Module CNAP:

Complex numbers and Phasors

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Where innovation starts

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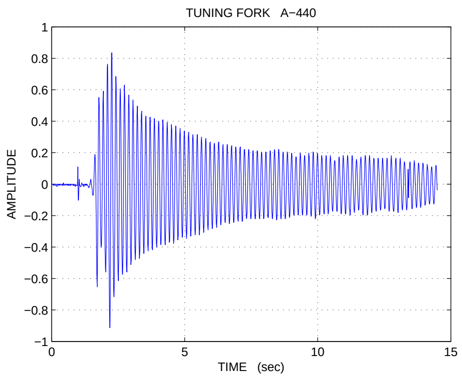
Importance sinusoidal signals (sinusoids)

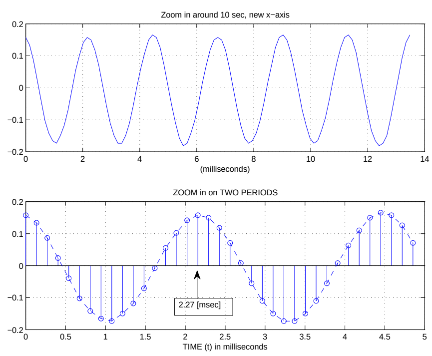
- Many signals represented by sum of sinusoidals (Fourier, 1807)

  Beethoven5 Birdsounds Vowels Male Female Racecar

- Sinusoids most basic in theory of signals and systems.

Example: Tuning fork





440 Hz? From detailed figure $\Rightarrow f_0 = \frac{1}{2.27 \cdot 10^{-3}} = 440$ Hz

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Basic properties sine, cosine

Property	Equation
Equivalency	$\sin(\theta) = \cos(\theta - \pi/2)$ or $\cos(\theta) = \sin(\theta + \pi/2)$
Periodicity	$\cos(\theta + k \cdot 2\pi) = \cos(\theta)$, for k integer
Evenness cos	$\cos(-\theta) = \cos(\theta)$
Oddness sin	$\sin(-\theta) = -\sin(\theta)$
Zeros sin	$\sin(k \cdot \pi) = 0$ for k integer
Ones cos	$\cos(k \cdot 2\pi) = 1$ for k integer
Minus ones cos	$\cos((k + \frac{1}{2}) \cdot 2\pi) = -1$ for k integer
Cos: slope sin	$\frac{d \sin(\theta)}{d\theta} = \cos(\theta)$
Sin: negative slope cos	$\frac{d \cos(\theta)}{d\theta} = -\sin(\theta)$
	$\sin^2(\theta) + \cos^2(\theta) = 1$
...	...

Many other basic trigonometric identities
Can be derived simply by using phasor description (see furtheron)

Review sine- cosine- functions

Mathematical formula continuous-time cosine:

$$x(t) = A\cos(\omega_0 \cdot t + \phi) = A\cos(2\pi \cdot f_0 \cdot t + \phi)$$

Symbol	Name	Dimension
A	Amplitude	-
ω_0	Radian frequency	rad/sec
ϕ	phase	rad
f_0	(cyclic) frequency	sec^{-1}= Hz

Relation period - frequency:

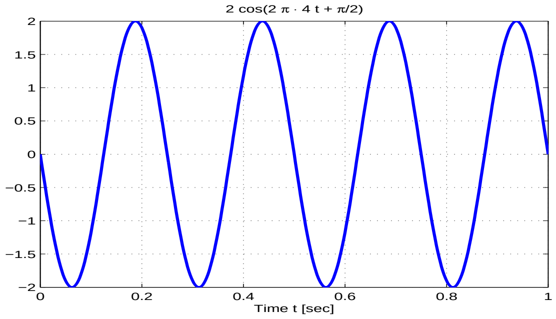
$$\begin{aligned} x(t + T_0) &= x(t) \\ A\cos(\omega_0 \cdot (t + T_0) + \phi) &= A\cos(\omega_0 \cdot t + \phi) \end{aligned}$$

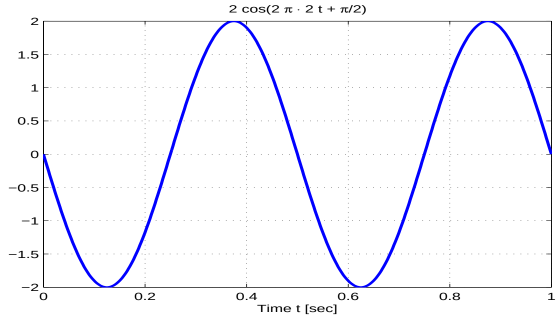
$$\omega_0 T_0 = 2\pi \quad \Rightarrow \quad T_0 = \frac{2\pi}{\omega_0} \quad \text{or} \quad (2\pi f_0)T_0 = 2\pi \quad \Rightarrow \quad T_0 = \frac{1}{f_0}$$

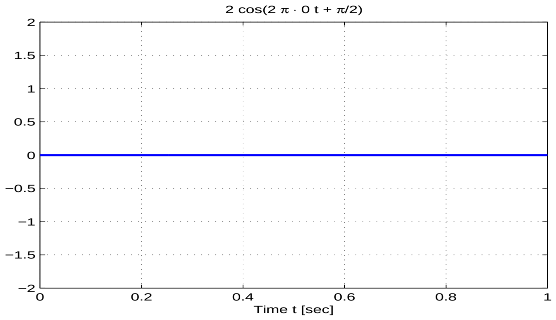
Example

Amplitude $A = 2$, Phase $\phi = \pi/2$, Frequency f_0 varies $4 \to 0$ [Hz]

$$x(t) = 2\cos(2\pi \cdot f_0 \cdot t + \pi/2)$$







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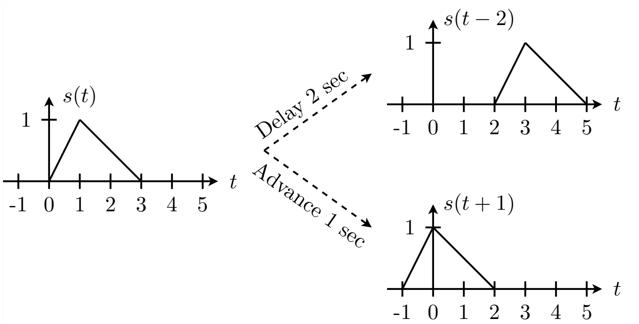
Phase and Time shift

Positive and negative time shift:



Conversion time- to phase shift:

$$x(t) = \cos(\omega_0 t) \quad \Rightarrow \quad x(t - t_1) = \cos(\omega_0(t - t_1)) = \cos(\omega_0 t + \phi)$$

$$\phi = -\omega_0 t_1 \quad \Leftrightarrow \quad t_1 = -\frac{\phi}{\omega_0} = -\frac{\phi}{2\pi f_0}$$

Notes:

- Phase negative for positive time shift (delay)
- Phase can always be chosen in $-\pi < \phi \leq \pi$. Why?

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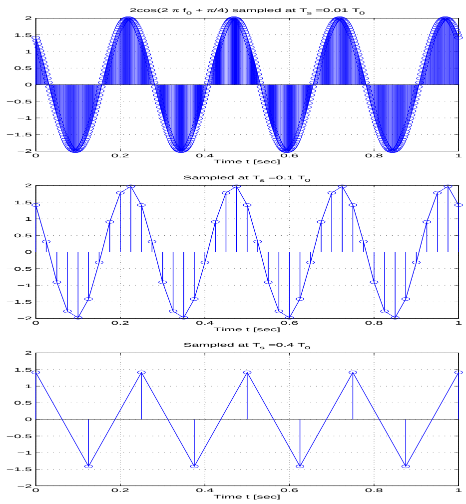
Sampling and plotting sinusoids

$x(t) = 2\cos(2\pi \cdot 4 \cdot t + \pi/4) \rightarrow$ One period? T_0=1/4 =0.25 [sec]

$T_s = 0.0025$ [sec] $\rightarrow$ 100 samples in T_0

$T_s = 0.025$ [sec] $\rightarrow$ 10 samples in T_0

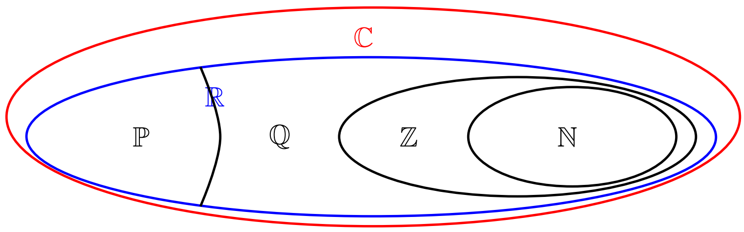
$T_s = 0.1$ [sec] $\rightarrow$ 2.5 samples in T_0



Notes:

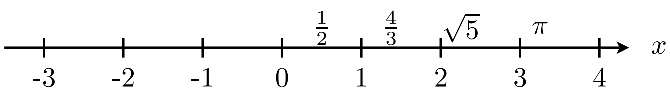
- Choice of T_s depends on frequency cosine
- String of numbers at sampling space T_s (`stem`)
- Matlab `plot` connects samples by lines
- How large T_s for accurate reconstruction?

From real numbers towards complex numbers



$\mathbb{R}$: Set of real numbers is extension of:

- $\mathbb{N}$: Natural numbers, e.g. $0, 1, 2, \cdots$
- $\mathbb{Z}$: Interger numbers, e.g. $-2, -1, 0, 1, 2, \cdots$
- $\mathbb{Q}$: Rational numbers, e.g. $\frac{1}{2}, \frac{4}{5}, \cdots$
- $\mathbb{P}$: Irrational numbers, e.g. $\pi, \sqrt{5}, \cdots$



$\Rightarrow \mathbb{R}$ is set of numbers x for which $x^2 \geq 0$

- **Is there a set $\mathbb{C}$ for which $x^2 < 0$, which is extension of $\mathbb{R}$?**
- **Physical meaning of such a set $\mathbb{C}$?**

- Two examples: *Tuning fork:* *Harmonic oscillator:*

General solution 2^{nd} order DE

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\{f(\theta)\} + f(\theta) = 0$$

General solution $f(\theta) = \mathrm{e}^{c\cdot\theta}$ with c some unknown constant

$$\Rightarrow \quad c^2(\mathrm{e}^{c\cdot\theta}) + (\mathrm{e}^{c\cdot\theta}) = 0 \quad \Rightarrow \quad c^2 + 1 = 0 \quad \Rightarrow \quad c^2 = -1 \quad \Rightarrow \quad c = \pm\sqrt{-1}$$

Define complex number : $\mathrm{j} \stackrel{\mathrm{Def}}{=} \sqrt{-1}$

$\Rightarrow$ General solution 2^{nd} order DE writes as complex exponential:

$\mathrm{e}^{\pm \mathrm{j}\theta}$ and all linear combinations

However we know physical solution is sinusoidal!

Relation $\mathrm{e}^{\mathrm{j}\theta}$ with sinusoidal

$$\mathrm{e}^{x} = 1 + \frac{(x)}{1!} + \frac{(x)^2}{2!} + \frac{(x)^3}{3!} + \frac{(x)^4}{4!} + \frac{(x)^5}{5!} + \cdots$$

$$\mathrm{e}^{\mathrm{j}\theta} = 1 + \frac{(\mathrm{j}\theta)}{1!} + \frac{(\mathrm{j}\theta)^2}{2!} + \frac{(\mathrm{j}\theta)^3}{3!} + \frac{(\mathrm{j}\theta)^4}{4!} + \frac{(\mathrm{j}\theta)^5}{5!} + \cdots$$

With $\mathrm{j} \stackrel{\mathrm{Def}}{=} \sqrt{-1}$ we obtain: $\mathrm{j}^2 = -1$, $\mathrm{j}^3 = -\mathrm{j}$, $\mathrm{j}^4 = +1$, $\mathrm{j}^5 = \mathrm{j}$ etc.

$$\mathrm{e}^{\mathrm{j}\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \mathrm{j}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

With geometric series expansion of sine- and cosine function:

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \quad \text{and} \quad \sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

- **Alternative expression $\mathrm{e}^{\mathrm{j}\theta}$:**

$\Rightarrow \quad \mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$: Euler

Relation $e^{j\theta}$ with physical solution of DE

- **Linear combinations $e^{\pm j\theta}$ also valid solutions of DE:**

$$\begin{aligned}\frac{e^{j\theta}+e^{-j\theta}}{2} &= \frac{(\cos(\theta)+j\sin(\theta))+(\cos(\theta)-j\sin(\theta))}{2} = \cos(\theta) \quad \in \mathbb{R}\\ \frac{e^{j\theta}-e^{-j\theta}}{2j} &= \frac{(\cos(\theta)+j\sin(\theta))-(\cos(\theta)-j\sin(\theta))}{2j} = \sin(\theta) \quad \in \mathbb{R}\end{aligned}$$

- **Notes on complex exponential $e^{j\theta} = \cos(\theta) + j\sin(\theta)$:**
  - Real part of $e^{j\theta}$: $\Re e\{e^{j\theta}\} = \cos\theta$
  - Imaginary part of $e^{j\theta}$ ("*part after symbol* j"): $\Im m\{e^{j\theta}\} = \sin\theta$
  - $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ in which both $\cos(\theta)$ and $\sin(\theta)$ are real $\Rightarrow$

Set of complex numbers : $\mathbb{C} = \{z = x + jy | x \in \mathbb{R}$ and $y \in \mathbb{R}\}$

Basic trigonometric identities via Euler

1	$\sin^2(\theta) + \cos^2(\theta) = 1$
2	$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
3	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
4	$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
5	$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

Simple proof via Euler, e.g. property 1:

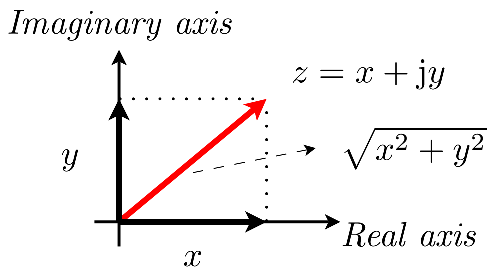
$$\begin{aligned}\sin^2(\theta) + \cos^2(\theta) &= \left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)^2 + \left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)^2\\ &= \left(\frac{e^{j2\theta} + e^{-j2\theta} - 2}{4(j)^2}\right) + \left(\frac{e^{j2\theta} + e^{-j2\theta} + 2}{4}\right)\\ &= \left(\frac{-e^{j2\theta} - e^{-j2\theta} + 2}{4}\right) + \left(\frac{e^{j2\theta} + e^{-j2\theta} + 2}{4}\right) = 1\end{aligned}$$

Complex numbers

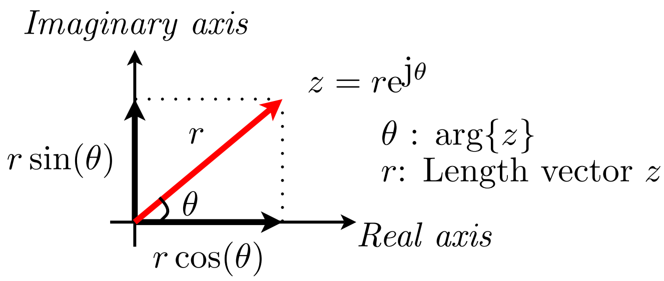
Set of complex numbers : $\mathbb{C} = \{z = x + \mathrm{j}y | x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

- **Visualization: 2-Dimensional vector representation (Animations)**

Cartesian representation:



Polar representation:



- **Conversion:**

Polar $\rightarrow$ Cartesian : $x = r\cos(\theta)$ and $y = r\sin(\theta)$

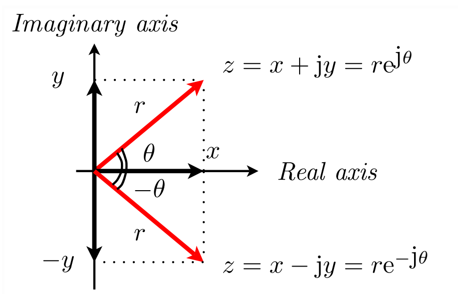
Cartesian $\rightarrow$ polar : $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})(+\pi, \text{ for } x < 0)$

Complex numbers: Conjugation and rules

- **Complex conjugation z, denoted by z^*:** Replace $+\mathrm{j}$ by $-\mathrm{j}$

$$z = r\mathrm{e}^{+\mathrm{j}\theta} \quad \Rightarrow \quad z^* = r\mathrm{e}^{-\mathrm{j}\theta}$$

$$z = x + jy \quad \Rightarrow \quad z^* = x - \mathrm{j}y$$



- **Calculation rules:**

Equality : $x_1 + \mathrm{j}y_1 = x_2 + \mathrm{j}y_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$

Addition : $(x_1 + \mathrm{j}y_1) + (x_2 + \mathrm{j}y_2) = (x_1 + x_2) + \mathrm{j}\,(y_1 + y_2)$

Scaling : $c \cdot (x + \mathrm{j}y) = c \cdot x + \mathrm{j}c \cdot y$

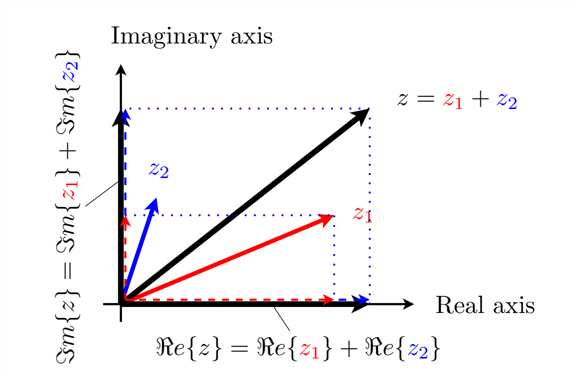
Multiplication : $z_1 \cdot z_2 = (x_1 + \mathrm{j}y_1) \cdot (x_2 + \mathrm{j}y_2) =$

$= (x_1 \cdot x_2 - y_1 \cdot y_2) + \mathrm{j}\,(x_1 \cdot y_2 + x_2 \cdot y_1)$

Alternative $z_1 \cdot z_2 = r_1\mathrm{e}^{\mathrm{j}\theta_1} \cdot r_2\mathrm{e}^{\mathrm{j}\theta_2} = r_1 \cdot r_2\mathrm{e}^{\mathrm{j}(\theta_1+\theta_2)}$

Complex numbers: Vectorial addition

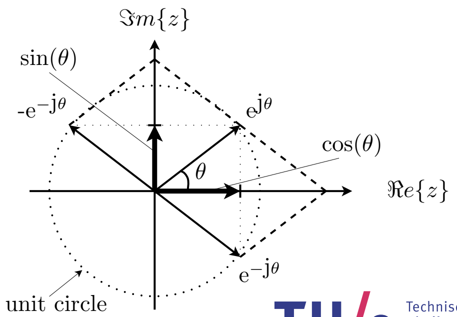
$$
\begin{aligned}
z &= z_1 + z_2 \\
&= (\Re e\{z_1\} + \Re e\{z_2\}) + \\
&\quad + \mathrm{j}\,(\Im m\{z_1\} + \Im m\{z_2\}) \\
&= \Re e\{z\} + \mathrm{j}\Im m\{z\}
\end{aligned}
$$



Inverse Euler via complex addition rule:

$$
\cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2}
$$

$$
\sin(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2\mathrm{j}}
$$



Complex numbers: Some important facts

- **Multiplication by its own complex conjugate**

$$
\begin{aligned}
\text{Polar} &: \quad z \cdot z^* = r\mathrm{e}^{\mathrm{j}\theta} \cdot r\mathrm{e}^{-\mathrm{j}\theta} = r^2 \\
\text{Cartesian} &: \quad z \cdot z^* = (x + \mathrm{j}y) \cdot (x - \mathrm{j}y) = (x^2 + y^2)
\end{aligned}
$$

- **Length complex vector z, denoted by $|z|$:**

$$
|z| \overset{\text{Def}}{=} \sqrt{z \cdot z^*} \hat{=} r \hat{=} \sqrt{x^2 + y^2} \geq 0
$$

- **Division complex numbers:**

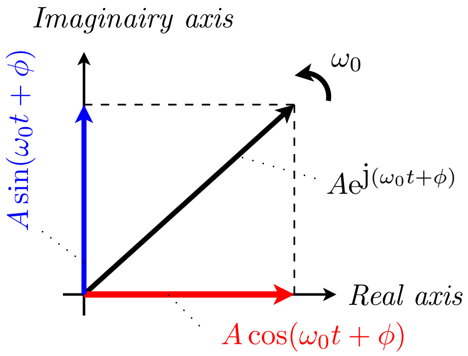
$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{x + \mathrm{j}y}{u + \mathrm{j}v} = \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} = \frac{x + \mathrm{j}y}{u + \mathrm{j}v} \cdot \frac{u - \mathrm{j}v}{u - \mathrm{j}v} = \frac{(xu + yv) + \mathrm{j}(yu - xv)}{u^2 + v^2} \\
&= \left(\frac{xu + yv}{u^2 + v^2}\right) + \mathrm{j}\left(\frac{yu - xv}{u^2 + v^2}\right) \overset{?}{=} \hat{=} \alpha + \mathrm{j}\beta
\end{aligned}
$$

Time-dependent complex exponential: phasor

$$A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}$$

$$\Re e\{A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}\} = A\cos(\omega_0 t+\phi)$$

$$\Im m\{A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}\} = A\sin(\omega_0 t+\phi)$$



(Animations)

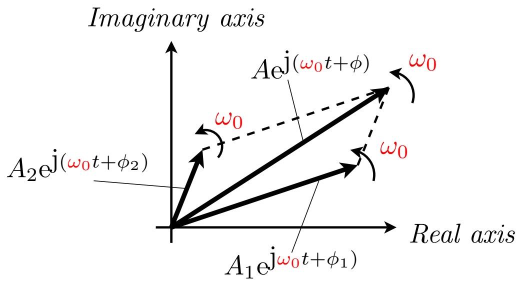
$$\boxed{\textsf{Phasor} \quad : \quad z(t) = A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}}$$

Sum of phasors with same frequency

- **Example:** (Animations)

$$A_1\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi_1)} + A_2\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi_2)}$$
$$= \left\{A_1\mathrm{e}^{\mathrm{j}\phi_1} + A_2\mathrm{e}^{\mathrm{j}\phi_2}\right\}\cdot \mathrm{e}^{\mathrm{j}\omega_0 t}$$
$$= \left\{A\mathrm{e}^{\mathrm{j}\phi}\right\}\cdot \mathrm{e}^{\mathrm{j}\omega_0 t} = A\mathrm{e}^{\mathrm{j}(\omega_0 t+\phi)}$$

with $A\mathrm{e}^{\mathrm{j}\phi} = A_1\mathrm{e}^{\mathrm{j}\phi_1} + A_2\mathrm{e}^{\mathrm{j}\phi_2}$



- **Phasor addition rule:**

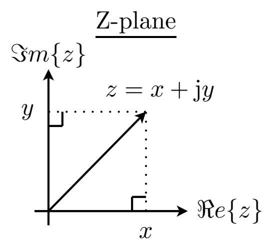
Sum cosine signals, same frequency $\omega_o \hat{=}$ Single cosine, frequency ω_o:

$$x(t) = \sum_{k=1}^{N} A_k\cos(\omega_0 t+\phi_k) \hat{=} A\cos(\omega_0 t+\phi) = \Re e\{A\mathrm{e}^{\mathrm{j}\phi}\mathrm{e}^{\mathrm{j}\omega_0 t}\}$$

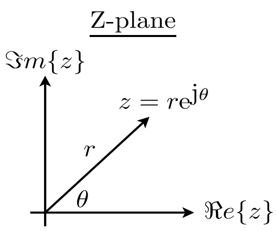
with $$A\mathrm{e}^{\mathrm{j}\phi} = \sum_{k=1}^{N} A_k\mathrm{e}^{\mathrm{j}\phi_k}$$

Summary Module CNAP

- **Periodic sine (cosine) wave:**
  $x(t) = A\cos(\omega_0 \cdot t + \phi) = A\cos(2\pi \cdot f_0 \cdot t + \phi)$
- **Relation period sine wave and frequency:** $T_0 = 1/f_0$
- **Relation phase- and time shift (delay):**
  $x(t) = \cos(\omega_0 t)\,;\, x(t-\tau) = \cos(\omega_0 t + \phi) \Rightarrow \tau = -\frac{\phi}{\omega_0} = -\frac{\phi}{2\pi f_0}$
- **Representation complex exponential in Z-plane** ($\mathrm{j} = \sqrt{-1}$)



Cartesian representation



Polar representation

$$\begin{aligned}
\Re e\{r\mathrm{e}^{\mathrm{j}\theta}\} &= x = r\cos(\theta) \\
\Im m\{r\mathrm{e}^{\mathrm{j}\theta}\} &= y = r\sin(\theta) \\
\theta &= \arctan\left(\frac{y}{x}\right) \\
\text{if } x < 0 &= \arctan\left(\frac{y}{x}\right) + \pi
\end{aligned}$$

- **Euler and inverse Euler:**
  $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta) \leftrightarrow \cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2}\,;\, \sin(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2\mathrm{j}}$

Summary Module CNAP

- **Addition rule complex numbers:** $z_1 = x_1 + \mathrm{j}y_1\,;\, z_2 = x_2 + \mathrm{j}y_2$

  $\Rightarrow z_3 = z_1 + z_2 = (x_1 + x_2) + \mathrm{j}(y_1 + y_2)$
- **Multiplication rule complex numbers:** $z_1 = r_1\mathrm{e}^{\mathrm{j}\theta_1}\,;\, z_2 = r_2\mathrm{e}^{\mathrm{j}\theta_2}$

  $\Rightarrow z_3 = z_1 \cdot z_2 = r\mathrm{e}^{\mathrm{j}\theta}$ with $r = r_1 \cdot r_2$ and $\theta = \theta_1 + \theta_2$
- **Complex exponential (phasor):** $z(t) = A\mathrm{e}^{\mathrm{j}(\omega_0 t + \phi)} = A\mathrm{e}^{\mathrm{j}\phi} \cdot \mathrm{e}^{\mathrm{j}\omega_0 t}$

  $\Re e\{z(t)\} = A\cos(\omega_0 t + \phi)\,;\, \Im m\{z(t)\} = A\sin(\omega_0 t + \phi)$
- **Sum cosine signals with same frequency (phasor addition rule):**

  $\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi) = \Re e\{A\mathrm{e}^{\mathrm{j}\phi}\mathrm{e}^{\mathrm{j}\omega_0 t}\}$

  with $A\mathrm{e}^{\mathrm{j}\phi} = \sum_{k=1}^{N} A_k \mathrm{e}^{\mathrm{j}\phi_k}$