SABANCI UNIVERSITY
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NOTES ON THE ANATOMY OF A *PDA*

Let $P = (Q, \Sigma, \Gamma, \delta, s, Z_0, F)$ be a given ***PDA*** and let

$(q , w , \gamma) \mid\!\text{---}^{\,n} \ (p , e , e)$ …………(1)

be a computation of P . In what follows all computations follow the same path as in *(1),* that is, the same transition sequence used in *(1)* is used .

(Fact 1) For any $u \in \Sigma^*$, $\beta \in \Gamma^*$ we have $(q , wu , \gamma\beta) \mid\!\text{---}^{\,n} \ (p , u , \beta)$

Proof of Fact 1

Given

$(q , w , \gamma) \mid\!\text{---}^{\,n} \ (p , e , e)$ …………(1)

we prove the result by induction on n

n=1 Case

If $n=1$ then we must have $w = a$ where $a \in \Sigma$ or $a = e$; $\gamma = X \in \Gamma$ and $(p,e) \in \delta(q,a,X)$ which implies that $(q , au , X\beta) \mid\!\text{---}^{\,n} (p , u , \beta)$ by definition of the δ function.

Now assume that $(q , w , \gamma) \mid\!\text{---}^{\,n} (p , e , e)$ implies that $(q , wu , \gamma\beta) \mid\!\text{---}^{\,n} \ (p , u , \beta)$ (induction hypothesis) and prove that :

$(q , w , \gamma) \mid\!\text{---}^{\,n+1} \ (p , e , e)$ implies that $(q , wu , \gamma\beta) \mid\!\text{---}^{\,n+1} \ (p , u , \beta)$

But letting $(q , w , \gamma) := (q, aw' , Y\gamma') \mid\!\text{---} (q', w',\xi\gamma')$ where we assume that $(q',\xi) \in \delta (q,a,Y)$ we have

$(q , w , \gamma) := (q, aw' , Y\gamma') \mid\!\text{---} (q', w',\xi\gamma')) \mid\!\text{---}^{\,n} (p , e , e)$ and by induction hypothesis $(q', w'u ,\xi\gamma'\beta)) \mid\!\text{---}^{\,n} (p , u , \beta)$ and for the case $n=1$ applied to the first step of the computation above

$(q, wu , \gamma\beta) \mid\!\text{---} (q', w'u,\xi\gamma'\beta)$ we conclude that $(q , wu , \gamma\beta) \mid\!\text{---}^{\,n+1} \ (p , u , \beta)$

(Fact 2)

Conversely if

$(q , wu , \gamma\beta) \mid\!\text{---}^{\,n} \ (p , u , \beta)$ …………(2)

Then again using the same path in *(2)* we conclude that

$(q , w , \gamma) \mid\!\text{---}^{\,n} \ (p , e , e)$ …………(3) provided that :

$(q , wu , \gamma\beta) \mid\!\!-\!\!-\!\!-^{k} \ (q_k , v_k u , \gamma_k\beta)$ with $| \gamma_k | > 0$ for all $0 \le k < n$ …………(4)

(Condition (4) is referred to as the *no top-of-the-stack exposure of* β throughout the computation path)

Proof of Fact 2

To prove the converse fact we again use induction on n. The case for $n=1$ is trivial and we omit the proof. Hence given

$(q , wu , \gamma\beta) \mid\!\!-\!\!-\!\!-^{n+1} \ (p , u , \beta)$ …………(2)

using the same mechanism in the proof of ***Fact 1*** and the implication $|\gamma| > 0$ of the additional assumption (*no top-of-the-stack exposure of* β) of ***Fact 2*** we have

$(q , wu , \gamma\beta) := (q, aw'u , Y\gamma'\beta) \mid\!\!-\!\!-\!\!- (q', w'u , \xi\gamma'\beta)) \mid\!\!-\!\!-\!\!-^{n} (p , u , \beta)$

and hence by the induction hypothesis

$(q', w' , \xi\gamma')) \mid\!\!-\!\!-\!\!-^{n} (p , e , e)$

Also using the $n=1$ case

$(q , w , \gamma) := (q, aw' , Y\gamma') \mid\!\!-\!\!-\!\!- (q', w' , \xi\gamma')) \mid\!\!-\!\!-\!\!-^{n} (p , e , e)$ which proves the result.

(Fact 3)

Let $(q , w , X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{n} \ (p , e , e)$ …………(5)

where $X_j \in \Gamma$ for $j = 1, 2, \ldots , m$

then along the computational path defined by *(5)* above, for each $j = 1, 2, \ldots, m$ there exists a ***unique*** integer $n(j)$, a segmentation $u_j v_j$ of w (i.e. $w = u_j v_j$) and a state q_j such that

$(q , w , X_1 X_2 \ldots X_m) \mid\!\!-\!\!-^{n(j)} \ (q_j , v_j , X_j X_{j+1} \ldots X_m)$ …………(6)

and

$(q , w , X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{k} \ (q' , w' v_j , \gamma X_j X_{j+1} \ldots X_m)$ …………(7)

with $| \gamma | > 0$ for all $0 \le k < n(j)$. Note $n(1) = 0$ and $q_1 = q$.

Proof of Fact 3

Given

$(q , w , X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{n} \ (p , e , e)$ …………(3)

we use induction on n for the proof.

For $n=1$ case , *(3)* implies that $w = a$ where $a \in \Sigma$ or $a = e$; $m = 1$ and $(p,e) \in \delta(q,a,X_1)$

The conclusion then trivially holds where $n(1)=0$; $q_1 = q$ and $w = a = e\, a$ ($u_1 = e$, $v_1 = a$) since $(q , a , X_1) \mid\!\!-\!\!-^{0} (q , a , X_1)$. Now assume the conclusion holds for n and prove it for $n+1$. Thus we assume that

$(q , w , X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{n+1} \ (p , e , e)$ …………(4)

and expand the first transition in *(4)* as

$(q\ ,\ w\ ,\ X_1\ X_2\ \dots\ X_m\ ) = (q\ ,\ aw'\ ,\ X_1\ X_2\dots\ X_m) \mid\!\text{---}\ (q'\ ,\ w'\ ,\ Y_1\ Y_2\ \dots\ Y_p\ X_2\ X_3\ \dots\ X_m)$ where in general $(q',\ Y_1 Y_2\ \dots\ Y_p) \in \delta\,(q\ ,\ a\ ,\ X_1)$. Now we apply the induction hypothesis to the case

$(q'\ ,\ w'\ ,\ Y_1\ Y_2\ \dots\ Y_p\ X_2\ X_3\ \dots\ X_m) \mid\!\text{---}^{\,n}\ \ (p\ ,\ e\ ,\ e)$ …………….*(5)*

with n transitions. If we let the corresponding unique integers for $Y_1\ ,Y_2\ ,\dots,Y_p,\ X_2\ ,X_3,\dots,X_m$ as $n'(1)$ to $n'(p+m-1)$ then the unique integer corresponding to X_j *(from state q)* is $n'(p+j\ -1)+1$ which implies that $n(j) = n'(p+j-1)+1$ for $j=2,\dots,m$ and similarly if q'_k are the states and u'_k and v'_k are the segmentations of w' for each $k=1,\dots,\ p+m-1$ we conclude that , using also the initial step before the last n steps, $q_j = q'_{p+j-1}$ and $w = a\ u'_{p+j-1}\ v'_{p+j-1}$ and hence $u_j = a\ u'_{p+j-1}$; $v_j = v'_{p+j-1}$ for $j = 2,\ \dots,\ m$. The remaining details follow from the induction hypothesis.

(Fact 4)

Suppose that *(5)* above holds then there is a unique segmentation

$w = w_1\ w_2\ \dots\ w_m$ such that for $j = 1,\ 2\ ,\ \dots\ ,\ m$

$(q_j\ ,\ w_j\ ,\ X_j\ ) \mid\!\text{---}^{\,k(j)}\ \ (q_{j+1}\ ,\ e,\ \ e)$ …………*(8)*

with $k(j) := n(j+1) - n(j)$ and $q_{m+1} :=\ p$ and $n(m+1) := n$ and w_j are related to u_j and v_j (see ***Fact 3*** above) by :

$u_j = w_1\ \dots w_j$ and $v_j = w_{j+1}\ \dots\ w_m$

<u>Proof of Fact 4</u>

We use ***Fact (3)*** to prove ***Fact (4)*** . This time we use induction on m , that is, the number of active stack symbols. For $m=1$ the result trivially follows from ***Fact (3).*** Assume the result holds for m and prove it for $m+1$. Now using ***Fact (3)***

$(q\ ,\ w\ ,\ X_1\ X_2\ \dots\ X_{m+1}\ ) \mid\!\text{---}^{\,n(2)}\ \ (q_2\ ,\ v_1\ ,\ X_2\ X_3\ \dots\ X_{m+1}\ )$ …………………..*(6)*

where $w = u_1\ v_1$. Now we can use ***Fact (2)*** with $u = v_1$ and $\beta = X_2 X_3\ \dots\ X_m$ to conclude that $(q\ ,\ u_1\ ,\ X_1\ ) \mid\!\text{---}^{\,n(2)}\ \ (q_2\ ,\ e\ ,\ e\ )$ where $k(1) = n(2) - n(1) = n(2) - 0 = n(2)$ as required for $j=1$.

Now using the induction hypothesis we conclude that the original premiss

$(q_2\ ,\ v_1\ ,\ X_2\ \dots\ X_{m+1}\ ) \mid\!\text{---}^{\,n-n(2)}\ \ (p\ ,\ e,\ e\ )$

yields the required results for $j =2,\ \dots,m+1$ where $k(j) = n(j+1) - n(j)$.

The input segmentation for v_1 is $v_1 = w_2\ \ \dots.\ w_{m+1}$ which follows from the induction hypothesis and the fact that $w_1 = u_1$ completes the entire segmentation. Remaining minor details are left to the reader.

(Fact 5)

If for $j = 1, 2, \dots, m$

$(q_j , w_j , X_j) \ |\text{---}^{k(j)} \ (q_{j+1} , e, e)$ ………… *(9)*

then

$(q , w , X_1 X_2 \dots X_m) \ |\text{---}^{n} \ (p , e , e)$

where $w := w_1 w_2 \dots w_m$, $p := q_{m+1}$ and $q := q_1$.

Proof of Fact 5

We are given that

$(q_j , w_j , X_j) \ |\text{---}^{k(j)} \ (q_{j+1} , e , e)$ ………… *(7)*

for $j = 1, 2,\dots,m$ and asked to prove that

$(q , w , X_1 X_2 \dots X_m) \ |\text{---}^{n} \ (p , e , e)$

where $n := k(1) + k(2) + \dots + k(m)$, $q_1 := q$, $q_{m+1} := p$ and $w : = w_1 w_2 \dots w_m$

We prove this by using induction on m. For $m=1$ the result is trivial since the conclusion and the premiss are identical. Now assume the statement is true for m and assuming *(7)* holds for $j=1,2,\dots,m+1$ we shall prove that

$(q , w , X_1 X_2 \dots X_{m+1}) \ |\text{---}^{n} \ (p , e , e)$

Using ***Fact (2)*** and *(7)* for $j=1$

$(q , w_1 u , X_1 \beta) \ |\text{---}^{k(1)} (q_2, u , \beta) = (q_2 , w_2 w_3 \dots w_{m+1} , X_2 X_3 \dots X_{m+1})$ ………. *(8)*

where we chose $u = w_2 w_3 \dots w_{m+1}$ and $\beta = X_2 X_3 \dots X_{m+1}$ in applying ***Fact (2)***

Now we can invoke the induction hypothesis since the depth of the stack is m and write

$(q_2 , w_2 w_3 \dots w_{m+1} , X_2 X_3 \dots X_{m+1}) \ |\text{---}^{n'} \ (p , e , e)$ …………. *(9)*

where $n' = k(2) + k(3) + \dots + k(m+1)$. Combining *(8)* and *(9)* we arrive at

$(q , w_1 w_2 \dots w_{m+1} , X_1 X_2 \dots X_{m+1}) \ |\text{---}^{k(1)+n'} \ (p , e , e)$

which proves ***Fact (5).***

The facts below show that the constructed equivalent CFG to a PDA is indeed equivalent !

For the given *PDA* $P = (Q, \Sigma, \Gamma, \delta, s, Z_0, F)$ define a CFG , $G = (V, \Sigma, R, S)$ where

$V = \{S\} \cup \{ [\, q\, X\, p\,] \mid p,q \in Q ,\, X \in \Gamma \}$ ....*(10)*

In addition to the initial productions namely

$S \rightarrow [s\, Z_0\, p]$

where s is the initial state and p takes values over Q and we have the following productions in R :

whenever $(q_1 , X_1 X_2 \dots X_m) \in \delta (q, a , X)$ there are productions

$[\, q\, X\, p\,] \rightarrow a\, [\, q_1\, X_1\, q_2\,] \dots [q_m\, X_m\, p]$ ....*(11)*

for all possible $q_2 ,\dots, q_m , p \in Q$ and if $(q_1 , e) \in \delta (a, q , X)$

$[\, q\, X\, q_1\,] \rightarrow a$ is a production.

(Fact 6)

If $(q , w , X) \mid\!--_P{}^* (p , e , e)$ then $[\, q\, X\, p\,] \Rightarrow_G{}^* w$

Proof of Fact 6

Again we use induction on n . For $n=1$

$(q , w , X) \mid\!-- (p , e , e)$ implies $w=a$ or $w=e$ and $(p,e) \in \delta (q,a,X)$ hence

$[qXp] \rightarrow a$ which proves the result.

Now given $(q , w , X) \mid\!--^{n+1} (p , e , e)$ in order to use induction on n we first evaluate the first step of the computation as below :

$(q , aw' , X) \mid\!--- (q_1 , w' , X_1 X_2 \dots X_m)$ where we assumed that

$(q_1, X_1 X_2 \dots X_m) \in \delta (q , a , X).$ Now by definition of the productions of G we have

$[qXp] \rightarrow a\, [q_1\, X_1\, q_2] \dots [q_m\, X_m\, p\,]$ …………………….*(12)*

are productions for all possible $q_2 , q_3,\dots,q_m , p \in Q$. Since

$(q_1, w' , X_1 X_2 \dots X_m) \mid\!--^{n} (p , e , e)$

which by ***Fact 4*** implies for $j = 1, \dots, m$ there are states q'_j for $j =1, ..., m+1$ where

$q'_{m+1} := p$, $q'_1 = q_1$ such that

$(q'_j , w'_j , X_j) \mid\!--^{k(j)} (q'_{j+1} , e , e)$ where each $k(j) < n+1$ and by the induction hypothesis and choosing $q_j := q'_j$ for $j=1,2,\dots,m$ and $p := q'_{m+1}$ in *(12)* $[q'_j\, X_j\, q'_{j+1}\,] \rightarrow w'_j$ and the result follows by recalling that $w' = w'_1\, w'_2 \dots w'_m$; $w = a\, w'$ and *a leftmost derivation* of *(12)* yields w.

(Fact 7)

If $[\,q\,X\,p\,] \Rightarrow_G^* w$ then $(q\,,w\,,X) \mid\!\!---_P^* \;(p\,,e\,,e\,)$

Proof of Fact 7

The proof is similar to that of ***Fact 6.*** This time we use induction on the steps of a leftmost derivation that yields w. For $n=1$ we have $[\,q\,X\,p\,] \Rightarrow w$ which implies that $w=a$ or $w=e$ and $(p.e) \in \delta\,(q,\,a,X)$ that further implies $(q,\,a,\,X) \mid\!\!--- (p,e)$

Now we prove the case for $[\,q\,X\,p\,] \Rightarrow^{n+1} w.$ We perform a first step of the derivation which in general yields

$[\,q\,X\,p] \Rightarrow a\,[q_1\,X_1\,q_2]\,[q_2\,X_2\,q_3]\,\ldots\,[q_m\,X_m\,p]$

where $q_2,\,\ldots,\,q_m\,,p$ take values in Q and q_1 takes a value dictated by a transition

$(q_1,\,X_1\,X_2\,\ldots\,X_m) \in \delta(q,\,a,\,X)$

Hence we have

$(q,\,aw'\,,X) \mid\!\!--- (q_1\,,\,w'\,,\,X_1\,X_2\,\ldots\,X_m\,)$

Now since the steps of derivation for each

$[\,q_j\,X_j\,q_{j+1}] \Rightarrow w'_j\;,\; j=1\,,\,\ldots,\,m\text{-}1$ and $[\,q_m\,X_m\,p] \Rightarrow w'_m$

is necessarily less than $n+1$, the induction hypothesis can be applied which leads to

$(q_j\,,\,w'_j\,,\,X_j) \mid\!\!--- (q_{j+1}\,,e,\,e)$ for $j=1\,,\,\ldots,\,m\text{-}1$ and

$(q_m,\,w'_m\,,\,X_m) \mid\!\!--- (p\,,e,\,e)$

which by ***Fact 4*** implies that

$(q_1,\,w'_1\,w'_2\,\ldots w'_m\,,\,X_1\,X_2\,\ldots X_m\,) \mid\!\!-- (p,e,e)$ and therefore

$[q\,X\,p] \Rightarrow aw'_1\,w'_2\,\ldots w'_m = w$ implies that

$(q,\,w\,,X\,)= (q,\,aw',\,X) \;\mid\!\!-- (q_1\,,\,w'\,,\,X_1\,X_2\,\ldots\,X_m\,) \mid\!\!-- \;(p,\,e,e)$

which proves ***Fact*** *7*

(Fact 8)

$L_P = L_G$

Proof of Fact 8

To prove that $L_P = L_G$ we first use ***Fact 6*** with $q=s$ (initial state) and $X = Z_0$ (initial stack) and use the *empty stack acceptance criterion* which proves that $L_P \subseteq L_G$. Conversely we use ***Fact*** *7* after the first derivation step which is $S \Rightarrow [s\,Z_0\,p]$ for an appropriate p which yields a string w **,** that is , $S \Rightarrow [s\,Z_0\,p] \Rightarrow^* w$ **,** and apply ***Fact*** *7* which proves that $L_G \subseteq L_P$.

This concludes the proofs for all the ***Facts.***

An Illustrative Example

Consider the single state *PDA*

$P = (\{q\}, \{a,b\}, \{a,b,Z_0\}, \delta, q, Z_0, \{q\})$

that accepts by empty stack the language

$L = \{\omega \in (a+b)^* \mid \#a\text{s} = \#b\text{s}\}$ with the following δ - transitions :

$(q, a, Z_0) \rightarrow (q, aZ_0)$ ....(1)

$(q, b, Z_0) \rightarrow (q, bZ_0)$ ....(2)

$(q, a, a) \rightarrow (q, aa)$ .... (3)

$(q, b, b) \rightarrow (q, bb)$ .... (4)

$(q, a, b) \rightarrow (q, e)$ .... (5)

$(q, b, a) \rightarrow (q, e)$ .... (6)

$(q, e, Z_0) \rightarrow (q, e)$ .... (7)

Using the definition justified by ***Fact 1*** to ***Fact 8*** above $G = (V,T,P,S)$ is

$T = \{a,b\}$

$V = \{S, [q Z_0 q], [qaq], [qbq]\}$

Letting $Z := [q Z_0 q]$, $A := [qaq]$, $B := [qbq]$

$V = \{S, Z, A, B\}$

and ***P*** becomes the set :

$S \rightarrow Z$; $Z \rightarrow a A Z \mid b B Z \mid e$ *(using (1),(2) and (7))* ; $A \rightarrow aAA \mid b$ *(using (3) and (5))*; $B \rightarrow bBB \mid a$ *(using (4) and (6))*