Frequency Response of FIR filter

PRELIMINARY VERSION

P. Sommen and B. van Erp

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Contents

1 Frequency response FIR

Besides the impulse response, denoted by $h[n]$, and the Difference Equation (DE) we will see that the frequency response is an alternative description of an FIR filter. Obviously all these different descriptions are related to each other and each of the descriptions has its own advantage. The advantage of the frequency response description is that we can fairly easily see the response of the FIR to frequencies.

1.1 Response FIR to phasor with single frequency

Since each single frequency can be viewed as the sum of two phasor components, we will first study the response of a very simple FIR filter, a delay, as a result of an input signal $x[n]$ which consists of one phasor frequency θ_1, amplitude A and phase ϕ, thus $x[n] = A\mathrm{e}^{\mathrm{j}(\theta_1 n+\phi)}$. This is depicted in Fig. 1. The output can be calculated as follows:

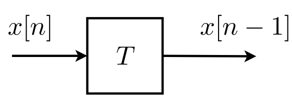

Figure 1: Response of a delay to signal $x[n]$.

$$y[n] = x[n-1] = A\mathrm{e}^{\mathrm{j}(\theta_1(n-1)+\phi)} = A\mathrm{e}^{\mathrm{j}(\theta_1 n+\phi-\theta_1)} = \mathrm{e}^{-\mathrm{j}\theta} \cdot A\mathrm{e}^{\mathrm{j}(\theta_1 n+\phi)}$$

From this result it follows that when applying a single phasor with frequency θ_1 to a delay, than the output $y[n]$ contains exact the same phasor (denoted in red) with the same frequency θ_1. Only the phase of the phasor has changed from ϕ to $\phi-\theta_1$.

Example:

Given an FIR filter which consists of 4 delays, or equivalently, the impulse response equals $h[n] = \delta[n-4]$. Calculate the response of this filter when the input is a phasor with frequency $\theta_1 = \frac{\pi}{8}$: $x[n] = 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n+\frac{\pi}{3})}$.
Solution:
The output of this FIR filter with 4 delays is as follows:

$$\begin{aligned} y[n] &= x[n-4] = 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}(n-4)+\frac{\pi}{3})} = 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n-4\cdot\frac{\pi}{8}+\frac{\pi}{3})} \\ &= \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}} \cdot 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n+\frac{\pi}{3})} = 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n-\frac{\pi}{6})}. \end{aligned}$$

Thus a system which contains 4 delays does only change the phase of the input phasor (from $+\frac{\pi}{3}$ into $-\frac{\pi}{6}$).

$\square$

Example:

Calculate the response of a simple FIR filter, as depicted in Fig. 2, when the input is the following phasor with frequency $\theta_1 = \frac{\pi}{8}$: $x[n] = 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n+\frac{\pi}{3})}$.

Solution:

The output of this FIR filter is as follows:

$$\begin{aligned} y[n] &= x[n] + x[n-4] = 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n+\frac{\pi}{3})} + 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}(n-4)+\frac{\pi}{3})} \\ &= (1 + \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}) \cdot 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n+\frac{\pi}{3})} = \sqrt{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}} \cdot 2\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n+\frac{\pi}{3})} = 2\sqrt{2}\mathrm{e}^{\mathrm{j}(\frac{\pi}{8}n+\frac{\pi}{12})}. \end{aligned}$$

So the phasor at the output has the same frequency ($\frac{\pi}{8}$) as the input, only the amplitude has changed from 2 into $2\sqrt{2}$ and the phase has changed from $+\frac{\pi}{3}$ to $+\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$.

$\square$

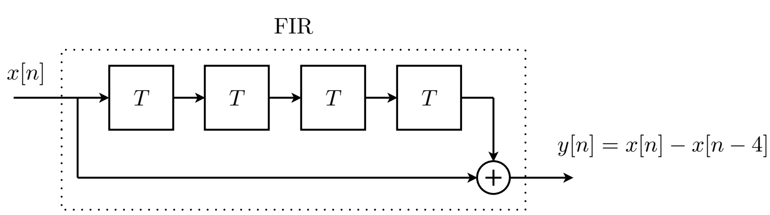

Figure 2: Simple FIR filter.

The previous results can by generalizrd by using the following equation which describes the convolution sum of an FIR:

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k],$$

in which $h[n]$, for $n = 0, 1, \cdots, M-1$ are the impulse response coefficients of the FIR filter. From this convolution sum we can calculate the following response of this system to the phasor $x[n] = Ae^{j(\theta_1 n+\phi)}$:

$$y[n] = \sum_{k=0}^{M-1} h[k]Ae^{j(\theta_1(n-k)+\phi)} = \left(\sum_{k=0}^{M-1} h[k]e^{-jk\theta_1}\right) \cdot \left(Ae^{j(\theta_1 n+\phi)}\right).$$

Thus we can split the result as a product of a complex quantity (in blue) multiplied with the phasor description of the input signal (in red). The complex quantity (in blue) only depends on the FIR impulse response $h[n]$ and the phasor frequency θ_1 of the input signal. Because of the fact that we can use any frequency for the frequency θ_1, we can generalize this complex quantity as a function of θ which leads to the following definition: [1]

Frequency Response of FIR: $$\boxed{H(e^{j\theta}) = \sum_{k=0}^{M-1} h[k]e^{-j\theta k}} \tag{1}$$

The frequency response $H(e^{j\theta})$ of an FIR describes how a phasor frequency of an input signal is changed in amplitude and phase by the FIR filter. E.g. for an input signal with phasor frequency θ_1 the change in amplitude is a result of the magnitude of the frequency response evaluated at frequency $\theta = \theta_1$, denoted by $|H(e^{j\theta_1})|$. Furthermore the change in phase is a result of the phase of the frequency response evaluated at frequency $\theta = \theta_1$, denoted by $\angle H(e^{j\theta_1})$. This is depicted in Fig. 3.

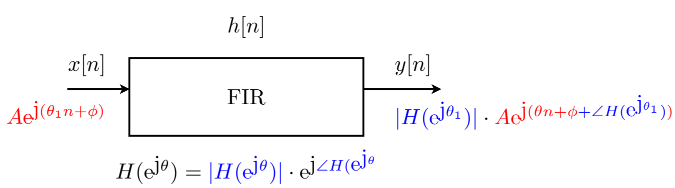

Figure 3: Response of an FIR to a phasor input signal with frequency θ_1.

[1] We denote the (discrete) impulse response with a small letter h, square brackets $[\cdot]$ and it is a function of discrete time function variable n. The frequency response is denoted by a capital letter H, round brackets $(\cdot)$ and it is a function of the continuous frequency variable θ. In fact we could denote the frequency response as $H(\theta)$. The reason that it is denoted as $H(e^{j\theta})$ will be explained in one of the following sections.

Example:

First calculate the Frequency response $H(e^{j\theta})$ of the FIR filter with impulse response $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$ and then evaluate the response of this FIR filter to the phasor frequency $x[n] = e^{j(\frac{\pi}{3}n+\frac{\pi}{4})}$.
Solution:
With the given impulse response $h[n]$ we obtain from definition (1) the following expression for the frequency response:

$$H(e^{j\theta}) = 1 \cdot e^{-j0\cdot\theta} + 2 \cdot e^{-j1\cdot\theta} + 1 \cdot e^{-j2\cdot\theta} = 1 + 2e^{-j\theta} + e^{-j2\theta}. \qquad (2)$$

With the input signal $x[n] = e^{j(\frac{\pi}{3}n+\frac{\pi}{4})}$ the resulting output signal can be found as follows:

$$\begin{aligned} y[n] &= x[n] + 2x[n-1] + x[n-2] \\ &= e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} + 2e^{j(\frac{\pi}{3}(n-1)+\frac{\pi}{4})} + e^{j(\frac{\pi}{3}(n-2)+\frac{\pi}{4})} \\ &= \left(e^{-j0\cdot\frac{\pi}{3}} + 2e^{-j1\cdot\frac{\pi}{3}} + e^{-j2\cdot\frac{\pi}{3}}\right) \cdot e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} \end{aligned}$$

Alternatively, we can obtain the first part (in blue) of this last equation by evaluation the general description of the frequency response $H(e^{j\theta})$, as derived in equation (2), for frequency $\theta = \frac{\pi}{3}$. This will be denoted by $H(e^{j\theta})|_{\theta=\frac{\pi}{3}}$.

Thus for an input signal $x[n] = e^{j(\frac{\pi}{3}n+\frac{\pi}{4})}$, which consists of a single phasor frequency $\theta = \frac{\pi}{3}$, with amplitude $A = 1$ and phase $\phi = \frac{\pi}{4}$, we can alternatively obtain the output $y[n]$ of this FIR filter as follows:

$$\begin{aligned} y[n] &= \left(H(e^{j\theta})|_{\theta=\frac{\pi}{3}}\right) \cdot e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} \\ &= \left(e^{-j0\cdot\frac{\pi}{3}} + 2e^{-j1\cdot\frac{\pi}{3}} + e^{-j2\cdot\frac{\pi}{3}}\right) \cdot e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} \\ &= \left(3e^{-j\frac{\pi}{3}}\right) \cdot e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} = 3e^{j(\frac{\pi}{3}n-\frac{\pi}{12})}. \end{aligned}$$

Thus the output signal $y[n]$ is also a phasor with the same frequency $\theta = \frac{\pi}{3}$ as the input. However the amplitude and phase of the input signal have been changed in a way which can be obtained by evaluating the frequency response $H(e^{j\theta})$ for the frequency $\theta = \frac{\pi}{3}$ of the input signal.

□

1.2 Response FIR to sinusoidal signal with single frequency

By using the result of the previous subsection we can now find into a few steps, which are depicted in Fig. 4, the response of an FIR filter to a sinusoidal signal with a single frequency $x[n] = A\cos(\theta_1 n + \phi)$. The first step

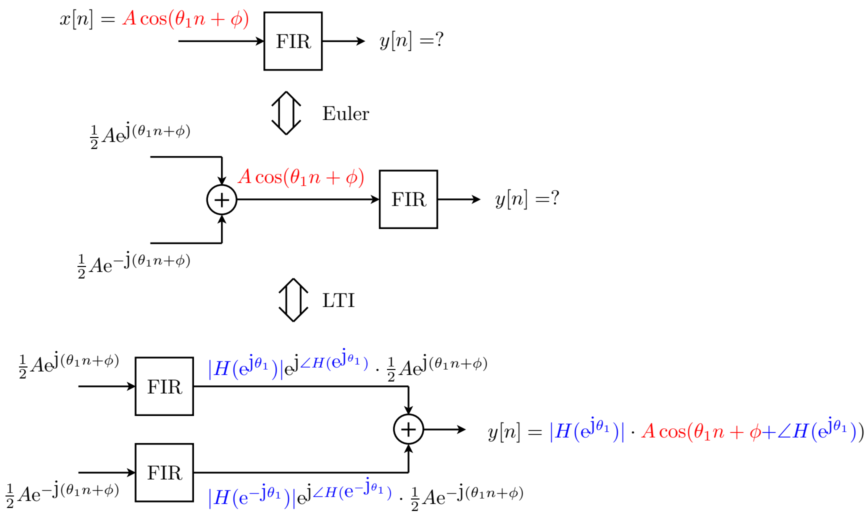

Figure 4: Response FIR to sinusoidal signal with single frequency.

is to split the sinusoidal signal by using Euler, into two phasor components. Then we can use the Linear Time Invariant (LTI) property of an FIR. This step implies that when applying the sum of 2 signals to an FIR filter is the same as first applying the individual signals to the same FIR filters and than add the results. This is depicted in the lower part of Fig. 4. In the upper branch of this figure the FIR filter is 'triggered' by a phasor with frequency $+\theta_1$, while in the lower branch a copy of the same FIR filter is 'triggered' by a phasor with frequency $-\theta_1$. As explained in the previous subsection, the change in amplitude and phase of both upper- and lower branch input signals follows from the magnitude $|H(\mathrm{e}^{\mathrm{j}\theta})|$ and phase $\angle H(\mathrm{e}^{\mathrm{j}\theta})$ of the frequency response evaluated at frequency θ_1 and $-\theta_1$ respectively. When adding the results of upper- and lower-branch we obtain at the output of the FIR filter a sinusoidal signal, from which the frequency is the same as the frequency of the input signal. The amplitude A of the input signal has been changed by the magnitude $|H(\mathrm{e}^{\mathrm{j}\theta_1})|$ of the frequency response of the FIR, while the phase ϕ of the input signal has been changed by the phase $\angle H(\mathrm{e}^{\mathrm{j}\theta_1})$ of the frequency response. This is depicted in Fig. 5 Concluding we have the following general result:

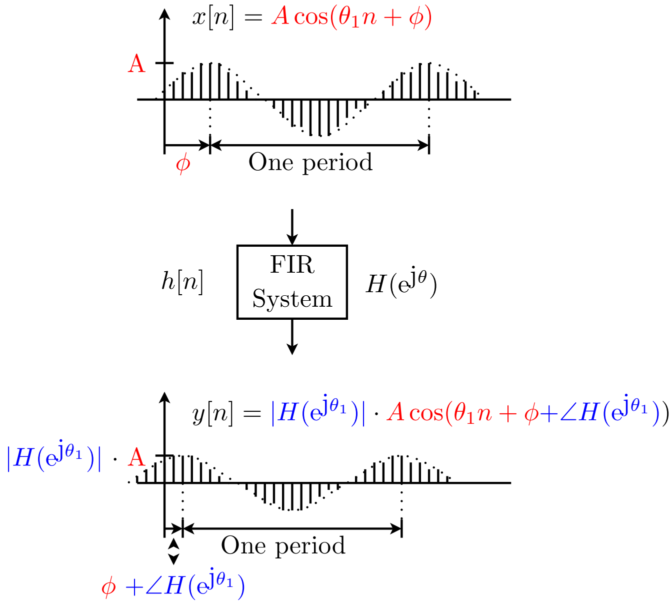

Figure 5: Response of a sinusoidal signal to FIR.

When applying a sinusoidal signal with a single frequency $\theta = \theta_1$ to an FIR filter then the output is a sinusoidal signal with the same frequency θ_1 as the input signal, only the amplitude and phase have changed. The change in amplitude is completely described by the magnitude response evaluated at frequency θ_1, thus $|H(e^{j\theta})|_{\theta=\theta_1}$. The change in phase is completely described by the phase response evaluated at frequency θ_1, thus $\angle H(e^{j\theta})|_{\theta=\theta_1}$.

Example:

In this example we use the same FIR filter as in the previous example. Thus $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$ and from this it follows that the frequency response is given by $H(e^{j\theta}) = 1 + 2e^{-j\theta} + e^{-j2\theta}$. Now the input signal is given by the following sinusoidal signal: $x[n] = \cos(\frac{\pi}{3}n)$. Calculate the output signal $y[n]$.
Solution:
Now we can use the superposition property which implies that we can

evaluate the output $y[n]$ to the input signal $x[n] = \cos(\frac{\pi}{3}n)$ by first splitting the cosine into its two phasor components $x_1[n] = e^{j\frac{\pi}{3}n}$ and $x_2[n] = e^{-j\frac{\pi}{3}n}$. Then evaluate the output of each of these two individual phasor components which results into:

$$\begin{aligned} x_1[n] = e^{j\frac{\pi}{3}n} \quad &\mapsto \quad y_1[n] = H(e^{j\frac{\pi}{3}}) \cdot e^{j\frac{\pi}{3}n} = 3e^{-j\frac{\pi}{3}} \cdot e^{j\frac{\pi}{3}n} = 3e^{j(\frac{\pi}{3}n-\frac{\pi}{3})} \\ x_2[n] = e^{-j\frac{\pi}{3}n} \quad &\mapsto \quad y_2[n] = H(e^{-j\frac{\pi}{3}}) \cdot e^{-j\frac{\pi}{3}n} = 3e^{j\frac{\pi}{3}} \cdot e^{-j\frac{\pi}{3}n} = 3e^{-j(\frac{\pi}{3}n-\frac{\pi}{3})} \end{aligned}$$

By adding the two phasor components and dividing by 2 we obtain the given input signal:

$$x[n] = \frac{1}{2}\left\{x_1[n] + x_2[n]\right\} = \frac{1}{2}\left\{e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right\} = \cos(\frac{\pi}{3}n)$$

Now because of the fact that an FIR filter is LTI, we can evaluate the output $y[n]$ by applying the same operations to the two individual output signals $y_1[n]$ and $y_2[n]$, thus:

$$y[n] = \frac{1}{2}\left\{y_1[n] + y_2[n]\right\}$$

which results into:

$$\begin{aligned} x[n] = \frac{1}{2}\left\{e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right\} \quad &\mapsto \quad y[n] = \frac{1}{2}\left\{3e^{j(\frac{\pi}{3}n-\frac{\pi}{3})} + 3e^{-j(\frac{\pi}{3}n-\frac{\pi}{3})}\right\} \\ \Rightarrow \quad x[n] = \cos(\frac{\pi}{3}n) \quad &\mapsto \quad y[n] = 3 \cdot \cos(\frac{\pi}{3}n - \frac{\pi}{3}) \end{aligned}$$

We can also obtain this result directly from the frequency response by evaluating the amplitude and phase of the frequency response for the input frequency $\theta = \frac{\pi}{3}$, as follows:

$$\begin{aligned} x[n] = \cos(\frac{\pi}{3}n) \quad \mapsto \quad y[n] &= |H(e^{j\theta})|_{\theta=\frac{\pi}{3}} \cdot \cos\left(\frac{\pi}{3}n + \angle H(e^{j\theta})|_{\theta=\frac{\pi}{3}}\right) \\ &= 3 \cdot \cos(\frac{\pi}{3}n - \frac{\pi}{3}) \end{aligned}$$

The result is depicted in Fig. 6.

□

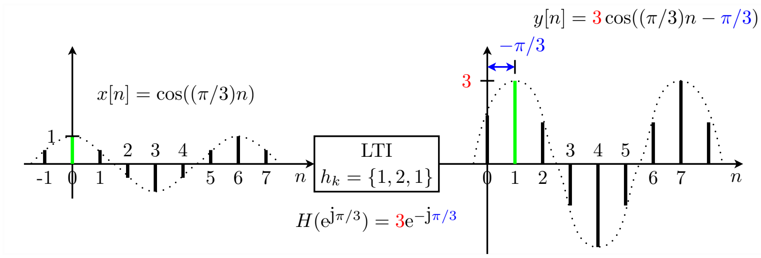

Figure 6: Example response of FIR to sinusoidal signal with one frequency.

2 Superposition

Because of the fact that an FIR filter is an Linear Time Invariant (LTI) system we can use the superposition rule when an input signal consists of more than one frequency. So let's assume the input signal of an FIR filter consists of a DC component and N different frequencies as follows:

$$x[n] = A_0 + \sum_{k=1}^{N} A_k \cos(\theta_k n + \phi_k).$$

Furthermore the frequency response of an FIR is given by:

$$H(\mathrm{e}^{\mathrm{j}\theta}) = |H(\mathrm{e}^{\mathrm{j}\theta})|\mathrm{e}^{\mathrm{j}\angle H(\mathrm{e}^{\mathrm{j}\theta})}$$

Each of the individual frequencies θ_k of the input signal will 'trigger' the FIR filter only at that particular frequency θ_k. This implies that the amplitude A_k of frequency θ_k is changed by $|H(\mathrm{e}^{\mathrm{j}\theta_k})|$ and the phase ϕ_k is changed by $\angle H(\mathrm{e}^{\mathrm{j}\theta_k})$. This results into the following output signal:

$$y[n] = |H(\mathrm{e}^{\mathrm{j}0})| + \sum_{k=1}^{N} |H(\mathrm{e}^{\mathrm{j}\theta_k})| A_k \cos(\theta_k n + \phi_k + \angle H(\mathrm{e}^{\mathrm{j}\theta_k}))$$

Example:

In this example we use again the same FIR filter as in the previous example. Thus $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$ and from this it follows that the frequency response is given by $H(\mathrm{e}^{\mathrm{j}\theta}) = 1 + 2\mathrm{e}^{-\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}2\theta}$. Now the input signal is given by the following frequencies:$x[n] = 1 +$

$\frac{4}{3}\cos(\frac{\pi}{3}n) + 2\cos(\frac{\pi}{2}n) + \cos(\pi n)$. Calculate the output signal $y[n]$.
Solution:
By using superposition we first calculate the response of the given FIR filter as a result of the individual frequency components of the input signal $x[n]$ and then we add the resulting outputs. With the individual frequency components $\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}$ and $\theta = \pi$, this results into:

$$\begin{aligned}
H(e^{j\theta})|_{\theta=0} &= 4 \\
H(e^{j\theta})|_{\theta=\frac{\pi}{3}} = H^*(e^{j\theta})|_{\theta=-\frac{\pi}{3}} &= 3e^{-j\frac{\pi}{3}} \\
H(e^{j\theta})|_{\theta=\frac{\pi}{2}} = H^*(e^{j\theta})|_{\theta=-\frac{\pi}{2}} &= 2e^{-j\frac{\pi}{2}} \\
H(e^{j\theta})|_{\theta=\pi} = H^*(e^{j\theta})|_{\theta=-\pi} &= 0.
\end{aligned}$$

Now the output becomes:

$$\begin{aligned}
y[n] &= 4 \cdot 1 + 3 \cdot \frac{4}{3}\cos(\frac{\pi}{3}n - \frac{\pi}{3}) + 2 \cdot \cos(\frac{\pi}{2}n - \frac{\pi}{2}) + 0 \cdot \cos(\pi n) \\
&= 4 + 4\cos(\frac{\pi}{3}n - \frac{\pi}{3}) + 4\cos(\frac{\pi}{2}n - \frac{\pi}{2}).
\end{aligned}$$

□

3 Properties Frequency Response

From the previous section we have found the following relation between the impulse response description in time domain $h[n]$ and the frequency response description in frequency domain $H(e^{j\theta})$:

$$\begin{array}{ccc}
\text{Time domain} & \circ\!-\!\circ & \text{Frequency domain} \\
h[n] = \sum_{k=0}^{M-1} h[k]\delta[n-k] & \circ\!-\!\circ & H(e^{j\theta}) = \sum_{k=0}^{M-1} h[k]e^{-j\theta k}
\end{array}$$

The first important property of the Frequency response $H(e^{j\theta})$ is that it is a periodic function with period 2π. This can be verified by adding an integer number l times 2π to the frequency θ, which result into:

$$H(e^{j\theta}) = \sum_{k=0}^{M-1} h[k]e^{-j(\theta k + l\cdot 2\pi)} = \sum_{k=0}^{M-1} h[k]e^{-j\theta k} \cdot e^{-jl\cdot 2\pi} = \sum_{k=0}^{M-1} h[k]e^{-j\theta k} \cdot 1.$$

The fact that the frequency response is a periodic function of θ, with period 2π, is the main reason that we do not denote this function as $H(\theta)$, but in a special way, namely by $H(\mathrm{e}^{\mathrm{j}\theta})$.

The second important property is that the Frequency response of a real impulse response $h[n]$ is a conjugate symmetric function, thus:

$$H(\mathrm{e}^{-\mathrm{j}\theta}) = H^*(\mathrm{e}^{\mathrm{j}\theta}).$$

A proof of this property goes as follows:
By using the fact that the impulse response $h[n]$ is real, thus the complex conjugate values of the impulse response coefficients are the same as the impulse response values themselves ($h[n] = h^*[n]$), we obtain:

$$H(\mathrm{e}^{-\mathrm{j}\theta}) = \sum_{k=0}^{M-1} h[k]\mathrm{e}^{\mathrm{j}\theta k} = \sum_{k=0}^{M-1} h[k](\mathrm{e}^{-\mathrm{j}\theta k})^* = (\sum_{k=0}^{M-1} h[k]\mathrm{e}^{\mathrm{j}\theta k})^* = H^*(\mathrm{e}^{\mathrm{j}\theta}).$$

Concluding the two most important properties of the frequency response of an FIR filter with real coefficients are:

Periodic	:	$H(\mathrm{e}^{\mathrm{j}\theta}) = H(\mathrm{e}^{\mathrm{j}(\theta+l\cdot 2\pi})$
Complex conjugated	:	$H(\mathrm{e}^{-\mathrm{j}\theta}) = (H(\mathrm{e}^{\mathrm{j}\theta}))^*$

Example:

Calculate the frequency response $H(\mathrm{e}^{\mathrm{j}\theta})$ of the FIR filter as depicted in Fig. 7 and show the two main basic properties as described above.
Solution:
From the signal flow graph (or realization scheme) of the FIR it follows that the impulse response is given by:

$$h[n] = -\delta[n] + 3\delta[n-1] - \delta[n-2]$$

and from this equation we obtain the following expression for the frequency response:

$$H(\mathrm{e}^{\mathrm{j}\theta} = -1\mathrm{e}^{-\mathrm{j}0\theta} + 3\mathrm{e}^{-\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}2\theta} = -1 + 3\mathrm{e}^{-\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}2\theta}.$$

The periodicity of this function can be showing by adding for example 2π to the frequency θ. Doing so results into:

$$\begin{aligned} H(\mathrm{e}^{\mathrm{j}(\theta+2\pi)} &= -1\mathrm{e}^{-\mathrm{j}0\theta}\cdot\mathrm{e}^{-\mathrm{j}2\pi} + 3\mathrm{e}^{-\mathrm{j}\theta}\cdot\mathrm{e}^{-\mathrm{j}2\pi} - \mathrm{e}^{-\mathrm{j}2\theta}\cdot\mathrm{e}^{-\mathrm{j}2\pi} \\ &= -1\mathrm{e}^{-\mathrm{j}0\theta} + 3\mathrm{e}^{-\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}2\theta} = -1 + 3\mathrm{e}^{-\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}2\theta} = H(\mathrm{e}^{\mathrm{j}\theta}). \end{aligned}$$

The complex conjugate property can be shown as follows:

$$\begin{aligned}H(\mathrm{e}^{-\mathrm{j}\theta}) &= -1+3\mathrm{e}^{\mathrm{j}\theta}-\mathrm{e}^{\mathrm{j}2\theta} = -1+3(\mathrm{e}^{-\mathrm{j}\theta})^{*}-(\mathrm{e}^{-\mathrm{j}2\theta})^{*}\\ &= (-1)^{*}+(3\mathrm{e}^{-\mathrm{j}\theta})^{*}+(-\mathrm{e}^{-\mathrm{j}2\theta})^{*}\\ &= (-1+3\mathrm{e}^{-\mathrm{j}\theta}-\mathrm{e}^{-\mathrm{j}2\theta})^{*} = H^{*}(\mathrm{e}^{\mathrm{j}\theta})\end{aligned}$$

□

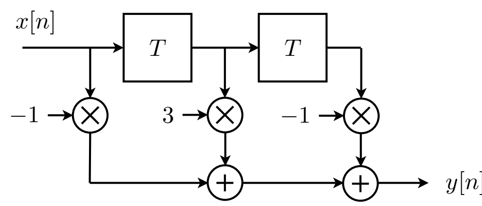

Figure 7: Example to show basic properties of Frequency response.

4 Graphical Representation

In the module Complex Numbers And Phasors (CNAP) we have seen that a complex number can be represented as a vector in the complex plain. Such a complex vector can be represented either in Cartesian way or in the Polar way. In order to represent the frequency response $H(\mathrm{e}^{\mathrm{j}\theta})$ we often choose the Polar way of representing this complex quantity. For this reason we write the frequency response as:

$$H(\mathrm{e}^{\mathrm{j}\theta}) = |H(\mathrm{e}^{\mathrm{j}\theta})| \cdot \mathrm{e}^{\mathrm{j}\angle H(\mathrm{e}^{\mathrm{j}\theta})}. \tag{3}$$

In order to represent the frequency response in a graphical way, we can make two plots: One plot representing the magnitude response $|H(\mathrm{e}^{\mathrm{j}\theta})|$ and the other plot representing the phase response $\angle H(\mathrm{e}^{\mathrm{j}\theta})$, both as a function of frequency θ.

In most signal processing software, like Matlab, the plot of the phase response is restricted to the range $-\pi \leq \angle H(\mathrm{e}^{\mathrm{j}\theta}) \leq \pi$. This is allowed because we may add or subtract 2π to the phase without changing the result, thus:

$$\mathrm{e}^{\mathrm{j}\angle H(\mathrm{e}^{\mathrm{j}\theta})\pm 2\pi} = \mathrm{e}^{\mathrm{j}\angle H(\mathrm{e}^{\mathrm{j}\theta})} \cdot \mathrm{e}^{\mathrm{j}\pm 2\pi} = \mathrm{e}^{\mathrm{j}\angle H(\mathrm{e}^{\mathrm{j}\theta})}.$$

Example:

Calculate the frequency response of a system which consists of two delays. Make a plot of both the magnitude- and phase-response as a function of relative frequency θ inside the Fundamental Interval (FI), thus with $|\theta| \leq \pi$.

Solution:

The impulse response of this system is given by $h[n] = \delta[n-2]$ and from this it follows that the frequency response equals $H(e^{j\theta}) = 1 \cdot e^{j(-2\theta)}$. Thus the magnitude response is equal to $|H(e^{j\theta})| = 1$ and the phase response is equal to $\angle H(e^{j\theta}) = -2\theta$. The graphical representation of the magnitude- and phase-response is depicted in Fig. 8. In the lower left hand side figure we can see that, within the FI, the plot of the phase response of this system is in the range: $-2\pi \leq e^{j\angle H(e^{j\theta})} \leq 2\pi$. However the phase does not change if we add or subtract 2π to it, since $e^{j2\pi} = e^{-j2\pi} = 1$. Thus in the graphical representation of the phase response we can subtract 2π in case the phase response becomes larger than 2π. On the other hand we can add 2π in case the value becomes smaller than 2π. The lower right hand plot depicts the phase response within these boundaries $|\angle H(e^{j\theta})| \leq \pi$. This is usually done in Matlab.

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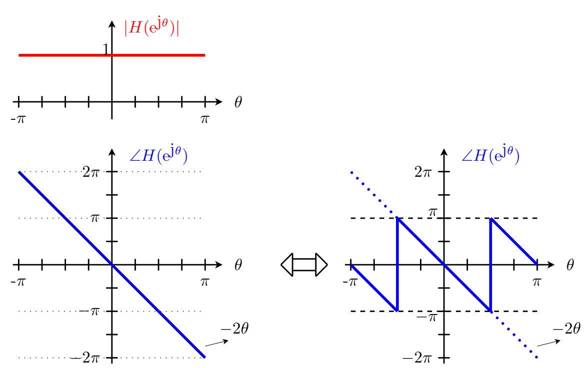

Figure 8: Magnitude- and phase-response of a system with 2 delays.

Furthermore the magnitude response $|H(e^{j\theta})|$ is a positive function of θ. However it may happen that by evaluation of the frequency response we obtain an expression which looks like:

$$H(e^{j\theta}) = A(e^{j\theta}) \cdot e^{j\angle H(e^{j\theta})},$$

in which $A(e^{j\theta})$ can become negative for some values of θ. When representing the magnitude response $|H(e^{j\theta})|$, which is the absolute value of $A(e^{j\theta})$, we need to compensate for the fact that we plot a negative value as a positive value. So for frequencies for which $A(e^{j\theta})$ is negative we obtain the magnitude response plot $|H(e^{j\theta})|$ by multiplying $A(e^{j\theta})$ by -1. This is equal to a phase correction of $\pm\pi$, since $-1 = e^{j\pi} = e^{-j\pi}$. Thus, because of the fact that we plot the magnitude response $|H(e^{j\theta})|$ we will see phase jumps of $\pm\pi$ in the phase response plot at those positions where we have zero crossings in the magnitude response. Concluding we have the following rules for the graphical representation of the magnitude- and phase-response plots of the frequency response:

- *Both Magnitude- and phase- response are depicted in the Fundamental Interval (FI): $|\theta| \leq \pi$.*
- *The phase response is depicted in the range $|\angle H(e^{j\theta})| \leq \pi$.*
- *A phase jump of $\pm\pi$ is depicted in the phase response plot for each zero crossing of the magnitude response.*

Example:

Calculate the frequency response $H(e^{j\theta})$ of a first order difference system, from which the Difference Equation is given by: $y[n] = x[n] - x[n-1]$, and make a graphical plot for θ in the Fundamental Interval ($|\theta| \leq \pi$)of the magnitude response $|H(e^{j\theta})|$ and phase response $\angle H(e^{j\theta})$, with $|\angle H(e^{j\theta})| \leq \pi$.

<u>*Solution:*</u>

The frequency response of this first order difference system is given by: $H(e^{j\theta}) = 1 - e^{-j\theta}$. In order to give the graphical representation of the magnitude- and phase-response we can rewrite the frequency response as follows:

$$\begin{aligned} H(e^{j\theta}) &= \left(e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}\right) \cdot e^{-j\frac{\theta}{2}} = 2\sin(\frac{\theta}{2}) \cdot j \cdot e^{-j\frac{\theta}{2}} \\ &= 2\sin(\frac{\theta}{2}) \cdot e^{j(\frac{\pi}{2}-\frac{\theta}{2})} = |H(e^{j\theta})| \cdot e^{j\angle H(e^{j\theta})}. \end{aligned}$$

Because of the fact that the function $2\sin(\frac{\theta}{2})$ becomes negative in the frequency range $-\pi \leq \theta < 0$, we have to compensate in this range the phase with $-\pi$. Thus the magnitude- and phase- response of this first order difference system are given by:

$$|H(e^{j\theta})| = \left|2\sin(\frac{\theta}{2})\right| \;\; ; \angle H(e^{j\theta}) = \begin{cases} \frac{\pi}{2} - \frac{\theta}{2} & \text{for } 0 \leq \theta \leq \pi \\ -\frac{\pi}{2} - \frac{\theta}{2} & \text{for } -\pi \leq \theta < 0 \end{cases}$$

These plots are depicted in Fig. 9 and we can see a phase jump of π at the zero crossing for frequency $\theta = 0$.

□

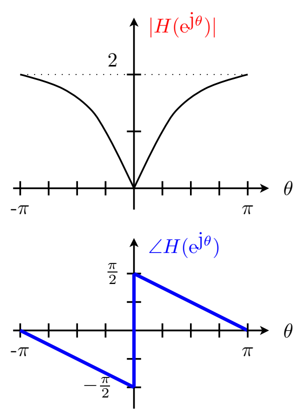

Figure 9: Magnitude- and phase-response of first order difference system.

4.1 Running average filter

A running average filter is used to evaluate the average value of succeeding input signal samples. By averaging we typically expect to attenuate higher frequencies. This effect can be shown by evaluating and making plots of the frequency response of such an running average filter. This goes as follows: The Difference Equation (DE) of an L-point running average filter is given by:

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

From this we obtain the frequency response[2]:

$$\begin{aligned}
H(\mathrm{e}^{\mathrm{j}\theta}) &= \frac{1}{L}\sum_{k=0}^{L-1}\mathrm{e}^{-\mathrm{j}k\theta} = \frac{1}{L}\frac{1-\mathrm{e}^{-\mathrm{j}L\theta}}{1-\mathrm{e}^{-\mathrm{j}\theta}} = \frac{1}{L}\frac{\mathrm{e}^{\mathrm{j}L\frac{\theta}{2}}-\mathrm{e}^{-\mathrm{j}L\frac{\theta}{2}}}{\mathrm{e}^{\mathrm{j}\frac{\theta}{2}}-\mathrm{e}^{-\mathrm{j}\frac{\theta}{2}}}\cdot\frac{\mathrm{e}^{-\mathrm{j}L\frac{\theta}{2}}}{\mathrm{e}^{-\mathrm{j}\frac{\theta}{2}}} \\
&= \frac{\sin(L\frac{\theta}{2})}{L\sin(\frac{\theta}{2})}\cdot\mathrm{e}^{-\mathrm{j}(L-1)\frac{\theta}{2}}
\end{aligned}$$

Thus the magnitude- and phase-response of a running average filter are given by the following expressions:

$$\begin{aligned}
|H(\mathrm{e}^{\mathrm{j}\theta})| &= \left|\frac{\sin(L\frac{\theta}{2})}{L\sin(\frac{\theta}{2})}\right| \\
\angle H(\mathrm{e}^{\mathrm{j}\theta}) &= -(L-1)\frac{\theta}{2}.
\end{aligned}$$

An important value of the magnitude-response is the value for $\theta = 0$. This value can be found by approximate the sin-functions by the first component of its geometric series.[3]

$$\frac{\sin(L\frac{\theta}{2})}{L\sin(\frac{\theta}{2})}\Big|_{\theta=0} = \lim_{\theta\to 0}\left\{\frac{\sin(L\theta)}{L\sin(\theta)}\right\} = 1.$$

The other important values of the phase-response are the zero crossings, because at the frequencies we will find phase jumps of $\pm\pi$ in the phase-response plot. These zero crossing can be found for those frequencies for which $|H(\mathrm{e}^{\mathrm{j}\theta})|$ becomes zero. Because of the fact that the frequency of the numerator of $|H(\mathrm{e}^{\mathrm{j}\theta})|$ is L times faster than the frequency of the denominator, we can find these zero crossings by evaluating those frequencies for

[2]Use the geometric series $\sum_{k=0}^{L-1} a^k = \frac{1-a^L}{1-a}$

[3]For small values of x the value of $\sin(x)$ can be approximated by its first term of the geometric series: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$. Thus for small values of x we can approximate the fraction $\frac{\sin(Lx)}{Lsin(x)}$ by $\frac{Lx}{Lx} = 1$.

which the numerator becomes zero. Besides $\theta = 0$ these values can be found as follows:

$$\sin(L\frac{\theta}{2}) = 0 \quad \Leftrightarrow \quad L\frac{\theta}{2} = k\pi \;\Rightarrow\; \theta = \frac{2\pi}{L}.$$

These results are depicted in Fig. 10, which is the graphical representation of the magnitude- and phase-response of an 11-point averaging filter. From

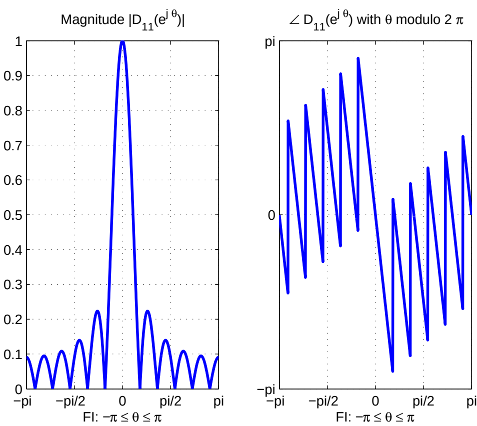

Figure 10: Magnitude- and phase-response of 11-point avering filter.

the left-hand figure we clearly can see that indeed this 11-point averaging filter attenuates higher frequencies. Furthermore we see in this plot 10 zero crossings at locations $\theta = k \cdot \frac{2\pi}{11}$, for $k = \pm 1, \pm 2, \cdots \pm 5$. In the right hand side phase-response plot we see that the slope of the plot is $-(L-1)\frac{\theta}{2} = -5\theta$. Furthermore we see in this plot at each of the 10 zero crossings of the magnitude-response a phase jump of $\pm\pi$.

5 Cascading FIR systems

In the previous module FIR we have seen that we can combine the impulse responses $h_1[n]$ and $h_2[n]$ of two cascaded FIR filters to one impulse response $h[n]$ which can be obtained as the convolution sum of the two individual impulse responses. Thus $h[n] = h_1[n] \star h_2[n]$.

Now let's see how this works if we use the frequency responses $H_1(e^{j\theta})$ and $H_2(e^{j\theta})$ of the two FIR filters. This is depicted in Fig. 11. When applying a phasor with frequency $e^{j\theta n}$ to the first filter, the resulting output signal can be described by $H_1(e^{j\theta_1}) \cdot e^{j\theta n}$. This is a phasor with frequency

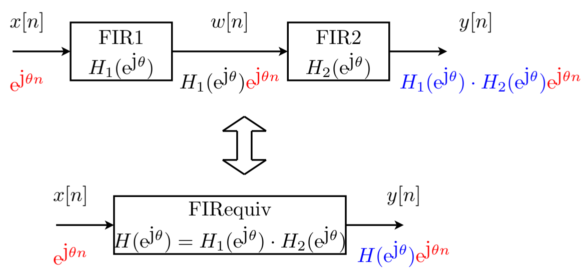

Figure 11: Cascading of two FIR filters.

θ_1 and the magnitude and phase of this phasor are given by the complex number $H_1(e^{j\theta_1})$. Now we apply this phasor to the second filter, which results into $H_1(e^{j\theta_1}) \cdot H_2(e^{j\theta}) \cdot e^{j\theta n}$. In other words the phasor $e^{j\theta n}$ of the input signal results is changed in amplitude and phase by the complex value $H_1(e^{j\theta_1}) \cdot H_2(e^{j\theta})$. From this we can conclude the following:

When cascading two different FIR filter, one with frequency response $H_1(e^{j\theta})$ and the other with frequency response $H_2(e^{j\theta})$, we may combine these two FIR filters to one FIR filter from which the frequency response is given by the product of the individual frequency responses: $H(e^{j\theta}) = H_1(e^{j\theta}) \cdot H_2(e^{j\theta})$.

Example:

The following two FIR filters are cascaded: $h_1[n] = \delta[n] + \delta[n+2]$ and $h_2[n] = \delta[n] - \delta[n+2]$. Calculate the impulse- and frequency- response when we cascade these two FIR filters.

<u>*Solution:*</u>

Via the impulse responses:

$$h[n] = h_1[n] \star h_2[n] = (\delta[n] + \delta[n-2]) \star (\delta[n] - \delta[n-2]) = \delta[n] - \delta[n-4]$$

Via frequency responses:

$$H(e^{j\theta}) = H_1(e^{j\theta}) \cdot H_2(e^{j\theta}) = \left(1 + e^{-j2\theta}\right) \cdot \left(1 - e^{-j2\theta}\right) = 1 - e^{-j4\theta}$$

□

In general we can conclude that a convolution in time domain is equivalent to multiplication in the frequency domain:

$$\boxed{h_1[n] * h_2[n] \quad \circ\!\!-\!\!\circ \quad H_1(\mathrm{e}^{\mathrm{j}\theta}) \cdot H_2(\mathrm{e}^{\mathrm{j}\theta})}$$

6 Steady State and transient behaviour

Until now we assumed the signals exist for all values of n in the range $-\infty < n < \infty$. In practice however this will not be the case. So a more realistic signal starts e.g. for $n = 0$, thus:

$$x[n] = A\mathrm{e}^{\mathrm{j}(\theta_1 n+\phi)} \cdot u[n] = \begin{cases} A\mathrm{e}^{\mathrm{j}(\theta_1 n+\phi)} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{cases}$$

We can use the convolution sum to find an expression for the output of an FIR filter with impulse response $h[n]$ to such a signal. This result into:

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k] = \sum_{k=0}^{M-1} h[k]A\mathrm{e}^{\mathrm{j}(\theta_1(n-k)+\phi)} \cdot u[n-k].$$

By using the fact that the unit step $u[n-k]$ is only 1 for indices $n-k \geq 0$, or equivalently for $k \leq n$, this equation can be rewritten as follows:

$$y[n] = \begin{cases} 0 & n < 0 & \\ \left(\sum_{k=0}^{n} h[k]\mathrm{e}^{-\mathrm{j}\theta_1 k}\right) \cdot A\mathrm{e}^{\mathrm{j}(\theta_1 n+\phi)} & 0 \leq n < M-1 & \textbf{Transient region} \\ \left(\sum_{k=0}^{M-1} h[k]\mathrm{e}^{-\mathrm{j}\theta_1 k}\right) \cdot A\mathrm{e}^{\mathrm{j}(\theta_1 n+\phi)} & n \geq M-1 & \textbf{Steady state region} \end{cases}$$

Concluding:

- **Transient region:** *The complex multiplier* $\left(\sum_{k=0}^{n} h[k]e^{-j\theta_1 k}\right)$ *depends on the index* n*.*
- **Steady state region:** *Complex multiplier* $\sum_{k=0}^{M-1} h[k]e^{-j\theta_1 k}$ *is constant. Output contains only input frequency (*θ_1 *in this example).*
- *If for* $n > M-1$ *the input changes to zero or to another frequency* $\theta_2 \neq \theta_1$*, then there will be a new transition- and steady state-region.*

Fig. 12 shows the transient-region (in red samples) and steady state-region when we apply the signal $x[n] = \cos(\frac{\pi}{3}n) \cdot u[n]$ to an FIR filter with impulse response $h[k] = \delta[n] + 2\delta[n-1] + \delta[n-2]$.

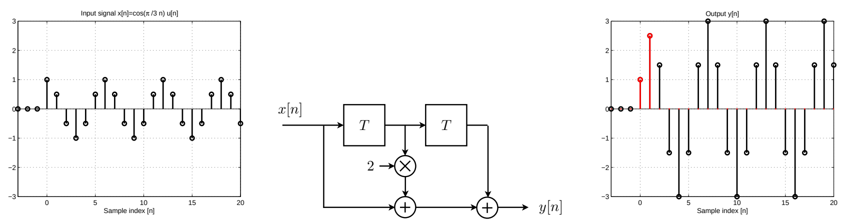

Figure 12: Example to show transient- and steady state-region.

7 Filtering continuous signal

In practice we often filter a continuous-time signal in the discrete-time domain as depicted in Fig. 13. The main reason is because of the fact that the implementation in discrete-time domain is much more flexible and cheaper compared to the implementation in the continuous-time domain. Of course

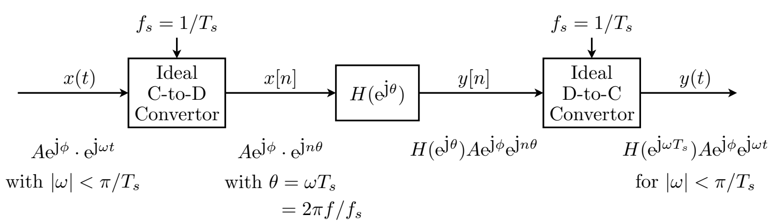

Figure 13: Filtering continuous-time signal in Discrete-time domain.

in practice we have to take care of the aliasing effect, as discussed in the module Sampling And Aliasing (SAA), so the sampling frequency of the C/D and D/C convertor are chosen in practice at least twice the largest frequency of the continuous-time signal $x(t)$.

Example:

Given the continuous-time signal $x(t) = 5 + 2\cos(2100\pi t + \frac{\pi}{4}) + 3\cos(2800\pi t - \frac{\pi}{6})$. We sample this signal with a C/D convertor which runs at a a sample rate $f_s = 1/T_s = 4200$ [samples/sec]. Then we filter the discrete-time signal samples $x[n]$ with an FIR filter from which the impulse response is given by $h[n] = \delta[n] + \delta[n-1]$. The discrete-time output signal samples $y[n]$ are converted by a D/C convertor to the continuous-time signal $y(t)$. The D/C convertor runs at the same sampling frequency $f_s 1/T_s = 4200$ [samples/sec] as the C/D convertor. Calculate the continuous-time signal $y(t)$.

<u>*Solution:*</u>

With a sample rate $f_s = 1/T_s = 4200$ [samples/sec] of the C/D convertor we obtain the following expression for the discrete-time input signal samples:

$$x[n] = 5 + 2\cos(\frac{\pi}{2}n + \frac{\pi}{4}) + 3\cos(\frac{2\pi}{3}n - \frac{\pi}{6}).$$

From the given impulse response it follows that the frequency response can be written as follows:

$$H(\mathrm{e}^{\mathrm{j}\theta}) = 1 + \mathrm{e}^{-\mathrm{j}\theta} = 2\cos(\frac{\theta}{2})\mathrm{e}^{-\mathrm{j}\frac{\theta}{2}}$$

The discrete-time input signal $x[n]$ consists of the frequencies: $\theta = 0, \frac{\pi}{2}$ and $\frac{2\pi}{3}$. So we have to evaluate the frequency response $H(\mathrm{e}^{\mathrm{j}\theta})$ for these frequencies in order to find the output signal $y[n]$. Thus:

$$\begin{aligned} H(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=0} &= 2 \\ H(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=\frac{\pi}{2}} = H^*(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=-\frac{\pi}{2}} &= \sqrt{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}} \\ H(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=\frac{2\pi}{3}} = H^*(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=-\frac{2\pi}{3}} &= \mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}. \end{aligned}$$

With these values the output samples $y[n]$ become as follows:

$$\begin{aligned} y[n] &= 5\cdot 2 + 2\cdot\sqrt{2}\cos(\frac{\pi}{2}n + \frac{\pi}{4} - \frac{\pi}{4}) + 3\cdot 1\cos(\frac{2\pi}{3}n - \frac{\pi}{6} - \frac{\pi}{3}) \\ &= 10 + 2\sqrt{2}\cos(\frac{\pi}{2}n) + 3\cos(\frac{2\pi}{3}n - \frac{\pi}{2}) \end{aligned}$$

With a sample rate $f_s = 1/T_s = 4200$ [samples/sec] of the D/C convertor the resulting continuous-time output signal becomes:

$$y(t) = 10 + 2\sqrt{2}\cos(2100\pi t) + 3\cos(2800\pi t - \frac{\pi}{2}).$$

□

8 Summary Frequency response FIR filter

- Relation impulse response and frequency response:

Time domain		Frequency domain
$h[n] = \sum_{k=0}^{M-1} h[k]\delta[n-k]$	○–○	$H(e^{j\theta}) = \sum_{k=0}^{M-1} h[k]e^{-j\theta k}$

- Basic FIR property:

When applying a sinusoidal signal with a single frequency $\theta = \theta_1$ to an FIR filter then the output is a sinusoidal signal with the same frequency θ_1 as the input signal, only the amplitude and phase have changed. The change in amplitude is completely described by the magnitude response evaluated at frequency θ_1, thus $|H(e^{j\theta})|_{\theta=\theta_1}$. The change in phase is completely described by the phase response evaluated at frequency θ_1, thus $\angle H(e^{j\theta})|_{\theta=\theta_1}$.

- When applying a sum of frequencies $x[n] = A_0 + \sum_{k=1}^{N} A_k \cos(\theta_k n + \phi_k)$ to an FIR filter with frequency response $H(e^{j\theta}) = |H(e^{j\theta})|e^{j\angle H(e^{j\theta})}$ then the output can be written as follows:

$$y[n] = |H(e^{j0})| + \sum_{k=1}^{N} |H(e^{j\theta_k})| A_k \cos(\theta_k n + \phi_k + \angle H(e^{j\theta_k}))$$

- Properties frequency response:

Periodic	:	$H(e^{j\theta}) = H(e^{j(\theta + l\cdot 2\pi)})$
Complex conjugated	:	$H(e^{-j\theta}) = (H(e^{j\theta}))^*$

- Graphical representation:

> - *Both Magnitude- and phase- response are depicted in the Fundamental Interval (FI):* $|\theta| \leq \pi$.
> - *The phase response is depicted in the range* $|\angle H(e^{j\theta})| \leq \pi$.
> - *A phase jump of* $\pm\pi$ *is depicted in the phase response plot for each zero crossing of the magnitude response.*

- Cascading FIR filters:

> *When cascading two different FIR filter, one with frequency response* $H_1(e^{j\theta})$ *and the other with frequency response* $H_2(e^{j\theta})$*, we may combine these two FIR filters to one FIR filter from which the frequency response is given by the product of the individual frequency responses:* $H(e^{j\theta}) = H_1(e^{j\theta}) \cdot H_2(e^{j\theta})$.

- Convolution in time- is equivalent to multiplication in frequency-domain:

$$\boxed{h_1[n] * h_2[n] \quad \circ\!\!-\!\!\circ \quad H_1(e^{j\theta}) \cdot H_2(e^{j\theta})}$$

- Steady state and transient behaviour:

> - **Transient region:** *The complex multiplier* $\left(\sum_{k=0}^{n} h[k]e^{-j\theta_1 k}\right)$ *depends on the index* n.
> - **Steady state region:** *Complex multiplier* $\sum_{k=0}^{M-1} h[k]e^{-j\theta_1 k}$ *is constant. Output contains only input frequency* (θ_1 *in this example*).
> - *If for* $n > M-1$ *the input changes to zero or to another frequency* $\theta_2 \neq \theta_1$*, then there will be a new transition- and steady state- region.*