bringing the North Pole of the R-sphere to $(0, 0, 1)$

We are now ready to introduce a slightly different set of coordinates for $\mathbb{R}^3$, three-dimensional Euclidean space. To see why we do this, suppose we are standing at the North Pole

$$N = (0, 0, R)$$

of the sphere

$$\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2 \tag{14}$$

of radius R. As R increases (but we stay our same size, the sphere around us becomes more and more like a flat, plane surface. However it can never get completely flat because we are zooming out the positive $\hat{z}$-axis and we would have to be 'at infinity' for our surface to become exactly flat. We remedy that unfortunate situation by considering another copy of $\mathbb{R}^3$, whose coordinates we denote as (x, y, z) and make the following rule in order to pass between the two $\mathbb{R}^3$'s:

$$\begin{aligned} \hat{x} &= x \\ \hat{y} &= y \\ \hat{z} &= Rz. \end{aligned} \tag{15}$$

We think of the (x, y, z)-coordinates as simply being a different set of addresses for the points in Euclidean 3-space, for example,

$$(x, y, z) = (0, 0, 1)$$

tells me that the point in Euclidean 3-space that I'm referring to is

$$(\hat{x}, \hat{y}, \hat{z}) = (0, 0, R) = N,$$

and the sphere of radius R in Euclidean 3-space is given by

$$\begin{aligned} R^2 &= \hat{x}^2 + \hat{y}^2 + \hat{z}^2 \\ &= x^2 + y^2 + R^2 z^2 \end{aligned}$$

that is, by the equation

$$1 = \frac{1}{R^2}\left(x^2 + y^2\right) + z^2. \tag{16}$$

The quantity

$$K = \frac{1}{R^2}$$

is called the curvature of the R-sphere. So in (x, y, z)-coordinates, as R goes to infinity, K goes to 0. The formula (16) is rewritten as

$$1 = K\left(x^2 + y^2\right) + z^2, \tag{17}$$

and so goes to

$$1 = z^2$$

as R goes to infinity. So, in the (x, y, z)-coordinates, our 'R-geometry' does indeed go to something finite and flat as R goes to infinity, namely the set given by the formula

$$z = \pm 1$$

which is in fact (two copies of) a plane!

Exercise 106 *a) Sketch the solution set in (x, y, z)-coordinates representing the sphere*

$$R^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 2^2$$

of radius 2 in Euclidean three-space.

b) Sketch the solution set in the same (x, y, z)-coordinates representing the sphere

$$R^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 10^2$$

of radius 10 in Euclidean three-space.

c) Sketch the solution set in the same (x, y, z)-coordinates representing the sphere

$$R^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 10^{-2}$$

of radius 10^{-1} in Euclidean three-space.

8.2 K-geometry: Formulas for Euclidean lengths and angles in terms of (x, y, z)-coordinates

To prepare ourselves to do hyperbolic geometry, which has no satisfactory model in Euclidean three-space, we will 'practice' by doing spherical geometry (which *does* have a completely satisfactory model in Euclidean three-space) using these 'slightly strange' (x, y, z)-coordinates. Gradually throughout this course we will discover that the same rules that govern spherical geometry, expressed in (x, y, z)-coordinates, also govern flat and hyperbolic geometry! In all three cases, the space in (x, y, z)-coordinates that we will study is

$$1 = K\left(x^2 + y^2\right) + z^2. \tag{18}$$

If $K > 0$, the geometry we will be studying is the geometry of the the Euclidean sphere of radius

$$R = \frac{1}{\sqrt{K}}.$$

If $K = 0$ we will be studying flat (plane) geometry. If $K < 0$, we will be studying hyperbolic geometry. The number K, in all cases, is called the *curvature* of the geometry.

In short, we want to use (x, y, z)-coordinates to compute with, but we want lengths and angles to be the usual Euclidean ones in $(\hat{x}, \hat{y}, \hat{z})$-coordinates.

Exercise 107 *a) Suppose we have functions*

$$\left(\hat{x}\left(x, y, z\right), \hat{y}\left(x, y, z\right), \hat{z}\left(x, y, z\right)\right)$$

where

$$\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t). \end{aligned}$$

State the Chain Rule for

$$\begin{aligned} \frac{d\hat{x}}{dt} &= \\ \frac{d\hat{y}}{dt} &= \\ \frac{d\hat{z}}{dt} &= . \end{aligned}$$

b) Rewrite the Chain Rule in matrix notation

$$\begin{pmatrix} \frac{d\hat{x}}{dt} & \frac{d\hat{y}}{dt} & \frac{d\hat{z}}{dt} \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{pmatrix} \cdot \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

Exercise 108 *Recalling that R is a positive constant, use* (15) *and the Chain Rule to show that, for any path $\hat{X}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t))$ in Euclidean* 3*-space,*

$$\frac{d\hat{x}}{dt} = \frac{dx}{dt}$$
$$\frac{d\hat{y}}{dt} = \frac{dy}{dt}$$
$$\frac{d\hat{z}}{dt} = R\frac{dz}{dt}.$$

Exercise 109 *Use matrix multiplication [DS,307] and Exercise 108 to show that*

$$\frac{d\hat{X}(t)}{dt} = \left(\frac{dX(t)}{dt}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}$$
$$\frac{dX(t)}{dt} = \left(\frac{d\hat{X}(t)}{dt}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{pmatrix}.$$

NB: This last computation shows that, if

$$\hat{V}_1 = \left(\hat{a}_1, \hat{b}_1, \hat{c}_1\right)$$
$$\hat{V}_2 = \left(\hat{a}_2, \hat{b}_2, \hat{c}_2\right)$$

are tangent vectors in $(\hat{x}, \hat{y}, \hat{z})$-coordinates and

$$V_1 = (a_1, b_1, c_1)$$
$$V_2 = (a_2, b_2, c_2)$$

are their transformations into (x, y, z)-coordinates, then

$$\hat{V}_1 = (V_1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}$$
$$\hat{V}_2 = (V_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}$$

and

$$\begin{aligned}
\hat{V}_1 \bullet \hat{V}_2 &= \left(\hat{V}_1\right) \cdot \left(\hat{V}_2\right)^t \\
&= (V_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \left((V_2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \right)^t \\
&= (V_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} (V_2)^t \\
&= (V_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} (V_2)^t .
\end{aligned}$$

This last computation says that we can compute the Euclidean dot $\hat{V}_1 \bullet \hat{V}_2$ without ever referring to Euclidean coordinates. We incorporate that fact into the following definition.

Definition 110 *"K-dot-product" of vectors:*

$$\begin{aligned}
V_1 \bullet_K V_2 &= (V_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} (V_2)^t \qquad (19) \\
&= \begin{pmatrix} a_1 & b_1 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} .
\end{aligned}$$

So, suppose we have a curve on the R-sphere in Euclidean 3-space but it is given to us in $X(t) = (x(t), y(t), z(t))$-coordinates. Then the length of that curve in Euclidean 3-space is

$$\int_b^e \sqrt{\frac{dX}{dt} \bullet_K \frac{dX}{dt}} dt.$$

Exercise 111 *Use Exercise 96 to show that, if we have any two vectors in Euclidean three-space that are tangent to the R-sphere at some point on it, but the two vectors are given to us in (x, y, z)-coordinates as*

$$\begin{aligned}
V_1 &= (a_1, b_1, c_1) \\
V_2 &= (a_2, b_2, c_2) ,
\end{aligned}$$

then the area of the parallelogram spanned by those two vectors in Euclidean 3-space is

$$\sqrt{\begin{vmatrix} V_1 \bullet_K V_1 & V_2 \bullet_K V_1 \\ V_1 \bullet_K V_2 & V_2 \bullet_K V_2 \end{vmatrix}} = \sqrt{\left| \begin{pmatrix} (V_1) \\ (V_2) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \begin{pmatrix} (V_1)^t & (V_2)^t \end{pmatrix} \right|}.$$

Moral of the story: The dot-product rules! That is, if you know the dot-product you know everything there is to know about a geometry, lengths, areas, angles, everything. And the set (18) continues to make sense even when K is negative. And as we will see later on, the definition of the K-dot product also makes sense for tangent vectors to that set when K is negative.The geometry we get, when the constant K is chosen to be negative is called a hyperbolic geometry. The geometry we get, when the constant K is just chosen to be non-zero is called a non-euclidean geometry. In fact all the non-euclidean 2-dimensional geometries are either spherical or hyperbolic.

Coming attractions: A big idea is that in hyperbolic geometry K^{-1} in (19) becomes negative, so that the third coordinate of velocity, that is, the c-direction, actually *contracts* lengths. It was the understanding of this mysterious fact that allowed Einstein to discover (special) relativity.

9 Congruences, that is, rigid motions

9.1 Transformations of Euclidean 3-space

Consider the following mapping of Euclidean 3-space to itself:

$$\left(\begin{array}{ccc} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \end{array}\right) = \left(\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array}\right) \cdot \hat{M} \tag{20}$$

where $\hat{M}$ is an invertible 3×3 matrix. Then by matrix multiplication

$$\left(\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array}\right) = \left(\begin{array}{ccc} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \end{array}\right) \cdot \hat{M}^{-1}$$

so that this mapping is $1-1$ and onto.

Definition 112 *The mapping of Euclidean* 3*-space to itself given by the rule* (20) *is called a rigid motion if the distance between any two points in Euclidean* 3*-space is left unchanged by the mapping, that is, for any two points, $\hat{X}_1$ and $\hat{X}_2$ in Euclidean* 3*-space*

$$d\left(\left(\hat{X}_1\right) \cdot \hat{M}, \left(\hat{X}_2\right) \cdot \hat{M}\right) = d\left(\hat{X}_1, \hat{X}_2\right).$$

We saw in formula (12) that the square of the distance between $\hat{X}_1$ and $\hat{X}_1$ is just the dot-product of the vector

$$\hat{V} = \hat{X}_2 - \hat{X}_1$$

with itself. So the transformation given by the matrix $\hat{M}$ will leave distances unchanged if and only if, for all vectors $\hat{V}$,

$$\left(\left(\hat{V}\right) \cdot \hat{M}\right) \bullet \left(\left(\hat{V}\right) \cdot \hat{M}\right) = \hat{V} \bullet \hat{V}$$

that is

$$\left(\left(\hat{V}\right) \cdot \hat{M}\right) \cdot \left(\left(\hat{V}\right) \cdot \hat{M}\right)^t = \left(\hat{V}\right) \cdot \left(\hat{V}\right)^t.$$

We can rewrite this requirement as

$$\left(\hat{V}\right) \cdot \hat{M} \cdot \hat{M}^t \cdot \left(\hat{V}\right)^t = \left(\hat{V}\right) \cdot \left(\hat{V}\right)^t \tag{21}$$

for all vectors $\hat{V}$. Condition (21) is certainly satisfied for all vectors $\hat{V}$ if

$$\hat{M} \cdot \hat{M}^t = I = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right). \tag{22}$$

Exercise 113 *Suppose $\hat{M}$ is such that*

$$\hat{M} \cdot \hat{M}^t = \left(\begin{array}{ccc} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

Find a vector $\hat{V} = \left(\hat{a}, \hat{b}, \hat{c}\right)$ *such that*

$$\left(\hat{V}\right) \cdot \hat{M} \cdot \hat{M}^t \cdot \left(\hat{V}\right)^t \neq \left(\hat{V}\right) \cdot \left(\hat{V}\right)^t.$$

In fact, reasoning as in this last Exercise, one can show that, if a matrix $\hat{M}$ satisfies the condition (21) for *all* vectors $\hat{V}$, then the matrix $\hat{M}$ also satisfies (22). A matrix $\hat{M}$ satisfying (22) is called an *orthogonal matrix.* [DS,316-321]

Exercise 114 *Show that the matrix*

$$\hat{M} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is orthogonal. Can you describe geometrically what this rigid motion is doing to the points in Euclidean 3*-space?*

Exercise 115 *Show that the matrix*

$$\hat{M} = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}$$

is orthogonal. Can you describe geometrically what this rigid motion is doing to the points in Euclidean 3*-space?*

For any curve

$$\hat{X}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t)), \ b \leq t \leq e,$$

its length is

$$\int_b^e \sqrt{\left(\frac{d\hat{X}}{dt}\right) \cdot \left(\frac{d\hat{X}}{dt}\right)^t} dt.$$

Suppose now that the curve is moved by a transformation given by an orthogonal matrix $\hat{M}$. After it is moved, its length is given by

$$\int_b^e \sqrt{\left(\frac{d\left(\hat{X} \cdot \hat{M}\right)}{dt}\right) \cdot \left(\frac{d\left(\hat{X} \cdot \hat{M}\right)}{dt}\right)^t} dt.$$

But

$$\int_b^e \sqrt{\left(\frac{d\left(\hat{X}\cdot\hat{M}\right)}{dt}\right)\cdot\left(\frac{d\left(\hat{X}\cdot\hat{M}\right)}{dt}\right)^t}\,dt = \int_b^e \sqrt{\left(\frac{d\hat{X}}{dt}\cdot\hat{M}\right)\cdot\left(\frac{d\hat{X}}{dt}\cdot\hat{M}\right)^t}\,dt$$

$$= \int_b^e \sqrt{\left(\frac{d\hat{X}}{dt}\cdot\hat{M}\right)\cdot\left(\hat{M}^t\cdot\frac{d\hat{X}}{dt}^t\right)}\,dt$$

$$= \int_b^e \sqrt{\left(\frac{d\hat{X}}{dt}\right)\cdot\left(\frac{d\hat{X}}{dt}\right)^t}\,dt.$$

Corollary 116 *If a curve*

$$\hat{X}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t)), \ b \le t \le e$$

is moved by a transformation given by an orthogonal matrix $\hat{M}$*, its length is unchanged.*

9.2 Formula in (x, y, z)-coordinates for rigid motions of Euclidean 3-space

We now wish to figure out how to write the transformation (20) in (x, y, z)-coordinates. This is a simple substitution problem:

$$\left(\begin{array}{ccc} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \end{array} \right) = \left(\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \right) \cdot \hat{M} \tag{23}$$

$$\left(\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \right) = \left(\begin{array}{ccc} x & y & Rz \end{array} \right) = \left(\begin{array}{ccc} x & y & z \end{array} \right) \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{array} \right)$$
$$\left(\begin{array}{ccc} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \end{array} \right) = \left(\begin{array}{ccc} \underline{x} & \underline{y} & R\underline{z} \end{array} \right) = \left(\begin{array}{ccc} \underline{x} & \underline{y} & \underline{z} \end{array} \right) \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{array} \right)$$

So we have the diagram

$$\begin{array}{ccc} \left(\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \right) \in \mathbb{R}^3 & \overset{\cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{array} \right)}{\longleftarrow} & \left(\begin{array}{ccc} x & y & z \end{array} \right) \in \mathbb{R}^3 \\ \downarrow \cdot \hat{M} & & \downarrow \cdot M =? \\ \left(\begin{array}{ccc} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \end{array} \right) & \overset{\cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{array} \right)}{\longrightarrow} & \left(\begin{array}{ccc} \underline{x} & \underline{y} & \underline{z} \end{array} \right) \end{array}$$

Exercise 117 *Starting from the equality* (23) *describing the transformation in Euclidean coordinates, explain why*

$$\left(\begin{array}{ccc} \underline{x} & \underline{y} & \underline{z} \end{array} \right) = \left(\begin{array}{ccc} x & y & z \end{array} \right) \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{array} \right) \cdot \hat{M} \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{array} \right).$$

So, if we let

$$M = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{array} \right) \cdot \hat{M} \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{array} \right),$$

then

$$\left(\begin{array}{ccc} \underline{x} & \underline{y} & \underline{z} \end{array} \right) = \left(\begin{array}{ccc} x & y & z \end{array} \right) \cdot M, \tag{24}$$

that is M is the matrix that gives the transformation (20) in (x, y, z)-coordinates.

So how would we check whether a transformation given in (x, y, z)-coordinates by a matrix M preserves distances in Euclidean 3-space? Again, starting from

(22) this is just a substitution problem:

$$\hat{M} \cdot \hat{M}^t = I$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \cdot \hat{M} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{pmatrix} \cdot M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} = \hat{M}$$

Exercise 118 *Finish the matrix algebra computations just above to show that the condition that a transformation M in (x, y, z)-coordinates preserves distances in Euclidean 3-space is the condition that*

$$M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix}. \tag{25}$$

This is the condition (in (x, y, z)-coordinates) which affirms that the transformation which takes the path $(x(t), y(t), z(t))$ to the path $(x(t), y(t), z(t)) \cdot M$ preserves lengths of tangent vectors at corresponding points. Therefore, by integrating, the (total) length of the curve $\{(x(t), y(t), z(t)) \cdot M : b \leq t \leq e\}$ is the same as the total length of the curve $\{(x(t), y(t), z(t)) : b \leq t \leq e\}$.

Exercise 119 *Check that* (25) *is the correct condition by showing that any* 3×3 *matrix M that satisfies* (25) *also satisfies*

$$((V) \cdot M) \bullet_K ((V) \cdot M) = V \bullet_K V$$

where

$$V = X_2 - X_1.$$

That is, the transformation given in (x, y, z)-coordinates by a matrix M that satisfies (25) *preserves the K-dot product.*

Part V
K-geometry

10 Uniform coordinates for the two-dimensional geometries

10.1 The two-dimensional geometries in (x, y, z)-coordinates

At the beginning of this book we changed the coordinates on Euclidean three-space so that the equations for the sphere of radius R became

$$1 = K\left(x^2 + y^2\right) + z^2 \tag{26}$$

where

$$K = \frac{1}{R^2}.$$

In these new (x, y, z)-coordinates, the set of points (x, y, z) satisfying (26) when $K > 0$ matched up in $1-1$ fashion with the R-sphere

$$\left\{(\hat{x}, \hat{y}, \hat{z}) \in \mathbb{R}^3 : \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2\right\}$$

in the usual coordinates $(\hat{x}, \hat{y}, \hat{z})$ of 3-dimensional Euclidean space.

If we have a curve $(\hat{x}(t), \hat{y}(t), \hat{z}(t))$ lying in the R-sphere in Euclidean space, then for all $t \in [b, e]$,

$$\hat{x}(t)^2 + \hat{y}(t)^2 + \hat{z}(t)^2 = R^2.$$

Differentiating both sides with respect to t we obtain

$$2\hat{x}(t)\frac{d\hat{x}}{dt} + 2\hat{y}(t)\frac{d\hat{y}}{dt} + 2\hat{z}(t)\frac{d\hat{z}}{dt} = 0$$

which we can rewrite as

$$(\hat{x}(t), \hat{y}(t), \hat{z}(t)) \bullet \frac{d\hat{X}(t)}{dt} = 0.$$

[DS,105ff] Said another way, vectors $\hat{V}$ are tangent to the R-sphere at $\hat{X}(t)$ if and only if

$$\hat{X}(t) \bullet \hat{V} = 0.$$

[DS,106,109]

Repeating the same calculation in (x, y, z)-coordinates, the corresponding curve $(x(t), y(t), z(t))$ lies in the set (26) so that

$$\begin{aligned}
1 &= K\left(x(t)^2 + y(t)^2\right) + z(t)^2 \\
0 &= K\left(2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt}\right) + 2z(t)\frac{dz}{dt}.
\end{aligned}$$

That is, a vector $V=(a,b,c)$ is tangent to the set (26) if and only if

$$(x(t), y(t), z(t)) \cdot \begin{pmatrix} 2K & 0 & 0 \\ 0 & 2K & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot V^t = 0. \tag{27}$$

Exercise 120 *For $K \neq 0$, show that the condition* (27) *on V is exactly the same condition as*

$$(x(t), y(t), z(t)) \bullet_K V = 0.$$

We will call the set of (x,y,z) satisfying (26) K-geometry. Its tangent vectors at a point (x,y,z) in the set are the vectors $V=(a,b,c)$ such that

$$(x,y,z) \bullet_K V = 0.$$

If you get nervous using these weird coordinates to compute things that are clearer in $(\hat{x},\hat{y},\hat{z})$-coordinates, just go through each construction in Part V in the special case $K=1$ first. In that special case

$$(x,y,z) = (\hat{x},\hat{y},\hat{z})$$

and your calculations reduce to the usual ones on the unit sphere in ordinary Euclidean 3-space.

10.2 Rigid motions in (x, y, z)-coordinates

We are now going to study K-geometry using only (x, y, z)-coordinates. If we have a curve $X(t) = (x(t), y(t), z(t))$ on the surface given in K-coordinates as

$$1 = K\left(x^2 + y^2\right) + z^2, \tag{28}$$

we have seen that we measure its length L by the formula

$$L = \int_b^e l(t)\, dt \tag{29}$$

where

$$l(t)^2 = \frac{dX}{dt} \bullet_K \frac{dX}{dt} \tag{30}$$

and that we measure angles θ between tangent vectors V_1 and V_2 at a point on the surface by the formula

$$\theta = \arccos\left(\frac{V_1 \bullet_K V_2}{|V_1|_K \cdot |V_2|_K}\right)$$

where

$$|V|_K^2 = V \bullet_K V.$$

We now want to explore the condition that a transformation

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot M$$

take the surface (28) to itself andpreserve the length of any curve $(x(t), y(t), z(t))$ lying on the surface. Rewriting the transformation as

$$(\underline{X}) = (X) \cdot M$$

the formulas (29) and (30) show that all we have to worry about is that

$$\frac{d\underline{X}}{dt} \bullet_K \frac{d\underline{X}}{dt} = \frac{dX}{dt} \bullet_K \frac{dX}{dt}$$

for all values t of the parameter of the curve. But

$$\left(\frac{d\underline{X}}{dt}\right) = \left(\frac{dX}{dt}\right) \cdot M$$

by the product rule since M is a constant matrix. So the transformation given by the matrix $\hat{M}$ will preserve the length of any path and will preserve the measure of any angle if

$$\left(\frac{dX}{dt}\right) \cdot M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \left(\left(\frac{dX}{dt}\right) \cdot M\right)^t = \left(\frac{dX}{dt}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \left(\frac{dX}{dt}\right)^t. \tag{31}$$

Exercise 121 *a) Show that this last equality is always true if*

$$M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix}. \tag{32}$$

b) Show that, if M *satisfies the identity 32,then the transformation* $\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot M$ *takes the set of points* (x, y, z) *such that*

$$1 = K\left(x^2 + y^2\right) + z^2,$$

to the set of points $\left(\underline{x}, \underline{y}, \underline{z}\right)$ *such that*

$$1 = K\left(\underline{x}^2 + \underline{y}^2\right) + \underline{z}^2.$$

That is, M *gives a* $1-1$*, onto mapping of* K*-geometry to itself.*

Hint: For $K \neq 0$*, write the equation* $1 = K\left(\underline{x}^2 + \underline{y}^2\right) + \underline{z}^2$ *in matrix notation as*

$$\begin{pmatrix} \underline{x} & \underline{y} & \underline{z} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \begin{pmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{pmatrix} = \frac{1}{K}.$$

Definition 122 *A* 3×3 *matrix* M *is called* K*-orthogonal if*

$$M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix}.$$

Definition 123 *A* K*-distance-preserving transformation of* K*-geometry is called a* K***-rigid motion*** *or a* K***-congruence****.*

So K-orthogonal matrices give K-rigid motions.

Exercise 124 *For* $K \neq 0$*, show that the set of* K*-orthogonal matrices* M *form a group. That is, show that*

a) the product of two K*-orthogonal matrices is* K*-orthogonal,*

b) the identity matrix is K*-orthogonal,*

c) the inverse matrix M^{-1} *of a* K*-orthogonal matrix* M *is* K*-orthogonal.*

Hint: Write

$$M \cdot M^{-1} = I = M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{pmatrix}$$

and use matrix multiplication to reduce to showing that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot M^t \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{pmatrix}$$

is K*-orthogonal.*

10.3 Why use K-coordinates?

We saw that we could measure the usual Euclidean lengths of curves $\hat{X}(t)$ on the usual Euclidean R-sphere just in terms of the formulas $X(t)$ for their paths in (x, y, z)-coordinates using the K-dot product, since lengths depended only on lengths of tangent vectors and

$$\frac{d\hat{X}(t)}{dt} \bullet \frac{d\hat{X}(t)}{dt} = \frac{dX(t)}{dt} \bullet_K \frac{dX(t)}{dt}$$

where

$$\frac{dX(t)}{dt} \bullet_K \frac{dX(t)}{dt} = \left(\frac{dX(t)}{dt}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \left(\frac{dX(t)}{dt}\right)^t.$$

In other words, the usual geometry of the sphere of radius R is simply the geometry of the set (26) with $K = 1/R^2$ and with lengths (and areas) given by the K-dot product. Said another way, we can do all of spherical geometry in (x, y, z)-coordinates. All we need is the set (26) and the K-dot product. But the set (26) continues to exist even if $K = 0$ or $K < 0$, and the K-dot product formula continues to make sense even if $K < 0$. In short we have the following table:

$$\begin{array}{ccc} \text{Spherical } (K > 0) & \text{Euclidean } (K = 0) & \text{Hyperbolic } (K < 0) \\ \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2 & & \\ \hat{V} \bullet \hat{V} & & \\ 1 = K\left(x^2 + y^2\right) + z^2 & 1 = K\left(x^2 + y^2\right) + z^2 & 1 = K\left(x^2 + y^2\right) + z^2 \\ V_1 \bullet_K V_2 & & V_1 \bullet_K V_2 \end{array} \tag{33}$$

This table tells us that 'there is something else out there,' that is, some other type of two-dimensional geometry beyond plane geometry and spherical geometry. But the gap in the bottom row of the table is a bit disturbing. If we can't express the usual dot-product in plane geometry as the K-dot product for $K = 0$, we can't pass smoothly from spherical through plane geometry to hyperbolic geometry using (x, y, z)-coordinates. We now examine two ways to produce coordinates uniformly for spherical, plane and hyperbolic geometry that overcome this difficulty.

11 Central projection

11.1 Central projection coordinates

Let's project K-geometry, that is, the set

$$1 = K\left(x^2 + y^2\right) + z^2 \tag{34}$$

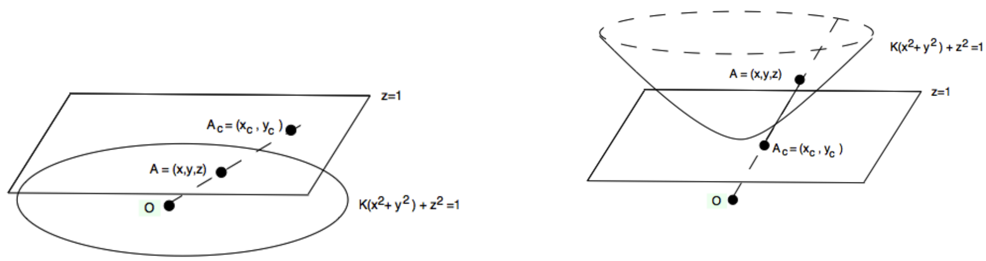
onto the set

$$z = 1$$

using the origin

$$O = (0, 0, 0)$$

as the center of projection:



That is,

$$r \cdot (x_c, y_c, 1) = (x, y, z)\,. \tag{35}$$

So

$$r = z$$

and, from the equation (34)

$$K\left((rx_c)^2 + (ry_c)^2\right) + r^2 = 1$$
$$r^2 = \frac{1}{K\left(x_c^2 + y_c^2\right) + 1}.$$

Notice that, when $K < 0$ this last formula only makes sense when

$$K\left(x_c^2 + y_c^2\right) > -1 \tag{36}$$
$$x_c^2 + y_c^2 < \frac{-1}{K}.$$

Exercise 125 *a) For the projection of the set* (34) *onto the* $z = 1$ *plane with center of projection* O*, write* (x_c, y_c) *as a function of* (x, y, z)*.*

b) For the projection of the set (34) *onto the* $z = 1$ *plane with center of projection* O*, write* (x, y, z) *as a function of* (x_c, y_c)*.*

11.2 Rigid motion in central projection coordinates

Suppose now we have a K-rigid motion

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot M$$

of K-geometry, given by a K-orthogonal matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

To see what this K-rigid motion looks like in central projection coordinates we simply do the matrix multiplication

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

$$= \left(\left(m_{11}x + m_{21}y + m_{31}z\right), \left(m_{12}x + m_{22}y + m_{32}z\right), \left(m_{13}x + m_{23}y + m_{33}z\right)\right).$$

Then

$$\begin{aligned} \underline{x_c} &= \frac{\underline{x}}{\underline{z}} \\ &= \frac{m_{11}x + m_{21}y + m_{31}z}{m_{13}x + m_{23}y + m_{33}z} \\ &= \frac{m_{11}\left(x/z\right) + m_{21}\left(y/z\right) + m_{31}}{m_{13}\left(x/z\right) + m_{23}\left(y/z\right) + m_{33}} \\ &= \frac{m_{11}x_c + m_{21}y_c + m_{31}}{m_{13}x_c + m_{23}y_c + m_{33}} \end{aligned} \tag{37}$$

and similarly

$$\underline{y_c} = \frac{m_{12}x_c + m_{22}y_c + m_{32}}{m_{13}x_c + m_{23}y_c + m_{33}}.$$

So we write

$$\left(\underline{x_c}, \underline{y_c}\right) = M_c\left(x_c, y_c\right)$$

$$= \left(\frac{m_{11}x_c + m_{21}y_c + m_{31}}{m_{13}x_c + m_{23}y_c + m_{33}}, \frac{m_{12}x_c + m_{22}y_c + m_{32}}{m_{13}x_c + m_{23}y_c + m_{33}}\right).$$

11.3 Length and angle in central projection coordinates

Exercise 126 *For the K-geometry coordinates*

$$X = (x, y, z)$$

use the formulas you derived in Exercise 125b) to calculate

$$dX = \left(\frac{\partial X}{\partial x_c}\right) dx_c + \left(\frac{\partial X}{\partial y_c}\right) dy_c$$

That is, calculate the 2×3 matrix

$$D_c = \begin{pmatrix} \frac{\partial x}{\partial x_c} & \frac{\partial y}{\partial x_c} & \frac{\partial z}{\partial x_c} \\ \frac{\partial x}{\partial y_c} & \frac{\partial y}{\partial y_c} & \frac{\partial z}{\partial y_c} \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial X}{\partial x_c}\right) \\ \left(\frac{\partial X}{\partial y_c}\right) \end{pmatrix}.$$

Hint: Use logarithmic differentiation:

$$\begin{aligned} dx &= d\left(r x_c\right) = x_c dr + r dx_c \\ r^{-1} dx &= x_c d\ln\left(r\right) + dx_c \end{aligned}$$

and similarly for y and z since it is easier to compute $r^{-1}\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ than $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$. Next use that

$$\begin{aligned} 2d\ln\left(r\right) &= d\ln\left(r^2\right) = -d\ln\left(K\left(x_c^2 + y_c^2\right) + 1\right) \\ &= -\frac{1}{K\left(x_c^2 + y_c^2\right) + 1} d\left(K\left(x_c^2 + y_c^2\right) + 1\right) \\ &= -r^2 K\left(2x_c dx_c + 2y_c dy_c\right). \end{aligned}$$

Exercise 127 *Now suppose we have a path,*

$$\left(x_c\left(t\right), y_c\left(t\right)\right), \ a \leq t \leq b$$

in the (x_c, y_c)-plane, that is, in the central projection plane

$$\left(x_c, y_c, 1\right).$$

Use the formula you derived in Exercise 125b) to write the corresponding path

$$x\left(x_c\left(t\right), y_c\left(t\right)\right), y\left(x_c\left(t\right), y_c\left(t\right)\right), z\left(x_c\left(t\right), y_c\left(t\right)\right)$$

in the K-geometry space of (x, y, z) such that $K\left(x^2 + y^2\right) + z^2 = 1$.

Exercise 128 *For the path $\left(x\left(t\right), y\left(t\right), z\left(t\right)\right)$ in Exercise 127 lying on the set* (34)*, use the Chain Rule from calculus of several variables to compute*

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{dx_c\left(t\right)}{dt}, \frac{dy_c\left(t\right)}{dt}\right) \cdot D_c.$$

This last Exercise allows us to do something very nice. Namely now, not only can we use the coordinates (x_c, y_c) for our geometry but we can also compute the K-dot product in terms of these coordinates. By the Chain Rule from calculus of several variables

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{dx_c}{dt}, \frac{dy_c}{dt}\right) \cdot D_c.$$

So

$$\begin{aligned}\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \bullet_K \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) &= \left(\begin{array}{ccc} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{array} \right) \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{array} \right) \left(\begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{array} \right) \\ &= \left(\begin{array}{cc} \frac{dx_c}{dt} & \frac{dy_c}{dt} \end{array} \right) \cdot D_c \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{array} \right) \cdot D_c^t \cdot \left(\begin{array}{c} \frac{dx_c}{dt} \\ \frac{dy_c}{dt} \end{array} \right),\end{aligned}$$

Exercise 129 *Compute the* 2×2 *matrix*

$$P_c = D_c \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{array} \right) \cdot D_c^t,$$

that that gives the K*-dot product in* (x_c, y_c)*-coordinates. That is, use matrix multiplication to show that*

$$P_c = \left(\begin{array}{cc} r^2\left(1 - r^2 K x_c^2\right) & -r^4 K x_c y_c \\ -r^4 K x_c y_c & r^2\left(1 - r^2 K y_c^2\right) \end{array} \right).$$

Hint: For example

$$\begin{aligned}\frac{\partial x}{\partial x_c} &= r\left(x_c \frac{\partial ln\left(r\right)}{\partial x_c} + 1\right) = -r^3 K x_c^2 + r \\ \frac{\partial y}{\partial x_c} &= r\left(y_c \frac{\partial ln\left(r\right)}{\partial x_c}\right) = -r^3 K x_c y_c \\ \frac{\partial z}{\partial x_c} &= r\left(\frac{\partial ln\left(r\right)}{\partial x_c}\right) = -r^3 K x_c\end{aligned}$$

so that

$$\begin{aligned}&\left(\frac{\partial x}{\partial x_c}, \frac{\partial y}{\partial x_c}, \frac{\partial z}{\partial x_c}\right) \bullet_K \left(\frac{\partial x}{\partial x_c}, \frac{\partial y}{\partial x_c}, \frac{\partial z}{\partial x_c}\right) \\ &= r^6 K^2 x_c^4 - 2r^4 K x_c^2 + r^2 + r^6 K^2 x_c^2 y_c^2 + r^6 K x_c^2 \\ &= \left(r^6 K^2 x_c^4 + r^6 K^2 x_c^2 y_c^2 + r^6 K x_c^2\right) - 2r^4 K x_c^2 + r^2 \\ &= r^4 K x_c^2 - 2r^4 K x_c^2 + r^2 = r^2\left(1 - r^2 K x_c^2\right).\end{aligned}$$

So, if, if $K > 0$ and you have a path on the sphere of radius $R = K^{-1/2}$ in Euclidean 3-space given in (x_c, y_c)-coordinates as $(x_c(t), y_c(t))$ for $t \in [b, e]$, you can trace back everything we have done with coordinate changes to see that the length of the path on the sphere of radius $R = K^{-1/2}$ in Euclidean 3-space is given by

$$\int_b^e l(t)\, dt$$

where

$$\begin{aligned} l(t)^2 &= \left(\frac{dx_c}{dt}, \frac{dy_c}{dt}\right) \bullet_c \left(\frac{dx_c}{dt}, \frac{dy_c}{dt}\right) \\ &= \begin{pmatrix} \frac{dx_c}{dt} & \frac{dy_c}{dt} \end{pmatrix} \cdot P_c \cdot \begin{pmatrix} \frac{dx_c}{dt} \\ \frac{dy_c}{dt} \end{pmatrix}. \end{aligned}$$

Notice that the matrix P_c still makes sense when $K = 0$ and when K becomes negative. So we do have

Spherical $(K > 0)$	*Euclidean* $(K = 0)$	*Hyperbolic* $(K < 0)$
$\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2$		
$\hat{V} \bullet \hat{V}$		
$1 = K\left(x^2 + y^2\right) + z^2$	$1 = K\left(x^2 + y^2\right) + z^2$	$1 = K\left(x^2 + y^2\right) + z^2$
$V_1 \bullet_K V_2$		$V_1 \bullet_K V_2$
$V_1^c \bullet_c V_2^c$	$V_1^c \bullet_c V_2^c$	$V_1^c \bullet_c V_2^c$

where

$$V_1^c \bullet_c V_2^c = (V_1^c) \cdot P_c \cdot (V_2^c)^t.$$

Of course if $K > 0$, we again have Euclidean angles θ between vectors $\hat{V}_1$ and $\hat{V}_2$ tangent to the R-sphere at some point computed by

$$\begin{aligned} \hat{V}_1 \bullet \hat{V}_2 &= \left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \cos(\theta) \\ &= V_1^c \bullet_c V_2^c. \end{aligned}$$

11.4 Area in central projection coordinates

Suppose you were given a region G_c in the (x_c, y_c)-coordinate plane. Also suppose that $K > 0$. If you trace back everything we have done with coordinate changes, you can see how G_c gives you a region $\hat{G}$ on the sphere of radius $R = K^{-1/2}$ in Euclidean 3-space via the formulas

$$\begin{aligned}(\hat{x}, \hat{y}, \hat{z}) &= (x, y, Rz) \\ &= r \cdot (x_c, y_c, R) \\ &= \left(\frac{x_c}{\sqrt{K\left(x_c^2 + y_c^2\right) + 1}}, \frac{y_c}{\sqrt{K\left(x_c^2 + y_c^2\right) + 1}}, \frac{R}{\sqrt{K\left(x_c^2 + y_c^2\right) + 1}} \right).\end{aligned}$$

Now there is a formula in several variable calculus for computing the area of the region $\hat{G}$ on the sphere of radius R in Euclidean 3-space in terms of the parameters (x_c, y_c). [DS,49,231]. It is

$$\int_{G_c} \hat{a}\left(\frac{d\hat{X}}{dx_c}, \frac{d\hat{X}}{dy_c} \right) dx_c dy_c \tag{38}$$

where $\hat{a}\left(\frac{d\hat{X}}{dx_c}, \frac{d\hat{X}}{dy_c} \right)$ is the (Euclidean) area of the parallelogram spanned by the two vectors $\frac{d\hat{X}}{dx_c}$ and $\frac{d\hat{X}}{dy_c}$ in Euclidean 3-space. Thus

$$\hat{a}\left(\frac{d\hat{X}}{dx_c}, \frac{d\hat{X}}{dy_c} \right) = \left| \frac{d\hat{X}}{dx_c} \right| \cdot \left| \frac{d\hat{X}}{dy_c} \right| \cdot \sin(\theta)$$

where θ is the angle between the two vectors.

Exercise 130 *Using Exercise 96 and Exercise 111 show that*

$$\begin{aligned}\hat{a}\left(\frac{d\hat{X}}{dx_c}, \frac{d\hat{X}}{dy_c} \right)^2 &= \left| \begin{matrix} \frac{d\hat{X}}{dx_c} \bullet \frac{d\hat{X}}{dx_c} & \frac{d\hat{X}}{dy_c} \bullet \frac{d\hat{X}}{dx_c} \\ \frac{d\hat{X}}{dx_c} \bullet \frac{d\hat{X}}{dy_c} & \frac{d\hat{X}}{dy_c} \bullet \frac{d\hat{X}}{dy_c} \end{matrix} \right| \\ &= \left| \begin{matrix} \frac{dX}{dx_c} \bullet_K \frac{dX}{dx_c} & \frac{dX}{dy_c} \bullet_K \frac{dX}{dx_c} \\ \frac{dX}{dx_c} \bullet_K \frac{dX}{dy_c} & \frac{dX}{dy_c} \bullet_K \frac{dX}{dy_c} \end{matrix} \right| \\ &= \left| \left(\begin{matrix} \left(\frac{dX}{dx_c} \right) \\ \left(\frac{dX}{dy_c} \right) \end{matrix} \right) \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{matrix} \right) \left(\begin{matrix} \left(\frac{dX}{dx_c} \right)^t & \left(\frac{dX}{dy_c} \right)^t \end{matrix} \right) \right| \\ &= |P_c|.\end{aligned}$$

Exercise 131 *Use Exercise 129 to show that*

$$\hat{a}\left(\frac{d\hat{X}}{dx_c}, \frac{d\hat{X}}{dy_c} \right)^2 = r^6 = \frac{1}{\left(K\left(x_c^2 + y_c^2\right) + 1\right)^3}$$

as a function of (x_c, y_c).

Hint: Notice that the matrix D_c *in Exercise 126 is simply the* 2×3 *matix whose rows are the vectors* $\frac{dX}{dx_c}$ *and* $\frac{dX}{dy_c}$. *So referring to Exercise 129, we know that*

$$\left(\begin{array}{cc} \frac{d\hat{X}}{dx_c}\bullet\frac{d\hat{X}}{dx_c} & \frac{d\hat{X}}{dy_c}\bullet\frac{d\hat{X}}{dx_c} \\ \frac{d\hat{X}}{dx_c}\bullet\frac{d\hat{X}}{dy_c} & \frac{d\hat{X}}{dy_c}\bullet\frac{d\hat{X}}{dy_c} \end{array}\right)=\left(\begin{array}{cc} r^2\left(1-r^2Kx_c^2\right) & -r^4Kx_cy_c \\ -r^4Kx_cy_c & r^2\left(1-r^2Ky_c^2\right) \end{array}\right).$$

Since all these computations can be extended to K-geometry for all K, we define the K-area of a region G_c in the (x_c, y_c)-coordinate plane by first computing the K-area of the parallelogram spanned by $\frac{dX}{dx_c}$ and $\frac{dX}{dy_c}$ at each point of G_c as

$$\begin{aligned} a_K\left(\frac{dX}{dx_c},\frac{dY}{dy_c}\right) &= \left|\frac{dX}{dx_c}\right|_K\cdot\left|\frac{dX}{dy_c}\right|_K\cdot\sin\left(\theta_K\right) \\ &= \sqrt{\left|\begin{array}{cc} \frac{dX}{dx_c}\bullet_K\frac{dX}{dx_c} & \frac{dX}{dy_c}\bullet_K\frac{dX}{dx_c} \\ \frac{dX}{dx_c}\bullet_K\frac{dX}{dy_c} & \frac{dX}{dy_c}\bullet_K\frac{dX}{dy_c} \end{array}\right|} \end{aligned}$$

and then integrating this area over G_c to get

$$\begin{aligned} A_K\left(G_c\right) &= \int_{G_c} a_K\left(\frac{dX}{dx_c},\frac{dY}{dy_c}\right)dx_cdy_c \\ &= \int_{G_c}\sqrt{\left|\begin{array}{cc} \frac{dX}{dx_c}\bullet_K\frac{dX}{dx_c} & \frac{dX}{dy_c}\bullet_K\frac{dX}{dx_c} \\ \frac{dX}{dx_c}\bullet_K\frac{dX}{dy_c} & \frac{dX}{dy_c}\bullet_K\frac{dX}{dy_c} \end{array}\right|}dx_cdy_c. \end{aligned}$$

12 Stereographic projection

12.1 Stereographic projection coordinates

On the other hand we can project the set

$$1 = K\left(x^2 + y^2\right) + z^2$$

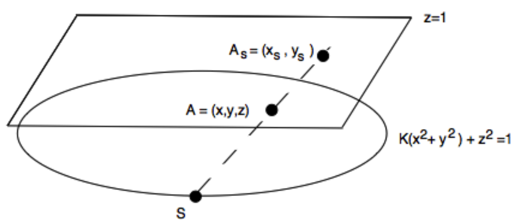
onto the set

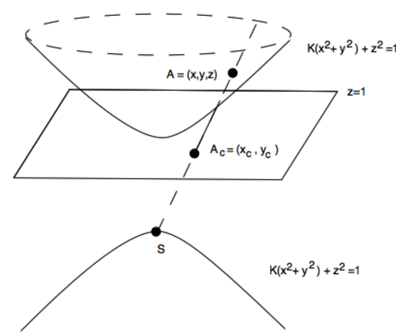
$$z = 1$$

using the 'South Pole'

$$S = (0, 0, -1)$$

as the center of projection:





That is,

$$\rho \cdot (x_s, y_s, 1 - (-1)) = (x, y, z - (-1)).$$

So

$$\rho = \frac{z+1}{2}$$
$$z = 2\rho - 1$$

and, from the equation (34)

$$K\left((\rho x_s)^2 + (\rho y_s)^2\right) + (2\rho - 1)^2 = 1$$
$$K\left((\rho x_s)^2 + (\rho y_s)^2\right) + 4\rho^2 - 4\rho = 0$$
$$\rho K\left(x_s^2 + y_s^2\right) + 4\rho = 4$$
$$\rho = \frac{1}{\frac{K}{4}\left(x_s^2 + y_s^2\right) + 1}.$$

Notice that, when $K < 0$ this last formula only makes sense when

$$\frac{K}{4}\left(x_s^2 + y_s^2\right) > -1$$
$$x_s^2 + y_s^2 < \frac{-4}{K}.$$

Exercise 132 *a) For the projection of the set* (34) *onto the* $z = 1$ *plane with center of projection* S*, write* (x_s, y_s) *as a function of* (x, y, z)*.*

b) For the projection of the set (34) *onto the* $z = 1$ *plane with center of projection* S*, write* (x, y, z) *as a function of* (x_s, y_s)*.*

12.2 Length and angle in stereographic projection coordinates

Exercise 133 *Suppose we have a path*

$$X\left(x_s\left(t\right),y_s\left(t\right)\right)=\left(x\left(x_s\left(t\right),y_s\left(t\right)\right),y\left(x_s\left(t\right),y_s\left(t\right)\right),z\left(x_s\left(t\right),y_s\left(t\right)\right)\right)$$

lying on the set (34) *given in terms of its projection* $\left(x_s\left(t\right),y_s\left(t\right)\right)$ *in the plane* $z=1$. *Use the formula you derived in Exercise 132b) and the Chain Rule from calculus of several variables to find the* 2×3 *matrix*

$$D_s=\begin{pmatrix} \left(\frac{\partial X}{\partial x_s}\right) \\ \left(\frac{\partial X}{\partial y_s}\right) \end{pmatrix}$$

such that

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=\left(\frac{dx_s\left(t\right)}{dt},\frac{dy_s\left(t\right)}{dt}\right)\cdot D_s.$$

Hint: Use logarithmic differentiation:

$$\begin{aligned} dx &= d\left(\rho x_s\right)=x_s d\rho+\rho dx_s \\ \rho^{-1}dx &= x_s d\ln\left(\rho\right)+dx_s \end{aligned}$$

and similarly for y. *Also*

$$\begin{aligned} d\ln\left(\rho\right) &= -d\ln\left(\frac{K}{4}\left(x_s^2+y_s^2\right)+1\right) \\ &= -\frac{1}{\frac{K}{4}\left(x_s^2+y_s^2\right)+1}d\left(\frac{K}{4}\left(x_s^2+y_s^2\right)+1\right) \\ &= -\rho\frac{K}{4}\left(2x_s dx_s+2y_s dy_s\right). \end{aligned}$$

This last Exercise allows us to do something very nice. Namely now, not only can we use the coordinates (x_s, y_s) for our geometry but we can also compute the K-dot product in terms of these coordinates.:

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=\left(\frac{dx_s}{dt},\frac{dy_s}{dt}\right)\cdot D_s$$

so that

$$\begin{aligned} \left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)\bullet_K\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right) &= \begin{pmatrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix}\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} \\ &= \begin{pmatrix} \frac{dx_s}{dt} & \frac{dy_s}{dt} \end{pmatrix}\cdot D_s\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix}\cdot D_s^t\cdot\begin{pmatrix} \frac{dx_s}{dt} \\ \frac{dy_s}{dt} \end{pmatrix}, \end{aligned}$$

Exercise 134 *Use matrix multiplication to compute the* 2×2 *matrix*

$$P_s = D_s \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot D_s^t,$$

that is, to compute the K*-dot product in* (x_s, y_s)*-coordinates. (You may be surprised at the answer! It is quite simple and only involves the quantity* ρ*.)*

So, if, if $K > 0$ and you have a path on the sphere of radius $R = K^{-1/2}$ in Euclidean 3-space given in (x_s, y_s)-coordinates as $(x_s(t), y_s(t))$ for $t \in [b, e]$, you can trace back everything we have done with coordinate changes to see that the length of the path on the sphere of radius $R = K^{-1/2}$ in Euclidean 3-space is given by

$$\int_b^e l(t)\, dt$$

where

$$\begin{aligned} l(t)^2 &= \left(\frac{dx_s}{dt}, \frac{dy_s}{dt}\right) \bullet_s \left(\frac{dx_s}{dt}, \frac{dy_s}{dt}\right) \\ &= \begin{pmatrix} \frac{dx_s}{dt} & \frac{dy_s}{dt} \end{pmatrix} \cdot P_s \cdot \begin{pmatrix} \frac{dx_s}{dt} \\ \frac{dy_s}{dt} \end{pmatrix} \end{aligned}$$

and that the measure θ of an angle between vectors $\hat{V}_1$ and $\hat{V}_2$ on the R-sphere is computed by

$$\arccos\left(\frac{V_1^s \cdot P_s \cdot (V_2^s)^t}{|V_1^s|_s\, |V_2^s|_s}\right).$$

Notice that the matrix P_s still makes sense when $K = 0$ and when K becomes negative.

Exercise 135 *Write the formula for the* K*-dot product* (x_s, y_s)*-coordinates when* $K = 0$*. Does it look familiar?*

So we do have

Spherical $(K > 0)$	Euclidean $(K = 0)$	Hyperbolic $(K < 0)$
$\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = R^2$		
$\hat{V} \bullet \hat{V}$		
$1 = K\left(x^2 + y^2\right) + z^2$	$1 = K\left(x^2 + y^2\right) + z^2$	$1 = K\left(x^2 + y^2\right) + z^2$
$V_1 \bullet_K V_2$		$V_1 \bullet_K V_2$
$V_1^c \bullet_c V_2^c$	$V_1^c \bullet_c V_2^c$	$V_1^c \bullet_c V_2^c$
$V_1^s \bullet_s V_2^s$	$V_1^s \bullet_s V_2^s$	$V_1^s \bullet_s V_2^s$

where

$$V_1^s \bullet_s V_2^s = (V_1^s) \cdot P_s \cdot (V_2^s)^t.$$

Of course if $K > 0$, we again have Euclidean angles θ between vectors $\hat{V}_1$ and $\hat{V}_2$ tangent to the R-sphere at some point computed by

$$\begin{aligned} \hat{V}_1 \bullet \hat{V}_2 &= \left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right| \cdot \cos(\theta) \\ &= V_1^s \bullet_s V_2^s. \end{aligned}$$

12.3 Area in stereographic projection coordinates

Suppose you were given a region G_s in the (x_s, y_s)-coordinate plane. Also suppose that $K > 0$. If you trace back everything we have done with coordinate changes, you can see how G_s gives you a region $\hat{G}$ on the sphere of radius $R = K^{-1/2}$ in Euclidean 3-space via the formulas

$$\begin{aligned}(\hat{x}, \hat{y}, \hat{z}) &= (x, y, Rz) \\ &= \rho \cdot (x_s, y_s, R(2\rho - 1)) \\ &= \left(\frac{x_s}{\frac{K}{4}(x_s^2 + y_s^2) + 1}, \frac{y_s}{\frac{K}{4}(x_s^2 + y_s^2) + 1}, \frac{R\left(1 - \frac{K}{4}\left(x_s^2 + y_s^2\right)\right)}{1 + \frac{K}{4}(x_s^2 + y_s^2)} \right).\end{aligned}$$

Now there is a formula in several variable calculus for computing the area of the region $\hat{G}$ on the sphere of radius R in Euclidean 3-space in terms of the parameters (x_s, y_s). [DS,49,231]. It is

$$\int_{G_c} \hat{a}\left(\frac{d\hat{X}}{dx_s}, \frac{d\hat{X}}{dy_s}\right) dx_s dy_s$$

where $\hat{a}\left(\frac{d\hat{X}}{dx_s}, \frac{d\hat{X}}{dy_s}\right)$ is the (Euclidean) area of the parallelogram spanned by the two vectors $\frac{d\hat{X}}{dx_s}$ and $\frac{d\hat{X}}{dy_s}$ in Euclidean 3-space. That is

$$\hat{a}\left(\frac{d\hat{X}}{dx_s}, \frac{d\hat{X}}{dy_s}\right) = \left|\frac{d\hat{X}}{dx_s}\right| \cdot \left|\frac{d\hat{X}}{dy_s}\right| \cdot \sin(\theta)$$

where θ is the angle between the two vectors $\frac{d\hat{X}}{dx_s}$ and $\frac{d\hat{X}}{dy_s}$.

Exercise 136 *As in Exercise 96 show that*

$$\begin{aligned}\hat{a}\left(\frac{d\hat{X}}{dx_s}, \frac{d\hat{X}}{dy_s}\right)^2 &= \begin{vmatrix} \frac{d\hat{X}}{dx_s} \bullet \frac{d\hat{X}}{dx_s} & \frac{d\hat{X}}{dy_s} \bullet \frac{d\hat{X}}{dx_s} \\ \frac{d\hat{X}}{dx_s} \bullet \frac{d\hat{X}}{dy_s} & \frac{d\hat{X}}{dy_s} \bullet \frac{d\hat{X}}{dy_s} \end{vmatrix} \\ &= \begin{vmatrix} \frac{dX}{dx_s} \bullet_K \frac{dX}{dx_s} & \frac{dX}{dy_s} \bullet_K \frac{dX}{dx_s} \\ \frac{dX}{dx_s} \bullet_K \frac{dX}{dy_s} & \frac{dX}{dy_s} \bullet_K \frac{dX}{dy_s} \end{vmatrix}\end{aligned}$$

Now notice the matrix D_s in Exercise 133 is simply the 2×3 matrix whose rows are the vectors $\frac{dX}{dx_s}$ and $\frac{dX}{dy_s}$.

Exercise 137 *Use Exercise 134 to show that*

$$\hat{a}\left(\frac{d\hat{X}}{dx_s}, \frac{d\hat{X}}{dy_s}\right)^2 = \rho^4 = \frac{1}{\left(\frac{K}{4}(x_s^2 + y_s^2) + 1\right)^4}.$$

13 Relationship between central and stereographic projection coordinates

13.1 Two addresses for the same point in K-geometry

Finally we should compare the relationship between the two kinds of coordinates for the set

$$\left\{(x,y,z) \in \mathbb{R}^3 : 1 = K\left(x^2+y^2\right)+z^2\right\} \tag{39}$$

that we have been exploring, namely central projection coordinates

$$\begin{aligned} x_c &= \frac{x}{z} \\ y_c &= \frac{y}{z} \end{aligned} \tag{40}$$

and stereographic projection coordinates

$$\begin{aligned} x_s &= \frac{2x}{z+1} \\ y_s &= \frac{2y}{z+1}. \end{aligned} \tag{41}$$

To do this we use the Exercises in which we wrote (x,y,z) in the set (39) as functions of (x_c, y_c) and (x_s, y_s) respectively. Namely

$$\begin{aligned} x &= \frac{x_c}{\sqrt{K\left(x_c^2+y_c^2\right)+1}} \\ y &= \frac{y_c}{\sqrt{K\left(x_c^2+y_c^2\right)+1}} \\ z &= \frac{1}{\sqrt{K\left(x_c^2+y_c^2\right)+1}} \end{aligned}$$

and

$$\begin{aligned} x &= \frac{x_s}{1+\frac{K}{4}\left(x_s^2+y_s^2\right)} \\ y &= \frac{y_s}{1+\frac{K}{4}\left(x_s^2+y_s^2\right)} \\ z &= \frac{1-\frac{K}{4}\left(x_s^2+y_s^2\right)}{1+\frac{K}{4}\left(x_s^2+y_s^2\right)}. \end{aligned}$$

The rest is simple algebra. [MJG,258]

Exercise 138 *Do the algebra to write the explicit formulas for*

$$\left(x_s\left(x_c,y_c\right), y_s\left(x_c,y_c\right)\right)$$

and

$$\left(x_c\left(x_s,y_s\right), y_c\left(x_s,y_s\right)\right).$$

13.2 Plane sections of K-geometry

Exercise 139 *Suppose we intersect K-geometry* (39) *with a plane*

$$ax + by + z = 0.$$

a) Find the equation for the resulting path in central projection coordinates.

b) Show that the equation for the resulting path in stereographic projection coordinates is

$$\left(x_s - \frac{2a}{K}\right)^2 + \left(y_s - \frac{2b}{K}\right)^2 = \frac{4\left(K + a^2 + b^2\right)}{K^2}.$$

c) What is the equation for the resulting path in stereographic projection coordinates if we intersect the K-geometry with a plane given by

$$ax + by = 0,$$

that is, a plane containing the z-axis?

13.3 When K is negative, there are asymptotic cones

Notice that, if $K < 0$, the equation of K-geometry becomes

$$z^2 - |K|\left(x^2 + y^2\right) = 1.$$

Thus K-geometry forms a 2-sheeted hyperboloid with the z-axis as major axis. The hyerboloid is obtained by rotating the hyperbola

$$z^2 - |K|\, x^2 = 1 \tag{42}$$

in the (x, z)-plane around the z-axis. We will only consider the sheet on which z is positive as forming the K-geometry.

Now the asymptotes of this last hyperbola are the pair of lines

$$\left(z + |K|^{1/2}\, x\right)\left(z + |K|^{1/2}\, x\right) = z^2 - |K|\, x^2 = 0.$$

Rotating the asymptotes around the z-axis we obtain the *asymptotic cone*

$$z^2 - |K|\left(x^2 + y^2\right) = 0$$

for K-geometry.

Since the entire K-geometry lies inside the asymptotic cone, the central projection coordinates (x_c, y_c) only correspond to points in K-geometry when $(x_c, y_c, 1)$ lies *inside* the asymptotic cone, that is, when

$$1 - |K|\left(x^2 + y^2\right) > 0.$$

Said otherwise, the disk of radius $|K|^{-1/2}$ around $(0, 0)$ in the (x_c, y_c)-plane captures all the points (x, y, z) of K-geometry under central projection.

Exercise 140 *a) Show that the slopes of the asymptotes to the hyperbola* (42) *are the limits as x goes to $\pm\infty$ of the slopes of the lines through* $(0, 0)$ *and the point (x, z) on the hyperbola.*

b) Show that the slopes of the asymptotes to the hyperbola (42) *are the limits as x goes to $\pm\infty$ of the slopes of the lines through* $(0, -1)$ *and the point* (x, z) *on the hyperbola.*

c) Use b) to compute the radius of the disk around $(0, 0)$ *in the* (x_s, y_s)*-plane that captures all the points* (x, y, z) *of K-geometry under stereographic projection.*

Part VI

Spherical geometry II

14 Rigid motions in spherical geometry

14.1 Rigid motions in Euclidean coordinates

If we have a curve $\hat{X}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t))$ on the R-sphere, we have seen that we measure its length L by the formula

$$L = \int_b^e l(t)\, dt \tag{43}$$

where

$$l(t)^2 = \frac{d\hat{X}}{dt} \bullet \frac{d\hat{X}}{dt} = \frac{dX}{dt} \bullet_K \frac{dX}{dt} \tag{44}$$

and that we measure angles θ between tangent vectors $\hat{V}_1$ and $\hat{V}_2$ at a point on the R-sphere by the formula

$$\theta = \arccos\left(\frac{\hat{V}_1 \bullet \hat{V}_2}{\left|\hat{V}_1\right| \cdot \left|\hat{V}_2\right|}\right) = \arccos\left(\frac{V_1 \bullet_K V_2}{|V_1|_K \cdot |V_2|_K}\right)$$

where

$$|V|_K^2 = V \bullet_K V.$$

Let $\hat{M}$ denote an invertible 3×3 matrix. We begin by noting the condition that a transformation

$$\left(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\right) = (\hat{x}, \hat{y}, \hat{z}) \cdot \hat{M}$$

preserve the length of any curve lying on the R-sphere. Rewriting the transformation as

$$\underline{\hat{X}} = \hat{X} \cdot \hat{M} \tag{45}$$

the formulas (43) and (44) show that all we have to worry about is that

$$\frac{d\underline{\hat{X}}}{dt} \bullet \frac{d\underline{\hat{X}}}{dt} = \frac{d\hat{X}}{dt} \bullet \frac{d\hat{X}}{dt}$$

for all values t of the parameter of the curve. But

$$\frac{d\underline{\hat{X}}}{dt} = \frac{d\hat{X}}{dt} \cdot \hat{M} \tag{46}$$

by the product rule since $\hat{M}$ is a constant matrix. So the transformation given by the matrix $\hat{M}$ will preserve the length of any path and will preserve the measure of any angle if

$$\frac{d\hat{X}}{dt} \cdot \hat{M} \cdot \left(\frac{d\hat{X}}{dt} \cdot \hat{M}\right)^t = \frac{d\hat{X}}{dt} \cdot \left(\frac{d\hat{X}}{dt}\right)^t. \tag{47}$$

Exercise 141 *(**SG**) a) Show that the transformation* (45) *takes the R-sphere to itself if*

$$\hat{M} \cdot \hat{M}^t = I$$

where I is the 3×3 *identity matrix. (A matrix $\hat{M}$ satisfying this condition is called an orthogonal matrix.)*

b) Show that (47) *also holds if $\hat{M}$ is orthogonal.*

Exercise 142 *(**SG**) Show that the matrix*

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is orthogonal. Describe geometrically what this transformation is doing to the R-sphere. [DS,316ff]

Exercise 143 *(**SG**) Show that the matrix*

$$\begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

is orthogonal. Describe geometrically what this transformation is doing to the R-sphere.

Definition 144 *A distance-preserving transformation of a space or geometry is called a **rigid motion** or a **congruence**.*

So, by Exercise 141, every orthogonal matrix $\hat{M}$ corresponds to a rigid motion of the R-sphere. (It can be shown that every rigid motion of the R-sphere is given by an orthogonal matrix, but we will not treat that subtlety in this book.)

Exercise 145 *(**SG**) Show that the set of orthogonal matrices $\hat{M}$ form a group. That is, show that*

a) the product of two orthogonal matrices is orthogonal,

b) the identity matrix is orthogonal,

c) the inverse matrix $\hat{M}^{-1}$ of an orthogonal matrix $\hat{M}$ is orthogonal.

Hint: Write

$$\hat{M} \cdot \hat{M}^{-1} = I = \hat{M} \cdot \hat{M}^t$$

and use matrix multiplication to reduce to showing that the transpose of an orthogonal matrix is orthogonal.) [MJG,311]

Our conclusion from this exercise is that the set of rigid motions of the Euclidean R-sphere form a group.

14.2 Orthogonal and K-orthogonal matrices

Recalling again the fact that the Euclidean R-sphere is a K-geometry with $K = 1/R^2$ we should compare the K-orthogonal transformations M in Definition 122 with the orthogonal ones just above.

Exercise 146 *(**SG**) Referring to Definition 122, show that the transformations*

$$\left(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\right) = (\hat{x}, \hat{y}, \hat{z}) \cdot \hat{M}$$

and

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot M$$

give the same rigid motion of the R-sphere if

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix} \cdot \hat{M} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^{-1} \end{pmatrix}.$$

Exercise 147 *(**SG**) Show that the matrix*

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is both orthogonal and K-orthogonal and gives the same transformation of the Euclidean R-sphere. Describe geometrically what this transformation is doing to the R-sphere.

Exercise 148 *(**SG**) Show that the matrix*

$$\hat{M} = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

is orthogonal and that the matrix

$$M = \begin{pmatrix} \cos\varphi & 0 & R^{-1}\cdot\sin\varphi \\ 0 & 1 & 0 \\ -R\cdot\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

is the K-orthogonal matrix describing the same transformation of the R-sphere. Describe geometrically what this transformation is doing to the R-sphere.

15 Spherical geometry is homogeneous

15.1 Moving a point to the North Pole by a rigid motion

As the heading suggests, we are next going to move any point on the R-sphere to the North Pole by a rigid motion. However, we are going to describe the entire process in (x, y, z)-coordinates, that is, in K-geometry. This will allow us to use all the computations we did in Part V since **SG** is a K-geometry in the sense of Part V. Recll that, in (x, y, z)-coordinates, the equation for the R-sphere becomes

$$K\left(x^2 + y^2\right) + z^2 = 1 \tag{48}$$

with

$$K = \frac{1}{R^2}$$

and the Euclidean dot product is given by the K-dot product. Again, if you get nervous using these weird coordinates to compute things that are clearer in $(\hat{x}, \hat{y}, \hat{z})$-coordinates, just go through the constructions in the case $K = 1$ first. In that special case

$$(x, y, z) = (\hat{x}, \hat{y}, \hat{z})$$

and your calculations (as well as all those in Part V above reduce to the usual ones on the unit sphere in ordinary Euclidean 3-space.

So, first of all, in (x, y, z)-coordinates the North Pole is the point

$$N = (0, 0, 1).$$

Suppose we start with a point

$$X_0 = (x_0, y_0, z_0)$$

in the geometry, that is, satisfying the equation (48).

Exercise 149 ***(SG)*** *Write an explicit K-rigid motion*

$$M_1 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes the point X_0 to a point $X_1 = (x_1, 0, z_0)$.

Hint: Start from the identity

$$\frac{-y_0}{\sqrt{x_0^2 + y_0^2}} \cdot x_0 + \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \cdot y_0 = 0$$
$$\sin\theta \cdot x_0 + \cos\theta \cdot y_0 = 0$$

and then show that there is a θ so that

$$\cos\theta = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}$$
$$\sin\theta = \frac{-y_0}{\sqrt{x_0^2 + y_0^2}}.$$

Exercise 150 ***(SG)*** *Write an explicit K-rigid motion*

$$M_2 = \begin{pmatrix} \cos\varphi & 0 & R^{-1}\cdot\sin\varphi \\ 0 & 1 & 0 \\ -R\cdot\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

that takes the point $X_1 = (x_1, 0, z_0)$ to $N = (0, 0, 1)$.

Using these last two Exercises we conclude that the transformation

$$(\underline{x}, \underline{y}, \underline{z}) = (x, y, z) \cdot (M_1 \cdot M_2)$$

is a K-rigid motion (why?) and that

$$N = (x_0, y_0, z_0) \cdot (M_1 \cdot M_2)$$

(why?).

15.2 Moving a (point, direction) to any other (point, direction) by a rigid motion

Let

$$V_2 = (a_2, b_2, 0)$$

be a tangent vector to K-geometry at the North Pole N.

Exercise 151 *(**SG**) Write an explicit K-rigid motion*

$$M_3 = \begin{pmatrix} \cos\theta' & \sin\theta' & 0 \\ -\sin\theta' & \cos\theta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes V_2 to the vector

$$\left(\sqrt{a_2^2 + b_2^2}, 0, 0\right) = \left(\sqrt{V_2 \bullet_K V_2}, 0, 0\right).$$

Why does the transformation given by M_3 leave the North Pole N fixed?

Now suppose we have any point

$$X_0 = (x_0, y_0, z_0)$$

in K-geometry and any K-tangent vector

$$V_0 = (a_0, b_0, c_0)$$

at that point.

Exercise 152 *(**SG**) Explain why the K-rigid motion*

$$(\underline{x}, \underline{y}, \underline{z}) = (x, y, z) \cdot (M_1 \cdot M_2 \cdot M_3)$$

constructed over the last couple of sections takes the point X_0 to N and the tangent vector V_0 to $\left(\sqrt{V_0 \bullet_K V_0}, 0, 0\right)$

Now suppose that (X_0, V_0) gives a point X_0 in K-geometry and a tangent direction V_0 to K-geometry at X_0. Suppose that (X_0', V_0') gives another point in K-geometry and a tangent direction to K-geometry at X_0'. Finally suppose that

$$V_0 \bullet_K V_0 = V_0' \bullet_K V_0'.$$

As above, find a K-rigid motion given by

$$M = (M_1 \cdot M_2 \cdot M_3)$$

taking X_0 to the North Pole and V_0 to $\left(\sqrt{V_0 \bullet_K V_0}, 0, 0\right)$. Similarly find a K-rigid motion given by

$$M' = (M_1' \cdot M_2' \cdot M_3')$$

taking X_0' to the North Pole and V_0' to $\left(\sqrt{V_0' \bullet_K V_0'}, 0, 0\right)$.

Exercise 153 *(**SG**) Explain why the K-rigid motion given by*

$$M \cdot (M')^{-1}$$

takes (X_0, V_0) *to* $\left(X_0', \left(\frac{|V_0|_K}{|V_0'|_K}\right) \cdot V_0'\right)$.

By completing this Exercise we have shown that K-geometry looks the same at each point and in each direction at that point. That is, we have shown that each K-geometry is homogeneous.

16 Lines in Spherical Geometry

16.1 Spherical coordinates, a shortest path from the North Pole

We next will figure out what is the shortest path you can take between two points on the Euclidean R-sphere. Again we will do our calculation using only (x, y, z)-coordinates (since, as we have seen in (33) we won't have $(\hat{x}, \hat{y}, \hat{z})$-coordinates when we get to Hyperbolic Geometry. For our purposes, it will be convenient to use yet another set of coordinates for K-geometry, namely what are commonly known as spherical coordinates:

$$\begin{aligned} x(\sigma, \tau) &= R \cdot \sin\sigma \cdot \cos\tau \\ y(\sigma, \tau) &= R \cdot \sin\sigma \cdot \sin\tau \\ z(\sigma, \tau) &= \cos\sigma \end{aligned} \tag{49}$$

Exercise 154 *(**SG**) Show that these spherical coordinates do actually parametrize the R-sphere, that is, that*

$$K\left(x(\sigma, \tau)^2 + y(\sigma, \tau)^2\right) + z(\sigma, \tau)^2 \equiv 1$$

for all (σ, τ).

Notice that you can write a path on the R-sphere by giving a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane. In fact, you can use σ as the parameter t and just write

$$(\sigma, \tau(\sigma)) \tag{50}$$

where τ is a function of σ. To write a path that starts at the North Pole, just write

$$(\sigma, \tau(\sigma)),\ 0 \leq \sigma \leq \varepsilon$$

and demand that

$$\tau(0) = 0.$$

If you want the path to end on the plane $y = \hat{y} = 0$, demand additionally that

$$\tau(\varepsilon) = 0.$$

But if we are going to describe paths on the R-sphere by paths in the (σ, τ)-plane we are going to need to figure out the K-dot product in (σ, τ)-coordiates so that we can compute the lengths of paths in these coordinates.

Exercise 155 *(**SG**) a) Refering to* (49)*compute the* 2×3 *matrix*

$$D_{sph} = \begin{pmatrix} \frac{dx}{d\sigma} & \frac{dy}{d\sigma} & \frac{dz}{d\sigma} \\ \frac{dx}{d\tau} & \frac{dy}{d\tau} & \frac{dz}{d\tau} \end{pmatrix}.$$

b) Show that, if a path in K-geometry is given by a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane,

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \cdot D_{sph}.$$

c) For two paths in K-geometry given by paths $(\sigma_1(t), \tau_1(t))$ and $(\sigma_2(t), \tau_2(t))$ in the (σ, τ)-plane, use a) and b) to show that

$$\left(\frac{d\hat{x}_1}{dt}, \frac{d\hat{y}_1}{dt}, \frac{d\hat{z}_1}{dt}\right) \cdot \left(\frac{dx_2}{dt}, \frac{d\hat{y}_2}{dt}, \frac{d\hat{z}_2}{dt}\right)^t =$$

$$\left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right)^t =$$

$$\left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot \begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1}\sin^2\sigma \end{pmatrix} \cdot \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right)^t$$

d) Explain why the definition

$$\left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \bullet_{sph} \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right) = \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot \begin{pmatrix} K^{-1} & 0 \\ 0 & K^{-1}\sin^2\sigma \end{pmatrix} \cdot \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right)^t$$

allows us to compute the dot product of two tangent vectors to the R-sphere in Euclidean space if we just know the values of the two corresponding vectors in the (σ, τ)-plane.

Exercise 156 ***(SG)*** *Show that the length L of any path on the R-sphere given by*

$$(\sigma, \tau(\sigma)),\ 0 \leq \sigma \leq \varepsilon$$

with

$$\tau(0) = 0.$$

and

$$\tau(\varepsilon) = 0$$

is given by the formula

$$L = R\int_0^{\varepsilon} \sqrt{\left(1, \frac{d\tau}{d\sigma}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\sigma \end{pmatrix} \cdot \left(1, \frac{d\tau}{d\sigma}\right)^t} d\sigma.$$

Hint: Use Exercise 155 with $t = \sigma$.

This last formula for L lets us figure out the shortest path from $N = (R{\cdot}\sin 0{\cdot}\cos 0, R{\cdot}\sin 0{\cdot}\sin 0, \cos 0)$ to $(R{\cdot}\sin\varepsilon, 0, \cos\varepsilon) = (R{\cdot}\sin\varepsilon{\cdot}\cos 0, R{\cdot}\sin 0{\cdot}\sin 0, \cos\varepsilon)$. Since

$$L = R\int_0^{\varepsilon} \sqrt{1 + \sin^2\sigma{\cdot} \left(\frac{d\tau}{d\sigma}\right)^2} d\sigma$$

and $\sin^2\sigma$ is is positive for almost all $\sigma \in [0, \varepsilon]$, L is minimal only when $\frac{d\tau}{d\sigma}$ is identically 0. But this means that $\tau(\sigma)$ is a constant function. Since $\tau(0) = 0$, this means that $\tau(\sigma)$ is identically 0. So we have the shown the following result.

Theorem 157 *(**SG**) The shortest path on the R-sphere from the North Pole to a point* $(x, y, z) = (R{\cdot}\sin\varepsilon, 0, \cos\varepsilon)$ *is the path lying in the plane* $y = 0$.

16.2 Shortest path between any two points

We next prove the theorem that shows that shortest path on the surface of the earth from Rio de Janeiro to Los Angeles is the one cut on the surface of the earth by the plane that passes through the center of the earth and through Rio and through Los Angeles. That is usually the route an airplane would take when flying between the two cities.

Theorem 158 *(**SG**) Given any two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in K-geometry, the shortest path between the two points is the path cut out by the two equations*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

$$\left| \begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \right| = 0, \tag{51}$$

that is, the plane containing $(0,0,0)$ and X_1 and X_2.

Proof. By Exercise 153 there is a K-rigid motion M that takes X_1 to the North Pole N and X_2 to $\left(K^{-1/2} sin\varepsilon, 0, cos\varepsilon\right)$ for some ε. That is

$$X_2 \cdot M = (R{\cdot}\sin\varepsilon, 0, \cos\varepsilon)$$

for some ε since all points in K-geometry with $\underline{y}= 0$ can be written as $(R{\cdot}\sin\varepsilon, 0, \cos\varepsilon)$ for some ε. But

$$\left| \begin{pmatrix} \underline{x} & \underline{y} & \underline{z} \\ 0 & 0 & 1 \\ K^{-1/2} sin\varepsilon & 0 & cos\varepsilon \end{pmatrix} \right| = \left| \begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \cdot M \right|$$

$$= \left| \begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \right| \cdot |M| = 0,$$

Since $|M| \neq 0$ and $K^{-1/2} sin\varepsilon \neq 0$ if $\varepsilon < \pi$, (x, y, z) lies in the plane (51) if and only if

$$\underline{y} = 0.$$

Since M is a K-rigid motion it must take the shortest path from X_1 to X_2 to the shortest path from $X_1 \cdot M = N$ to $X_2 \cdot M = (R{\cdot}\sin\varepsilon, 0, \cos\varepsilon)$. But we already know that the shortest path from $X_1 \cdot M$ to $X_2 \cdot M$ is the one cut out by the plane $y = 0$. But that path comes from the path cut out by the plane given by equation (51). This path is called the *great circular arc* between X_1 and X_2. ■

Definition 159 *A **line** in **SG** will be a curve that extends infinitely in each direction and has the property that, given any two points X_1 and X_2 on the path, the shortest path between X_1 and X_2 lies along that curve. Lines in **SG** are usually called great circles on the R-sphere. They are the intersections of the R-sphere with planes through $(0,0,0)$.*

Letting X_2 approach X_1 along the great circular arc joining X_1 and X_2 we see that the solution set to equation (51) does not change. Taking a limit as X_2 approaches X_1, this set can also be expressed as the solution set of the equation

$$\left| \begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ a_1 & b_1 & c_1 \end{pmatrix} \right| = 0$$

where (a_1, b_1, c_1) is a tangent vector at the point X_1 pointing in the direction of X_2.

17 Central projection in SG

17.1 Central projection preserves lines

We all probably realize that you can't make a perfect map of the world; that is, you can't make a map so that angles on the map are equal to the corresponding angles on the sphere and straight lines on the map correspond to great circular arcs on the sphere. We do the next best thing–we make two maps of the sphere, one that has the property that angles are faithfully represented and the other for which straight lines on the map correspond to shortest paths on the sphere. We start with a simple way to make a map for which straight lines on the map correspond to shortest paths on the sphere. The map coordinates we use to do this are the *central projection coordinates* we learrned about in Part V.

Now **SG** is a K-geometry in the sense of Part V since, in (x, y, z)-coordinates, the equation for the R-sphere becomes

$$K\left(x^2 + y^2\right) + z^2 = 1$$

with

$$K = \frac{1}{R^2}$$

and the Euclidean dot product is given by the K-dot product. So all the computations of Part V hold for Spherical Geometry as long as we understand that we are computing it in (x, y, z)-coordinates.

Exercise 160 *Show that central projection of a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space to the plane $\hat{z} = R$ is the same as central projection of the corresponding point in (x, y, z)-coordinates to the plane $z = 1$.*

Hint: Recall (35) *and write the corresponding relation $\hat{r}(\hat{x}_c, \hat{y}_c, R) = (\hat{x}, \hat{y}, \hat{z})$ in $(\hat{x}, \hat{y}, \hat{z})$-coordinates. Conclude that $\hat{r} = r$. (Why?)*

Exercise 161 *(**SG**) Show that lines (i.e. shortest paths in **SG**) correspond under central projection to straight lines in the (x_c, y_c)-coordinates.*

Hint: See Exercise 139a). Or just write the equation for a line in (x_c, y_c)-coordinates and substitute (40)*. Then reverse the process.*

17.2 Spherical area computed in central projection coordinates

Recall that in Part V we learned how to compute the K-area in K-geometry of a region G given by a region G_c in the (x_c, y_c)-plane. That is, we learned how to compute the area of a region $\hat{G}$ on the R-sphere given by a region G_c in the (x_c, y_c)-plane

Exercise 162 *(**SG**) Show that, if a region $\hat{G}$ on the R-sphere is parametrized by a region G_c in (x_c, y_c)-coordinates, then the area $\hat{A}$ of $\hat{G}$ is given by the formula*

$$\hat{A} = \int_{G_c} \left(K\left(x_c^2 + y_c^2\right) + 1\right)^{-3/2} dx_c dy_c.$$

18 Stereographic projection in SG

18.1 Stereographic projection preserves angles

We now turn to a simple way to make a map of the R-sphere in such a way that the measure of any angle on the map is exactly the same as the measure of the corresponding angle on the R-sphere. The map coordinates that do the job are the *stereographic projection coordinates* that again we learned about in Part V.

Exercise 163 ***(SG)*** *a) Compute the stereographic projection of a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space to the plane $\hat{z} = R$.*

b) Show that the coordinates $(\hat{x}_s, \hat{y}_s)$ of the stereographic projection of a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space to the plane $\hat{z} = R$ are the same as the coordinates (x_s, y_s) of the stereographic projection of the corresponding point in (x, y, z)-coordinates to the plane $z = 1$.

Hint: Reduce to showing that

$$R\left(\frac{2\hat{x}}{\hat{z}+R}, \frac{2\hat{y}}{\hat{z}+R}\right) = \left(\frac{2x}{z+1}, \frac{2y}{z+1}\right).$$

Exercise 164 ***(SG)*** *a) Show that stereographic projection is conformal, that is, that the angle between two paths through a point on the R-sphere in $(\hat{x}, \hat{y}, \hat{z})$-space is the same as the usual (Euclidean) angle between the corresponding two paths through the corresponding point in the (x_s, y_s)-plane.*

Hint: From Exercise 134 we know that, for tangent vectors $\hat{V}_1$ and $\hat{V}_2$ emanating from the same point on the R-sphere

$$\begin{aligned} \hat{V}_1 \bullet \hat{V}_2 &= V_1 \bullet_K V_2 \\ &= V_1^s \bullet_s V_2^s \\ &= (V_1^s) \cdot \begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix} \cdot (V_2^s)^t. \end{aligned}$$

b) Draw a picture of an angle between two paths through a point on the Euclidean R-sphere and the stereographic projection of that angle onto the plane $\hat{z} = R$. Try to give an intuitive geometric explanation for why it should have the same measure as the original angle.

18.2 Areas of spherical triangles in stereographic projection coordinates

Using Exercise 139b) to show that lines in **SG** become circles under stereographic projection unless the line in **SG** passes through the North Pole (in which case it corresponds to a line through $(x_s, y_s) = (0,0)$ in the (x_s, y_s)-plane). Suppose a spherical triangle T corresponds to a region T_s in (x_s, y_s)-coordinates and the vertices of T correspond to $(x_s, y_s) = (-2, 0)$, $(x_s, y_s) = (2, 0)$, and $(x_s, y_s) = (0, 2)$. So one side of T_s lies on the line $y_s = 0$.

Exercise 165 *a) Use Exercise 139b) to compute the equations for the other two sides of* T_s.

b) In the (x_s, y_s)*-plane, draw* T_s *as accurately as you can when* $K = 4$*, then then when* $K = \frac{1}{4}$.

c) Compute the area of T *in both cases in b).*

Hint: You will need the radian measure of the angle at each of the vertices of T_s*. Why? To calculate these angles, calculate* $\frac{dy_s}{dx_s}$ *by implicit differentiation of the equations in a), then take* $\arctan\left(\frac{dy_s}{dx_s}\right)$ *in radians. Your job will be easier if you notice that the* y_s*-axis divides* T_s *into two congruent isosceles triangles.*

Exercise 166 *Explain why we know from an Exercise in Part V that in all of the cases in Exercise 165 the area of the spherical triangle* T *is also given by the formula*

$$\int_{T_s} \frac{1}{\left(1 + \frac{K}{4}\left(x_s^2 + y_s^2\right)\right)^2} dx_s dy_s.$$

Part VII
Hyperbolic geometry

19 The curvature K becomes negative

19.1 The world sheet and the light cone

We now turn to the case in which the radius R of the Euclidean R-sphere goes to infinity and beyond! Of course that doesn't make any sense in $(\hat{x}, \hat{y}, \hat{z})$-space but if we look at the R-sphere in (x, y, z)-coordinates, it makes perfect sense because there the equation of the R-sphere is

$$K\left(x^2+y^2\right)+z^2=1 \tag{52}$$

for $K=\frac{1}{R^2}$ so that R going to infinity means that K goes to 0 and 'beyond' simply means that K becomes negative. We have seen that all we need to have a geometry with lengths, angles, areas and congruences is to have a smooth set and a dot-product between vectors tangent to that set. Now if K becomes negative, our geometry becomes a hyperboloid of two sheets (obtained by rotating a hyperbola in the (x, z)-plane with major axis the z-axis around that axis). So that we have a connected universe, we will only consider the 'top' sheet (where $z>0$) as our K-geometry. (In special relativity, this sheet might be called something like the 'world sheet.') If, instead of rotating a hyperbola around the z-axis we rotate the asymptotes of the hyperbola around the z-axis, we obtain a cone given by the equation

$$K\left(x^2+y^2\right)+z^2=0.$$

(Again this might be called something like the 'light cone.')

There is one potential problem we need to worry about when $K<0$, and it is regarding the length of tangent vectors. Namely, our formulas for lengths invove taking square roots of dot products of tangent vectors with themselves, so those dot-products had better be positive (and only zero if the tangent vector itself is the zero-vector.) Our K-dot product is given by the formula

$$\left(\begin{array}{lll}a & b & c\end{array}\right) \cdot\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1}\end{array}\right) \cdot\left(\begin{array}{l}a \\ b \\ c\end{array}\right).$$

So, when $K<0$, it seems entirely possible that some tangent vector V has the property that $V \bullet_K V<0$. (Indeed that will always happen if c is sufficiently big and x and y are sufficiently small. [MJG,241-242]

19.2 Non-zero tangent vectors in HG have positive length

Exercise 167 *Suppose that V emanates from $(0,0,0)$ in (x,y,z)-space.*

a) Show that $V \bullet_K V = 0$ if and only if V points in a direction of the light cone.

b) Show that $V \bullet_K V < 0$ if and only if V points in a direction inside the light cone.

Hint: Use that the (Euclidean) angle θ that the light cone makes with the plane $z = 0$ is given by taking any point (x,y,z) on the light cone with $z > 0$ and computing

$$\tan(\theta) = \frac{z}{\sqrt{x^2+y^2}} = |K|^{1/2}.$$

Compute the angle that V makes with the plane $z = 0$ in a similar way.

Now our world sheet lies *inside* the light cone but tangent vectors to it point *outside* the light cone. That is what saves our K-dot product, as we see in the next Lemma.

Lemma 168 *(**HG**) Let $V = (a,b,c)$ denote a vector that is tangent to our K-geometry, that is, to the set* (52)*. Then*

$$V \bullet_K V \geq 0$$

and $V \bullet_K V = 0$ if and only if $V = 0$.

Proof. If $c = 0$, then the assertion of the Lemma is obviously true. So we can assume $c \neq 0$. Notice, since V is assumed to be a tangent vector at (x,y,z), this means that (x,y,z) is not the North Pole so that

$$x^2 + y^2 > 0.$$

Next replacing V with $\frac{1}{c}(V)$ just multiplies $V \bullet_K V$ by $\frac{1}{c^2}$ so it suffices to consider the case in which

$$V = (a,b,1)$$

and we must show that

$$\left(a^2+b^2\right) + K^{-1} > 0.$$

Since V is tangent to our K-geometry at some point (x,y,z), we know by Exercise 120 that $(x,y,z) \bullet_K V = 0$, that is,

$$ax + by + \frac{z}{K} = 0.$$

On the other hand

$$K\left(x^2+y^2\right) + z^2 = 1.$$

Substituting this becomes

$$K\left(x^2+y^2\right) + K^2(ax+by)^2 = 1.$$

On the other hand

$$(ay - bx)^2 \geq 0$$
$$(ay)^2 + (bx)^2 \geq 2abxy$$

so that

$$K\left(x^2 + y^2\right) + K^2\left((ax)^2 + (by)^2 + (ay)^2 + (bx)^2\right) \geq 1$$
$$K\left(x^2 + y^2\right) + K^2\left(a^2 + b^2\right)\left(x^2 + y^2\right) \geq 1$$
$$K^{-1} + \left(a^2 + b^2\right) \geq \frac{1}{K^2\left(x^2 + y^2\right)}.$$

■

20 Hyperbolic geometry is homogeneous

20.1 Rigid motions in (x, y, z)-coordinates

Now **HG** is a K-geometry in the sense of Part V since, in (x, y, z)-coordinates, the equation for the K-geometry becomes

$$K\left(x^2+y^2\right)+z^2=1 \tag{53}$$

with

$$K<0$$

and the K-dot product. If we have a curve $X(t)=(x(t), y(t), z(t))$ on the R-sphere given in K-coordinates as

$$1=K\left(x^2+y^2\right)+z^2,$$

we have seen that we measure its length L by the formula

$$L=\int_b^e l(t)\, dt \tag{54}$$

where

$$l(t)^2=\frac{dX}{dt} \bullet_K \frac{dX}{dt} \tag{55}$$

and that we measure angles θ between tangent vectors $\hat{V}_1$ and $\hat{V}_2$ at a point on the R-sphere by the formula

$$\theta=\arccos\left(\frac{V_1 \bullet_K V_2}{\left|V_1\right|_K \cdot\left|V_2\right|_K}\right)$$

where

$$|V|_K^2=V \bullet_K V.$$

We again want to explore the condition that a transformation

$$\left(\underline{x}, \underline{y}, \underline{z}\right)=(x, y, z) \cdot M$$

preserve the length of any curve $(x(t), y(t), z(t))$ lying on the R-sphere. Rewriting the transformation as

$$\underline{X}=X \cdot M$$

all we have to worry about is that

$$\frac{d\underline{X}}{dt} \bullet_K \frac{d\underline{X}}{dt}=\frac{dX}{dt} \bullet_K \frac{dX}{dt}.$$

So, referring to Definition 122 a transformation given by a matrix M will preserve the length of any path and will preserve the measure of any angle if M is K-orthogonal.

Exercise 169 *(**HG**) Show that the matrix*

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is K-orthogonal. Describe geometrically what this transformation is doing to the K-geometry.

For our second K-rigid motion in **HG** we will need a pair of functions

$$\left(\cosh\sigma = \frac{e^{\sigma} + e^{-\sigma}}{2}, \sinh\sigma = \frac{e^{\sigma} - e^{-\sigma}}{2}\right)$$

that parametrize the unit hyperbola

$$z^2 - x^2 = 1$$

in the same way that $(\cos\sigma, \sin\sigma)$ parametrize the unit circle. That is

$$\cosh^2\sigma - \sinh^2\sigma \equiv 1.$$

Exercise 170 *(**HG**) Show that the matrix*

$$\begin{pmatrix} \cosh\varphi & 0 & |K|^{1/2} \cdot\sinh\varphi \\ 0 & 1 & 0 \\ |K|^{-1/2} \cdot\sinh\varphi & 0 & \cosh\varphi \end{pmatrix}$$

is K-orthogonal. Describe geometrically what this transformation is doing to the the K-geometry.

Notice that, when $K > 0$, we had the relation

$$R^2 \cdot K = 1$$

where R was the radius of the sphere. The last exercise suggests that when $K < 0$, we should define R by the relation

$$R^2 \cdot |K| = 1$$

so that

$$R = |K|^{-1/2}.$$

(Now compare this last Exercise with the corresponding Exercise in Spherical Geometry.) In what follows, you will find the quantity $|K|^{-1/2}$ occurring throughout. Feel free to use the (simpler) notation R for this quantity as you work through the Exercises.

20.2 Moving a point to the North Pole by a rigid motion

So, first of all, the North Pole is the point

$$N = (0, 0, 1) .$$

Suppose we start with a point

$$X_0 = (x_0, y_0, z_0)$$

in the geometry, that is, satisfying the equation (53).

Exercise 171 *(**HG**) Write an explicit K-rigid motion*

$$M_1 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes the point X_0 to a point $X_1 = (x_1, 0, z_0)$.

Exercise 172 *(**HG**) Write an explicit K-rigid motion*

$$M_2 = \begin{pmatrix} \cosh\varphi & 0 & |K|^{1/2} \cdot \sinh\varphi \\ 0 & 1 & 0 \\ |K|^{-1/2} \cdot \sinh\varphi & 0 & \cosh\varphi \end{pmatrix}$$

that takes the point $X_1 = (x_1, 0, z_0)$ to $N = (0, 0, 1)$.

Hint: Notice that

$$Kx_1^2 + z_0^2 = 1 = -\left(-|K|^{1/2} \cdot x_1\right)^2 + z_0^2.$$

So there is a φ with

$$\cosh\varphi = z_0$$

and

$$\sinh\varphi = -|K|^{1/2} \cdot x_1.$$

Using these last two Exercises we conclude that the transformation

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot (M_1 \cdot M_2)$$

is a K-rigid motion (why?) and that

$$N = (x_0, y_0, z_0) \cdot (M_1 \cdot M_2)$$

(why?).

20.3 Moving a (point, direction) to any other (point, direction) by a rigid motion

Let

$$V_2 = (a_2, b_2, 0)$$

be a tangent vector to K-geometry at the North Pole N.

Exercise 173 *(**HG**) Write an explicit K-rigid motion*

$$M_3 = \begin{pmatrix} \cos\theta' & \sin\theta' & 0 \\ -\sin\theta' & \cos\theta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes V_2 to the vector

$$\left(\sqrt{a_2^2 + b_2^2}, 0, 0\right) = \left(\sqrt{V_2 \bullet_K V_2}, 0, 0\right).$$

Why does the transformation given by M_3 leave the North Pole N fixed?

Now suppose we have any point

$$X_0 = (x_0, y_0, z_0)$$

in K-geometry and any K-tangent vector

$$V_0 = (a_0, b_0, c_0)$$

at that point.

Exercise 174 *(**HG**) Explain why the K-rigid motion*

$$(\underline{x}, \underline{y}, \underline{z}) = (x, y, z) \cdot (M_1 \cdot M_2 \cdot M_2)$$

constructed over the last couple of sections takes the point X_0 to N and the tangent vector V_0 to $\left(\sqrt{V_0 \bullet_K V_0}, 0, 0\right)$

Now suppose that (X_0, V_0) gives a point X_0 in K-geometry and a tangent direction V_0 to K-geometry at X_0. Suppose that (X_0', V_0') gives another point in K-geometry and a tangent direction to K-geometry at X_0'. Finally suppose that

$$V_0 \bullet_K V_0 = V_0' \bullet_K V_0'.$$

As above, find a K-rigid motion given by

$$M = (M_1 \cdot M_2 \cdot M_3)$$

taking X_0 to the North Pole and V_0 to $\left(\sqrt{V_0 \bullet_K V_0}, 0, 0\right)$. Similarly find a K-rigid motion given by

$$M' = (M_1' \cdot M_2' \cdot M_3')$$

taking X_0' to the North Pole and V_0' to $\left(\sqrt{V_0' \bullet_K V_0'}, 0, 0\right)$.

Exercise 175 *(**HG**) Explain why the K-rigid motion given by*

$$M \cdot (M')^{-1}$$

takes (X_0, V_0) *to* (X_0', V_0') *as long as* $\sqrt{V_0 \bullet_K V_0} = \sqrt{V_0' \bullet_K V_0'}$.

By completing this Exercise we have shown that **HG** looks the same at each point and in each direction at that point. That is, we have shown that **HG** is homogeneous.

21 Lines in Hyperbolic Geometry

21.1 Hyperbolic coordinates, a shortest path from the North Pole

We next will figure out what is the shortest path you can take between two points in **HG**. Again we will do our calculation using only (x, y, z)-coordinates (since, as we have seen in (33) we don't have $(\hat{x}, \hat{y}, \hat{z})$-coordinates). The (x, y, z)-coordinates for **SG**, namely

$$\begin{aligned} x(\sigma, \tau) &= R \cdot \sin\sigma \cdot \cos\tau \\ y(\sigma, \tau) &= R \cdot \sin\sigma \cdot \sin\tau \\ z(\sigma, \tau) &= \cos\sigma \end{aligned}$$

won't work this time because they involve R which has gone off to infinity. Fortunately there are hyperbolic coordinates

$$\left(\cosh\sigma = \frac{e^{\sigma} + e^{-\sigma}}{2}, \sinh\sigma = \frac{e^{\sigma} - e^{-\sigma}}{2} \right)$$

that parametrize the 'unit' hyperbola just like $(\cos\sigma, \sin\sigma)$ parametrize the unit circle. So we define

$$\begin{aligned} x(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh\sigma \cdot \cos\tau \\ y(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh\sigma \cdot \sin\tau \\ z(\sigma, \tau) &= \cosh\sigma \end{aligned}$$

Exercise 176 *(**HG**) Show that these hyperbolic coordinates do actually parametrize the K-geometry, that is, that*

$$K \left(x(\sigma, \tau)^2 + y(\sigma, \tau)^2 \right) + z(\sigma, \tau)^2 \equiv 1$$

for all (σ, τ).

Again notice that you can write a path on the R-sphere by giving a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane. In fact, you can use σ as the parameter t and just write

$$(\sigma, \tau(\sigma))$$

where τ is a function of σ. To write a path that starts at the North Pole, just write

$$(\sigma, \tau(\sigma)), \ 0 \leq \sigma \leq \varepsilon$$

and demand that

$$\tau(0) = 0.$$

If you want the path to end on the plane $y = 0$, demand additionally that

$$\tau(\varepsilon) = 0.$$

But if we are going to describe paths on **HG** by paths in the (σ,τ)-plane we are going to need to figure out the K-dot product in (σ,τ)-coordiates so that we can compute the lengths of paths in these coordinates.

Exercise 177 *(**HG**) a) Compute the 2×3 matrix D_{hyp} such that*

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=\left(\frac{d\sigma}{dt},\frac{d\tau}{dt}\right)\cdot D_{hyp}$$

when a path in K-geometry is given by a path in the (σ,τ)-plane.

Hint: By the Chain Rule from several variable calculus

$$D_{hyp}=\begin{pmatrix} \frac{dx}{d\sigma} & \frac{dy}{d\sigma} & \frac{dz}{d\sigma} \\ \frac{dx}{d\tau} & \frac{dy}{d\tau} & \frac{dz}{d\tau} \end{pmatrix}.$$

b) Use a) to compute the K-dot product in (σ,τ)-coordinates, namely compute the matrix P_{hyp} in the equation

$$\begin{aligned}\left(\frac{d\sigma_1}{dt},\frac{d\tau_1}{dt}\right)\bullet_{hyp}\left(\frac{d\sigma_2}{dt},\frac{d\tau_2}{dt}\right) &= \left(\frac{dx_1}{dt},\frac{dy_1}{dt},\frac{dz_1}{dt}\right)\bullet_K\left(\frac{dx_2}{dt},\frac{dy_2}{dt},\frac{dz_2}{dt}\right)\\ &= \left(\frac{dx_1}{dt},\frac{dy_1}{dt},\frac{dz_1}{dt}\right)\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&K^{-1}\end{pmatrix}\cdot\left(\frac{dx_2}{dt},\frac{dy_2}{dt},\frac{dz_2}{dt}\right)^t\\ &= \left(\frac{d\sigma_1}{dt},\frac{d\tau_1}{dt}\right)\cdot D_{hyp}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&K^{-1}\end{pmatrix}\cdot D_{hyp}^t\cdot\left(\frac{d\sigma_1}{dt},\frac{d\tau_1}{dt}\right)^t\\ &= \left(\frac{d\sigma_1}{dt},\frac{d\tau_1}{dt}\right)\cdot P_{hyp}\cdot\left(\frac{d\sigma_1}{dt},\frac{d\tau_1}{dt}\right)^t.\end{aligned}$$

Exercise 178 *(**HG**) Show that the length L of any path in our K-geometry is given by*

$$(\sigma,\tau(\sigma)),\ 0\leq\sigma\leq\varepsilon$$

with

$$\tau(0)=0.$$

and

$$\tau(\varepsilon)=0$$

is given by the formula

$$L=|K|^{-1/2}\int_0^\varepsilon\sqrt{\left(1,\frac{d\tau}{d\sigma}\right)\cdot\begin{pmatrix}1&0\\0&\sinh^2\sigma\end{pmatrix}\cdot\left(1,\frac{d\tau}{d\sigma}\right)^t}\,d\sigma.$$

This last formula for L lets us figure out the shortest path from $N=(\sinh 0\cdot\cos 0, R\cdot\sinh 0\cdot\sin 0,\cosh 0)$ to $\left(|K|^{-1/2}\cdot\sinh\varepsilon,0,\cosh\varepsilon\right)=\left(|K|^{-1/2}\cdot\sin\varepsilon\cdot\cos 0,|K|^{-1/2}\cdot\sin 0\cdot\sin 0,\cosh\varepsilon\right)$. Since

$$L=|K|^{-1/2}\cdot\int_0^\varepsilon\sqrt{1+\sinh^2\sigma\cdot\left(\frac{d\tau}{d\sigma}\right)^2}\,d\sigma$$

and $\sinh^2\sigma$ is is positive for almost all $\sigma \in [0, \varepsilon]$, L is minimal only when $\frac{d\tau}{d\sigma}$ is identically 0. But this means that $\tau(\sigma)$ is a constant function. Since $\tau(0) = 0$, this means that $\tau(\sigma)$ is identically 0. So we have the shown the following result.

Theorem 179 *(**HG**) The shortest path in K-geometry from the North Pole to a point $(x, y, z) = \left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$ is the path lying in the plane $y = 0$. The K-length of that shortest path is*

$$|K|^{-1/2} \cdot \varepsilon.$$

21.2 Shortest path between any two points

Theorem 180 *(**HG**) Given any two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in K-geometry, the shortest path between the two points is the path cut out by the two equations*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

and the plane

$$\left|\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}\right| = 0, \qquad (56)$$

that is, the plane containing $(0,0,0)$ *and* X_1 *and* X_2*.*

Proof. Let $V_1 = (a_1, b_1, c_1)$ be the tangent vector at X_1 of K-length 1 that is tangent to the path cut out by the plane given by equation (56). Then $(x, y, z) = (a_1, b_1, c_1)$ also satisfies equation (56) and so the equation for that plane can also be written

$$\left|\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ a_1 & b_1 & c_1 \end{pmatrix}\right| = 0. \qquad (57)$$

By Exercise 153 there is a K-rigid motion M that takes X_1 to the North Pole N and V_1 to $(1,0,0)$. So M takes the plane (57) to the plane given by the equation

$$\left|\begin{pmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right| = 0,$$

namely the plane.

$$y = 0.$$

So $X_2 \cdot M$ must also line in this plane since X_2 lies in the plane (57). So

$$X_2 \cdot M = \left(|K|^{-1/2} \cdot \text{sinh}\varepsilon, 0, \text{cosh}\varepsilon\right)$$

for some ε since all points in K-geometry with $y = 0$ can be written as $\left(|K|^{-1/2} \cdot \text{sinh}\varepsilon, 0, \text{cosh}\varepsilon\right)$ for some ε. Since M is a K-rigid motion it must take the shortest path from X_1 to X_2 to the shortest path from $X_1 \cdot M = N$ to $X_2 \cdot M = \left(|K|^{-1/2} \cdot \text{sinh}\varepsilon, 0, \text{cosh}\varepsilon\right)$. But we already know that the shortest path from $X_1 \cdot M$ to $X_2 \cdot M$ is the one cut out by the plane $y = 0$. But that path comes from the path cut out by the plane given by equation (57), or, what is the same thing, the plane given by the equation (56). This path is called the *great hyperbolic arc* between X_1 and X_2. ■

Definition 181 *A **line** in **HG** will be a curve that extends infinitely in each direction and has the property that, given any two points X_1 and X_2 on the path, the shortest path between X_1 and X_2 lies along that curve. Lines in **HG** are the intersections of the K-geometry with planes through* $(0,0,0)$*. The length of the shortest path between two points in K-geometry will be called the K-distance.*

22 Central projection in HG

22.1 The edge of the universe

Again **HG** is a K-geometry in the sense of Part V since, in (x, y, z)-coordinates, the equation for the K-geometry

$$K\left(x^2 + y^2\right) + z^2 = 1 \tag{58}$$

with

$$K < 0$$

and the K-dot product. So all the calculations in Part V hold, in particular (36). So the (x_c, y_c)-coordinates that parametrize the entire K-geometry are the (x_c, y_c) such that

$$x_c^2 + y_c^2 < \frac{1}{|K|}.$$

So we will call the circle

$$x_c^2 + y_c^2 = \frac{1}{|K|}$$

the *edge of the universe*. (The (x_c, y_c)-coordinates are called Klein coordinates for hyperbolic geometry and the disk of radius $|K|^{-1/2}$ is called the *Klein model* for **HG** in honor of the famous German geometer, Felix Klein.)

22.2 Lines go to chords

Again all the calculations in Part V hold, in particular Exercise 139a). We conclude that lines in **HG** correspond to chords on the Klein (x_c, y_c)-disk that connect two points on the edge of the universe.lines in the (x_c, y_c)-disk.

Exercise 182 *(**HG**) a) Explain why the K-line $y = 0$ is given by the x_c-axis and the North Pole N is given by $(x_c, y_c) = (0, 0)$.*

b) Explain why the point $\left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$ in K-geometry is given by the point

$$(x_c, y_c) = \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}}, 0\right).$$

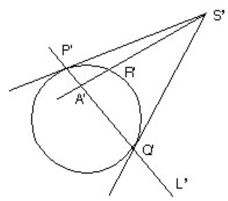
c) Explain why the K-distance between $(x_c, y_c) = (0, 0)$ and $(x_c, y_c) = \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)$ is $|K|^{-1/2} \cdot \varepsilon$.

Exercise 183 *(**HG**) Explain why lines in HG extend infinitely in each direction.*

Hint: There is a K-rigid motion that takes any two points to $(0, 0)$ and $\left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)$ for some $\varepsilon > 0$.

22.3 K-perpendicularity in the Klein model for HG

Suppose we are given any three distinct points P', R' and Q' on the edge of the universe of the Klein K-disk. We construct the line L' through P' and Q' and mark a point A' on it as shown in the figure below.



We know that there is a K-rigid motion M_c that takes A' to $(0,0)$ and L' to the x_c-axis. (Why?) Viewed as a transformation

$$\left(\underline{x_c}, \underline{y_c}\right) = M_c\left(x_c, y_c\right)$$
$$= \left(\frac{m_{11}x_c + m_{21}y_c + m_{31}}{m_{13}x_c + m_{23}y_c + m_{33}}, \frac{m_{12}x_c + m_{22}y_c + m_{32}}{m_{13}x_c + m_{23}y_c + m_{33}}\right).$$

of the entire (x_c, y_c)-plane, this transformation takes the tangent line to the edge of the universe at P' to the tangent line T_- to the edge of the universe at $\left(-|K|^{-1/2}, 0\right)$ and the tangent line to the edge of the universe at Q' to the tangent line T_+ to the edge of the universe at $\left(|K|^{-1/2}, 0\right)$. Since the tangent lines at $\left(-|K|^{-1/2}, 0\right)$ and $\left(|K|^{-1/2}, 0\right)$ are vertical, the point S'must have gone to infinity under the K-rigid motion. So the line through A' and R' must go to a line that goes through $(0,0)$ and that lies between T_- and T_+. But here is only one such line, namely the y_c-axis.

Exercise 184 *Explain why there is a K-rigid motion M_c that takes any three points P', R' and Q' in order along the edge of the universe to any other three points P'', R'' and Q'' in order along the edge of the universe.*

Hint: Use that the set of K-rigid motions form a group under the composition operation.

Exercise 185 *(**HG**) Explain why the above discussion implies that the angles $\angle P'A'R'$ and $\angle Q'A'R'$ must both be K-right angles, that is, their K-measures must each be $90°$. So the line segments $\overline{P'Q'}$ and $\overline{A'R'}$ are K-perpendicular. [MJG,238-239]*

Hint: You may need to use the fact that, since there is a K-rigid motion that interchanges $\left(-|K|^{-1/2}, 0\right)$ and $\left(|K|^{-1/2}, 0\right)$ and leaves $(0,0)$ fixed, the x_c-axis and the y_c-axis are K-perpendicular.

Exercise 186 *(**HG**) Use the previous Exercise and the fact that A' can be any point along the chord $\overline{P'Q'}$ in the figure above to explain why the Klein model is not conformal, that is, it does not faithfully represent the measure of angles in **HG**.*

22.4 Quadrilaterals in HG, in fact, in any NG

Exercise 187 *(**HG**) Use (x_c, y_c)-coordinates to show that **HG** satisfies the four Euclidean postulates E1, E2, E3, and E4. Thus hyperbolic geometry is a Neutral Geometry (**NG**).*

The next Exercise will help us appreciate some important properties of **HG** that are properties of any Neutral Geometry. So do the Exercise assuming that the context is any Neutral Geometry, that is, any two-dimensional geometry satisfying E1-E4.

Exercise 188 *(**NG**): Let $ABCD$ be a quadrilateral with $\angle ABC = \angle BCD$ right angles. (We denote polygons by naming their vertices in counterclockwise order.) [MJG,164-165]*

*a) Show in **NG** that*
$|AB| = |CD|$ implies that $\angle BAD = \angle ADC$,
$|AB| > |CD|$ implies that $\angle BAD < \angle ADC$,
$|AB| < |CD|$ implies that $\angle BAD > \angle ADC$.
b) Use a) and pure logic to show that
$\angle BAD < \angle ADC$ implies that $|AB| > |CD|$,
$\angle BAD = \angle ADC$ implies that $|AB| = |CD|$,
$\angle BAD > \angle ADC$ implies that $|AB| < |CD|$.

Hint: For the first implication in a) show that quadrilateral $ABCD$ is (self-)congruent to the quadrilateral $DCBA$. Now suppose that the second implication in a) is false for some quadrilateral $ABCD$. Construct A' on $\overline{AB}$ so that $|A'B| = |CD|$. By Exercise 13

$$\angle BAD < \angle BA'D.$$

By the (already proved) first implication

$$\angle BA'D = \angle CDA'.$$

Finally

$$\angle A'DC < \angle ADC$$

since the segment DA' lies between the segment DA and the segment DC. The proof of the third implication in a) fis the same as the proof of the second implication–just interchange A and D and interchange B and C.

For b), just use pure logic: If the first implication is false, then $\angle BAD < \angle ADC$ and either $|AB| < |CD|$ or $|AB| = |CD|$. If $|AB| < |CD|$, then by a)

$$\angle BAD > \angle ADC.$$

Contradiction! Etc., etc.

Exercise 189 *Use (x_c, y_c)-coordinates to show that **HG** does not satisfy Euclid's postulate E5. That is, through a point not on a line, it is not true that there passes a unique parallel (i.e. non-intersecting) line.*

22.5 Computing K-distances in Klein coordinates

In fact the tool that will let us compute all K-distances in (x_c, y_c)-coordinates is the cross-ratio from Definition 67. Let $d_K(A_c, B_c)$ denote the K-distance between two points A_c and B_c in the Klein K-disk. Now we know that

$$d_K\left((0,0), \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)\right) = |K|^{-1/2} \cdot \varepsilon.$$

To see what this has to do with cross-ratio, we begin by computing the cross ratio

$$\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}} : |K|^{-1/2}\right)$$

given by the two points $(0,0)$, $\left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)$ and the two points $\left(-|K|^{-1/2}, 0\right)$ and $\left(|K|^{-1/2}, 0\right)$ where the x_c-axis intersects the edge of the universe.

Exercise 190 ***(HG)*** *a) Draw a picture of the Klein K-disk, the edge of the universe, and the four points on the x_c-axis.*

b) Show that

$$\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}} : |K|^{-1/2}\right) = \left(0 : -1 : \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}} : 1\right).$$

In particular, notice that the computation doesn't depend on K.

c) Show that

$$\left(0 : -1 : \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}} : 1\right) = e^{-2\varepsilon}.$$

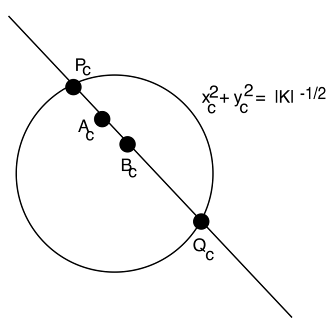
From these two Exercises we conclude that

$$d_K\left((0,0), \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)\right) = \frac{|K|^{-1/2}}{2} \cdot \left|\ln\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}} : |K|^{-1/2}\right)\right|. \tag{59}$$

Now suppose we are given any two A_c and B_c in the Klein K-disk. They determine a line

$$\alpha x_c + \beta y_c + \gamma = 0 \tag{60}$$

and so points P_c and Q_c where that line meets the edge of the universe as shown in the figure below.



We are now ready to prove the following Theorem.

Theorem 191 ***(HG)*** *For any two points A_c and B_c on the Klein K-disk, the K-distance between them $d_K(A_c, B_c)$ is given by the formula*

$$d_K(A_c, B_c) = \frac{|K|^{-1/2}}{2} \cdot |\ln(x_c(A_c) : x_c(P_c) : x_c(B_c) : x_c(Q_c))|$$

where P_c and Q_c are the endpoints of the chord through A_c and B_c. (Compare with [MJG,268].)

Proof. We know that there is a K-rigid motion $\left(\underline{x_c}, \underline{y_c}\right) = M_c(x_c, y_c)$ of the Klein disk that takes A_c to $(0,0)$ and B_c to some point $\left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)$ on the positive x_c-axis. From (37) we know that

$$\underline{x_c} = \frac{m_{11}x_c + m_{21}y_c + m_{31}}{m_{13}x_c + m_{23}y_c + m_{33}}.$$

But from (60) we know that for our four points A_c, B_c, P_c, and Q_c

$$\alpha x_c + \beta y_c + \gamma = 0$$
$$y_c = \frac{-\alpha x_c - \gamma}{\beta}.$$

So if we calculate M_c only for these four points we have

$$\begin{aligned}\underline{x_c} &= \frac{m_{11}x_c + m_{21}\left(\frac{-\alpha x_c - \gamma}{\beta}\right) + m_{31}}{m_{13}x_c + m_{23}\left(\frac{-\alpha x_c - \gamma}{\beta}\right) + m_{33}} \\ &= \frac{\left(m_{11} - \frac{m_{21}\alpha}{\beta}\right)x_c + \left(m_{31} - \frac{m_{21}\gamma}{\beta}\right)}{\left(m_{13} - \frac{m_{23}\alpha}{\beta}\right)x_c + \left(m_{33} - \frac{m_{23}\gamma}{\beta}\right)}.\end{aligned}$$

That is, the function $x_c \mapsto \underline{x_c}$ is a linear fractional transformation! So by Exercise 66

$$\begin{aligned}(x_c(A_c) : x_c(P_c) : x_c(B_c) : x_c(Q_c)) &= \left(\underline{x_c}(A_c) : \underline{x_c}(P_c) : \underline{x_c}(B_c) : \underline{x_c}(Q_c)\right) \\ &= \left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}} : |K|^{-1/2}\right) \\ &= e^{-2\varepsilon}.\end{aligned}$$

Therefore

$$\begin{aligned}d_K(A_c, B_c) &= d_K\left((0,0), \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}}, 0\right)\right) \\ &= \frac{|K|^{-1/2}}{2} \cdot \left|\ln\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}} : |K|^{-1/2}\right)\right| \\ &= \frac{|K|^{-1/2}}{2} \cdot |\ln(x_c(A_c) : x_c(P_c) : x_c(B_c) : x_c(Q_c))|.\end{aligned}$$

■

Exercise 192 *For $K = -1$, calculate the K-distance between the two points given in (x_c, y_c)-coordinates by $(0, 0)$ and $(1/2, 0)$*

22.6 Areas of hyperbolic lunes

Finally there is one K-area computation that it is convenient to do in Klein coordinates. Namely suppose that we take the ordinary triangle T_c in the (x_c, y_c)-plane with vertices $(0,0)$, $\left(|K|^{-1/2}\cos\beta, |K|^{-1/2}\sin\beta\right)$ and $\left(|K|^{-1/2}\cos\beta, -|K|^{-1/2}\sin\beta\right)$. Notice that two of the three vertices lie on the edge of the universe of the Klein K-disk and that the K-angle at the third vertex is

$$\alpha = 2\beta.$$

(In fact $(0,0)$ is the *one* point in the Klein K-disk where K-angels *are* faithfully represented. (Why?)) We will call the interior of this triangle, or any K-rigid motion of it, an α-**lune**. So we wish to compute the K-area of an α-lune. Since **HG** is a K-geometry we know from Exercise 131 that this area $A_K(\alpha)$ is given by the formula

$$A_K(\alpha) = \int_{T_c} \frac{1}{\left(1 - |K|\left(x_c^2 + y_c^2\right)\right)^{3/2}} dx_c dy_c.$$

Exercise 193 *Show that*

$$A_K(\alpha) = |K|^{-1}(\pi - \alpha).$$

Hint: Use the substitution

$$\underline{x_c} = |K|^{1/2} x_c$$
$$\underline{y_c} = |K|^{1/2} y_c$$

to reduce the computation to the computation in the case that $|K| = 1$*. Then use polar coordinates to get*

$$A_K(\alpha) = |K|^{-1} \int_{\theta=-\beta}^{\theta=\beta} \int_{r=0}^{r=\frac{\cos\beta}{\cos\theta}} \frac{1}{\left(1-r^2\right)^{3/2}} r dr d\theta$$

Then do the substitution

$$u = 1 - r^2$$
$$du = -2rdr$$

to compute

$$\int_{r=0}^{r=\frac{\cos\beta}{\cos\theta}} \frac{1}{\left(1-r^2\right)^{3/2}} r dr.$$

In the final step use the substitution

$$t = \sin(\theta).$$

23 Stereographic projection in HG

23.1 The Poincaré K-disk

Under stereographic projection, the center of projection is the South Pole $(0,0,-1)$. So if $K<0$ a point $(x,0,z)$ on the K-geometry goes out the hyperboloid to infinity in the (x,z)-plane, the line joining $(0,0,-1)$ to that point becomes parallel to an asymptote of the hyperbola

$$Kx^2+z^2=1.$$

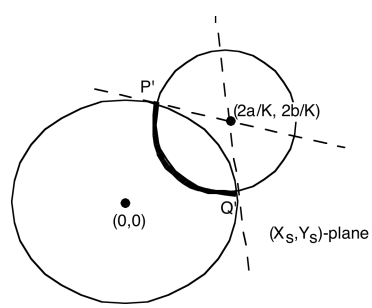
So the line approaches a line of slope $\pm|K|^{1/2}$ in the (x,z)-plane, that is the line $z=\pm|K|^{1/2}x-1$. So the intersection of that line with the line $z=1$ in the (x,z)-plane approaches the point with $x=\pm 2|K|^{-1/2}$. Therefore under stereographic projection, the edge of the universe is given by the circle

$$x_s^2+y_s^2=4|K|^{-1}.$$

The interior of this circle, that is, the image of K-geometry under stereographic projection, is called the Poincaré model of Hyperbolic Geometry, of course again after a famous geometer, Henrí Poincaré. Again, since **HG** is a K-geometry, all the rules of Part V apply. So by Exercise 139b) line in the K-geometry are given by circles of the form

$$\left(x_s-\frac{2a}{K}\right)^2+\left(y_s-\frac{2b}{K}\right)^2=\frac{4\left(K+a^2+b^2\right)}{K^2}$$

in the Poincaré K-disk. The darker arc below



(61)

represents the K-line

$$ax+by+1=0$$

in the Poincaré K-disk.

23.2 Stereographic projection preserves angles

Exercise 194 *(**HG**) a) Show that stereographic projection is conformal, that is, that the measure of K-angles between K-lines on K-geometry is just the ordinary Euclidean measure of angles formed.by (the circles that are) their stereographic projections.*

Hint: From Exercise 134 we know that, for tangent vectors V_1 and V_2 emanating from the same point on the K-geometry,

$$\begin{aligned} V_1 \bullet_K V_2 &= V_1^s \bullet_s V_2^s \\ &= (V_1^s) \cdot \begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix} \cdot (V_2^s)^t . \end{aligned}$$

b) For $K = -1$, construct the K-line in (x_s, y_s)-coordinates that meets the K-line

$$(x_s - 2)^2 + (y_s - 2)^2 = 4$$

perpendicularly in the point $\left(2-\sqrt{2}, 2-\sqrt{2}\right)$.

To get a more precise idea of what K-lines look like under stereographic projection, consider the picture (61) again. The equations of the circles in the picture are

$$x_s^2 + y_s^2 = \frac{4}{|K|} \tag{62}$$

and

$$\left(x_s - \frac{2a}{K}\right)^2 + \left(y_s - \frac{2b}{K}\right)^2 = \frac{4\left(K + a^2 + b^2\right)}{K^2} \tag{63}$$

Construct a third circle whose diameter is the line segment from $(0,0)$ to $\left(\frac{2a}{K}, \frac{2b}{K}\right)$, namely the circle $\left(x_s - \frac{a}{K}\right)^2 + \left(y_s - \frac{b}{K}\right)^2 = \left(\frac{a}{K}\right)^2 + \left(\frac{b}{K}\right)^2$ which can be rewritten

$$x_s^2 + y_s^2 - \frac{2a}{K} x_s - \frac{2b}{K} y_s = 0. \tag{64}$$

Lemma 195 *The circles* (62)*,* (63)*, and* (64) *all three pass through two common points.*

Proof. From (62) and (63) we get by addition that

$$x_s^2 + y_s^2 + \left(x_s - \frac{2a}{K}\right)^2 + \left(y_s - \frac{2b}{K}\right)^2 = \frac{4}{|K|} + \frac{4\left(K + a^2 + b^2\right)}{K^2}.$$

Simplifying this last equation and dividing both sides by 2 we obtain the equation (64). So the two points P' and Q' in picture (61) that satisfy both equations (62) and (63) also satisfy equation (64). ■

The Lemma tells us that that the angle formed by the segments $\overline{(0,0)\,P'}$ and $\overline{P'\left(\frac{2a}{K}, \frac{2b}{K}\right)}$ is a right angle since it is an inscribed angle in the circle (64) whose associated central angle is a diameter of that circle. But $\overline{(0,0)\,P'}$ is a radius of circle (62) and so $\overline{P'\left(\frac{2a}{K},\frac{2b}{K}\right)}$ is tangent to circle (62). Similarly $\overline{P'\left(\frac{2a}{K},\frac{2b}{K}\right)}$ is a radius of circle (63) and so $\overline{(0,0)\,P'}$ is tangent to circle (63). So we conclude the following Theorem.

Theorem 196 ***(HG)*** *In the Poincaré model for K-geometry, the K-lines are represesented by circular arcs that meet the edge of the universe perpendicularly.*

23.3 Infinite triangles in the Poincaré K-disk

By Exercise 139b) and 139c), lines in **HG** become circles under stereographic projection unless the line in **HG** passes through the North Pole (in which case it corresponds to a line through $(x_s, y_s) = (0,0)$ in the (x_s, y_s)-plane). Suppose a hyperbolic triangle T corresponds to a region T_s in (x_s, y_s)-coordinates and the vertices of T correspond to $(x_s, y_s) = (-2, 0)$, $(x_s, y_s) = (2, 0)$, and $(x_s, y_s) = (0, 2)$. So one side of T_s lies on the line $y_s = 0$.

Exercise 197 *(**HG**) a) Use Exercise 139b) to compute the equations for the other two sides of T_s.*

b) In the (x_s, y_s)-plane, draw T_s as accurately as you can when $K = -\frac{1}{4}$, then when $K = -1$.

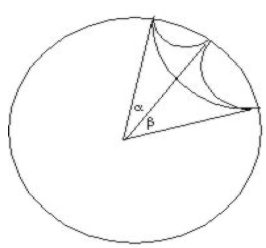
The area of a hyperbolic triangle T is given by the formula

$$\int_{T_s} \frac{1}{\left(1 + \frac{K}{4}\left(x_s^2 + y_s^2\right)\right)^2} dx_s dy_s.$$

However we do not as yet have a way to calculate the area numerically for any given triangle T. The last topic in this book will remedy that situation. Analogously to the case of spherical triangles, we start from the fact that we do know the area of α-lunes. From Exercise 193 the K-area of one an α-lune with vertex at $(0,0)$ in (x_c, y_c)-coordinates is

$$|K|^{-1}(\pi - \alpha).$$

Since rotation of the (x_c, y_c)-plane around $(0,0)$ is a K-rigid motion, this formula holds for any K-lune with vertex at $(0,0)$. Now represent the *same* lunes in the (x_s, y_s)-plane. Below is a picture in the (x_s, y_s)-plane of some of these K-lunes.



(65)

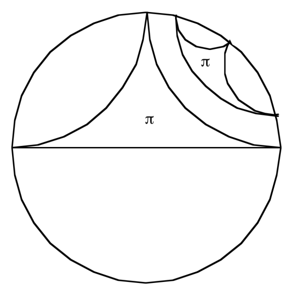
Exercise 198 *(**HG**) Use Exercise 193 to show that the area in the picture* (65) *that lies in the union of the α-lune and the β-lune but does not lie in the $(\alpha + \beta)$-lune has K-area $|K|^{-1}\pi$.*

Definition 199 *(**HG**) An infinite K-triangle is the figure given in stereographic projection coordinates by the stereographic projection of three K-lines such that any two meet the edge of the universe in a common point.*

Exercise 200 *a) (**HG**) Use Exercise 184 to show that the area of (the interior of) any infinte triangle has K-area*

$$|K|^{-1} \cdot \pi.$$

For example, if $K = -1$ we have

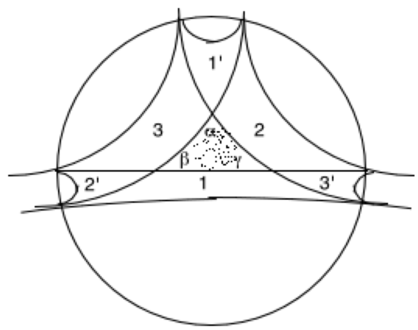


*b) Use a) to give a formula for the K-area of any infinite n-gon in **HG**, that is, a figure described by a set of n disjoint K-lines that is the limit of a family of finite n-gons, all of whose vertices have gone to infinity. In particular, what is the area of any infinite hexagon?*

Hint: Divide the infinite n-gon into infinite triangles.

23.4 Areas of polygons in HG

Consider the picture below in the Poincaré model for **HG**. Find six lunes that cover an infinte hexagon. Notice that the six lunes cover the shaded hyperbolic triangle two extra times.



Exercise 201 *(**HG**) a) Use the picture and remarks just above to explain why the K-area of the hyperbolic triangle is*

$$|K|^{-1} \cdot (\pi - (\alpha + \beta + \delta)).$$

b) Use a) to give a formula for the K-area of a hyperbolic n-gon.