Exercises

Module Spectrum And Fourier Series

Notes:

- Only the answers are available.
- The symbol [P] in the margin of an exercise denotes there is a pencast available.

Exercise 1
A signal composed of sinusoids is given by the equation

$$x(t) = 3\cos(50\pi t - \pi/8) - 5\cos(150\pi t + \pi/6)$$

Make a plot of the spectrum of this signal. Plot on the horizontal axis the frequency in [Hz] and indicate for each frequency of the signal a bar indicating the complex amplitude (magintude and phase)of each frequency component.

Exercise 2
Fig. 1 shows several signals along with their corresponding spectra. However, they are in a random order. For each spectrum plot (a)-(e), determine the correct signal (1)-(5).

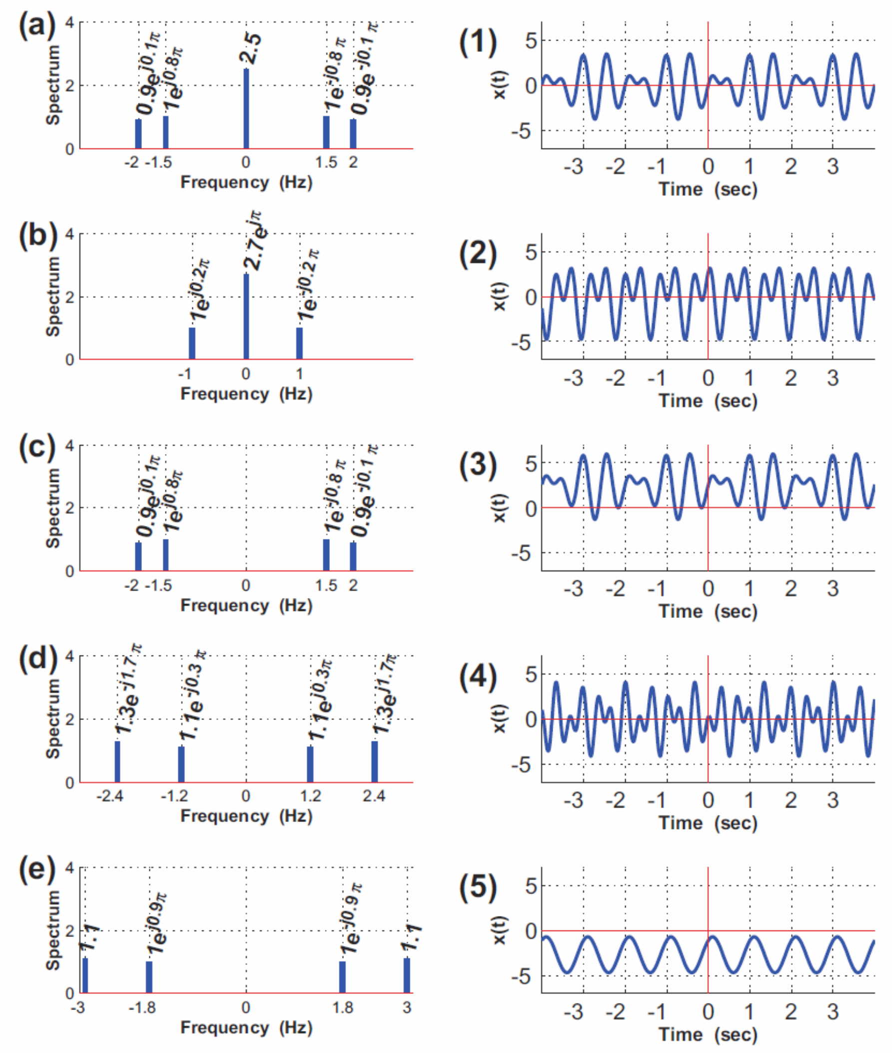

Figure 1: Five signals with their corresponding spectra.

Exercise 3
The frequency spectrum of the signal $x(t)$ is shown in Fig. 2.

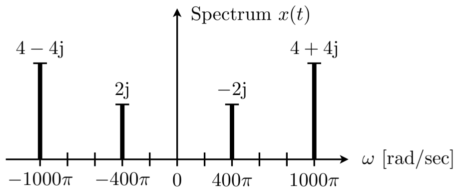

Figure 2: Frequency spectrum of $x(t)$.

Obtain a formula for the signal $x(t)$ as a sum of sinusoidal signals, i.e., in the form

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \phi_k).$$

Notes: Make sure that the amplitudes A_k are real-valued. Furthermore note that in Fig. 2 the horizontal axis of the spectral plot denotes the frequency in [rad/sec], with $\omega_k = 2\pi f_k$ and that the values of the bars are not given in Polar notation but in Cartesian notation.

Exercise 4

[P1] The incomplete spectrum of the *real* signal $x(t)$ is shown in Fig. 3.

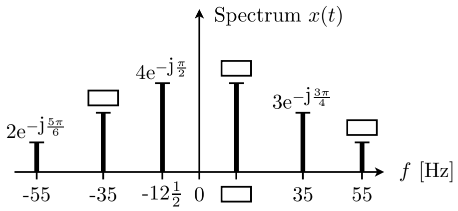

Figure 3: Spectrum of $x(t)$.

a. Fill in the empty boxes for the missing components.

b. Write an equation for $x(t)$ in terms of sinusoidal signals:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k).$$

Exercise 5

Given the spectrum of signal $x(t)$ in Fig. 4. Draw the spectrum of the following signals.

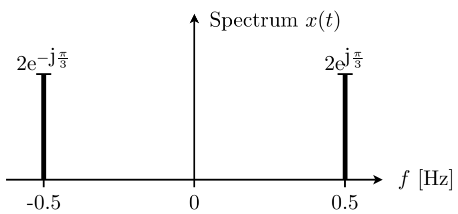

Figure 4: Spectrum of $x(t)$.

Note: Remember to label your axes and indicate the complex amplitudes in polar notation as in Fig. 4. Try to obtain your answers using as few mathematical derivations as possible.

a. $y_1(t) = 3x(t) - 1$,

b. $y_2(t) = x(t-1)$,

c. $y_3(t) = x(t) \cdot \cos(2.4\pi t)$.

Exercise 6

[P2] The signal $x(t)$ is formed from the signal $v(t)$ by amplitude modulation. Assume that

$$v(t) = 3 + 3\cos(10\pi t + \pi/3), \qquad \text{and} \qquad x(t) = v(t) \cdot \cos(40\pi t).$$

a. Draw the spectrum for $v(t)$.

b. Draw the spectrum for $x(t)$.

Exercise 7

Fig. 5 is the spectral plot of signal $x(t)$.

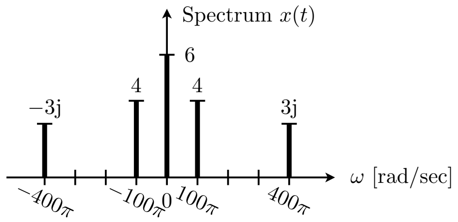

Figure 5: Spectrum of $x(t)$.

a. Write an equation for $x(t)$ in terms of sinusoidal signals:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k).$$

b. Determine the fundamental period T_0 of $x(t)$.

c. Write this signal as a Fourier series of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 k t}$$

in which F_0 denotes the fundamental frequency $F_0 = 1/T_0$. Determine which coefficients α_k (spectral weights) have non-zero value. List these Fourier series coefficients and their values.

Exercise 8

A periodic signal $x(t)$ is given by

$$x(t) = 1 + 3\cos(300\pi t) + 2\sin(500\pi t - \pi/4)$$

a. This signal is a periodic signal. Thus we can write it as a Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 k t}$, with the fundamental frequency $F_0 = 1/T_0$. What is the fundamental period T_0 of $x(t)$?

b. Find the Fourier series coefficients α_k of $x(t)$.

Exercise 9
The frequency spectrum of the signal $x(t)$ is shown in Fig. 2. This signal is a periodic signal. Thus we can write it as a Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}\omega_0 kt}$, with the fundamental frequency ω_0. Determine ω_0 as well as the Fourier coefficients α_k of $x(t)$.

Exercise 10
An amplitude-modulated signal $x(t)$ can be written as

$$x(t) = s(t) \cdot g(t).$$

The carrier signal $g(t)$, with carrier frequency $f_c = 10000$ [Hz], and the message signal $s(t)$ are given as

$$g(t) = \cos(2\pi f_c t) \qquad \text{and} \qquad s(t) = 1 + \cos(500\pi t + \pi/2)$$

a. Draw the frequency spectrum of $x(t)$, with on the horizontal axis the frequency f in [Hz].

b. Since $x(t)$ is periodic we are able to write it as a Fourier series $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{\mathrm{j}2\pi F_0 kt}$ with the fundamental frequency $F_0 = 1/T_0$ and the Fourier coefficients α_k. Evaluate F_0 and the coefficients α_k.

Exercise 11
A signal composed of sinusoidal signals is given by the equation:

$$x(t) = 3\cos(50\pi t - \pi/8) - 5\cos(150\pi t + \pi/6)$$

a. Is $x(t)$ periodic? If so, what is the fundamental period $T_{0,x}$? Which harmonics are present?

b. Now consider a new signal:

$$y(t) = x(t) + 7\cos(160\pi t - \pi/3).$$

How is the spectrum changed? Is $y(t)$ periodic? If so, what is the fundamental period $T_{0,y}$?

c. Finally, consider another new signal

$$w(t) = x(t) + \cos\left(5\sqrt{2}\pi t + \pi/3\right).$$

How is the spectrum changed? Is $w(t)$ periodic? If so, what is the fundamental period $T_{0,w}$? If not, why not?

Exercise 12
[P3] A periodic signal $x(t)$ with a period $T_0 = 4$ is described *over one period*, $0 \le t \le T_0$, by the equation

$$x(t) = \begin{cases} 2 & 0 \le t \le 2 \\ 0 & 2 < t \le 4 \end{cases}$$

a. Sketch the periodic function $x(t)$ for $-4 < t < 8$.

b. Determine the DC coefficient α_0 of the Fourier Series.

c. Use the Fourier analysis integral (for $k \neq 0$)

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{e}^{-\mathrm{j}2\pi F_0 kt}\mathrm{d}t \quad \text{with fundamental frequency } F_0 = 1/T_0$$

to find the Fourier series coefficients, α_k.

d. This periodic signal $x(t)$ can be expressed with the Fourier series as:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}$$

In practice we approximate such a periodic signal with a finite number of harmonics as follows:

$$\hat{x}(t) = \sum_{k=-N}^{N} \alpha_k e^{j2\pi F_0 kt}$$

Make a sketch of $\hat{x}(t)$ for $N = 1$ showing that this approximation with one harmonic is a reasonable approximation of $x(t)$.

e. Now we replace this periodic signal $x(t)$ with another related periodic signal $y(t)$ which is defined as:

$$y(t) = 2x(t + \frac{T_0}{2}) - 1$$

Since $y(t)$ is again periodic with the same period T_0 we can write it as the following Fourier series:

$$y(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{j2\pi F_0 kt}$$

How are the Fourier coefficients β_k of signal $y(t)$ related to the Fourier coefficients α_k of signal $x(t)$? Try to give a physical explanation of this result.

Exercise 13

Let $x(t)$ be the periodic signal shown in Fig. 6.

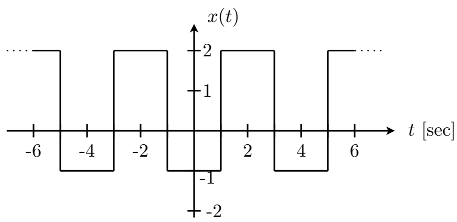

Figure 6: Plot of periodic signal $x(t)$.

Since $x(t)$ is a periodic signal with fundamental period $T_0 = 1/F_0$ we can write it by its Fourier series expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}$$

a. Consider the signal $y(t)$ shown in Fig. 7, which is related to $x(t)$ by $y(t) = 2x(t) + 3$. This signal is clearly again a periodic signal with the same fundamental period T_0 as $x(t)$ so we can write this signal by its Fourier series expansion:

$$y(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{j2\pi F_0 kt}$$

Express the Fourier series coefficients for this signal, β_k, in terms of the coefficients α_k for $x(t)$.

Hint: This is a simple relationship, and finding it should not require that you compute any of the coefficients explicitly.

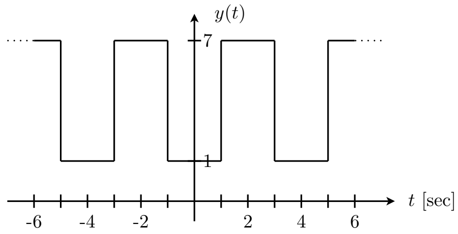

Figure 7: Plot of periodic signal $y(t)$.

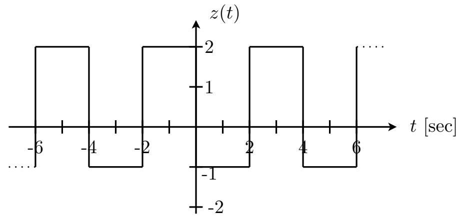

Figure 8: Plot of periodic signal $z(t)$.

b. Consider the periodic signal $z(t)$ shown in Fig. 8, which is related to $x(t)$ by $z(t) = x(t-1)$. This signal $z(t)$ has again the same fundamental period T_0 as $x(t)$ so we can write this signal by its Fourier series expansion:

$$z(t) = \sum_{k=-\infty}^{\infty} \gamma_k e^{j2\pi F_0 kt}$$

Express the Fourier series coefficients for this signal, γ_k, in terms of the coefficients α_k for $x(t)$. Again, this is a simple relationship, and finding it should not require that you compute any coefficients explicitly.

Exercise 14

Write the signal $x(t) = \cos^3(100\pi t)$ as a Fourier series, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\omega_0 kt}$$

with fundamental frequency $\omega_0 = 2\pi F_0$.

Exercise 15

Consider the time-domain plots and frequency spectra shown below, as well as the time-domain formulas and Fourier series coefficients listed below the figures. Together, these eight signal representations (R1-R8) describe four different signals. Each signal is characterized by two of these representations. Link the corresponding signal representations.

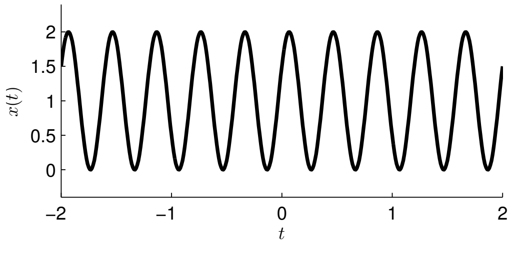

Figure 9: R1: time-domain plot

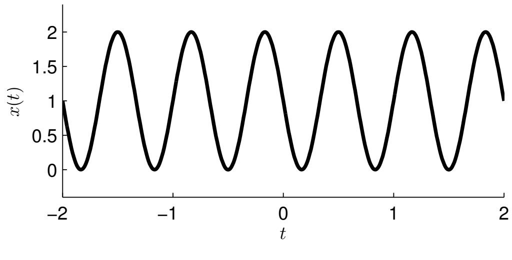

Figure 10: R2: time-domain plot

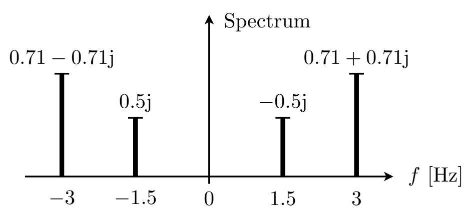

Figure 11: R3: frequency spectrum

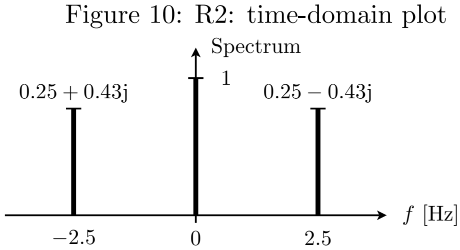

Figure 12: R4: frequency spectrum

- R5: time-domain formula $x(t) = \cos\left(2\pi 1.5t - \dfrac{\pi}{2}\right) + 2\cos\left(2\pi 3t + \dfrac{\pi}{4}\right)$
- R6: time-domain formula $x(t) = 2\cos\left(2\pi 1.5t + \dfrac{\pi}{4}\right) + \cos\left(2\pi 2.5t - \dfrac{3\pi}{2}\right)$
- R7: Fourier series with $\omega_0 = 3\pi$ and coefficients $\alpha_k = \begin{cases} -j\frac{1}{2} & k=-1 \\ 1 & k=0 \\ j\frac{1}{2} & k=1 \\ 0 & \text{otherwise} \end{cases}$
- R8: Fourier series with $\omega_0 = \pi$ and coefficients $\alpha_k = \begin{cases} -j\frac{1}{2} & k=-5 \\ -\frac{1}{2}\sqrt{2} - j\frac{1}{2}\sqrt{2} & k=-3 \\ \frac{1}{2}\sqrt{2} + j\frac{1}{2}\sqrt{2} & k=3 \\ j\frac{1}{2} & k=5 \\ 0 & \text{otherwise} \end{cases}$

Exercise 16
The signal $x(t)$ is a periodic triangular signal, for which $x(t) = x(t+T_0)$ holds. A complete description of $x(t)$ is given by the following formula for one period of $x(t)$

$$x(t) = \begin{cases} \frac{2t}{T_0} & \text{for } 0 \le t \le \frac{T_0}{2} \\ 2 - \frac{2t}{T_0} & \text{for } \frac{T_0}{2} < t < T_0 \end{cases}$$

Since $x(t)$ is a periodic signal with fundamental period $T_0 = 1/F_0$ we can write it by its Fourier series expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi F_0 kt}.$$

a. Make a sketch of the periodic function $x(t)$ for $|t| \le 2T_0$.

b. Determine the DC coefficient of the Fourier Series, α_0.

c. Use the Fourier analysis integral

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi F_0 kt} dt$$

to determine a general formula for the Fourier Series coefficients α_k. Your final result for α_k should depend on k.
Note: You can use the following integral:

$$\int_A^B x e^{-x} dx = -(x+1)e^{-x}|_A^B$$

d. In practice we approximate such a periodic signal with a finite number of harmonics as follows:

$$\hat{x}(t) = \sum_{k=-N}^{N} \alpha_k e^{j2\pi F_0 kt}$$

Make a sketch of $\hat{x}(t)$ for $N = 1$ showing that this approximation with one harmonic is a reasonable approximation of $x(t)$.