Topics in Applied Optimization

Optimization Algorithms for ML and Data Sciences

Pawan Kumar

CSTAR, IIIT-H

Conjugate Function

Conjugate Function

$y^T x - f(x)$

Conjugate Function: Let $f : \mathbb{R}^n \to \mathbb{R}$. The function $f^* : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f^*(y) = \sup_{x \in \mathbf{dom}\ f} (y^T x - f(x))$$

is called **conjugate** of the function f. The **domain** of the conjugate function consists of $y \in \mathbb{R}^n$ for which the sum is **finite**, i.e., the difference $y^T x - f(x)$ is **bounded** above.

Conjugate Function

For a fixed x,
Is $y^T x - f(x)$ cvx ?

$\sup\{y^T x_1 - f(x_1), y^T x_2 - f(x_2), \ldots\}$

x_1 x_2

$\{g_1, g_2, \ldots\}$ If g_i's are cvx then $\sup\{g_1, g_2, \ldots\}$ is cvx.

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$f(x) = Ax + b$

$\nabla^2 f(x) \geq 0$

- f^* is a convex function since it is a pointwise supremum of a family of convex functions

Linear fn of y.

$h(y) = y^T x - \cancel{f(x)}$

$h(\theta y_1 + (1-\theta) y_2) \leq \theta h(y_1) + (1-\theta) h(y_2)$

$\nabla_y h(y) = x$, $\boxed{\nabla_y^2 h(y) = 0}$

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$x^{i+1} = x^i - \alpha \nabla f(x^i)$

$\max\{f_1(x), f_2(x)\}$

not necessarily diff.

- f^* is a convex function since it is a pointwise supremum of a family of convex functions
- f^* is convex regardless of whether f is convex or not

(Non-smooth opt problem). We need sub-differential

$f(x)$ $f_2(x)$ $\Rightarrow$ not diff. corner/kink

History and Geometric Intuition of Conjugates

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Recall supporting hyperplanes:

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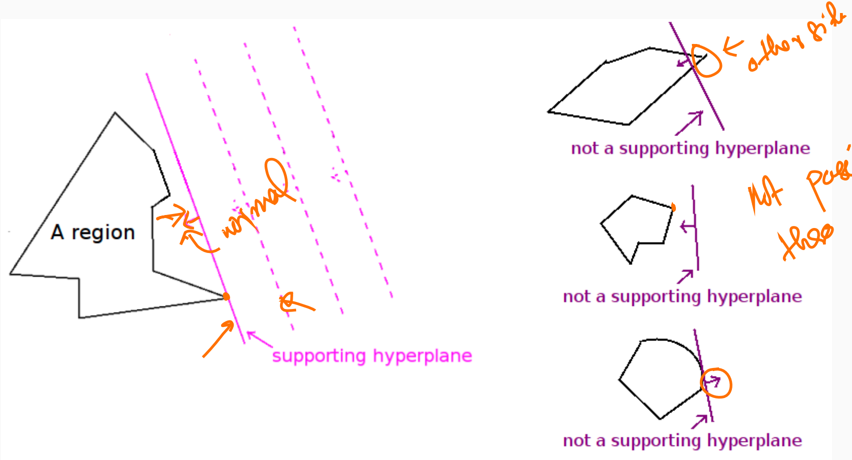
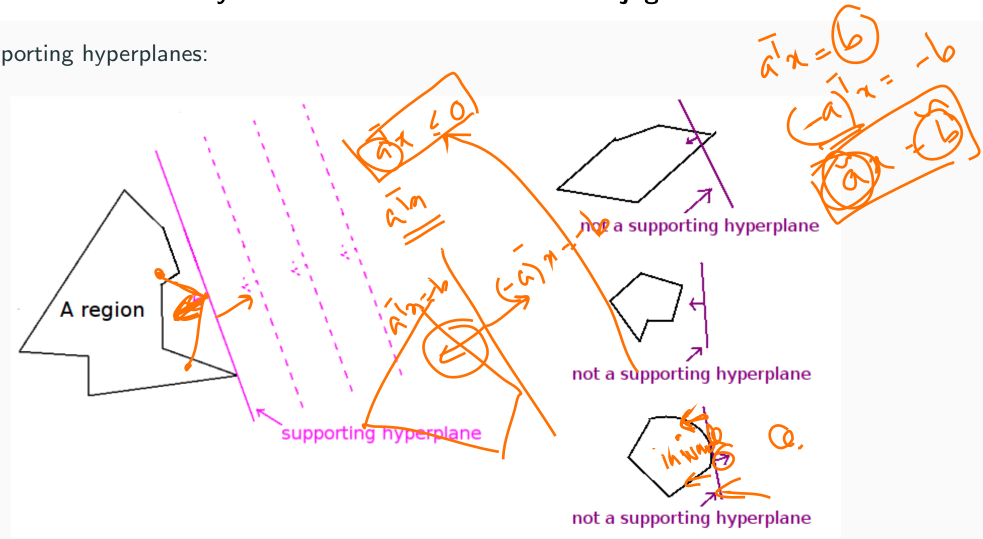


Not passing thro one of the pts.

History and Geometric Intuition of Conjugates

Recall supporting hyperplanes:

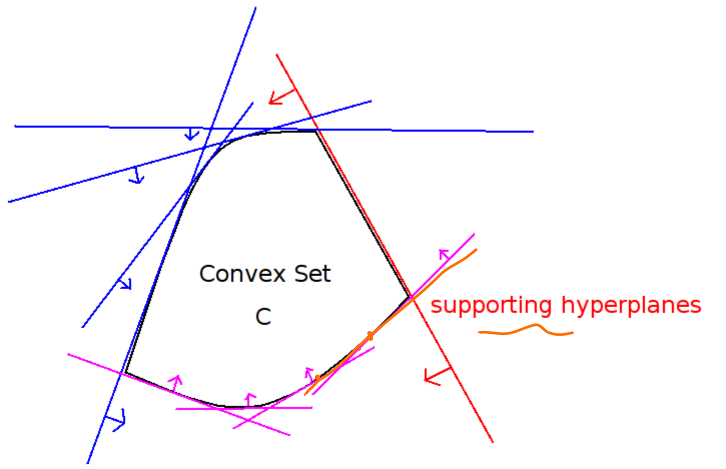


- Last one is not, because normal is pointing outwards

History and Geometric Intuition of Conjugates

History and Geometric Intuition of Conjugates

A closed convex set can be represented by hyperplanes:



History and Geometric Intuition of Conjugates

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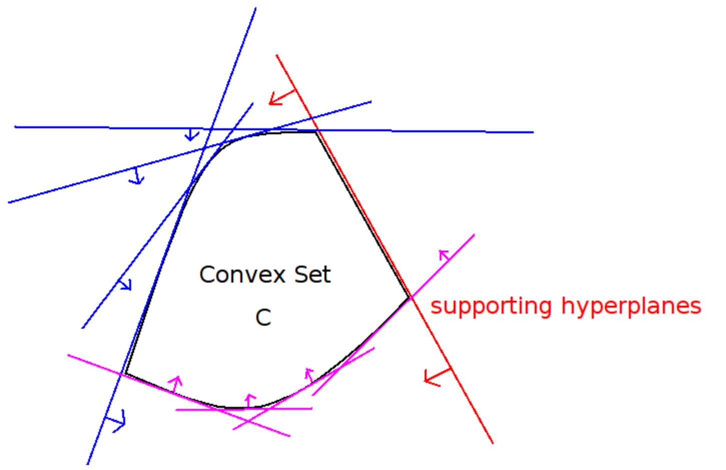
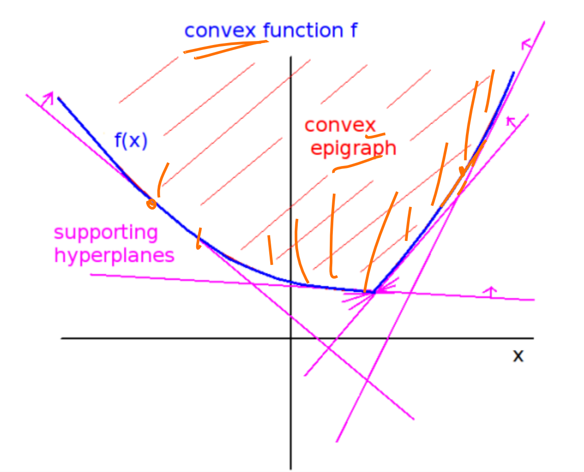


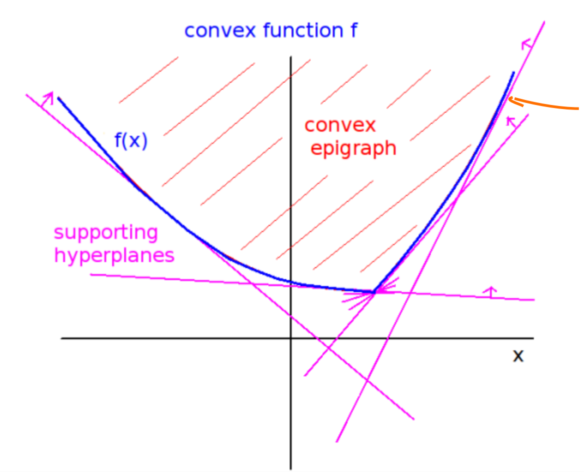
Note: At each point of a convex set, there is a unique supporting hyperplane

Supporting Hyperplanes for Convex Sets

Supporting Hyperplanes for Convex ~~Sets~~ Function



Supporting Hyperplanes for Convex Sets

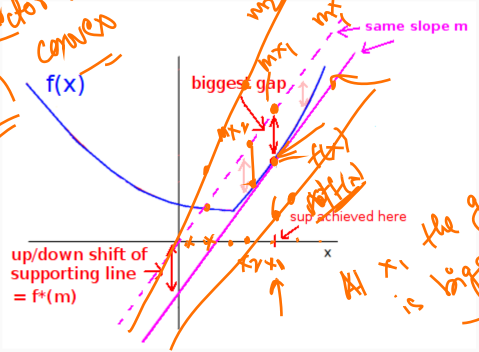


- A closed convex set is uniquely determined by lower hyperplanes

Geometric intuition of Fenchel/Legendre's Transform

In 1D, Fenchel/Legendre's tranform is:

$$f^*(m) = \sup_{x \in \mathbb{R}} (mx - f(x))$$



- Pick a plane with slope m and passing through origin
- Move the plane parallel to above plane until it becomes supporting hyperplane

Conjugates of Some Convex Functions on $\mathbb{R}$

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Find the conjugates of the following functions:

Conjugates of Some Convex Functions on $\mathbb{R}$

Find the conjugates of the following functions:

- Affine function: $f(x) = ax + b$.
- Negative logarithm: $f(x) = -\log x$
- Exponential. $f(x) = e^x$
- Negative Entropy. $f(x) = x \log x$
- Inverse. $f(x) = 1/x$

See classnotes for solutions.

Solution to previous problem...

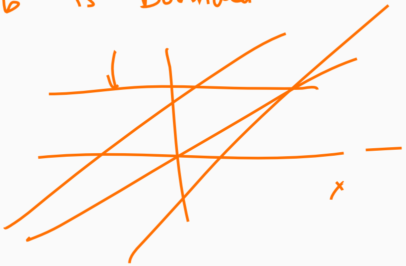
$f^*(y) = \sup_x \left(y^T x - f(x)\right)$

$y^T x - f(x)$ must be bounded.

Find conjugate of the affine function $f(x) = ax + b$ $x \in \mathbb{R}$.

$yx - f(x) = yx - ax - b$ is bounded

Are linear fns bounded?



if $y = a$

dom $f^* : \{a\}$

$f^*(a) = -b$.

Solution to previous problem...

Find conjugate of the negative logarithm $f(x) = -\log x$,

$\sup_x (g) \; (h) \; \cdot \; 0 \to \infty$

$\operatorname{dom} f = \mathbb{R}_{++}$

$f^*(y) = y\cdot\left(-\frac{1}{y}\right) + \log\left(-\frac{1}{y}\right)$

$= -\log(-y) - 1 \quad \forall y < 0$

$f(y) = yx - f(x) = yx + \log x$

Case-I. $y \geq 0$

$y + \frac{1}{x} = \geq 0 + > 0 = > 0$

$\Rightarrow f'(y) = \cancel{x}$ $\Rightarrow f(y)$ is strictly incr. fn $\forall y \geq 0$

unbounded case

Case-II $\boxed{y < 0}$

$f'(x) = y + \frac{1}{x}$

$\leq 0 \quad \geq 0$

Critical point

$f'(x) = 0 \Rightarrow y + \frac{1}{x} = 0 \Rightarrow \boxed{y = -\frac{1}{x}}$

$\Rightarrow \boxed{x = -\frac{1}{y}}$

$\boxed{\operatorname{dom} f^* = \{y \mid y < 0\}}$

$y_1 \geq 0$, $y_2 \geq 0$ $= \mathbb{R}_{++}$

$f''(x) = -\frac{1}{x^2} < 0$

$\Rightarrow f''(x)\big|_{x=-1/y} = -\frac{1}{(-1/y)^2} = -y^2 < 0$