Homework exercises

Module FREQ: Frequency Response FIR filter

Course: Signals Processing Basics (5ESE0)

Notes:

- Only the answers are available for students. No pdf document with complete workout available for students.
- During the contact hours complete workout of exercises can be explained on request.
- The symbol [P] in the margin of an exercise denotes there is a pencast available.

Exercise 1
An LTI system is described by the following DE:

$$y[n] = x[n] - 3x[n-1] + 9x[n-2] - 3x[n-3] + x[n-4].$$

Evaluate the frequency response $H(e^{j\theta})$ of this system and write this frequency response in the general form:

$$H(e^{j\theta}) = |H(e^{j\theta})| \cdot e^{j\angle\{H(e^{j\theta})\}}$$

in which $|H(e^{j\theta})|$ is the magnitude and $\angle\{H(e^{j\theta})\}$ the phase characteristic.
Hint: In this case $|H(e^{j\theta})|$ can be written as a real valued function and $\angle\{H(e^{j\theta})\} = -K\theta$ with K an integer number.

Exercise 2
An LTI system is described by the following DE:

$$y[n] = -x[n] + 2x[n-1] - x[n-2].$$

a. Obtain an expression for the frequency response $H(e^{j\theta})$ of this system.

b. What is the output if the input is $x[n] = 5\cos(\frac{\pi}{2}n + \frac{\pi}{3})$?

c. What is the output if the input is the *unit impulse sequence*, thus $x[n] = \delta[n]$.

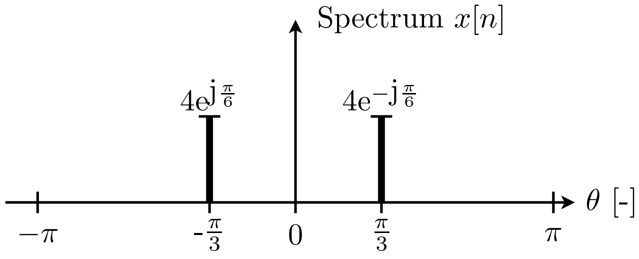
d. What is the output if the input is the *unit step sequence*, thus $x[n] = u[n]$.

Exercise 3
An FIR filter is characterized by the following frequency response:

$$H(e^{j\theta}) = e^{-j\theta}\left(1 + \cos(\theta)\right).$$

a. Give the signal flow graph (realization scheme) of this filter.

b. Suppose the input to this filter is a signal whose spectrum is shown in the figure. Determine the output $y[n]$ for $-\infty < n < \infty$.



Exercise 4
An LTI system is described by the following DE:

$$y[n] = -x[n] + x[n-1] - x[n-2].$$

Obtain an expression for the frequency response $H(e^{j\theta})$ of this system and make a sketch of the magnitude $|H(e^{j\theta})|$ and phase $\angle\{H(e^{j\theta})\}$ both in the Findamental Interval (FI) $|\theta| \leq \pi$.
Notes: $|H(e^{j\theta})|$ is a positive function and the phase $\angle\{H(e^{j\theta})\}$ has to be plotted in the range $\pi \leq \angle\{H(e^{j\theta})\} \leq \pi$

Exercise 5
A discrete-time system is described by the the following DE:

$$y[n] = 2x[n+2] + 6x[n] + 2x[n-2].$$

a. Obtain an expression for the frequency response $H(e^{j\theta})$ of this system.

b. Make a sketch of the magnitude $|H(e^{j\theta})|$ and phase $\angle\{H(e^{j\theta})\}$ both in the Fundamental Interval (FI) $|\theta| \leq \pi$.

c. Determine the output $y[n]$ when the input is $x[n] = 10 - 10\cos(\frac{\pi}{2}(n-1))$.
Hint: Use the frequency response and superposition to solve this problem.

Exercise 6
[P1] Consider the linear time-invariant system described by the difference equation

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] = \sum_{k=0}^{3} x[n-k]$$

a. Find an expression for the frequency response $H(e^{j\theta})$ of the system and show that it can be expressed in the form:

$$H(e^{j\theta}) = \frac{\sin(2\theta)}{\sin\left(\frac{\theta}{2}\right)} e^{-j\frac{3\theta}{2}}$$

b. Make a sketch of the magnitude $|H(e^{j\theta})|$ and phase $\angle\{H(e^{j\theta})\}$ both in the Fundamental Interval (FI) $|\theta| \leq \pi$.

c. Suppose that the input is $x[n] = 1 + 2\cos(n\theta_0)$ for $-\infty < n < \infty$. Find the non-zero frequency θ_0, with $0 < \theta_0 < \pi$, for which the output $y[n]$ is a constant for all n, i.e., $y[n] = c$, and find the value for c. In other words, determine the frequency for which the sinusoid is removed by the filter. Also, compute the output of the filter.

Exercise 7
In this exercise, you will design a relatively simple FIR filter based on a desired frequency response. All coefficients must be real-valued, and the design constraints are as follows:

$$|H(e^{j\theta})|_{\theta=0} = 1 \; ; \; |H(e^{j\theta})|_{\theta=\frac{\pi}{2}} = 0 \; ; \; |H(e^{j\theta})|_{\theta=\pi} = 1$$

a. Which type of filter meets these constraints from the following types: low-pass, high-pass, band-pass, band-stop?

b. The function $|H(e^{j\theta})| = |\cos(2\theta)|$ meets the requirements. Derive a phase response $\angle\{H(e^{j\theta})\}$ such that the filter is *causal*. Give an expression for the impulse response $h[n]$ of this filter and draw a signal flow graph (realization scheme) of the filter with the minimal number of delays.

Exercise 8
Suppose that three systems S_1, S_2 and S_3 are cascaded. In other words, the output of S_1 is the input to S_2, and the output of S_2 is the input to S_3. Furthermore the output of system S_i is $y_i[n]$ and the input is $x_i[n]$. The three systems are specified as follows:

$$\begin{aligned} S_1 &: \; y_1[n] = x_1[n] + x_1[n-2] \\ S_2 &: \; y_2[n] = 7x_2[n-5] + 7x_2[n-6] \\ S_3 &: \; H_3(e^{j\theta}) = e^{-j\theta} - e^{-j2\theta} \end{aligned}$$

The objective in this problem is to determine the equivalent system that is a single operation from the input $x[n]$ (into S_1) to the output $y[n]$ which is the output of S_3. Thus $x[n] = x_1[n]$ and $y[n] = y_3[n]$.

a. Determine the difference equation and impulse response $h_3[n]$ for S_3.

b. Determine the frequency response of the first two systems: $H_i(e^{j\theta})$ for $i = 1, 2$.

c. Determine the frequency response $H(e^{j\theta})$ of the overall cascaded system.

d. Determine the difference equation and impulse response of the overall system.

Exercise 9

[P2] The frequency response of a linear time-invariant filter is given by the formula:

$$H(e^{j\theta}) = \left(1 + e^{-j\theta}\right) \cdot \left(1 - e^{-j\frac{\pi}{3}} e^{-j\theta}\right) \cdot \left(1 - e^{j\frac{\pi}{3}} e^{-j\theta}\right)$$

a. Determine the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$ and impulse response $h[n]$ of this system.

b. If the input is of the form $x[n] = Ae^{j(\theta n + \phi)}$, for what values of the relative frequency θ, with $-\pi \leq \theta \leq \pi$, will $y[n] = 0$ for all n?
Hint: In this part, the answer is most obvious in the given factored form of the frequency response.

c. Use superposition to determine the output of this system when the input is

$$x[n] = 3 + \delta[n-2] + \cos(\frac{\pi}{2}n + \frac{\pi}{4}) \qquad \text{for } -\infty < n < \infty.$$

Hint: Because of the fact that the filter is LTI you may use superposition and split the input signal $x[n]$ into three different parts and find the outputs separately each by the easiest method and then add the results.

Exercise 10

Consider a cascade system containing two FIR filters. System S_1 is characterized by its impulse response $h_1[n] = \delta[n] - \alpha\delta[n-1]$, in which α is some number. Its output is the input to system S_2, which has an impulse response $h_2[n] = \sum_{k=0}^{5} \alpha^k \delta[n-k]$.
In this exercise, you will have to investigate the combination of S_1 and S_2. The input signal $x[n]$ is the input to S_1, the output signal $y[n]$ is the output of S_2.

a. Determine $H_1(e^{j\theta})$, the frequency response of the first system S_1.

b. Show that the frequency response of the second system S_2 equals

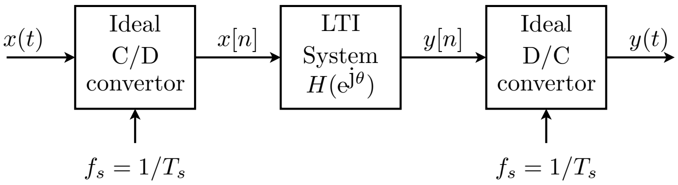
$$H_2(e^{j\theta}) = \frac{1 - \alpha^6 e^{-j6\theta}}{1 - \alpha e^{-j\theta}}.$$

c. Calculate the frequency response $H(e^{j\theta})$ of the combined system and derive from this result the impulse response $h[n]$ of the combined system.

d. Show that the impulse response $h[n]$ of the combined system can also be found as the convolution of the impulse responses $h_1[n]$ and $h_2[n]$ of the individual sub systems. Thus show tha $h[n]$ of the previous sub exercise equals $h[n] = h_1[n] \star h_2[n]$.

Exercise 11
The input to the C-to-D converter shown in the figure is

$$x(t) = 10 + 4\cos\left(4000\pi t - \frac{\pi}{8}\right) + 6\cos\left(14000\pi t - \frac{\pi}{3}\right).$$



The system function of the LTI system is:

$$H(\mathrm{e}^{\mathrm{j}\theta}) = 1 + \mathrm{e}^{-\mathrm{j}2\theta}$$

For $f_s = 8000$ [Hz], determine an expression for $y(t)$, the output of the D-to-C converter.

Exercise 12
Consider again the DSP system of the figure of the previous exercise. The input signal $x(t)$ is given as

$$x(t) = 1 + \cos(400\pi t) + \cos(600\pi t).$$

Your task is to design a causal discrete-time FIR filter such that the output signal $y(t)$ is given as

$$y(t) = A\cos(400\pi t + \varphi)$$

in which the nonzero amplitude A and phase φ are variable.

a. Suppose the sampling frequency $f_s = 800$ [Hz]. Give a rough sketch of the magnitude response $|H(\mathrm{e}^{\mathrm{j}\theta})|$ of an FIR filter that you would choose to obtain the desired output $y(t)$.

b. Now, both the sampling frequency f_s and the FIR filter are design parameters. You have the following options:

- The sampling frequency of the C-to-D and D-to-C converters f_s can be selected from

$$f_{s,a} = 300 \text{ [Hz]}, \quad f_{s,b} = 500 \text{ [Hz]}, \quad f_{s,c} = 600 \text{ [Hz]}.$$

- For the FIR filter $H(e^{j\theta})$, you can choose one of the following:

$$\left|H_1(\mathrm{e}^{\mathrm{j}\theta})\right| = |1 - \cos(\theta)|, \quad H_2(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{e}^{-\mathrm{j}\theta}(1 - \sin(2\theta)), \quad h_3[n] = \begin{cases} 1 & n = 1, 3 \\ 2 & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Which sampling frequency and which filter would you use to obtain the correct output signal $y(t)$? Explain your choices.