Module SPFS:

Spectrum representation and Fourier Series

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Where innovation starts

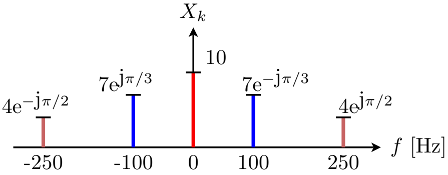
Example: Two-sided spectrum

$$
\begin{aligned}
x(t) &= 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \\
&= 10 + 7\mathrm{e}^{-\mathrm{j}\pi/3}\mathrm{e}^{\mathrm{j}2\pi(100)t} + 7\mathrm{e}^{\mathrm{j}\pi/3}\mathrm{e}^{-\mathrm{j}2\pi(100)t} \\
&\quad + 4\mathrm{e}^{\mathrm{j}\pi/2}\mathrm{e}^{\mathrm{j}2\pi(250)t} + 4\mathrm{e}^{-\mathrm{j}\pi/2}\mathrm{e}^{-\mathrm{j}2\pi(250)t}
\end{aligned}
$$

Spectrum is set of 5 rotating phasors (one with frequency 0):

$$
\left\{(0, 10), (100, 7\mathrm{e}^{-\mathrm{j}\pi/3}), (-100, 7\mathrm{e}^{\mathrm{j}\pi/3}), (250, 4\mathrm{e}^{\mathrm{j}\pi/2}), (-250, 4\mathrm{e}^{-\mathrm{j}\pi/2})\right\}
$$

Graphical plot (frequency-domain representation):



Why two-sided spectrum symmetric and why spectral lines?

Spectrum of real signal

What is the spectrum of a signal?

Example: Spectrum of constant + sum of N sinusoids

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

$$= X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} \mathrm{e}^{\mathrm{j}2\pi f_k t} + \frac{X_k^*}{2} \mathrm{e}^{-\mathrm{j}2\pi f_k t} \right\} \text{ with } X_0 = A_0 \text{ ; } X_k = A_k \mathrm{e}^{\mathrm{j}\phi_k}$$

Two-sided spectrum $\rightarrow$ Set of $2N+1$ complex amplitudes + frequencies:

$$\left\{ (f_0, X_0), (f_1, \frac{1}{2}X_1), (-f_1, \frac{1}{2}X_1^*), \cdots, (f_k, \frac{1}{2}X_k), (-f_k, \frac{1}{2}X_k^*), \cdots \right\}$$

$\Rightarrow$ Contribution @ f_k: $\frac{1}{2}X_k$ and @ $-f_k$: $\frac{1}{2}X_k^*$ (Complex conjugation)

Procedure to compute and plot spectrum

If signal composed of DC + sum of sinusoids:

- Express signal as sum of phasors
- Plot complex amplitude and phase @ corresponding frequency.

Remarks:

- Which signals can be represented by sum of exponentials?
  Any periodic waveform: Frequencies all integer multiples of *fundamental frequency* $\Rightarrow$ *Fourier series (Ch. 3, this course)*
- Spectrum of arbitrary signal $\Rightarrow$ Fourier analysis (Ch.11, not this course)

Beat note waveform

Spectral content multiplication of two sinusoids?

Example: Product sinusoids of 0.5 Hz and 5 Hz

$$
\begin{aligned}
x(t) &= \cos(\pi t)\cdot\sin(10\pi t) \\
&= \left(\frac{\mathrm{e}^{\mathrm{j}\pi t}+\mathrm{e}^{-\mathrm{j}\pi t}}{2}\right)\cdot\left(\frac{\mathrm{e}^{\mathrm{j}10\pi t}-\mathrm{e}^{-\mathrm{j}10\pi t}}{2\mathrm{j}}\right) \\
&= \cdots \\
&= \frac{1}{2}\cos(11\pi t-\pi/2)+\frac{1}{2}\cos(9\pi t-\pi/2)
\end{aligned}
$$

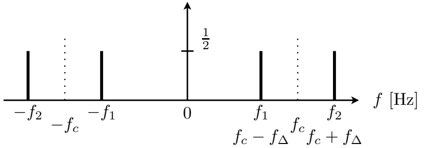
Beat note: Adding two sinusoids with nearly same frequency (e.g. playing two neighboring piano keys)

Previous example suggests product of two sinusoids equals sum

Beat note waveform

Adding two closely spaced sinusoids:

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) \text{ with } f_2 > f_1$$



Show that sum is equivalent to product:

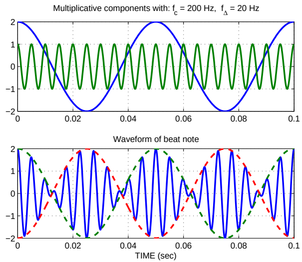
With $f_c = \frac{1}{2}(f_1 + f_2)$ and $f_\Delta = \frac{1}{2}(f_2 - f_1)$
$\Rightarrow f_1 = f_c - f_\Delta$ and $f_2 = f_c + f_\Delta \Rightarrow$

$$x(t) = \cdots = 2\cos(2\pi f_\Delta t) \cdot \cos(2\pi f_c t)$$

Note: Product easy to draw time plot, sum easy to plot spectrum

Beat note waveform

$$x(t) =$$
$$= 2\cos(2\pi(20)t) \cdot \cos(2\pi(200)t)$$
$$= \cos(2\pi(180)t) + \cos(2\pi(220)t)$$



Remarks:

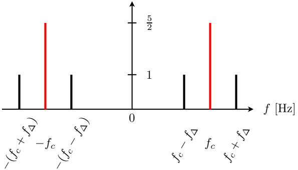
- Q: Plot spectrum $x(t) = 2\cos(2\pi(20)t) \cdot \cos(2\pi(200)t)$
- Low frequency causes envelope (dashed lines in plot)
- Interval between nulls: $1/(2f_\Delta)$
- f_Δ causes signal to fade in and out $\Rightarrow$ "beating"
- Musicians use beat phenomenon tune instruments to same pitch
- Example: Beat note $f_c = 660$ Hz; $f_\Delta = 12$ Hz :

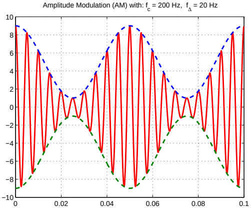
Amplitude modulation (AM)

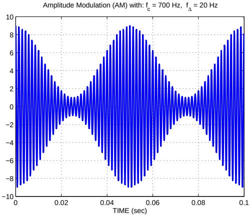
$$x(t) = m(t) \cdot \cos(2\pi f_c t)$$

$m(t)$: Voice/ music to transmit ; f_c: Carrier signal $\gg$ frequencies in $m(t)$

$$\begin{aligned} x(t) &= \{5 + 4\cos(40\pi t)\} \cdot \cos(400\pi t) = \cdots \\ &= \mathrm{e}^{-\mathrm{j}440\pi t} + \frac{5}{2}\mathrm{e}^{-\mathrm{j}400\pi t} + \mathrm{e}^{-\mathrm{j}360\pi t} + \mathrm{e}^{\mathrm{j}360\pi t} + \frac{5}{2}\mathrm{e}^{\mathrm{j}400\pi t} + \mathrm{e}^{\mathrm{j}440\pi t} \end{aligned}$$



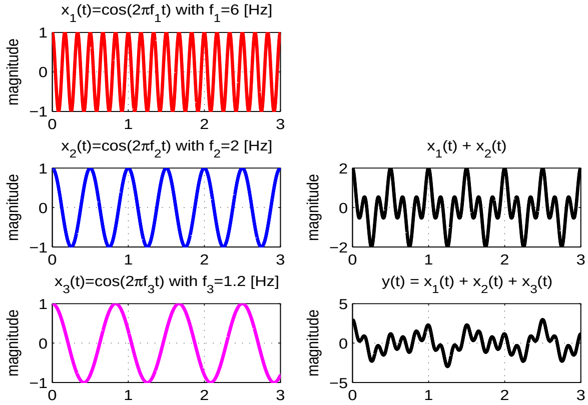




- Difference with beatnotes?
- Spectrum: 2 identical shifted versions of two-sided spectrum $m(t)$

Periodic waveforms

$$y(t) = \cos(2\pi 6t) + \cos(2\pi 2t) + \cos(2\pi 1.2t) \;\; ; F_0 = \;\; 0.4[\text{Hz}]$$



⇒ Any periodic signal with period $T_0 = 1/F_0$ consists of sum of sinusoidals all with related frequencies (multiples of F_0)

The other way around: Which frequencies?

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Periodic waveforms

Assume $x(t)$ real and consists of DC + sum of N frequencies $f_1, \cdots, f_N$ with $f_1 < \cdots < f_N$

When $x(t)$ periodic with period T_0, thus $x(t) = x(t + T_0)$:

- Fundamental frequency $F_0 = 1/T_0 = \text{gcd}\{f_1, \cdots, \cdots, f_N\}$
- Largest frequency $f_N = M \cdot F_0$,

$\Rightarrow x(t)$ can be written as:

$$x(t) = \sum_{k=-M}^{M} \alpha_k \mathrm{e}^{\mathrm{j}2\pi k \cdot F_0 t} = \sum_{k=-M}^{M} \alpha_k \mathrm{e}^{\mathrm{j}k \cdot \frac{2\pi}{T_0} t}$$

Notes:

- α_k represents amplitude and phase @ frequency $k \cdot F_0$
- When $x(t)$ DC + N frequencies $\Rightarrow$ Only α_0 and N out of M α_k's $\neq 0$
- Try previous example: $y(t) = \cos(2.4\pi t) + \cos(4\pi t) + \cos(12\pi t)$

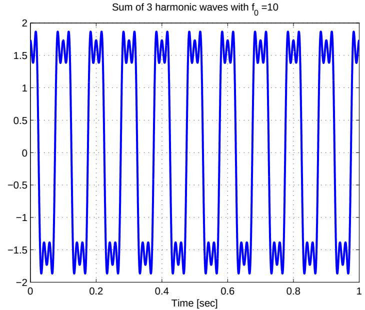
Nonperiodic signal

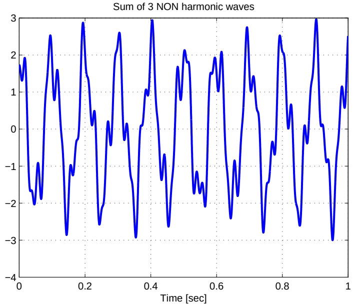
Periodic signal $1/T_0 = F_0 = 10$ [Hz] (first 3 harmonics square wave):

$$x_{per}(t) = 2\cos(20\pi t) - \frac{2}{3}\cos(20\pi(3)t) + \frac{2}{5}\cos(20\pi(5)t)$$

Non periodic signal ($\sqrt{8} = 2.8284 \approx 3$ and $\sqrt{27} = 5.1962 \approx 5$)

$$x_{nonper}(t) = 2\cos(20\pi t) - \frac{2}{3}\cos(20\pi(\sqrt{8})t) + \frac{2}{5}\cos(20\pi(\sqrt{27})t)$$





Fourier series

Any periodic signal (period T_0) writes as sum harmonic related sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}$$

Fourier analysis: $x(t) \to \{\alpha_k\}$; Fourier synthesis: $\{\alpha_k\} \to x(t)$

Real function of time: $\alpha_{-k} = \alpha_k^*$ (amplitudes conjugate-symmetric)

Alternative notation: $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos((2\pi/T_0)kt + \phi_k)$

with $A_0 = \alpha_0$ and $\alpha_k = \frac{1}{2} A_k \mathrm{e}^{\mathrm{j}\phi_k}$

Note: Even discontinuous periodic signal (e.g. square wave) can be represented with infinite number of sinusoids (!)

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Fourier series: Analysis

How come from $x(t)$ to α_k? $\rightarrow$ Fourier series integral:

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t$$

with T_0 fundamental period of $x(t)$.

Note: for DC we have

$$\alpha_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{d}t$$

Example: What will be the result for $x(t) = \cos((2\pi/T_0)t)$?

$$\begin{aligned}\alpha_k &= \frac{1}{T_0}\int_0^{T_0}\left(\frac{\mathrm{e}^{\mathrm{j}(2\pi/T_0)t}+\mathrm{e}^{-\mathrm{j}(2\pi/T_0)t}}{2}\right)\mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t \\ &= \cdots = \begin{cases}\frac{1}{2} & \text{for } k=\pm 1\\ 0 & \text{elsewhere}\end{cases}\end{aligned}$$

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Fourier series: Derivation

Important property: for $k \neq 0$

$$\int_0^{T_0} \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t \;=\; \frac{\mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}}{\mathrm{j}(2\pi/T_0)k}\bigg|_0^{T_0} = \frac{\mathrm{e}^{\mathrm{j}(2\pi/T_0)kT_0} - 1}{\mathrm{j}(2\pi/T_0)k} = 0$$

Exercise: Show yourself same result via Euler

Define $v_k(t)$ as complex exponents of frequency $\omega_k = (2\pi/T_0)k$:

$$v_k(t) = \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}$$

$v_k(t)$ has (also) period T_0, thus $v_k(t) = v_k(t + T_0)$:

$$\begin{aligned} v_k(t+T_0) &= \mathrm{e}^{\mathrm{j}(2\pi/T_0)k(t+T_0)} = \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt} \cdot \mathrm{e}^{\mathrm{j}(2\pi/T_0)kT_0} \\ &= \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt} \cdot \mathrm{e}^{\mathrm{j}2\pi k} = \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt} = v_k(t) \end{aligned}$$

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Orthogonality property Animation

$$\int_0^{T_0} v_k(t)v_l^*(t)\mathrm{d}t = \begin{cases} 0 & \text{if } k \neq l \\ T_0 & \text{if } k = l \end{cases} \quad \text{with } v_k(t) = \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}$$

<u>Proof:</u>

$$\begin{aligned} \int_0^{T_0} v_k(t)v_l^*(t)\mathrm{d}t &= \int_0^{T_0} \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}\mathrm{e}^{-\mathrm{j}(2\pi/T_0)lt}\mathrm{d}t \\ &= \int_0^{T_0} \mathrm{e}^{\mathrm{j}(2\pi/T_0)(k-l)t}\mathrm{d}t \end{aligned}$$

Case $k = l$:

$$\int_0^{T_0} \mathrm{e}^{\mathrm{j}(2\pi/T_0)(k-l)t}\mathrm{d}t = \int_0^{T_0} \mathrm{e}^{\mathrm{j}0t}\mathrm{d}t = \int_0^{T_0} 1\mathrm{d}t = T_0$$

Case $k \neq l$, thus $k - l = m \neq 0$:

$$\int_0^{T_0} \mathrm{e}^{\mathrm{j}(2\pi/T_0)(k-l)t}\mathrm{d}t = \int_0^{T_0} \mathrm{e}^{\mathrm{j}(2\pi/T_0)mt}\mathrm{d}t = 0$$

Fourier analysis/ synthesis

With $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}$

Multiply both sides with $v_l^*(t) = \mathrm{e}^{-\mathrm{j}(2\pi/T_0)lt}$ and integrate over T_0:

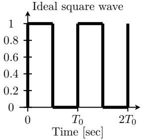
$$\begin{aligned}
\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}(2\pi/T_0)lt}\mathrm{d}t &= \int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt} \right) \mathrm{e}^{-\mathrm{j}(2\pi/T_0)lt}\mathrm{d}t \\
&= \sum_{k=-\infty}^{\infty} \alpha_k \left(\int_0^{T_0} \mathrm{e}^{\mathrm{j}(2\pi/T_0)(k-l)t}\mathrm{d}t \right) = \alpha_l T_0
\end{aligned}$$

Fourier analysis and synthesis equations:

$$\boxed{\alpha_k = \frac{1}{T_0} \int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t \quad \circ\!\!-\!\!\circ \quad x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}}$$

Example: Square wave

Definition one period of 50% duty cycle square wave:



$$\Rightarrow \qquad x(t) = \begin{cases} 1 & \text{for } 0 \leq t < \frac{1}{2}T_0 \\ 0 & \text{for } \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

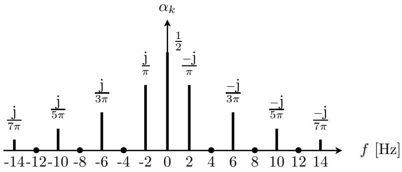
$$\begin{aligned}
\alpha_k &= \left(\frac{1}{T_0}\right) \int_0^{T_0/2} (1)\, \mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt} \mathrm{d}t = \left(\frac{1}{T_0}\right) \frac{\mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}}{-\mathrm{j}(2\pi/T_0)k} \Big|_0^{\frac{1}{2}T_0} \\
&= \left(\frac{1}{T_0}\right) \frac{\mathrm{e}^{-\mathrm{j}(2\pi/T_0)k(\frac{1}{2}T_0)} - \mathrm{e}^{-\mathrm{j}(2\pi/T_0)k0}}{-\mathrm{j}(2\pi/T_0)k} = \frac{\mathrm{e}^{-\mathrm{j}\pi k} - 1}{-\mathrm{j}2\pi k} = \frac{1-(-1)^k}{\mathrm{j}2\pi k}
\end{aligned}$$

$$\Rightarrow \ \alpha_k = \begin{cases} 1/2 & k = 0 \\ 1/\mathrm{j}\pi k & k = \pm 1, \pm 3, \pm 5, \cdots \\ 0 & k = \pm 2, \pm 4, \pm 6, \cdots \end{cases}$$

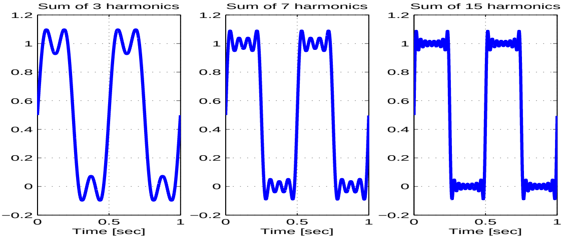
Note: $\alpha_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\mathrm{d}t$ = average value (DC) of signal $x(t)$

Spectrum and synthesis of square wave

Spectrum 50% duty cycle square wave of $F_0 = 2$ [Hz]



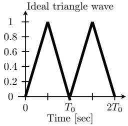
Synthesis via approximation $\rightarrow x_N(t) = \sum_{k=-N}^{N} \alpha_k e^{j2\pi k F_0 t}$



Note: Gibbs phenomenon

Triangular wave: *Homework*

Definition one period of triangular wave:



$$\Rightarrow \quad x(t) = \begin{cases} 2t/T_0 & \text{for } 0 \le t < \frac{1}{2}T_0 \\ 2(T_0 - t)/T_0 & \text{for } \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

Use $\int_A^B x\mathrm{e}^{-x}\,dx = -(x+1)|_A^B$

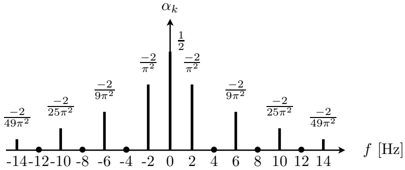
$$\alpha_k = \left(\frac{1}{T_0}\right) \cdot \left\{ \int_0^{T_0/2} (2t/T_0)\,\mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t + \int_{T_0/2}^{T_0} (2(T_0 - t)/T_0)\,\mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t \right\} = \cdots = \frac{(-1)^k - 1}{\pi^2 k^2}$$

$$\Rightarrow \alpha_k = \begin{cases} 1/2 & k = 0 \\ -2/(\pi^2 k^2) & k = \pm 1, \pm 3, \pm 5, \cdots \\ 0 & k = \pm 2, \pm 4, \pm 6, \cdots \end{cases}$$

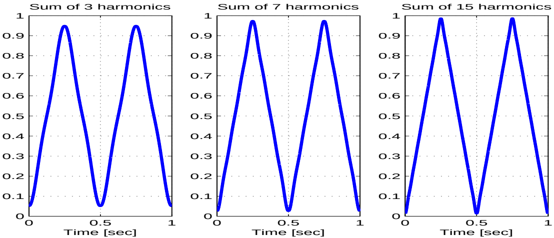
Note: $\alpha_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{d}t = 1/2$ = average value (DC) of signal $x(t)$

Spectrum, synthesis triangle wave: Homework

Spectrum triangle wave of $F_0 = 2$ [Hz]

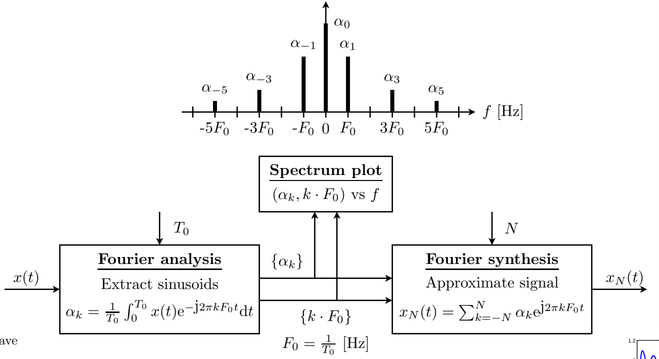


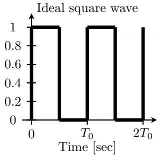
Synthesis via approximation $\rightarrow x_N(t) = \sum_{k=-N}^{N} \alpha_k \mathrm{e}^{\mathrm{j}2\pi k F_0 t}$

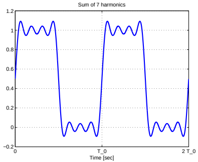


Fourier analysis periodic signals

Any periodic signal writes as sum of (harmonically related) frequencies







Convergence of Fourier synthesis

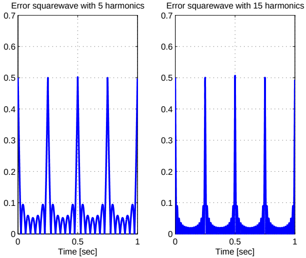
$$x(t) \approx x_N(t) = \sum_{k=-N}^{N} \alpha_k \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}$$

Error: $e_N(t) = x(t) - x_N(t)$

Important feature or error is worst-case error::

$$E_{wc} = \max_{t\in[0,T_0]} |x(t) - x_N(t)|$$

Square wave: E_{wc} half the jump and 9% overshoot (Gibbs)



Example without Fourier integral

Example: $x(t) = \sin^3(3\pi t)$ (*Use* $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$)

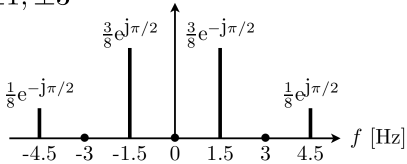
Via Fourier integral: $\alpha_k = \frac{1}{T_0}\int_0^{T_0} \sin^3(3\pi t)\cdot \mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t$

or via inverse Euler (easiest for this type of signal):

$$x(t) = \left(\frac{\mathrm{e}^{\mathrm{j}3\pi t} - \mathrm{e}^{-\mathrm{j}3\pi t}}{2\mathrm{j}}\right)^3 = \frac{\mathrm{j}}{8}\mathrm{e}^{\mathrm{j}9\pi t} + \frac{-3\mathrm{j}}{8}\mathrm{e}^{\mathrm{j}3\pi t} + \frac{3\mathrm{j}}{8}\mathrm{e}^{-\mathrm{j}3\pi t} + \frac{-\mathrm{j}}{8}\mathrm{e}^{-\mathrm{j}9\pi t}$$

Fundamental freq. $\rightarrow F_0 = \mathrm{gcd}(1.5; 4.5) = 1.5$ [Hz]

$\Rightarrow \alpha_k \neq 0$ for $k = \pm 1; \pm 3$



Time-frequency spectrum

Wide range of waveforms (constant, $\cdots$, (not) periodic) synthesized by:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

Examples:

Sine	Square	Saw	Beat

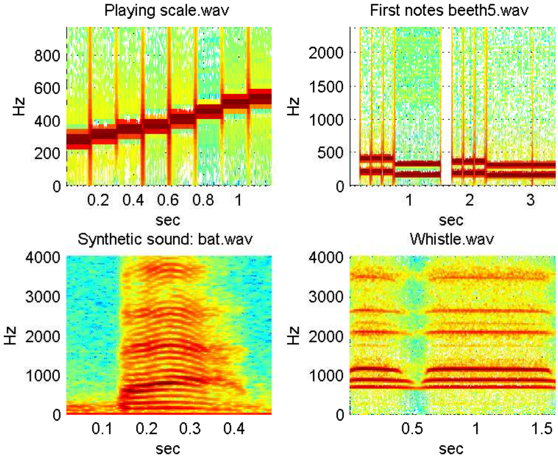
Assumption made: amplitude, phase, frequencies $\rightarrow$ **constant**

Most real-world signals: Frequency changes over time

Examples:

Scale	Beeth5	Bat	Whistle

Real sounds with changing frequencies



Summary Module SPFS

- **Sum of DC and N sinusoids**: $x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$

  With $\alpha_0 = A_0$, $f_0 = 0$, $\alpha_k = \frac{1}{2} A_k \mathrm{e}^{\mathrm{j}\phi_k}$, $\alpha_{-k} = \alpha_k^*$, $f_{-k} = -f_k$

  $\Leftrightarrow x(t) = \sum_{k=-N}^{N} \alpha_k \mathrm{e}^{\mathrm{j}2\pi f_k t} = \alpha_0 + 2\Re e\{\sum_{k=1}^{N} \alpha_k \mathrm{e}^{\mathrm{j}2\pi f_k t}\}$

  - **Spectrum**: $\{(f_0, \alpha_0), (f_1, \alpha_1), (-f_1, \alpha_1^*), \cdots, (f_N, \alpha_N), (-f_N, \alpha_N^*)\}$
  - **Periodic waveform**: When $\gcd\{f_1, \cdots, f_N\} = F_0 = 1/T_0$

    $\Rightarrow x(t) = x(t + T_0) \Rightarrow x(t) = \sum_{k=-M}^{M} \alpha_k \mathrm{e}^{\mathrm{j}2\pi \cdot k \cdot F_0 t}$

    with $f_N = M \cdot F_0$ and N out of M α_k's $\neq 0$

- **Adding two sinusoids (beat notes)**: $x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$

  $f_c = \frac{1}{2}(f_1 + f_2)$; $f_\Delta = \frac{1}{2}(f_1 - f_2) \Rightarrow x(t) = 2\cos(2\pi f_\Delta t) \cdot \cos(2\pi f_c t)$

  AM: $x(t) = m(t) \cdot \cos(2\pi f_c t)$

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Summary Module SPFS

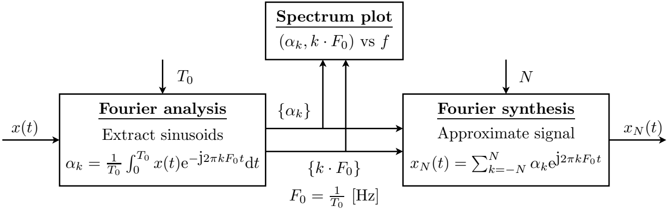
- **Orthogonality property**:

$$\int_0^{T_0} v_k(t)v_l^*(t)\mathrm{d}t = \begin{cases} 0 & \text{if } k \neq l \\ T_0 & \text{if } k = l \end{cases} \quad \text{with } v_k(t) = \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}$$

- **Fourier analysis/ synthesis**:

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}(2\pi/T_0)kt}\mathrm{d}t \circ\!\!-\!\!\circ\; x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}(2\pi/T_0)kt}$$

- **Fourier spectral analysis periodic signals**:



OUT OF SCOPE

Chirp: Signal with sweeping frequency.

Book: Par. 3.7 + 3.8

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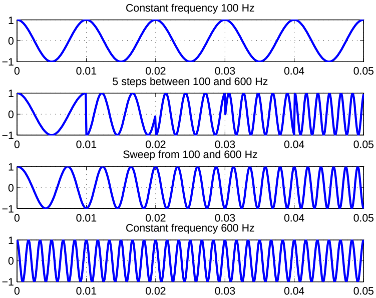
OUT OF SCOPE: Chirp



OUT OF SCOPE: Instantaneous frequency

Constant-frequency sinusoid:

$$x(t) = \Re e\{A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}\} = A\cos(\omega_0 t + \phi)$$

Angle function $(\omega_0 t + \phi)$ changes linearly with time

$\Rightarrow$ Time derivative of angle function ω_0 constant

Generalization:

$$\begin{aligned} x(t) &= \Re e\{A\mathrm{e}^{\mathrm{j}\psi(t)}\} = A\cos(\psi(t)) \\ \text{Angle frequency} &: \psi(t) \text{ [rad]} \\ \text{Instantaneous frequency} &: \omega_{inst}(t) = \frac{d}{dt}\psi(t) \text{ [rad/sec]} \\ \text{or} &\quad f_{inst}(t) = \frac{1}{2\pi}\frac{d}{dt}\psi(t) \text{ [Hz]} \end{aligned}$$

OUT OF SCOPE: Instantaneous frequency

Example (FM, chirp): $\psi(t) = 2\pi\mu t^2$ [rad] (quadratic with time)

$\Rightarrow f_{inst}(t) = 2\mu t + f_0$ (linearly with time)

Example: Desired sweep from $f_0 = 220$ to $f_1 = 2320$ in $T = 3$ sec.

Instantaneous frequency: $f_{inst}(t) = \frac{f_1 - f_0}{T} \cdot t + f_0 = 700 \cdot t + 220 \Rightarrow$

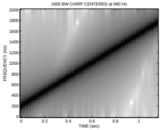
$$\begin{aligned}\psi(t) &= \int_0^t \omega_{inst}(u)\,\mathrm{d}u = \int_0^t 2\pi\,(700 \cdot u + 220)\,\mathrm{d}u \\ &= 700\pi t^2 + 440\pi t + \phi\end{aligned}$$

with ϕ some arbitrary constant.

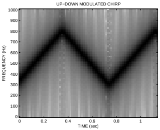
Chirp signal: $\Rightarrow x(t) = \cos(\psi(t)) = \cos(700\pi t^2 + 440\pi t + \phi)$

OUT OF SCOPE: Chirp examples

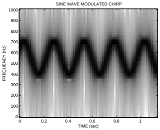
Chirp1



Chirp2



Chirp3



Birds