1 Euclid's postulates for plane geometry

In this activity we explore neutral geometries.

In Western civilization, the primary source of our understanding of this geometry comes from Euclid's *Elements*. The treatise is of transcendant importance well beyond geometry itself, because it is among the first, and perhaps the most influential single example of organized, formal logical deductive reasoning. Certain fundamentals, that are called *axioms*, are postulated or 'given,' providing the platform on which a 'geometry' is built, that is, a mathematical entity modeling a physical 'reality'. Its properties are arrived at by applying the laws of logic to the given fundamentals. Euclid gives five axioms for plane geometry, the first four of which seem to be 'obvious' reflections of physical reality. In paraphrased form, they are:

Axiom 1 *(E1) Through any point P and any other point Q, there lies a unique line.*

Axiom 2 *(E2) Given any two segments $\overline{AB}$ and $\overline{CD}$, there is a segment $\overline{AE}$ such that B lies on $\overline{AE}$ and $|CD| = |BE|$*

(NB: In plane geometry we often use the notation $|CD|$ to denote the distance between two points A and B rather than the notation $d(A, B)$ used previously.

Axiom 3 *(E3) Given and point P and any positive real number r, there exists a (unique) circle of radius r and center P. (Said another way, if you move away from P along a line in any direction, you will encounter a unique point at distance r from P.)*

Axiom 4 *(E4) All right angles are congruent. (A right angle is defined as follows. Let C be the midpoint on the segment $\overline{AB}$. Let E be any point not equal to C. The angle $\angle ACE$ is called a right angle if $\angle ACE$ is congruent to $\angle BCE$.) [MJG,17-18]*

Definition 5 *If we are only given axioms E1–E4, we will call our geometry **neutral geometry (NG)**.*

Definition 6 *In **NG**, two distinct lines are called parallel if and only if they don't intersect.*

One implicit assumption of two-dimensional neutral (and Euclidean) geometry is the existence of (a group of) rigid motions or congruences. That is, it is assumed that given any point $\hat{A}$ and any tangent vector $\hat{V}$ emanating from $\hat{A}$ and given any second point $\hat{B}$ in the geometry and any tangent vector $\hat{W}$ emanating from $\hat{B}$, then there is a transformation $\hat{M}$ of the geometry such that

(a) $\hat{M}$ takes $\hat{A}$ to $\hat{B}$,

(b) $\hat{M}$ takes $\hat{V}$ and to a positive scalar multiple times $\hat{W}$ to $\hat{M}\left(\hat{V}\right)$,

(c) for all points $\hat{A}', \hat{A}''$ in the geometry, $\hat{M}$ leaves the distance between them unchanged, that is,

$$\left|\hat{M}\left(\hat{A}'\right)\hat{M}\left(\hat{A}''\right)\right| = \left|\hat{A}'\hat{A}''\right|,$$

(d) for any two tangent vectors $\hat{V}'$ and $\hat{V}''$ emanating from $\hat{A}$, the angle between $\hat{M}\left(\hat{V}'\right)$ and $\hat{M}\left(\hat{V}''\right)$ is the same as the angle between $\hat{V}'$ and $\hat{V}''$.

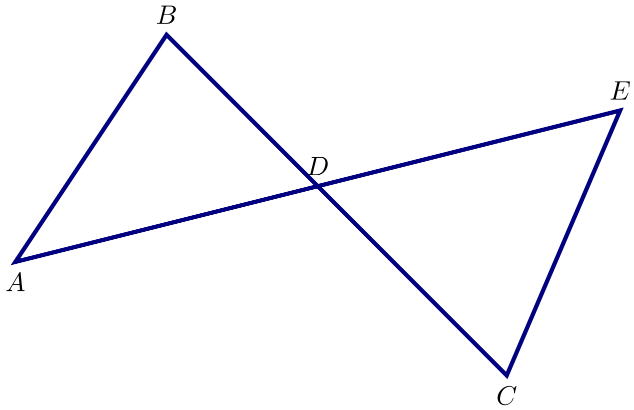
Exercise 1 Using a sketch on grid paper **and** an algebraic formulation in the Euclidean plane, give a concrete example of a rigid motion that takes $(1,2)$ to $(3,5)$ and the tangent vector $(1,0)$ emanating from $(1,2)$ to a positive multiple of the tangent vector $(0,2)$ emanating from $(3,5)$. Where is the point $(0,0)$ mapped to by this rigid motion?

Exploration 2 Think back to high school days and write the congruence rules SSS, SAS, and ASA. Be very careful with your wording—it had better be that triangles can be **moved onto each other by a rigid motion** if and only if they satisfy any one (and hence all) of the three properties (SSS, SAS, ASA).

Question 3 Give a counterexample to show that there is no universal SSA law. Can you find a restriction that will allow for an "SSA-type" law?

Although it is a bit tedious to show (and we will not ask you to do it here), using only E1–E4 you can derive the usual rules for congruent triangles (SSS, SAS, ASA). Thus these laws hold in any neutral geometry, that is, in any geometry satisfying E1–E4.

Question 4 Suppose, in the diagram below that $|BD| = |CD|$ and $|AD| = |ED|$.

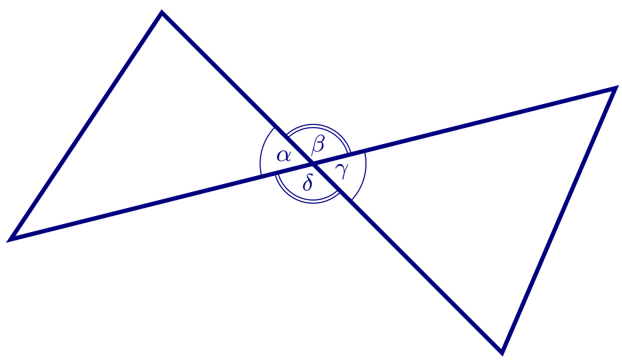


Show that triangle $\triangle BDA$ and triangle $\triangle CDE$ are congruent. [MJG,138]
Solution

Hint: *First you should explain why $\angle BDA = \angle CDE$.*

Hint: *Next you should use one of the congruence properties above.*

To start, we claim that $\angle BDA = \angle CDE$. Labeling our diagram above,



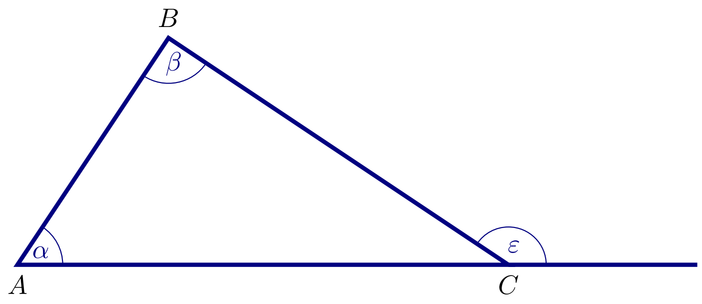
we see that

$$\alpha + \beta = 180^\circ$$
$$\beta + \gamma = 180^\circ.$$

Subtracting the equations above we fine that $\alpha = \gamma = 0$. This means that $\alpha = \gamma$ and hence $\angle BDA = \angle CDE$. Since we know that $|BD| = |CD|$ and $|AD| = |ED|$ we may now apply SAS to prove that triangle $\triangle BDA$ and triangle $\triangle CDE$ are congruent.

Question 5

(a) Show in neutral geometry that, for $\triangle ABC$



the exterior angle ε of the triangle at C is greater than either remote interior angle α or β. [MJG,119]

(b) Use **??** to show that the sum of any two angles of a triangle is less than $180°$.

Exploration 6 Show in **NG** that, if two lines cut by a transversal line have a pair of congruent alternate interior angles, then they are parallel. [MJG,117]

Hint: Suppose the assertion is false for some pair of lines. Find a triangle that violates the conclusion of Exercise **??** part **??**.