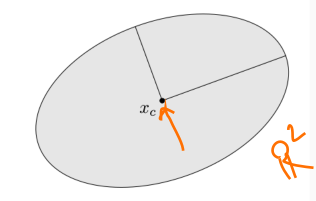
Example of convex set: Ellipsoid

Ellipsoid: $\{x \mid (x - x_c)^T P (x - x_c) \le 1\}$

- P is symmetric positive definite



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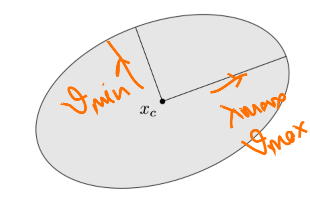

$\mathbb{R}^2$

$P \in \mathbb{R}^{2\times 2}$

λ_{max}

λ_{min}

- Matrix P determines how far the ellipsoid extends in every direction from center x_c

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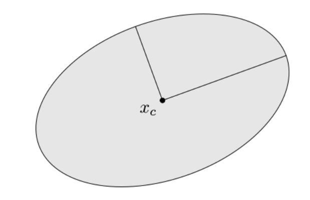

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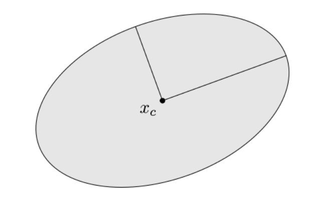

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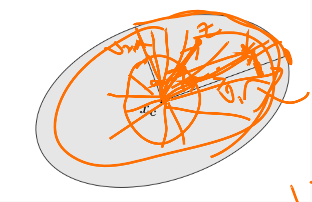
$\|x\|_P$

$x^T P x - x_c^T P x + x_c^T P x_c$

- P is symmetric positive definite



$x - x_c \neq 0$

$z = \alpha v_1 + \beta v_2$

$\alpha \geq \beta$

$(x - x_c)^T P (x - x_c) \geq 0$

We have a basis

- Matrix P determines how far the ellipsoid extends in every direction from center x_c
- The length of semi-axes of ellipsoid are given by $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P
- A ball is an ellipsoid with $P = r^2 I$
- Another representation of Ellipsoid

$$\{x_c + Au \mid \|u\|_2 \leq 1\}, \quad A \text{ is } SPD$$

P is sym. $\Rightarrow$ P has n-lin. ind. eig. vec.

Example of Convex Set: Norm Cone

The norm cone associated with a norm $\|\cdot\|$ is the set

$$C = \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$$

$\in \mathbb{R}^{n+1}$

$x \in \mathbb{R}^n, \; t \in \mathbb{R}$

$(x, t) \in \mathbb{R}^{n+1}$

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$n \quad n+1$

Ex This is convex

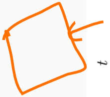
(n, t)

$n : 2$

C in $\mathbb{R}^3$

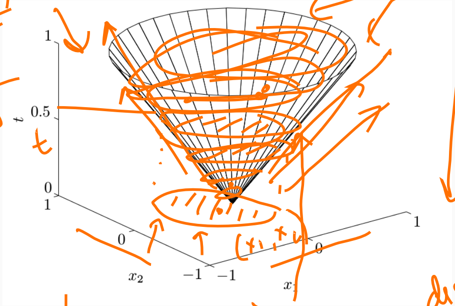
(x, t)

(y, t)

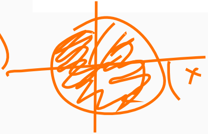


Norm Ball

$B_2 = \{x \mid \|x\|_2 \le 1\}$



(x_1, x_2)



(x_1, x_2, t)

(x_1, x_2, t)

x

$\sqrt{x_1^2 + x_2^2} \le t$

$\boxed{x_1^2 + x_2^2 \le t^2}$ $\le t^2$

disc / ~~Circle~~ with center $(0,0)$

radius $= t$

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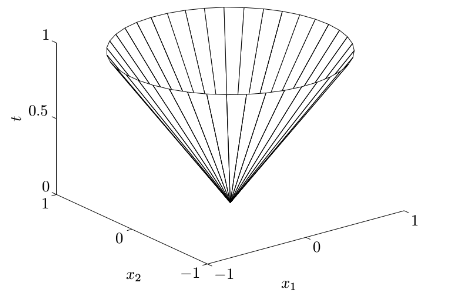

- Second order cone: Norm cone with $\|\cdot\|_2$ norm

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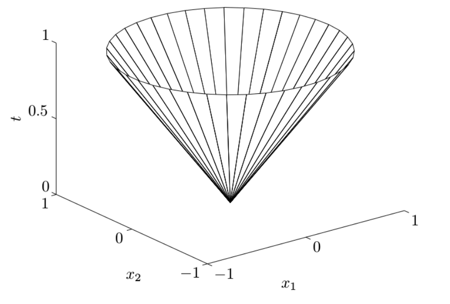

- Second order cone: Norm cone with $\|\cdot\|_2$ norm
- Other names of second order cone: quadratic cone, Lorentz-cone, ice-cream cone

Example of convex sets: Polyhedron

Polyhedron is defines as a set of finite number of equalities and inequalities:

$$P = \{x \mid \quad a_j^T x \leq b_j,\, j = 1, \ldots, m, \quad c^T x = d_j,\, j = 1, 2, \ldots, p\}$$

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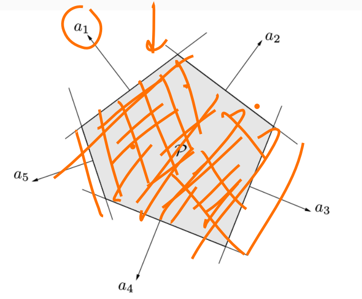

$$\left.\begin{array}{l} a_1^T x \leq b_1 \\ a_2^T x \leq b_2 \\ \vdots \\ a_5^T x \leq b_5 \end{array}\right\}$$

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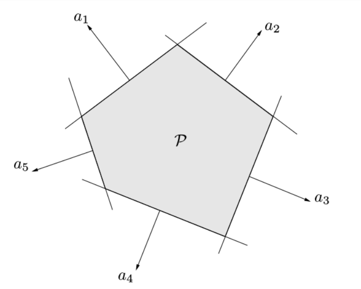

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$a^T x \leq b$

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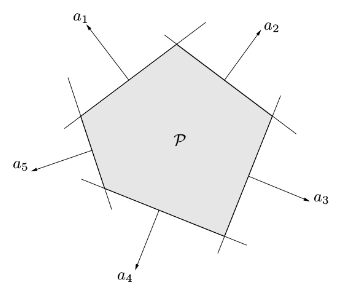


Quiz Which of the following are polyhedron?

- hyperplanes, lines, half-spaces

Example of convex sets: Polyhedron

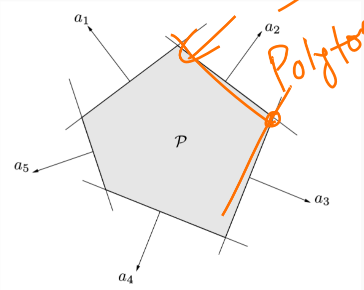
$$\underbrace{\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}}_{A} [x] \le \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

Polyhedron is defines as a set of finite number of equalities and inequalities:

Can't be circle

$$P = \{x \mid a_j^T x \le b_j,\ j = 1, \ldots, m,\quad \cancel{c_j^T x = d_j},\ j = 1, 2, \ldots, p\}$$

- Intersection of finite number of halfspaces and hyperplanes



Polytope

Quiz Which of the following are polyhedron?

- hyperplanes, lines, half-spaces

A bounded polyhedron is usually called polytope

Example of Convex Sets: Polyhedron

A polyhedron can be more compactly represented as

$$P = \{x \mid Ax \leq b, Cx = d\},$$

where

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_p^T \end{bmatrix}$$

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$\boxed{x_1 \geq 0}$

$x_2 \geq 0$

Example: The nonnegative orthant is the set of points with nonnegative components

$\vdots$

$x_n \geq 0$

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, 2, \ldots, n\} = \{x \in \mathbb{R}^n \mid x \geq 0\}$$

Example of Convex set: Simplexes

Simplexes are another family of polyhedra.

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The simplex determined by $v_0, v_2, \dots, v_k$ is

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix}$$

$$C = conv\{v_0, v_1, \dots, v_k\} = \{\theta_0 v_0 + \theta_1 v_1 + \cdots + \theta_k v_k \mid \theta \succeq 0,\ \mathbf{1}^T\theta = 1\}$$

- Here $\mathbf{1}$ is the vector of all ones

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k+1

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simplex → polyhedron

Polyhedron

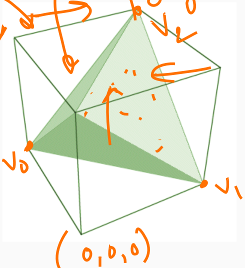

Figure 1: Tetrahedron

- Here 1 is the vector of all ones
- The affine dimension of this simplex is k

$\{v_1 - v_0,\ v_2 - v_0\}$ L.I

Other Examples of Simplexes

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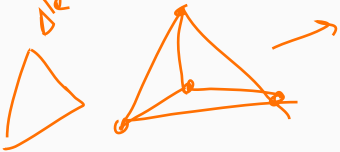
Other Examples of Simplexes

$k \to k+1$



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2. 2D simplex = triangle: This simplex can be generated by considering affinely independent points x_1, x_2, and x_3
3. 3D simplex = tetrahedron: This simplex is generated by considering affinely independent points x_1, x_2, x_3, and x_4

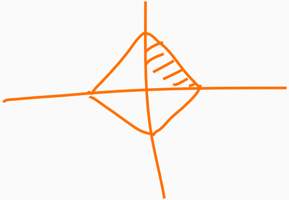
1. Unit Simplex: $n-$dimensional simplex determined by zero vector and the unit vectors, i.e., $0, e_1, \ldots, e_n \in \mathbb{R}^n$. It can be expressed as

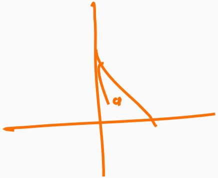
$0, e_1, e_2$

$$x \geq 0, \quad 1^T x \leq 1$$

$\mathbb{R}^2$

$\|x\|_1 \leq 1$





Other simplexes

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$$x \geq 0, \quad 1^T x \leq 1$$

$x_1 + x_2 = 1$

$x_1, x_2 \geq 0$

2. Probability Simplex: $n-1$ dimensional simplex determined by the unit vectors $e_1, \dots, e_n \in \mathbb{R}^n$. It is the set of vectors that satisfy

$$x \geq 0, \quad 1^T x = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \sum_{i=1}^{n} x_i = 1$$

- Vectors in probability simplex correspond to probability distributions on a set with n elements
- x_i intreprated as probability of the ith element

Cone: A set C is called a cone if for every $x \in C$, and $\theta \geq 0$, we have $\theta x \in C$.

$C = \{\alpha x_1\} \cup \{\beta x_2\}$ is a cone.

θx

Let $y \in C$

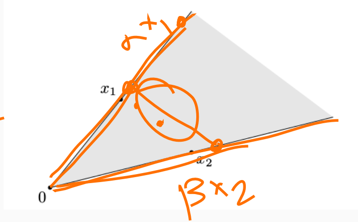
$\Rightarrow y \in \{\alpha x_1\}$

$y = \beta x_1 \in C$

~~$1'\theta = 1$~~

Cone is not necessarily convex.

Figure 2: Cone



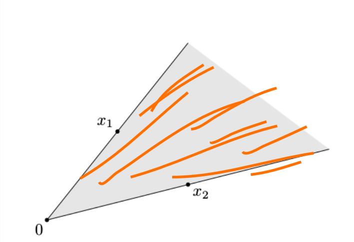
Cones, Convex Cones

Cone: A set C is called a cone if for every $x \in C$, and $\theta \geq 0$, we have $\theta x \in C$.

Convex Cone: A set C is called convex cone if it is convex and a cone, i.e.,

$$\theta_1 x_1 + \theta_2 x_2 \in C, \quad x_1, x_2 \in C, \theta_1, \theta_2 \geq 0$$

Figure 2: Cone



Conic Combination and Conic Hull

$\cancel{1^T\theta = 1}$

Conic Combination: A point of the form $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$ with $\theta_1, \theta_2, \ldots, \theta_k \geq 0$ is called the conic combination of $x_1, x_2, \ldots, x_k$

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Conic Hull: The conic hull of a set C is the set of all conic combinations of points in C, i.e.,

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_k \in C, \quad \theta_i \geq 0, i = 1, \ldots, k\}$$

$C = \{x_1, x_2\}$

$C\{\theta_1 x_1 + \theta_2 x_2\}$ $\theta_1, \theta_2 \geq 0$

$= \{\theta_1 x_1\} \cup \{\theta_2 x_2\}$

Q: is Conic Hull(C) Cone

smallest cone containing C.

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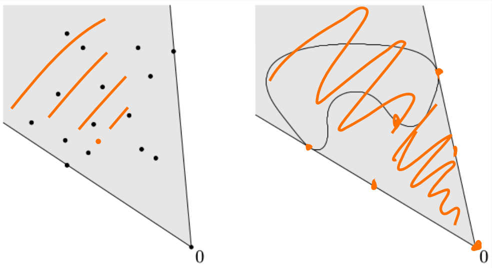

Positive Semi-definite Cone

Let S^n denote set of symmetric $n \times n$ matrices

$$S^n = \{X \in \mathbb{R}^n \mid X = X^T\}$$