Deterministic Finite Automata (DFA)

$$A = (Q , \Sigma, \delta, q_0 , F)$$

Q = a finite set (of states)

Σ = a finite (input alphabet) set

δ = the transition function (full function) where :

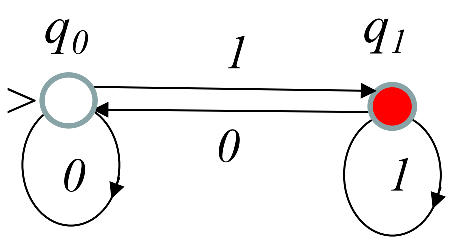
$$\delta : Q \times \Sigma \rightarrow Q ; (q, \sigma) \rightarrow \delta(q,\sigma) \in Q$$

q_0 = initial state, $q_0 \in Q$

F = final state set , $F \subseteq Q$

Simple Representations of **DFA**

(1) Visual (Graphical) : Transition Diagrams



strings (event sequences) that end up in

colored (final) state

(2) Tabular : Transition Tables

state	input	state'
q_0	0	q_0
q_0	1	q_1
q_1	0	q_0
q_1	1	q_1

no. of columns in transition table = 3

in general how many rows are there ?

answer → $|\Sigma| \times |Q|$ rows

δE = Extended Transition Function

$\delta E : Q \times \Sigma^* \rightarrow Q \,;\, (q, s) \rightarrow \delta E(q,s) \in Q$

Inductive Definition (**e** =*empty string*)

$\delta E(q, e) := q$, *Basis*

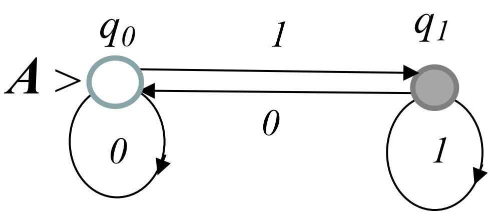
$\delta E(q , s.a) = \delta(\delta E(q, s) , a)$, *Induction*

L(A) **:=** *the language* ***accepted*** *by* ***A***

$s \in L(A) \leftrightarrow$ (*if and only if*) $\delta E(q_0 , s) \in F$; **or :**

$L(A) = (s \in \Sigma^* \mid \delta E(q_0 , s) \in F)$

Examples **1-** *Describe in simple natural language* ***L(A)*** *= the language accepted by* ***A***

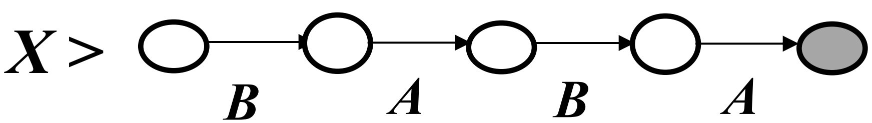


$$L(A) = (s \in \{0,1\}^* \mid \delta E(q_0 , s) \in \{q_1\})$$

Answer : all strings in {0,1}* that terminate with a 1

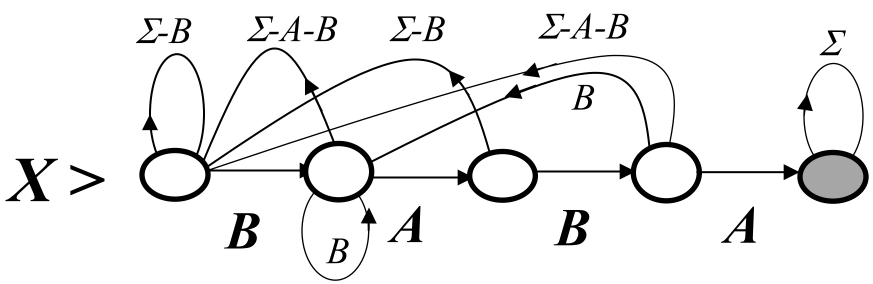
2- *Design a DFA* ***X*** *that accepts the string of letters in Turkish alphabet in which the substring* ***'BABA'*** *occurs at least once !*

Let Σ *denote the set of all capital letters in the Turkish alphabet*



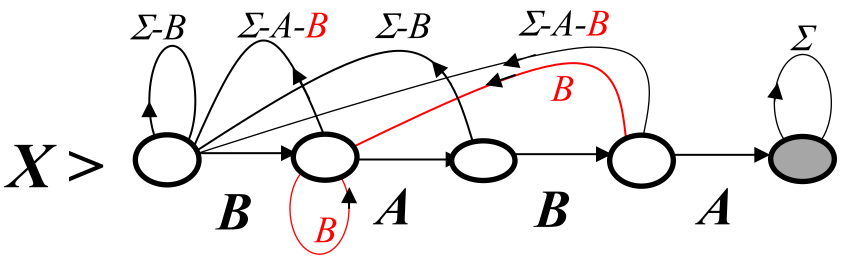
NOT *a DFA ! ! Why ?*

δ ***NOT*** *a full function ; missing transitions !*



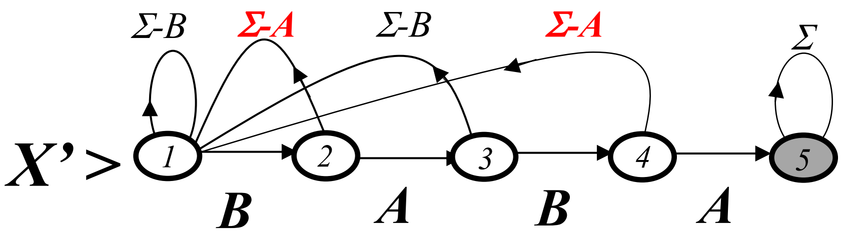
This is a DFA !

Discussion slide on Example 2



Is this or the next DFA

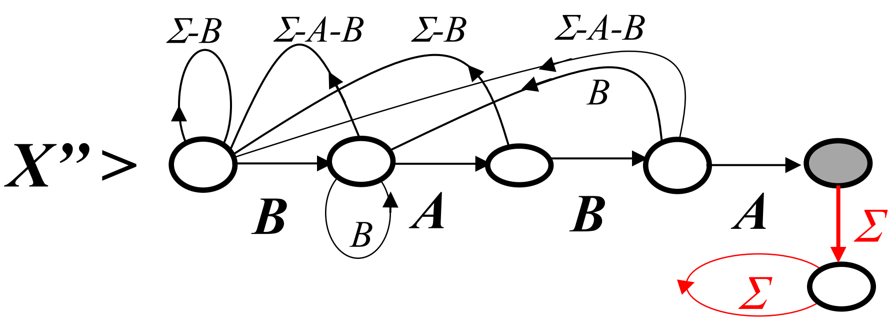
the true solution ?



Does not accept strings

'BBABA'** or **; 'BABBABA'

Both strings get stuck in state 3
which is not a final state



*What language does **X"** accept ?*

*All strings where **'BABA'** occurs as a substring*

precisely once** and only as a **postfix

Nondeterministic Finite Automata (NFA)

set of all subsets of Q

Same as ***DFA*** *except :*

(1) $\delta : Q \times \Sigma \rightarrow 2^Q$ (where $2^Q := P(Q) =$ *power set of* Q)

(2) initial state (is a set !) $Q_0 \subseteq Q$ *(differs from main text !)*

Distinction in graphical representation (transition diagram) :

In ***DFA*** *for every* $\sigma \in \Sigma$ *there* ***is*** *exactly* ***one*** *outgoing transition edge from every state* $q \in Q$

In ***NFA*** *for every* $\sigma \in \Sigma$ *there may be* ***multiple*** *(including* ***none*** *!) outgoing transition edges from every state* $q \in Q$

Extended Transition Function for NFA

$\delta E : 2^{Q} \times \Sigma^* \rightarrow 2^{Q}\ ; (X, s) \rightarrow \delta E(X,s) \in 2^{Q}$

Inductive Definition

$\delta E(X , e) := X$, *Basis*

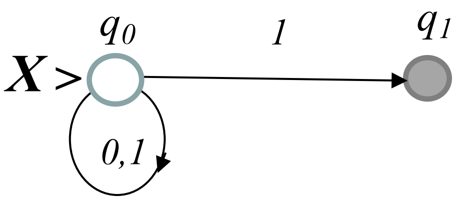
$\delta E(X, s.a) = \cup_{q\ \in\ \delta E(X,\ s)}\ \delta\,(q , a)$, *Induction*

L(A) := the language ***accepted*** by ***A***

$s \in L(A) \leftrightarrow$ *(if and only if)* $\delta E\big(Q_0, s) \cap F \neq \varnothing$; *or* :

$L(A) := \{ s \in \Sigma^* \mid \delta E\big(Q_0, s) \cap F \neq \varnothing \}$; $\varnothing :=$ *null set*

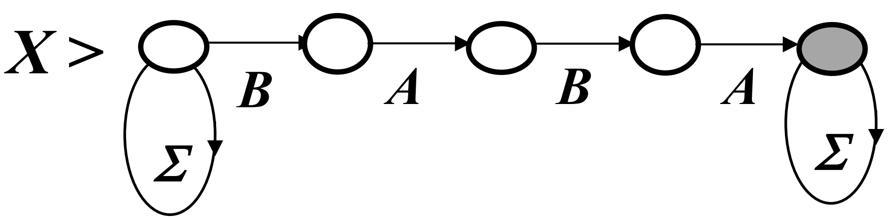
Examples *1 - Describe in simple natural language* **L(X)** *= the language accepted by* **X**



$$L(X) = (s \in \{0,1\}^* \mid \delta E(q_0 , s) \cap \{q_1\} \neq \emptyset)$$

Answer : all strings in {0,1}* that terminate with a 1

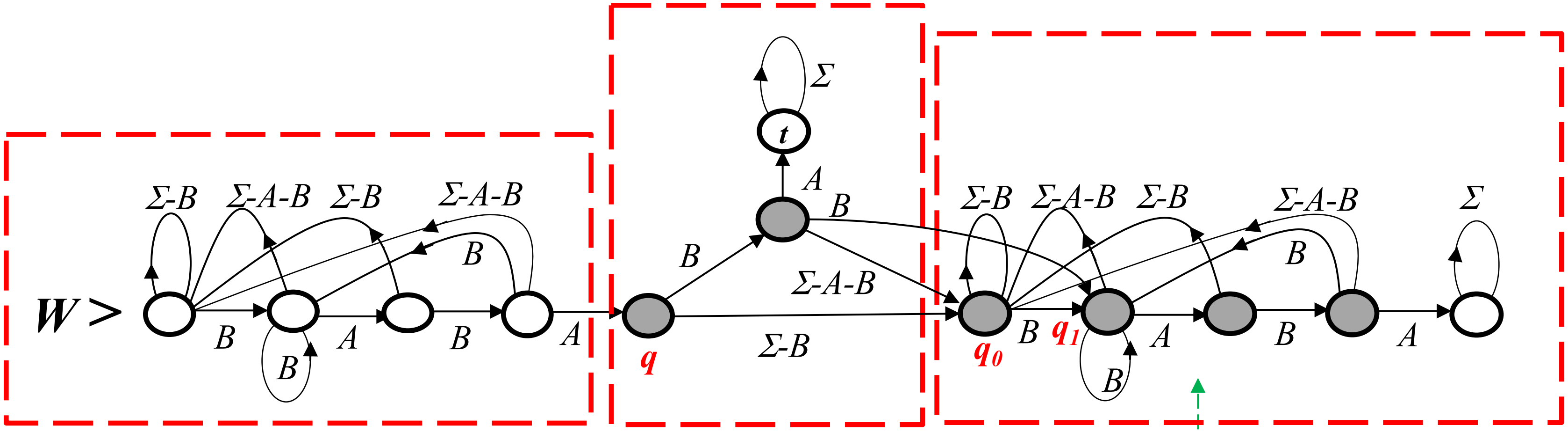
2 - Design an NFA **X** *that accepts the string of letters in Turkish alphabet in which the substring* **'BABA'** *occurs* ***AT LEAST*** *once !*



3 - Design a NFA **X** *that accepts the string of letters in Turkish alphabet in which the substring* **'BABA'** *occurs* ***PRECISELY*** *once !* ***Exercise***

Hint for exercise : use an NFA that ***rejects*** *all strings in which 'BABA' is a* ***substring***

Solution to the Exercise on slide no. 8 is automaton W below



The automaton that generates ALL strings in which the substring 'BABA' occurs precisely once as a postfix upon arrival at state *q*

__ALL__ sequences of length __1__ or __2__ that differ from __BA__ reach from __q__ to q_0 or q_1. To avoid __BABA__ through a second __BA__ sequence,a trap state __t__ is placed.

The automaton starting at initial state q_0 accepts ALL strings that do NOT have the substring __'BABA'__ in it

FACT *: If a DFA __X__ accepts the language __L(X)__ then the DFA that accepts the complement language __Σ*-L(X)__ is same as __X__ except __F__ is replaced with __Q-F__*

Construction of Equivalent DFA D from a given NFA N

Problem : Given an NFA $\boldsymbol{N = (Q, \Sigma, \delta_N, Q_0, F_N)}$ construct a DFA $\boldsymbol{D = (2^Q, \Sigma, \delta_D, Q_0, F_D)}$ such that **L(N) = L(D)**

Solution :

(1) $\delta_D(X, \sigma) := \cup_{\{v \in X\}} \delta_N(v, \sigma)$; $\delta_D(\emptyset, \sigma) := \emptyset$, $\forall \sigma \in \Sigma$

(2) $F_D := \{ Y \subseteq Q \mid Y \cap F_N \neq \emptyset \}$

To prove that **L(D) = L(N)** first show that $\boldsymbol{\delta_D E(Q_0, s) = \delta_N E(Q_0, s)}$ using induction on the length of **s**

$\boldsymbol{\delta_D E(Q_0, e) = \delta_N E(Q_0, e) = Q_0}$ by definition **(basis ; s=e case)**

$\boldsymbol{\delta_D E(Q_0, s.a) = \delta_D(\delta_D E(Q_0, s), a)} = \cup_{\{v \in X\}} \delta_N(v, a)$;
where $\boldsymbol{X = \delta_D E(Q_0, s)}$

But by induction hypothesis : $\boldsymbol{\delta_D E(Q_0, s) = \delta_N E(Q_0, s) = X}$; hence

$\boldsymbol{\delta_D E(Q_0, s.a)} = \cup_{\{v \in X\}} \delta_N(v, a) = \boldsymbol{\delta_N E(Q_0, s.a)}$; by **def.** of $\boldsymbol{\delta_N E(Q_0, s.a)}$

Finally **$L(N)=L(D)$** *is proved as follows :*

$s \in L(N) \Leftrightarrow \delta_N E(Q_0, s) \cap F_N \neq \varnothing$; *by def. of* **$L(N)$**

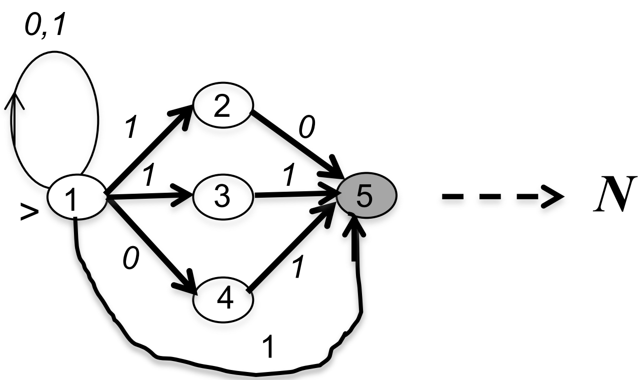
$\Leftrightarrow \delta_N E(Q_0, s) \in F_D$; *since* $F_D := \{ Y \subseteq Q \mid Y \cap F_N \neq \varnothing \}$

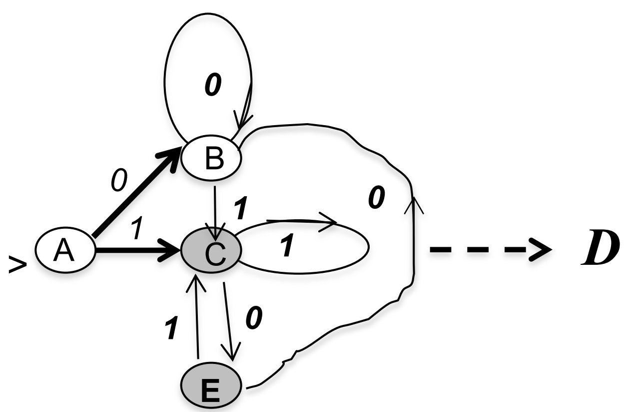
$\Leftrightarrow \delta_D E(Q_0, s) \in F_D$; *since* $\delta_N E(Q_0, s) = \delta_D E(Q_0, s)$

$\Leftrightarrow s \in L(D)$; *by def. of* **$L(D)$**

*Example for DFA equivalent **D** for an NFA **N***

***L**=(s$\in$ {0,1}* | s = u v ; |v| $\le$ 2 ; v has at least one **1**}*



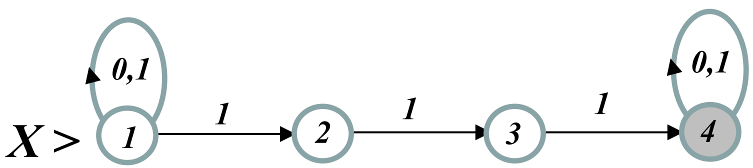


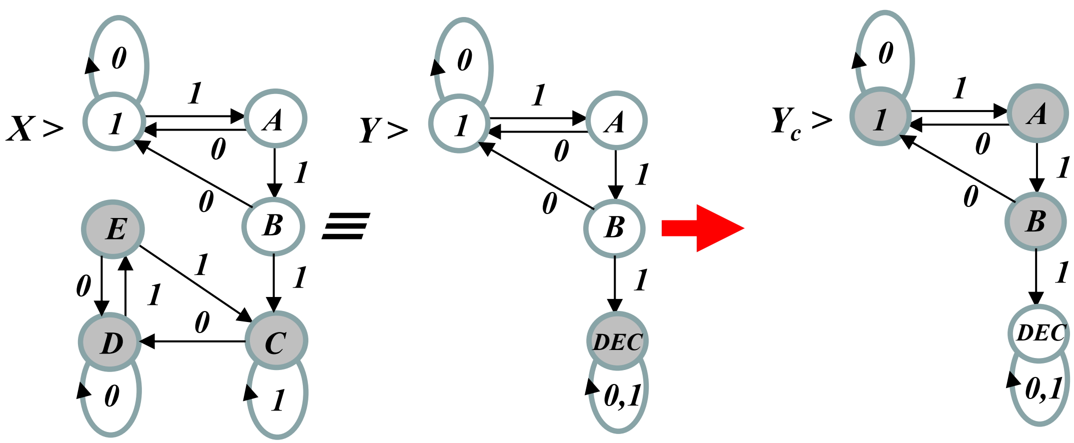
	state	input	next
> A →	1	0	1,4 **(B)**
	1	1	1,2,3,5 **(C)**
B →	1,4	0	1,4 **(B)**
	1,4	1	1,2,3,5 **(C)**
final C →	1,2,3,5	0	1,4,5 **(E)**
	1,2,3,5	1	1,2,3,5 **(C)**
final E →	1,4,5	0	1,4 **(B)**
	1,4,5	1	1,2,3,5 **(C)**

*Another example for DFA equivalent **D** for an NFA **X***

$L=(s \in \{0,1\}^* \mid s$ does **NOT** have a substring $1.1.1)$

$L^c=(s \in \{0,1\}^* \mid s = u.1.1.1.v\ ;\ u,v \in \{0,1\}^*)$



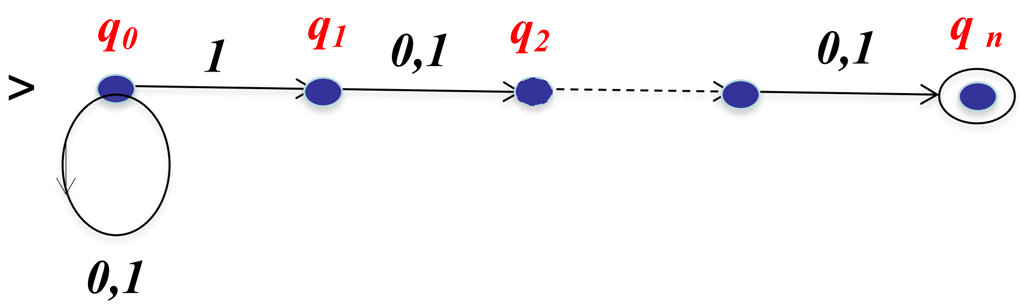


q	*σ*	*q'*
X > 1	*0*	*1*
1	*1*	*1,2*
1,2 =A	*0*	*1*
1,2	*1*	*1,2,3*
1,2,3 =B	*0*	*1*
1,2,3	*1*	*1,2,3,4*
*1,2,3,4 = C**	*0*	*1,4*
1,2,3,4	*1*	*1,2,3,4*
*1,4, = D**	*0*	*1,4*
1,4,	*1*	*1,2,4*
*1,2,4 = E**	*0*	*1,4*
1,2,4	*1*	*1,2,3,4*

A 'bad case' example for NFA-to-DFA conversion

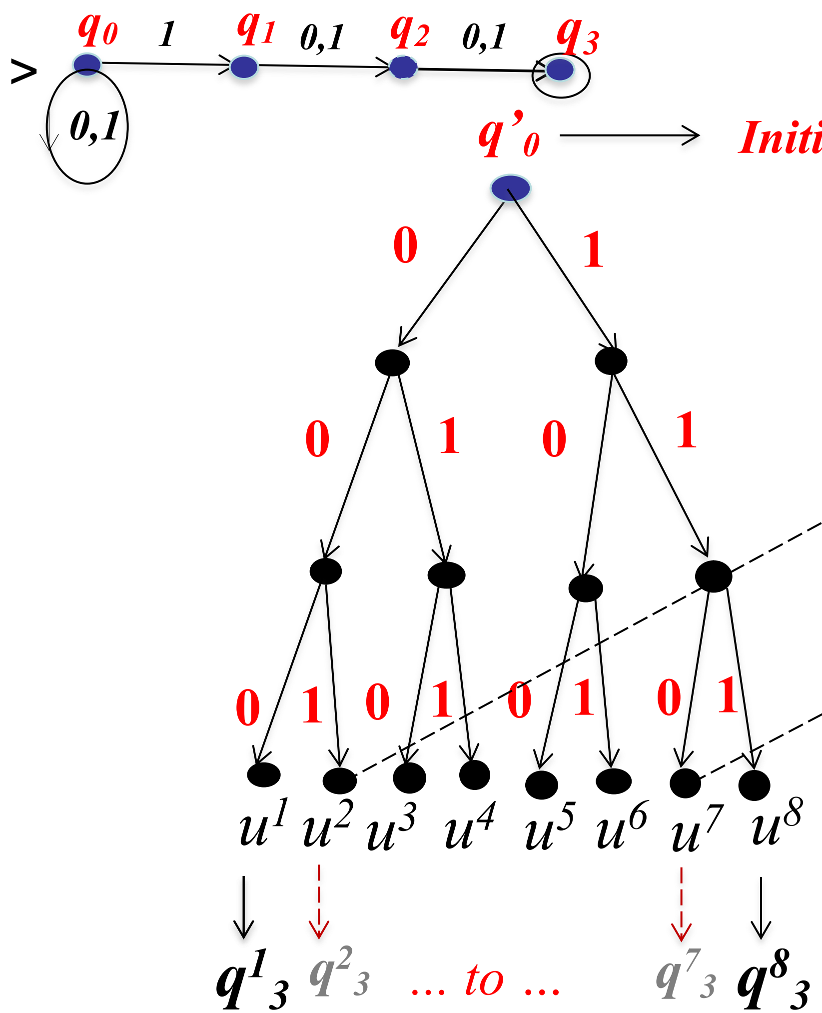
$L = (s \in \{0,1\}^* \mid s = u.1.v \; ; \; |v| = n-1 \; , \; n > 1$, *a fixed integer*)

An **n+1** *state NFA to accept* **L**



Fact : *Any DFA* **D** *to accept* **L** *has at least* 2^n *states*

A special case : n=3



Initial state of a DFA equivalent

Suppose not !!

say $q^2_3 = q^7_3$ *driven by*

u^2 **=001** *and* u^7 **=110** *respectively*

*But that is impossible since inputs **001** (to be rejected) and **110** (to be accepted) **are** either both accepted or both rejected violating the design !*

must all be 8 distinct states !

Proof of Fact

*(1) Consider all (2^n) sequences of **0** and **1**s of length **n** **;** denote each by $\mathbf{u^k}$ for $\mathbf{k=1,\ldots,2^n}$ and **j**th input of $\mathbf{u^k}$ by $\mathbf{u_j^k}$ for $\mathbf{j=1,\ldots,n}$.*

*(2) Apply each sequence $\mathbf{u^k}$ starting from the initial state $\mathbf{q'_0}$ of **D** and let $\mathbf{q_n^k}$ be the state of **D** arrived at the end of the application of $\mathbf{u^k}$.*

*Claim $\mathbf{k \neq p}$ implies $\mathbf{q_n^k \neq q_n^p}$ **!***

*(3) Suppose the claim is false for some $\mathbf{k \neq p}$ **(i.e.** $\mathbf{q_n^k = q_n^p}$ **!)** then let **j** be the first (smallest) index for which $\mathbf{u_j^k = 1}$ and $\mathbf{u_j^p = 0}$*

*(4) Then after **n-j** steps the corresponding states **merge** at the same value $\mathbf{q_n^k = q_n^p}$*

*(5) But then it becomes impossible to differentiate inputs of length **n+j** starting with $\mathbf{u^k}$ and $\mathbf{u^p}$ although at **j**th stage one continues with **1** (to be accepted by **D**) and the other with **0** (to be rejected by **D**) ! **A contradiction !***

NFA with ε-transitions

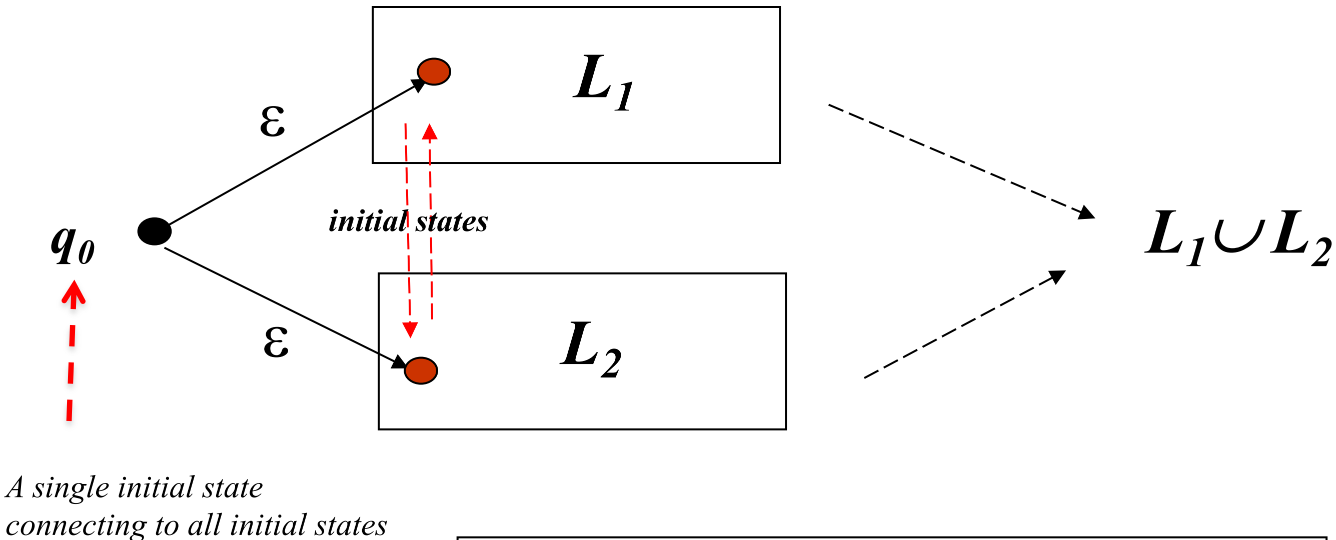
$N\varepsilon = (Q , \Sigma, \delta_{N\varepsilon}, Q_0 , F)$

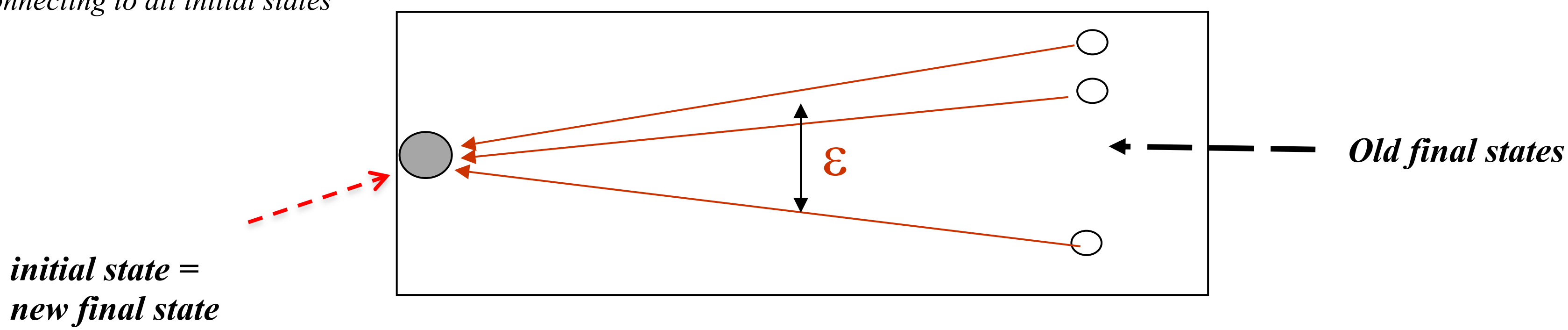
Difference is in $\delta_{N\varepsilon} : Q \times (\Sigma \cup \varepsilon) \rightarrow 2^Q$

$\delta_{N\varepsilon}(q, \varepsilon) \in 2^Q$ is called *(a bundle of)* ***ε-transitions***

In computing the language accepted, ***L(Nε)****, ε-transitions do not count , i.e., they are defined as invisible and erased !*

Typical Applications of ε - transitions





L to L*

Eliminating ε-transitions

Idea : define ε-closures inductively (recursively)

Let $\boldsymbol{X} \subseteq \boldsymbol{Q}$ *and compute* $\boldsymbol{ECLOSE(X)} \subseteq \boldsymbol{Q}$ *recursively as below :*

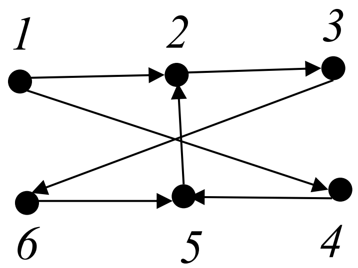
$\boldsymbol{ECLOSE\ (X) = X}$, ***basis***

If $\boldsymbol{y} \in \boldsymbol{ECLOSE(X)}$ *then set :*

$\boldsymbol{ECLOSE(X)\ := ECLOSE(X)} \cup \delta_{N\varepsilon}(\boldsymbol{y}, \varepsilon)$, ***recursion***

Example for computing E-closures

All transitions are epsilon-transitions



Progress in inductive steps →

E-CLOSURE (1) → (1) → (1,2,4) → (1,2,4,3,5) → (1,2,4,3,5,6)

E-CLOSURE (4) → (4) → (4,5) → (4,5,2) → (4,5,2,3) → (4,5,2,3,6)

The language L(Nε) accepted by an automaton Nε with ε-transitions

Extended Transition Function for $N\varepsilon$ **:**

$\delta_{N\varepsilon}E(X, e) := ECLOSE(X)$; ***basis***

$\delta_{N\varepsilon}E(X, s.a) := \cup_{y \in Y} ECLOSE(\delta_{N\varepsilon}(y, a)), Y=\delta_{N\varepsilon}E(X, s)$: ***induction***

$L(N\varepsilon)$ = *language accepted by* $N\varepsilon$

$$= \{ s \in \Sigma^* \mid \delta_{N\varepsilon}E(Q_0, s) \cap F \neq \varnothing \}$$

~N := NFA-*equivalent* *for* $N\varepsilon$ *with no ε-transitions*

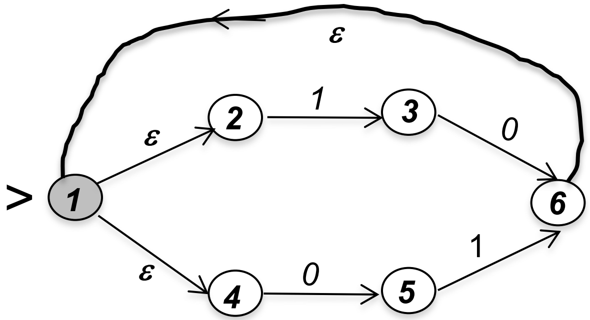
$\sim N := (Q, \Sigma, \delta_{\sim N}, Q'_0, F)$

where : $\delta_{\sim N}(q, a) := \delta_{N\varepsilon}E(\{q\}, a)$; $Q'_0 := ECLOSE(Q_0)$

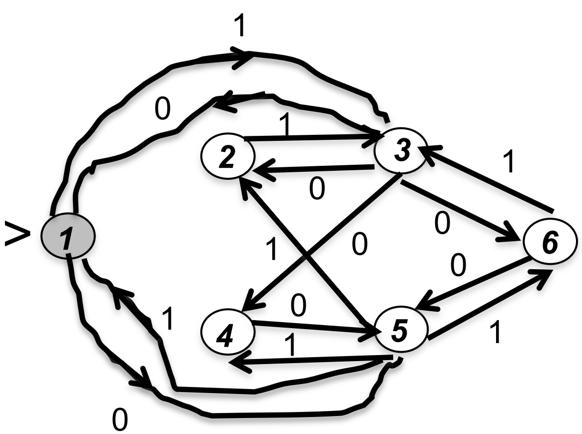
Fact : $L(\sim N) = L(N\varepsilon)$

Example for ε-NFA to NFA without ε-transitions transformation

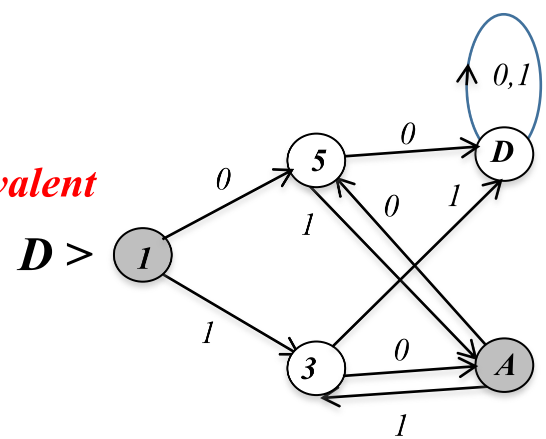
$L_1 = \{01\}$, $L_2 = \{10\}$ $\quad\quad$ $L := \{ L_1 \cup L_2 \}^*$



NFA Equivalent



DFA Equivalent



q	σ	q'
1*	0	5
1	1	3
2	0	ϕ
2	1	3
3	0	1,2,4,6 =A
3	1	ϕ
4	0	5
4	1	ϕ
5	0	ϕ
5	1	1,2,4,6
6	0	5
6	1	3
A*	0	5
A*	1	3

A Resume of equivalence formulas for DFA , NFA and ε -NFA

(1) $\delta_A : Q \times \Sigma \rightarrow Q$; $\delta_A E : Q \times \Sigma^* \rightarrow Q$; $s \in L(A) \Leftrightarrow \delta_A E(q_0, s) \in F$

(2) $\delta_N : Q \times \Sigma \rightarrow 2^Q$; $\delta_N E : 2^Q \times \Sigma^* \rightarrow 2^Q$; $s \in L(N) \Leftrightarrow \delta_N E(Q_0, s) \cap F \neq \emptyset$

(3) *Deterministic Equivalent* **D** *of an NFA* **N** *such that* **L(N) = L(D)**

$D = (2^Q, \Sigma, \delta_D, Q_0, F_D)$; $\delta_D(X, \sigma) := \cup_{\{v \in X\}} \delta_N(v, \sigma)$; $\delta_D(\emptyset, \sigma) := \emptyset$

$F_D := \{ Y \subseteq Q \mid Y \cap F_N \neq \emptyset \}$

(4) $\delta_{N\varepsilon} : Q \times \Sigma \cup \{\varepsilon\} \rightarrow 2^Q$; $\delta_{N\varepsilon} E : 2^Q \times \Sigma^* \rightarrow 2^Q$; $s \in L(N\varepsilon) \Leftrightarrow \delta_{N\varepsilon} E(Q_0, s) \cap F \neq \emptyset$

(5) *Equivalent* ~**N** *without* ε*-transitions of an* ε*-NFA* **N**ε *such that* **L(~N) = L(Nε)**

$\sim N := (Q, \Sigma, \delta_{\sim N}, Q'_0, F)$; $\delta_{\sim N}(q,a) := \delta_{N\varepsilon} E(\{q\}, a)$; $Q'_0 := ECLOSE\ (Q_0)$