Module SAA

Sampling and Aliasing

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Technische Universiteit
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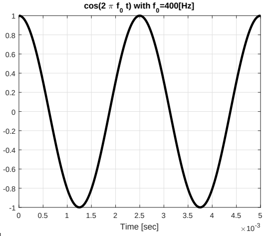
Where innovation starts

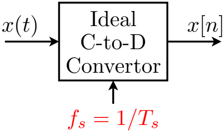
C-to-D: Sample continuous-time signal

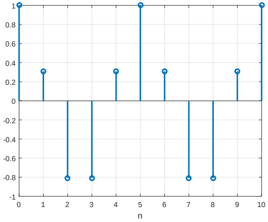
E.g. $x(t) = \cos(2\pi \cdot 400t)$ @ $f_s = 1/T_s$=2000 [Samples/sec]

$$\Rightarrow \; x[n] = x(t)|_{t=nT_s} = \cos(2\pi \cdot 400 \cdot n \cdot \frac{1}{2000}) = \cos(0.4\pi n)$$

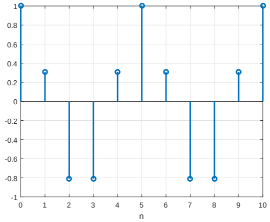
Result: Set of numbers $x[n]$ @ inter sample distance T_s[sec]

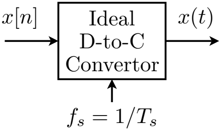


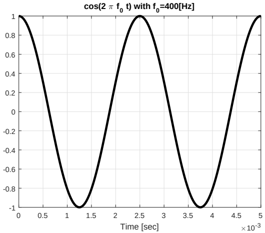




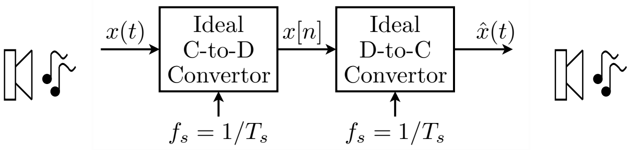
D-to-C: Reconstruct continuous time signal



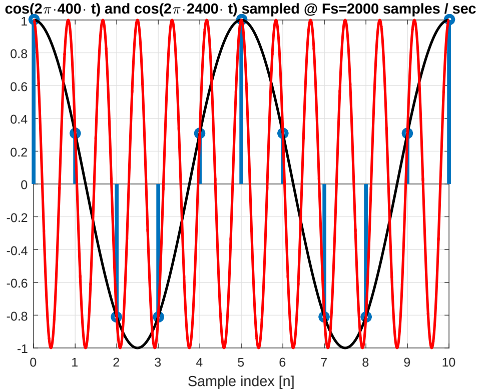




Amazing example



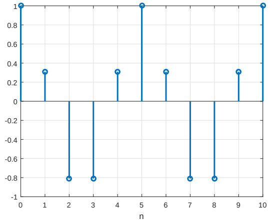
Discrete-time domain: Uniqueness issue

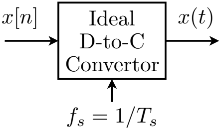


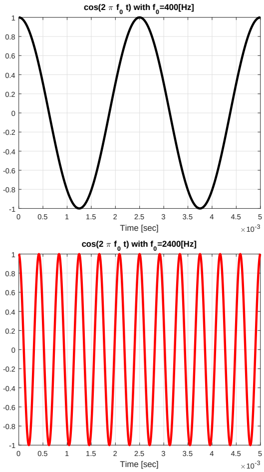
$$x[n] = x(t)|_{t=nT_s} = \cos(2\pi \cdot {\color{red}2400} \cdot n \cdot \frac{1}{2000}) = \cos({\color{red}2.4\pi} n) = \cos(0.4\pi n)$$

$$= {\color{blue}\cdots, \overset{\color{green}{n=0 \atop \downarrow}}{1}, 0.3, -0.8, -0.8, 0.3, \cdots}$$

Uniqueness issue: Implication D-to-C conversion







Absolute- and Relative- frequency

Continuous-time signal $x(t) = \cos(\omega \cdot t) = \cos(2\pi f \cdot t)$

Absolute frequency: $\omega = 2\pi f$ [rad/sec]; f [Hz]

"continuous-time domain"

Conversion to discrete-time @ $f_s = 1/T_s$ [samples/sec] or [Hz]:

$$x[n] \quad = \quad x(t)|_{t=n \cdot T_s} = \cos(\omega \cdot n \cdot T_s) = \cos((\omega \cdot T_s) \cdot n) = \cos(\theta \cdot n)$$

Relative frequency: θ [rad] (dimensionless)

"discrete-time domain"

$$\boxed{\theta = \omega \cdot T_s = 2\pi \left(\frac{f}{f_s}\right)}$$

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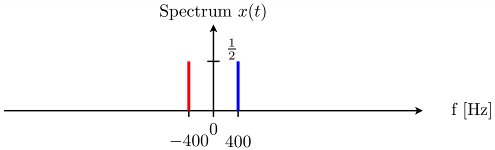
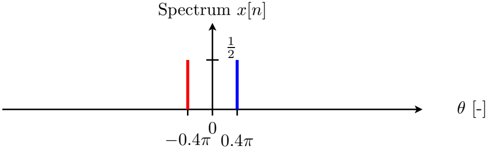
Uniqueness issue: Frequency-domain view

$$x(t) = \cos(2\pi \cdot 400t) = \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi\cdot 400t} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi\cdot 400t}$$

Sample @ f_s=2000[Hz]

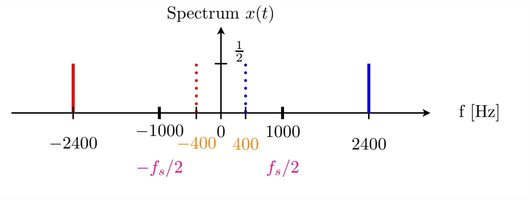


$$\theta = 2\pi\frac{400}{2000} = 0.4\pi \quad \Rightarrow \quad x[n] = \cos(0.4\pi n) = \frac{1}{2}\mathrm{e}^{\mathrm{j}0.4\pi n} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}0.4\pi n}$$

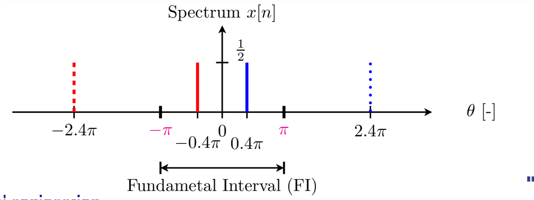


Uniqueness issue: Frequency-domain view

$$x(t) = \cos(2\pi \cdot 2400t) = \frac{1}{2}\mathrm{e}^{\mathrm{j}2\pi\cdot 2400t} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}2\pi\cdot 2400t}$$ Sample @ f_s=2000[Hz]



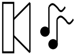
$$\theta = 2\pi\frac{2400}{2000} = 2.4\pi \quad \Rightarrow \quad x[n] = \frac{1}{2}\mathrm{e}^{\mathrm{j}0.4\pi n} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}0.4\pi n}$$

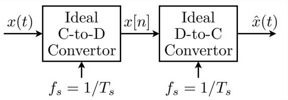


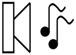
D-to-C: Choose frequencies below $f_s/2$

Reconstruction by D-to-C convertor $\rightarrow$ Select frequency $< \frac{f_s}{2}$

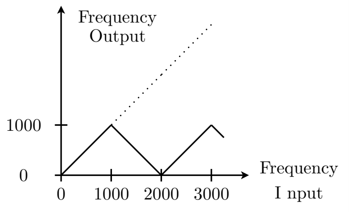
Chirp







Explanation:



Conclusion:

No aliasing when input frequency $< \frac{f_s}{2}$ (= in FI)

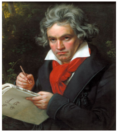
The sampling theorem

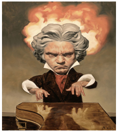
Which sample rate to reconstruct **original** continuous time signal?

Sampling Theorem

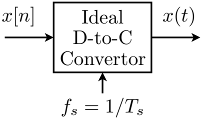
Continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(t)|_{t=n \cdot T_s}$, if samples are taken at a rate $f_s = 1/T_s$, that is greater than $2f_{max}$

If we don't sample fast enough $\rightarrow$ Aliasing





D-to-C conversion: Sinusoidal signal



$x(t) = x[n]|_{n=f_s t}$ **possible or correct?**

Only when mathematical expression of **sinusoidal** available:

For $\theta_0 (= 2\pi \frac{f_0}{f_s}) < \pi \Rightarrow$ Ideal D-to-C 'replaces' n by $f_s \cdot t$

$$x[n] = \cos(\theta_0 n) = \cos(2\pi \frac{f_0}{f_s} n)$$

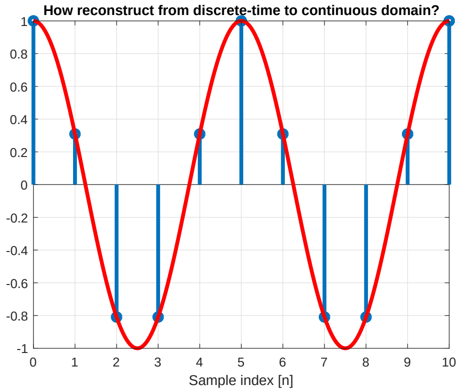
$$\Rightarrow \quad \hat{x}(t) = x[n]|_{n=f_s t} = \cos(2\pi f_0 t)$$

What if $\theta_0 (= 2\pi \frac{f_0}{f_s}) > \pi$? ($\Leftrightarrow$ original frequency $f_0 > \frac{f_s}{2}$)

Ideal D-to-C converts to **aliased frequency** less than $\frac{f_s}{2}$

D-to-C conversion: Sinusoidal signal

How interpolate continuous-time function through samples $x[n]$?

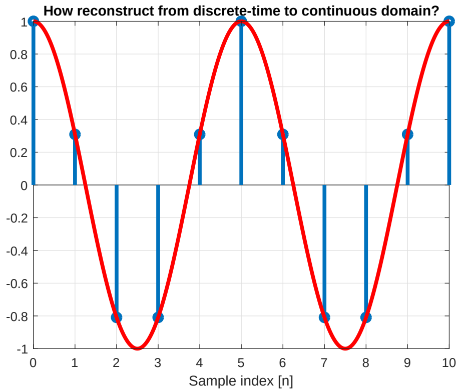


In theory for sinusoidal use mathematical expression of samples

If original $\theta_0 > \pi$ $(f_0 > \frac{f_s}{2})$ $\Rightarrow$ convert to frequency $< \frac{f_s}{2}$

D-to-C conversion: Practice

How interpolate continuous-time function through samples $x[n]$?



In practice: D-to-A convertor (= **approximation**)

Ideal

D-to-C conversion: Interpolation procedure

- Use values $x[n]$ which are sampled @ $f_s = 1/T_s$ [Hz]
- Generate each $n \cdot T_s$ a pulse $p(t - nT_s)$ with amplitude $x[n]$
- Add, possible overlapping, shifted pulses:

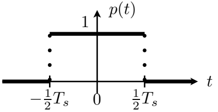
$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n]p(t - nT_s)$$

- Pulse $p(t)$ has characteristic shape for each D-to-C convertor.

D-to-C conversion: Examples pulse shapes

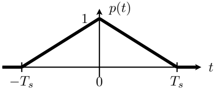
Zero Order Hold (ZOH) Interpolator

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$$



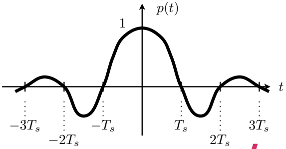
Linear Interpolation

$$p(t) = \begin{cases} 1 - \frac{|t|}{T_s} & -T_s < t \leq T_s \\ 0 & \text{otherwise} \end{cases}$$



Ideal Interpolation

$$p(t) = \frac{\sin(\frac{\pi}{T_s}t)}{\frac{\pi}{T_s}t} \quad -\infty < t < \infty$$



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D-to-C conversion: Reconstruction animations

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Summary Module SAA

- Sampling sinusoid: With $x(t) = A\cos(\omega t + \phi)$ and $T_s = 1/f_s$
  $x[n] = x(t)|_{t=n \cdot T_s} = A\cos((\omega \cdot T_s) \cdot n + \phi) = A\cos(\theta \cdot n + \phi)$
  - Absolute frequency: $\omega = 2\pi f$ [rad/sec] (f [Hz])
  - Relative frequency: $\theta = \omega \cdot T_s = 2\pi(f/f_s)$ [rad] (dimensionless)
- Aliasing frequencies $\cos(\theta_0 n + \phi)$ with relative frequency θ_0:
  $\Rightarrow (\theta_0 + 2\pi l)n + \phi$ or $(2\pi l - \theta_0)n - \phi$ with integer l
- Spectrum discrete-time signal: Fundamental Interval (FI)
  FI $= \{-\pi, \pi\} \simeq \{-\frac{f_s}{2}, \frac{f_s}{2}\}$, periodic with ∞ components
- Ideal reconstruction sinusoidal signal: Each discrete-time frequency θ mapped via $\omega = \theta \cdot f_s$ to continuous-time frequency. Take care of alias term by selecting one period $\Rightarrow$ Choose one FI
- Sampling theorem:
  Continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(t)|_{t=n \cdot T_s}$, if samples are taken at a rate $f_s = 1/T_s$, that is greater than $2f_{max}$