Topics in Applied Optimization

Optimization Algorithms for ML and Data Sciences

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CSTAR, IIIT-H

Convex Optimization Problems

Convex Optimization Problems

Optimization problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

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- Find an x that minimizes $f_0(x)$ among all x that satisfy
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- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is called objective function or cost function

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Convex Optimization Problems

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$$\begin{aligned} \text{minimize} \quad & f_0(x) && (1) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m && (2) \\ & h_i(x) = 0, i = 1, \ldots, p && (3) \end{aligned}$$

Convex Optimization Problems

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$$\text{minimize} \quad f_0(x) \tag{1}$$

$$\text{subject to} \quad f_i(x) \leq 0, i = 1, \ldots, m \tag{2}$$

$$h_i(x) = 0, i = 1, \ldots, p \tag{3}$$

- If no constraints, i.e., $m = p = 0$, then called unconstrained problem

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- If no constraints, i.e., $m = p = 0$, then called unconstrained problem
- Domain of opt. problem: where objective and constraint are defined

$$\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom } f_i \quad \cap \quad \bigcap_{i=1}^{p} \textbf{dom } h_i$$

f_0

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- The optimization problem is called feasible if there exists atleast one feasible point

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- Feasible set or constraint set: set of all feasible points
- Optimal Value: The optimal value p^* defined as

$$p^* = \inf \{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$$

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- p^* is allowed to take extended values $\pm\infty$

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$\inf\{\phi\}$

- p^* is allowed to take extended values $\pm\infty$
- Infeasible problem: problem is called infeasible when $p^* = \infty$

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- p^* is allowed to take extended values $\pm\infty$
- Infeasible problem: problem is called infeasible when $p^* = \infty$
  - Note: we used the fact that $\inf \phi = \infty$
- Unbounded below: Problem is unbounded below if $f_0(x_k) \to -\infty$ as $k \to \infty$

Optimal and locally optimal

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- Optimal Set: The set of all optimal points is called optimal set
- If there exists an optimal point, then we say that optimal value is achieved and the problem is solvable
- If the optimal set is empty, then we say that optimal value is not attained

Optimal and Locally Optimal Points

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- $\epsilon-$suboptimal: $\bar{x}$ is $\epsilon-$suboptimal if $f(\bar{x}) \leq p^* + \epsilon$, $\epsilon > 0$

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- Active constraint: If x is feasible and $f_i(x) = 0$, then ith inequality is active

Optimal and Locally Optimal Points

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$<$ or $=$

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- **Active constraint:** If x is feasible and $f_i(x) = 0$, then ith inequality is active
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- Active constraint: If x is feasible and $f_i(x) = 0$, then ith inequality is active
- Inactive constraint: If x is feasible and $f_i(x) < 0$, then this constraint is inactive
- Redundant constraint: A constraint is redundant if removing it does not change the feasible set

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Define the feasible set

$$\Omega = \{x \mid \underline{f_i(x) \leq} \, 0, \quad i = 1, \ldots, m, \quad h_i(x) \underline{= 0} \quad i = 1, \ldots, p\}$$

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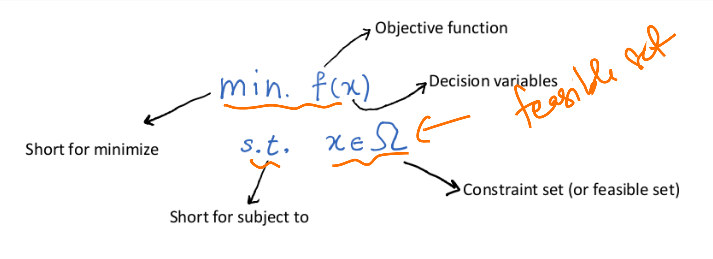
More compactly, we we can write:



Examples: $1/x$

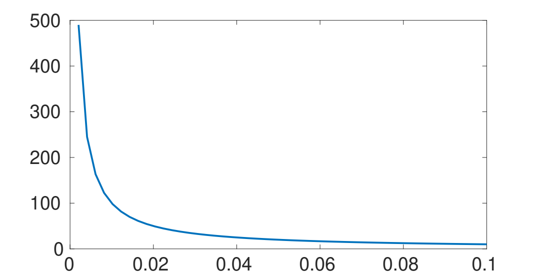
Consider the optimization problem:

$$\begin{aligned} \text{minimize } & f_0(x) = 1/x, \\ \text{subject to } & x \in \mathbb{R} \end{aligned}$$

where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set?
Quiz: What is p^*?
Quiz: Is the optimal value acheived?

Figure 1: Plot of $1/x$



Examples: $-\log x$

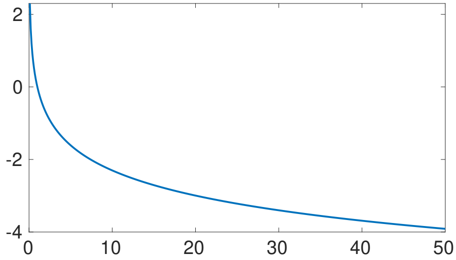
Consider the optimization problem:

$$\begin{aligned} &\text{minimize } f_0(x) = -\log x, \\ &\text{subject to } x \in \mathbb{R} \end{aligned}$$

where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set?
Quiz: What is p^*?
Quiz: Is the optimal value acheived?
Quiz: Is this problem bounded below?

Figure 2: Plot of $-\log x$



Examples: $x \log x$

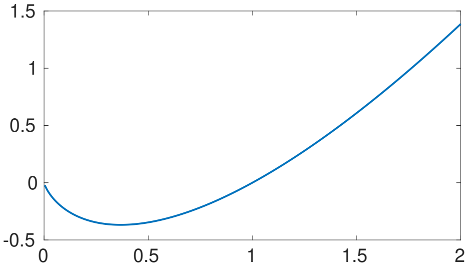
Consider the optimization problem:

$$\begin{aligned} \text{minimize } & f_0(x) = x \log x, \\ \text{subject to } & x \in \mathbb{R} \end{aligned}$$

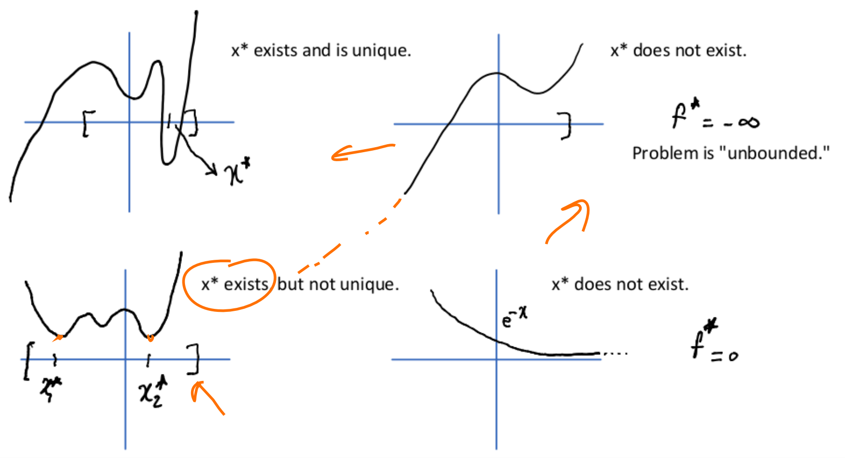
where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set?
Quiz: What is p^*?
Quiz: Is the optimal value acheived?
Quiz: Is this problem bounded below?
Quiz: What is optimal point?

Figure 3: Plot of $x \log x$



Examples: Graphically



Expressing Problems in Standard Form

Optimization problem (Standard Form):

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\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\
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Convention for standard form:

- righthand side of the inequality and equality constraints are zero

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$h_i = g_i(x) - \tilde{g}_i(x) = 0$

Convention for standard form:

- righthand side of the inequality and equality constraints are zero
  - For example, $g_i(x) = \tilde{g}_i(x)$ can be written as $h_i(x) = 0$

Expressing Problems in Standard Form

only $\leq$ is allowed

Optimization problem (Standard Form):

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Convention for standard form:

- righthand side of the inequality and equality constraints are zero
  - For example, $g_i(x) = \tilde{g}_i(x)$ can be written as $h_i(x) = 0$
- $f_i(x) \geq 0$ as $-f_i(x) \leq 0$

(Box Constraints). Consider the following

$$
\begin{array}{rl}
\text{minimize} & f_0(x) \\
\text{subject to} & l_i \le x_i \le u_i, \quad i = 1, \ldots, n
\end{array}
$$

$l_i \le x_i$

$u_i \ge x_i$

$x_i \le u_i$

$-x_i \le -l_i$

- constraints here are called **variable bounds** or **box constraints**

(Box Constraints). Consider the following

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where $f_i(x) = l_i - x_i$, $i = 1, \ldots, n$ and $f_i(x) = x_{i-n} - u_{i-n}$, $i = n+1, \ldots, 2n$

Maximization Problems Seen as Minimization Problems

Note: Maximization problem can be solved by minimization. Consider

$$
\begin{array}{ll}
\text{maximize} & f_0(x) \\
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$= -\inf\{-f_0(x) \mid \ldots\}$

- Note the constraints remain same!
- Obviously, the optimal value p^* is

$$p^* = \sup\{f_0(x) \mid \quad f_i(x) \leq 0,\ i = 1, \ldots, m, \quad h_i(x) = 0,\ i = 1, \ldots, p\}$$

Equivalent Optimization Problems

Two problems are equivalent if from a solution of the one, a solution of the other can be found

OPT 1

OPT 2

obtain a sol^n of one from another

$2x-3$

Equivalent Optimization Problems

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Example: Consider the following problem

$$\begin{array}{rl} \text{minimize} & \alpha_0 f_0(x) \\ \text{subject to} & \alpha_i f_i(x) \le 0,\ i = 1, \dots, m \\ & \beta_i h_i(x) = 0,\ i = 1, \dots, p \end{array}$$

where $\alpha_i > 0,\ i = 0, \dots, m, \quad \beta_i \ne 0,\ i = 1, \dots, p.$

$\alpha_0 > 0$

$\alpha_i f_i(x) \le 0 \iff f_i(x) \le 0$

$\beta_i h_i(x) = 0 \iff h_i(x) = 0$

If x^* is opt val

$\Rightarrow \boxed{p^* = \alpha_0 f_0(x^*)}$. don't forget

Change?

$\min \alpha_0 f_0(x) \iff \min f_0(x)$

Only optimal value remains same. Optimal pt

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f^{-1}

f

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x^* Same

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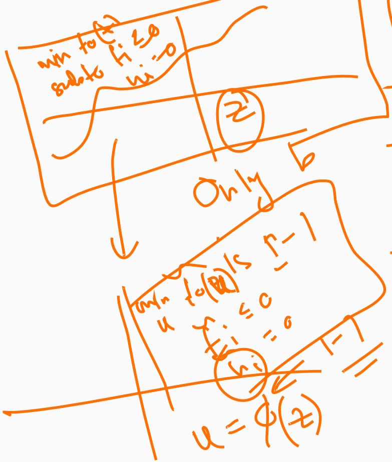

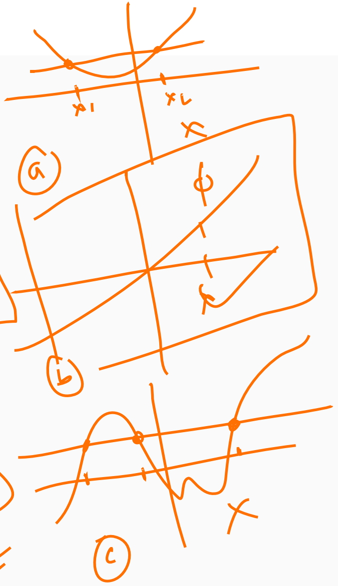

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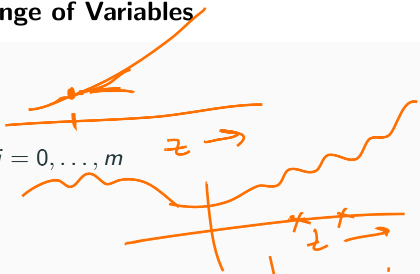

- Define $\tilde{h}_i$ as

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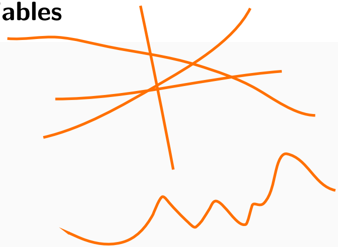
Is $z \uparrow$ $\downarrow$ ϕ being 1-1

$\phi(z)$ $\uparrow$

Equivalent Problems: Change of Variables



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Consider

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$$\left\{ \begin{array}{ll} \min & f_0(x) \\ \text{s.t} & f_i(x) \le 0 \\ & h_i(x) = 0 \end{array} \right.$$

$z^* = \phi^{-1}(x^*)$

Consider

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(z) \\ \text{subject to} & \tilde{f}_i(z) \le 0, \; i = 1, \ldots, m \\ & \tilde{h}_i(z) = 0, \; i = 1, \ldots, p \end{array}$$

$\phi(z^*)$

x^*

- If z solves above, then $x = \phi(z)$ solves the standard optimization problem
- Similarly, if x solves original opt problem, then $z = \phi^{-1}(x)$ solves above

Equivalent Problems: Transformation of objective and constraint function

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Given the optimization problem in standard form

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equality constr and

≥ 0

$-f_i(x) \geq 0$

Quiz: Is it possible to replace inequality constraints by equality constraints and non-negaivity constraints?

$s_i(x)$

$f_i + s_i = 0$

$s_i \geq 0$

$h_i(x) = 0$

$f_i(x) \leq 0 \quad \forall x$

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Ans: Yes. Key observation is: $f_i(x) \leq 0$, if and only if there is an $s_i \geq 0$ such that $f_i(x) + s_i = 0$

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Note: Here s_i are called slack variable. Is this equivalent?

Convex Optimization Problem in Standard Form

Opt. Prob

Convex Optimization Problem (Standard Form):

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & a_i^T x = b_i, \quad i = 1, \dots, p \end{array}$$

where $f_0, \dots, f_m$ are convex functions.

$h_i(x)$ is affine

h_i is cvx

$\Rightarrow h_i$

Comparing this with the standard form

Optimization problem (Standard Form):

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p \end{array}$$

- objective function must be convex
- inequality constraint functions must be convex
- equality constraint functions $h_i(x) = a_i^T x - b_i$ must be affine

Convex Optimization Problem

Consider the following optimization problem with $x \in \mathbb{R}^2$

$$
\begin{array}{ll}
\text{maximize} & f_0(x) = x_1^2 + x_2^2 \\
\text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\
& h_1(x) = (x_1 + x_2)^2 = 0,
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which is in standard form.

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Quiz: Is this problem a convex optimization problem?

$\|x\|_2^2$

Observation

① $h_1(x)$ is not affine

② Is f_0 cvx?

③ Is f_1 cvx?

Construct Hessian H

check $x^T H x \ge 0 \ \forall x$

$(x_1+x_2)^2 = 0 \Leftrightarrow \boxed{x_1 + x_2 = 0}$

$\dfrac{x_1}{1+x_2^2}$ is cvx ??

$\dfrac{x_1}{1+x_2^2} \le 0$

$1 + x_2^2 > 0$

$\Updownarrow$

$x_1 \le 0$ cvx

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Quiz: Can you rewrite this in convex optimization problem?

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f_1 is cvx

affin

Note: This is now a convex optimization problem

Convex Optimization Problem: Local Optima = Global Optima

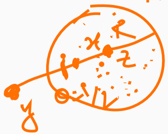
Fact: For a convex optimization problem, any local optima is a global optima

Ans: Proof on chalkboard!

Scratch Space : For Convex opt : Local opt = Global opt.

Suppose that x is locally optimal $\Rightarrow$ x is feasible and $B_R(x)$

$$f_0(x) = \inf\{ f_0(z) \mid z \text{ is feasible}, \|z-x\|_2 \le R \}$$



for some $R > 0$.

Assume on contrary: Spse x is not globally optimal, i.e., there is a feasible y for which $\boxed{f_0(y) \le f_0(x)}$

Evidently, $\|y-x\|_2 > R \longrightarrow (*)$

Consider the point z given by

$$z = (1-\theta)x + \theta y, \quad \theta = \frac{R}{2\|y-x\|_2} < \frac{1}{2}$$

Scratch Space

Look at $z - x :=-\theta x+\theta y$

We have $\|z-x\|_2^2=(z-x)^T(z-x)=(-\theta x+\theta y)^T(-\theta x+\theta y)=\theta^2(y-x)^T(y-x)$

$$=\theta^2\|y-x\|_2^2=\frac{R^2}{2^2\cancel{\|y-x\|_2^2}}\cdot\cancel{\|y-x\|_2^2}=\frac{R}{2}<R \Rightarrow z\in B_R(x)$$

Feasible set cvx $\Rightarrow$ $z=(1-\theta)x+\theta y\in$ feasible set.

$$f_0(z)=f_0((1-\theta)x+\theta y)\leq(1-\theta)f_0(x)+\theta\underline{f_0(y)}$$
$$\leq(1-\theta)f_0(x)+\theta\underline{f_0(x)}$$
$$=f_0(x)$$

$\Rightarrow z\in B_R(x)$, z is feasible, & $f_0(z)\leq f_0(x)$

$\Rightarrow$ it contradicts that x was locally optimal

Scratch Space

Hence our assumpt. that x is ~~not~~ globally opt - is wrong, i.e., x is globally optimal.

f diff f cvx

$$f(y) \geq f(x) + \nabla^T f(x)(y-x) \rightarrow (1)$$

$$\forall x, y$$

If x^* is locally optimal

$\Rightarrow \nabla f(x^*) = 0$

from (1) $\Rightarrow$ $f(y) \geq f(x^*)$ $\forall y \in$ feasible set

$\Rightarrow x^*$ is globally optimal.

Convex Optimization Problem: Optimilaty Criteria

Fact: If f_0 in a convex optimization problem is differentiable, then the point x is optimal if f

$$\nabla f_0(x)^T(y-x) \geq 0 \quad \text{for all} \quad y \in X$$

Proof: On chalkboard!

$$f(y) \geq f(x) + \underbrace{\nabla f(x)^T(y-x)}_{\forall y}$$

If x is optimal $\Rightarrow$ $\underbrace{f(y) \geq f(x)}$

$b \geq c -$

Scratch Space

X = feasible set

Suppose $x \in X$ and satisfies $\nabla f_0(x)^T (y - x) \geq 0$

Since f is cvx & diff: 1st order.

$$f_0(y) \geq f_0(x) + \underbrace{\nabla f_0(x)^T (y - x)}_{\geq 0}$$

dropping a positive term keeps the ineq.

$$f_0(y) \geq f_0(x) \quad ?$$

$\Rightarrow$ x is optimal.