08.09.2020

Digital Image Processing (CSE/ECE 478)

Lecture-9: Frequency Domain Processing

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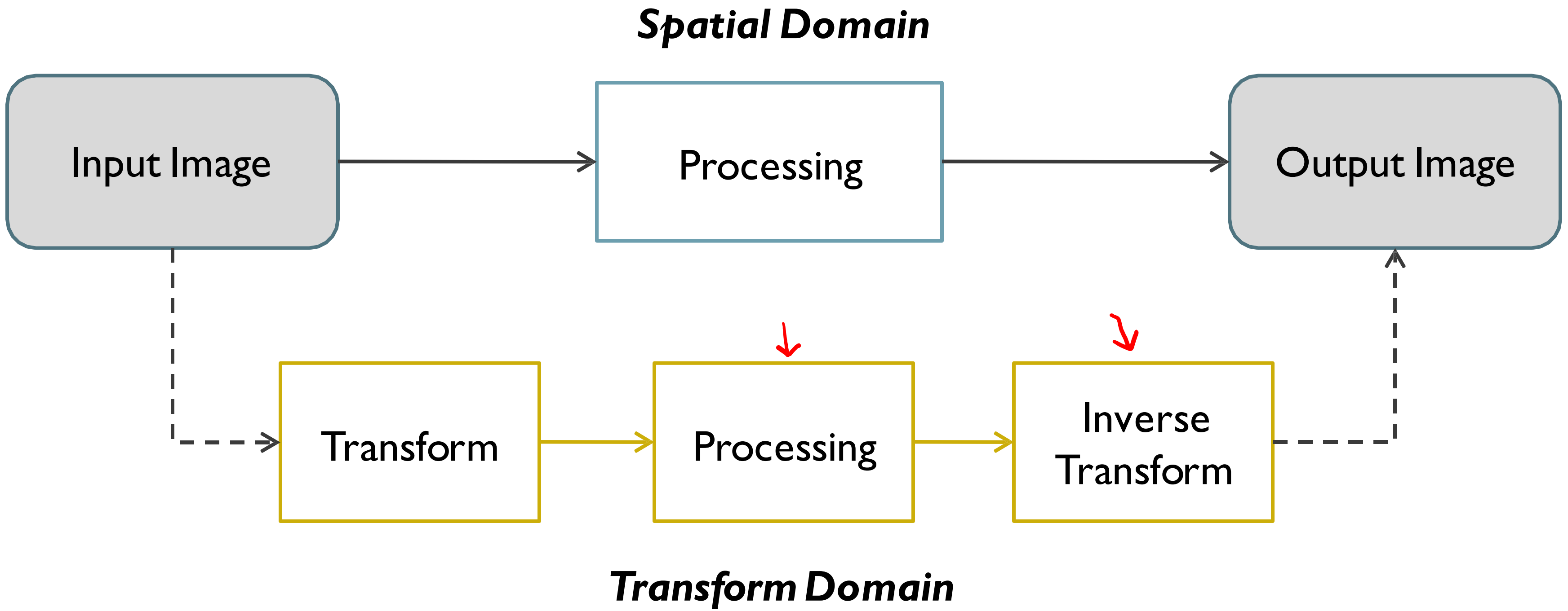
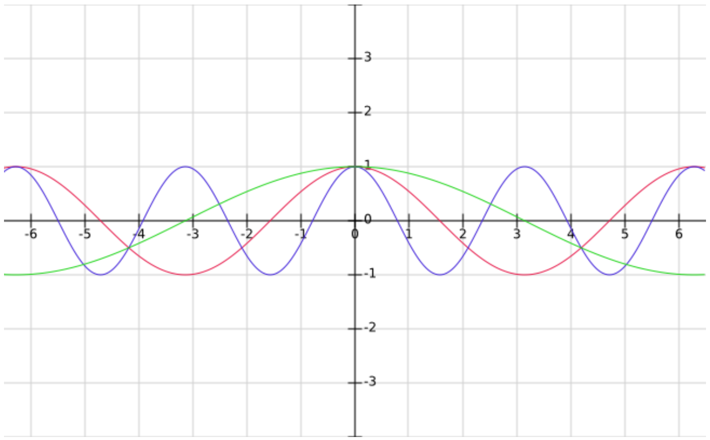
Spatial vs. Transform Domain Processing



Simple periodic signals

- $x(t) = A\cos(t)$
- $x(t) = A\cos(2t)$
- $x(t) = A\cos(t/2)$



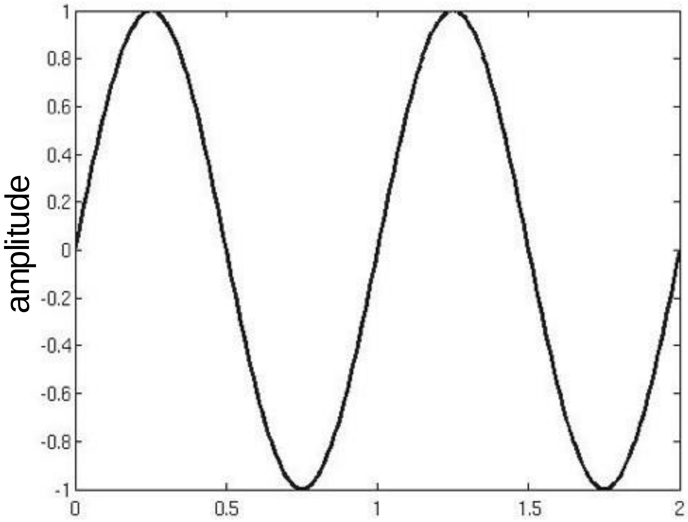
- $x(t) = A\cos(\omega t) = A\,cos(2\pi f t) = A\,cos(\frac{2\pi}{T} t)$

$f = \frac{1}{T}$

Angular frequency

Signal and Frequency Domains

$$y = 1\sin(2\pi s)$$

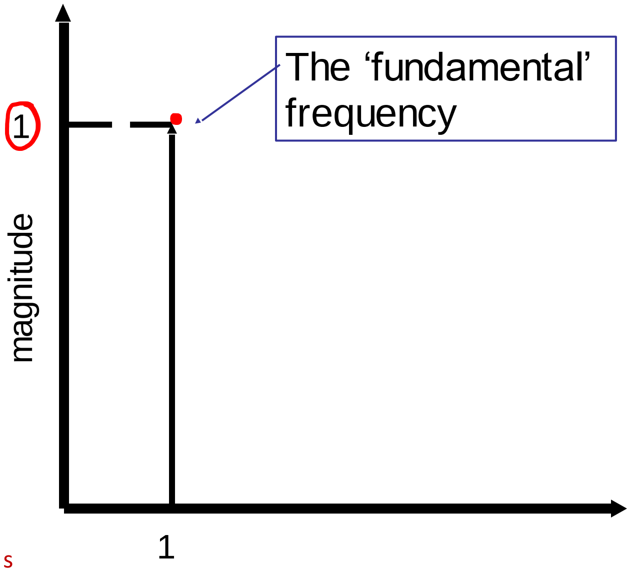


NOTE: s=0 is location from where signal is considered

spatial distance (s)

Signal domain

NOTE: s is bounded only for physical signals



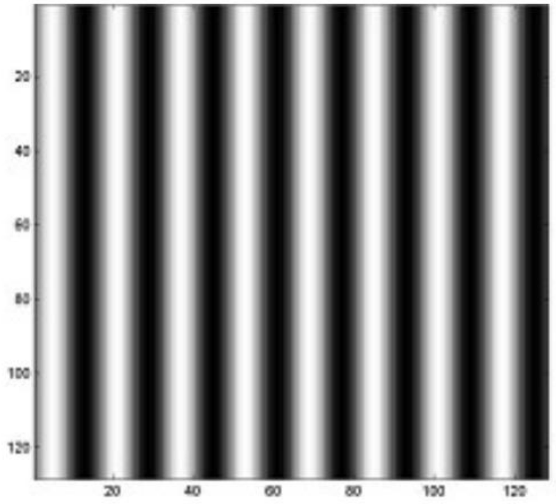
Frequency (# of cycles per unit distance)

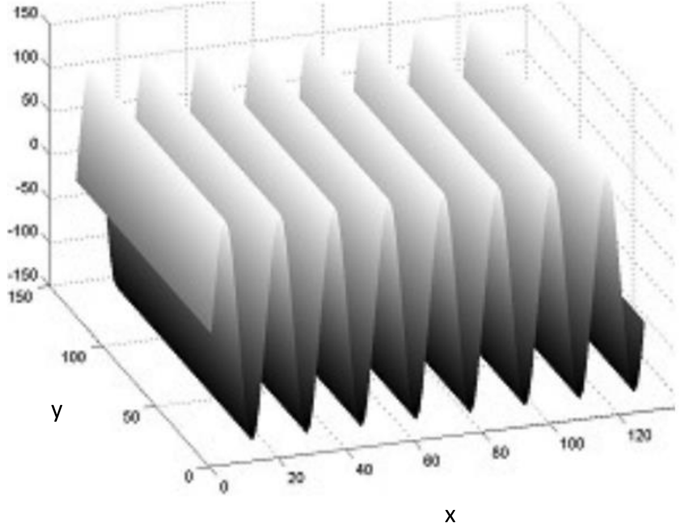
Frequency domain

Periodic Images

$I(x, y) = \underline{128}\ sin(2\pi x/16)$

128 x 128 grayscale image

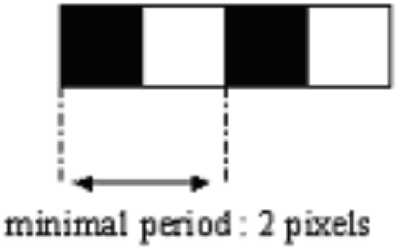




Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel

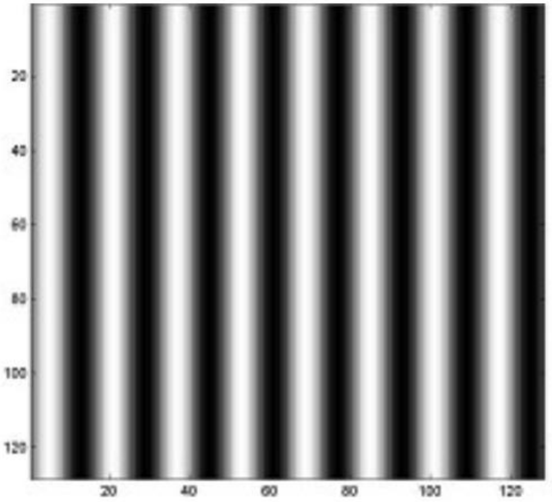
Periodic Images

Spatial period = Minimal # of pixels between two identical patterns in a "periodic" image



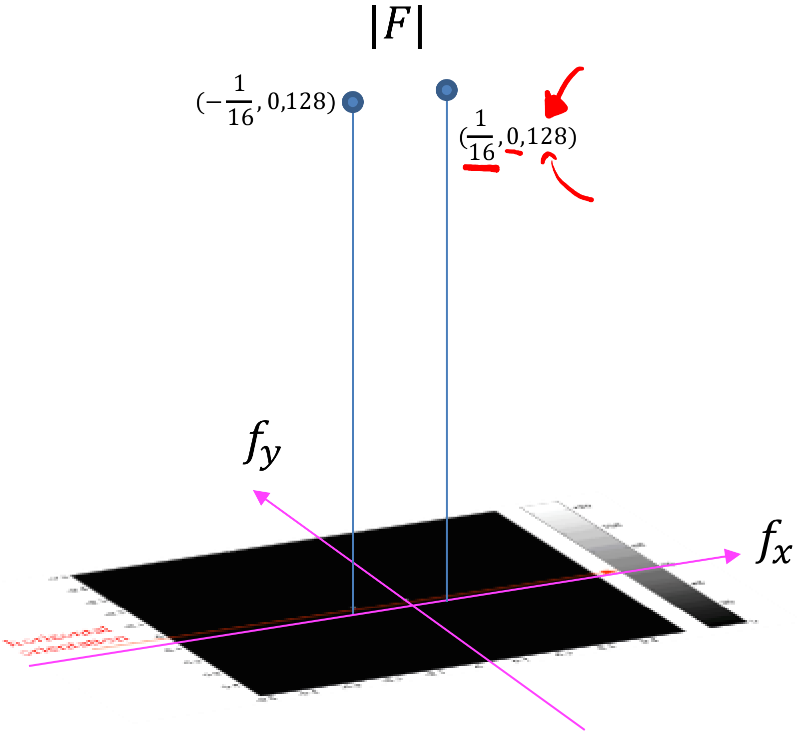
$$\Rightarrow \quad \nu_{max} = \frac{1}{\text{minimal period}} = \frac{1}{2}$$

$$I(x, y) = 128\ sin(2\pi x/16)$$



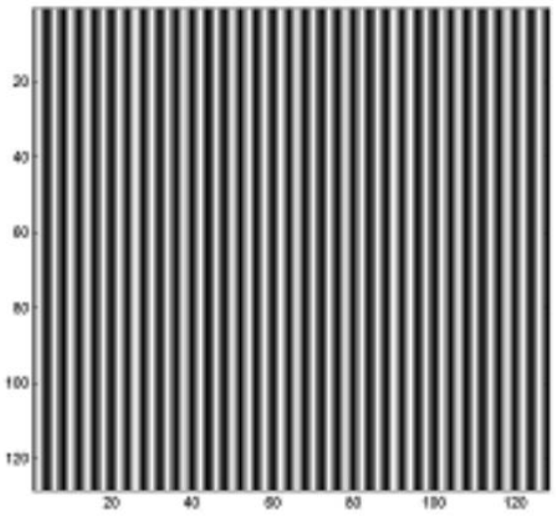
Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel

Spatial domain



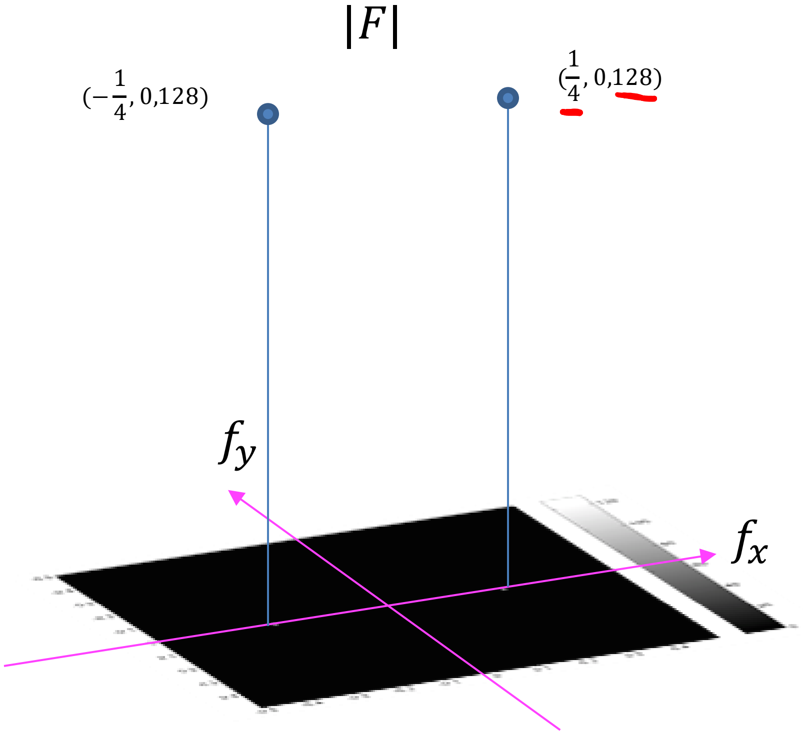
Frequency domain

$I(x, y) = 128\ sin(2\pi x/4)$



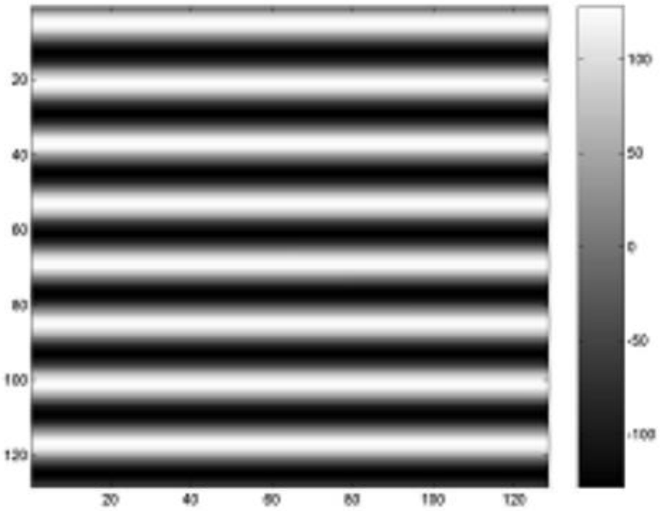
Sinusoid pattern repeats every 4 pixels
f = 1/4 cycles/pixel

Spatial domain



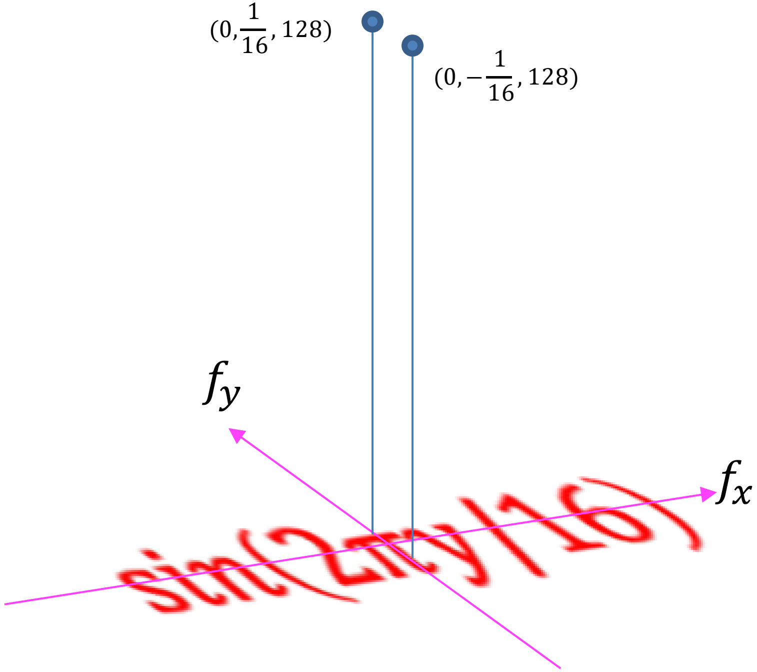
Frequency domain

$I(x, y) = 128\ sin(2\pi y/16)$



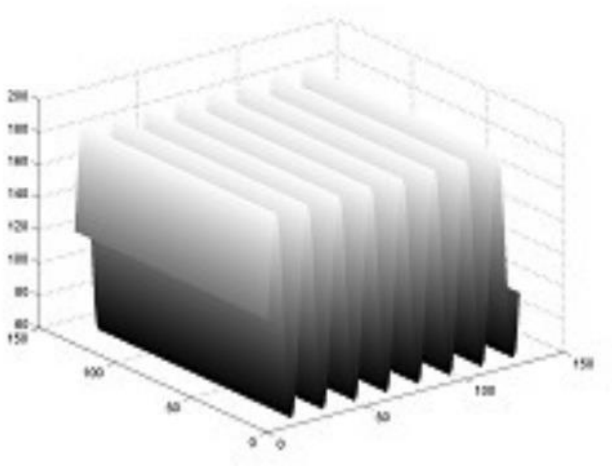
Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel

Spatial domain



Frequency domain

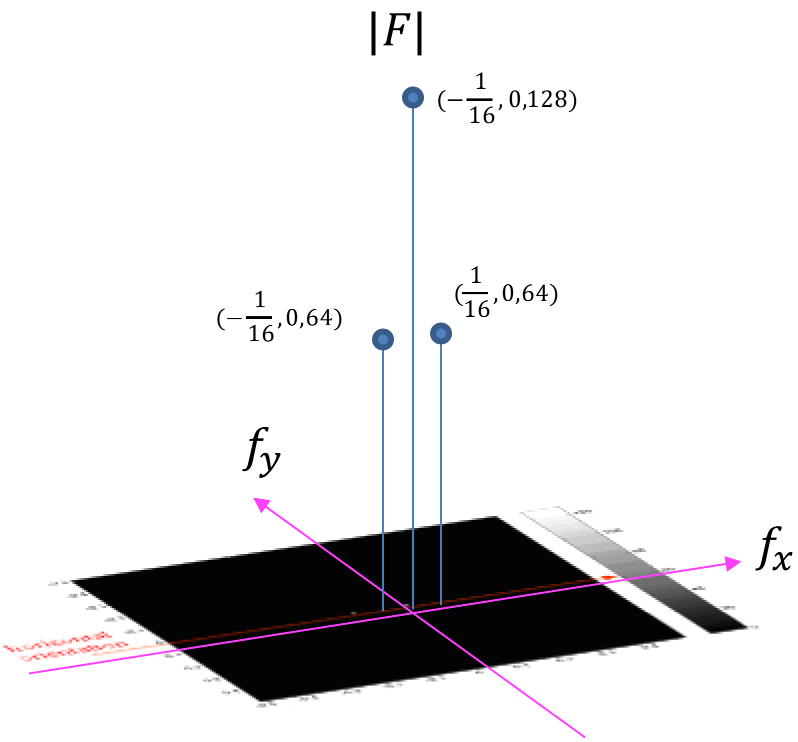
$$I(x, y) = 128 + 64\ sin(2\pi x / 16)$$



Sinusoid pattern repeats every 16 pixels

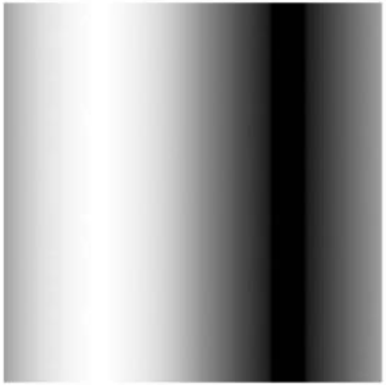
f = 1/16 cycles/pixel

Spatial domain



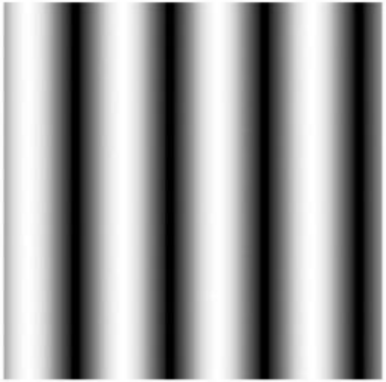
Frequency domain

- Intensity images for $s(x,y)=\sin\left[2\pi\left(\underline{u_0}x+\underline{v_0}y\right)\right]$

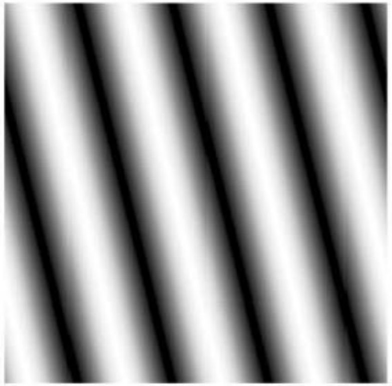


$u_0 = 1, v_0 = 0$

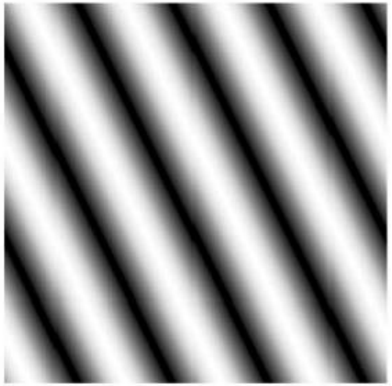
$u_0 = 2, v_0 = 0$



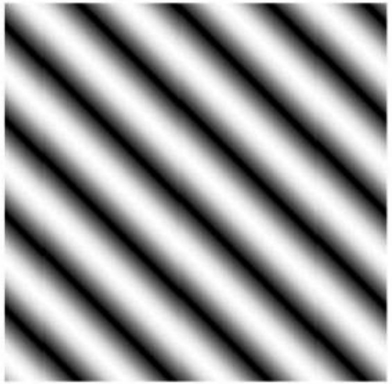
$u_0 = 4, v_0 = 0$



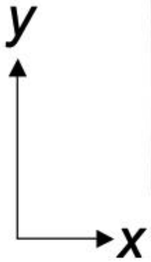
$u_0 = 4, v_0 = 1$

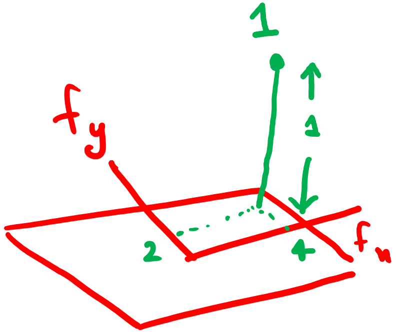


$u_0 = 4, v_0 = 2$ ✓

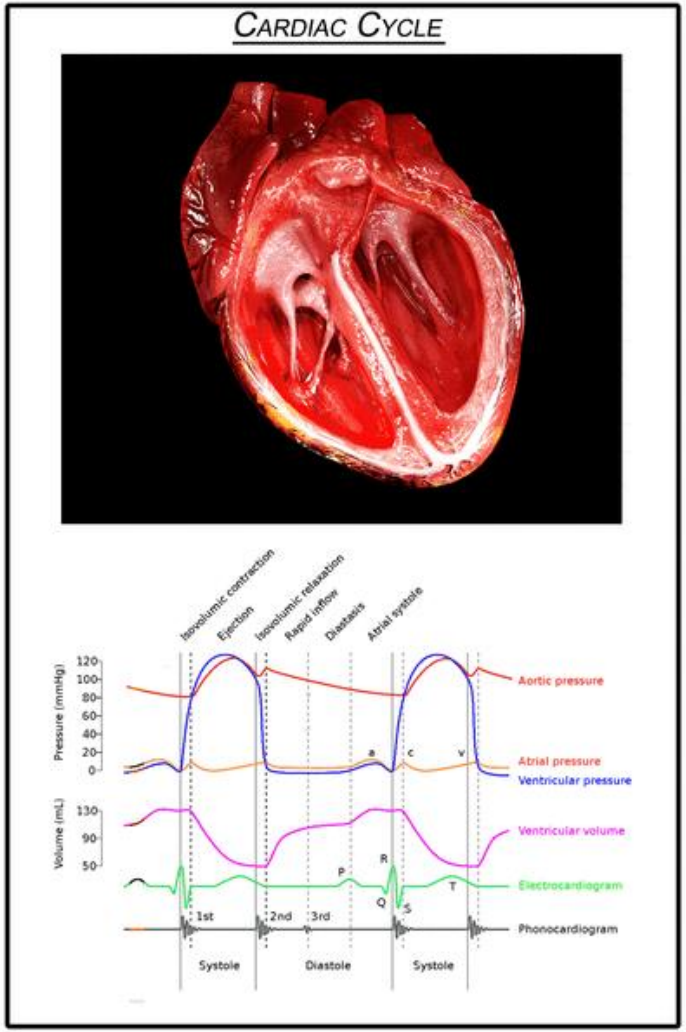


$u_0 = 4, v_0 = 4$





Many natural phenomena (signals) are periodic but not necessarily sinusoidal



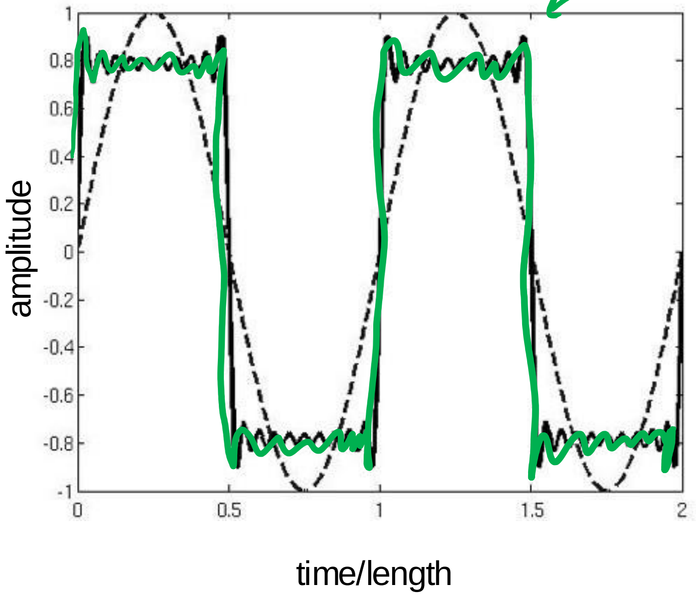
https://commons.wikimedia.org/wiki/File:Cardiac-Cycle-Animated.gif

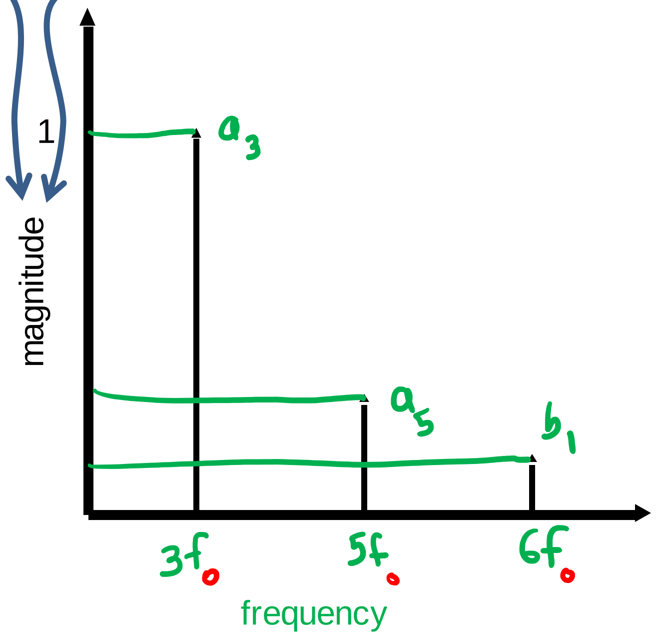
Fourier Series

Approximate **periodic signals** with sines and cosines

Fourier Series

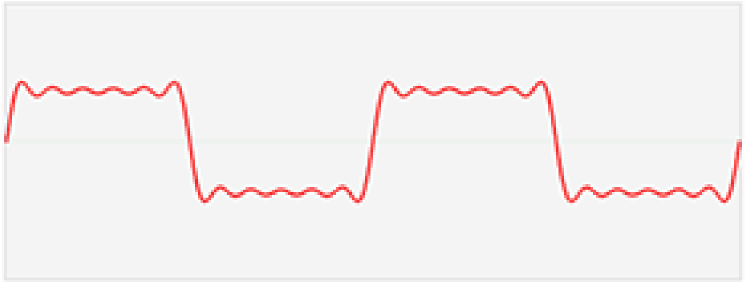
$$y(t)=\sum_n a_n \sin(nf \times 2\pi t)+\sum_n b_n \cos(nf \times 2\pi t)$$



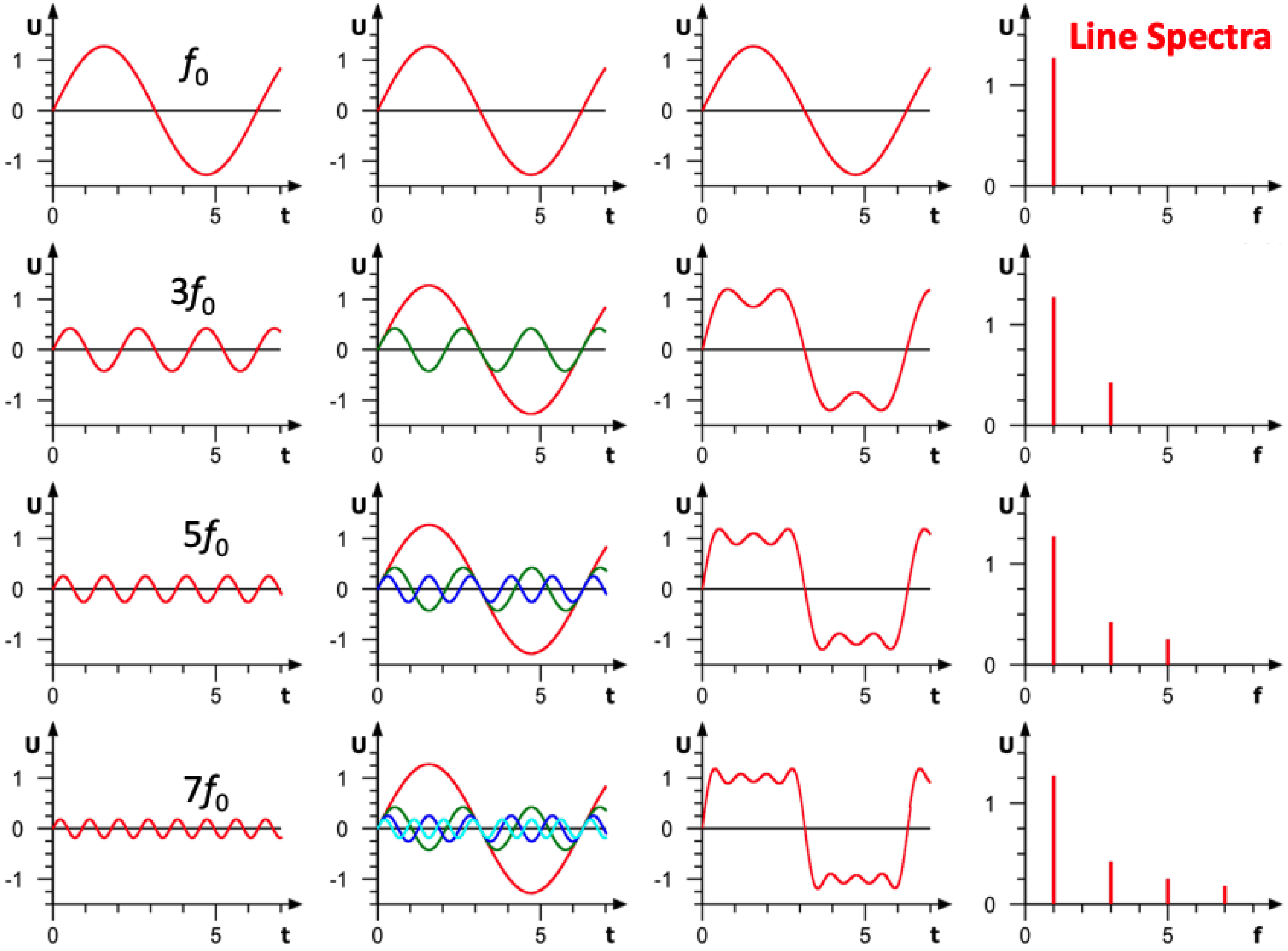


Signal ⟷ **Frequency Spectrum**

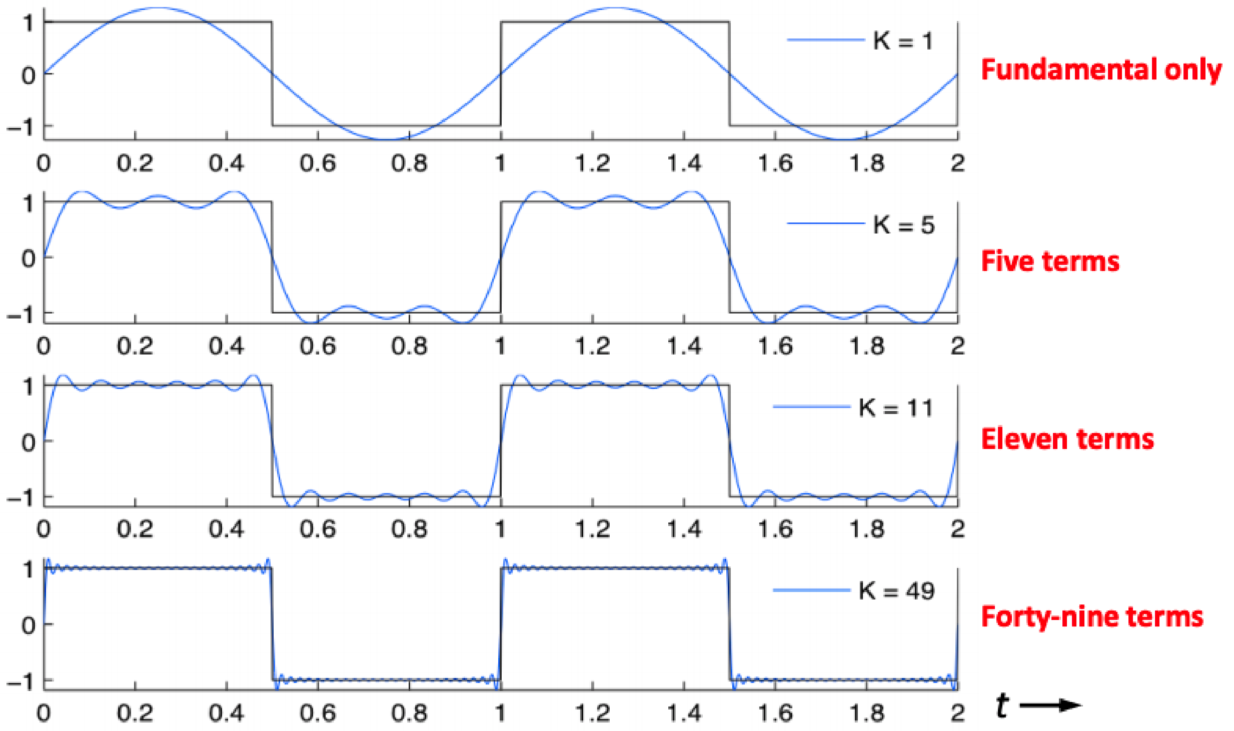
Fourier Series, visually



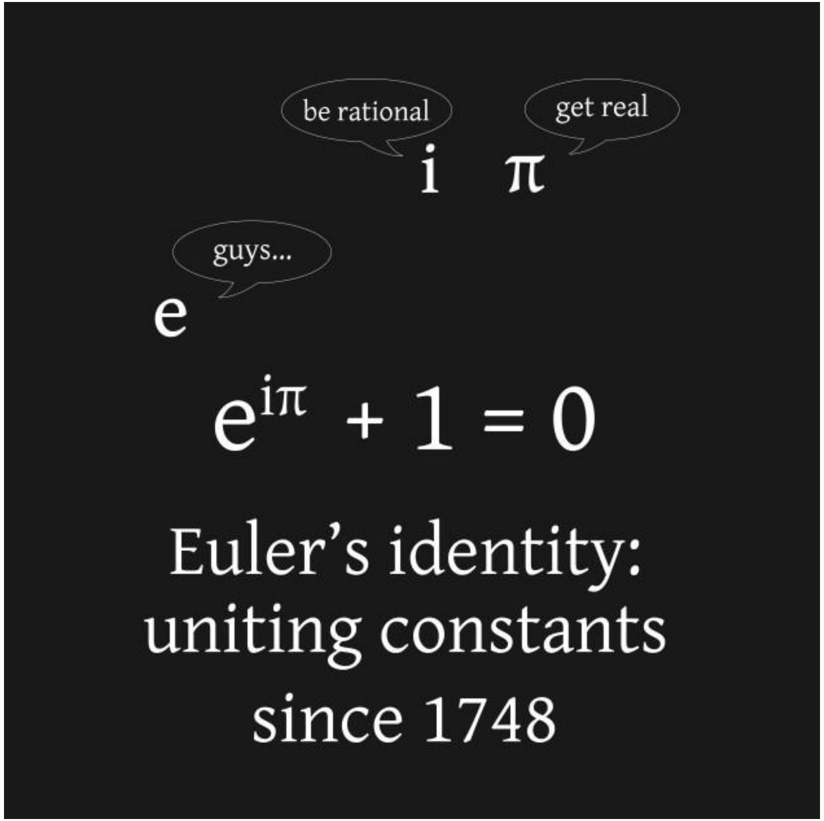
Example: Periodic Square Wave as Sum of Sinusoids



$$f(t)=\frac{4}{\pi}\left[\sin(\pi t)+\tfrac{1}{3}\sin(3\pi t)+\tfrac{1}{5}\sin(5\pi t)+\tfrac{1}{7}\sin(7\pi t)+\cdots\right]$$

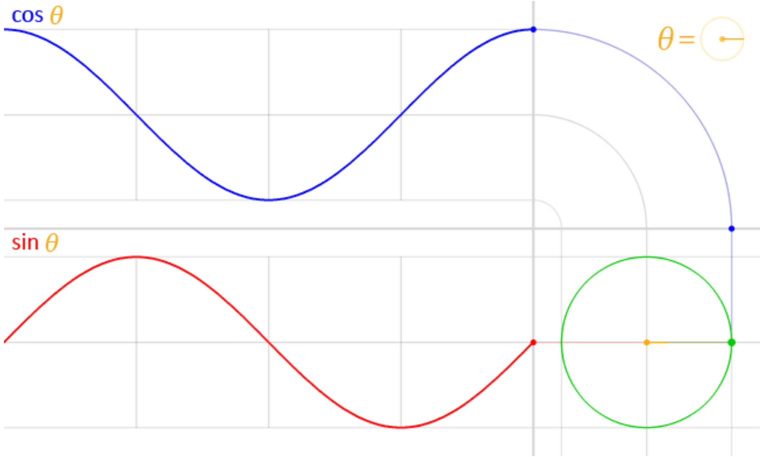


http://ceng.gazi.edu.tr/dsp/fourier_series/description.aspx



$$e^{it} = \cos t + i \sin t$$

$$i = \sqrt{-1}$$



https://res.cloudinary.com/teepublic/image/private/s--s4GX0hG3--/t_Preview/b_rgb:191919,c_limit,f_jpg,h_630,q_90,w_630/v1486157785/production/designs/1177382_1.jpg

Complex sinusoid

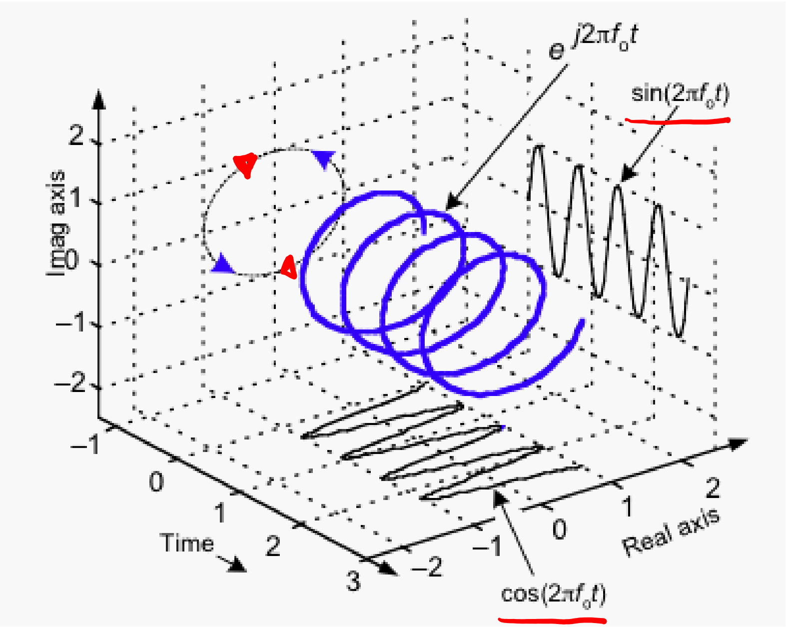
$t \to 0$ to ∞

$$e^{it} = \cos t + i \sin t$$

$$i = \sqrt{-1}$$

$$\cos t = \frac{e^{it} + e^{-it}}{2}$$

$$\sin t = \frac{e^{it} - e^{-it}}{2i}$$



Fourier Series in terms of complex coefficients

$\sin(2\pi x)$

$$\boxed{f(t)} = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi mt}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-i\frac{2\pi nt}{T}}\,dt$$

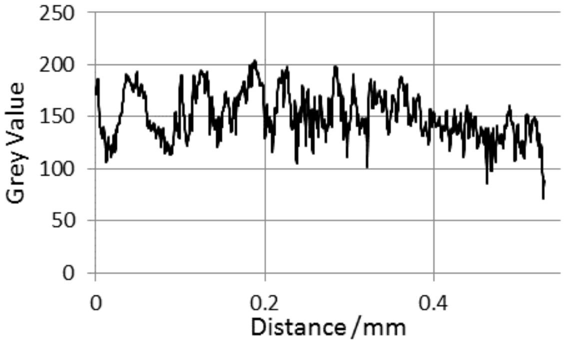
$n = -\infty$ to ∞

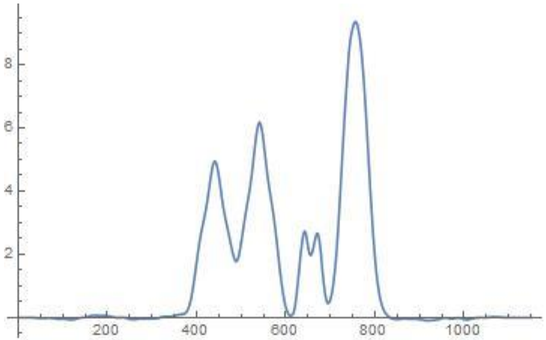
T

$$\underbrace{c_n}\ \underbrace{e^{-i\frac{2\pi nt}{T}}}$$

http://www.thefouriertransform.com/series/complexcoefficients.php

What if f(t) is non-periodic ?





$$\boxed{\cancel{f(t)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n t}{T}}}}$$

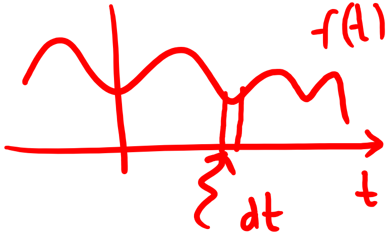
https://i.stack.imgur.com/fKznF.jpg

Fourier Transform

Approximate **non-periodic signals** with complex sinusoids

Definition: Fourier Transform

$\{Re(\omega)\}^2 + Im(\omega)^2$



- the Fourier Transform of a function $\underline{f(t)}$ is defined by:

complex

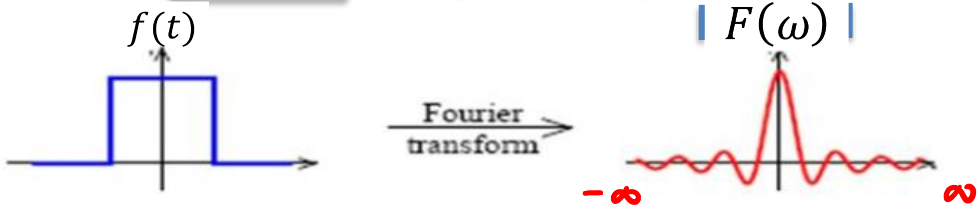
$[Re(\omega), Im(\omega)]$

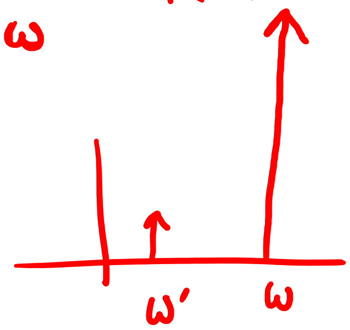
$$\underline{F(\omega)} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

f(t)

ω

- The result is a $\boxed{\text{function}}$ of ω (frequency).





$\sqrt{Re^2 + Im^2}$

→ ω

→ magnitude spectrum

Intuition for FT

- *f(t)* = Single number
- How much of frequency ω signal is present for all values of t ?

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$F(\omega) = \mathcal{F}[f(t)]$$

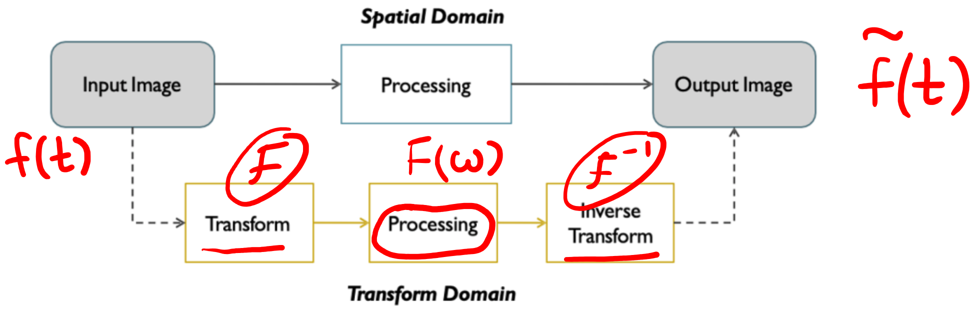
Fourier Transform and Inverse Fourier Transform

- Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad F(\omega) = \mathcal{F}[f(t)]$$

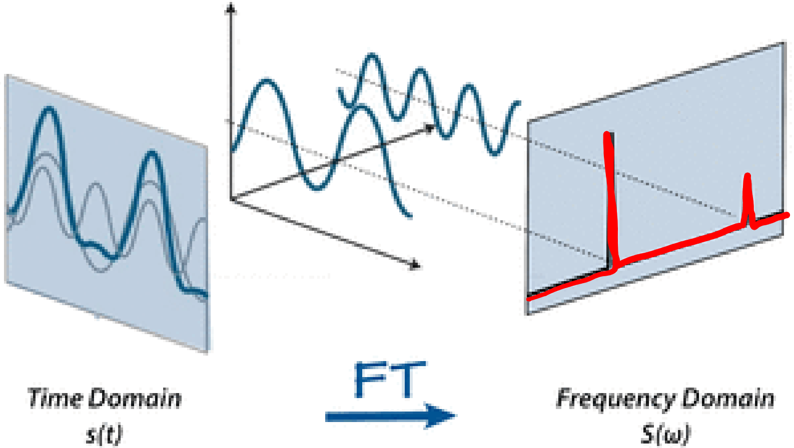
- Inverse Fourier Transform

$$f(t) = \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{2\pi} F(\omega) e^{i\omega t} d\omega \qquad f(t) = \mathcal{F}^{-1}[F(\omega)]$$



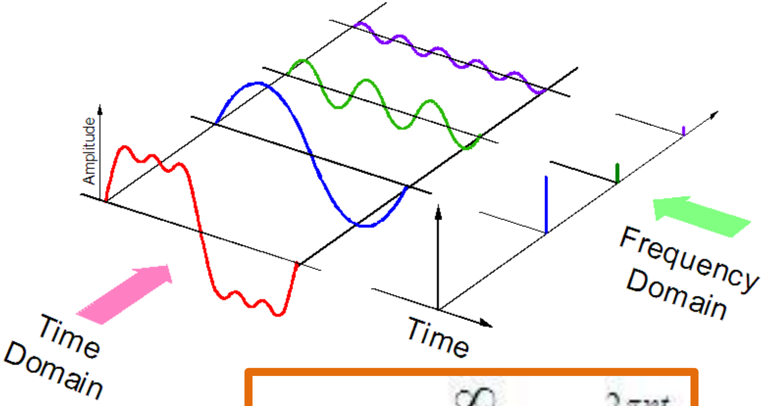
Fourier Transform vs Series

Fourier Transform



$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Fourier Series (periodic only)



$e^{-i\omega t}$

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

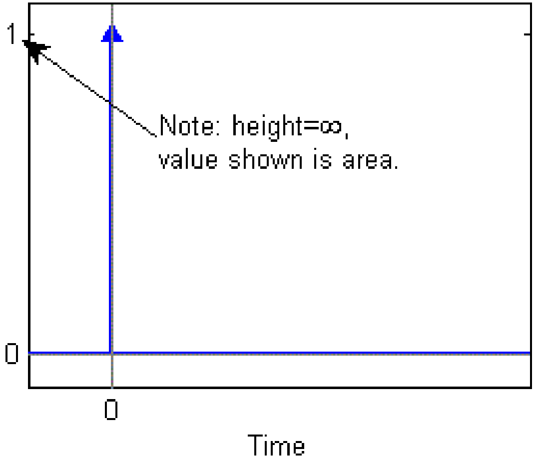
$$c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi nt}{T}}dt$$

Impulse Function

$\delta(t) = 0, \ for \ t \neq 0.$

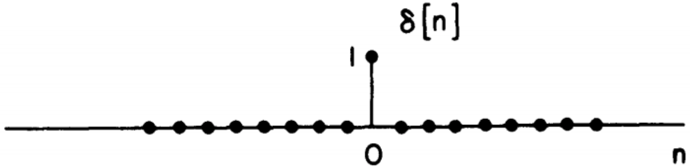
$\delta(0) = +\infty$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



Discrete Impulse Function

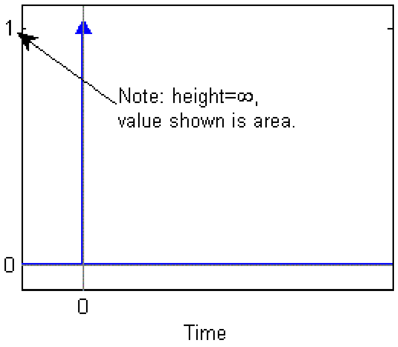
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

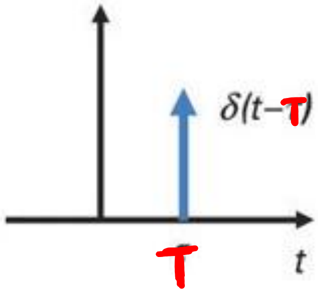
$$\delta(t) = 0, \quad for\ t \neq 0.$$
$$\delta(0) = +\infty$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

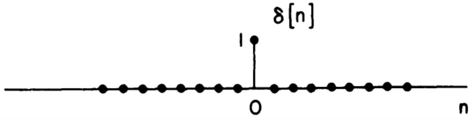


$$\begin{aligned} \delta(t) &= 0, \quad for\ t \neq T \\ &= \infty \quad for\ t = T \end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(t-T)dt = 1$$



Discrete Impulse Function



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

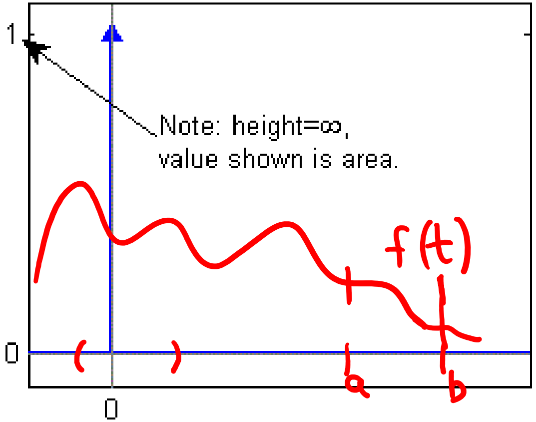
http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

Impulse Function – Some properties

$\delta(t) = 0, \ for \ t \neq 0.$

$\delta(0) = +\infty$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



$$\int_a^b \delta(t)dt = \begin{cases} 1, & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$

$$\int_a^b \delta(t) \cdot f(t)dt = \int_a^b \delta(t) \cdot f(0)dt$$

$$= f(0) \cdot \int_a^b \delta(t)dt$$

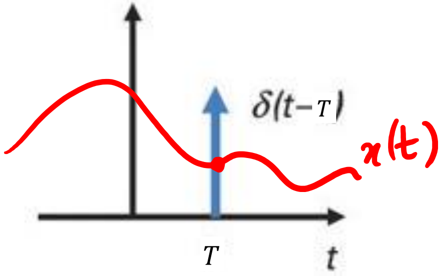
$$= \begin{cases} f(0), & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

Impulse Function – Some properties

Shifted impulse

$$\delta(t) = 0, \quad for\ t \neq T$$
$$= \infty \quad for\ t = T$$



$$\int_{-\infty}^{\infty} \delta(t-T)dt = 1$$

Sifting Property

x(T)

$$\int_a^b \delta(t-T)\,x(t)dt = x(T), \qquad a < T < b$$
$$= 0\ otherwise$$

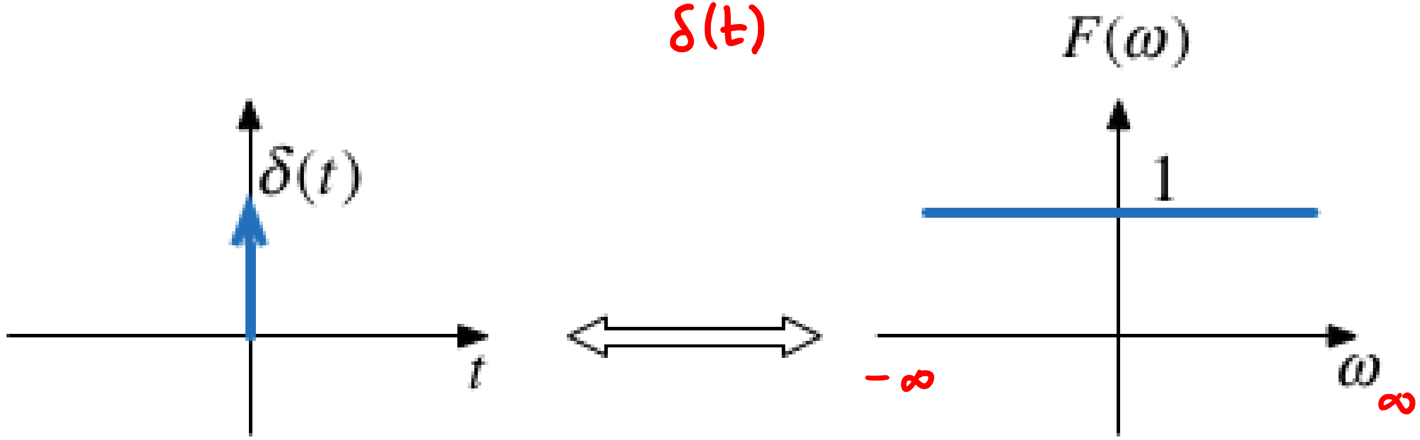
http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

FT of impulse function

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\underbrace{e^{-i\omega t}} dt$$

$$e^{-i\omega(0)} = 1$$

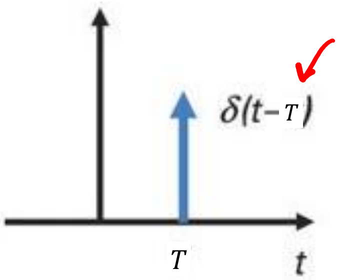
$\delta(t)$



$F(\omega) = 1$

$$\delta(t) \longleftrightarrow 1$$

FT of time-shifted impulse



$\delta(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= e^{-i\omega T}$$

$$\int_a^b \delta(t - T)\, x(t) dt = x(T), \qquad a < T < b$$

$$= 0 \; otherwise$$

$$\delta(t - T) \longleftrightarrow e^{-i\omega T}$$

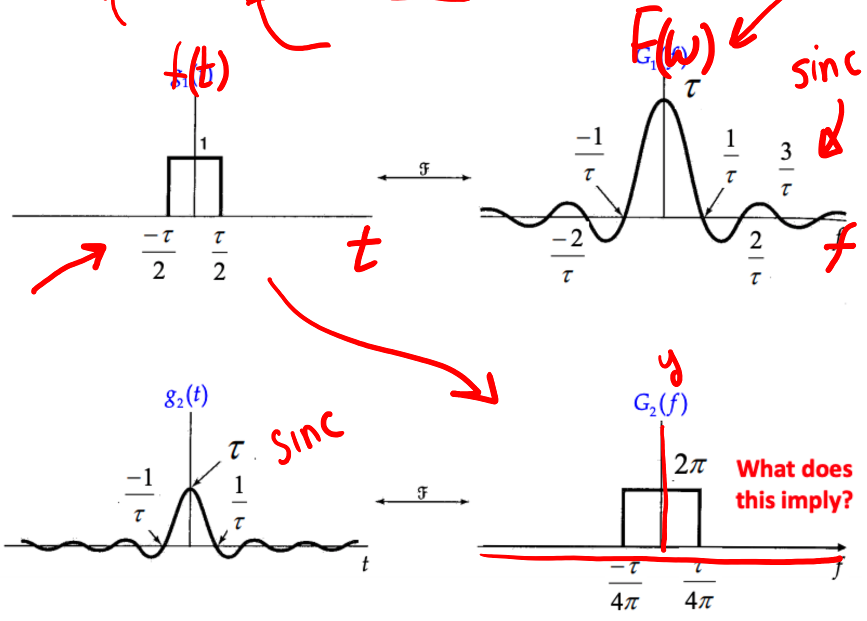
Symmetry property of FT

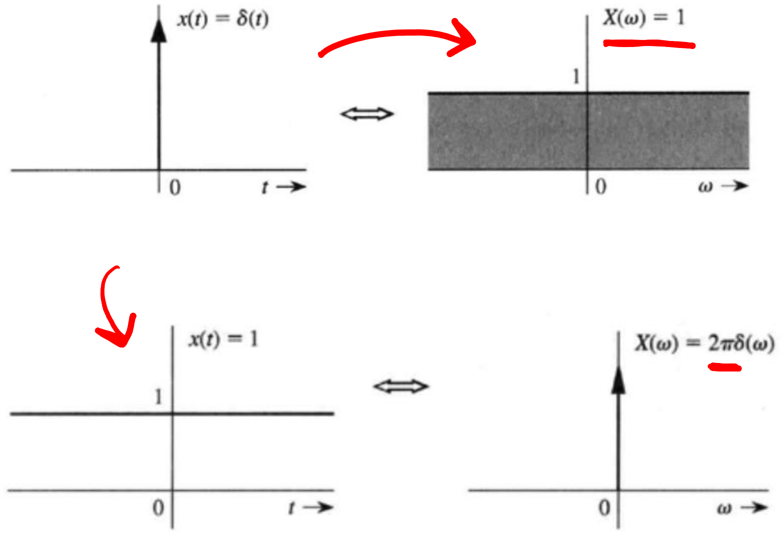
$$\mathcal{F}[f(t)] = F(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} F(\omega) e^{i\omega t} d\omega$$

$$\Longrightarrow \mathcal{F}[F(t)] = \underbrace{2\pi}\, \overbrace{f(-\omega)}$$





Symmetry property of FT

$$\mathcal{F}[f(t)] = F(\omega)$$

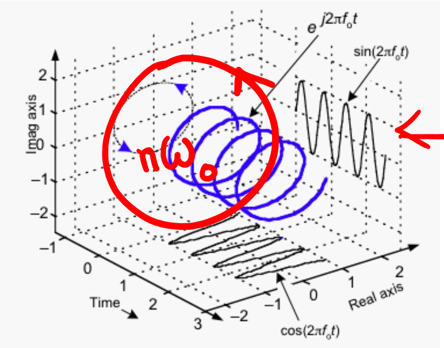
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

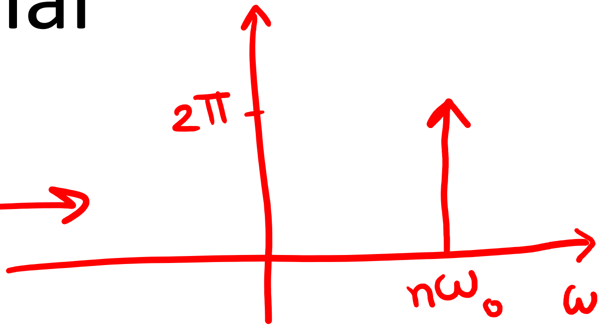
$$\Longrightarrow \mathcal{F}[F(t)] = 2\pi f(-\omega)$$

$$f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} F(\omega)e^{i\omega t}d\omega$$

FT of complex exponential

$$\underline{e^{jn\omega_0 t}} \xleftrightarrow{F} \underline{2\pi\delta(\omega - n\omega_0)}$$



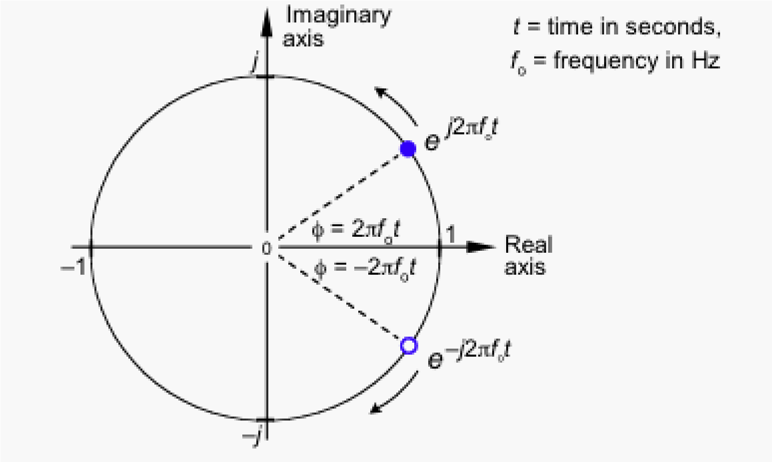


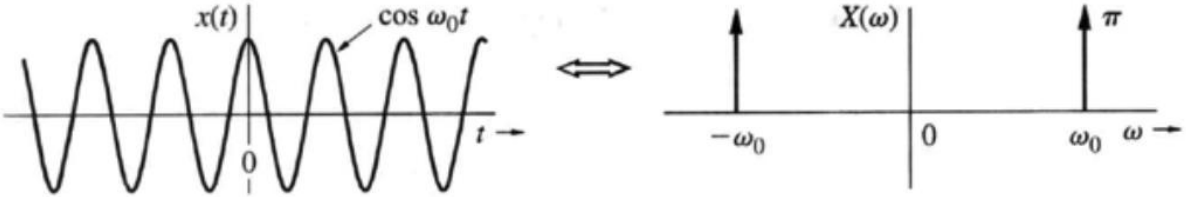
FT of cosine

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega - \omega_0)$$

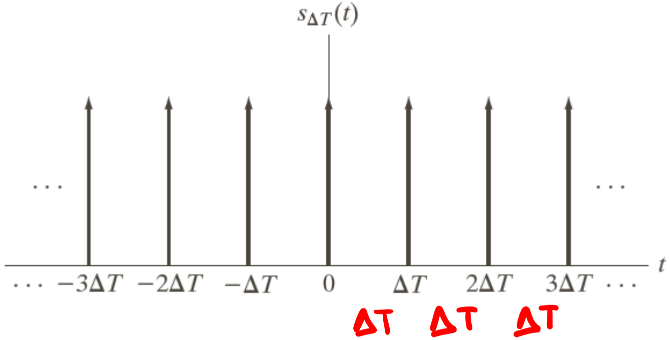
$$e^{-j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega + \omega_0)$$

$$\cos\omega_0 t = \frac{1}{2}\left(e^{-j\omega_0 t} + e^{j\omega_0 t}\right)$$





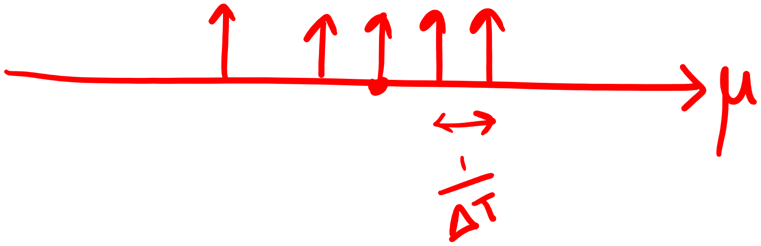
FT of impulse train(G&W, 4.2.4)



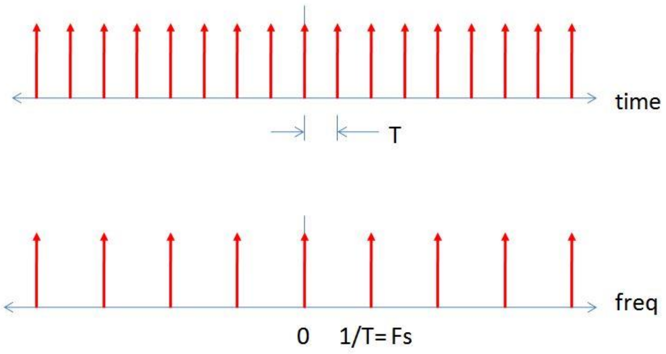
$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

$\mu = 2\pi\omega_0$

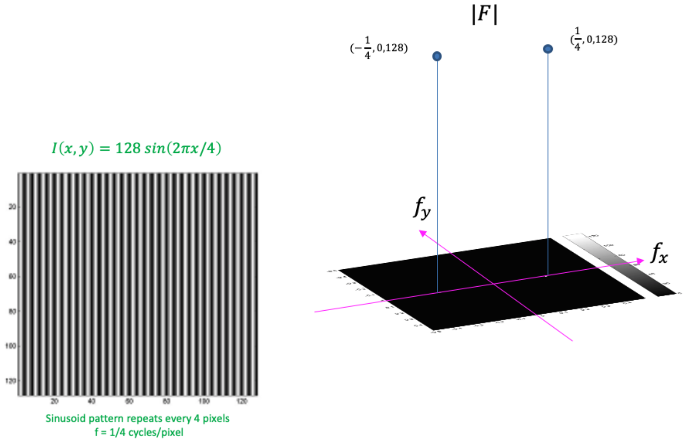
$$\mathcal{F}(\ ) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

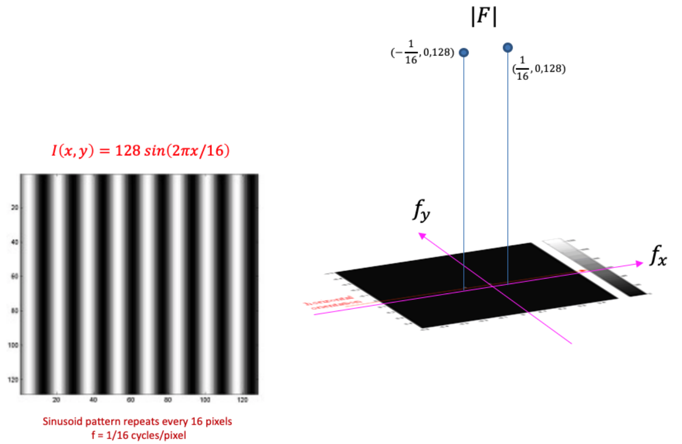


FT of impulse train



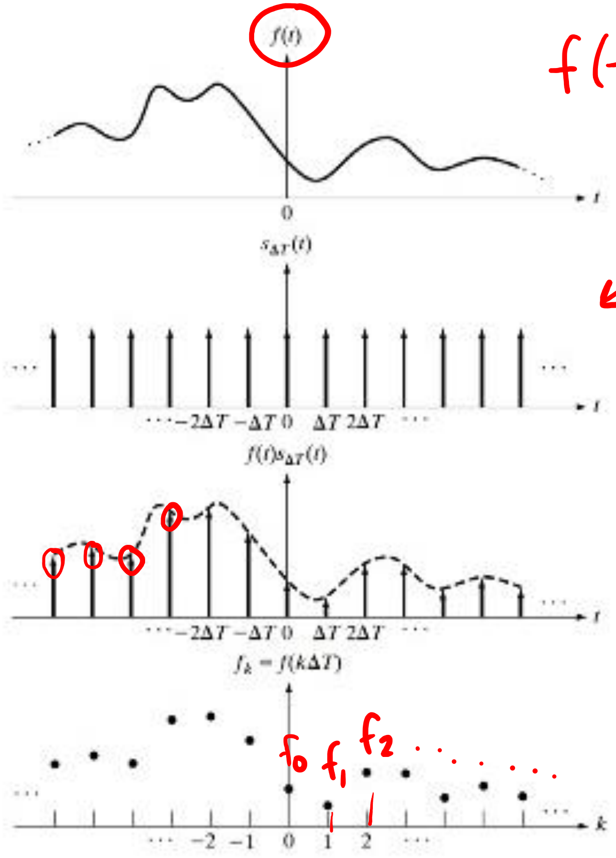
Impulses in time $\longrightarrow \mathcal{F}\{\} \longrightarrow$ Impulses in frequency





Sampling = f(t) x Impulse Train

$f(t)$ $\quad$ $\delta(t)$



$$s_{\Delta T}(t) = \sum_{n=-\infty}^{n=\infty} \underbrace{\delta(t - n\Delta T)}$$

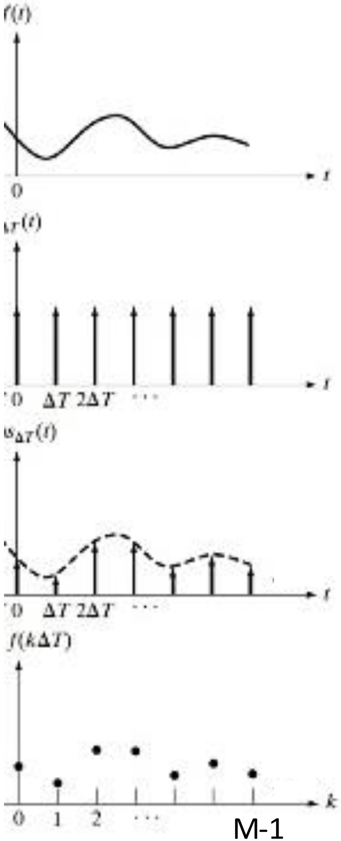
$\delta[n]$

$f_n \cdot$ (Area = 1)

$$\boxed{\tilde{f}(t)} = \sum_{n=-\infty}^{n=\infty} f_n \, \underbrace{\delta(t - n\Delta T)}$$

digital signal

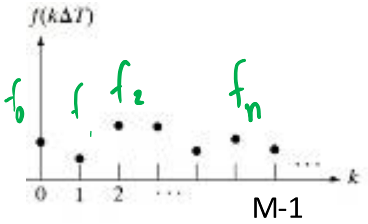
Sampling = f(t) x Impulse Train



1-D

$$\tilde{f}(t) = \sum_{n=0}^{n=(M-1)\Delta T} f_n \, \delta(t - n\Delta T)$$

FT of sampled function



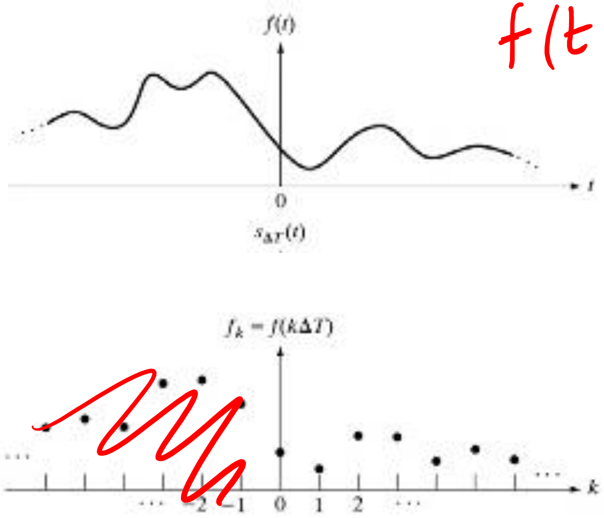
$$\mathcal{F}\left(\tilde{f}(t)\right) = \sum_{n=0}^{n=(M-1)\Delta T} f_n \, \mathcal{F}\left(\delta(t - n\Delta T)\right)$$

$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}$$

$\tilde{F}(\mu)$ $\mu \in \mathbb{R}$

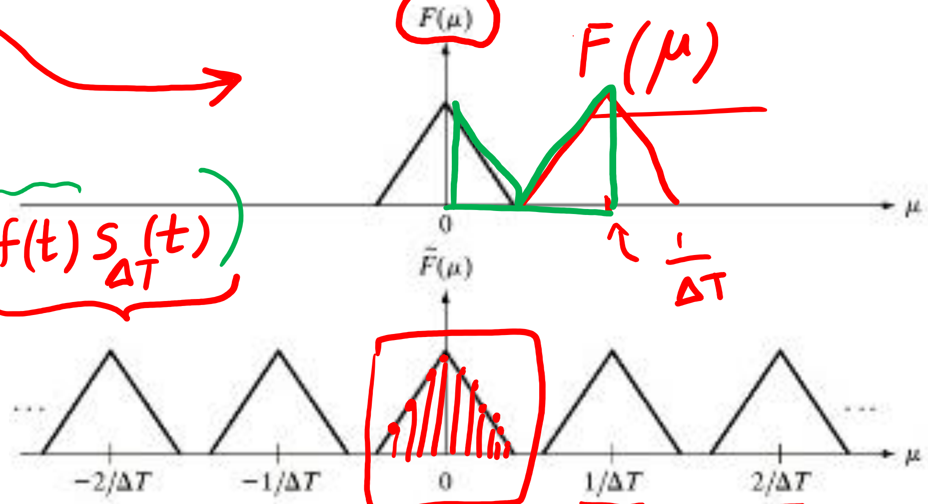
μ

FT of sampled function (G&W 4.2.4)



f(t)

$\tilde{f}(t) = \mathcal{F}(f(t)\, s_{\Delta T}(t))$



F(μ)

$\frac{1}{\Delta T}$

- Continuous
- Periodic (copies of f(t)'s FT)

$\frac{1}{\Delta T}$

$\mu = \frac{1}{\Delta T}$

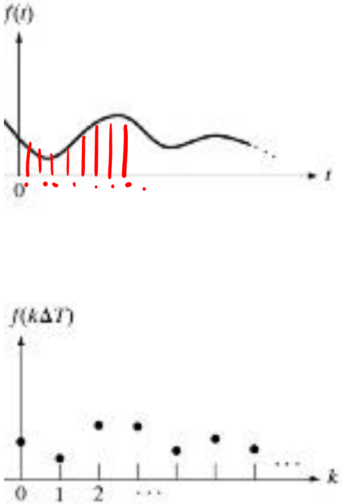
$\frac{1}{\Delta T}$

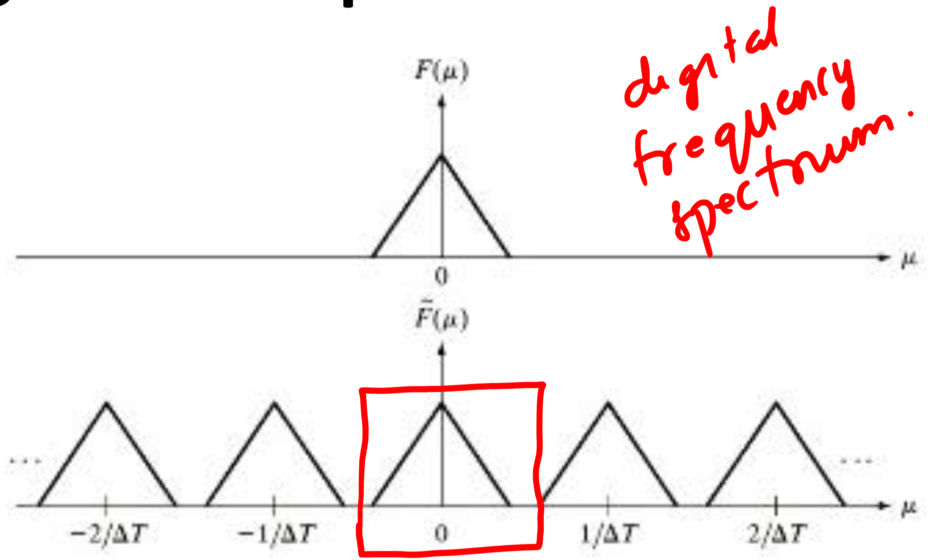
$\mu = 0$

$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n\, \delta(t - n\Delta T)$$

$$\tilde{F}(\mu) = \boxed{\frac{1}{\Delta T}} \sum_{n=-\infty}^{n=\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

Digital processing of frequencies





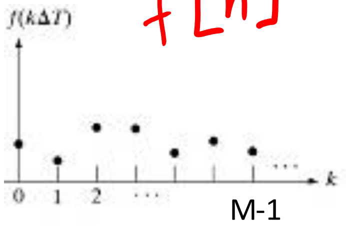
$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n\, \delta(t - n\Delta T)$$

$$\boxed{\tilde{F}(\mu)} = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}$$

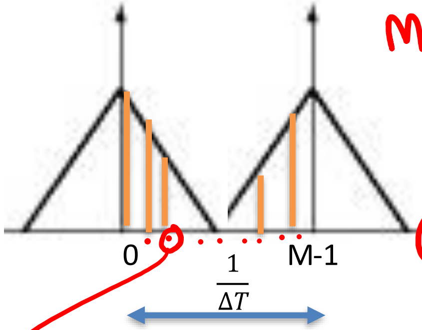
$\mu \in \mathbb{R}$

- Need discrete frequency samples, but FT of sampled function is continuous
- OBSERVATION: Characterizing one period $(\frac{1}{\Delta T})$ is enough
- How do we get frequency 'samples' ?

FT of sampled function (G&W 4.4.1)



f[n]



M samples

$\mu = 0$ $\mu = \frac{1}{\Delta T}$

M

F[m]

$$\tilde{f}(t) = \sum_{n=0}^{n=(M-1)\Delta T} f_n \, \delta(t - n\Delta T)$$

Freq spec μ

$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}, \mu \in R$$

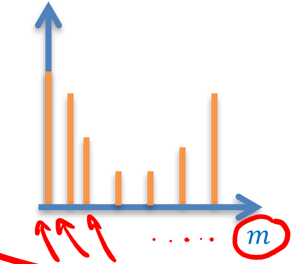
- Substituting

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

$m = 0$ $\mu = 0$

$m = (M-1), \mu = \frac{1}{\Delta T}$

$\mu = \frac{m}{(M-1)\Delta T}$



DFT

$$F[m] = \sum_{n=0}^{n=(M-1)} f_n \, e^{\frac{-j2\pi mn}{M}}, \text{m} = 0, 1, \ldots (\text{M} - 1)$$

(Some) Properties of FT

	Name:	Condition:	Property:
1	Amplitude scaling	$f(t) \leftrightarrow F(\omega)$, constant K	$Kf(t) \leftrightarrow KF(\omega)$
2	Addition	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$, $\cdots$	$f(t) + g(t) + \cdots \leftrightarrow F(\omega) + G(\omega) + \cdots$
3	Hermitian	Real $f(t) \leftrightarrow F(\omega)$	$F(-\omega) = F^*(\omega)$
4	Even	Real and even $f(t)$	Real and even $F(\omega)$
5	Odd	Real and odd $f(t)$	Imaginary and odd $F(\omega)$
6	Symmetry	$f(t) \leftrightarrow F(\omega)$	$F(t) \leftrightarrow 2\pi f(-\omega)$
7	Time scaling	$f(t) \leftrightarrow F(\omega)$, real s	$f(st) \leftrightarrow \frac{1}{\lvert s \rvert} F(\frac{\omega}{s})$
8	Time shift	$f(t) \leftrightarrow F(\omega)$	$f(t - t_o) \leftrightarrow F(\omega) e^{-j\omega t_o}$
9	Frequency shift	$f(t) \leftrightarrow F(\omega)$	$f(t) e^{j\omega_o t} \leftrightarrow F(\omega - \omega_o)$
10	Modulation	$f(t) \leftrightarrow F(\omega)$	$f(t) \cos(\omega_o t) \leftrightarrow \frac{1}{2} F(\omega - \omega_o) + \frac{1}{2} F(\omega + \omega_o)$
11	Time derivative	Differentiable $f(t) \leftrightarrow F(\omega)$	$\frac{df}{dt} \leftrightarrow j\omega F(\omega)$
12	Freq derivative	$f(t) \leftrightarrow F(\omega)$	$-jtf(t) \leftrightarrow \frac{d}{d\omega} F(\omega)$
13	Time convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t) * g(t) \leftrightarrow F(\omega) G(\omega)$
14	Freq convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t) g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega)$
15	Compact form	Real $f(t)$	$f(t) = \frac{1}{2\pi} \int_0^\infty 2 \lvert F(\omega) \rvert \cos(\omega t + \angle F(\omega)) d\omega$
16	Parseval, Energy W	$f(t) \leftrightarrow F(\omega)$	$W \equiv \int_{-\infty}^{\infty} \lvert f(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert F(\omega) \rvert^2 d\omega$

https://hkn.illinois.edu/wiki/_media/sidebar:screen_shot_2016-04-11_at_12.35.36_pm.png?w=800&tok=1f0b2e

References & Fun Reading/Viewing

- GW DIP textbook, 3rd Ed.
  - 4.1 to 4.2
  - 4.2.4, 4.2.5,4.3.1,4.3.2, 4.4.1

- https://betterexplained.com/articles/intuitive-understanding-of-sine-waves/
- A visual introduction to Fourier Transform: https://www.youtube.com/watch?v=spUNpyF58BY
- Fourier Transform, Fourier Series and Frequency Spectrum: https://www.youtube.com/watch?v=r18Gi8lSkfM

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