Spectrum and Fourier Series

P. Sommen and B. van Erp

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1 Spectrum

Many practical signals can be described as a set of sinusoidal signals. In this section we will first show how such such a set can be rewritten as a set of weighted phasor components. From this description it follows that these weights represent the spectral information of the signal.

1.1 Real signal as sum of phasors

Let's start with a general description of a signal $x(t)$ which consists of a DC component, with value A_0, and a sum of N sinusoidal components, each with a different frequency f_k, amplitude A_k and phase ϕ_k, as follows:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi \cdot f_k t + \phi_k) \tag{1}$$

Now by using Euler we can write this equation as the following sum of rotating phasors:

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{N} \left\{ \left(\frac{A_k}{2} \mathrm{e}^{\mathrm{j}\phi_k} \right) \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot f_k t} + \left(\frac{A_k}{2} \mathrm{e}^{-\mathrm{j}\phi_k} \right) \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot f_k t} \right\} \\ &= X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot f_k t} + \frac{X_k^*}{2} \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot f_k t} \right\} \qquad (2) \\ &\quad \text{with } X_0 = A_0 \text{ and } X_k = A_k \mathrm{e}^{\mathrm{j}\phi_k} \end{aligned}$$

From this description we can derive the following general property:

> *For* **real** *signals the values of the complex weights of the phasors with negative frequencies are the complex conjugated versions of the complex weights of the phasors with positive frequencies.*

1.2 Spectral plot

By ordering the frequencies from low to high we can represent the phasor description (2) in a frequency spectrum plot. In such a plot we denote the frequencies on the horizontal axis. The frequency on this axis can either be denoted by the values of f_k in [Hz] or by the values of $\omega_k = 2\pi f_k$ in [rad/sec]. For each of these frequencies f_k we plot a bar, denoting the complex weights

$\frac{X_k}{2}$ as described in equation (2). These complex weights are related to the magnitude and phase of the original sinusoidal signal.

Example:

Find the complex weights of the phasor components of the signal

$$x(t) = 10 + 14\cos(200\pi t - \frac{\pi}{3}) + 8\cos(500\pi t + \frac{\pi}{2})$$

and make a spectral plot of this signal.

Solution:

By using Euler we can rewrite the expression for $x(t)$ as follows:

$$\begin{aligned} x(t) &= 10 + 14\cos(200\pi t - \frac{\pi}{3}) + 8\cos(500\pi t + \frac{\pi}{2}) \\ &= 10 \cdot 1 + 14 \left\{ \frac{\mathrm{e}^{\mathrm{j}(200\pi t - \frac{\pi}{3})} + \mathrm{e}^{-\mathrm{j}(200\pi t - \frac{\pi}{3})}}{2} \right\} \\ &\quad +8 \left\{ \frac{\mathrm{e}^{\mathrm{j}(500\pi t + \frac{\pi}{2})} + \mathrm{e}^{-\mathrm{j}(500\pi t + \frac{\pi}{2})}}{2} \right\} \\ &= (10) \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot 0 \cdot t} + \left(7\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}\right) \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot 100 \cdot t} + \left(7\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}\right) \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot 100 \cdot t} \\ &\quad + \left(4\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\right) \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot 250 \cdot t} + \left(4\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\right) \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot 250 \cdot t} \end{aligned}$$

So the signal contains the following phasor frequencies:

$$f_0 = 0, f_1 = 100, f_2 = 250, -f_1 = -100 \text{ and } -f_2 = -250 \text{ [Hz]},$$

while the magnitude and phase of each of these phasor frequencies are represented by the complex numbers:

$$X_0 = 10, \frac{X_1}{2} = 7\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}, \frac{X_2}{2} = 4\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}, \frac{X_1}{2}^* = 7\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}, \frac{X_2}{2}^* = 4\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$$

The spectral plot of the signal $x(t)$ is given in Fig. 1.

□

In the spectral plot, as depicted in Fig. 1, the spectral components of the signal are denoted by complex numbers which are represented by bars. When denoting these complex numbers in Polar notation it becomes very easy to 'read from the spectral plot' the time domain representation of the

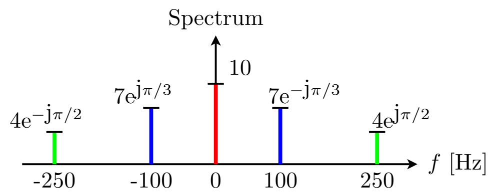


Figure 1: Spectral representation of a signal consisting of DC plus two sinusoidals

real signal $x(t)$: As an example we see from Figure 1 that the spectrum is symmetric and consists of a DC term, which equals 10, and two sinusoidal frequencies. One of the frequencies is at 100 [Hz] and has an amplitude of $2 \times 7 = 14$ and a phase of $-\frac{\pi}{3}$. The other frequency is at 250 [Hz] and has an amplitude of $2 \times 4 = 8$ and a phase of $\frac{\pi}{2}$. All together the spectral plot of Fig. 1 belongs to the following time-domain representation of signal $x(t)$:

$$x(t) = 10 + 14\cos(2\pi 100t - \frac{\pi}{3}) + 8\cos(2\pi 250t + \frac{\pi}{2}).$$

So we can conclude:

> *It is easiest to denote the complex numbers of the bars in the spectral plot in Polar notation.*

1.3 Product vs sum of frequencies

In the previous section we have seen that a spectral plot represents the spectral content of a signal. When a signal consists of a sum of sinusoidals we have seen which spectral components belongs to which sinusoid. But what is the frequency content of a signal which consists of a multiplication of sinusoidals? In practice this question becomes important when we apply "Amplitude Modulation" (AM) to transmit a baseband signal, with relative low frequency content, over a long distance. In such a case the baseband signal is multiplied with a carrier signal which has a relative high frequency f_c [Hz].

So lets start our discussion by assuming that a signal $x(t)$ consists of the multiplication of two sinusoids, one with frequency f_c and the other with frequency f_Δ (typically with $f_c \gg f_\Delta$), as follows:

$$x(t) = 2\cos(2\pi f_c t) \cdot \cos(2\pi f_\Delta t) \tag{3}$$

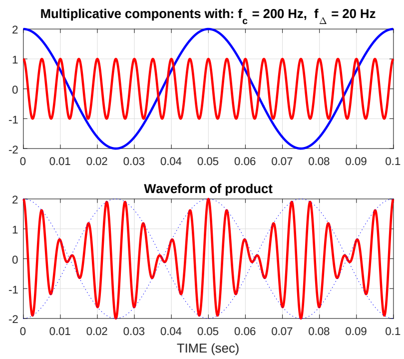


Figure 2: Time domain plot of a product of two sinusoidal signals

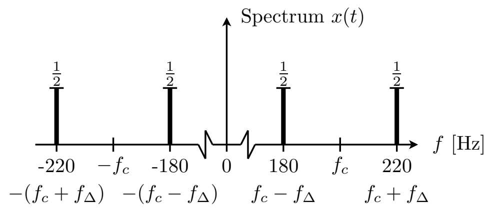


Figure 3: Spectral plot of a product of two sinusoidal signals

By using Euler we can rewrite this equation as follows:

$$\begin{aligned} x(t) &= 2\left(\frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi f_c t} + \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi f_c t}\right) \cdot \left(\frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi f_\Delta t} + \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi f_\Delta t}\right) \\ &= \frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi(f_c+f_\Delta)t} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi(f_c-f_\Delta)t} + \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi(f_c-f_\Delta)t} + \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi(f_c+f_\Delta)t} \\ &= \cos(f_c - f_\Delta) + \cos(f_c + f_\Delta) \end{aligned} \tag{4}$$

From this expression we can conclude that a product of two sinusoidal signals with frequencies f_c and f_Δ can be written as the sum of two sinusoidal signals with frequencies $f_c - f_\Delta$ and $f_c + f_\Delta$. Fig. 2 shows an example of the time domain waveforms of the individual components, with frequencies $f_c = 200$ and $f_\Delta = 20$ [Hz], and the product signal $x(t)$. The spectral components of this product are: $f_c - f_\Delta = 180$ [Hz] and $f_c + f_\Delta = 220$ [Hz], which are shown in the spectral plot of Fig. 3. In general we can conclude:

> *We can write the* **product** *of two sinusoidal signals with frequencies f_c and f_Δ as the* **sum** *of two sinusoidal signals with frequencies $f_c - f_\Delta$*

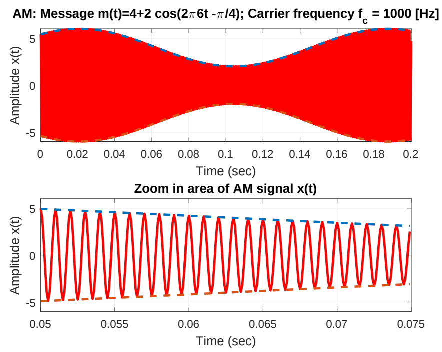


Figure 4: Waveform of AM signal $x(t) = c(t) \cdot m(t)$

and $f_c + f_\Delta$.

One of the examples where we use this concept is when tuning a guitar or piano. With the previous example in mind: Assume a guitar string produces a, wrongly tuned, 180 [Hz] signal and we want it to be tuned at 220 [Hz]. Then by playing an external signal generator, which produces a 220 [Hz] pure tone, while playing the, wrongly tuned, 180 [Hz] guitar string, we will not hear a clear sound. The waveform of the sound (as a product) is shown in Fig. 3: The amplitude of the sound is not constant but it changes. This effect is called "beat note". Now we can tune the string by tightening it until the amplitude does not change any more and we will hear one pure tone of 220 [Hz]. At that point our guitar string is tuned to 220 [Hz].

Another application of the previous concept is when transmiting a baseband signal, with relative low frequency content, over a long distance. An example of such a baseband signal is audio, e.g. speech or music, which has frequencies up to 20 [kHz]. It is impossible to transmit such an audio signal over a long distance in the air. In such a case we can apply "Amplitude modulation": The baseband signal is multiplied with a carrier signal which has a relative high frequency f_c [Hz].

Example:

Assume the baseband (message) signal $m(t) = 4 + 2\cos(2\pi 6t - \frac{\pi}{4})$ is multiplied (modulated) with the carrier signal $c(t) = \cos(2\pi 1000t)$ to produce the Amplitude Modulated (AM) signal $x(t) = c(t) \cdot m(t)$. The

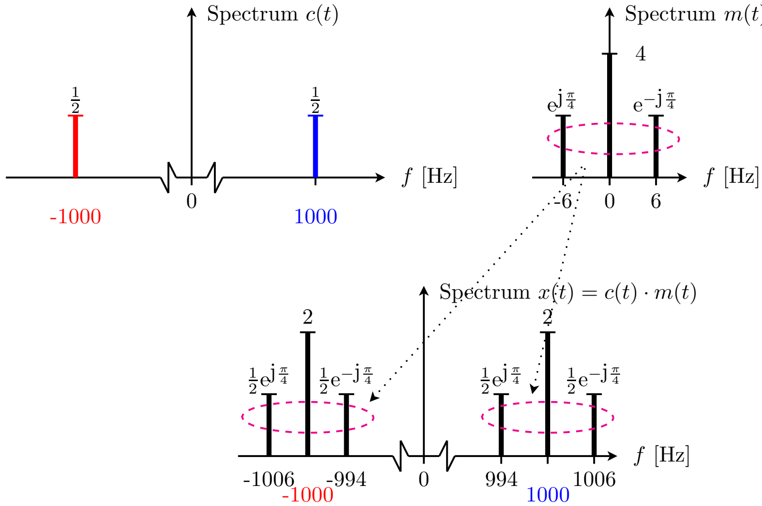


Figure 5: Spectral representation of $x(t) = c(t) \cdot m(t)$

time domain waveform of this AM signal is depicted in Fig. 4. Make the spectral plots of all signals.

<u>*Solution:*</u>

To investigate the spectral content of signal $x(t)$ we can apply Euler as follows:

$$
\begin{aligned}
x(t) &= \left(\frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi 1000t} + \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi 1000t}\right) \cdot \left(4 + \mathrm{e}^{\mathrm{j}\frac{\pi}{4}}\mathrm{e}^{-\mathrm{j}2\pi 6t} + \mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}\mathrm{e}^{\mathrm{j}2\pi 6t}\right) \\
&= \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}\mathrm{e}^{-\mathrm{j}2\pi(1000+6)t} + 2\mathrm{e}^{-\mathrm{j}2\pi 1000t} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}\mathrm{e}^{-\mathrm{j}2\pi(1000-6)t} \\
&\quad + \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}\mathrm{e}^{\mathrm{j}2\pi(1000-6)t} + 2\mathrm{e}^{\mathrm{j}2\pi 1000t} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}\mathrm{e}^{\mathrm{j}2\pi(1000+6)t}
\end{aligned}
$$

With this result we obtain the spectral plot of the signals $c(t)$, $m(t)$ and their product $x(t)$ as depicted in Fig. 5. From this spectral plot it follows that the whole spectral content of the message signal $m(t)$ has been moved ('modulated') by the carrier frequency $f_c = 1000$ [Hz] to both the positive and negative frequency component of the the carrier signal $c(t)$.

□

From this example we can conlude the following:

When a message signal $m(t)$ is modulated by a carrier signal $c(t)$ with frequency f_c, the whole spectral content of the message is moved to both the positive and negative frequency component of the the carrier signal.

2 Fourier series

In the previous section we studied the spectral behavior of the sum of sinusoidal signals. In this section we will study signals which are composed of a sum of harmonic related sine waves.

2.1 Periodic signals

In this subsection we will show that any periodic signal with a, so called, Fundamental period T_0 can be composed as the sum of sinusoidal signals from which each of these frequencies is related to the, so called, Fundamental frequency $F_0 = 1/T_0$ by an integer number.

In order to show this Fig. 6 depicts a signal $y_1(t)$ which is composed of the sum of two sinusoidal signals $x_1(t)$ of frequency $f_1 = 6$[Hz] and $x_2(t)$ of frequency $f_2 = 2$ [Hz]. From this figure it follows that $y_1(t)$ is periodic and we can measure that the period equals $T_0 = 1/F_0 = 0.5$ [sec]. The reason for this periodicity of signal $y_1(t)$ is that exactly **three** periods of $x_1(t)$ and **one** period of $x_2(t)$ fit into **one** period T_0 of signal $y_1(t)$. Thus after each $T_0 = 0.5$ [sec] the same composition of the signals $x_1(t)$ and $x_2(t)$ appears in the signal $y_1(t)$. The figure also shows another signal $y(t)$ which is composed of the sum three sinusoidal signals. The same sinusoidal signals $x_1(t), x_2(t)$ and a new sinusoidal signal $x_3(t)$, with frequency $f_3 = 1.2$ [Hz]. Again it follows from the figure that $y(t)$ is periodic but now we can measure that the period equals $T_0 = 1/F_0 = 2.5$ [sec]. This is because of the fact that exactly **fifteen** periods of $x_1(t)$, **five** periods of $x_2(t)$ and **three** periods of $x_3(t)$ fit into **one** period T_0 of signal $y(t)$. Mathematically we can calculate $F_0 = 1/T_0$ of $y(t)$ as the Greatest Common Divisor (gcd) of the frequencies from the individual sinusoidal components, since $F_0 = \text{gcd}\{f_1, f_2, f_3\} = \text{gcd}\{6, 2, 1.2\} = 0.4$ [Hz]. From this it follows that the the Fundamental period of $y(t)$ equals $T_0 = 1/F_0 = 2.5$ [sec]. This leads to the following general statement:

A real signal $x(t)$, which consists of DC plus a sum of N different frequencies $f_1, f_2, \cdots f_N$, with $f_1 < f_2 < \cdots < f_N$, is periodic with

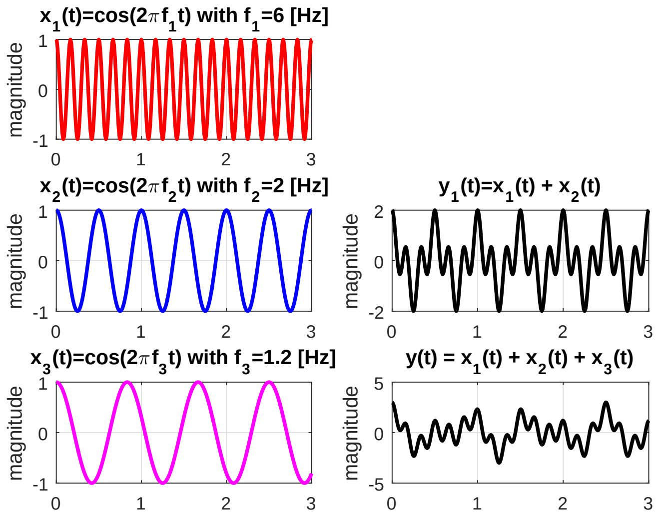


Figure 6: Sum of 2 and 3 harmonic related sinusoidal signals

> *Fundamental period* $T_0 = 1/F_0$ *when it has a Fundamental frequency* $F_0 = gcd\{f_1, f_2, \cdots f_N\}$.

In case this general statement is true, all N different frequencies of signal $x(t)$ are related by an integer number times the Fundamental frequency F_0. Thus we can define the largest frequency by $f_N = M \cdot F_0$ with integer $M \geq N$. With this definition we can write such a periodic signal $x(t)$ as a weighted sum of phasor components as follows:

$$x(t) = \sum_{k=-M}^{M} \alpha_k \mathrm{e}^{\mathrm{j}2\pi k F_0 t} \tag{5}$$

in which the complex weights α_k represent the amplitude and phase of the frequency components at frequency $k \cdot F_0$. Finally note that, eventually, α_0 and only N out of M complex weights α_k are not equal to zero.

Example:

Is the following signal periodic:

$$x(t) = 2 + 2\cos(102\pi t - \frac{\pi}{8}) - \sin(238\pi t + \frac{\pi}{3}) + 3\cos(340\pi t + \frac{\pi}{2})$$

If so calculate the Fundamental frequency $F_0 = 1/T_0$ and give the general expression for $x(t)$ as in equation (5) and make spectral plot of $x(t)$.

<u>*Solution:*</u>

By using Euler we can write $x(t)$ as follows:

$$\begin{aligned} x(t) &= \frac{3}{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\mathrm{e}^{-\mathrm{j}2\pi 170t} + \frac{1}{2\mathrm{j}}\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}\mathrm{e}^{-\mathrm{j}2\pi 119t} + \mathrm{e}^{\mathrm{j}\frac{\pi}{8}}\mathrm{e}^{-\mathrm{j}2\pi 51t} + 2 \\ &\quad + \mathrm{e}^{-\mathrm{j}\frac{\pi}{8}}\mathrm{e}^{\mathrm{j}2\pi 51t} + \frac{-1}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}\mathrm{e}^{\mathrm{j}2\pi 119t} + \frac{3}{2}\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\mathrm{e}^{\mathrm{j}2\pi 170t}. \end{aligned}$$

The signal consists of a sum of DC equal to 2 and three sinusoidal components with different frequencies $f_1 = 51$, $f_2 = 119$ and $f_3 = 170$ [Hz]. We can evaluate the Fundamental frequency as $F_0 =$gcd$\{51, 119, 170\}=$ 17 [Hz], thus signal $x(t)$ is a periodic signal with Fundamental period $T_0 = 1/F_0 = 1/17$ [sec]. The relation of the individual frequencies with F_0 is:

$$f_1 = 51[\text{Hz}] = \mathbf{3} \times F_0 \ ; \ f_2 = 119[\text{Hz}] = \mathbf{7} \times F_0 \ ; \ f_3 = 170[\text{Hz}] = \mathbf{10} \times F_0$$

Thus we can write $x(t)$ as:

$$x(t) = \sum_{k=-10}^{10} \alpha_k \mathrm{e}^{\mathrm{j}2\pi k F_0 t}$$

with $F_0 = 17$ [Hz] and all $\alpha_k = 0$ except for $\alpha_0 = 2$ and

$$\alpha_3 = \mathrm{e}^{-\mathrm{j}\frac{\pi}{8}} = \alpha^*_{-3} \ ; \ \alpha_7 = \frac{-1}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}\frac{\pi}{3}} = \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{5\pi}{6}} = \alpha^*_{-7} \ ; \ \alpha_{10} = \frac{3}{2}\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} = \alpha^*_{-10}$$

The spectral plot of $x(t)$ is depicted in Fig. 7. The horizontal axis denotes the absolute frequency f [Hz]. Alternatively we can denote on the horizontal axis the integer index number k of the summation above. In this case the inter frequency distance in between each step on the horizontal axis represents $F_0 = 17$ [Hz].

$\square$

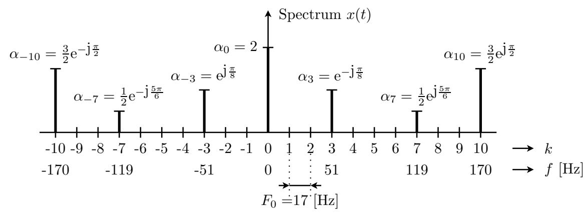


Figure 7: Spectral plot of periodic signal

2.2 Fourier series Synthesis

In the previous subsection we have seen that every periodic signal $x(t)$, with Fundamental period $T_0 = 1/F_0$, can be written as a sum of weighted harmonic related phasor components. In theory the summation can contain an infinite amount of components which leads to the following Fourier series expansion:

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k F_0 t}} \tag{6}$$

Thus when the spectral weights, which are represented by the complex numbers α_k, are known we can construct (synthesize) the periodic signal $x(t)$ according the above general expression (6).

2.3 Fourier series Analysis

In this subsection we will show the other way around: when we are given the waveform in time domain of a period signal $x(t)$, how can we derive the spectral weights α_k? In order to do so we need the following basic property of a phasor function:

> *The integral over one period $T_0 = 1/F_0$ of a phasor with a harmonic related frequency $k \cdot F_0$ results always in zero except for the case $k = 0$.*

Mathematically this can be shown as follows:

$$\int_0^{T_0} e^{j2\pi k F_0 t} dt = \frac{e^{j2\pi k F_0 t}}{j2\pi F_0 k}\Big|_{t=0}^{T_0} = \frac{e^{j2\pi F_0 k T_0} - 1}{j2\pi F_0 k} = \frac{e^{j2\pi k} - 1}{j2\pi F_0 k} = \frac{1-1}{j2\pi F_0 k} = 0$$

Furthermore if we evaluate this integral for $k = 0$ we obtain:

$$\int_0^{T_0} e^{j2\pi 0 F_0 t} dt = T_0$$

Thus when normalizing the integral by $1/T_0$, we obtain the following basic property of phasors:

$$\boxed{\frac{1}{T_0}\int_0^{T_0} \mathrm{e}^{\mathrm{j}2\pi k F_0 t}\mathrm{d}t = \left\{\begin{array}{ll} 1 & \text{for } k=0 \\ 0 & \text{for } k\neq 0 \end{array}\right.} \tag{7}$$

Since we are sure that a periodic signal $x(t)$, with Fundamental period $T_0 = 1/F_0$, only consists of harmonic related frequencies, as denoted in equation (6), we can use property (7) as follows to analyse **which harmonic related frequencies** are present in $x(t)$ and **what are the complex values of the spectral weights**. For this we evaluate the normalized integral over one period T_0 of $x(t)$ multiplied by a phasor with harmonic related frequency $l \cdot F_0$. Thus with both k and l integer we obtain:

$$\begin{aligned}
\frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}2\pi l F_0 t}\mathrm{d}t &\overset{\text{Use (6)}}{=} \frac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty}\alpha_k \mathrm{e}^{\mathrm{j}2\pi k F_0 t}\right)\mathrm{e}^{-\mathrm{j}2\pi l F_0 t}\mathrm{d}t \\
&= \sum_{k=-\infty}^{\infty}\alpha_k\left(\frac{1}{T_0}\int_0^{T_0}\mathrm{e}^{\mathrm{j}2\pi(k-l)F_0 t}\mathrm{d}t\right) \\
&\overset{\text{Use (7)}}{=} \alpha_l.
\end{aligned}$$

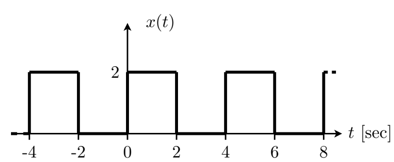
Thus when we are given a periodic signal $x(t)$ with period $T_0 = 1/F_0$ then we can find the spectral components α_k by using the following Fourier series analysis equation:

$$\boxed{\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}2\pi k F_0 t}\mathrm{d}t} \tag{8}$$

In many practice situations we approximate a periodic signal $x(t)$ with only a limited amount of term as follows:

$$\hat{x}(t) = \sum_{k=-N}^{N}\alpha_k \mathrm{e}^{\mathrm{j}\frac{\pi}{2}kt}. \tag{9}$$

It is obvious that the approximation becomes better and better for larger N.



Example:

Given the periodic signal $x(t)$ as depicted in the figure. Evaluate the spectral weights α_k. Furthermore, show the approximated periodic signal $\hat{x}(t)$ when using only up to the first 5 terms.

<u>*Solution:*</u>

From the figure it follows that $x(t)$ is a periodic signal with period $T_0 = 1/F_0 = 4$ [sec]. In most cases it is convenient to evaluate first the DC component α_0 of the periodic signal. For this example this goes as follows:

$$\alpha_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{4}\left\{\int_0^2 2\mathrm{d}t + \int_2^4 0\mathrm{d}t\right\} = \frac{1}{4}\{4+0\} = 1$$

Furthermore with $\omega_0 = 2\pi F_0 = \frac{2\pi}{4} = \frac{\pi}{2}$, we obtain:

$$\begin{aligned}\alpha_k &= \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}\omega_0 kt}\mathrm{d}t = \frac{1}{4}\int_0^2 2\mathrm{e}^{-\mathrm{j}\omega_0 kt}\mathrm{d}t = \frac{1}{4}\cdot 2\cdot\frac{1}{-\mathrm{j}\omega_0 k}\cdot \mathrm{e}^{-\mathrm{j}\omega_0 kt}|_0^2 \\ &= \frac{\mathrm{j}}{2\omega_0 k}\left(\mathrm{e}^{-\mathrm{j}2\omega_0 k} - 1\right) = \frac{\mathrm{j}}{k\pi}\left(\mathrm{e}^{-\mathrm{j}\pi k} - 1\right) = \frac{(-1)^k - 1}{k\pi}\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} \\ &= \frac{1-(-1)^k}{k\pi}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\end{aligned}$$

Thus except for $k = 0$ all Fourier coefficients with even index k are equal to zero. The Fourier coefficients with odd index are:

$$\alpha_1 = \alpha_{-1}^* = \frac{2}{\pi}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\ ;\ \alpha_3 = \alpha_{-3}^* = \frac{2}{3\pi}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\ ;\ \alpha_5 = \alpha_{-5}^* = \frac{2}{5\pi}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}\ \text{etc.}$$

When approximating this periodic signal with the first N terms we obtain:

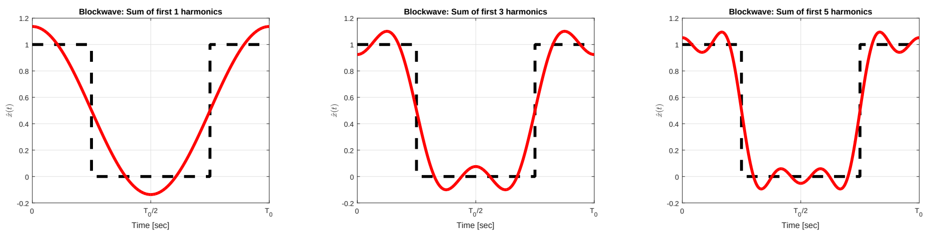
$$\begin{aligned}\hat{x}(t) &= \alpha_0 + \sum_{k=1}^{N}\left(\alpha_k \mathrm{e}^{\mathrm{j}k\frac{\pi}{2}t} + \alpha_{-k}\mathrm{e}^{-\mathrm{j}k\frac{\pi}{2}t}\right) \\ &= 1 + \frac{4}{\pi}\sum_{\substack{k=1\\ k\ \text{odd}}}^{N}\cos(k\frac{\pi}{2}t - \frac{\pi}{2}) \\ &= 1 + \frac{4}{\pi}\sum_{\substack{k=1\\ k\ \text{odd}}}^{N}\sin(k\frac{\pi}{2}t)\end{aligned}$$

This for example for $N = 1$ we have:

$$\hat{x}(t) = \alpha_{-1}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}t} + \alpha_0 + \alpha_1\mathrm{e}^{\mathrm{j}\frac{\pi}{2}t} = 1 + \frac{4}{\pi}\cos(\frac{\pi}{2}t - \frac{\pi}{2}) = 1 + \frac{4}{\pi}\sin(\frac{\pi}{2}t)$$

The figure depicts the results for $N = 1, 3$ and 5.

□



3 Summary spectrum and Fourier series

- Sum of DC and N sinusoids written as sum of phasors:

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k) \\ &= X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot f_k t} + \frac{X_k^*}{2} \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot f_k t} \right\} \\ &\quad \text{with } X_0 = A_0 \text{ and } X_k = A_k \mathrm{e}^{\mathrm{j}\phi_k} \end{aligned}$$

> *For* **real** *signals the values of the complex weights of the phasors with negative frequencies are the complex conjugated versions of the complex weights of the phasors with positive frequencies.*

- By ordering the frequencies from low to high we can represent the phasor description in a frequency spectrum plot. In such a plot we denote the frequencies on the horizontal axis. The frequency on this axis can either be denoted by the values of f_k in [Hz] or by the values of $\omega_k = 2\pi f_k$ in [rad/sec]. For each of these frequencies f_k we plot a bar, denoting the complex weights $\frac{X_k}{2}$. These complex weights are related to the magnitude and phase of the original sinusoidal signal.

> *It is easiest to denote the complex numbers of the bars in the spectral plot in Polar notation.*

- A product of sinusoidals is related to a sum of sinusoidals:

> *We can write the* **product** *of two sinusoidal signals with frequencies f_c and f_Δ as the* **sum** *of two sinusoidal signals with frequencies $f_c - f_\Delta$ and $f_c + f_\Delta$.*

- An Amplitude Modulated (AM) signal has the following property:

> *When a message signal $m(t)$ is modulated by a carrier signal $c(t)$ with frequency f_c, the whole spectral content of the message is moved to both the positive and negative frequency component of the the carrier signal.*

- A periodic signal has the following property:

> *A real signal $x(t)$, which consists of DC plus a sum of N different frequencies $f_1, f_2, \cdots f_N$, with $f_1 < f_2 < \cdots < f_N$, is periodic with Fundamental period $T_0 = 1/F_0$ when it has a Fundamental frequency $F_0 = gcd\{f_1, f_2, \cdots f_N\}$.*

- The integral over one period $T_0 = 1/F_0$ of a phasor with a harmonic related frequency $k \cdot F_0$ results always in zero except for the case $k = 0$. Mathematically this writes as follows:

$$\boxed{\frac{1}{T_0}\int_0^{T_0} \mathrm{e}^{\mathrm{j}2\pi k F_0 t}\mathrm{d}t = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}} \tag{10}$$

- Fourier analysis and synthesis equations for periodic signal $x(t) = x(t + T_0)$ with $T_0 = 1/F_0$:

$$\boxed{\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}2\pi F_0 k t}\mathrm{d}t \quad \circ\!\!-\!\!\circ \quad x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}2\pi F_0 k t}}$$