Regular Expressions vs Finite State Automata

- *Regular Expressions* ***(RE**s**)*** *are an* ***algebraic*** *means for defining languages.*

- *Languages accepted by* ***DFA's*** *and* ***NFA's*** *vs* ***RE's***

Definition of a set RE of regular expressions

(over a finite set $\Sigma := \{\sigma_1, \sigma_2, \ldots \sigma_K\}$ *)*

Recursive Formal Definition

(A) *(Basis)* $\mathbf{e}$, $\varnothing$ *and* $\sigma_1, \sigma_2, \ldots, \sigma_K$ *are all elements of* ***RE***

(B) *(Recursion)*

(1) If ***E*** *and* ***F*** *are in* ***RE*** *then so is* ***E+F***

(2) If ***E*** *and* ***F*** *are in* ***RE*** *then so is* ***E.F***

(3) If ***E*** *is in* ***RE*** *then so is* ***E****

(4) If ***E*** *is in* ***RE*** *then so is* ***(E)***

We call each element of the set ***RE*** *a* ***regular expression*** *!*

Example of a RE (over the set $\Sigma := \{0,1\}$)

Is $\mathbf{1+(1.0^*).(1^*.0)+e}$ *an element of* **RE** *?*

$$\mathbf{1+(1.0^*).(1^*.0)+e} \xrightarrow{0,1,e\ \in\ \mathbf{RE}} \mathbf{E+(E.E^*).(E^*.E)+E}$$

$$\xrightarrow{E^* \in\ \mathbf{RE}} \mathbf{E+(E.E).(E.E)+E} \xrightarrow{E.E\ \in\ \mathbf{RE}} \mathbf{E+(E).(E)+E}$$

$$\xrightarrow{(E)\ \in\ \mathbf{RE}} \mathbf{E+E.E+E} \xrightarrow{E.E\ \in\ \mathbf{RE}} \mathbf{E+E+E} \xrightarrow{E+E\ \in\ \mathbf{RE}} \mathbf{E+E}$$

$$\xrightarrow{E+E\ \in\ \mathbf{RE}} \mathbf{E} \in \mathbf{RE}$$

Language interpretation is a mapping $L : RE \rightarrow 2^{\Sigma^*}$ given by :

$L(e) := \{e\}$ *where* e := *empty string*

$L(\varnothing) := \varnothing$ *where* $\varnothing$:= *null language (language with no strings)*

$L(\sigma_j) := \{\sigma_j\}$, $j=1, \ldots, K$

$L(E+F) := L(E) \cup L(F)$

$L(E.F) := L(E).L(F)$

$L(E^*) := L(E)^*$

$L((E)) := (L(E))$

Relation of Basic Operations on Languages to REs

(1) **Union** : $L = L_1 \cup L_2$ $\Longrightarrow$ $E + E$

(2) **Concatenation** : $L = L_1 . L_2$ $\Longrightarrow$ $E.E$

formal *logical notation for AND =* ***conjunction***

$$L_1 . L_2 := (s \in \Sigma^* \mid s = u.v \,;\, u \in L_1 \wedge v \in L_2)$$

informal *logical notation for AND =* ***conjunction***

(3) **Closure** (star or Kleene closure) $L^* = \cup_{k=0,\infty} L^k$ $\Longrightarrow$ E^*

$$L^k := (s \in \Sigma^* \mid s = u_1 . u_2 \ldots u_k \,;\, u_j \in L \text{ for } j=1,\ldots k)$$

Definition *: A language L is called a* ***regular language*** *if it is the language interpretation of a* ***regular expression***

Main Theorem

A language is ***regular*** *if and and only if it is accepted by some* ***finite state automaton*** *.*

Proof of the Main Theorem

(if) **Idea :**

(1) Let a DFA $\mathbf{D = (Q, \Sigma, \delta, 1, F)}$ with $\mathbf{Q = \{1,2, \ldots, n\}}$

(2) Let $\mathbf{R_{ij}^{k}}$ denote the language corresponding to strings covering **all** paths of **D** that start at state $\mathbf{i}$; end at state $\mathbf{j}$; and visit intermediate states with numbers $\mathbf{p \le k}$

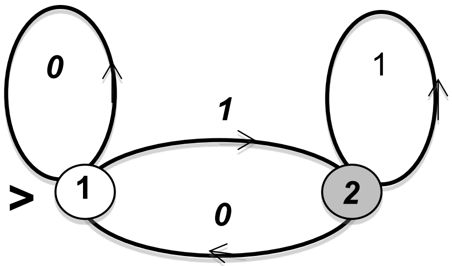
(3) Note that $\mathbf{L(D) = \cup_{(m \in F)} R_{1m}^{n}}$ where **1** is the initial state

(4) Prove by induction on $\mathbf{k}$ that $\mathbf{R_{ij}^{k}}$ is a **RE** for all $\mathbf{i,j = 1,\ldots,n}$ and $\mathbf{k = 0,\ldots, n}$.

(see the next slide first formula)

(5) Conclude that **L(D)** is a **RE.**

Illustration of the language R_{ij}^{k}



R_{11}^{0} = *start at **1** and terminate at **1** (no intermediate visit is allowed)* = $0+e$

R_{12}^{0} = *start at **1** and terminate at **2** (no intermediate visit is allowed)* = 1

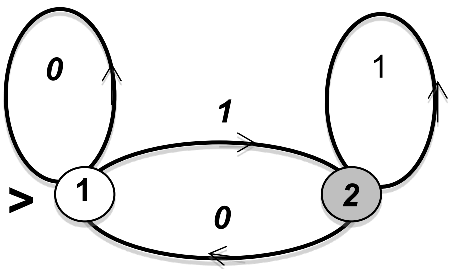
R_{11}^{1} = *start at **1** , move to allowed intermediate state **1** as desired and terminate at **1*** = 0^*

R_{12}^{1} = *start at **1** , move to intermediate state **1** as desired and finally terminate at **2*** = $0^*.1$

The Inductive Formula for DFA → RE

$$R_{ij}^{k} = R_{ij}^{k-1} + R_{ik}^{k-1}.\,(R_{kk}^{k-1})^{*}.\,R_{kj}^{k-1} \quad ; \; i,j=1,\ldots,n \; ; \; k=0,\ldots,n$$

Example



$$R_{11}^{0} = 0+e \; ; \; R_{22}^{0} = 1+e \; ; \; R_{21}^{0} = 0 \; ; \; R_{12}^{0} = 1$$

$$R_{11}^{1} = 0^{*} \; ; \; R_{22}^{1} = 0.0^{*}.1+1+e \; ; \; R_{21}^{1} = 0.0^{*} \; ; \; R_{12}^{1} = 0^{*}.1$$

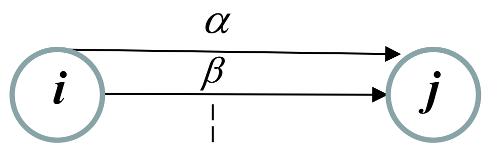
$$R_{11}^{2} = \ldots \; ; \; R_{22}^{2} = \ldots \; ; \; R_{21}^{2} = \ldots \; ; \; R_{12}^{2} = \ldots$$

After Simplification : $L=R_{12}^{2} = (0^{*}.1.1^{*}.0)^{*}.0^{*}.1.1^{*}$

Continue with the Proof (by induction on the superscript **k**)

Basis (k=0)

$R_{ij}^{0} = \alpha+\beta+\ldots$ ***if***

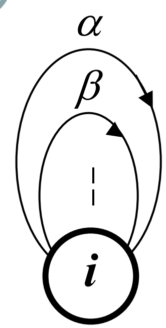


$E \rightarrow E+E$

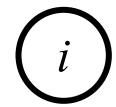
$= \varnothing$ ***if***

$\alpha,\beta,\ldots, e,\varnothing \in E$

$R_{ii}^{0} = \alpha+\beta+\ldots+e$ ***if***



$R_{ii}^{0} = e$ ***if***



*Induction (true for **k-1** , show for **k**)*

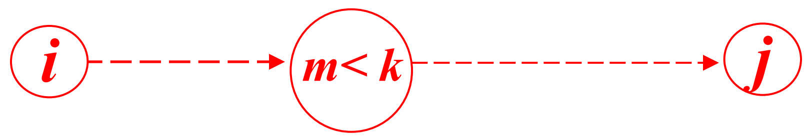
$R_{ij}^{k} = R_{ij}^{k-1} + R_{ik}^{k-1}.\,(R_{kk}^{k-1})^*.\,R_{kj}^{k-1}$; $i,j=1,\ldots,n$; $k=0,\ldots,n$

$(E) \rightarrow E$ $\quad E^* \rightarrow E$ $\quad E.E \rightarrow E$ **(twice)** $\quad E+E \rightarrow E$

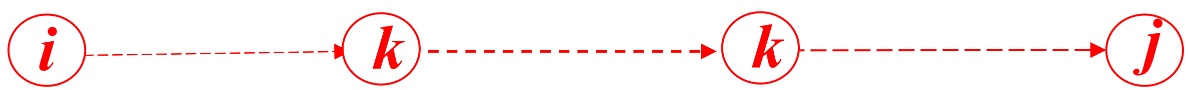
Interpreting the induction formula :

$$R_{ij}^{k} = R_{ij}^{k\text{-}1} + R_{ik}^{k\text{-}1}.\ (R_{kk}^{k\text{-}1})^{*}.\ R_{kj}^{k\text{-}1}\ ;\ \ i,j=1,\ldots,n\ ;\ k=0,\ldots,n$$

A path (string) $\mathbf{s}$ *in* R_{ij}^{k} *can be expressed in terms of a sequence of states as shown below :*



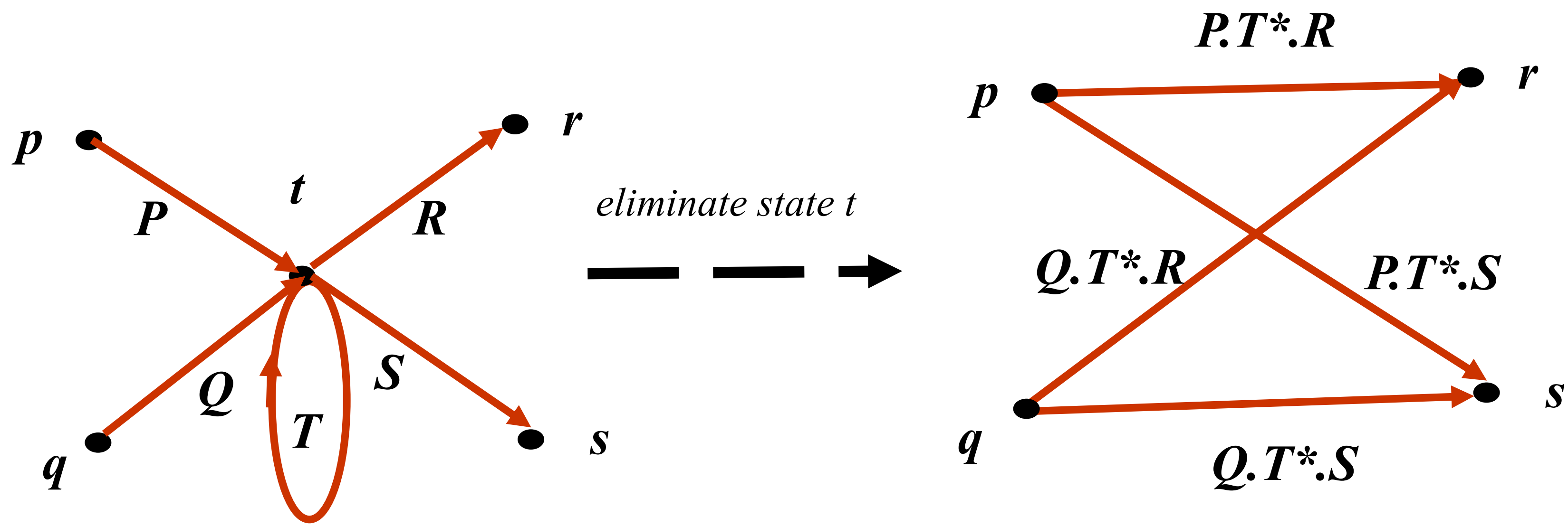
$s \in R_{ij}^{k\text{-}1}$



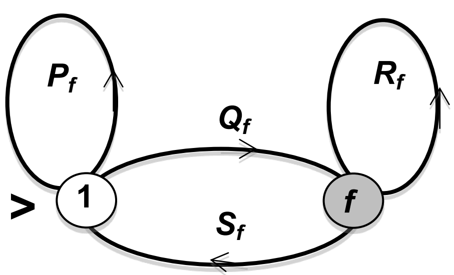
(OR) $s=u.v.w$ $\quad u \in R_{ik}^{k\text{-}1} \quad v \in (R_{kk}^{k\text{-}1})^{*} \quad w \in R_{kj}^{k\text{-}1}$

First occurrence of **k** *Last occurrence of* **k**

Alternative Proof of the Main Theorem (State Elimination)



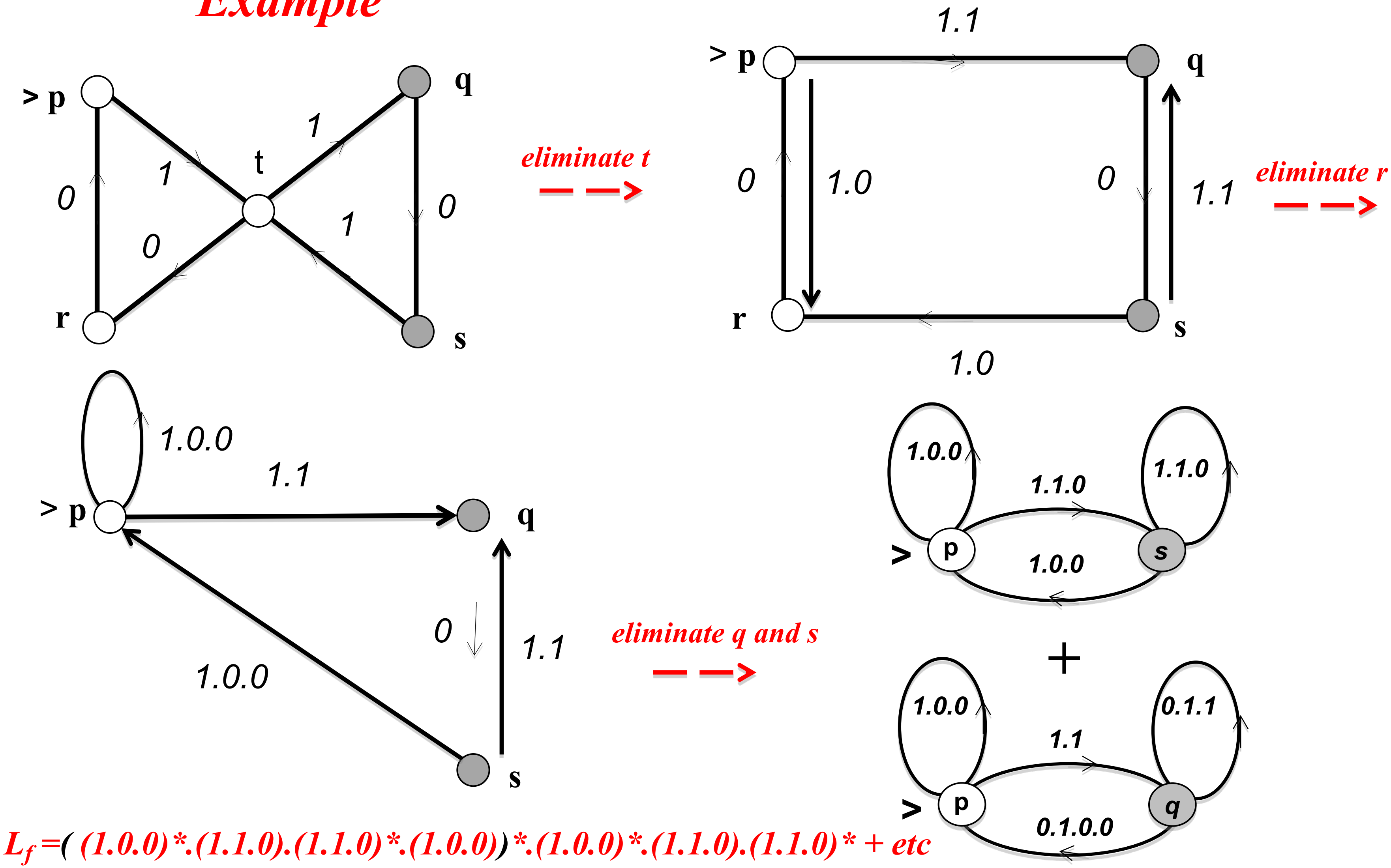
*After eliminating all non-initial and non-final states ; start eliminating all final states except one **f** in **F** and repeat this for each distinct **f** in **F**. Then the following picture(s) prevail*



$$L_f = (P_f^*.Q_f.R_f^*.S_f)^*.P_f^*.Q_f.R_f^*$$

$$L = \Sigma_{(f \in F)} L_f$$

Example



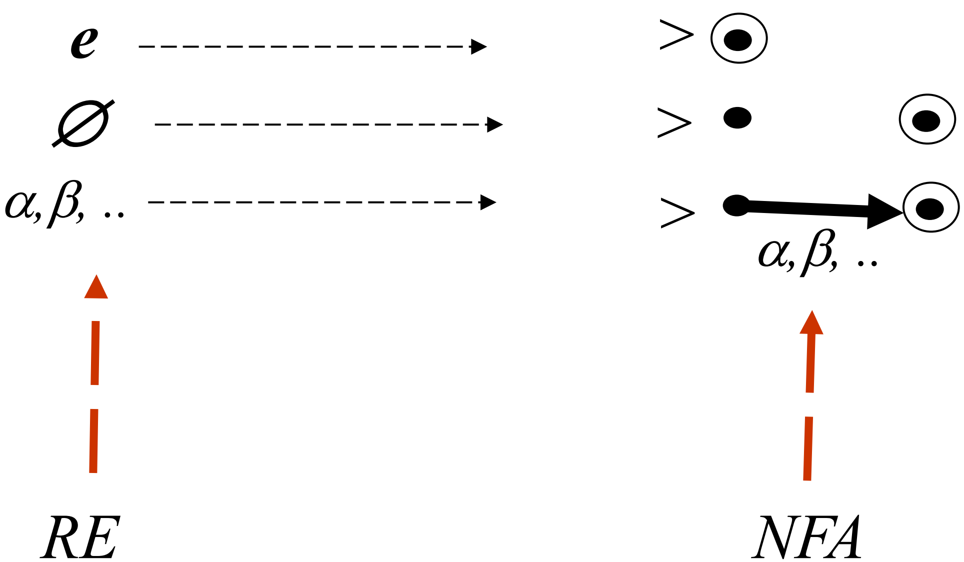
L_f =((1.0.0)*.(1.1.0).(1.1.0)*.(1.0.0))*.(1.0.0)*.(1.1.0).(1.1.0)* + etc

Proof of the Main Theorem

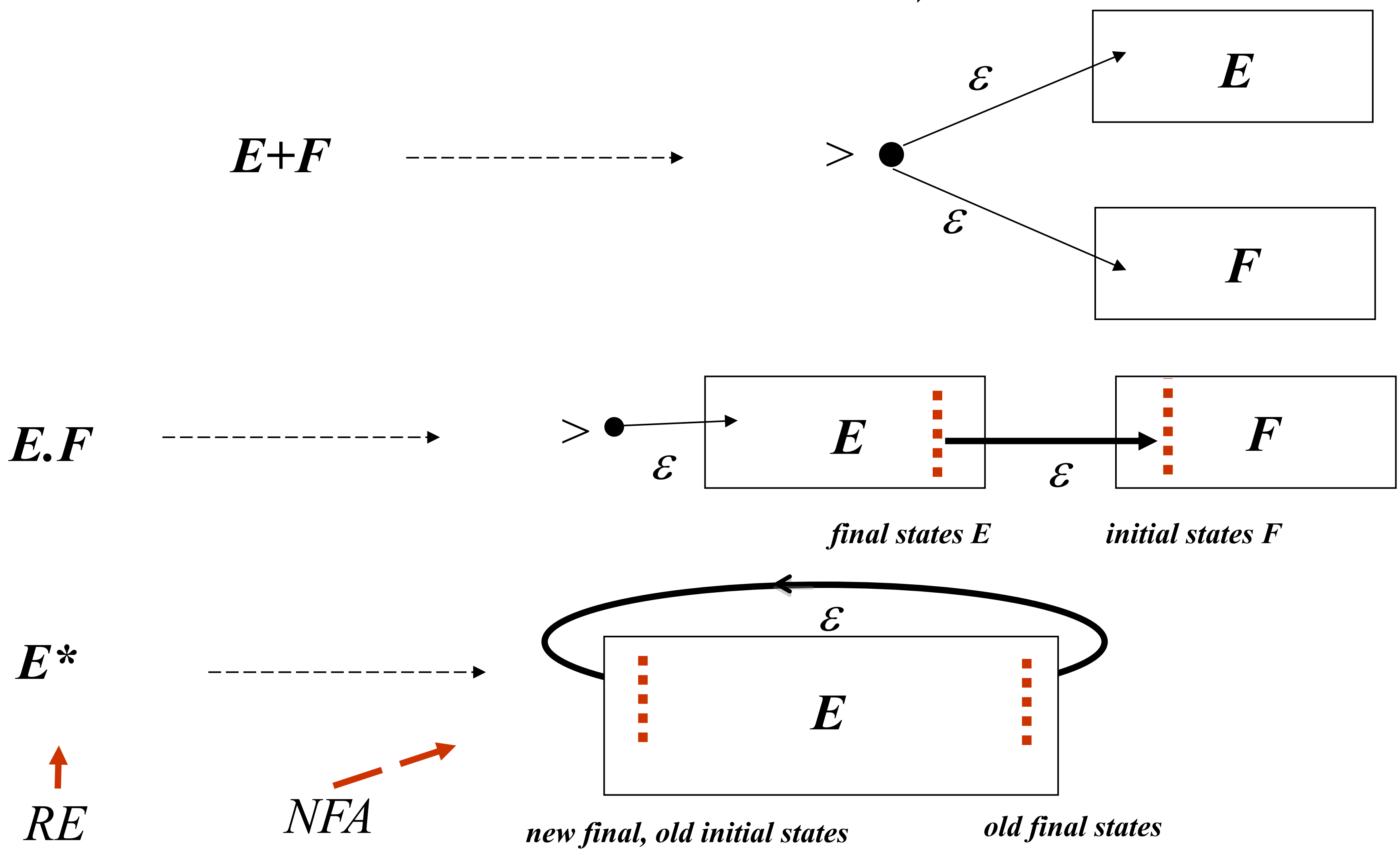
⇒ *(only if)* ***Idea :***

given ***RE****s over the set* $\Sigma = (\alpha, \beta, \gamma, \ldots,)$

Basis

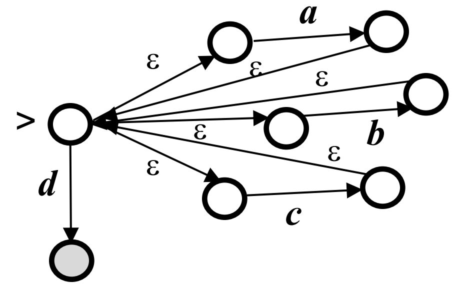


Proof of the **Main Theorem** (continued) ⇒ (only if)

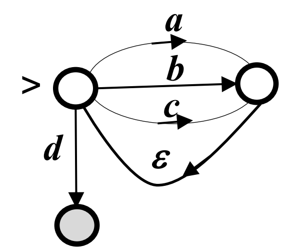


Some short cuts !

(a+b+c).d*

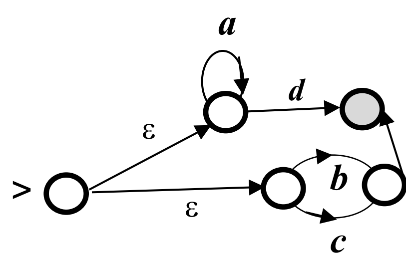


Simplified !

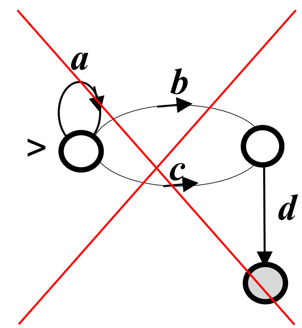


But !

(a+b+c).d*



NOT!



Algebraic Laws For REs

Trivial Laws

(1) $L+M = M+L$; $(L+M) + N = L + (M+N)$; $(L.M).N = L.(M.N)$

(2) $\phi + L = L$; $e.L = L.e = L$; $\phi.L = \phi$

(3) $L.(M+N) = L.M + L.N$; $(L+M).N = L.N + M.N$; $L+L = L$

Non-trivial Laws

(4) $(L+M)^* = (L^*+M^*)^* = (L^*.M^*)^*$

(5) $(L.M)^* \subseteq (L^*.M^*)^*$ and $(L.M)^* = (L^*.M^*)^*$ iff $e \in L$ *and* $e \in M$

Proof of (4) $\rightarrow (L+M)^* = (L^*. M^*)^*$

Two steps : (1) $(L+M)^* \subseteq (L^*.M^*)^*$; (2) $(L^*.M^*)^* \subseteq (L+M)^*$

(1) *Let* $u \in (L+M)^*$ *then* $u = u_1 . u_2 . \ldots . u_k$ for some integer $k \geq 0$ where for each j , $u_j \in L+M$;

but $L \subseteq L^* \subseteq L^*. e \subseteq L^* . M^*$ *and* $M \subseteq M^* \subseteq e . M^* \subseteq L^*. M^*$;

hence $u_j \in L^*. M^* + L^*. M^* = L^*. M^*$ and therefore $(L+M)^* \subseteq (L^* . M^*)^*$

(2) *Conversely let* $u \in (L^*.M^*)^*$ *then by definition* $u = u_1 . u_2 . \ldots . u_k$ *where* $u_j \in L^*.M^*$;

hence $u_j = v_j^1 . v_j^2 \ldots . v_j^{l(j)} . w_j^1 . w_j^2 \ldots . w_j^{p(j)}$ *where* $v_j^m \in L \subseteq L+M$ and $w_j^m \in M \subseteq L+M$;

thus $u = z_1 . z_2 . \ldots z_q$ *where* $q = \sum_{j=1,k} l(j)+p(j)$ *and each* $z_i \in L+M$. *Hence* $u \in (L+M)^*$;

this proves that $(L^*.M^*)^* \subseteq (L+M)^*$

Proof of $(L+M)^* = (L^*+M^*)^*$ given (4) $\rightarrow (L+M)^* = (L^*.\ M^*)^*$

Since $L \subseteq L^*$ *and* $M \subseteq M^*$ *it follows that* $(L+M)^* \subseteq (L^*+M^*)^*$

Conversely let $u \in (L^*+M^*)^*$ *then* $u = (v_1+w_1).\ \ldots\ .(v_k+w_k)$ *where for each* j $v_j \in L^*$ *and* $w_j \in M^*$.

We show that $u \in (L^*.\ M^*)^*$ *by using induction on* k.

For $k=1$ $\quad v_1 \in L^* \subseteq L^*.\ e \subseteq L^*.M^* \subseteq (L^*.\ M^*)^*$

similarly $w_1 \in M^* \subseteq e.\ M^* \subseteq L^*.M^* \subseteq (L^*.\ M^*)^*$ *hence* $v_1+w_1 \subseteq (L^*.\ M^*)^*$.

Now assume statement holds for $k-1$, *hence* $z := (v_1+w_1).\ \ldots\ .(v_{k-1} + w_{k-1}) \in (L^*.\ M^*)^*$

But using the above reasoning for v_1+w_1 *it follows that* $v_k+w_k \in (L^*.\ M^*)^*$

and therefore $u = z\ .\ (v_k+w_k) \in (L^*.\ M^*)^*.\ (L^*.\ M^*)^* = (L^*.\ M^*)^*$ *using the obvious identity* $K^*\ .\ K^* = K^*$ *for any language* K. *This proves that* $(L^*+M^*)^* \subseteq (L^*.\ M^*)^*$

but by (4) $(L+M)^* = (L^*.\ M^*)^*$ *hence* $(L^*+M^*)^* \subseteq (L + M)^*$ *and result follows*