Complex numbers and phasors

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1 The set of real numbers

In mathematics, sets are defined as collections of objects. Recalling from primary school, one particular set of numbers that was first introduced: The natural numbers. The set of natural numbers contains all positive integers (including 0) and is denoted as:

$$\mathbb{N} = \{0, 1, 2, ...\}. \tag{1}$$

However, this set contains only a small part of the numbers everyone is familiar to. Expanding this set with, in addition to all positive integers in $\mathbb{N}$, negative integers the set with integers numbers $\mathbb{Z}$ is created. This set with both positive and negative integer numbers is denoted as:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}. \tag{2}$$

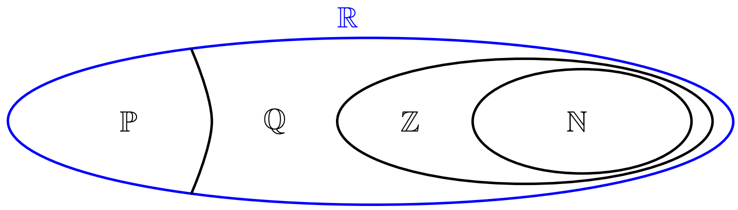
With the numbers in this set, many operations like addition, subtraction and multiplication can be performed. However, division of the numbers in this set might result in a number which is not an integer number (e.g. one divided by 2 results in $1/2 = 0.5$, where 0.5 is not an integer). To allow for division of all numbers in $\mathbb{Z}$, the set of all rational numbers $\mathbb{Q}$ is introduced, which in addition to all numbers in set $\mathbb{Z}$ also contains all fractions. This set is denoted as

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}. \tag{3}$$

This set, however, does not include numbers which cannot be written as a fraction, for example the numbers $\sqrt{2}$ and π. These numbers belong to the set of irrational numbers $\mathbb{P}$. Expanding the set of rational numbers $\mathbb{Q}$ with irrational numbers $\mathbb{P}$, the set of all real numbers $\mathbb{R}$ is created. The nice thing about the set of real numbers is that the numbers of this set can be represented graphically on a line of numbers. Furthermore the squared value of a real number x is always greater or equal than zero, thus $x^2 \geq 0$, when $x \in \mathbb{R}$. A visual representation of the discussed number sets is shown in Fig. 1.

2 The imaginary unit j= $\sqrt{-1}$

In engineering it is quite common to describe a practical problem by a mathematical model. For example a mathematical model that describes the movement $f(\theta)$, as a function of the position θ, of a harmonic oscillator is



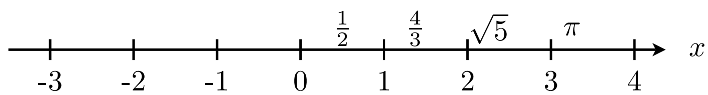
$\mathbb{R}$: Set of real numbers is extension of:

$\mathbb{N}$: Natural numbers, e.g. $0, 1, 2, \cdots$

$\mathbb{Z}$: Interger numbers, e.g. $-2, -1, 0, 1, 2, \cdots$

$\mathbb{Q}$: Rational numbers, e.g. $\frac{1}{2}, \frac{4}{5}, \cdots$

$\mathbb{P}$: Irrational numbers, e.g. $\pi, \sqrt{5}, \cdots$



$\Rightarrow$ $\mathbb{R}$ is set of numbers x for which $x^2 \geq 0$

Figure 1: Visual representations of the number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$

given by the following (simplified) second order differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\{f(\theta)\} + f(\theta) = 0 \tag{4}$$

A general solution of this equation is given by the function $f(\theta) = \mathrm{e}^{c\theta}$, in which $\theta \in \mathbb{R}$ is the function variable and $c \in \mathbb{R}$ is some unknown constant. Filling in this general solution in equation (4) results in the following:

$$c^2(\mathrm{e}^{c\cdot\theta}) + (\mathrm{e}^{c\cdot\theta}) = 0 \quad \Rightarrow \quad c^2 + 1 = 0 \quad \Rightarrow \quad c^2 = -1 \quad \Rightarrow \quad c = \pm\sqrt{-1}$$

Since the set of real numbers $\mathbb{R}$ does not include the value $\sqrt{-1}$ the imaginary unit j is introduced [1] , which is defined as:

$$\boxed{\mathrm{j} = \sqrt{-1}} \tag{5}$$

As a result we can write the general solution of equation (4) as the following complex exponential:

$$f(\theta) = \mathrm{e}^{\pm\mathrm{j}\theta} \quad \text{and all linear combinations} \tag{6}$$

[1] In literature the symbol i is often used. However this might create confusion in some fields, like in electrical engineering and automotive, where the symbol i is used for current.

Example:

Find the roots of the following equation: $x^2 - 2x + 4 = 0$
Solution:
We can use the standard quadratic rule for determining the two root(s) x_1 and x_2 of this quadratic equation:

$$x_{1,2} = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{-3}$$

Now by using the definition of the imaginary unit j=$\sqrt{-1}$ we can write $\sqrt{-3} = \sqrt{-1} \cdot \sqrt{3} = \mathrm{j} \cdot \sqrt{3}$ which results in the following two roots:

$$x_{1,2} = 1 \pm \mathrm{j} \cdot \sqrt{3}$$

Check:
We can verify if $x_1 = 1 + \mathrm{j} \cdot \sqrt{3}$ and $x_2 = 1 - \mathrm{j} \cdot \sqrt{3}$ are indeed roots of the original equation by filling in these values into the original equation. By doing so and using the fact that $\mathrm{j}^2 = (\sqrt{-1})^2 = -1$, this results into the following:

$$\begin{aligned}
x^2 - 2x + 4|_{x=x_1} &= (1 + \mathrm{j} \cdot \sqrt{3})^2 - 2(1 + \mathrm{j} \cdot \sqrt{3}) + 4 \\
&= 1 + 2\mathrm{j}\sqrt{3} + 3\mathrm{j}^2 - 2 - 2\mathrm{j}\sqrt{3} + 4 \\
&= 1 + 2\mathrm{j}\sqrt{3} - 3 - 2 - 2\mathrm{j}\sqrt{3} + 4 = 0 \\
\text{and} & \\
x^2 - 2x + 4|_{x=x_2} &= (1 - \mathrm{j} \cdot \sqrt{3})^2 - 2(1 - \mathrm{j} \cdot \sqrt{3}) + 4 \\
&= 1 - 2\mathrm{j}\sqrt{3} + 3\mathrm{j}^2 - 2 + 2\mathrm{j}\sqrt{3} + 4 \\
&= 1 - 2\mathrm{j}\sqrt{3} - 3 - 2 + 2\mathrm{j}\sqrt{3} + 4 = 0
\end{aligned}$$

Thus we can conclude that we can split the original equation into the following product:

$$x^2 - 2x + 4 = (x - x_1) \cdot (x - x_2) = \Big(x - (1 + \mathrm{j} \cdot \sqrt{3})\Big) \cdot \Big(x - (1 - \mathrm{j} \cdot \sqrt{3})\Big)$$

3 Relation complex exponential with real sinusoidal

From practical experience we know that the solution $f(\theta)$, which describes the movement of a harmonic oscillator, behaves as a real sinusoidal function. So the question is how does the general complex exponential solution (6) relate to, the expected, real sinusoidal solution. For this we use the following geometric Taylor expansion for an exponential function:

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \dots \tag{7}$$

When evaluating this function for $x = \mathrm{j}\theta$, we can write the complex exponential (6) as follows:

$$\mathrm{e}^{\mathrm{j}\theta} = 1 + \mathrm{j}\theta + \frac{(\mathrm{j}\theta)^2}{2!} + \frac{(\mathrm{j}\theta)^3}{3!} + \frac{(\mathrm{j}\theta)^4}{4!} + \frac{(\mathrm{j}\theta)^5}{5!} + \dots \tag{8}$$

From definition (5) it follows:

$$\mathrm{j} = \sqrt{-1} \;\;\Rightarrow\;\; \mathrm{j}^2 = -1 \;;\; \mathrm{j}^3 = -\mathrm{j} \;;\; \mathrm{j}^4 = +1 \;;\; \mathrm{j}^5 = \mathrm{j} \text{ etc.}$$

By using this we can simplify the previous expression as follows:

$$\mathrm{e}^{j\theta} = 1 + \mathrm{j}\theta - \frac{\theta^2}{2!} - \mathrm{j}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \mathrm{j}\frac{\theta^5}{5!} \dots \tag{9}$$

After reorganizing the terms this expression can be split in two parts:

$$e^{j\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots) + \mathrm{j}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots) \tag{10}$$

Again recall the following two Taylor expansions:

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots \quad \text{and} \quad \sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \tag{11}$$

and substituting this equation into (10) yields the following expression

$$\boxed{\mathrm{e}^{j\theta} = \cos(\theta) + \mathrm{j} \cdot \sin(\theta)} \tag{12}$$

which is the so called **Euler equation**. Now we can use Euler's equation to find alternative expressions for the sin and cos functions. For this we first evaluate equation (12) for $-\theta$, which results in

$$\mathrm{e}^{-\mathrm{j}\theta} = \cos(-\theta) + \mathrm{j}\sin(-\theta) = \cos(\theta) - \mathrm{j}\sin(\theta) \tag{13}$$

Now by adding or subtracting equations (12) and (13) we obtain the following two equations:

$$\begin{aligned} e^{j\theta} + e^{-j\theta} &= (\cos(\theta) + j\sin(\theta)) + (\cos(\theta) - j\sin(\theta)) = 2\cos(\theta) \\ e^{j\theta} - e^{-j\theta} &= (\cos(\theta) + j\sin(\theta)) - (\cos(\theta) - j\sin(\theta)) = 2j\sin(\theta) \end{aligned}$$

From this we obtain the following **alternative expressions for the** sin- **and** cos- **functions** as sums of complex exponentials:

$$\boxed{\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}} \quad \text{and} \quad \boxed{\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}} \tag{14}$$

As mentioned before, all linear combinations of the complex exponential are also valid solutions of the second order differential equation (4). As a result it follows from (14) that the sinusoidal functions sin and cos are the real solutions of the second order differential equation (4) and we can generalize these real solutions to a complex exponential solution (6).
Example:

Prove the following identity: $e^{j\pi} + 1 = 0$
Solution:
By using Euler's equation (12) for $\theta = \pi$, results in:

$$e^{j\pi} + 1 = \cos(\pi) + j\sin(\pi) + 1 = -1 + j\cdot 0 + 1 = 0$$

4 The set of complex numbers $\mathbb{C}$

From the previous discussion it follows that in practice there is need to extend the set of real numbers to a set of complex numbers. In this section we will first introduce the set of complex numbers in two different ways and we will show how complex numbers can be visualized by vectors. Then we will introduce the concept of complex conjugation, which is important to obtain the definition of the length of a complex vector. Furthermore we will introduce calculation rules for complex numbers, which are defined in such a way that these rules are generalizations of the calculation rules for real numbers. Finally we show how the addition of two complex numbers can be visualized, which leads to the vectorial addition rule.

4.1 Polar and Cartesian representation

In the first approach we use the fact the complex exponential $z = r\mathrm{e}^{\mathrm{j}\theta}$, with both $\theta \in \mathbb{R}$ and $r \in \mathbb{R}$, is also a valid solution of the second order differential equation (4). This notation of a complex number is the so called polar notation. By using this the first definition of the set of complex numbers is as follows:

$$\textbf{Polar representation:} \quad \boxed{\mathbb{C} = \{z = r\mathrm{e}^{\mathrm{j}\theta} | r \in \mathbb{R} \text{ and } \theta \in \mathbb{R}\}} \tag{15}$$

In the second approach it follows from the Euler equation (12) that we can write the complex exponential as the sum of two independent terms $z = r\mathrm{e}^{\mathrm{j}\theta} = r\cos(\theta) + \mathrm{j}\sin(\theta)$. The first term is the so called real part, denoted by $\Re e$. The second term, which is the part after the unit j, is the so called imaginary part which is denoted by $\Im m$. Both real- and imaginary-part are real numbers and are given by the following expressions:

$$\Re e\{z\} = r\cos(\theta) \in \mathbb{R} \quad \text{and} \quad \Im m\{z\} = r\sin(\theta) \in \mathbb{R} \tag{16}$$

From this it follows that in general we can write a complex number as $z = x + \mathrm{j}y$, in which x represents the real part of the complex number z and y the imaginary part and both numbers x and y belong to the set of real numbers. With this in mind the second definition of the set of complex numbers $\mathbb{C}$ is as follows:

$$\textbf{Cartesian representation:} \quad \boxed{\mathbb{C} = \{z = x + \mathrm{j}y | x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}} \tag{17}$$

From these definitions it follows that a real number can be viewed as a complex number from which the imaginary part $\Im m$ is zero. Thus the set of complex numbers $\mathbb{C}$ is a generalization of the set of real numbers $\mathbb{R}$. In section 1 we have seen that real numbers can be visualized graphically by a one dimensional line with real numbers, as depicted in Fig. 1. From the Cartesian- or Polar-representation of complex numbers it follows that we need two independent axis to represent a complex number graphically. In Fig. 2 we visualized the complex number, both in Cartesian and Polar notation, as a complex **vector** z in a two dimensional plane. The horizontal axis represents the real part of the complex vector z while the projection on the vertical axis represents the imaginary part. Thus, when using the Cartesian representation $z = x + \mathrm{j}y$, the parameter x represents the real part $\Re e\{z\}$ and y represents the imaginary part $\Im m\{z\}$. On the other hand, when using the polar representation $z = r\mathrm{e}^{\mathrm{j}\theta}$, the parameter r represents

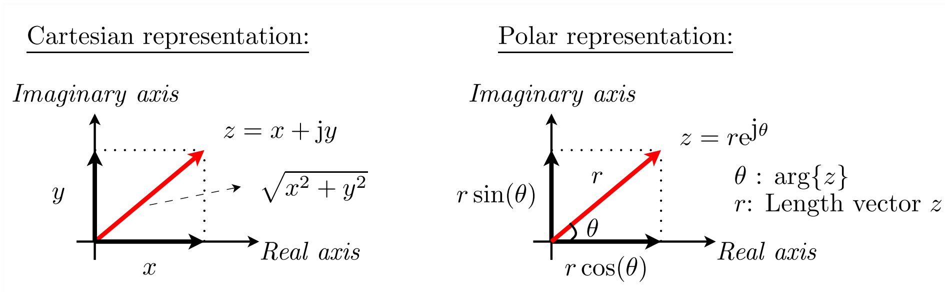


Figure 2: Cartesian- and Polar- representations of complex number z

the length of the complex vector z and θ the angle with the (positive) real axis.

From this it follows that for the Cartesian representation the length of the complex vector $z = x + \mathrm{j}y$ is equal to $\sqrt{x^2+y^2}$. The angle with the positive real axis can be calculated, for positive values of x, by $\arctan(\frac{y}{x})$ or by $\arctan(\frac{y}{x}) + \pi$ for negative values of x. In other words if we need to convert from Cartesian to Polar notation we can use the following equations:

$$\mathrm{C} \;\Rightarrow\; \mathrm{P}: \boxed{r = \sqrt{x^2+y^2} \text{ and } \theta = \begin{cases} \arctan(\frac{y}{x}) & \text{for } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{for } x < 0 \end{cases}} \tag{18}$$

On the other hand from the polar representation as depicted in Fig. 2 it follows that the projection of the complex vector $z = r\mathrm{e}^{\mathrm{j}\theta}$ on the real axis equals $r\cos(\theta)$, while the projection on the imaginary axis is given by $r\sin(\theta)$. Thus if we need to convert from Polar to Cartesian notation we can use the following equations:

$$\mathrm{P} \;\Rightarrow\; \mathrm{C}: \boxed{x = r\cos(\theta)} \text{ and } \boxed{y = r\sin(\theta)} \tag{19}$$

Example:

Convert $z = 4\mathrm{e}^{-\mathrm{j}\frac{3\pi}{4}}$ to Cartesian notation $z = x + \mathrm{j}y$.
Solution:

By using Euler's equation (12) we obtain:

$$\begin{aligned} 4\mathrm{e}^{-\mathrm{j}\frac{3\pi}{4}} &= 4\cos\left(\frac{-3\pi}{4}\right) + \mathrm{j}4\sin\left(\frac{-3\pi}{4}\right) \\ &= 4\cdot -\frac{1}{2}\sqrt{2} + \mathrm{j}4\cdot -\frac{1}{2}\sqrt{2} \\ &= -2\sqrt{2} - \mathrm{j}2\sqrt{2} \end{aligned}$$

Example:

Convert $z = -\frac{3}{2}\sqrt{3} + \mathrm{j}\frac{3}{2}$ to polar notation $z = r\mathrm{e}^{\mathrm{j}\theta}$.
Solution:
First calculate the length r of the complex vector z as follows:

$$r = |z| = \sqrt{\left(-\frac{3}{2}\sqrt{3}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4} + \frac{9}{4}} = 3$$

Since the real part of the complex vector z is negative the argument can be found with the following equation:

$$\theta = \arctan\left(\frac{\frac{3}{2}}{-\frac{3}{2}\sqrt{3}}\right) + \pi = \arctan\left(-\frac{1}{\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5}{6}\pi$$

Concluding we obtain:

$$z = -\frac{3}{2}\sqrt{3} + \mathrm{j}\frac{3}{2} = 3\mathrm{e}^{\mathrm{j}\frac{5}{6}\pi}$$

4.2 Complex conjugation

In this subsection we will show that we can obtain the length of a complex vector by using the so called complex conjugated vector, denoted by z^*, which is constructed by replacing the imaginary number j by -j, thus:

$$z = \mathrm{e}^{\mathrm{j}\theta} \quad \Rightarrow \quad \boxed{z^* = \mathrm{e}^{-\mathrm{j}\theta}} \tag{20}$$

$$z = x + \mathrm{j}y \quad \Rightarrow \quad \boxed{z^* = x - \mathrm{j}y} \tag{21}$$

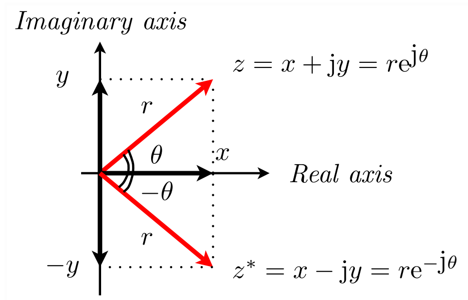


Figure 3: Complex conjugation

This conjugation operation is visualized in Fig. 3. We have seen that the length of a complex vector is either given by the parameter r, when using polar notation, or by $\sqrt{x^2+y^2}$, when using the Cartesian notation. Since the length of a vector is always a positive real number, we will denote it by using the absolute value sign $|\cdot|$. Now by using the definition of complex conjugation we can define the length of a complex vector z as follows:

$$\boxed{|z| = \sqrt{z \cdot z^*}} \quad \text{real and } \geq 0 \tag{22}$$

This definition can be shown either by using polar or Cartesian notation as follows:

$$\begin{aligned} |z| &= \sqrt{r\mathrm{e}^{\mathrm{j}\theta} \cdot r\mathrm{e}^{-\mathrm{j}\theta}} = \sqrt{r^2\mathrm{e}^{\mathrm{j}(\theta-\theta)}} = r \\ &= \sqrt{(x+\mathrm{j}y)\cdot(x-\mathrm{j}y)} = \sqrt{x^2 - \mathrm{j}xy + \mathrm{j}yx - \mathrm{j}^2y^2} = \sqrt{x^2+y^2} \end{aligned}$$

4.3 Calculation rules

Similar to each extension from one set of numbers to a more general set of numbers, the calculation rules for complex numbers are defined in such a way that these rules generalize the calculation rules of real numbers.

Because of the fact that we can represent a complex number in Cartesian or in Polar notation, each of the calculation rules can be performed by using both of these representations, however a general rule of thumb for the calculation rules is:

It is easiest to use Cartesian notation when adding or subtraction complex numbers, while polar notation works easiest for multiplication and division

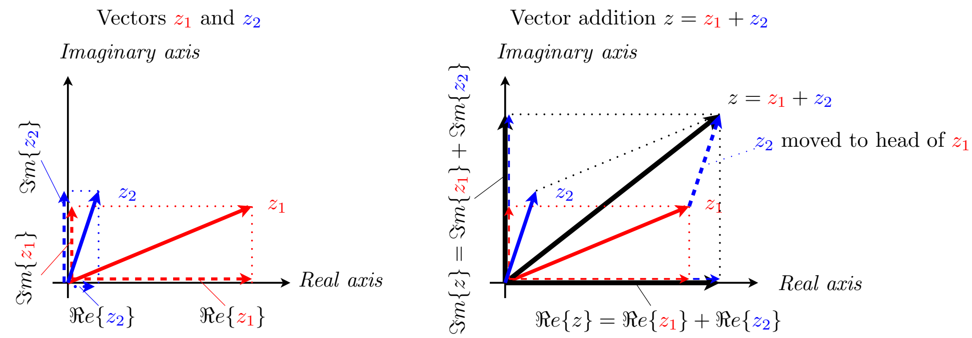


Figure 4: Vector addition rule

4.3.1 Addition & subtraction

When adding two complex numbers $z_1 = x_1 + \mathrm{j}y_1$ and $z_2 = x_2 + \mathrm{j}y_2$ we have to add **separately** their real parts and imaginary parts. Thus when evaluating the real- and imaginary- part of $z = z_1 + z_2$ we obtain:

$$z = z_1 + z_2 = (x_1 + \mathrm{j}y_1) + (x_2 + \mathrm{j}y_2) = (x_1 + x_2) + \mathrm{j}(y_1 + y_2) \tag{23}$$

from which it follows that:

$$\Re e\{z\} = x_1 + x_2 \quad \text{and} \quad \Im m\{z\} = y_1 + y_2$$

As a result of the separate addition of the real and the imaginary parts we can visualize the addition of two complex vector by the so called vector addition rule which is depicted in Fig. 4. The left hand side of the figure represents two individual complex vectors z_1 and z_2. Each of these vectors originates in the origin. Now because of the fact that we have to add the real and imaginary parts separately the construction of the addition of two complex vectors in a resulting complex vector z is as follows: Move one of the vectors, say z_2, as in a parallelogram from the origin to the head of the other vector z_1. Via the vector addition rule the resulting vector z is pointing from the origin to the head of the moved vector z_2.

4.3.2 Multiplication

The result $z = r\mathrm{e}^{\mathrm{j}\theta}$ of multiplying two complex vectors $z_1 = r_1\mathrm{e}^{\mathrm{j}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{j}\theta_2}$ is:

$$z = z_1 \cdot z_2 = r_1\mathrm{e}^{\mathrm{j}\theta_1} \cdot r_2\mathrm{e}^{\mathrm{j}\theta_2} = (r_1 \cdot r_2)\mathrm{e}^{\mathrm{j}(\theta_1+\theta_2)} = r\mathrm{e}^{\mathrm{j}\theta} \tag{24}$$

with $r = r_1 \cdot r_2$ and $\theta = \theta_1 + \theta_2$.
Example:

Use the Cartesian notation to find the muliplication result.
Solution:
When using the Cartesian notation the result becomes as follows:

$$\begin{aligned} z &= z_1 \cdot z_2 = (x_1 + \mathrm{j}y_1) \cdot (x_2 + \mathrm{j}y_2) \\ &= x_1x_2 + \mathrm{j}^2 y_1y_2 + \mathrm{j}x_1y_2 + \mathrm{j}y_1x_2 \\ &= (x_1x_2 - y_1y_2) + \mathrm{j}(x_1y_2 + x_2y_1) = \Re e\{z\} + \mathrm{j}\Im m\{z\} \end{aligned}$$

4.3.3 Division

The result $z = r\mathrm{e}^{\mathrm{j}\theta}$ of dividing two complex vectors $z_1 = r_1\mathrm{e}^{\mathrm{j}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{j}\theta_2}$ is:

$$z = \frac{z_1}{z_2} = \frac{r_1\mathrm{e}^{\mathrm{j}\theta_1}}{r_2\mathrm{e}^{\mathrm{j}\theta_2}} = \frac{r_1}{r_2} \cdot \mathrm{e}^{\mathrm{j}(\theta_1 - \theta_2)} = r\mathrm{e}^{\mathrm{j}\theta} \tag{25}$$

with $r = \frac{r_1}{r_2}$ and $\theta = \theta_1 - \theta_2$.
Example:

Use the Cartesian notation to find the division result.
Solution:
When using the Cartesian notations $z_1 = x_1 + \mathrm{j}y_1$ and $z_2 = x_2 + \mathrm{j}y_2$ for dividing two complex numbers the main problem is that there is a complex number in the denominator. So we have to find a way to get rid of this complex number in the denominator. Now we have seen before that when multiplying a complex number by its complex conjugate the result is real. We can use this fact to rewrite the original fraction of two complex numbers as a fraction of a complex number divided by a real number.This strategy goes as follows:

$$z = \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot 1 = \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} = \frac{z_1 \cdot z_2^*}{|z_2|^2}$$

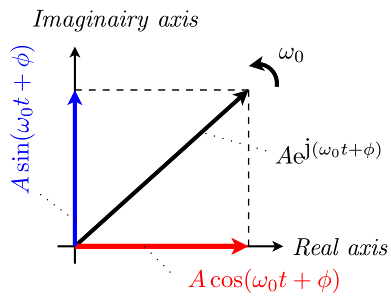


Figure 5: Phasor: Time dependent complex exponential

Now the division works out as follows:

$$\begin{aligned} z &= \frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{|z_2|^2} = \frac{x_1x_2 - \mathrm{j}^2 y_1y_2 - \mathrm{j}x_1y_2 + \mathrm{j}y_1x_2}{x_2^2+y_2^2} \\ &= \frac{(x_1x_2+y_1y_2)+\mathrm{j}(y_1x_2-x_1y_2)}{x_2^2+y_2^2} \\ &= \left(\frac{x_1x_2+y_1y_2}{x_1^2+y_1^2}\right) + \mathrm{j}\left(\frac{y_1x_2-x_1y_2}{x_1^2+y_1^2}\right) = \Re e\{z\} + \mathrm{j}\Im m\{z\} \end{aligned}$$

5 Phasors: Time dependent complex exponentials

Many real world signals can be described by a time depending sinusoidal signals such as:

$$x(t) = A\cos(\omega_o t + \phi) \tag{26}$$

In this equation is A represents the amplitude, from which the dimension is equal to the dimension of the signal $x(t)$, ω_0 [rad/sec] the radian frequency and ϕ [rad] the phase. By using the Euler equation (12) and replacing θ by $\omega_o t+\phi$ we obtain the following time depending complex exponential function $z(t)$:

$$z(t) = Ae^{\mathrm{j}(\omega_o t+\phi)} = A\cos(\omega_o t+\phi) + \mathrm{j}A\sin(\omega_o t+\phi) \tag{27}$$

Such time depending complex exponential, which is depicted as a time depending complex vector in Fig. 5, is called a phasor. The projection of the phasor on the real axis behaves like a cos signal, while the projection on the imaginary axis behaves like a sin function. In other words we can generalize a time depending sinusoidal signal by a phasor, since it describes both a sin and cos function at the same time. Alternatively we write the cos and sin in phasor notation as follows:

$$\boxed{\begin{aligned} A\cos(\omega_o t+\phi) &= \Re e\{(A\mathrm{e}^{\mathrm{j}\omega_o t+\phi)}\} = \frac{A\mathrm{e}^{\mathrm{j}(\omega_o t+\phi)}+A\mathrm{e}^{-\mathrm{j}(\omega_o t+\phi)}}{2} \quad (28)\\ A\sin(\omega_o t+\phi) &= \Im m\{(A\mathrm{e}^{\mathrm{j}(\omega_o t+\phi)}\} = \frac{A\mathrm{e}^{\mathrm{j}(\omega_o t+\phi)}-A\mathrm{e}^{-\mathrm{j}(\omega_o t+\phi)}}{2\mathrm{j}} \quad (29)\end{aligned}}$$

5.1 Phasor addition rule

We can write a phasor as the following product:

$$A\mathrm{e}^{\mathrm{j}(\omega_o t+\phi)} = A\mathrm{e}^{\mathrm{j}\phi}\cdot\mathrm{e}^{\mathrm{j}\omega_0 t} \tag{30}$$

The second part $\mathrm{e}^{\mathrm{j}\omega_0 t}$ is the phasor component, containing the radian frequency ω_0. The first part $A\mathrm{e}^{\mathrm{j}\phi}$ is a complex number, in which A represents the amplitude and ϕ the phase of the phasor. By using this fact it becomes obvious that the addition of two phasors, $A_1\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi_1)}$ and $A_2\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi_2)}$, with the same frequency ω_0 result in one new phasor $A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}$ having the same frequency ω_0. This can be shown as follows:

$$A_1\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi_1)}+A_2\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi_2)} = A_1\mathrm{e}^{\mathrm{j}\phi_1}\cdot\mathrm{e}^{\mathrm{j}\omega_0}+A_2\mathrm{e}^{\mathrm{j}\phi_2}\cdot\mathrm{e}^{\mathrm{j}\omega_0} = \left(A_1\mathrm{e}^{\mathrm{j}\phi_1}+A_2\mathrm{e}^{\mathrm{j}\phi_2}\right)\cdot\mathrm{e}^{\mathrm{j}\omega_0}$$

The amplitude A and phase ϕ of the resulting phasor $A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}$ can be calculated by using the complex addition rule for the complex numbers $A_1\mathrm{e}^{\mathrm{j}\phi_1}$ and $A_2\mathrm{e}^{\mathrm{j}\phi_2}$ of the individual phasors as follows:

$$A\mathrm{e}^{\mathrm{j}\phi} = A_1\mathrm{e}^{\mathrm{j}\phi_1}+A_2\mathrm{e}^{\mathrm{j}\phi_2} \tag{31}$$

This phasor addition example is depicted in Fig. 6. We can use this property when a signal $x(t)$ consists of the sum of two sinusoidal signals, both with the same frequency ω_0, which goes as follows:

$$\begin{aligned} x(t) &= A_1\cos(\omega_0 t+\phi_1)+A_2\cos(\omega_0 t+\phi_2)\\ &= \left(\frac{A_1}{2}\mathrm{e}^{\mathrm{j}\phi_1}+\frac{A_2}{2}\mathrm{e}^{\mathrm{j}\phi_2}\right)\cdot\mathrm{e}^{\mathrm{j}\omega_0 t}+\left(\frac{A_1}{2}\mathrm{e}^{-\mathrm{j}\phi_1}+\frac{A_2}{2}\mathrm{e}^{-\mathrm{j}\phi_2}\right)\cdot\mathrm{e}^{-\mathrm{j}\omega_0 t}\end{aligned}$$

By defining

$$A\mathrm{e}^{\mathrm{j}\phi} = A_1\mathrm{e}^{\mathrm{j}\phi_1}+A_2\mathrm{e}^{\mathrm{j}\phi_2}$$

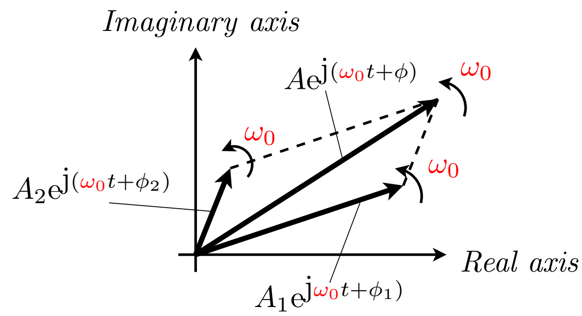


Figure 6: Adding two phasors with same frequency

we obtain:

$$x(t) \quad = \quad \left(\frac{A}{2}e^{j\phi}\right) \cdot e^{j\omega_0 t} + \left(\frac{A}{2}e^{-j\phi}\right) \cdot e^{-j\omega_0 t} = A\cos(\omega_0 t + \phi)$$

Thus the amplitude A and phase ϕ of the resulting sinusoidal signal $x(t)$ are found by complex addition of the amplitude and phase of the individual sinusoidal components.
This example can be extended to the addition of N sinusoidal signals, all with the same radian frequency ω_0, resulting in one sinusoidal signal with the same radian frequency ω_0 as follows:

$$\boxed{x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi)} \tag{32}$$

in which the amplitude A and phase ϕ can be calculated by using complex addition rules as follows:

$$\boxed{Ae^{j\phi} = \sum_{k=1}^{N} A_k e^{j\phi_k}} \tag{33}$$

In words this results in the so called phasor addition rule:

The sum of original sinusoidal signals, all with the same radian frequency ω_0, results in one sinusoidal signal with the same radian frequency ω_0. Amplitude and phase of this resulting signal can be found by adding the complex representation of amplitude and phase of the individual original sinusoidal signals.

Example:

The function $x(t)$ consists of the sum of the following 3 sinusoidal signals:

$$x(t) = 5\cos\left(\omega_0 t + \frac{3}{2}\pi\right) + 4\cos\left(\omega_0 t + \frac{2}{3}\pi\right) + 4\cos\left(\omega_0 t + \frac{1}{3}\pi\right)$$

Express $x(t)$ in the form $x(t) = A\cos(\omega t + \phi)$ by finding the numerical values of A and ϕ.

<u>*Solution:*</u>

The different cosine functions can be written as the real part of phasors.

$$x(t) = \Re e\{5\mathrm{e}^{\mathrm{j}\omega_0 t + \frac{3}{2}\pi}\} + \Re e\{4\mathrm{e}^{\mathrm{j}\omega_0 t + \frac{2}{3}\pi}\} + 4\Re e\{\mathrm{e}^{\mathrm{j}\omega_0 t + \frac{1}{3}\pi}\}$$

The phasor component ($\mathrm{e}^{\mathrm{j}\omega_0 t}$) can be separated:

$$x(t) = \Re e\{\mathrm{e}^{\mathrm{j}\omega_0 t} \cdot \left(5\mathrm{e}^{\mathrm{j}\frac{3\pi}{2}} + 4\mathrm{e}^{\mathrm{j}\frac{2\pi}{3}} + 4\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}\right)\}$$

The complex numbers, containing the amplitude and phase of the 3 individual sinusoidal signals, can be added together by using complex addition rules. Since we have to add complex numbers it is easiest to rewrite these complex numbers in Cartesian notation:

$$\begin{aligned} x(t) &= \Re e\{\mathrm{e}^{\mathrm{j}\omega_0 t} \cdot \left(-\mathrm{j}5 + 4(-\frac{1}{2} + \mathrm{j}\frac{1}{2}\sqrt{3}) + 4(\frac{1}{2} + \mathrm{j}\frac{1}{2}\sqrt{3})\right)\} \\ &= \Re e\{\mathrm{e}^{\mathrm{j}\omega_0 t} \cdot \left((4\sqrt{3} - 5)\mathrm{j}\right)\} = \Re e\{\mathrm{e}^{\mathrm{j}\omega_0 t} \cdot \left((4\sqrt{3} - 5)\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\right)\} \\ &= \Re e\{(4\sqrt{3} - 5)\mathrm{e}^{\mathrm{j}\omega_0 t + \frac{\pi}{2}}\} \end{aligned}$$

Now this answer can be written as the final answer:

$$x(t) = (4\sqrt{3} - 5)\cos\left(\omega t + \frac{\pi}{2}\right)$$

6 Summary complex numbers and phasors

- The imaginary unit: $\boxed{\mathrm{j} = \sqrt{-1}}$
- Euler equation: $\boxed{\mathrm{e}^{j\theta} = \cos(\theta) + \mathrm{j} \cdot \sin(\theta)}$
- Alternative expression for sin- and cos- function:

$$\boxed{\cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2}} = \Re e\{\mathrm{e}^{\mathrm{j}\theta}\} \quad \text{and} \quad \boxed{\sin(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2j}} = \Im m\{\mathrm{e}^{\mathrm{j}\theta}\}$$

- Polar representation set of complex numbers: $\boxed{\mathbb{C} = \{z = r\mathrm{e}^{\mathrm{j}\theta} | r \in \mathbb{R} \text{ and } \theta \in \mathbb{R}\}}$
- Cartesian representation set of complex numbers: $\boxed{\mathbb{C} = \{z = x + \mathrm{j}y | x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}}$
- Conversion Cartesion to Polar representation:

$$\boxed{r = \sqrt{x^2 + y^2} \text{ and } \theta = \begin{cases} \arctan(\frac{y}{x}) & \text{for } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{for } x < 0 \end{cases}}$$

- Conversion Polar to Cartesion representation:

$$\boxed{x = r\cos(\theta)} \text{ and } \boxed{y = r\sin(\theta)}$$

- Complex conjugation: Replace j by $-\mathrm{j}$
- Length of complex vector: $\boxed{|z| = \sqrt{z \cdot z^*}}$ real and ≥ 0
- It is easiest to use Cartesian notation when adding or subtraction, and polar notation for multiplication and division.
- Phasor representation of sinusoidal signals:

$$\boxed{\begin{aligned} A\cos(\omega_o t + \phi) &= \Re e\{(A\mathrm{e}^{\mathrm{j}\omega_o t + \phi)}\} = \frac{A\mathrm{e}^{\mathrm{j}(\omega_o t + \phi)} + A\mathrm{e}^{-\mathrm{j}(\omega_o t + \phi)}}{2} \\ A\sin(\omega_o t + \phi) &= \Im m\{(A\mathrm{e}^{\mathrm{j}(\omega_o t + \phi)}\} = \frac{A\mathrm{e}^{\mathrm{j}(\omega_o t + \phi)} - A\mathrm{e}^{-\mathrm{j}(\omega_o t + \phi)}}{2\mathrm{j}} \end{aligned}}$$

- The sum of original sinusoidal signals, all with the same radian frequency ω_0, results in one sinusoidal signal with the same radian frequency ω_0. Amplitude and phase of this resulting signal can be found by adding the complex representation of amplitude and phase of the individual original sinusoidal signals.

$$\boxed{x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi)} \text{ with } \boxed{A\mathrm{e}^{\mathrm{j}\phi} = \sum_{k=1}^{N} A_k \mathrm{e}^{\mathrm{j}\phi_k}}$$