Deterministic Finite Automata (DFA)

$$A = (Q , \Sigma , \delta , q_0 , F)$$

Q = a finite set (of states)

Σ = a finite (input alphabet) set

δ = the transition function (full function) where :

$$\delta : Q \times \Sigma \rightarrow Q \; ; (q, \sigma) \rightarrow \delta(q,\sigma) \in Q$$

q_0 = initial state, $q_0 \in Q$

F = final state set , $F \subseteq Q$

*Simple Representations of **DFA***

(1) Visual (Graphical) : Transition Diagrams

(2) Tabular : Transition Tables

δE = Extended Transition Function

$\delta E : Q \times \Sigma^* \rightarrow Q \; ; \; (q, s) \rightarrow \delta E(q,s) \in Q$

Inductive Definition (e =*empty string*)

$\delta E(q, e) := q$, *Basis*

$\delta E(q , s.a) = \delta(\delta E(q, s) , a)$, *Induction*

$L(A)$:= *the language* ***accepted*** *by* A

$s \in L(A) \leftrightarrow$ (*if and only if*) $\delta E(q_0 , s) \in F$; **or :**

$L(A) = \{s \in \Sigma^* \mid \delta E(q_0 , s) \in F \}$

Nondeterministic Finite Automata (NFA)

*Same as **DFA** except :*

(1) $\boldsymbol{\delta : Q \times \Sigma \rightarrow 2^Q}$ (where $2^Q := P(Q) =$ *power set of* $\boldsymbol{Q}$)

(2) *initial state (is a set !)* $\boldsymbol{Q_0 \subseteq Q}$ *(differs from main text !)*

Distinction in graphical representation (transition diagram) :

*In **DFA** for every* $\boldsymbol{\sigma \in \Sigma}$ *there **is** exactly **one** outgoing transition edge from every state* $\boldsymbol{q \in Q}$

*In **NFA** for every* $\boldsymbol{\sigma \in \Sigma}$ *there may be **multiple** (including **none** !) outgoing transition edges from every state* $\boldsymbol{q \in Q}$

Extended Transition Function for NFA

$\delta E : 2^Q \times \Sigma^* \rightarrow 2^Q \; ; \; (X, s) \rightarrow \delta E(X,s) \in 2^Q$

Inductive Definition

$\delta E(X, e) := X$, *Basis*

$\delta E(X, s.a) = \cup_{q \in \delta E(X, s)} \delta(q, a)$, *Induction*

L(A) := the language ***accepted*** by ***A***

$s \in L(A) \leftrightarrow$ *(if and only if)* $\delta E(Q_0, s) \cap F \neq \varnothing$; *or* :

$L(A) := \{ s \in \Sigma^* \mid \delta E(Q_0, s) \cap F \neq \varnothing \}$; $\varnothing :=$ *null set*

Construction of Equivalent DFA D from a Given NFA N

Problem : *Given an NFA* $\boldsymbol{N = (Q\, ,\, \Sigma\, ,\, \delta_N\, ,\, Q_0\, ,\, F_N\,)}$ *construct a DFA* $\boldsymbol{D = (2^Q\, ,\, \Sigma,\, \delta_D\, ,\, Q_0\, ,\, F_D\,)}$ *such that* $\boldsymbol{L(N) = L(D)}$

Solution :

$\boldsymbol{(1)\ \delta_D\,(X\, ,\, \sigma\,) := \cup_{\{v\, \in X\}}\, \delta_N\,(v\, ,\, \sigma\,)\ ;\ \delta_D\,(\varnothing\, ,\, \sigma\,) := \varnothing,\ \ \forall\, \sigma \in \Sigma}$

$\boldsymbol{(2)\ F_D := \{\, Y \subseteq Q \mid Y \cap F_N \neq \varnothing\, \}}$

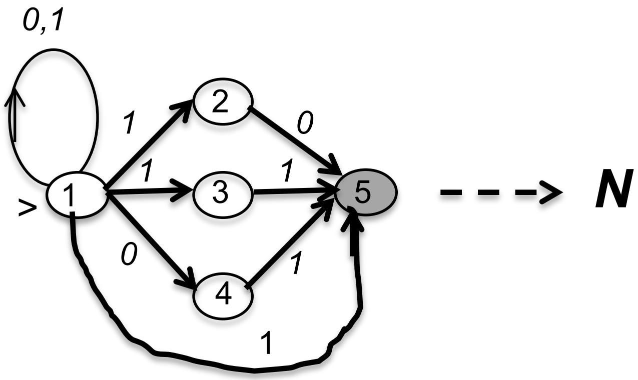
In order to prove that $\boldsymbol{L(N) = L(D)}$ *proceed as follows :*

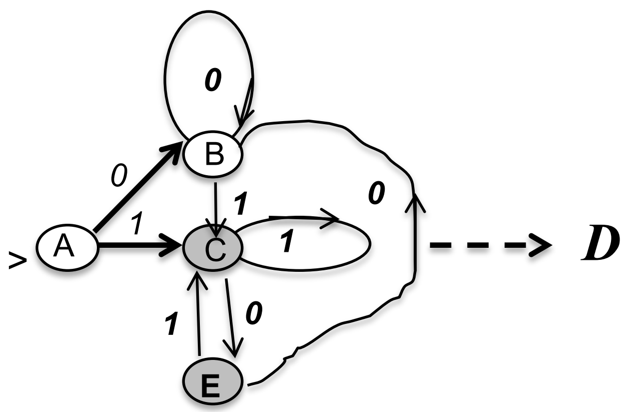
(1) Show that $\boldsymbol{\delta_D E(Q_0\, ,s) = \delta_N E(Q_0\, ,s)}$ *using induction on the length of* $\boldsymbol{s}$

(2) $\boldsymbol{s \in L(N)}$ $\leftrightarrow$ (*if and only if*) $\boldsymbol{\delta_N E(Q_0\, ,\, s) \cap F_N \neq \varnothing}$ $\leftrightarrow$ $\boldsymbol{\delta_D E(Q_0\, ,\, s) \cap F_N \neq \varnothing}$
$\leftrightarrow$ $\boldsymbol{\delta_D E(Q_0\, ,\, s) \in F_D}$ $\leftrightarrow$ $\boldsymbol{s \in L(D)}$

*Example for DFA equivalent **D** for an NFA **N***

L=(s∈{0,1}* | s = u v ; |v| ≤ 2 ; v** has at least one **1}



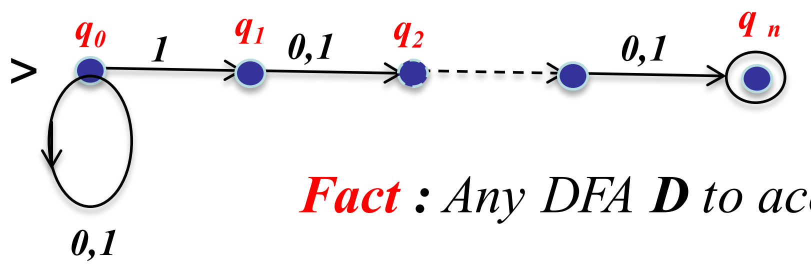


	state	**input**	**next**
> A→	*1*	*0*	*1,4* ***(B)***
	1	*1*	*1,2,3,5* ***(C)***
B→	*1,4*	*0*	*1,4* ***(B)***
	1,4	*1*	*1,2,3,5* ***(C)***
final *C→*	*1,2,3,5*	*0*	*1,4,5* ***(E)***
	1,2,3,5	*1*	*1,2,3,5* ***(C)***
final *E→*	*1,4,5*	*0*	*1,4* ***(B)***
	1,4,5	*1*	*1,2,3,5* ***(C)***

A 'bad case' example for NFA-to-DFA conversion

$$L = (s\in \{0,1\}^* \mid s=u.1.v\ ;\ |v|=n-1\ ,\ n>1\ ,\ \text{a fixed integer})$$

*$n+1$ state NFA to accept **L***



Fact **:** *Any DFA **D** to accept **L** has at least 2^n states*

Proof of Fact

*(1) Consider all (2^n) sequences of **0** and **1**s of length **n** ; denote each by u^k for $k=1,\ldots,2^n$ and **j**th input of u^k by u_j^k for $j=1,\ldots,n$.*

*(2) Apply each sequence u^k starting from the initial state q_0 of **D** and let q_j^k be the state of **D** arrived at after the application of **j**th input of u^k. Claim $k\neq p$ implies $q_n^k \neq q_n^p$!*

*(3) Suppose the claim is false for some **k** and **p** then let **j** be the first (smallest) index for which $u_j^k=1$ and $u_j^p=0$ then after **n-j** steps the corresponding states merge at the same value $q_n^k= q_n^p$ and it becomes impossible to differentiate inputs of length **n+j** starting with u^k and u^p although at **j**th stage one continues with **1** (to be accepted by **D**) and the other with **0** (to be rejected by **D**) !*

NFA with ε-transitions

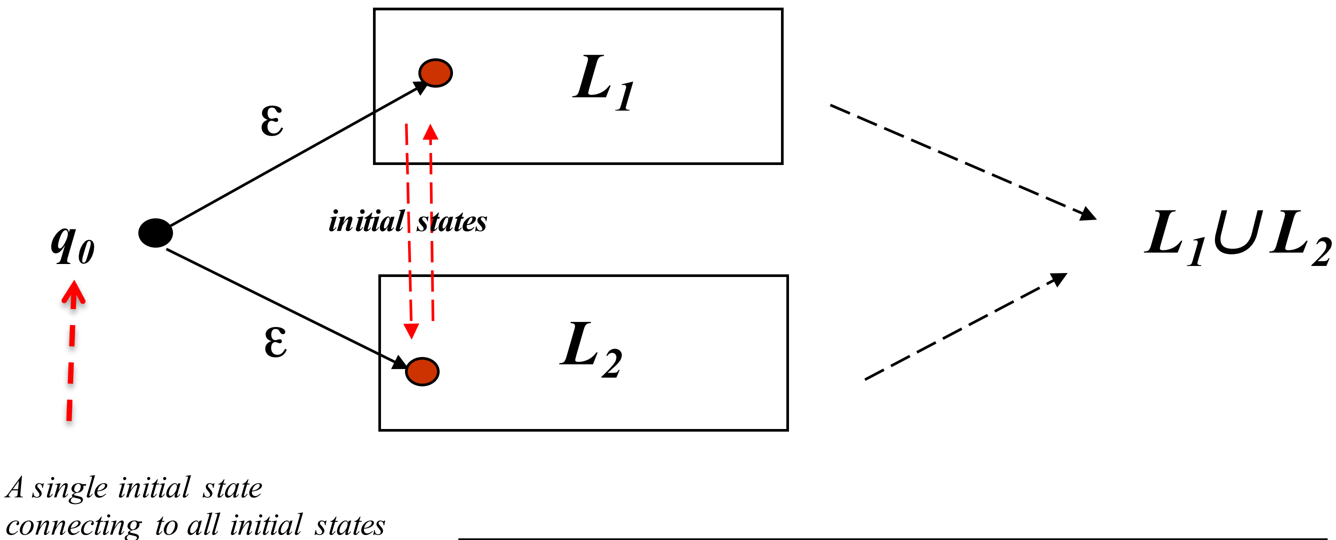
$N\varepsilon = (Q\ , \Sigma, \delta_{N\varepsilon}\ , Q_0\ , F\ )$

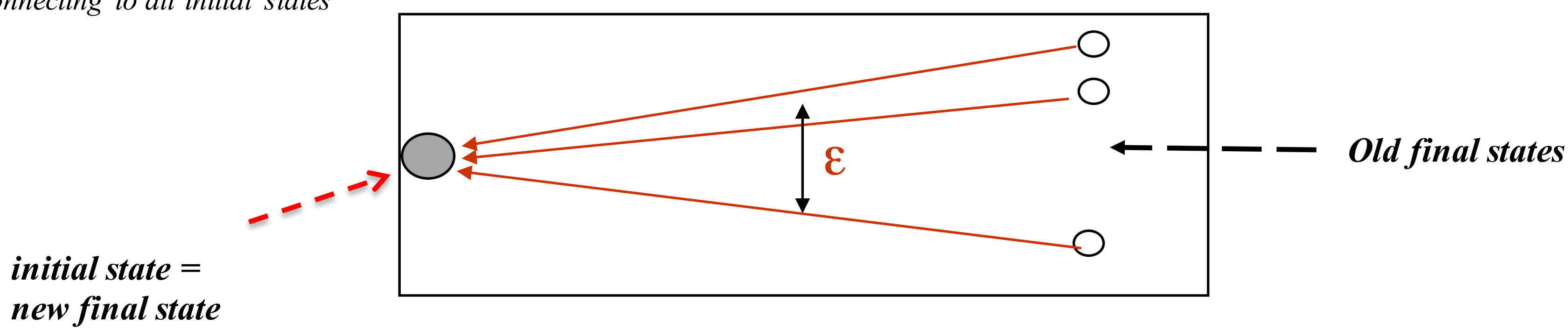
Difference is in $\delta_{N\varepsilon} : Q \times (\Sigma \cup \varepsilon) \rightarrow 2^Q$

$\delta_{N\varepsilon}(q, \varepsilon\ ) \in 2^Q$ is called *(a bundle of)* ***ε-transitions***

In computing the language accepted, ***L(Nε)****, ε-transitions do not count , i.e., they are defined as invisible and erased !*

Typical Applications of ε - transitions





L to *L**

Eliminating ε-transitions

Idea : define ε-closures inductively (recursively)

Let $\boldsymbol{X} \subseteq \boldsymbol{Q}$ *and compute* $\boldsymbol{ECLOSE(X)} \subseteq \boldsymbol{Q}$ *recursively as below :*

$\boldsymbol{ECLOSE\,(X) = X}$ *,* ***basis***

If $\boldsymbol{y} \in \boldsymbol{ECLOSE(X)}$ *then set :*

$\boldsymbol{ECLOSE(X) := ECLOSE(X)} \cup \delta_{N\varepsilon}(\boldsymbol{y}, \boldsymbol{\varepsilon})$ *,* ***recursion***

The language $L(N\varepsilon)$ accepted by an automaton $N\varepsilon$ with ε-transitions

Extended Transition Function for $\boldsymbol{N\varepsilon}$ **:**

$\delta_{N\varepsilon}E(X, e) := ECLOSE(X)$; ***basis***

$\delta_{N\varepsilon}E(X, s.a) := \cup_{y \in Y} ECLOSE\ (\delta_{N\varepsilon}(y, a)),\ Y=\delta_{N\varepsilon}E(X, s)$: ***induction***

$L(N\varepsilon) =$ *language accepted by* $\boldsymbol{N\varepsilon}$

$$= \{ s \in \Sigma^* \mid \delta_{N\varepsilon}E(Q_0, s) \cap F \neq \varnothing \}$$

$\sim N :=$ **NFA-*equivalent*** *for* $\boldsymbol{N\varepsilon}$ *with no* ε*-transitions*

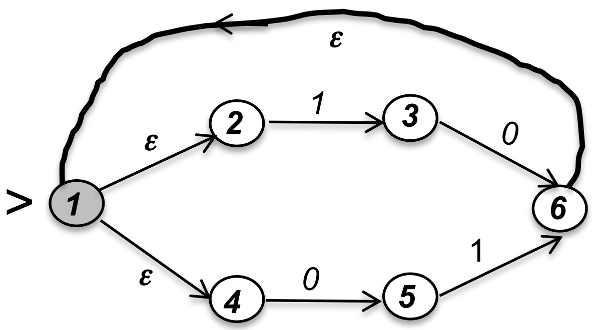
$\sim N := (Q, \Sigma, \delta_{\sim N}, Q'_0, F)$

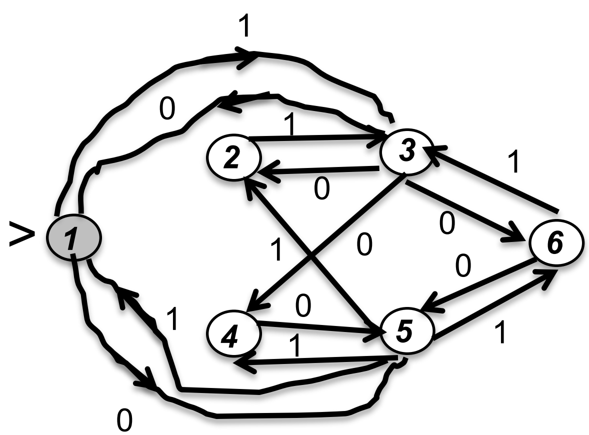
where : $\delta_{\sim N}(q, a) := \delta_{N\varepsilon}E(\{q\}, a)$; $Q'_0 := ECLOSE\ (Q_0)$

Fact : $L(\sim N) = L(N\varepsilon)$

Example for ε-NA to NA without ε-transitions transformation

$L_1 = \{01\}$, $L_2 = \{10\}$ $\qquad$ $L := \{ L_1 \cup L_2 \}^*$





q	*σ*	*Q'*
1	0	5
1	1	3
2	0	ϕ
2	1	3
3	0	1,2,4,6
3	1	ϕ
4	0	5
4	1	ϕ
5	0	ϕ
5	1	1,2,4,6
6	0	5
6	1	3

A Resume of equivalence formulas for DFA, NFA and ε-NFA

(1) $\delta_A : Q \times \Sigma \rightarrow Q$; $\delta_A E : Q \times \Sigma^* \rightarrow Q$; $s \in L(A) \Leftrightarrow \delta_A E(q_0, s) \in F$

(2) $\delta_N : Q \times \Sigma \rightarrow 2^Q$; $\delta_N E : 2^Q \times \Sigma^* \rightarrow 2^Q$; $s \in L(N) \Leftrightarrow \delta_N E(Q_0, s) \cap F \neq \varnothing$

(3) *Deterministic Equivalent* **D** *of an NFA* **N** *such that* **L(N) = L(D)**

$D = (2^Q, \Sigma, \delta_D, Q_0, F_D)$; $\delta_D(X, \sigma) := \cup_{\{v \in X\}} \delta_N(v, \sigma)$; $\delta_D(\varnothing, \sigma) := \varnothing$

$F_D := \{ Y \subseteq Q \mid Y \cap F_N \neq \varnothing \}$

(4) $\delta_{N\varepsilon} : Q \times \Sigma \cup \{\varepsilon\} \rightarrow 2^Q$; $\delta_{N\varepsilon} E : 2^Q \times \Sigma^* \rightarrow 2^Q$; $s \in L(N\varepsilon) \Leftrightarrow \delta_{N\varepsilon} E(Q_0, s) \cap F \neq \varnothing$

(5) *Equivalent* **~N** *without* **ε**-*transitions of an* **ε**-*NFA* **Nε** *such that* **L(~N) = L(Nε)**

$\sim N := (Q, \Sigma, \delta_{\sim N}, Q'_0, F)$; $\delta_{\sim N}(q, a) := \delta_{N\varepsilon} E(\{q\}, a)$; $Q'_0 := ECLOSE\ (Q_0)$