Module FREQ

Frequency response of FIR filters

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Where innovation starts

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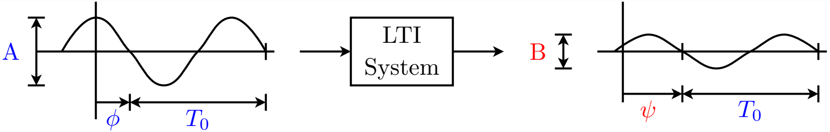
Basic LTI property



Frequency response of FIR filters

Example on blackboard:

$$e^{\pm j\frac{\pi}{3}n} \mapsto \text{FIR } \{1,2,1\}$$

General complex sinusoid input $x[n] = A \cdot e^{j\phi} \cdot e^{j\theta_1 n}$ to LTI system (FIR)

$$\begin{aligned}
y[n] &= \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k A e^{j\phi} e^{j\theta_1 (n-k)} \\
&= \left(\sum_{k=0}^{M} b_k e^{-j\theta_1 k} \right) \cdot A e^{j\phi} e^{j\theta_1 n} = \left(H(e^{j\theta_1}) \right) \cdot A e^{j\phi} e^{j\theta_1 n} \\
&= \left(|H(e^{j\theta_1})| e^{j\angle H(e^{j\theta_1})} \right) \cdot A e^{j\phi} e^{j\theta_1 n} \\
&= \left(|H(e^{j\theta_1})| A \right) \cdot e^{j\left(\phi + \angle H(e^{j\theta_1})\right)} \cdot e^{j\theta_1 n}
\end{aligned}$$

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Frequency response of FIR filters

Previous equation holds $\forall\theta_1$, thus

Frequency response of FIR system

$$\boxed{H(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{k=0}^{M} b_k \mathrm{e}^{-\mathrm{j}\theta k} = \sum_{k=0}^{M} h[k]\mathrm{e}^{-\mathrm{j}\theta k}}$$

<u>Conjugate-symmetry property</u> (for real coefficients $h[k]$):

$$\begin{aligned} H(\mathrm{e}^{-\mathrm{j}\theta}) &= \sum_{k=0}^{M} h[k]\mathrm{e}^{\mathrm{j}\theta k} = \sum_{k=0}^{M} h[k]\left(\mathrm{e}^{-\mathrm{j}\theta k}\right)^* \\ &= \left(\sum_{k=0}^{M} h[k]\mathrm{e}^{-\mathrm{j}\theta k}\right)^* = H^*(\mathrm{e}^{\mathrm{j}\theta}) \end{aligned}$$

$\Rightarrow$ Amplitude $|H(\mathrm{e}^{\mathrm{j}\theta})|$ symmetric and phase $\angle H(\mathrm{e}^{\mathrm{j}\theta})$ anti-symmetric

Example blackboard via Frequency response FIR

Coefficients LTI: $\{b_k\} = \{h[k]\} = \{1, 2, 1\} \Rightarrow$

$$\begin{aligned} H(\mathrm{e}^{\mathrm{j}\theta}) &= 1 + 2\mathrm{e}^{-\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}2\theta} = \mathrm{e}^{-\mathrm{j}\theta} \cdot \left(\mathrm{e}^{\mathrm{j}\theta} + 2 + \mathrm{e}^{-\mathrm{j}\theta}\right) \\ &= \mathrm{e}^{-\mathrm{j}\theta} \cdot (2 + 2\cos(\theta)) \end{aligned}$$

Complex input signal: $x[n] = \mathrm{e}^{\mathrm{j}\theta_1 n} = \mathrm{e}^{\mathrm{j}(\pi/3)\cdot n}$

$$y[n] = H(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=\pi/3} \cdot \left\{\mathrm{e}^{\mathrm{j}(\pi/3)\cdot n}\right\}$$

With $H(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=\pi/3} = H(\mathrm{e}^{\mathrm{j}\pi/3}) = \mathrm{e}^{-\mathrm{j}\pi/3} \cdot (2 + 2\cos(\pi/3)) = 3\mathrm{e}^{-\mathrm{j}\pi/3}$

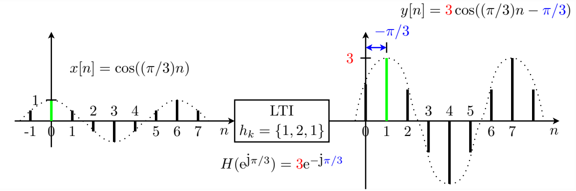
$$\Rightarrow \quad y[n] = 3\mathrm{e}^{-\mathrm{j}\pi/3} \cdot \left\{\mathrm{e}^{\mathrm{j}(\pi/3)\cdot n}\right\}$$

$\Leftrightarrow$ "Output = 3 $\cdot$ Input, with phase shifted over $-\pi/3$"

Superposition and frequency response

Superposition of 2 complex frequencies $\theta_1 = \frac{\pi}{3}n$ and $-\theta_1 = -\frac{\pi}{3}n$

$$\begin{aligned}
\mathrm{e}^{\mathrm{j}\frac{\pi}{3}n} &\mapsto H(\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}) \cdot \mathrm{e}^{\mathrm{j}\frac{\pi}{3}n} = 3\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}} \cdot \mathrm{e}^{\mathrm{j}\frac{\pi}{3}n} = 3\mathrm{e}^{\mathrm{j}(\frac{\pi}{3}n - \frac{\pi}{3})} \\
\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}n} &\mapsto H(\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}}) \cdot \mathrm{e}^{-\mathrm{j}\frac{\pi}{3}n} = 3\mathrm{e}^{\mathrm{j}\frac{\pi}{3}} \cdot \mathrm{e}^{-\mathrm{j}\frac{\pi}{3}n} = 3\mathrm{e}^{-\mathrm{j}(\frac{\pi}{3}n - \frac{\pi}{3})} \\
\mathrm{e}^{\mathrm{j}\frac{\pi}{3}n} + \mathrm{e}^{-\mathrm{j}\frac{\pi}{3}n} &\mapsto 3\mathrm{e}^{\mathrm{j}(\frac{\pi}{3}n - \frac{\pi}{3})} + 3\mathrm{e}^{-\mathrm{j}(\frac{\pi}{3}n - \frac{\pi}{3})} \\
\Rightarrow \quad \cos(\frac{\pi}{3}n) &\mapsto 3 \cdot \cos(\frac{\pi}{3}n - \frac{\pi}{3}) \\
&= |H(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=\frac{\pi}{3}} \cdot \cos\left(\frac{\pi}{3}n + \angle H(\mathrm{e}^{\mathrm{j}\theta})|_{\theta=\frac{\pi}{3}}\right)
\end{aligned}$$



Superposition and frequency response

Input LTI system is following real signal:

$$x[n] = X_0 + \sum_{k=1}^{N} \left(\frac{X_k}{2} \mathrm{e}^{\mathrm{j}\theta_k n} + \frac{X_k^*}{2} \mathrm{e}^{-\mathrm{j}\theta_k n} \right) = X_0 + \sum_{k=1}^{N} |X_k| \cos(\theta_k n + \angle X_k)$$

With conjugate-symmetry property $H(\mathrm{e}^{\mathrm{j}\theta})$ output writes

$$\begin{aligned} y[n] &= H(\mathrm{e}^{\mathrm{j}0}) X_0 + \sum_{k=1}^{N} \left(H(\mathrm{e}^{\mathrm{j}\theta_k}) \frac{X_k}{2} \mathrm{e}^{\mathrm{j}\theta_k n} + H(\mathrm{e}^{-\mathrm{j}\theta_k}) \frac{X_k^*}{2} \mathrm{e}^{-\mathrm{j}\theta_k n} \right) \\ &= \cdots \textit{homework} \cdots \\ &= H(\mathrm{e}^{\mathrm{j}0}) X_0 + \sum_{k=1}^{N} |H(\mathrm{e}^{\mathrm{j}\theta_k})| |X_k| \cos\left(\theta_k n + \angle X_k + \angle H(\mathrm{e}^{\mathrm{j}\theta_k}) \right) \end{aligned}$$

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Superposition and frequency response

Example superposition of 4 frequencies:

$$x[n] = 1 + \frac{4}{3}\cos((\pi/3)n) + 2\cos((\pi/2)n) + \cos((\pi)n)$$

LTI: $\quad h[k] = \{1, 2, 1\} \quad \leftrightarrow \quad H(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{e}^{-\mathrm{j}\theta} \cdot (2 + 2\cos(\theta))$

$$H(\mathrm{e}^{\mathrm{j}0}) = 4\,;\ H(\mathrm{e}^{\mathrm{j}\pi/3}) = H^*(\mathrm{e}^{-\mathrm{j}\pi/3}) = 3\mathrm{e}^{-\mathrm{j}\pi/3}\,;$$
$$H(\mathrm{e}^{\mathrm{j}\pi/2}) = H^*(\mathrm{e}^{-\mathrm{j}\pi/2}) = 2\mathrm{e}^{-\mathrm{j}\pi/2}\,;\ H(\mathrm{e}^{\mathrm{j}\pi}) = H^*(\mathrm{e}^{-\mathrm{j}\pi}) = 0$$
$$\Rightarrow \quad y[n] \quad = \quad 4 \cdot 1 + 3 \cdot \frac{4}{3}\cos((\pi/3)n - \pi/3) + $$
$$+2 \cdot 2\cos((\pi/2)n - \pi/2) + 0 \cdot \cos((\pi)n)$$

Conclusion:

Different frequency components are amplified differently.

Frequency $\theta = \pi$ vanishes at the output!

Properties of frequency response

Time domain	$\circ\!\!-\!\!\bullet$	Frequency domain
$h[n] = \sum_{k=0}^{M-1} h[k]\delta[n-k]$	$\circ\!\!-\!\!\bullet$	$H(e^{j\theta}) = \sum_{k=0}^{M-1} h[k]e^{-j\theta k}$

Examples relation impulse response and difference equation:

Example: $h[n] = -\delta[n] + 3\delta[n-1] - \delta[n-2] \Rightarrow$

Difference equation : $y[n] = -x[n] + 3x[n-1] - x[n-2]$

Frequency response : $H(e^{j\theta}) = -1 + 3e^{-j\theta} - e^{-j2\theta}$

Example: $H(e^{j\theta}) = e^{-j\theta}(3 - 2\cos(\theta)) \Rightarrow$

$$\begin{aligned} H(e^{j\theta}) &= \cdots = -1 + 3e^{-j\theta} - e^{-j2\theta} \\ \Rightarrow y[n] &= -x[n] + 3x[n-1] - x[n-2] \end{aligned}$$

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Properties of frequency response

Periodicity: $H(e^{j(\theta+2\pi)}) = H(e^{j\theta})$

$$H(e^{j(\theta+2\pi)}) = \sum_{k=0}^{M} b_k e^{-j(\theta+2\pi)k} = \sum_{k=0}^{M} b_k e^{-j\theta k} \cdot e^{-j2\pi k} = H(e^{j\theta})$$

- Sampling in time domain $\leftrightarrow$ periodicity in frequency domain
- Fundamental Interval $-\pi < \theta \leq \pi$ is sufficient

Conjugate symmetry: $H(e^{-j\theta}) = H^*(e^{j\theta})$

$$H^*(e^{j\theta}) = \left(\sum_{k=0}^{M} b_k e^{-j\theta k}\right)^* = \sum_{k=0}^{M} b_k^* e^{j\theta k} = \sum_{k=0}^{M} b_k e^{-j(-\theta)k} = H(e^{-j\theta})$$

- Even functions of θ:
  $|H(e^{-j\theta})| = |H(e^{j\theta})|$ and $\Re e\{H(e^{-j\theta})\} = \Re e\{H(e^{j\theta})\}$
- Odd functions of θ:
  $\angle H(e^{-j\theta}) = -\angle H(e^{j\theta})$ and $\Im m\{H(e^{-j\theta})\} = -\Im m\{H(e^{j\theta})\}$

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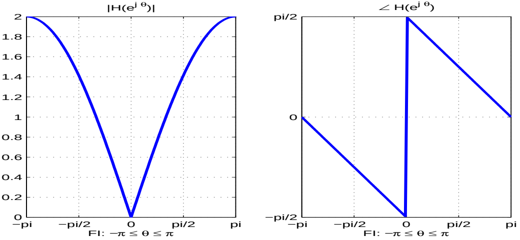
First order difference system

$$y[n] = x[n] - x[n-1] \quad \Rightarrow \quad H(e^{j\theta}) = 1 - e^{-j\theta}$$

Note: Coefficients satisfy symmetry condition $b_k = b_{M-k}$ $\Rightarrow$

$$\begin{aligned} H(e^{j\theta}) &= 1 - e^{-j\theta} = e^{-j\theta/2}\left(e^{j\theta} - e^{-j\theta/2}\right) \\ &= 2je^{-j\theta/2}\sin(\theta/2) = 2\sin(\theta/2)e^{j(\pi/2-\theta/2)} \end{aligned}$$

$$\Rightarrow \quad |H(e^{j\theta})| = 2|\sin(\theta/2)| \quad ; \quad \angle H(e^{j\theta}) = \begin{cases} \pi/2 - \theta/2 & 0 < \theta < \pi \\ -\pi + \pi/2 - \theta/2 & -\pi < \theta < 0 \end{cases}$$



Highpass character

$\Rightarrow$

Removes DC

First order difference system

Input DC + $\theta_0 = 0.3\pi$: $x[n] = 4 + 2\cos(0.3\pi n - \pi/4) \Rightarrow$

$$\begin{aligned} y[n] &= x[n] - x[n-1] \\ &= (4 + 2\cos(0.3\pi n - \pi/4)) - (4 + 2\cos(0.3\pi(n-1) - \pi/4)) \\ &= 2\cos(0.3\pi n - \pi/4) - 2\cos(0.3\pi n - 0.55\pi) \\ &= \cdots \text{phasor addition} \cdots = 1.816\cos(0.3\pi n + 0.1\pi) \end{aligned}$$

$\Rightarrow$ No DC!

Simpler via frequency response $H(\mathrm{e}^{\mathrm{j}\theta}) = 2\sin(\theta/2)\mathrm{e}^{\mathrm{j}(\pi/2-\theta/2)} \Rightarrow$

$$y[n] = 4|H(\mathrm{e}^{\mathrm{j}0})| + 2|H(\mathrm{e}^{\mathrm{j}0.3\pi})|\cos(0.3\pi n - \pi/4 + \angle H(\mathrm{e}^{\mathrm{j}0.3\pi}))$$

with $H(\mathrm{e}^{\mathrm{j}0}) = 0$; $H(\mathrm{e}^{\mathrm{j}0.3\pi}) = 0.908\mathrm{e}^{\mathrm{j}(0.35\pi)}$

$$\Rightarrow \quad y[n] = 1.816\cos(0.3\pi n + 0.1\pi)$$

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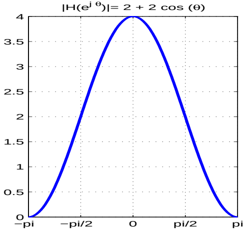
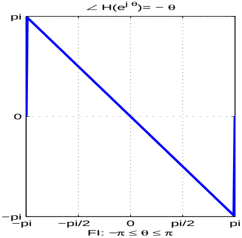
A simple lowpass filter

Input $x[n] = 4 + 3\cos((\pi/3)n - (\pi/2)) + 3\cos((7\pi/8)n)$ to filter with

coefficients $b_k = \{1, 2, 1\} \Rightarrow h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2] \Rightarrow$

$$H(e^{j\theta}) = 1 + 2e^{-j\theta} + e^{-j2\theta} = e^{-j\theta}\left(e^{j\theta} + 2 + e^{-j\theta}\right) = (2 + 2\cos(\theta))e^{-j\theta}$$



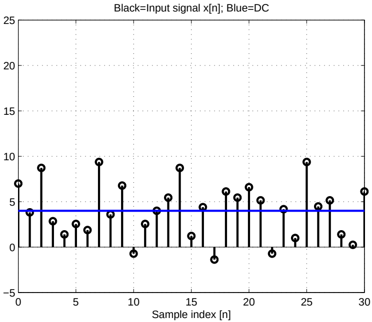


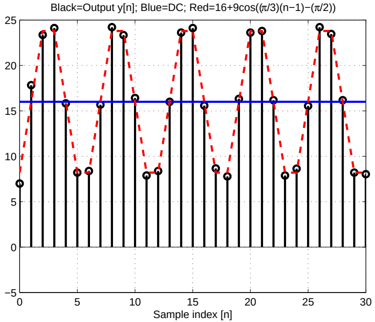
$$H(e^{j0}) = 4$$
$$H(e^{j\pi/3}) = 3e^{-j\pi/3}$$
$$H(e^{j7\pi/8}) = 0.1522e^{-j7\pi/8}$$

$$\begin{aligned} y[n] &= 4 \cdot 4 + 3 \cdot 3\cos(\frac{\pi}{3}n - \frac{\pi}{2} - \frac{\pi}{3}) + 0.1522 \cdot 3\cos(\frac{7\pi}{8}n - \frac{7\pi}{8}) \\ &= 16 + 9\cos(\frac{\pi}{3}(n-1) - \frac{\pi}{2}) + 0.4567\cos(\frac{7\pi}{8}(n-1)) \end{aligned}$$

A simple lowpass filter

$$x[n] = 4 + 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7\pi}{8}n\right)$$





$$y[n] = 16 + 9\cos\left(\frac{\pi}{3}(n-1) - \frac{\pi}{2}\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right)$$

Note: Linear phase slope of -1 $\Rightarrow$

All frequencies experience a time delay of one sample

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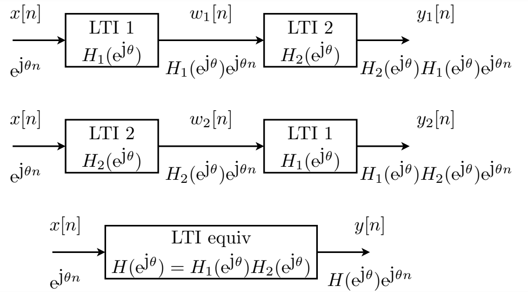
Cascaded LTI systems

Three equivalent LTI systems:



From $H_1(e^{j\theta}) \cdot H_2(e^{j\theta}) = H_2(e^{j\theta}) \cdot H_1(e^{j\theta}) \Rightarrow y[n] = y_1[n] = y_2[n]$

Conclusion:

Convolution	$\leftrightarrow$	Multiplication
$h_1[n] * h_2[n]$	$\leftrightarrow$	$H_1(e^{j\theta}) \cdot H_2(e^{j\theta})$

Example cascaded LTI systems

$$h_1[n] = 2\delta[n] + 4\delta[n-1] + 4\delta[n-2] + 2\delta[n-3]$$

$$\leftrightarrow \quad H_1(\mathrm{e}^{\mathrm{j}\theta}) = 2 + 4\mathrm{e}^{-\mathrm{j}\theta} + 4\mathrm{e}^{-\mathrm{j}2\theta} + 2\mathrm{e}^{-\mathrm{j}3\theta}$$

$$h_2[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$

$$\leftrightarrow \quad H_2(\mathrm{e}^{\mathrm{j}\theta}) = 1 - 2\mathrm{e}^{-\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}2\theta}$$

$$\begin{aligned} H(\mathrm{e}^{\mathrm{j}\theta}) &= H_1(\mathrm{e}^{\mathrm{j}\theta}) \cdot H_2(\mathrm{e}^{\mathrm{j}\theta}) \\ &= \cdots = 2 + 0\mathrm{e}^{-\mathrm{j}\theta} - 2\mathrm{e}^{-\mathrm{j}2\theta} - 2\mathrm{e}^{-\mathrm{j}3\theta} + 0\mathrm{e}^{-\mathrm{j}4\theta} + 2\mathrm{e}^{-\mathrm{j}5\theta} \\ &\leftrightarrow \\ h[n] &= 2\delta[n] - 2\delta[n-2] - 2\delta[n-3] + 2\delta[n-5] \\ &\equiv h_1[n] * h_2[n] \end{aligned}$$

Running average filtering

L-point running averager: $y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$

$$\begin{aligned} H(\mathrm{e}^{\mathrm{j}\theta}) &= \frac{1}{L}\cdot\sum_{k=0}^{L-1}\mathrm{e}^{-\mathrm{j}k\theta} = \frac{1}{L}\cdot\frac{1-\mathrm{e}^{-\mathrm{j}L\theta}}{1-\mathrm{e}^{-\mathrm{j}\theta}} = \frac{1}{L}\cdot\frac{\mathrm{e}^{\mathrm{j}L\theta/2}-\mathrm{e}^{-\mathrm{j}L\theta/2}}{\mathrm{e}^{\mathrm{j}\theta/2}-\mathrm{e}^{-\mathrm{j}\theta/2}}\cdot\frac{\mathrm{e}^{-\mathrm{j}L\theta/2}}{\mathrm{e}^{-\mathrm{j}\theta/2}} \\ &= \frac{\sin(L\theta/2)}{L\sin(\theta/2)}\cdot\mathrm{e}^{-\mathrm{j}(L-1)\theta/2} = D_L(\mathrm{e}^{\mathrm{j}\theta})\cdot\mathrm{e}^{-\mathrm{j}(L-1)\theta/2} \end{aligned}$$

Dirichlet function:

$$\boxed{D_L(\mathrm{e}^{\mathrm{j}\theta}) = \frac{\sin(L\theta/2)}{L\sin(\theta/2)}}$$

Properties $D_L(\mathrm{e}^{\mathrm{j}\theta})$:

- Even periodic function of θ
- Maximum value at $\theta = 0$
- Zeros at nonzero integer multiples of $2\pi/L$

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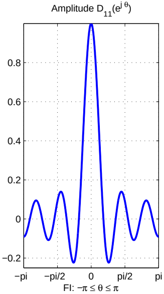
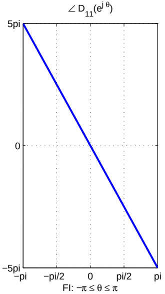
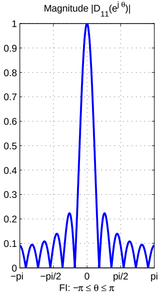
Frequency response 11-point running average

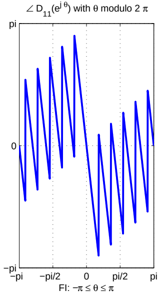
$$H(\mathrm{e}^{\mathrm{j}\theta}) = D_{11}(\mathrm{e}^{\mathrm{j}\theta}) \cdot \mathrm{e}^{-\mathrm{j}5\theta} = \frac{\sin(11\theta/2)}{11\sin(\theta/2)} \cdot \mathrm{e}^{-\mathrm{j}5\theta}$$





$\Leftrightarrow$





Phase: Straight line, slope -5

Main properties $D_{11}(\mathrm{e}^{\mathrm{j}\theta})$*:*

- Even function of θ
- periodic with period 2π

- Max at $\theta = 0$
- Decaying for $\theta \uparrow$
- Smallest nonzero at $\theta = \pm\pi$
- Zeros at $k \cdot (2\pi/L)$ $(k = \pm 1, \pm 2, \cdots)$

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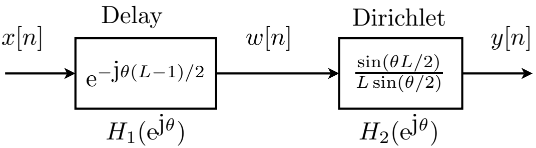
Running average filtering

Cascade of Magnitude and phase: $H(e^{j\theta}) = H_2(e^{j\theta}) \cdot H_1(e^{j\theta})$



- H_1 only contributes to phase: $\angle H_1(e^{j\theta}) = -\theta(L-1)/2$
- Linear phase shift $\leftrightarrow$ time delay $w[n] = x[n-(L-1)/2]$
- H_2 has low-pass character
- H_2 cannot be implemented (why?)
- Overall system implemented by: $y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$

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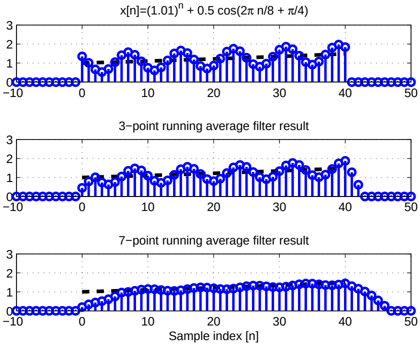
Experiment: Find trend

(Same example as in Chapter 5)

$$x[n] = \begin{cases} (1.01)^n + \frac{1}{2}\cos(2\pi n/8 + \pi/4) & 0 \le n \le 40 \\ 0 & \text{otherwise} \end{cases}$$

Use L-point averaging filter $\rightarrow$ $$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$



Observations:

- Run-in and -out area $y[n]$
- Length $y[n] \uparrow$
- For $L \uparrow \Rightarrow$ Less fluctuations
- Fluctuations reduced not eliminated

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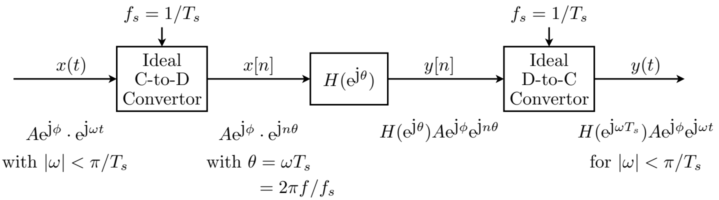
Filtering sampled continuous-time signals

Common practise: Filtering continuous signal with discrete-time filter:



If $|\omega| > \pi/T_s \Rightarrow$ Take care of aliasing!

Exercise

Steady-state and transient response

Until now we assumed $x[n] = A\mathrm{e}^{\mathrm{j}\theta_1 n}$ for $-\infty < n < \infty$, not practical!
More realistic, complex exponential input starting at $n = 0$:

$$x[n] = A\mathrm{e}^{\mathrm{j}\theta_1 n} \cdot u[n] = \begin{cases} A\mathrm{e}^{\mathrm{j}\theta_1 n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Original input signal $x[n]$ multiplied by unit-step function $u[n]$.
Applying this signal to LTI results in

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k A\mathrm{e}^{\mathrm{j}\theta_1 (n-k)} u[n-k]$$

Using the fact that $u[n-k] = 0$ for $n < k \Rightarrow$

$$y[n] = \begin{cases} 0 & n < 0 \\ \left(\sum_{k=0}^{\color{red}{n}} b_k \mathrm{e}^{-\mathrm{j}\theta_1 k}\right) A\mathrm{e}^{\mathrm{j}\theta_1 n} & 0 \leq n < M \quad \text{Transient region} \\ \left(\sum_{k=0}^{\color{blue}{M}} b_k \mathrm{e}^{-\mathrm{j}\theta_1 k}\right) A\mathrm{e}^{\mathrm{j}\theta_1 n} & n \geq M \quad \text{Steady-state region} \end{cases}$$

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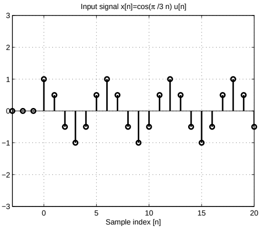
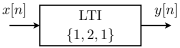
Steady-state and transient response

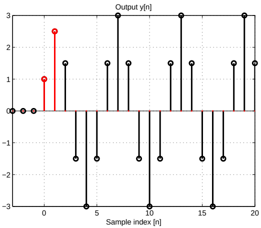
Notes:

- Transient region: Complex multiplier of $e^{j\theta_1 n}$ depends on n
- Steady-state region: Remains as long as input equal to $Ae^{j\theta_1 n}$
- If for $n > M$ input changes frequency or goes to zero $\rightarrow$ new transient and steady-state

<u>Example:</u> Input $x[n] = \cos(\frac{\pi}{3}n) \cdot u[n]$ applied to LTI with $b_k = \{1, 2, 1\}$







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Summary Module FREQ

- **Frequency response FIR**: $H(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{k=0}^{M} b_k \mathrm{e}^{-\mathrm{j}\theta k} = \sum_{k=0}^{M} h[k]\mathrm{e}^{-\mathrm{j}\theta k}$
- **Properties frequency response**:
  - $h[n] = \sum_{k=0}^{M} h[k]\delta[n-k] \circ\!\!-\!\!\circ H(\mathrm{e}^{\mathrm{j}\theta}) = \sum_{k=0}^{M} h[k]\mathrm{e}^{-\mathrm{j}\theta k}$
  - Periodicity: $H(\mathrm{e}^{\mathrm{j}(\theta+2\pi)}) = H(\mathrm{e}^{\mathrm{j}\theta})$
  - Conjugate symmetry: $H(\mathrm{e}^{-\mathrm{j}\theta}) = H^*(\mathrm{e}^{\mathrm{j}\theta})$
- **Superposition**: If $x[n] = A_0 + \sum_{k=1}^{N} A_k \cos(\theta_k n + \phi_k) \rightarrow$
  $y[n] = H(\mathrm{e}^{\mathrm{j}0})A_0 + \sum_{k=1}^{N} |H(\mathrm{e}^{\mathrm{j}\theta_k})| A_k \cos\left(\theta_k n + \phi_k + \angle H(\mathrm{e}^{\mathrm{j}\theta_k})\right)$
- **Graphical representation**: Mainly via $|H(\mathrm{e}^{\mathrm{j}\theta})|$ and $\angle H(\mathrm{e}^{\mathrm{j}\theta})$
- **Cascaded LTI**: $h_1[n] * h_2[n] = h_2[n] * h_1[n]$
- **Convolution property LTI**: $h_1[n] * h_2[n] \circ\!\!-\!\!\circ H_1(\mathrm{e}^{\mathrm{j}\theta}) \cdot H_1(\mathrm{e}^{\mathrm{j}\theta})$

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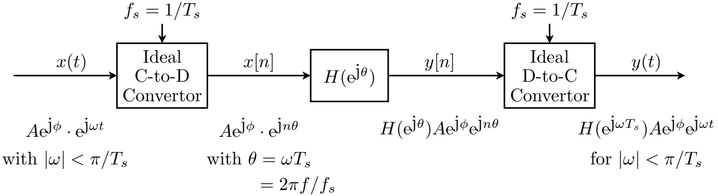
Summary Module FREQ

- L-point running average: $y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \rightarrow$ $H(\mathrm{e}^{\mathrm{j}\theta}) = \frac{1}{L}\cdot\sum_{k=0}^{L-1}\mathrm{e}^{-\mathrm{j}k\theta} = D_L(\mathrm{e}^{\mathrm{j}\theta})\cdot\mathrm{e}^{-\mathrm{j}(L-1)\theta/2}$ With Dirichlet function: $D_L(\mathrm{e}^{\mathrm{j}\theta}) = \frac{\sin(L\theta/2)}{L\sin(\theta/2)}$
- Filtering sampled continuous-time signal:



- Steady-state and transient response: $x[n] = A\mathrm{e}^{\mathrm{j}\theta_1 n}\cdot u[n] \rightarrow$

$$y[n] = \begin{cases} 0 & n < 0 \\ \left(\sum_{k=0}^{n} b_k \mathrm{e}^{-\mathrm{j}\theta_1 k}\right) A\mathrm{e}^{\mathrm{j}\theta_1 n} & 0 \leq n < M \quad \text{Transient region} \\ \left(\sum_{k=0}^{M} b_k \mathrm{e}^{-\mathrm{j}\theta_1 k}\right) A\mathrm{e}^{\mathrm{j}\theta_1 n} & n \geq M \quad \text{Steady-state region} \end{cases}$$

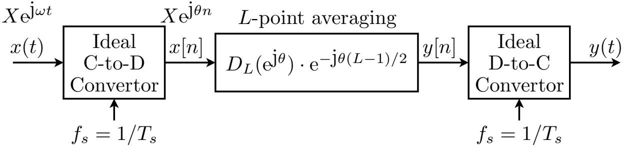
Interpretation of delay: OUT OF SCOPE

For integer n_0, $H(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{e}^{-\mathrm{j}\theta n_0}$ represents delay over n_0 samples:

$$H(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{e}^{-\mathrm{j}\theta n_0} \quad \circ\!\!-\!\!\circ \quad h[n] = \delta[n - n_0] \quad \rightarrow \quad y[n] = x[n - n_0]$$

Interpretation of $H(\mathrm{e}^{\mathrm{j}\theta}) = \mathrm{e}^{-\mathrm{j}\theta n_0}$ if n_0 not an integer?

<u>Example:</u>



$$y[n] = D_L(\mathrm{e}^{\mathrm{j}\theta}) \cdot \mathrm{e}^{-\mathrm{j}\theta(L-1)/2} X \mathrm{e}^{\mathrm{j}\theta n} \quad \rightarrow \quad y(t) = D_L(\mathrm{e}^{\mathrm{j}\omega T_s}) \cdot X \mathrm{e}^{\mathrm{j}\omega(t - T_s(L-1)}$$

Regardless or not $\frac{1}{2}(L-1)$ integer $\rightarrow$ delay $\frac{1}{2}T_s(L-1)$ [sec]

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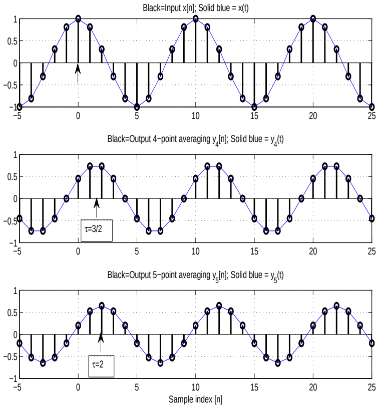
Interpretation of delay: OUT OF SCOPE

Now: $x(t) = \cos(200\pi t)$ **and** $f_s = 1000$ **[Hz]** $\to x[n] = \cos(0.2\pi n)$



$\underline{L=4\text{:}}$

$$y_4[n] = 0.7694\cos(0.2\pi(n - \frac{3}{2}))$$

$$\to$$

$$y_4(t) = 0.7694\cos(200\pi(t - 0.0015))$$

$\underline{L=5\text{:}}$

$$y_5[n] = 0.6472\cos(0.2\pi(n - 2))$$

$$\to$$

$$y_5(t) = 0.6472\cos(200\pi(t - 0.002))$$