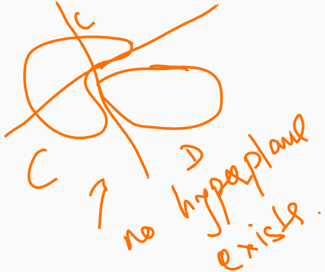
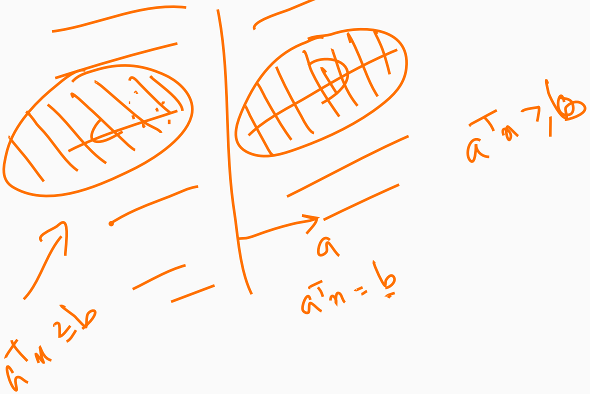
Seperating hyperplane theorem: Suppose C and D are nonempty disjoint (closed) convex sets, i.e., $C \cap D = \phi$. Then there exists $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$. The hyperplane $\{x \mid a^T x = b\}$ is called a separating hyperplane for the sets C and D.

C & D being convex is necessary, because





Separating and Supporting Hyperplanes

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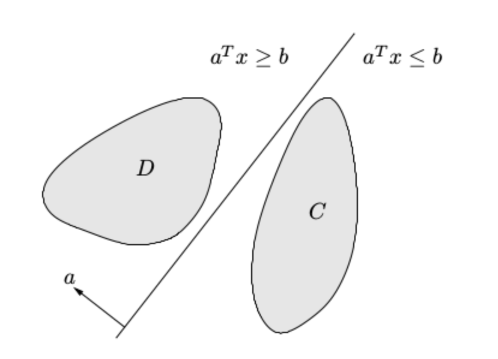

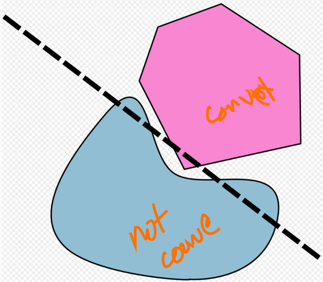

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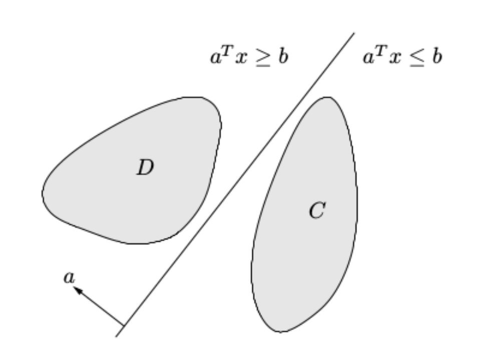
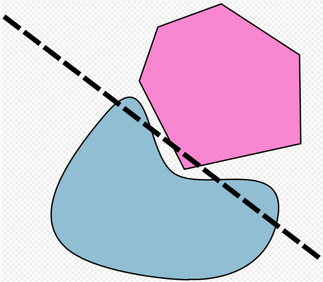




Note: It does not work, if the sets are non-convex

Scratch Space

$[0,1]$ $[2,3]$

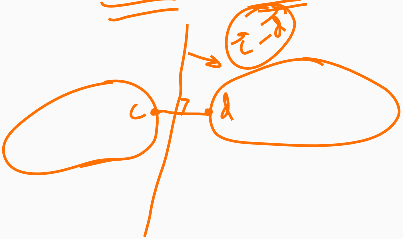
Proof of Sep. Hyp. Th.

Given $C \cap D = \phi$, C, D are cvx

$$\mathrm{dist}(C,D) = \inf\{\|u-v\|_2 \mid u \in C, v \in D\}$$

+ve

C & D are closed sets $\Rightarrow$ $\exists\, c \in C$, $d \in D$ that achieves $= \mathrm{dist}(C,D)$



$(0,1)$ $[0,1]$ 2.1

$C = (0,1)$, 0.9 $D = (2,3)$

$$\mathrm{dist}(C,D) = \inf\{\|u-v\|_2,\ u \in C, v \in D\}$$

$= \{\cancel{0.9, 0.99}, \ldots \cancel{0.999}, \ldots\}$

$2.5 \ldots$

$\{1.1, 1.01, \ldots 1.0001 \ldots\}$

1

Scratch Space

We define $a = d - c$, $b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}$

Claim $f(x) = a^T x + b$

$= (d-c)^T \left(x + \left(\frac{1}{2}\right)(d+c) \right)$

$(d-c)^T (d+c)$

$= d^T d - c^T d + d^T c - c^T c$

$\|d\|_2^2 - \|c\|_2^2$

≤ 0 on C ①

≥ 0 on D ②

② f is non-neg on D

Sppse $u \in D$ we have to prove that

$\boxed{f(u) \geq 0}$ Proof by contradiction:

Sppse $f(u) < 0$ ← assumption must be wrong.

$\Rightarrow (d-c)^T \left(u - d + \left(\frac{1}{2}\right)(d-c) \right) < 0$

$\Rightarrow \underbrace{(d-c)^T (u-d)} + \underbrace{\left(\frac{1}{2}\right) \|d-c\|_2^2}_{\geq 0} < 0$

$\Rightarrow (d-c)^T (u-d) < 0$

$f(x) \geq 0$ on D

Scratch Space

We had $(d-c)^T(u-d) < 0$

Consider:

$$\frac{d}{dt}\|d+t(u-d)-c\|_2^2$$

$$= \frac{d}{dt}\left(d+t(u-d)-c\right)^T\left(d+t(u-d)-c\right)$$

$$= \frac{d}{dt}\left((d-c)^T(d-c) + (d-c)^T t(u-d) + t(u-d)^T(d-c) + t^2(u-d)^T(u-d)\right)$$

$$= \frac{d}{dt}\left(2t(d-c)^T(u-d) + t^2(u-d)^T(u-d)\right)$$

$$= 2(d-c)^T(u-d) + 2t(u-d)^T(u-d)$$

$$\left.\frac{d}{dt}\|d+t(u-d)-c\|_2^2\right|_{t=0}$$

$$= 2(d-c)^T(u-d) < 0$$

$\Rightarrow \exists$ some ~~t~~ t small enough s.t. $\|d+t(u-d)-c\|_2 < \|d-c\|$

$u \in D,\ d \in D$

$d+t(u-d) \in D$

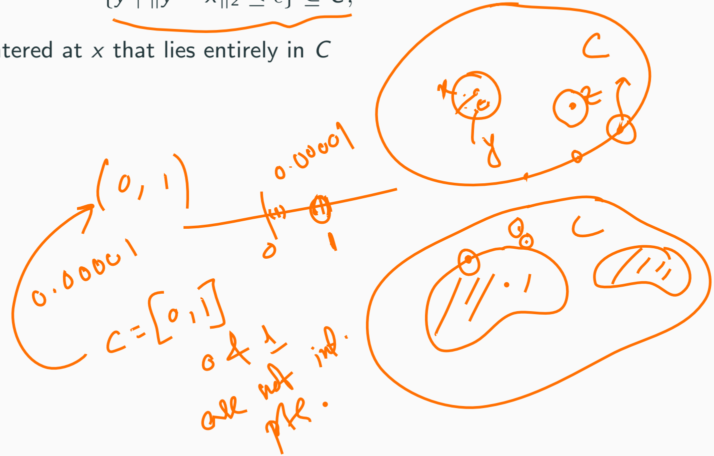
a new pt closer to centroid

c, a

Open Sets: An element $x \in C \subseteq \mathbb{R}^n$ is called **interior** point of C if there exists an $\epsilon > 0$ for which

$$\{y \mid \|y - x\|_2 \leq \epsilon\} \subseteq C,$$

that is, there exists a ball centered at x that lies entirely in C



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$\text{int}([0,1]) = (0,1).$

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$$\textbf{cl } C = \mathbb{R}^n \setminus \textbf{int}(\mathbb{R}^n \setminus C)$$

0, 1 bdry

closure

(0, 1) $\cup \{0, 1\}$ $\Rightarrow$ [0, 1]

$C \cup \text{bdry}(C)$

C

$x \in C$ is bdry pt.

if whenever we take a $B_\epsilon(x) \cap C \neq \phi$ "and" $B_\epsilon(x) \cap C^c \neq \phi$

Analysis and Topology: Boundary, Closure, and Interior

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Analysis and Topology: Boundary, Closure, and Interior

Open Sets: An element $x \in C \subseteq \mathbb{R}^n$ is called **interior** point of C if there exists an $\epsilon > 0$ for which

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$(1-\epsilon, 1+\epsilon)$

$\epsilon = 1$

$(0, 2) \not\subset Z$

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- The **boundary** of a set C is defined as **bd** $C =$ **cl** $C \setminus$ **int** C
- A point belonging to **bd** C is called **boundary point**

Illustration of interior, boundary points

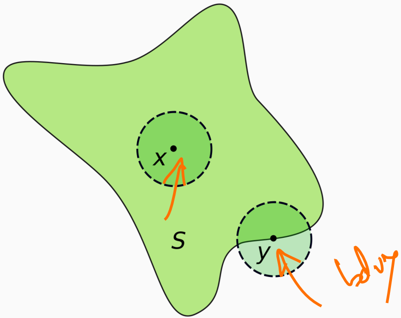

Figure 1: Source: Wiki

Illustration of interior, boundary points

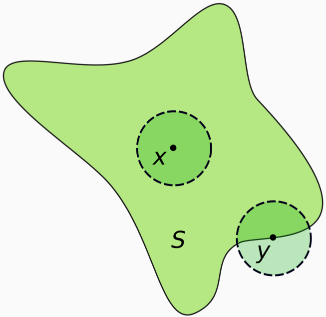

Figure 1: Source: Wiki

- Here x is the interior point

Illustration of interior, boundary points

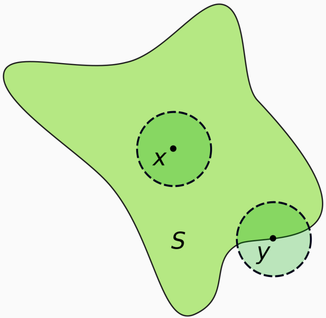

Figure 1: Source: Wiki

- Here x is the interior point
- Here y is the boundary point

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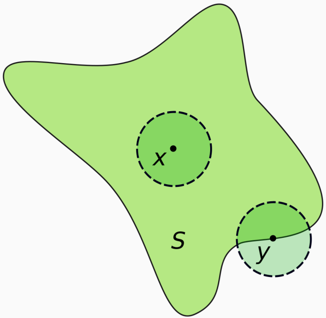


Figure 1: Source: Wiki

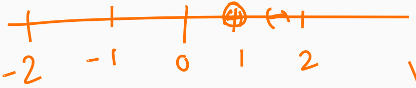
- Here x is the interior point
- Here y is the boundary point
    - Note: A boundary point y is a point in either the set or its complement such that whenever a ball is drawn around the point y of however small radius, there always exists a point $z \neq S, z \neq y$ and a point $u \in S^c, u \neq y$.

Examples: Open Sets

open set $\equiv$ all points are interior pts

Quiz: Which of the following are open sets as a subset of $\mathbb{R}$?

1. $\mathbb{Z}$
2. $\{1/n, n = 1, 2, 3, \dots\}$
3. $\{\sin x \mid 0 \le x \le 2\pi\}$
4. $\{\sin x \mid 0 < x < \pi\}$
5. $\{x \in \mathbb{R} \mid$ x is rational number$\}$



$0 < y \le 1$

$(0, 1]$

$(x-\epsilon, x+\epsilon) \not\subseteq \mathbb{Q}$

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$S = \{1, \frac{1}{2}, \frac{1}{3}, \dots \to 0\}$

$(1-\epsilon, 1+\epsilon)$

$(0,1) \cup (1,2)$ open

$(1-\epsilon, 1+\epsilon)$

$\epsilon = 1$

$-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

Quiz: Which of the following are open subsets of $\mathbb{R}^2$?

1. $\{(m, n) \mid m, n \in \mathbb{Z}\}$
2. $\{(1/m, 1/n) \mid m, n \in \mathbb{R}\}$

any (x,y) generalization

except $(0,0)$

Examples: Interior

Quiz: Find the interior of the following sets as a subset of $\mathbb{R}$.

1. $\{x \mid -1 \leq x \leq 1\}$ $(-1,1)$
2. $\{1/n \mid n \in \mathbb{N}\}$ $\emptyset$

Examples: Interior

Quiz: Find the interior of the following sets as a subset of $\mathbb{R}$.

1. $\{x \mid -1 \leq x \leq 1\}$
2. $\{1/n \mid n \in \mathbb{N}\}$

Quiz: Find the closure of the following sets as a subset of $\mathbb{R}$.

1. $\{x \mid 1 < x < 2\}$
2. $\{1, 2, 3\}$

$[1, 2]$

$\{1,2,3\} \cup$

pts in this set s.t $\exists$ any B_ϵ s.t

$1 \in B_\epsilon(1)$

$B_\epsilon(1) \cap \{1,2,3\} = \{1\}$

$S \cup Bry$

$B_\epsilon(1) \cap \{1,2,3\}^c$

$B_\epsilon(1) \cap \{1,2,3\} \neq \emptyset$

$B_\epsilon \cap \{1,2,3\}^c \neq \emptyset$

& $B_\epsilon \cap \{1,2,3\} \neq \emptyset$

1 2 3