Module FIR

Finite Impulse Response filter

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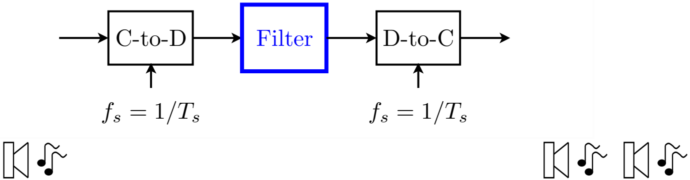
Where innovation starts

Filtering example



Example: The running average filter

Compute a moving (running) average of two or more consecutive samples, forming a new sequence of the average values

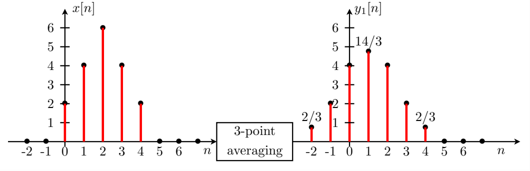
Note: FIR is generalization of running average

Averaging used when data fluctuates, thus smoothing prior interpretation (e.g. view trends). E.g. Stock-market prices, credit-card balances, etc.

Example:

Input-output relation (**=difference equation**) 3-point averaging

$\rightarrow y_1[n] = \frac{1}{3}\left(x[n] + x[n+1] + x[n+2]\right)$



The running average filter

Observations output signal samples:

Smooths input , longer than input , finite sequence ,starts (becomes nonzero) before input starts (=noncausal filter)

Last issue undesirable if input comes directly from A-to-D (e.g. audio).

Causal filter uses only present and past values

$$\text{Noncausal} \quad y_2[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right)$$

$$\text{Causal} \quad y_3[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

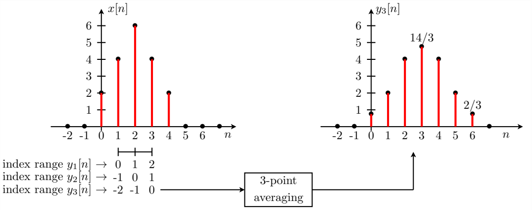


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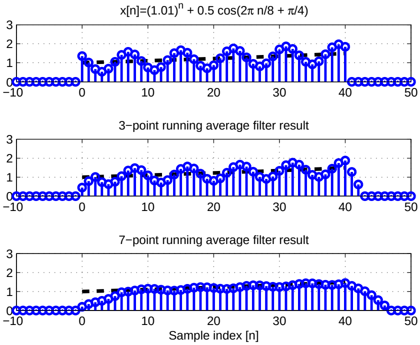
Illustration running average filter

Find trend in the following signal:

$$x[n] = \begin{cases} (1.01)^n + \frac{1}{2}\cos(2\pi n/8 + \pi/4) & 0 \le n \le 40 \\ 0 & \text{otherwise} \end{cases}$$

Use M-point averaging filter $\rightarrow \quad y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$



Observations:

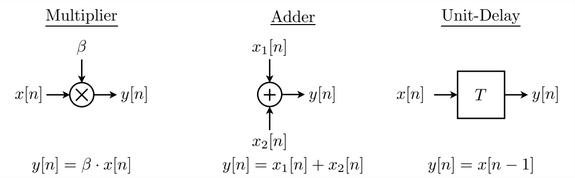
- Run-in and -out area $y[n]$
- Length $y[n] \uparrow$
- For $M \uparrow \Rightarrow$ Less fluctuations
- Fluctuations reduced not eliminated

General filter and basic building blocks

Difference equation (DE): $$y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$$

- Weighted running average of M samples
- Causal filter
- M is order of filter

$\Rightarrow$ Basic building blocks are: multiplier , adder , unit-delay operator



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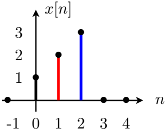
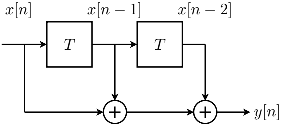
Convolution sum: Example

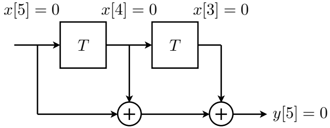
$$\text{DE} \quad : \quad y[n] = \sum_{k=0}^{2} h[k] \cdot x[n-k]$$

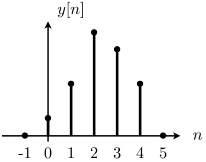
$$\text{Impulse response} \quad : \quad h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

$$\text{Input} \quad : \quad x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

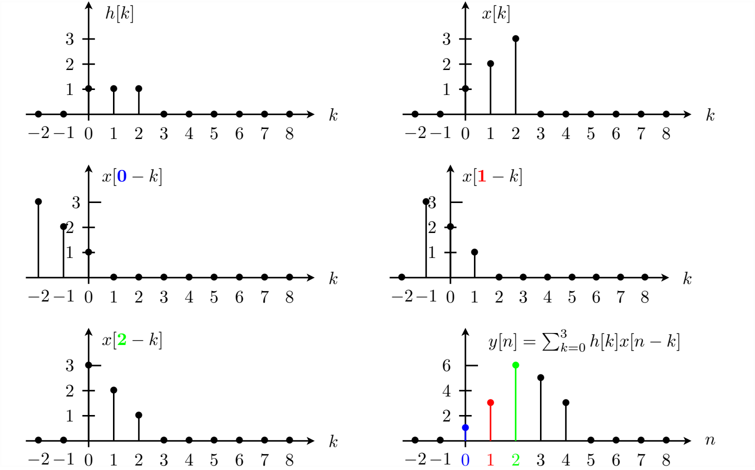






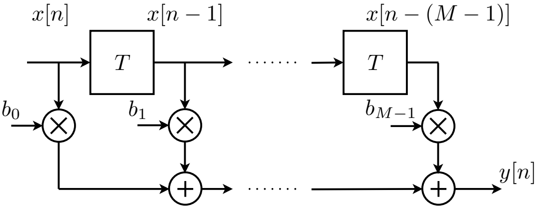


Convolution procedure via plot



Note for finite filter length : Length(y) = Length(x) + Length(h) -1

Convolution sum



$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$$

Procedure:

- Plot weights b_k as function of k
- Plot input signal samples $x[k]$ as function of k
- Mirror (reverse) $x[k]$ into $x[-k]$
- Shift mirrored signal $x[-k]$ to index n which results in $x[n-k]$.
- For each new time index n $y[n]$ equals the element by element multiplication of b_k and $x[n-k]$ in the range $k = 0, 1, \cdots M-1$.

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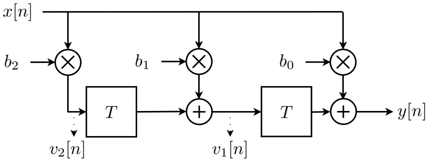
Difference equation from realization scheme



$$
\begin{aligned}
y[n] &= b_0 x[n] + v_1[n-1] \\
v_1[n] &= b_1 x[n] + v_2[n-1] \\
v_2[n] &= b_2 x[n]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad v_1[n] &= b_1 x[n] + b_2 x[n-1] \\
\Rightarrow \quad y[n] &= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]
\end{aligned}
$$

Implementation issues:

Constraints with VLSI or DSP architectures; Finite word-length effects; Memory length; etc

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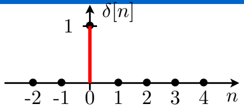
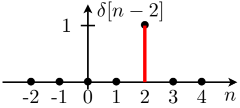
Basic signal: Delta pulse

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



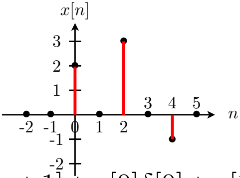
$$\delta[n-2] = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{for } n \neq 2 \end{cases}$$



n	$\cdots$	-2	-1	0	1	2	3	4	5	6	$\cdots$
$\delta[n]$	0	0	0	1	0	0	0	0	0	0	0
$\delta[n-2]$	0	0	0	0	0	1	0	0	0	0	0

Concept usefull to represent signals and system, e.g.

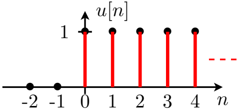
$$x[n] = 2\delta[n] + 3\delta[n-2] - \delta[n-4]$$



$$x[n] = \sum_k x[k]\delta[n-k] = \cdots + x[-1]\delta[n+1] + x[0]\delta[0] + x[1]\delta[n-1] + \cdots$$

Basic signal: Unit step

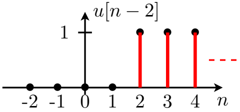
$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



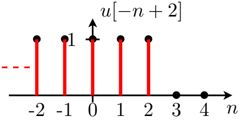
n	-2	-1	0	1	2	3	4	$\cdots$
$u[n]$	0	0	1	1	1	1	1	$\cdots$

- Examples:

$$u[n-2] = \begin{cases} 1 & \text{for } n \geq 2 \\ 0 & \text{for } n < 2 \end{cases}$$



$$u[-n+2] = \begin{cases} 1 & \text{for } n \leq 2 \\ 0 & \text{for } n > 2 \end{cases}$$



Impulse Response

Response for $x[n] = \delta[n]$

Impulse response $h[n]$ of order M FIR:

$$h[n] = \sum_{k=0}^{M-1} b_k \delta[n-k] = \begin{cases} b_n & n = 0, 1, \cdots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

In tabular form response to FIR (Finite Impulse Response):

n	$n < 0$	0	1	2	3	$\cdots$	$M-1$	M	$n > M$
$x[n] = \delta[n]$	0	1	0	0	0	0	0	0	0
$y[n] = h[n]$	0	b_0	b_1	b_2	b_3	$\cdots$	b_{M-1}	0	0

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Delay system

Delay or shift input $x[n]$ by n_0 samples, thus $y[n] = x[n - n_0]$

For order M FIR:

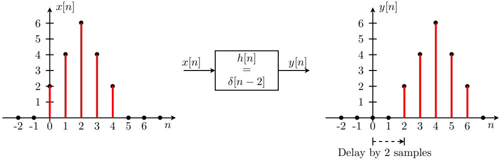
$$y[n] \quad = \quad b_0 x[n] + b_1 x[n-1] + \cdots b_{n_0} x[n - n_0] + \cdots + b_M x[n - (M-1)]$$

Now $y[n] = x[n - n_0]$ when $b_i = 0$ except for $i = n_0$, thus $b_{n_0} = 1$

$\Rightarrow$ Impulse response of delay by n_0 samples: $h[n] = \delta[n - n_0]$

Example:

Delay by 2 samples, thus
$h[n] = \delta[n-2]$ and
$y[n] = x[n-2]$



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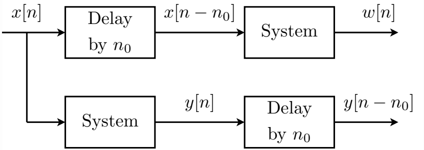
Linear Time-Invariant (LTI) Systems

Why LTI: Simplified mathematics and leads to greater insight

Input-output notation: $x[n] \mapsto y[n]$

Time Invariance: $x[n-n_0] \mapsto y[n-n_0]$



System Time-Invariant
$w[n] = y[n-n_0]$

Example: $y[n] = (x[n])^2$

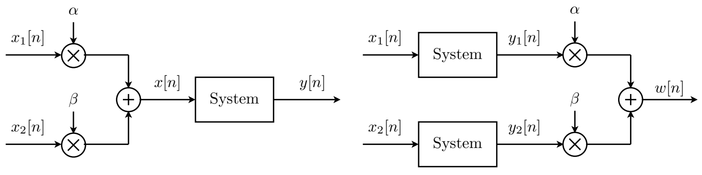
$$x[n] \mapsto y[n] = (x[n])^2 \Rightarrow \text{ Delay by one } \Rightarrow y[n-1] = (x[n-1])^2$$
$$x[n] \Rightarrow \text{ Delay by one } \Rightarrow x[n-1] \mapsto (x[n-1])^2 = y[n-1]$$

Not time-invariant: $y[n] = x[-n]$

Linearity

Linear system: If $x_1[n] \mapsto y_1[n]$ and $x_2[n] \mapsto y_2[n]$, then

$$x[n] = \alpha x_1[n] + \beta x_2[n] \mapsto y[n] = \alpha y_1[n] + \beta y_2[n]$$



System Linear iff $w[n] = y[n]$

Example: System $y[n] = (x[n])^2$ is **Non**linear since

$$y[n] = (\alpha x_1[n] + \beta x_2[n])^2 \neq \alpha (x_1[n])^2 + \beta (x_2[n])^2$$

Is $y[n] = x[-n]$ linear?

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FIR case

FIR both linear and time-invariant, proof:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \Rightarrow \quad y[n-n_0] = \sum_{k=0}^{M} b_k x[(n-n_0)-k]$$

On the other hand with $v[n] = x[n-n_0]$

$$w[n] = \sum_{k=0}^{M} b_k v[n-k] = \sum_{k=0}^{M} b_k x[(n-k)-n_0] = \sum_{k=0}^{M} b_k x[(n-n_0)-k]$$

FIR is Linear Time-Invariant (LTI) system

Main issues of LTI system:

- Impulse response is complete characterization
- Convolution is general formula to compute output from input

Convolution and LTI system

We can write $x[n] = \sum_l x[l]\delta[n-l]$

$$(\cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots)$$

For LTI system:

$$\begin{aligned} \delta[n] \mapsto h[n] \quad &\Rightarrow \quad \delta[n-l] \mapsto h[n-l] \quad \text{for any } l \\ &\Rightarrow \quad x[l]\delta[n-l] \mapsto x[l]h[n-l] \quad \text{for any } l \\ &\Rightarrow \quad x[n] = \sum_l x[l]\delta[n-l] \mapsto y[n] = \sum_l x[l]h[n-l] \end{aligned}$$

No assumptions made about length $x[n]$ or $h[n]$, $\Rightarrow$

The convolution sum formula: $$y[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$$

Thus all LTI systems can be represented by a convolution sum

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FIR convolution

Impulse response $h[n]$ of FIR only nonzero for $0 \leq n \leq M$. Thus

$$y[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$$

with index $h[n-l] \in \{0, M\}$ results in

$$y[n] = \sum_{l=n-M}^{n} x[l]h[n-l]$$

Simple example:

$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-3]$ and $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

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Some important properties LTI system

Convolution as operator:

$$y[n] = x[n] * h[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$$

Convolution with shifted impulse: $\Leftrightarrow$ Delay with n_0

$$x[n] * \delta[n-n_0] = x[n-n_0]$$

Commutative property of $*$:

$$x[n] * h[n] = h[n] * x[n]$$

Proof:

$$y[n] = x[n] * h[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l] \quad \text{with} \quad k = n - l$$

$$= \sum_{k=+\infty}^{-\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n]$$

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Some important properties LTI system

Associative property:

$$(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n])$$

Proof:

$$x_1[n] * (x_2[n] * x_3[n]) = \sum_{l=-\infty}^{\infty} x_1[l] \left(\sum_{k=-\infty}^{\infty} x_2[k] x_3[(n-l)-k] \right)$$

with $k = q - l \;\Rightarrow$ $$= \sum_{l=-\infty}^{\infty} x_1[l] \sum_{q=-\infty}^{\infty} x_2[q-l] x_3[n-q]$$

$$= \sum_{q=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} x_1[l] x_2[q-l] \right) x_3[n-q] = (x_1[n] * x_2[n]) * x_3[n]$$

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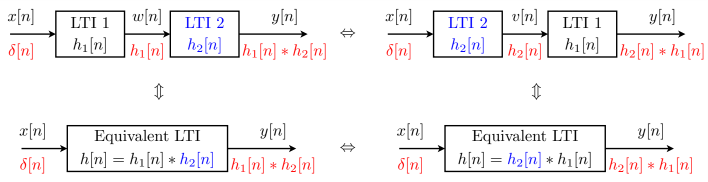
Cascade of LTI systems

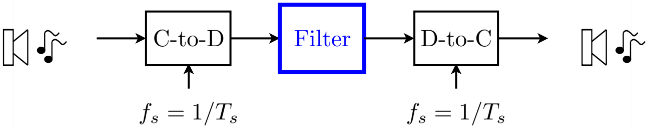


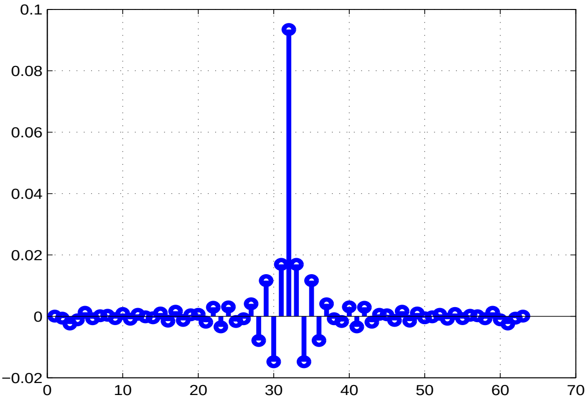
$$\begin{aligned} y[n] &= (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]) \\ &= x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n] \end{aligned}$$

Example: Evaluate $h[n] = h_1[n] * h_2[n]$ with

$$h_1[n] = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad h_2[n] = \begin{cases} 1 & 1 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Example FIR filtering





Summary Module FIR

- Finite Impulse Response (FIR) filters: Each output sample is sum of finite number of weighted samples of input sequence.
- Difference equation FIR: $y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$
  Weighted running average of M samples; Causal filter; M is order of FIR filter
- Unit Impulse Sequence: $\delta[n] = 1$ for $n = 0$ and zero elsewhere.
  Representation signal by unit impulses: $x[n] = \sum_k x[k]\delta[n-k]$
- Impulse response $h[n]$ of order M FIR:
  $h[n] = \sum_{k=0}^{M-1} b_k \delta[n-k] = b_n$
- Time-Invariance: If $x[n] \mapsto y[n]$ then $x[n-n_0] \mapsto y[n-n_0]$
- Linear system: If $x_1[n] \mapsto y_1[n]$ and $x_2[n] \mapsto y_2[n]$ then
  $x[n] = \alpha x_1[n] + \beta x_2[n] \mapsto y[n] = \alpha y_1[n] + \beta y_2[n]$

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Summary Module FIR

- **Main issues Linear Time-Invariance (LTI) system**: (FIR is LTI)
  Completely characterized by impulse response; Convolution is general formula to compute output from input.

- **Convolution sum FIR**: $y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$

- **Convolution operator**: $y[n] = x[n]*h[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$
  - Commutative property: $x[n]*h[n] = h[n]*x[n]$
  - Associative property: $(x_1[n]*x_2[n])*x_3[n] = x_1[n]*(x_2[n]*x_3[n])$

- **Cascade of LTI systems**: