SABANCI UNIVERSITY

Faculty of Engineering and Natural Sciences

CS 302 Automata Theory

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NOTES ON THE ANATOMY OF A *PDA*

Let $\boldsymbol{P = (Q, \Sigma, \Gamma, \delta, s, Z_0, F)}$ be a given ***PDA*** and let

$(q, w, \gamma) \vdash^{n} (p, e, e)$ …………*(1)*

be a computation of ***P***. In what follows all computations follow the same path as in *(1),* that is, the same transition sequence used in *(1)* is used .

(Fact 1) For any $u \in \Sigma^*, \beta \in \Gamma^*$ we have $(q, wu, \gamma\beta) \vdash^{n} (p, u, \beta)$

Proof of Fact 1

Given

$(q, w, \gamma) \vdash^{n} (p, e, e)$ …………*(1)*

we prove the result by induction on n

n=1 Case

If $n=1$ then we must have $w = a$ where $a \in \Sigma$ or $a = e; \gamma = X \in \Gamma$ and $(p,e) \in \delta(q,a,X)$ which implies that $(q, au, X\beta) \vdash^{n} (p, u, \beta)$ by definition of the δ function.

Now assume that $(q, w, \gamma) \vdash^{n} (p, e, e)$ implies that $(q, wu, \gamma\beta) \vdash^{n} (p, u, \beta)$ (induction hypothesis) and prove that:

$(q, w, \gamma) \vdash^{n+1} (p, e, e)$ implies that $(q, wu, \gamma\beta) \vdash^{n+1} (p, u, \beta)$

But letting $(q, w, \gamma) := (q, aw', Y\gamma') \vdash (q', w', \xi\gamma')$ where we assume that $(q', \xi) \in \delta(q,a,Y)$ we have

$(q, w, \gamma) := (q, aw', Y\gamma') \vdash (q', w', \xi\gamma') \vdash^{n} (p, e, e)$ and by induction hypothesis

$(q', w'u, \xi\gamma'\beta) \vdash^{n} (p, u, \beta)$ and for the case $n=1$ applied to the first step of the computation above

$(q, wu, \gamma\beta) \vdash (q', w'u, \xi\gamma'\beta)$ we conclude that $(q, wu, \gamma\beta) \vdash^{n+1} (p, u, \beta)$

(Fact 2)

Conversely if

$(q, wu, \gamma\beta) \vdash^{n} (p, u, \beta)$ …………*(2)*

Then again using the same path in *(2)* we conclude that

$(q, w, \gamma) \vdash^{n} (p, e, e)$ …………*(3)* provided that :

$(q, wu, \gamma\beta) \mid\!\!-\!\!-\!\!-^{k} (q_k, v_k u, \gamma_k\beta)$ with $|\gamma_k| > 0$ for all $0 \le k < n$ …………(4)

(Condition (4) is referred to as the *no top-of-the-stack exposure of* β throughout the computation path)

Proof of Fact 2

To prove the converse fact we again use induction on n. The case for $n=1$ is trivial and we omit the proof. Hence given

$(q, wu, \gamma\beta) \mid\!\!-\!\!-\!\!-^{n+1} (p, u, \beta)$ …………(2)

using the same mechanism in the proof of ***Fact 1*** and the implication $|\gamma| > 0$ of the additional assumption (*no top-of-the-stack exposure of* β) of ***Fact 2*** we have

$(q, wu, \gamma\beta) := (q, aw'u, Y\gamma'\beta) \mid\!\!-\!\!-\!\!- (q', w'u, \xi\gamma'\beta)) \mid\!\!-\!\!-\!\!-^{n} (p, u, \beta)$

and hence by the induction hypothesis

$(q', w', \xi\gamma')) \mid\!\!-\!\!-\!\!-^{n} (p, e, e)$

Also using the $n=1$ case

$(q, w, \gamma) := (q, aw', Y\gamma') \mid\!\!-\!\!-\!\!- (q', w', \xi\gamma')) \mid\!\!-\!\!-\!\!-^{n} (p, e, e)$ which proves the result.

(Fact 3)

Let $(q, w, X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{n} (p, e, e)$ …………(5)

where $X_j \in \Gamma$ for $j = 1, 2, \ldots, m$

then along the computational path defined by *(5)* above, for each $j = 1, 2, \ldots, m$ there exists a ***unique*** integer $n(j)$, a segmentation $u_j v_j$ of w (i.e. $w = u_j v_j$) and a state q_j such that

$(q, w, X_1 X_2 \ldots X_m) \mid\!\!-\!\!-^{n(j)} (q_j, v_j, X_j X_{j+1} \ldots X_m)$ …………(6)

and

$(q, w, X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{k} (q', w' v_j, \gamma X_j X_{j+1} \ldots X_m)$ …………(7)

with $|\gamma| > 0$ for all $0 \le k < n(j)$. Note $n(1) = 0$ and $q_1 = q$.

Proof of Fact 3

Given

$(q, w, X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{n} (p, e, e)$ …………(3)

we use induction on n for the proof.

For $n=1$ case, *(3)* implies that $w = a$ where $a \in \Sigma$ or $a = e$; $m = 1$ and $(p,e) \in \delta(q,a,X_1)$

The conclusion then trivially holds where $n(1)=0$; $q_1 = q$ and $w = a = e\,a$ ($u_1 = e$, $v_1 = a$) since $(q, a, X_1) \mid\!\!-\!\!-^{0} (q, a, X_1)$. Now assume the conclusion holds for n and prove it for $n+1$. Thus we assume that

$(q, w, X_1 X_2 \ldots X_m) \mid\!\!-\!\!-\!\!-^{n+1} (p, e, e)$ …………(4)

and expand the first transition in *(4)* as

$(q , w , X_1 X_2 \ldots X_m) = (q , aw' , X_1 X_2 \ldots X_m) \,|\!\!-\!\!-\!\!- (q' , w' , Y_1 Y_2 \ldots Y_p X_2 X_3 \ldots X_m)$ where in general $(q', Y_1Y_2 \ldots Y_p) \in \delta(q , a , X_1)$. Now we apply the induction hypothesis to the case

$(q' , w' , Y_1 Y_2 \ldots Y_p X_2 X_3 \ldots X_m) \,|\!\!-\!\!-\!\!-^{n} (p , e , e)$ ……………..*(5)*

with n transitions. If we let the corresponding unique integers for $Y_1 ,Y_2 ,\ldots,Y_p , X_2 ,X_3,\ldots,X_m$ as $n'(1)$ to $n'(p+m-1)$ then the unique integer corresponding to X_j *(from state q)* is $n'(p+j -1)+1$ which implies that $n(j) = n'(p+j-1)+1$ for $j=2,\ldots,m$ and similarly if q'_k are the states and u'_k and v'_k are the segmentations of w' for each $k=1,\ldots, p+m-1$ we conclude that , using also the initial step before the last n steps, $q_j = q'_{p+j-1}$ and $w = a\, u'_{p+j-1}\, v'_{p+j-1}$ and hence $u_j = a\, u'_{p+j-1}$; $v_j = v'_{p+j-1}$ for $j = 2, \ldots, m$. The remaining details follow from the induction hypothesis.

(Fact 4)

Suppose that *(5)* above holds then there is a unique segmentation

$w = w_1 w_2 \ldots w_m$ such that for $j = 1, 2 , \ldots , m$

$(q_j , w_j , X_j) \,|\!\!-\!\!-\!\!-^{k(j)} (q_{j+1} , e, e)$ …………*(8)*

with $k(j) := n(j+1) - n(j)$ and $q_{m+1} := p$ and $n(m+1) := n$ and w_j are related to u_j and v_j (see ***Fact 3*** above) by :

$u_j = w_1 \ldots w_j$ and $v_j = w_{j+1} \ldots w_m$

Proof of Fact 4

We use ***Fact (3)*** to prove ***Fact (4)*** . This time we use induction on m , that is, the number of active stack symbols. For $m=1$ the result trivially follows from ***Fact (3).*** Assume the result holds for m and prove it for $m+1$ **.** Now using ***Fact (3)***

$(q , w , X_1 X_2 \ldots X_{m+1}) \,|\!\!-\!\!-\!\!-^{n(2)} (q_2 , v_1 , X_2 X_3 \ldots X_{m+1})$ …………………..*(6)*

where $w = u_1 v_1$. Now we can use ***Fact (2)*** with $u = v_1$ and $\beta = X_2 X_3 \ldots X_m$ to conclude that $(q , u_1 , X_1) \,|\!\!-\!\!-\!\!-^{n(2)} (q_2 , e , e)$ where $k(1) = n(2) - n(1) = n(2) - 0 = n(2)$ as required for $j=1$.
Now using the induction hypothesis we conclude that the original premiss

$(q_2 , v_1 , X_2 \ldots X_{m+1}) \,|\!\!-\!\!-\!\!-^{n-n(2)} (p , e, e)$

yields the required results for $j =2, \ldots,m+1$ where $k(j) = n(j+1) - n(j)$.
The input segmentation for v_1 is $v_1 = w_2 \ldots. w_{m+1}$ which follows from the induction hypothesis and the fact that $w_1 = u_1$ completes the entire segmentation. Remaining minor details are left to the reader.

(Fact 5)

If for $j = 1, 2, \dots, m$

$(q_j, w_j, X_j) \;|\!\!-\!\!-\!\!-^{k(j)}\; (q_{j+1}, e, e)$ …………(9)

then

$(q, w, X_1 X_2 \dots X_m) \;|\!\!-\!\!-\!\!-^{n}\; (p, e, e)$

where $w := w_1 w_2 \dots w_m$, $p := q_{m+1}$ and $q := q_1$.

Proof of Fact 5

We are given that

$(q_j, w_j, X_j) \;|\!\!-\!\!-\!\!-^{k(j)}\; (q_{j+1}, e, e)$ …………(7)

for $j = 1, 2, \dots, m$ and asked to prove that

$(q, w, X_1 X_2 \dots X_m) \;|\!\!-\!\!-\!\!-^{n}\; (p, e, e)$

where $n := k(1) + k(2) + \dots + k(m)$, $q_1 := q$, $q_{m+1} := p$ and $w := w_1 w_2 \dots w_m$

We prove this by using induction on ***m.*** For $m=1$ the result is trivial since the conclusion and the premiss are identical. Now assume the statement is true for ***m*** and assuming *(7)* holds for $j=1,2,\dots,m+1$ we shall prove that

$(q, w, X_1 X_2 \dots X_{m+1}) \;|\!\!-\!\!-\!\!-^{n}\; (p, e, e)$

Using ***Fact (2)*** and *(7)* for $j=1$

$(q, w_1 u, X_1 \beta) \;|\!\!-\!\!-\!\!-^{k(1)}\; (q_2, u, \beta) = (q_2, w_2 w_3 \dots w_{m+1}, X_2 X_3 \dots X_{m+1})$ ……….(8)

where we chose $u = w_2 w_3 \dots w_{m+1}$ and $\beta = X_2 X_3 \dots X_{m+1}$ in applying ***Fact (2)***

Now we can invoke the induction hypothesis since the depth of the stack is ***m*** and write

$(q_2, w_2 w_3 \dots w_{m+1}, X_2 X_3 \dots X_{m+1}) \;|\!\!-\!\!-\!\!-^{n'}\; (p, e, e)$ ………….(9)

where $n' = k(2) + k(3) + \dots + k(m+1)$. Combining *(8)* and *(9)* we arrive at

$(q, w_1 w_2 \dots w_{m+1}, X_1 X_2 \dots X_{m+1}) \;|\!\!-\!\!-\!\!-^{k(1)+n'}\; (p, e, e)$

which proves ***Fact (5).***

The facts below show that the constructed equivalent CFG to a PDA is indeed equivalent !

For the given *PDA* $\boldsymbol{P = (Q, \Sigma, \Gamma, \delta, s, Z_0, F)}$ define a CFG , $\boldsymbol{G = (V, \Sigma, R , S)}$ where

$V = \{S\} \cup \{ [q\, X\, p] \mid p,q \in Q , X \in \Gamma\}$ ....*(10)*

In addition to the initial productions namely

$\boldsymbol{S \to [s\, Z_0\, p]}$

where $\boldsymbol{s}$ is the initial state and $\boldsymbol{p}$ takes values over $\boldsymbol{Q}$ and we have the following productions in $\boldsymbol{R}$:

whenever $(q_1 , X_1 X_2 \dots X_m) \in \delta(q, a , X)$ there are productions

$[q\, X\, p] \to a\, [q_1\, X_1\, q_2] \dots [q_m\, X_m\, p]$ ....*(11)*

for all possible $q_2 ,\dots, q_m , p \in Q$ and if $(q_1 , e) \in \delta(a, q , X)$

$[q\, X\, q_1] \to a$ is a production.

(Fact 6)

If $(q , w , X) \vdash_P^* (p , e , e)$ then $[q\, X\, p] \Rightarrow_G^* w$

Proof of Fact 6

Again we use induction on $\boldsymbol{n}$. For $\boldsymbol{n=1}$

$(q , w , X) \vdash (p , e , e)$ implies $\boldsymbol{w=a}$ or $\boldsymbol{w=e}$ and $(p,e) \in \delta(q,a,X)$ hence

$[qXp] \to a$ which proves the result.

Now given $(q , w , X) \vdash^{n+1} (p , e , e)$ in order to use induction on $\boldsymbol{n}$ we first evaluate the first step of the computation as below :

$(q , aw' , X) \vdash (q_1 , w' , X_1 X_2 \dots X_m)$ where we assumed that

$(q_1, X_1 X_2 \dots X_m) \in \delta(q , a , X)$. Now by definition of the productions of $\boldsymbol{G}$ we have

$[qXp] \to a\, [q_1\, X_1\, q_2] \dots [q_m\, X_m\, p]$ ……………………..*(12)*

are productions for all possible $q_2 , q_3,\dots,q_m , p \in Q$. Since

$(q_1, w' , X_1 X_2 \dots X_m) \vdash^{n} (p , e , e)$

which by ***Fact 4*** implies for $\boldsymbol{j = 1, \dots, m}$ there are states $\boldsymbol{q'_j}$ for $\boldsymbol{j =1, \dots, m+1}$ where

$q'_{m+1} := p$, $q'_1 = q_1$ such that

$(q'_j , w'_j , X_j) \vdash^{k(j)} (q'_{j+1} , e , e)$ where each $\boldsymbol{k(j) < n+1}$ and by the induction hypothesis and choosing $q_j := q'_j$ for $\boldsymbol{j=1,2,\dots,m}$ and $p := q'_{m+1}$ in *(12)* $[q'_j\, X_j\, q'_{j+1}] \to w'_j$ and the result follows by recalling that $w' = w'_1\, w'_2 \dots w'_m$; $w = a\, w'$ and *a leftmost derivation* of *(12)* yields $\boldsymbol{w}$.

(Fact 7)

If $[\, q\, X\, p\,] \Rightarrow_G^* w$ then $(q\, ,\, w\, ,\, X) \mid\!\!-\!\!-\!\!-_P^* \; (p\, ,\, e\, ,\, e\,)$

Proof of Fact 7

The proof is similar to that of ***Fact 6.*** This time we use induction on the steps of a leftmost derivation that yields w. For $n=1$ we have $[\, q\, X\, p\,] \Rightarrow w$ which implies that $w=a$ or $w=e$ and $(p.e) \in \delta(q,\, a,X)$ that further implies $(q,\, a,\, X) \mid\!\!-\!\!-\!\!- (p,e)$

Now we prove the case for $[\, q\, X\, p\,] \Rightarrow^{n+1} w.$ We perform a first step of the derivation which in general yields

$[\, q\, X\, p] \Rightarrow a\, [q_1\, X_1\, q_2]\, [q_2\, X_2\, q_3] \ldots [q_m\, X_m\, p]$

where $q_2, \ldots, q_m\, , p$ take values in Q and q_1 takes a value dictated by a transition

$(q_1, X_1\, X_2 \ldots X_m) \in \delta(q,\, a,\, X)$

Hence we have

$(q,\, aw'\, ,\, X) \mid\!\!-\!\!-\!\!- (q_1\, ,\, w'\, ,\, X_1\, X_2 \ldots X_m\,)$

Now since the steps of derivation for each

$[\, q_j\, X_j\, q_{j+1}] \Rightarrow w'_j\;$, $j=1\, , \ldots, m\text{-}1$ and $[\, q_m\, X_m\, p] \Rightarrow w'_m$

is necessarily less than $n+1$, the induction hypothesis can be applied which leads to

$(q_j\, ,\, w'_j\, ,\, X_j) \mid\!\!-\!\!-\!\!- (q_{j+1}\, ,e,\, e)$ for $j=1\, , \ldots, m\text{-}1$ and

$(q_m,\, w'_m\, ,\, X_m) \mid\!\!-\!\!-\!\!- (p\, ,e,\, e)$

which by ***Fact 4*** implies that

$(q_1,\, w'_1\, w'_2 \ldots w'_m\, ,\, X_1\, X_2 \ldots X_m\,) \mid\!\!-\!\!- (p,e,e)$ and therefore

$[q\, X\, p] \Rightarrow aw'_1\, w'_2 \ldots w'_m = w$ implies that

$(q,\, w\, ,X\,) = (q,\, aw',\, X) \;\mid\!\!-\!\!- (q_1\, ,\, w'\, ,\, X_1\, X_2 \ldots X_m\,) \mid\!\!-\!\!- \; (p,\, e,e)$

which proves ***Fact 7***

(Fact 8)

$L_P = L_G$

Proof of Fact 8

To prove that $L_P = L_G$ we first use ***Fact 6*** with $q=s$ (initial state) and $X = Z_0$ (initial stack) and use the *empty stack acceptance criterion* which proves that $L_P \subseteq L_G$. Conversely we use ***Fact 7*** after the first derivation step which is $S \Rightarrow [s\, Z_0\, p]$ for an appropriate p which yields a string w , that is , $S \Rightarrow [s\, Z_0\, p] \Rightarrow^* w$, and apply ***Fact 7*** which proves that $L_G \subseteq L_P$.

This concludes the proofs for all the ***Facts.***

An Illustrative Example

Consider the single state *PDA*

$\boldsymbol{P = (\{q\}, \{a,b\}, \{a,b,Z_0\}, \delta, q, Z_0, \{q\})}$

that accepts by empty stack the language

$\boldsymbol{L = \{\omega \in (a+b)^* \mid \#a\text{s} = \#b\text{s}\}}$ with the following $\boldsymbol{\delta}$**-** transitions :

$\boldsymbol{(q, a, Z_0) \to (q, aZ_0)}$ ....*(1)*

$\boldsymbol{(q, b, Z_0) \to (q, bZ_0)}$ ....*(2)*

$\boldsymbol{(q, a, a) \to (q, aa)}$ .... *(3)*

$\boldsymbol{(q, b, b) \to (q, bb)}$ .... *(4)*

$\boldsymbol{(q, a, b) \to (q, e)}$ .... *(5)*

$\boldsymbol{(q, b, a) \to (q, e)}$ .... *(6)*

$\boldsymbol{(q, e, Z_0) \to (q, e)}$ .... *(7)*

Using the definition justified by ***Fact 1*** to ***Fact 8*** above $\boldsymbol{G = (V,T,P,S)}$ is

$\boldsymbol{T = \{a,b\}}$

$\boldsymbol{V = \{S, [q Z_0 q], [qaq], [qbq]\}}$

Letting $\boldsymbol{Z := [q Z_0 q], A := [qaq], B := [qbq]}$

$\boldsymbol{V = \{S, Z, A, B\}}$

and ***P*** becomes the set :

$\boldsymbol{S \to Z\ ;\ Z \to a A Z \mid b B Z \mid e}$ *(using (1),(2) and (7))* $\boldsymbol{;\ A \to aAA \mid b}$ *(using (3) and (5));* $\boldsymbol{B \to bBB \mid a}$ *(using (4) and (6))*