Sampling and Aliasing

P. Sommen and B. van Erp

September 27, 2017

Contents

1 Sampling process of C-to-D convertor

In many practical situations signals are available as a continuous-time (or analog) signal $x(t)$. An example of such a continuous-time signal is the recording of an audio signal with a microphone. Manipulating such an continuous-time signal is, in most cases, done in the discrete-time (or digital) domain. For example equalizing an audio signal can easiest be performed in the discrete-time domain. So it is important to understand the sampling process, which is performed by a Continuous- to Discrete-time (C-to-D) convertor which converts the continuous-time signal $x(t)$ to the discrete-time domain signal samples $x[n]$. The continuous-time character of a signal is indicated by using round brackets $(\cdot)$ around the (continuous) variable t, while the discrete-time character of a signal is indicated by using square brackets $[\cdot]$ around the (integer) variable n.

In order to simplify the mathematical description of the sampling process we will assume the following sinusoidal continuous-time signal:

$$x(t) = A\cos(\omega \cdot t + \phi) = A\cos(2\pi f \cdot t + \phi) \tag{1}$$

In this equation ω represents the absolute frequency in [rad/sec], or f in [Hz], while A denotes the amplitude and ϕ, in [rad], the phase of $x(t)$. The result of the sampling process, by an C-to-D convertor, is a set of numbers, denoted by $x[n]$. In case of a sinusoidal signal as defined in (1), these numbers $x[n]$ can be obtained by evaluating $x(t)$ at regular time instances $n \cdot T_s$, in which the variable n is an integer and T_s is the inter sample time, typically denoted in seconds [sec]. The sampling frequency of the C-to-D convertor equals $f_s = 1/T_s$, typically denoted in samples per second [samples/sec] or Hertz [Hz]. Thus the relation between the discrete-time signal samples $x[n]$ and the continuous-time signal $x(t)$ can be described as follows:

$$x[n] = x(t)|_{t=n\cdot T_s} = A\cos(\omega \cdot n \cdot T_s + \phi) = A\cos(2\pi f \cdot n \cdot \frac{1}{f_s}) \quad -\infty < n < \infty \tag{2}$$

Example:

> The left hand side of Fig. 1 shows two periods of the continuous-time signal $x(t) = \cos(400\pi t)$. This continuous-time signal is sampled with an ideal C-to-D convertor, which runs at a sampling frequency of $f_s = 1000$ [samples/sec]. Evaluate and plot the samples $x[n]$ is the discrete-time domain.

Solution:
The samples $x[n]$ can be evaluated as follows:

$$x[n] = \cos(400\pi t)|_{t=n/f_s} = \cos(0.4\pi n)$$

which results in the following sequence: $\cdots, 1, 0.3, -0.8, -0.8, 0.3, 1, \cdots$ as depicted at the right hand side of Fig. 1.

Usually we represent the set of numbers $x[n]$ in a plot as depicted at the right hand side of Fig. 2, where the horizontal axis denotes the integer index n and the vertical axis represents the sample value $x[n]$.

$\square$

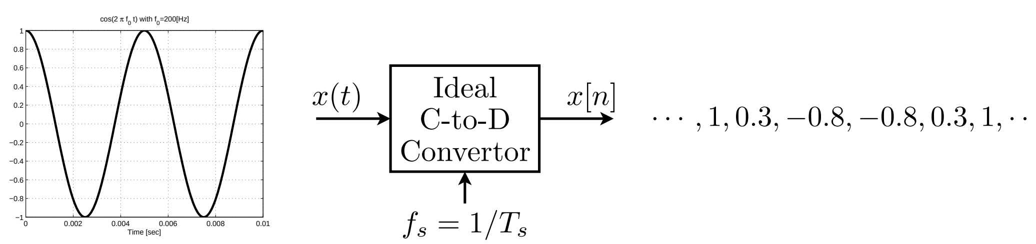

Figure 1: Output C-to-D convertor is a sequence of numbers (samples) $x[n]$.

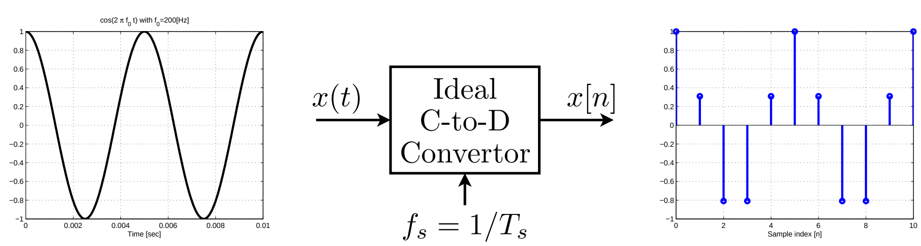

Figure 2: Example ideal C-to-D convertor: Output samples $x[n]$ represented in a plot.

2 Absolute vs relative frequency

When sampling the signal of equation (1) at a sampling frequency $f_s = 1/T_s$ we obtain equation (2), which can be rewritten as follows:

$$x[n] = A\cos(2\pi \frac{f}{f_s} n + \phi) = A\cos(\theta n + \phi) \tag{3}$$

In this equation we used the relative frequency symbol θ, which is a dimensionless quantity [-]. Thus the relation between relative- and absolute-frequency is:

$$\boxed{\theta = \omega \cdot T_s = 2\pi \frac{f}{f_s}} \tag{4}$$

and the frequency variables have the following dimensions:

Absolute frequency	:	ω [rad/sec] or f [Hz]
Relative frequency	:	θ [-]

For a continuous-time sinusoidal signal $x(t)$ we have seen before that a representation of the spectral content of such a signal can be found easily by using Euler, that is by rewriting $x(t)$ in terms of complex exponentials. The weights of these complex exponentials represent the spectral content of $x(t)$, as could have been obtained from the Fourier Series (FS) expansion. In a similar way we can use Euler to find a representation of the spectral content of a discrete-time sinusoidal signal $x[n]$.[1] In order to do so we rewrite equation (3) as follows:

$$x[n] = \left(\frac{A}{2}\mathrm{e}^{\mathrm{j}\phi}\right) \cdot \mathrm{e}^{\mathrm{j}\theta n} + \left(\frac{A}{2}\mathrm{e}^{-\mathrm{j}\phi}\right) \cdot \mathrm{e}^{-\mathrm{j}\theta n} \tag{5}$$

From this equation it follows that the spectral representation of the discrete-time sinusoidal signal $x[n]$ contains two bars: One with the complex value $\left(\frac{A}{2}\mathrm{e}^{\mathrm{j}\phi}\right)$ for the relative frequency $+\theta$ and another one with the complex value $\left(\frac{A}{2}\mathrm{e}^{-\mathrm{j}\phi}\right)$ for relative frequency $-\theta$.

[1]Such a description is only valid if the sequence of samples $x[n]$, as denoted in equation (3), is periodic. This is true if there exists an integer number N for which $x[n] = x[n+N]$, thus: $A\cos(2\pi\frac{f}{f_s}n + \phi) = A\cos(2\pi\frac{f}{f_s}(n+N) + \phi)$ or equivalently: $2\pi\frac{f}{f_s} \cdot N = \lambda \cdot 2\pi$ with integer λ. Thus $\frac{f}{f_s} = \frac{\lambda}{N}$, or in other words the fraction $\frac{f}{f_s}$ must be a rational number.

Example:

In this example we convert the 200 [Hz] continuous-time signal $x(t) = \cos(400\pi t)$ to the discrete-time domain signal samples $x[n]$ with an ideal C-to-D convertor which runs at a sample rate of $f_s = 1000$ [samples/sec]. Make a plot of the frequency content of both the continuous-time signal $x(t)$ and the discrete-time signal samples $x[n]$.
<u>*Solution:*</u>
By using the Euler equation we can write the input signal of the C-to-D convertor as follows:

$$x(t) = \cos(400\pi t) = \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi 200t} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi 200t} \tag{6}$$

The absolute frequency plot of $x(t)$ is depicted in the upper part of Fig.3. By using $\theta = 2\pi\frac{f}{f_s}$, the output signal samples of the C-to-D convertor can mathematically be described as:

$$x[n] = \cos(0.4\pi n) = \frac{1}{2}\mathrm{e}^{\mathrm{j}0.4\pi n} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}0.4\pi n} \tag{7}$$

In a similar way as with the absolute frequency plot, we can derive the relative frequency plot form this Euler equation, which results in the plot as depicted in the lower part of Fig.3.

□

3 Uniqueness issue discrete-time signal samples

In this section we will show that the C-to-D conversion will lead to an uniqueness issue in the discrete-time domain.

3.1 Uniqueness issue viewed in relative frequency domain

In this sub section we will show there is no unique spectral representation of the discrete-time signal samples $x[n]$. In order to show this let's start with the mathematical description of a sinusoidal signal as in equation (5). By using the fact that the number 1 can be written as a complex exponent:

$$1 = \left(\mathrm{e}^{\mathrm{j}2\pi}\right)^n = \left(\mathrm{e}^{-\mathrm{j}2\pi}\right)^n \qquad \text{with integer } n,$$

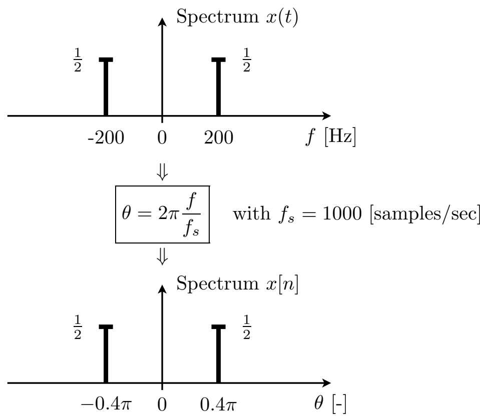

Figure 3: Frequency plot Absolute and Relative frequency of a 200 [Hz] signal sampled at $f_s = 1000$ [samples/sec].

we can change the mathematical representation of the sequence of samples $x[n]$ as follows:

$$\begin{aligned}
x[n] &= \left(\frac{A}{2}\mathrm{e}^{\mathrm{j}\phi}\right) \cdot \mathrm{e}^{\mathrm{j}\theta n} \cdot 1 + \left(\frac{A}{2}\mathrm{e}^{-\mathrm{j}\phi}\right) \cdot \mathrm{e}^{-\mathrm{j}\theta n} \cdot 1 \\
&= \left(\frac{A}{2}\mathrm{e}^{\mathrm{j}\phi}\right) \cdot \mathrm{e}^{\mathrm{j}\theta n} \cdot \mathrm{e}^{\mathrm{j}2\pi n} + \left(\frac{A}{2}\mathrm{e}^{-\mathrm{j}\phi}\right) \cdot \mathrm{e}^{-\mathrm{j}\theta n} \cdot \mathrm{e}^{-\mathrm{j}2\pi n} \\
&= \left(\frac{A}{2}\mathrm{e}^{\mathrm{j}\phi}\right) \cdot \mathrm{e}^{\mathrm{j}(\theta+2\pi)n} + \left(\frac{A}{2}\mathrm{e}^{-\mathrm{j}\phi}\right) \cdot \mathrm{e}^{-\mathrm{j}(\theta+2\pi)n} \qquad (8)
\end{aligned}$$

From this equation it follows that the spectral representation of $x[n]$ consists again of the same two bars, but now at the relative frequencies $\theta + 2\pi$ and $-(\theta + 2\pi)$. So we have found two different ways to represent the spectral information of the same sequence of samples $x[n]$. Moreover we can find an infinite amount of different spectral representations, just by adding or subtraction an integer times 2π to the relative frequency θ. Thus the same relative spectral content is repeated every 2π.

> *The spectral representation of a discrete-time signal $x[n]$ is not unique. The relative frequency representation is periodic with period 2π.*

Example:

Plot the spectal representation of the sequence of samples which are represented by $x[n] = 2\cos(0.4\pi n + \frac{\pi}{4})$ in the range $-3\pi < \theta < 3\pi$.
<u>*Solution:*</u>
By using Euler we can write $x[n]$ as follows:

$$x[n] = \mathrm{e}^{\mathrm{j}(0.4\pi n+\frac{\pi}{4})} + \mathrm{e}^{-\mathrm{j}(0.4\pi n+\frac{\pi}{4})} \ \left(= 2\cos(0.4\pi n + \frac{\pi}{4})\right)$$

So the spectral plot contains two bars: One at frequency 0.4π (in red) and one at -0.4π (in blue) as depicted in Fig. 4. When adding or subtracting 2π to both frequency components we obtain the following possibilities within the range $-3\pi < \theta < 3\pi$:
$0.4\pi + 2\pi = 2.4\pi$

$$\Rightarrow \ x[n] = \mathrm{e}^{\mathrm{j}(2.4\pi n+\frac{\pi}{4})} + \mathrm{e}^{-\mathrm{j}(2.4\pi n+\frac{\pi}{4})} \ \left(= 2\cos(2.4\pi n + \frac{\pi}{4})\right)$$

$0.4\pi - 2\pi = -1.6\pi$

$$\begin{aligned} \Rightarrow \ x[n] &= \mathrm{e}^{\mathrm{j}(-1.6\pi n+\frac{\pi}{4})} + \mathrm{e}^{-\mathrm{j}(-1.6\pi n+\frac{\pi}{4})} \\ &= \mathrm{e}^{\mathrm{j}(1.6\pi n-\frac{\pi}{4})} + \mathrm{e}^{-\mathrm{j}(1.6\pi n-\frac{\pi}{4})} \ \left(= 2\cos(1.6\pi n - \frac{\pi}{4})\right) \end{aligned}$$

Note that for this last case the phase has changed from sign.

All these possibilities are depicted in Fig. 4. From this Figure it follows that the spectral representation of $x[n]$ is indeed repeated outside the period $-\pi < \theta < \pi$.

□

3.2 Uniqueness issue viewed in discrete-time domain

In this subsection we will show by an example that different continuous-time signals can convert via an C-to-D convertor into the same sequence of samples.
Example:

In this example we convert two different continuous-time signals $x_1(t) = \cos(400\pi t)$ and $x_2(t) = \cos(2400\pi t)$ with a C-to-D convertor, which runs

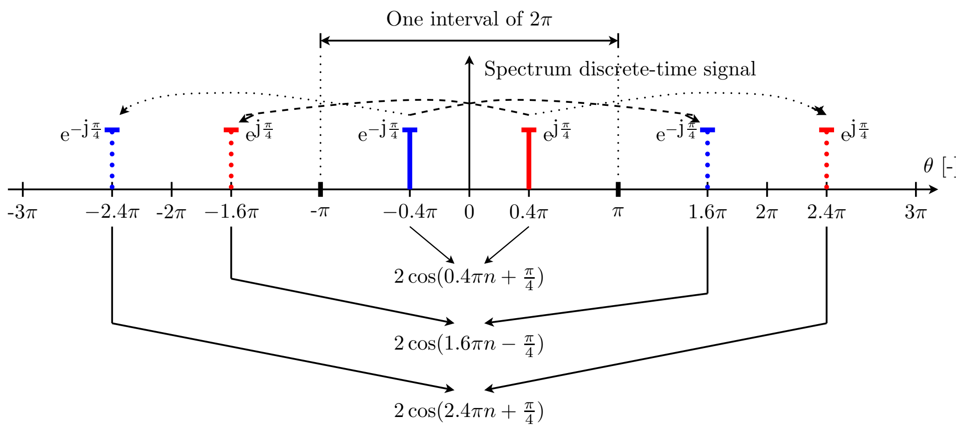

Figure 4: Uniqueness issue in frequency domain of discrete-time signal.

at a sample rate of $f_s = 1000$ [samples/sec], to the discrete-time domain. Calculate the sequence of samples $x_1[n]$ and $x_2[n]$ for two periods and make a plot which shows $x_1(t)$, $x_2(t)$ and the samples of $x_1[n]$ and $x_2[n]$.

<u>*Solution:*</u>

The converted signals are as follows:

$$\begin{aligned}
x_1[n] &= \cos(400\pi \cdot \frac{1}{1000}n) \\
&= \cos(0.4\pi n) = \cdots, 1, 0.3, -0.8, -0.8, 0.3, 1, \cdots \\
x_2[n] &= \cos(2400\pi \cdot \frac{1}{1000}n) \\
&= \cos(2.4\pi n) = \cdots, 1, 0.3, -0.8, -0.8, 0.3, 1, \cdots
\end{aligned}$$

Thus the samples which result from the C-to-D conversion of the two different continuous-time signals $x_1(t)$ and $x_2(t)$ result in the same sequence of samples $x_1[n] = x_2[n] = x[n] = \cdots, 1, 0.3, -0.8, -0.8, 0.3, 1, \cdots$. Fig. 5 shows a plot with two periods of the two different continuous-time $x_1(t)$ (in black) and $x_2(t)$ (in red) and the resulting discrete-time signal samples $x[n] = \cdots, 1, 0.3, -0.8, -0.8, 0.3, 1, \cdots$ (in blue).

$\square$

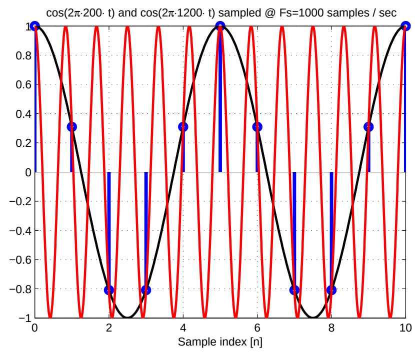

Figure 5: Two different continuous-time signals $x_1(t)$ (in black) and $x_2(t)$ (in red) result in the same sequence of samples $x[n]$ (in blue).

3.3 The Fundamental Interval (FI)

We have seen that the relative frequency representation of a discrete-time signal is periodic with period 2π. So in general only one interval of width 2π is of importance. In most cases we will use the interval of relative frequencies $-\pi < \theta \leq \pi$. This interval is called the Fundamental Interval (FI). Furthermore from the relation $\theta = 2\pi \frac{f}{f_s}$, as given in equation (4), the boundaries $\pm\pi$ of relative frequencies of the FI are related to the absolute frequencies $\pm\frac{f_s}{2}$.

The Fundamental Interval (FI) is the following range of frequencies:

$$\begin{array}{lcl} \textit{Relative frequencies} & : & -\pi < \theta \leq \pi \\ \textit{Absolute frequencies} & : & \frac{f_s}{2} < f \leq \frac{f_s}{2} \end{array}$$

Example:

In this example we convert the continuous-time signal $x(t) = x_1(t) + x_2(t)$, which consists of frequencies 200 [Hz] and 500 [Hz] with

$$x_1(t) = 4\cos(400\pi t + \frac{\pi}{4}) \text{ and } x_2(t) = 2\sin(1000\pi t - \frac{\pi}{3}),$$

to the discrete-time domain samples $x[n] = x_1[n] + x_2[n]$, by an ideal C-to-D convertor which runs at a sample rate of $f_s = 600$ [samples/sec].

Give a plot of the frequency representation of the continuous-time signal $x(t)$ and give also a plot of the frequency representation of the discrete-time signal $x[n]$ in the Fundamental Interval (FI).

Solution:

The spectral components of the continuous-time domain signal $x(t)$ are depicted in the upper part of Fig. 6. This figure also shows that frequencies smaller than 300 [Hz] map into the FI. So the result of the C-to-D conversion of $x_1(t)$ is as follows: $x_1(t) = 4\cos(400\pi t + \frac{\pi}{4}) \Rightarrow$

$$x_1[n] \quad = \quad 4\cos(\frac{2\pi}{3}n + \frac{\pi}{4}) = \left(2e^{j\frac{\pi}{4}}\right) e^{j\frac{2\pi}{3}n} + \left(2e^{-j\frac{\pi}{4}}\right) e^{-j\frac{2\pi}{3}n}$$

Furthermore the result of the C-to-D conversion of $x_2(t)$ is as follows: $x_2(t) = 2\sin(1000\pi t - \frac{\pi}{3}) \Rightarrow$

$$\begin{aligned} x_2[n] &= 2\sin(\frac{5\pi}{3}n - \frac{\pi}{3}) = \left(\frac{1}{j}e^{-j\frac{\pi}{3}}\right) e^{j\frac{5\pi}{3}n} + \left(\frac{-1}{j}e^{j\frac{\pi}{3}}\right) e^{-j\frac{5\pi}{3}n} \\ &= \left(\frac{1}{j}e^{-j\frac{\pi}{3}}\right) e^{-j\frac{\pi}{3}n} + \left(\frac{-1}{j}e^{j\frac{\pi}{3}}\right) e^{j\frac{\pi}{3}n} \\ &= -\left(\left(\frac{1}{j}e^{j\frac{\pi}{3}}\right) e^{j\frac{\pi}{3}n} + \left(\frac{-1}{j}e^{-j\frac{\pi}{3}}\right) e^{-j\frac{\pi}{3}n}\right) \\ &= -2\sin(\frac{\pi}{3}n + \frac{\pi}{3}) \end{aligned}$$

Thus the frequency component of $x_2(t)$ is converted outside the FI into relative frequency $\frac{5\pi}{3}$. When mapping this component back inside the FI it results in the relative frequency $\frac{\pi}{3}$ and the phase changes from sign. The lower part of Fig. 6 shows the spectral components. The components outside the FI, which are periodic components, are denoted in dashed lines.

□

4 Reconstruction of D-to-C convertor

In this section we will describe the reconstruction process of the D-to-C convertor which converts the sequence of discrete-time samples $x[n]$ into the continuous-time signal $x(t)$.

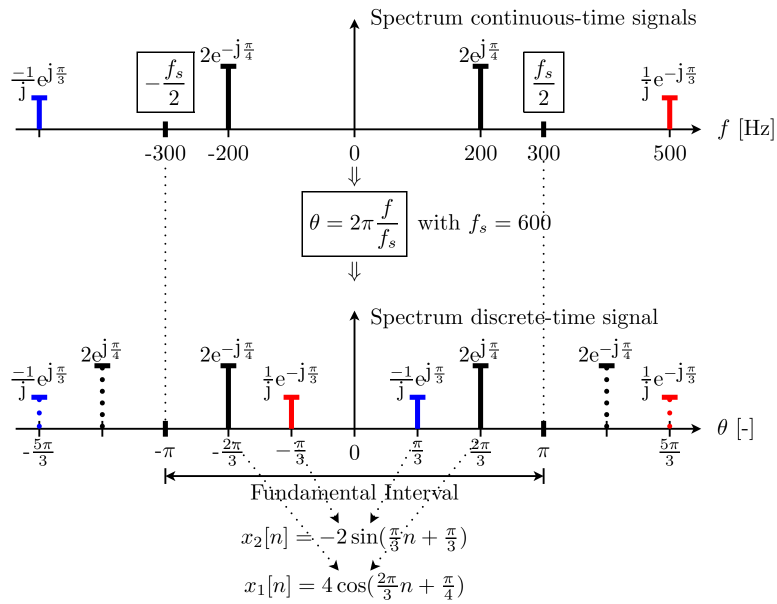

Figure 6: Result of uniqueness issue: Continuous-time signal frequencies which are converted outside the FI are mapped inside the FI.

4.1 Ideal D-to-C convertor

Because of the uniqueness issue in the discrete-time domain, the ideal D-to-C convertor makes a choice which frequencies are used for the conversion to the continuous-time domain. For this reason the ideal D-to-C convertor selects only frequencies which are inside the FI. Let's start the description

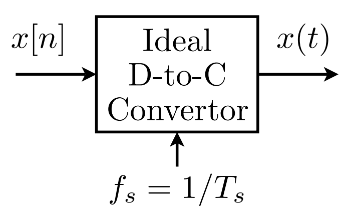

Figure 7: Ideal D-to-C convertor: Selects only frequencies in the FI

of the reconstruction from discrete-time to continuous-time by using again a simple sinusoidal signal from which the sequence of samples is described by equation (3), thus:

$$x[n] = A\cos(\theta n + \phi) \tag{9}$$

Because of the uniqueness issue we assume that the relative frequency θ is within the FI, thus $\theta < \pi$. In case this is not true, we first map the relative frequency into the FI. Furthermore we use the relation between relative- and absolute frequency as given in equation (4), $\theta = 2\pi \frac{f}{f_s}$, from which we assume as before that the ratio $\frac{f}{f_s}$ is a rational number. Finally, since we have available for this simple example the mathematical description of the signal samples as a sinusoidal function, we can obtain the mathematical description of the continuous-time signal $x(t)$ by replacing the integer index n by the continuous-time variable $t \cdot f_s$. Thus:

$$x(t) = x[n]|_{n=t\cdot f_s} = A\cos(2\pi \frac{f}{f_s} \cdot f_s \cdot t + \phi) = A\cos(2\pi f t + \phi)$$

Fig.8 shows an example where we converted the sequence of samples $x[n] = \cos(0.4\pi n) = \cdots, 1, 0.3, -0.8, -0.8, 0.3, 1, \cdots$ to the continuous-time signal $x(t)$, with an ideal D-to-C convertor, which runs at a sample rate of $f_s = 1000$ [samples/sec]. In Fig.9 the sequence of input signal samples $x[n]$ is represented in a plot.

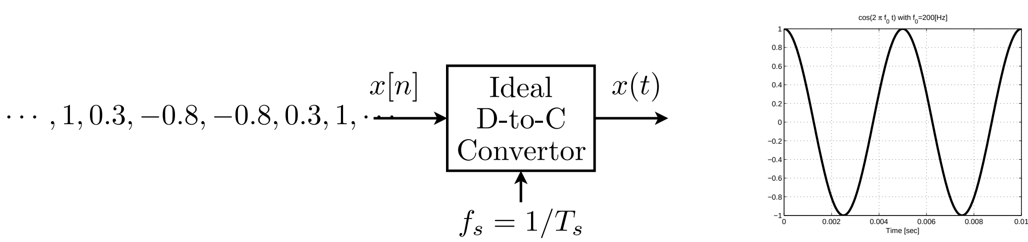

Figure 8: Input $x[n]$ ideal D-to-C convertor represented as sequence of numbers.

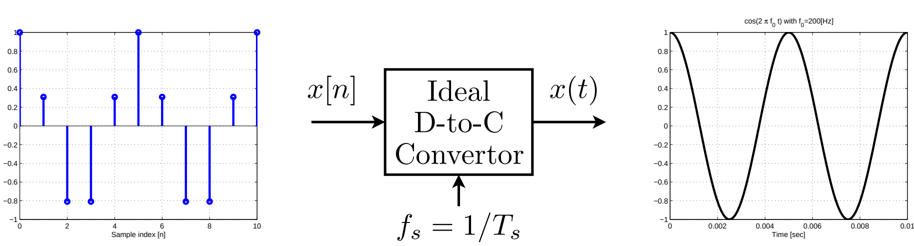

Figure 9: Input $x[n]$ ideal D-to-C convertor represented as a plot.

An ideal D-to-C convertor chooses the relative frequencies within the FI: $-\pi < \theta \leq \pi$. This implies that in case a continuous-time absolute frequency is converted with an C-to-D conversion outside the FI, such a frequency will always first be mapped inside the FI and the D-to-C convertor will use this frequency for the conversion of the discrete-time signal into a continuous-time signal. This mapping back into the FI is called **Aliasing**.

Example:

In this example we will study a system as depicted in Fig. 10. The continuous-time input signal $x(t)$ is converted to the discrete-time domain samples $x[n]$ with an ideal C-to-D convertor which runs at a sample rate of $f_{s,in}$ [samples/sec]. After that the samples $x[n]$ are converted back to the continuous-time signal $y(t)$ with an ideal D-to-C convertor which runs at a sample rate of $f_{s,out}$ [samples/sec]. The input signal $x(t)$ and the sample rate $f_{s,in}$, are the same as in the last example of the previous section. Thus: $x(t) = x_1(t) + x_2(t)$, with $x_1(t) = 4\cos(400\pi t + \frac{\pi}{4})$, $x_2(t) = 2\sin(1000\pi t - \frac{\pi}{3})$ and $f_{s,in} = 600$ [samples/sec]. In first instance we choose the sample rate of the D-to-C convertor equal to the sample rate of the C-to-D convertor, thus $f_{s,out} = f_{s,in} = 600$ [samples/sec]. After that we change the sample rate of the D-to-C convertor to $f_{s,out} = 900$ [samples/sec]. Calculate an expression for the output signal $y(t)$ in both cases.
Solution:
The first steps are the same of the last example of the previous subsection. The largest absolute frequency of the FI after the C-to-D convertor equals $f_{s,in}/2 = 300$ [Hz]. Thus the frequency of $x_1(t)$ (200 [Hz]) is converted inside the FI, at relative frequency $\frac{2\pi}{3}$, while the frequency of $x_2(t)$ (500 [Hz]) is converted outside the FI, at relative frequency $\frac{5\pi}{3}$. After mapping back this frequency inside the FI, into relative frequency $\frac{\pi}{3}$, we obtain the following mathematical description of the sequence of discrete-time samples $x[n]$:

$$x[n] = 4\cos(\frac{2\pi}{3}n + \frac{\pi}{4}) - 2\sin(\frac{\pi}{3}n + \frac{\pi}{3})$$

When converting the relative frequencies of this discrete-time signal into absolute frequencies of a continuous-time signal, we can make use of equation (4). In case the sample rate of the D-to-C convertor $f_{s,out} = f_{s,in} = 600$ [samples/sec] this leads to the following continuous-time output signal:

$$y(t) = 4\cos(400\pi t + \frac{\pi}{4}) - 2\sin(200\pi t + \frac{\pi}{3}).$$

Thus the frequency of the first 200 [Hz] component did not change while the frequency of the second 500 [Hz] component has changed due to aliasing. However if we change the sample rate of the D-to-C convertor into $f_{s,out} = 900$ [samples/sec] the resulting continuous-time signal becomes:

$$y(t) = 4\cos(600\pi t + \frac{\pi}{4}) - 2\sin(300\pi t + \frac{\pi}{3}).$$

Thus both frequencies are changed. The frequency of $x_2(t)$ has changed because of aliasing while above that both frequencies of $x_1(t)$ and $x_2(t)$ have changed because of the fact that the sample rates of C-to-D and D-to-C are different.

□

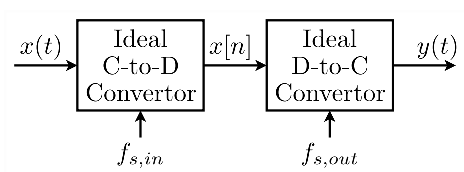

Figure 10: Example of C-to-D and D-to-C converter with different sample rates

4.2 Reconstruction D-to-C convertor based on samples $x[n]$

The D-to-C convertor constructs a continuous-time signal $x(t)$, based on the discrete-time samples $x[n]$. In the previous subsection we have seen that for sampled sinusoidal signals the D-to-C convertor in effect replaces the discrete-time integer variable n by the continuous-time variable $t \cdot f_s$. As a result we can construct, for such a case, the continuous-time signal $x(t)$ based on the mathematical description of the sinusoidal function. This procedure is only valid and possible in case of sampled sinusoidal signals and an example is shown at the left-hand side plot of Fig.11. The continuous-time sinusoidal signal $x(t)$ (in red) is constructed by replacing n into $t \cdot f_s$ in the sinusoidal description of the samples $x[n]$.

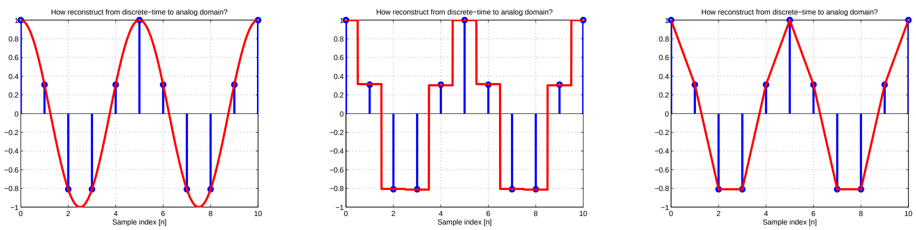

Figure 11: Reconstruction process D-to-C convertor based on samples $x[n]$

> *Only in case of sampled sinusoidal signals the ideal D-to-C convertor in effect replaces n by $t \cdot f_s$.*

In the more general case the continuous-time output signal $y(t)$ is constructed by placing a continuous-time function $p(t)$ at each new position of a sample $x[n]$ and adding together all the resulting continuous-time signals. This reconstruction is performed via the so called

$$\boxed{\textit{Interpolation equation:} \quad y(t) = \sum_{n=-\infty}^{\infty} x[n]p(t - n \cdot T_s)} \qquad (10)$$

in which the continuous-time function $p(t)$ represents the characteristic of the D-to-C convertor. In practice there are many different characteristic forms $p(t)$, e.g. a square pulse:

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{elsewhere} \end{cases}$$

which leads to the so called 'zero order hold' procedure, which is depicted in the middle-part of Fig.11. In this procedure the electronics of the D-to-C convertor are build in such a way that around each new discrete-time sample a continuous-time signal is constructed which has the same value as the discrete-time sample, while this value is kept constant during one sampling period of T_s [sec]. Thus the reconstructed continuous-time signal looks like a 'stair-case' approximation of the sinusoidal signal as depicted (in red) in the left-hand side figure. The electronics needed for this conversion procedure are very simple and cheap. For this reason the 'zero order hold' procedure is applied very often in practice.

Another characteristic form of $p(t)$ is a triangular pulse:

$$p(t) = \begin{cases} 1 - \frac{|t|}{T_s} & |t| \leq T_s \\ 0 & \text{elsewhere} \end{cases}$$

which leads to the conversion as shown in the right-hand side of Fig.11. The signal samples are interpolated in a linear way. The reconstructed continuous-time signal looks like a 'triangular-like' approximation of the sinusoidal signal as depicted (in red) in the left-hand side figure.

5 Sampling theorem

From the previous section it follows that for a system as depicted in Fig. 10, with $f_{s,in} = f_{s,out} = f_s$, the output $y(t)$ of the D-to-C convertor is equal to input $x(t)$ of the ideal C-to-D convertor if the continuous-time signal $x(t)$ contains no frequencies higher than f_{max} and the sampling frequency f_s is larger than $2 \cdot f_{max}$. This results into the following important **sampling theorem**[2]:

> *A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(t)|_{t=n \cdot T_s}$, if samples are taken at a rate $f_s = 1/T_s$, that is greater than $2f_{max}$*

[2]The sampling theorem is usualy assigned to Shannon, who introduced this theorem around 1940. Almost at the same time Kotelnikov did the same. However some years before the theoretical basis was created by E.T. and J.A. Whittaker.

6 Summary Sampling and Aliasing

Relation absolute- and relative- frequency: $\boxed{\theta = \omega \cdot T_s = 2\pi \frac{f}{f_s}}$

Absolute frequency	:	ω [rad/sec] or f [Hz]
Relative frequency	:	θ [-]

The spectral representation of a discrete-time signal $x[n]$ is not unique. The relative frequency representation is periodic with period 2π.

The Fundamental Interval (FI) is the following range of frequencies:

Relative frequencies	:	$-\pi < \theta \leq \pi$
Absolute frequencies	:	$\frac{f_s}{2} < f \leq \frac{f_s}{2}$

An ideal D-to-C convertor chooses the relative frequencies within the FI: $-\pi < \theta \leq \pi$. This implies that in case a continuous-time absolute frequency is converted with an C-to-D conversion outside the FI, such a frequency will always first be mapped inside the FI and the D-to-C convertor will use this frequency for the conversion of the discrete-time signal into a continuous-time signal. This mapping back into the FI is called **Aliasing**.

Only in case of sampled sinusoidal signals the ideal D-to-C convertor in effect replaces n by $t \cdot f_s$.

Interpolation equation: $y(t) = \sum_{n=-\infty}^{\infty} x[n]p(t - n \cdot T_s)$

Sampling theorem: *A continuous-time signal* $x(t)$ *with frequencies no higher than* f_{max} *can be reconstructed exactly from its samples* $x[n] = x(t)|_{t=n \cdot T_s}$, *if samples are taken at a rate* $f_s = 1/T_s$, *that is greater than* $2f_{max}$