Reader

Fourier Transform Continuous-time signals

P. Sommen and B. van Erp

May 29, 2019

Contents

1   Introduction

The Fourier Series (FS) representation is an extremely useful signal representation. Unfortunately, this signal representation can only be used for periodic signals. That's why we will introduce a tool for representing non-periodic signals, known as the Fourier Transform for Continuous time signals, abbreviated as FTC. One might wonder if we can somehow use the Fourier Series to develop a representation for non-periodic signals. As it turns out, this is possible. By viewing a non-periodicas a periodic signal with an infinitely large period, we can use the Fourier Series to develop a more general signal representation that can be used in the non-periodic case. By doing so we should realise that the development of the FTC is not completely rigorous. Instead it is plausible which suggests the correct form of the FTC equations and it provides a useful interpretation.

2   Fourier Series (FS)

The development of the Fourier Series was based on the following General Phasor Integral (GPI-FS) property:
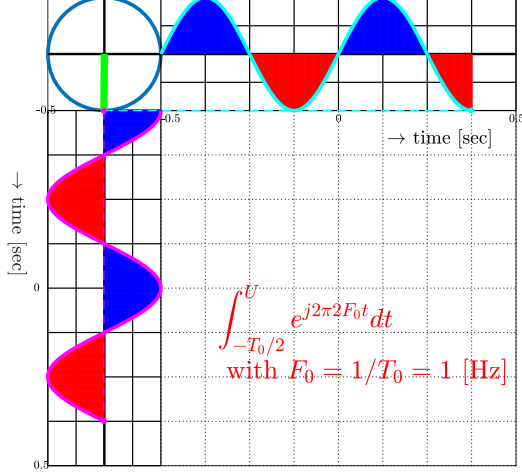
$$\int_{-(T_0/2)}^{T_0/2} e^{j2\pi k \cdot F_0 t} dt = \begin{cases} \neq 0 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{1}$$

An example of the calculation of this integral is shown in Figure 1. The figures show the projection of a phasor with frequency $2 \times F_0$, with $F_0 = 1/T_0 = 1$ [Hz], on both the real and imaginary axis. The result of the integral of this phasor is represented by a colored surface: blue for a positive and red for a negative surface. The upper bound U of the integral in the left hand figure equals $3T_0/8$ and in the right hand figure $U = T_0/2$. From the right hand figure it follows that the total amount of red and blue area is the same. This implies that the total surface (or integral), over 2 rotations, is equal to zero. It is clear that this is true for any frequency which equals an integer multiple of k times the fundamental frequency F_0. There is only one exception and that is when this multiple represents frequency 0 [Hz].
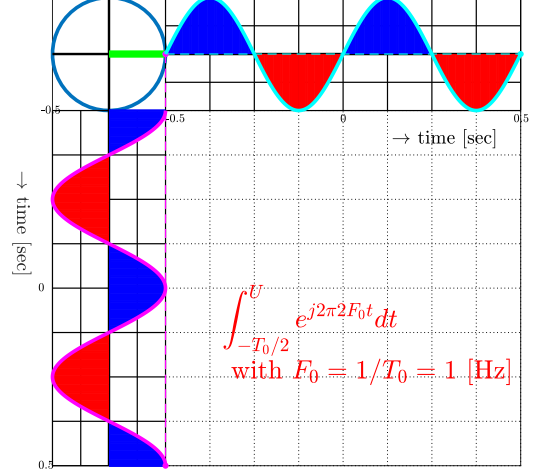
This GPI-FS property leads to the Fourier Series as given in the following equation:

$$\alpha_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t) e^{-j2\pi k F_0 t} dt \quad \circ\!\!-\!\!\circ \quad x_p(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k F_0 t}$$

(a) Upper bound intergral $U = 3T_0/8$ (b) Upper bound intergral $U = T_0/2$

Figure 1: Different stages General Phasor Integral property (GPI-FS)

By multiplying a periodic signal $x_p(t)$ with a phasor with frequency $k \cdot F_0$ and taking the integral over one period T_0, the result of the GPI-FS property is that only the frequency $k \cdot F_0$ of the periodic signal $x_p(t)$ is triggered. In case the frequency $k \cdot F_0$ is present, the integral results in a measure α_k, representing the frequency content of the periodic signal $x_p(t)$ at frequency $k \cdot F_0$. When this particular frequency $k \cdot F_0$ is not present in the periodic signal $x_p(t)$ the value of α_k will result in zero. This implies that a periodic signal can only have discrete (related) frequencies $k \cdot F_0$, which leads to the general expression of a periodic signal $x_p(t)$ which writes as a, possibly infinite, sum of weighted phasors. An example of a periodic signal $x_p(t)$, with fundamental period $T_0 = 1/F_0$, and its Fourier series representation is shown in Figure 2. The weights α_k represent the frequency content of the periodic signal $x_p(t)$.

3   FTC as limit of FS

The FTC and Inverse FTC are defined as follows:
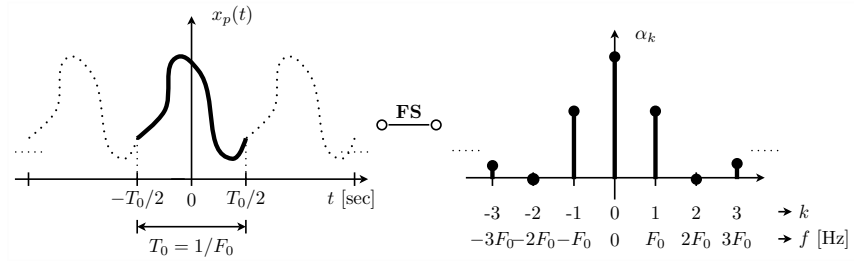
Figure 2: Example periodic signal $x_p(t)$ and its Fourier Series representation

$$X(f) = \mathscr{F}\{x(t)\}) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi f t}\mathrm{d}t \quad \circ\!\!-\!\!\circ \quad x(t) = \mathscr{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\mathrm{e}^{\mathrm{j}2\pi f t}\mathrm{d}f$$

In contrast to the Fourier Series the FTC equations are:

- based on **non-periodic** signals,

- based on **the entire** signal,

- its frequencies f are **continuous** and

- $x(t)$ can be expressed as an **integral** of phasor components.

The correctness of the FTC equations can be shown by relating these equations with the Fourier series expressions. For this we assume a finite length non-periodic signal $x(t)$, with length $\Delta T = 1/\Delta F$, as depicted in Figure 3. The right end side of the figure shows a sketch of its frequency distribution $X(f)$. On the one hand we can show that the non-periodic signal $x(t)$ can

Figure 3: Non-periodic signal $x(t)$ and its frequency distribution $X(f)$

5

be approximated by an infinite sum of weighted phasors, which is similar to the general representation of a periodic signal. In order to do s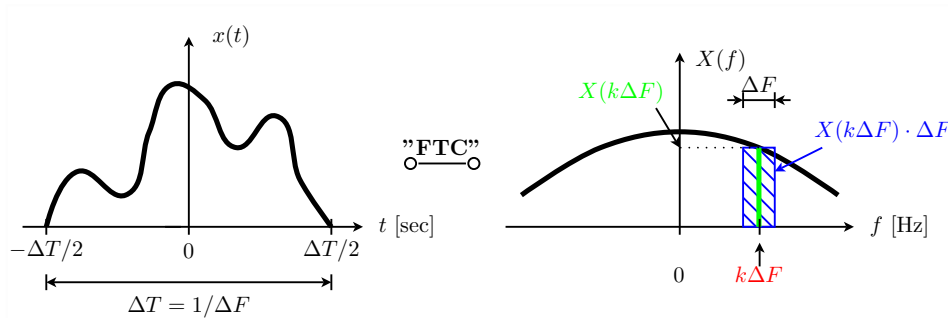o we write the non-periodic signal $x(t)$ as the Inverse FTC integral. By splitting the continuous frequency axis f of $X(f)$ of a small width ΔF and replacing $\mathrm{d}f$ of the integral with ΔF, it is possible to think of the integral as an infinite sum as represented in the following equation:

$$x(t) = \mathscr{F}^{-1}\{X(f)\} \overset{\text{def}}{=} \int_{-\infty}^{\infty} X(f)\mathrm{e}^{\mathrm{j}2\pi ft}\mathrm{d}f \overset{\Delta F \to 0}{\approx} \sum_{k=-\infty}^{\infty} \left(X(k\Delta F)\Delta F\right)\mathrm{e}^{\mathrm{j}2\pi k\Delta Ft}$$

As follows from Figure 3, the value of the product $X(k\Delta F)\Delta F$ (in blue) represents an approximation of the surface of the frequency distribution $X(f)$ at frequency $f = k \cdot \Delta F$. When comparing the resulting infinite sum of weighted phasors to the Fourier Series expression of a periodic signal, it follows that the surface $X(k\Delta F)\Delta F$ represent the weight α_k of the Fourier Series expression. On the other hand, when using the Fourier Series expression for these weights, multiplied by ΔT, we obtain the Fourier Series integral within the boundaries $-\Delta T/2$ until $+\Delta T/2$ as given in the following equation:

$$\left(X(k\Delta F)\Delta F\right) \cdot \Delta T \quad \widehat{=} \quad \alpha_k \cdot \Delta T \overset{\text{FS}}{=} \int_{-\Delta T/2}^{\Delta T/2} x_p(t)\mathrm{e}^{-\mathrm{j}2\pi(k\Delta F)t}\mathrm{d}t$$

$$\overset{k\Delta F \to f}{\approx} \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t \overset{\text{def}}{=} X(f) \tag{2}$$

When the length ΔT goes to infinite, or equivalently when ΔF goes to zero and $k\Delta F$ becomes close to the continuous frequency f, the non-periodic signal $x(t)$ can be viewed as the limiting case of the periodic signal $x_p(t)$ and the result is FTC equation.

As mentioned before we should realise that the given development of the FTC equations is not completely rigorous. Instead it is plausible which suggests the correct form of the FTC equations and it provides a useful interpretation.

3.1 Example rectangular signal

In order to further clarify the given insight we will evaluate the FTC, as an approximation of the Fourier Series, of the finite length non-periodic rectangular signal $x(t)$, which is depicted in Figure 4 and defined by the

following equation:

$$x(t) = \begin{cases} 1 & -\Delta T/2 < t < \Delta T/2 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{and} \qquad x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0) \quad (3)$$

With $T_0 > \Delta T$, we can interpret the non-periodic signal $x(t)$ as **one** pe-
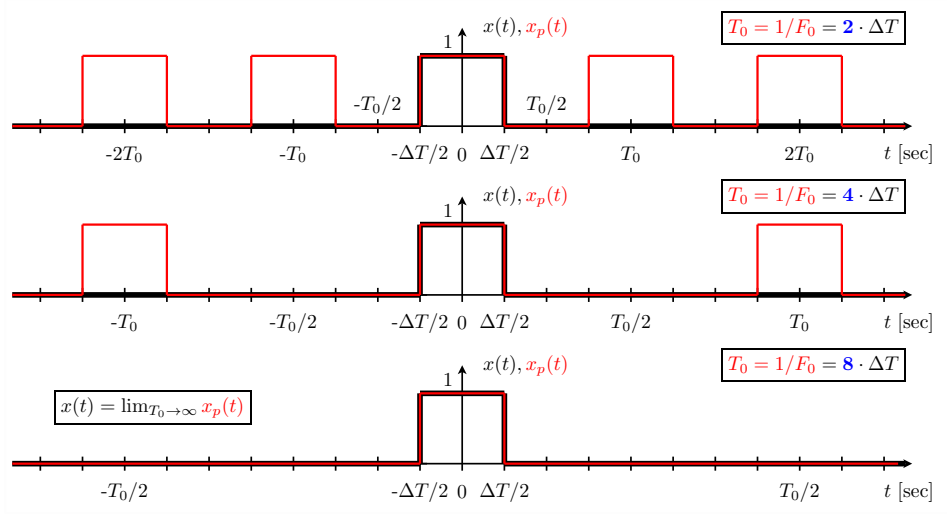
Figure 4: Similarity $x(t)$ and $x_p(t)$ for different period lengths T_0

riod of the periodic signal $x_p(t)$, with period T_0. The upper figure shows an example for $T_0 = 2\Delta T$. When increasing the period ΔT, as depicted in the other figures, it follows that the non-periodic signal $x(t)$ can be approximated by the periodic signal $x_p(t)$ when the period T_0 goes to infinity as follows:

$$x(t) = \lim_{T_0 \to \infty} x_p(t) \qquad (4)$$

From this it follows that the FTC of the non-periodic signal $x(t)$ can be viewed as a reasonable approximation of the Fourier Series of the periodic signal $x_p(t)$. The weights α_k multiplied by T_0 of the periodic signal $x_p(t)$ can be evaluated via the following Fourier Series:

$$\alpha_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t) e^{-j2\pi kF_0 t} dt \quad \overset{\text{FS}}{\circ\!\!-\!\!\circ} \quad x_p(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi kF_0 t} \qquad (5)$$

The block signal $x_p(t)$ equals 1 within the boundaries $-\Delta T/2$ and $+\Delta T/2$, which leads to the following result:

$$\alpha_k \cdot T_0 \quad = \quad \int_{-\Delta T/2}^{\Delta T/2} 1\mathrm{e}^{-\mathrm{j}2\pi kF_0 t}\mathrm{d}t = \frac{1}{-\mathrm{j}2\pi kF_0}\left.\mathrm{e}^{-\mathrm{j}2\pi kF_0 t}\right|_{-\Delta T/2}^{\Delta T/2} \qquad (6)$$

$$= \quad \frac{\mathrm{e}^{-\mathrm{j}2\pi kF_0\Delta T/2} - \mathrm{e}^{\mathrm{j}2\pi kF_0\Delta T/2}}{-\mathrm{j}2\pi kF_0} = \Delta T \cdot \frac{\sin(\pi kF_0\Delta T)}{\pi kF_0\Delta T} \qquad (7)$$

With $F_0 = 1/T_0$ and for the limiting case $F_0 \to 0$ and $kF_0 \to f$ we obtain:

$$\Rightarrow \quad \lim_{F_0 \to 0} \alpha_k \cdot T_0 \widehat{=} X(f) \to \Delta T \cdot \frac{\sin(\pi f\Delta T)}{\pi f\Delta T} \quad (\widehat{=} \text{ Sinc}) \qquad (8)$$

A plot of this function for increasing values of T_0 is depicted in Fig. 5. When
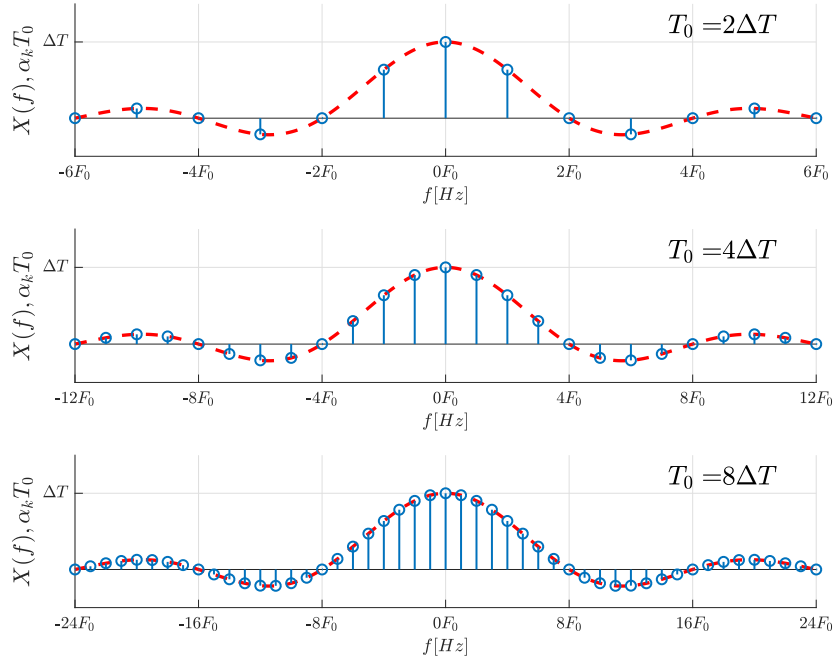

Figure 5: Plot of $X(f)$ and $\alpha_k \cdot T_0$ for increasing T_0

increasing T_0 we see that the values of the weights α_k multiplied by T_0 get closer and closer together and eventually become dense and approach the continuous envelope sinc function.

8

Until now we have discussed the FTC equations as a function of frequency f in [Hz]. In literature we also find the same FTC equations as a function of the radian frequency ω in [rad/sec] as defined in the following equation:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \quad \circ\!\!-\!\!\circ \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega$$

With $\omega = 2\pi f$, the replacement of the integral variable f by ω, in the Inverse FTC equation, results in the pre-multiplication factor of $1/2\pi$ in this case.

3.2 Existence and convergence FTC

Not every function $x(t)$ has an FTC representation. So it would be helpful to be able to determine whether the FTC exists or not. As a simple condition for convergence of the FTC integral, we can check the magnitude $|X(f)|$, as denoted by equation (9). By first using the fact that the absolute value of an integral is smaller or equal than the integral of the absolute value (10) and then that the absolute value of the given phasor equals one, we obtain the resulting integral (27).

$$|X(f)| = \left| \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t \right| \tag{9}$$

$$\leq \int_{-\infty}^{\infty} \left| x(t)\mathrm{e}^{-\mathrm{j}2\pi ft} \right| \mathrm{d}t = \int_{-\infty}^{\infty} |x(t)|\,\mathrm{d}t \tag{10}$$

$$\Rightarrow \quad |X(f)| < \infty \quad \leftrightarrow \quad \int_{-\infty}^{\infty} |x(t)|\,\mathrm{d}t < \infty \tag{11}$$

This implies that a sufficient, but not necessary, condition to check the existence of the FTC of a signal $x(t)$ is to evaluate the integral of the absolute value of $x(t)$ and verifying if the result is bounded.

4 FTC Examples

In this section we will discuss the FTC of some basic signals. As such we will discuss the FTC and IFTC of a delta pulse, a block signal, a sinusoidal signal and finally a pulse train.

4.1 (I)FTC delta pulse and General Phasor Integral (GPI)

The FTC of a delta pulse $\delta(t)$ can be found as follows:

$$X(f) = \mathscr{F}\{x(t)\} = \mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = 1 \ \ \forall f \qquad (12)$$

This implies that all frequencies, from $-\infty$ to $+\infty$, are present in a delta pulse.

The other way around it follows, according the definition of a delta pulse, that the IFTC of a delta pulse $\delta(f)$ in frequency domain can be found as follows:

$$x(t) = \mathscr{F}^{-1}\{X(f)\} = \mathscr{F}^{-1}\{\delta(f)\} = \int_{-\infty}^{\infty} \delta(f)e^{-j2\pi ft}df = 1 \ \ \forall t \qquad (13)$$

From this we can derive an important integral property which is basic for the development of the Fourier transform. This derivation goes as follows:

$$\mathscr{F}^{-1}\{\delta(f)\} = 1 \ \Leftrightarrow \ \delta(f) = \mathscr{F}\{1\} \ \Leftrightarrow \ \delta(f) = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi ft}dt$$

Thus a delta pulse $\delta(f)$ in frequency domain is the result of integrating a phasor with frequency f for time t from minus to plus infinity. In other words: The integral of a phasor is zero for all frequencies f except for frequency $f = 0$ (DC), then the result is infinite. By replacing the frequency f by $f \pm f_0$ we obtain the following so called General Phasor Integral (GPI):

$$\boxed{\int_{-\infty}^{\infty} e^{-j2\pi(f\pm f_0)t}dt \ \overset{\text{GPI}}{=} \ \delta(f \pm f_0)}$$

4.2 (I)FTC of block

In section 3.1 we have approximated the FTC of a block signal by viewing it as an approximation of a periodic block signal. By doing so we could evaluate the FTC of a block signal as an approximation of the Fourier Series of a periodic block signal. The following derivations show that the result is the same by applying the FTC equation.

The block signal $x(t)$ is shown in Fig. 6 and defined as:

$$x(t) = \begin{cases} 1 & \text{for } |t| \leq T_0/2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluating the FTC of this block signal results into the following:

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt = \int_{-T_0/2}^{T_0/2} 1 \cdot e^{-j2\pi ft} dt \\
&= \frac{1}{-2j\pi f} e^{-j2\pi ft} \Big|_{-T_0/2}^{T_0/2} = \frac{e^{-j2\pi fT_0/2} - e^{j2\pi fT_0/2}}{-2j\pi f} \\
&= T_0 \frac{\sin(\pi fT_0)}{\pi fT_0} = T_0 \ \text{sinc}(\pi fT_0) \qquad\qquad (14)
\end{aligned}
$$

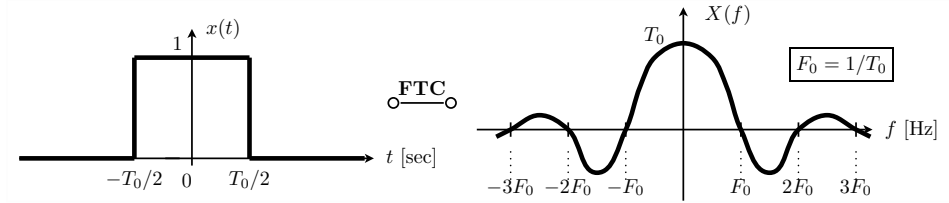In other words: The FTC of a block-function in time domain is a sinc-

Figure 6: FTC of time domain block results in sinc in frequency domain

function in frequency domain as depicted in Fig. 6.

By following similar steps we can evaluate the IFTC of a block signal $X(f)$ in the frequency domain, in which the block signal is defined as:

$$
X(f) = \begin{cases} 1 & \text{for } |f| \leq F_c \\ 0 & \text{elsewhere} \end{cases}
$$

and depicted in Fig. 7. The evaluation of the IFTC is as follows:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi ft} df \int_{-F_c}^{F_c} 1 \cdot e^{j2\pi ft} df = \frac{1}{2j\pi t} e^{j2\pi ft} \Big|_{-F_c}^{F_c} \\
&= \frac{e^{j2\pi F_c t} - e^{-j2\pi F_c t}}{2j\pi t} = 2F_c \frac{\sin(2\pi F_c t)}{2\pi F_c t} = 2F_c \ \text{sinc}(2\pi F_c t) \quad (15)
\end{aligned}
$$

In other words: The IFTC of a block-function in frequency domain is a sinc-function in time domain as depicted in Fig. 7.
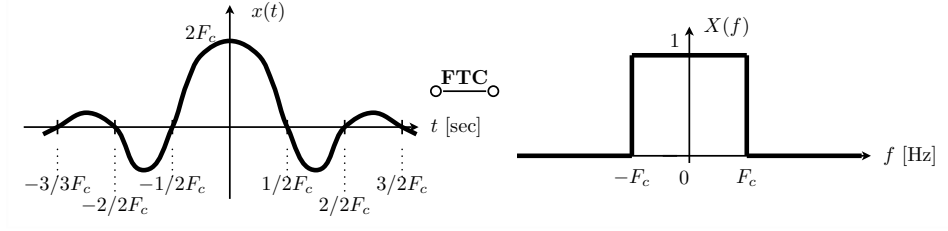
Figure 7: IFTC of frequency domain block results in sinc in time domain

4.3 FTC Sinusoidal

The following derivations describe the evaluation of the FTC of a sinuoidal function:

$$x(t) = \cos(2\pi F_0 t) = \frac{1}{2}e^{j2\pi F_0 t} + \frac{1}{2}e^{-j2\pi F_0 t}$$

$$\mathscr{F}\{e^{j2\pi F_0 t}\} = \int_{-\infty}^{\infty} e^{-j2\pi(f-F_0)t}\mathrm{d}t \overset{\mathrm{GPI}}{=} \delta(f - F_0)$$

$$\mathscr{F}\{e^{-j2\pi F_0 t}\} = \int_{-\infty}^{\infty} e^{-j2\pi(f+F_0)t}\mathrm{d}t \overset{\mathrm{GPI}}{=} \delta(f + F_0)$$

$$\Rightarrow \quad \mathscr{F}\{\cos(2\pi F_0 t)\} = \frac{1}{2}\delta(f - F_0) + \frac{1}{2}\delta(f + F_0) \tag{16}$$

In the first step we use Euler to split the signal into phasors. Then we can use for each of these phasor the GPI property, which results in a sum of two delta pulses in the frequency domain. In other words the frequency distribution of a sinusoidal signal, which runs from minus to plus infinity, is zero for all frequencies, except for the two frequencies of the phasor components of the sinusoidal signal the result is infinite. The result is depicted in Fig. 8.
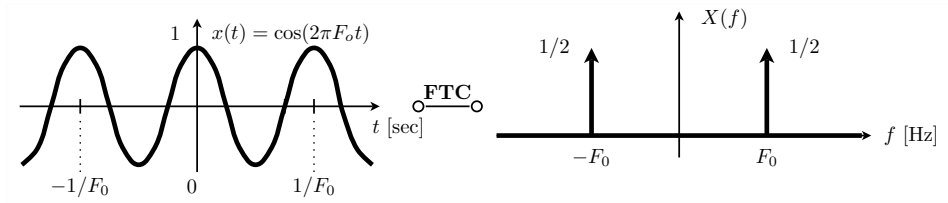

Figure 8: IFTC of frequency domain block results in sinc in time domain

4.3.1 Sufficient condition for existence of FTC

In section 3.2 we have shown that a sufficient condition for the FTC to exist is that the integral of the absolute value of the signal $x(t)$ should be bounded, thus $\int_{-\infty}^{\infty} |x(t)| \mathrm{d}t < \infty$. However, when we take the integral of the absolute value of the previous sinusoidal signal, the result is not bounded. In other words since we have found a valid FTC result for the sinusoidal signal, it follows that the given condition for the existence of the FTC is a sufficient but not a necessary condition.

4.3.2 Comparison with FS Sinusoidal

When viewing the sinusoidal signal $x(t)$ as a periodic signal $x_p(t)$, we can evaluate the frequency content by applying the Fourier Series of one period. The first steps of these derivations are as follows:

$$
\begin{aligned}
\alpha_k &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t) \mathrm{e}^{-\mathrm{j}2\pi k F_0 t} \mathrm{d}t = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \cos(2\pi F_0 t) \mathrm{e}^{-\mathrm{j}2\pi k F_0 t} \mathrm{d}t \\
&= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left(\frac{1}{2} \mathrm{e}^{\mathrm{j}2\pi F_0 t} + \frac{1}{2} \mathrm{e}^{-\mathrm{j}2\pi F_0 t} \right) \mathrm{e}^{-\mathrm{j}2\pi k F_0 t} \mathrm{d}t \\
&= \frac{1}{2T_0} \int_{-T_0/2}^{T_0/2} \mathrm{e}^{-\mathrm{j}2\pi(k-1)F_0 t} \mathrm{d}t + \frac{1}{2T_0} \int_{-T_0/2}^{T_0/2} \mathrm{e}^{-\mathrm{j}2\pi(k+1)F_0 t} \mathrm{d}t
\end{aligned}
$$

Using the General Phasor Integral Property for Fourier Series GPI-FS, as discussed in section 2, it follows that the result is zero except for frequencies $\pm F_0$. For these frequencies the result is finite, namely $1/2$ as denoted in the following equation:

$$
\Rightarrow \alpha_k \overset{\text{GPI-FS}}{=} \begin{cases} \frac{1}{2} & \text{for } f = (\pm 1) \cdot F_0 \\ 0 & \text{elsewhere} \end{cases} \tag{17}
$$

This result is also depicted in Fig 9 From this result it follows that the main difference between the FTC and FS of a sinusoidal signal is the finiteness of the value at the phasor frequencies $\pm F_0$ of the sinusoidal signal. The main "mathematical" reason for this difference is the fact that the integral of the FS is taken of one period, while the integral of the FTC is calculated from $-\infty$ to $+\infty$.

4.4 FTC Deltas

As a final example we will evaluate the FTC of an infinite train of delta pulses in time domain as depicted in Fig. 11. This result will be used for
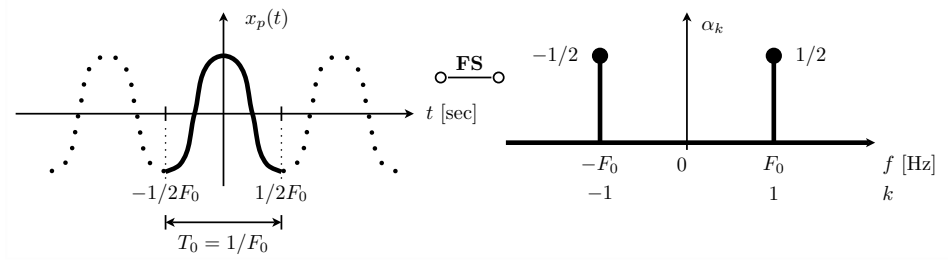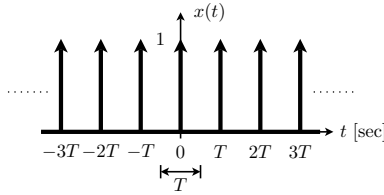
Figure 9: FS of sinusoidal signal

Figure 10: FTC of infinite train of delta pulses

the explanation of the conversion from continuous to discrete time domain.

The first step in evaluating the FTC is to find an alternative representation for the infinite pulse train. For this we view the infinite pulse train as a periodic signal with period T, such that we can write this signal as an infinite sum of weighted phasors. The weights α_n can be evaluated by the FS integral as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - n \cdot T) \stackrel{\mathrm{FS}}{\Longleftrightarrow} \sum_{n=-\infty}^{\infty} \alpha_n e^{\mathrm{j}2\pi \frac{1}{T}nt}$$

$$\text{with } \alpha_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-\mathrm{j}2\pi \frac{1}{T}nt} \mathrm{d}t = \frac{1}{T} \ \forall n$$

From this it follows that we obtain the following alternative expression for the train of delta pulses:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - n \cdot T) \hat{=} \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{\mathrm{j}2\pi \frac{1}{T}nt} \tag{18}$$

In other words the FTC of the infinite train of delta pulses can be evaluated by applying the FTC of the infinite sum of phasors which results in the

14

following derivation:

$$
\begin{aligned}
\mathscr{F}\{x(t)\} &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \delta(t - n \cdot T) \right) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{1}{T} nt} \right) e^{-j2\pi ft} dt \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{j2\pi \frac{1}{T} nt} e^{-j2\pi ft} dt \right) \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-j2\pi (f - \frac{n}{T})t} dt \right) \overset{\mathrm{GPI}}{=} \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T})
\end{aligned}
$$

First we changed the order of integration and summation and then we combined the two phasors. After that we used the GPI property which results in an infinite train of delta pulses in frequency domain as denoted in the following equation and depicted in Fig. 11.

$$
\sum_{n=-\infty}^{\infty} \delta(t - n \cdot T) \quad \overset{\text{FTC}}{\circ\!-\!\circ} \quad \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T})
$$
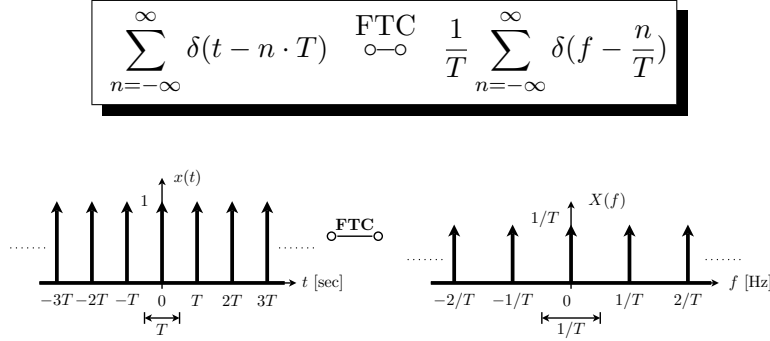
Figure 11: FTC of a train of delta pulses

Note that the resulting train of delta pulses in frequency domain is scaled by the factor 1/T and that the inter-pulse-distance is inversely proportional to the inter-pulse-distance in time domain. In other words, if we increase the period T in the time domain, by taking less pulses per time unit, the repetitions of the pulses in frequency domain become spaced more closely. This property is the basis for the so-called aliasing effect of an A-to-D convertor.

5 FTC Properties

In this section we will discuss the main properties of the FTC.

5.1 Linearity

As a result of the fact that the integration operation of the FTC is a linear operation it follows that the FTC is linear.

$$x(t) \circ\!\!-\!\!\circ X(f) \quad \text{and} \quad y(t) \circ\!\!-\!\!\circ Y(f)$$

$$\boxed{a \cdot x(t) + b \cdot y(t) \quad \circ\!\!-\!\!\circ \quad a \cdot X(f) + b \cdot Y(f)}$$

Proof

$$\mathscr{F}\{a \cdot x(t) + b \cdot y(t)\} = \int_{-\infty}^{\infty} (a \cdot x(t) + b \cdot y(t)) \, e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} (a \cdot x(t)) \, e^{-j2\pi ft} dt + \int_{-\infty}^{\infty} (b \cdot y(t)) \, e^{-j2\pi ft} dt$$

$$= a \cdot \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt + b \cdot \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \qquad (19)$$

$$= a \cdot X(f) + b \cdot Y(f)$$

The linearity property implies that of the sum of signals equals the sum of the FTC's of the signals.

Example Linearity

As an example of this linear property we calculate the FTC of a sinusoidal signal. By using Euler we can write the signal into two phasors. Each of these phasors transforms into a delta pulse in frequency domain.

$$\text{Euler} \rightarrow \cos(2\pi f_0 t) = \frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t}$$

$$e^{j2\pi f_0 t} \circ\!\!-\!\!\circ \delta(f - f_0)$$

$$e^{-j2\pi f_0 t} \circ\!\!-\!\!\circ \delta(f + f_0)$$

By using the linearity property we can now write the FTC of the sum of phasors as the sum of the FTC's of the two individual phasors.

$$\Rightarrow \mathscr{F}\{\cos(2\pi f_0 t)\} = \mathscr{F}\{\frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t}\} \qquad (20)$$

$$= \frac{1}{2} \mathscr{F}\{e^{j2\pi f_0 t}\} + \frac{1}{2} \mathscr{F}\{e^{-j2\pi f_0 t}\} \qquad (21)$$

$$= \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0) \qquad (22)$$

Thus the FTC of a cosine signal consists of a sum of two delta pulses in the frequency domain.

5.2   Conjugation

$$\boxed{x^*(t) \quad \circ\!\!-\!\!\circ \quad X^*(-f)}$$

Proof

The first step of the proof of the conjugation property is to move the conjugation sign outside the integral. Inside the brackets this results into an FTC from which the phasor has a positive sign, which writes as $X(-f)$.

$$
\begin{aligned}
y(t) = x^*(t) \circ\!\!-\!\!\circ Y(f) &= \int_{-\infty}^{\infty} x^*(t)e^{-j2\pi ft}dt = \left(\int_{-\infty}^{\infty} x(t)e^{j2\pi ft}dt\right)^* \\
&= (X(-f))^* = X^*(-f) \qquad\qquad (23)
\end{aligned}
$$

The final notation $X^*(-f)$ is just simpler way of the notation $(X(-f))^*$.

An important result of this conjugation property is that FTC $X(f)$ of a real signal $x(t)$, thus when $x(t) = x^*(t)$, has the following symmetry properties:

The magnitude response is symmetric, while the phase response is anti-symmetric, thus:

$$x(t) \text{ real} \;\Rightarrow\; x(t) = x^*(t) \;\Rightarrow\; X(f) = X^*(-f) \;\Rightarrow$$

$$|X(f)| = |X(-f)| \quad : \quad \textbf{Magnitude response symmetric} \quad (24)$$
$$\angle\{X(f)\} = -\angle\{X(f)\} \quad : \quad \textbf{Phase response anti-symmetric} \quad (25)$$

Example Conjugation

A simple example of these symmetry properties can be found in the FTC of a shifted delta pulse. For this case the time domain signal is a real and the FTC results in a complex exponent as follows:

$$
\begin{aligned}
x(t) = \delta(t - t_0) \quad \circ\!\!-\!\!\circ X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j2\pi ft}dt = e^{-j2\pi ft_0}
\end{aligned}
$$

From this result it folows that the magnitude response equals 1, while the phase response equals $-2\pi t_0$.

$$\textbf{Symmetric Magnitude response} \quad : \quad |X(f)| = |e^{-j2\pi ft_0}| = 1$$
$$\textbf{Anti-symmetric Phase response} \quad : \quad \angle\{X(f)\} = -2\pi t_0$$
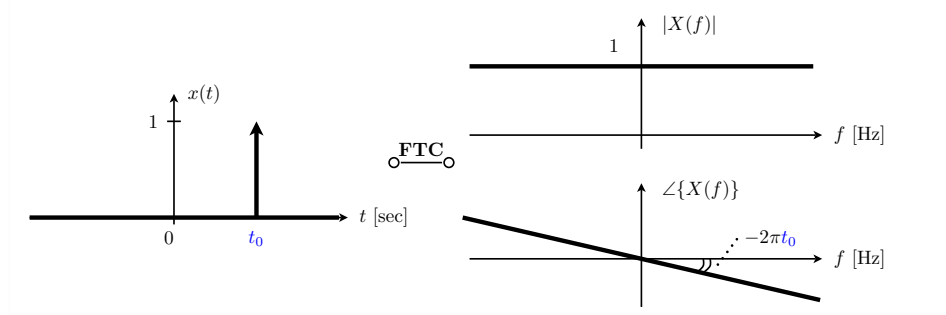
Figure 12: Magnitude and phase response of shifted delta pulse

A plot of this magnitude- and phase response is depicted in Fig. 12. From this plot the symmetry properties are clearly visible

Note finally that for this case the angle of the phase response is directly related to the shift in time domain.

5.3   Scaling

$$x(k \cdot t) \quad \circ\!\!-\!\!\circ \quad \frac{1}{|k|}X(f/k) \quad \text{with } k \neq 0$$

Proof

$$y(t) = x(k \cdot t) \circ\!\!-\!\!\circ Y(f) = \int_{-\infty}^{\infty} x(k \cdot t)e^{-\mathrm{j}2\pi f t}$$

For the proof of the scaling property it is the easiest to split the proof into two cases. For $k > 0$ we substitute for the scaled variable $k \cdot t$ a new variable τ which leads straightforward to the following result:

$$Y(f) \overset{\tau = k \cdot t}{=} \int_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}2\pi f \frac{1}{k}\tau} \cdot \frac{1}{k}\mathrm{d}\tau = \frac{1}{k}X(\frac{f}{k})$$

For $k < 0$ we follow the same procedure. But now the boundaries of the integral are changed. Changing these boundaries causes an extra minus sign. As a last step we take the absolute value of k which combines the negative sign together with the negative value of k.

$$Y(f) \overset{\tau = k \cdot t}{=} \int_{+\infty}^{-\infty} x(\tau)e^{-\mathrm{j}2\pi f \frac{1}{k}\tau} \cdot \frac{1}{k}\mathrm{d}\tau$$

$$= \frac{-1}{k} \int_{-\infty}^{+\infty} x(\tau)e^{-\mathrm{j}2\pi f \frac{1}{k}\tau}\mathrm{d}\tau = \frac{1}{|k|}X(\frac{f}{k})$$

18

Example Scaling

In subsection 4.2 we have shown that the FTC of a block signal results in a sinc function, as given in the following equation and depicted in Fig. 13.

$$x(t) = \begin{cases} 1 & \text{for } -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0 & \text{elsewhere} \end{cases} \quad \circ\!\!-\!\!\circ \quad X(f) = T_0 \frac{\sin(\pi f T_0)}{\pi f T_0}$$

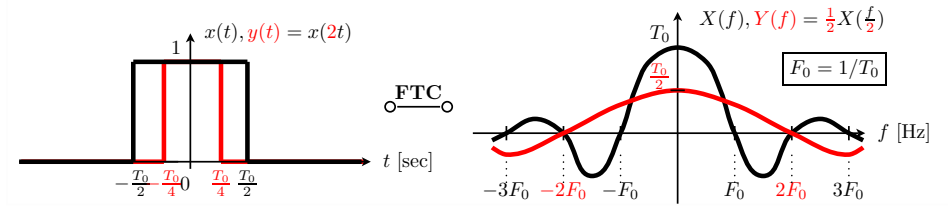The figure shows an example in which applied a scale factor of 2: The

Figure 13: Result of scale operation applied to a block function

time domain signal has shrunk by a factor of 2, while the frequency domain signal is stretched by a factor 2. This leads to the following important "rule of thumb":

Shrinking in one domain implies stretching in the other domain

5.4 Shift property

5.4.1 Time domain shift

$$\boxed{x(t - t_0) \quad \circ\!\!-\!\!\circ \quad e^{-j2\pi f t_0} \cdot X(f)}$$

Proof

The first step of the proof is to change everywhere the variable t by $t - t_0$.

$$x(t - t_0) \quad \circ\!\!-\!\!\circ \quad \int_{-\infty}^{\infty} x(t - t_0) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} x(t - t_0) e^{-j2\pi f (t - t_0 + t_0)} d(t - t_0)$$

In the next step we take out the negative exponent with the term t_0 and substitute $t - t_0$ by τ, which leads to the final result as follows:

$$\overset{\tau = t - t_0}{=} \quad e^{-j2\pi f t_0} \cdot \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \quad e^{-j2\pi f t_0} \cdot X(f)$$

19

5.4.2 Frequency domain shift

$$\boxed{e^{j2\pi f_0 t} \cdot x(t) \quad \circ\!\!-\!\!\circ \quad X(f - f_0)}$$

The steps of this proof are similar to the previous onesand are as follows:

$$
\begin{aligned}
\mathscr{F}^{-1}\{X(f - f_0)\} \quad &= \quad \int_{-\infty}^{\infty} X(f - f_0)e^{j2\pi ft}\mathrm{d}f \\
&= \quad \int_{-\infty}^{\infty} X(f - f_0)e^{j2\pi(f - f_0 + f_0)t}\mathrm{d}(f - f_0) \\
&\overset{\eta = f - f_0}{=\!=} \quad e^{j2\pi f_0 t} \cdot \int_{-\infty}^{\infty} X(\eta)e^{j2\pi\eta t}\mathrm{d}\eta = e^{j2\pi f_0 t} \cdot x(t)
\end{aligned}
$$

5.4.3 Modulation

A result of this frequency shift property is the following modulation property:

$$
\begin{aligned}
e^{j2\pi f_0 t} \cdot x(t) + e^{-j2\pi f_0 t} \cdot x(t) \quad &\circ\!\!-\!\!\circ \quad X(f - f_0) + X(f + f_0) \\
\Leftrightarrow \quad 2\cos(2\pi f_0 t)\cdot x(t) \quad &\circ\!\!-\!\!\circ \quad X(f - f_0) + X(f + f_0) \qquad (26)
\end{aligned}
$$

Thus when multiplying, or modulating, a signal $x(t)$ with a sinusoidal signal, the result in the frequency domain is a sum of two shifted versions of the FTC $X(f)$ of signal $x(t)$ which results into the following "rule of thumb":

Modulation in one domain is shifting in the other domain.

5.5 Convolution

5.5.1 Time domain convolution

$$\boxed{x(t) \star y(t) \quad \circ\!\!-\!\!\circ \quad X(f) \cdot Y(f)}$$

The first step of the proof of the time domain convolution property is to use the definition of the convolution integral into the FTC definition.

$$
\mathscr{F}\{x(t) \star y(t)\} \quad = \quad \int_{t=-\infty}^{\infty} \left(\int_{\tau=-\infty}^{\infty} x(\tau)y(t - \tau)\mathrm{d}\tau \right) e^{-j2\pi ft}\mathrm{d}t
$$

Then by interchanging the order of the integrals and introducing in the second integral everywhere the variable $t - \tau$, this results in an extra phasor component $\mathrm{e}^{-\mathrm{j}2\pi f\tau}$ in the first integral.

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \left(\int_{t=-\infty}^{\infty} y(t - \tau)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t \right) \mathrm{d}\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\mathrm{e}^{-\mathrm{j}2\pi f\tau} \left(\int_{t=-\infty}^{\infty} y(t - \tau)\mathrm{e}^{-\mathrm{j}2\pi f(t-\tau)}\mathrm{d}t \right) \mathrm{d}\tau$$

Finally by substituting $t - \tau$ by a new variable p we obtain the final result.

$$= \left(\int_{\tau=-\infty}^{\infty} x(\tau)\mathrm{e}^{-\mathrm{j}2\pi f\tau}\mathrm{d}\tau \right) \cdot \left(\int_{p=-\infty}^{\infty} y(p)\mathrm{e}^{-\mathrm{j}2\pi fp}\mathrm{d}p \right)$$

$$= X(f) \cdot Y(f)$$

5.5.2 Frequency domain convolution

$$\boxed{x(t) \cdot y(t) \quad \circ\!\!-\!\!\circ \quad X(f) \star Y(f)}$$

Proof
The first step of the proof of the frequency domain convolution property is to use the definition of Inverse FTC for the the signal $x(t)$.

$$\mathscr{F}\{x(t) \cdot y(t)\} = \int_{t=-\infty}^{\infty} x(t) \cdot y(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t$$

$$= \int_{t=-\infty}^{\infty} \left(\int_{\eta=-\infty}^{\infty} X(\eta)\mathrm{e}^{\mathrm{j}2\pi \eta t}\mathrm{d}\eta \right) \cdot y(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t$$

Then by interchanging the order of the integrals and combining the two phasor components of the second integral we obtain (in red) the shifted FTC of the signal $y(t)$. The resulting integral is the convolution integral of $X(f)$ with $Y(f)$.

$$= \int_{\eta=-\infty}^{\infty} X(\eta) \left(\int_{t=-\infty}^{\infty} y(t)\mathrm{e}^{\mathrm{j}2\pi \eta t}\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t \right) \mathrm{d}\eta$$

$$= \int_{\eta=-\infty}^{\infty} X(\eta) \left(\int_{t=-\infty}^{\infty} y(t)\mathrm{e}^{-\mathrm{j}2\pi (f-\eta)t}\mathrm{d}t \right) \mathrm{d}\eta$$

$$= \int_{\eta=-\infty}^{\infty} X(\eta)Y(f - \eta)\mathrm{d}\eta = X(f) \star Y(f)$$

This leads to the following important "rule of thumb":

Multiplication in one domain results in convolution in the other domain.

Example convolution
As an example of the convolution property we will use the modulation property as discussed before. In this example signal $x(t)$ is a sinc function as depicted in the upper left side of Fig. 14. In section 4.2 we have shown
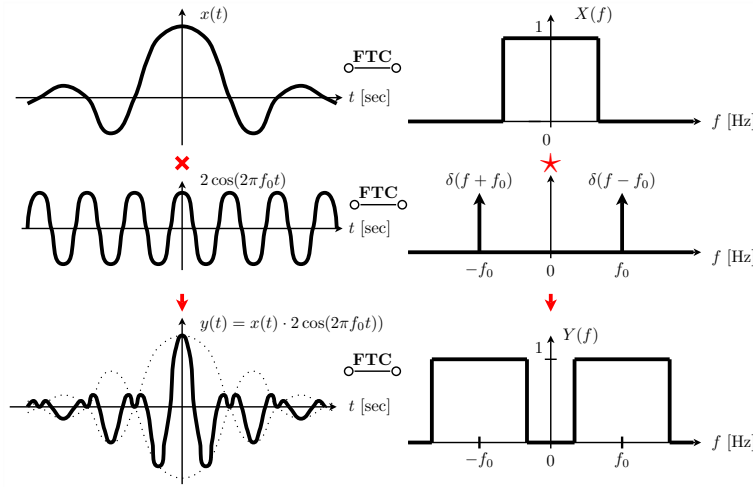
Figure 14: Modulation in time domain results in frequency shifts of the spectrum

that the sinc function is the result of the IFTC of a frequency domain block function, as depicted on in the upper right side of the figure. Multiplying signal $x(t)$ with a sinusoidal signal with frequency f_0, which is a modulation operation, is according to the last "rule of thumb" equivalent to the convolution of the FTC $X(f)$ with the FTC of the sinusoidal. The FTC of the sinusoidal consists of two delta pulses, $\delta(f - f_0)$ and $\delta(f + f_0)$, as depicted in the middle part of the figure. Convolving $X(f)$ with a delta pulse results in a shift, which can be shown as follows:

$$\delta(f \pm f_0) \star X(f) \quad = \quad \int_{\eta=-\infty}^{\infty} \delta(\eta \pm f_0) X(f - \eta) \mathrm{d}\eta = X(f \pm f_0)$$

The result is shown in the lower part of the figure.

5.6 Differentiation

5.6.1 Time domain differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \quad \circ\!\!-\!\!\circ \quad \mathrm{j}\omega X(f)$$

Proof

By using the IFTC definition of $x(t)$ and differentiating both sides with respect to t we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{\infty} X(f)\mathrm{e}^{\mathrm{j}2\pi ft}\mathrm{d}f\right) = \int_{-\infty}^{\infty} X(f)(\mathrm{j}2\pi f)\mathrm{e}^{\mathrm{j}2\pi ft}\mathrm{d}f = \mathscr{F}^{-1}\{\mathrm{j}\omega X(f)\}$$

Example time domain differentiation

Find the FTC of the signal $x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta(t)$. Taking the FTC of both sides gives:

$$X(f) = \mathscr{F}\{\frac{\mathrm{d}}{\mathrm{d}t}\delta(t)\} = \mathrm{j}\omega \cdot \mathscr{F}\{\delta(t)\} = \mathrm{j}\omega \cdot 1 = \mathrm{j}\omega$$

5.6.2 Frequency domain differentiation

$$tx(t) \quad \circ\!\!-\!\!\circ \quad \frac{\mathrm{j}}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}X(f)$$

Proof

Differentiation of both sides of the FTC difinition gives:

$$\frac{\mathrm{d}}{\mathrm{d}f}X(f) \;=\; \int_{-\infty}^{\infty} x(t)(-\mathrm{j}2\pi t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t = -\mathrm{j}2\pi \int_{-\infty}^{\infty} tx(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t = -\mathrm{j}2\pi\mathscr{F}\{tx(t)\}$$

Finally, multiplying both sides by the complex number j results into:

$$\frac{\mathrm{j}}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}X(f) = \mathscr{F}\{tx(t)\}$$

Example frequency domain differentiation

Find the FTC of $x(t) = t\cos(2\pi F_0 t)$

$$\begin{aligned}
X(f) \;&=\; \mathscr{F}\{t\cos(2\pi F_0 t)\} = \frac{\mathrm{j}}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left[\mathscr{F}\{\cos(2\pi F_0 t)\}\right] \\
&=\; \frac{\mathrm{j}}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left[\frac{1}{2}\delta(f - F_0) + \frac{1}{2}\delta(f + F_0)\right]
\end{aligned}$$

5.7   Theorem Parseval

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t \;\; = \;\; \int_{-\infty}^{\infty} |X(f)|^2 \mathrm{d}f}$$

In the first step of the proof we replace the complex conjugated version of $x(t)$ by the IFTC of the FTC of $x^*(t)$.

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t \;\; = \;\; \int_{-\infty}^{\infty} x(t)x^*(t)\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\mathscr{F}^{-1}\{\mathscr{F}\{x^*(t)\}\}\mathrm{d}t$$

Using the FTC property of complex conjugation and then applying the IFTC to this result gives:

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t \;\; &= \;\; \int_{-\infty}^{\infty} x(t)\mathscr{F}^{-1}\{X^*(-f)\}\mathrm{d}t \\
&= \;\; \int_{-\infty}^{\infty} x(t)\left(\int_{-\infty}^{\infty} X^*(-f)\mathrm{e}^{\mathrm{j}2\pi ft}\mathrm{d}f\right)\mathrm{d}t
\end{aligned}$$

Finally interchanging the variable $-f$ by f and then interchanging the order of the integrals and using the FTC definition leads to the final result.

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t \;\; &= \;\; \int_{-\infty}^{\infty} x(t)\left(\int_{-\infty}^{\infty} X^*(f)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}f\right)\mathrm{d}t \\
&= \;\; \int_{-\infty}^{\infty} X^*(f)\left(\int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t\right)\mathrm{d}f \\
&= \;\; \int_{-\infty}^{\infty} |X(f)|^2\mathrm{d}f
\end{aligned}$$

Concluding it follows that the theorem of Parseval states that the energy in time- and frequency-domain is the same. This implies that the FTC cannot introduce or lose energy, which leads to the following "rule of thumb":

Energy in one domain is the same as energy in the other domain.

Example Theorem Parseval
In this example the goal is to calculate the energy of a time domain sinc function as depicted on the left side of Fig.15. When using the definition of
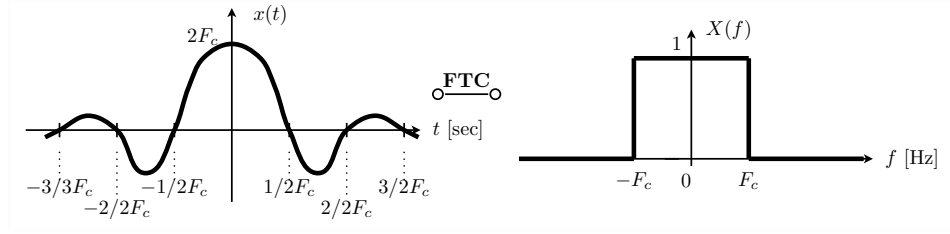
Figure 15

a sinc function we can calculate the energy as follows:

$$
\begin{aligned}
E &= \int_{t=-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{t=-\infty}^{\infty} |2F_c \mathrm{sinc}(2\pi F_c t)|^2 \mathrm{d}t \\
&= \int_{t=-\infty}^{\infty} |2F_c \frac{\sin(2\pi F_c t)}{2\pi F_c t}|^2 \mathrm{d}t
\end{aligned}
$$

The evaluation of this integral is not trivial. However, when using the theorem of Parseval we can find the same result by evaluating the energy in frequency domain. As shown before the FTC of a time domain sinc function is a block function in frequency domain, as depicted on the right side of Fig.15 and the result is as follows:

$$
\begin{aligned}
E &= \int_{t=-\infty}^{\infty} |x(t)|^2 \mathrm{d}t \\
&\stackrel{\text{Parseval}}{=} \int_{f=-\infty}^{\infty} |X(f)|^2 \mathrm{d}f = \int_{f=-F_c}^{F_c} |1|^2 \mathrm{d}f = 2F_c
\end{aligned}
$$

6 Summary

6.1 Definition FTC and IFTC

$$X(f) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\mathrm{d}t \quad \circ\!\!-\!\!\circ \quad x(t) = \int_{-\infty}^{\infty} X(f)\mathrm{e}^{\mathrm{j}2\pi ft}\mathrm{d}f$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \quad \circ\!\!-\!\!\circ \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega$$

6.2 Existence and convergence FTC

$$|X(f)| < \infty \quad \leftrightarrow \quad \int_{-\infty}^{\infty} |x(t)|\,\mathrm{d}t < \infty$$

6.3 General Phasor Integral property (GPI)

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\pi(f\pm f_0)t}\mathrm{d}t \quad \overset{\mathrm{GPI}}{=} \quad \delta(f \pm f_0)$$

6.4 Basic Fourier transform pairs

$$x(t) = \delta(t) \quad \circ\!\!-\!\!\circ \quad X(f) = 1 \ \forall f$$

$$x(t) = 1 \ \forall t \quad \circ\!\!-\!\!\circ \quad X(f) = \delta(f)$$

$$x(t) = \begin{cases} 1 & \text{for } |t| \leq T_0/2 \\ 0 & \text{elsewhere} \end{cases} \quad \circ\!\!-\!\!\circ \quad X(f) = T_0\frac{\sin(\pi f T_0)}{\pi f T_0}$$

$$2F_c\frac{\sin(2\pi F_c t)}{2\pi F_c t} \quad \circ\!\!-\!\!\circ \quad X(f) = \begin{cases} 1 & \text{for } |f| \leq F_c \\ 0 & \text{elsewhere} \end{cases}$$

$$x(t) = \cos(2\pi F_0 t) \quad \circ\!\!-\!\!\circ \quad X(f) = \frac{1}{2}\delta(f - F_0) + \frac{1}{2}\delta(f + F_0)$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - n \cdot T) \quad \circ\!\!-\!\!\circ \quad X(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T})$$

6.5 Main properties

- Linearity: $a \cdot x(t) + b \cdot y(t) \ \circ\!\!-\!\!\circ \ a \cdot X(f) + b \cdot Y(f)$

- Conjugation: $x^*(t) \ \circ\!\!-\!\!\circ \ X^*(-f)$

26

- Scaling: $x(k \cdot t) \; \circ\!\!-\!\!\circ \; \frac{1}{|k|} X(f/k)$ with $k \neq 0$

- Time domain shift: $x(t - t_0) \; \circ\!\!-\!\!\circ \; \mathrm{e}^{-\mathrm{j}2\pi f t_0} X(f)$

- Frequency domain shift: $\mathrm{e}^{\mathrm{j}2\pi f_0 t} x(t) \; \circ\!\!-\!\!\circ \; X(f - f_0)$

- Modulation: $2\cos(2\pi f_0 t) \cdot x(t) \; \circ\!\!-\!\!\circ \; X(f - f_0) + X(f + f_0)$

- Time domain convolution: $x(t) \star y(t) \; \circ\!\!-\!\!\circ \; X(f) \cdot Y(f)$

- Frequency domain convolution: $x(t \cdot y(t) \; \circ\!\!-\!\!\circ \; X(f) \star Y(f)$

- Time domain differentiation: $\frac{\mathrm{d}}{\mathrm{d}t} x(t) \; \circ\!\!-\!\!\circ \; \mathrm{j}\omega X(f)$

- Frequency domain differentiation: $t x(t) \; \circ\!\!-\!\!\circ \; \frac{\mathrm{j}}{2\pi} \frac{\mathrm{d}}{\mathrm{d}f} X(f)$

- Theorem of Parseval: $\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2 \mathrm{d}f$

6.6   Main "rules of thumb"

$$
\begin{aligned}
\text{Shrinking} &\leftrightarrow \text{Stretching} \\
\text{Multiplication} &\leftrightarrow \text{Convolution} \\
\text{Modulation} &\leftrightarrow \text{Shifting} \\
\text{Energy} &\leftrightarrow \text{Energy}
\end{aligned}
$$