Proof that S_+^n is a convex cone

(Recall) Convex Cone: A set C is called convex cone if it is convex **and** a cone, i.e.,

$$\theta_1 x_1 + \theta_2 x_2 \in C, \quad \text{for any } x_1, x_2 \in C, \quad \text{for any } \theta_1, \theta_2 \geq 0$$

Proof that S^n_+ is a convex cone

$S^n_+ = \{X \mid X$ is sym & semi pos def$\}$

(Recall) Convex Cone: A set C is called convex cone if it is convex **and** a cone, i.e.,

$$\theta_1 x_1 + \theta_2 x_2 \in C, \quad \text{for any } x_1, x_2 \in C, \quad \text{for any } \theta_1, \theta_2 \geq 0$$

$c x_1 \in C$, $c x_2 \in C$, $c \geq 0$

cone

Quiz: Prove that S^n_+ is a convex cone

Let $A, B \in S^n_+$

$x^T(\theta_1 A + \theta_2 B)x$

$= \underbrace{\theta_1}_{\geq 0} \underbrace{x^T A x}_{\geq 0} + \underbrace{\theta_2}_{\geq 0} \underbrace{x^T B x}_{\geq 0}$

≥ 0

$\Rightarrow \theta_1 A + \theta_2 B \in S^n_+$

$A + B$

$\mathbb{R}^{n \times n}$

$\mathbb{R}^n$

x^T

$n \times 1$

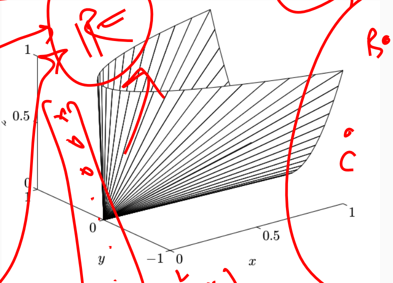

Figure 3: Semidefinite Cone

Scratch Space

$A \in \mathbb{R}^{2\times 2}$

$x^T A x \geq 0 \Rightarrow$ eig. val $(A) \geq 0$

Let $\underbrace{v_1, v_2}$ be the orthogon eig. vec of $\underline{\underline{A}}$

$\Rightarrow A v_1 = \lambda_1 v_1$ & $A v_2 = \lambda_2 v_2$

u_1

$u^T v = 0$

$u^T u = 1$

Clai: $\lambda_1 \geq 0$, $\underline{\underline{\lambda_2}} \geq 0$

$\underbrace{v_1^T A v_1} = \lambda_1 v_1^T v_1 = \lambda_1$

$\underline{\underline{\lambda_1 \geq 0}}$

Sin A is +ve def. $\Rightarrow v_1^T A v_1 \geq 0 \Rightarrow$

$v_1 \neq 0$ becau v_1 is an eig. vec.

Scratch Space

Scratch Space

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Quiz: Prove that S_+^n is a convex cone

Answer: Let $A, B \in S_+^n$. Let $\theta_1, \theta_2 \geq 0$.

Claim: $\theta_1 A + \theta_2 B$ is **SPD**

Proof on chalkboard...

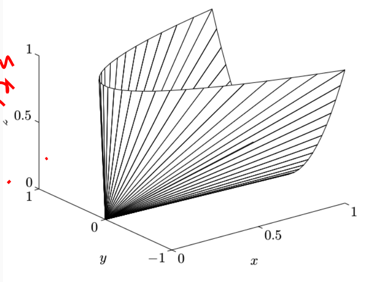
Example of V.S

$$P_n(x) = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R} \}$$

$u(x) = a_0 + a_1 x + \cdots + a_n x^n$

$v(x) = b_0 + b_1 x + \cdots + b_n x^n$

$(a_0 + b_0) + (a_1 + b_1) \cdots$



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Proof that S_+^n is a convex cone

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x — 0:1 z 0:1
1000
y — 0:1
1000

Quiz: Prove that S_+^n is a convex cone

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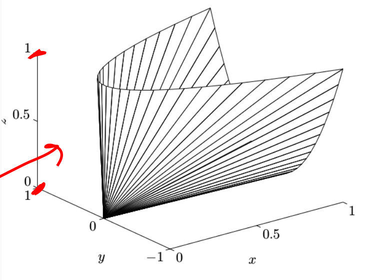


Example: Positive definite cone in $\mathbb{R}^2$

semi-pos. def

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2$$

$$\iff x \geq 0,\ z \geq 0,\ xz \geq y^2 \text{ (Why?)}$$

- What are positive definite cones in S_+^1, S_+^3?

Scratch Space

S^n_+

$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$

Let λ_1, λ_2 be two eig. values of X

For X to be Positive definite

$\text{eig}(X) \geq 0$

$\Rightarrow \det(X) = \underbrace{\lambda_1}_{\geq 0} \cdot \underbrace{\lambda_2}_{\geq 0} \geq 0$

$\Rightarrow \underbrace{xz - y^2}_{} \geq 0$

① $\Rightarrow xz \geq y^2$ ①

② $\underbrace{x \geq 0}_{}$, $\underbrace{z \geq 0}_{}$

On contrary $\boxed{x < 0}$ $\times$

$\Rightarrow x \geq 0$

$[1, 0]$

$[0, 1]$

$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= x < 0$

$z \leq 0$

$\Rightarrow X$ is not P.D a contr.

- $x^T A x \geq 0$ semi P.D $\forall x \neq 0$
- $x^T A x > 0$ P.D $\forall x \neq 0$

$\theta \geq 1$

Fact-1: If S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex

$\boxed{0 \leq \theta \leq 1}$

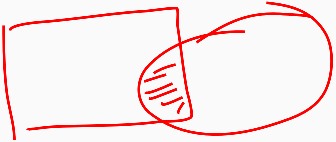
Proof: Let $x_1, x_2 \in S_1 \cap S_2$. Also, let $\theta \in \mathbb{R}$.

S_1 cvx, S_2 cvx

$$x_1, x_2 \in S_1 \Rightarrow \theta x_1 + (1-\theta) x_2 \in S_1$$
$$x_1, x_2 \in S_2 \Rightarrow \theta x_1 + (1-\theta) x_2 \in S_2$$

$$\Rightarrow \theta x_1 + (1-\theta) x_2 \in S_1 \cap S_2$$

$$\Rightarrow S_1 \cap S_2 \text{ is cvx}$$



Operations that preserve convexity: Intersection

Fact-1: If S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex

Proof: Let $x_1, x_2 \in S_1 \cap S_2$. Also, let $\theta \in \mathbb{R}$.
We have $x_1, x_2 \in S, x_1, x_2 \in S_2$. Since S_1 and S_2 are both convex, $\theta x_1 + (1 - \theta x_2)$ is in **both** S_1 and S_2.

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$(\underbrace{S_1 \cap S_2}) \cap S_3$

$\underline{C} \cap \underline{S_3}$

cvx

More generally,

Fact-2: If S_α is convex for every $\alpha \in \mathcal{A}$, then $\cap_{\alpha \in \mathcal{A}} S_\alpha$ is convex

Proof: By induction.

Example of Intersection of Convex Sets

Fact-1: Intersection of subspace, convex cones are also closed under intersection

C is cvx cone if C is cone & it is cvx.

C is cone if $\forall\, x \in C$, $\theta x \in C$, when $\theta \geq 0$

$\theta_1 x_1 + \theta_2 x_2 \in C$ for any $x_1, x_2 \in C$, $\theta_1, \theta_2 \geq 0$

Let C_1 & C_2 be two cones.

$x_1, x_2 \in C_1 \cap C_2 \Rightarrow x_1, x_2 \in C_1$

& $x_1, x_2 \in C_2$

$\Rightarrow \theta_1 x_1 + \theta_2 x_2 \in C_1$ & $\theta_1 x_1 + \theta_2 x_2 \in C_2$

$\Rightarrow \theta_1 x_1 + \theta_2 x_2 \in C_1 \cap C_2$

Example of Intersection of Convex Sets

Fact-1: Intersection of subspace, convex cones are also closed under intersection

Fact-2: The positive semi-definite cone is convex

$S_+^n = \{$ all +ve semi-def $n \times n$ matrices $\}$

$A, B \in S_+^n$

$\underset{\geq 0}{\theta} A + \underset{\geq 0}{(1-\theta)} B \in S_+^n$

$\boxed{0 \leq \theta \leq 1}$

Example of Intersection of Convex Sets

Fact-1: Intersection of subspace, convex cones are also closed under intersection

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Proof of Fact-2: The set S_+^n can be expressed as

$$\bigcap_{z \neq 0} \left\{ X \in S^n \mid z^T X z \geq 0 \right\} \tag{1}$$

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Since linear functions are convex, using Fact-1, the arbitrary intersections of linear functions in (1) is convex. Hence proved.

*Example of Intersection of Convex Sets

*Example: Consider the set

$$S = \{x \in \mathbb{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\},$$

where $p(t) = \sum_{k=1}^{m} x_k \cos kt$.

Quiz: Is this set convex?

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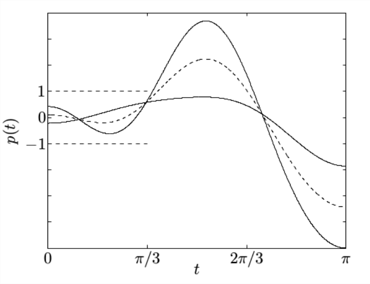

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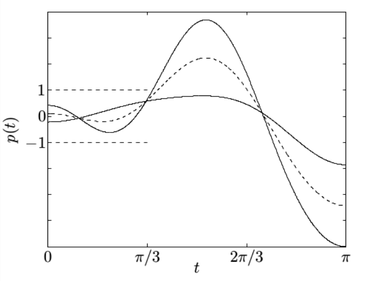

Proof: Can we write this set S as an intersection of convex sets?

$$S_t = \bigcap_{|t| \leq \pi/3} S_t,$$

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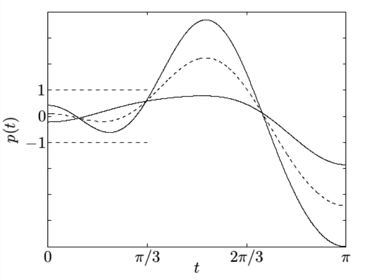

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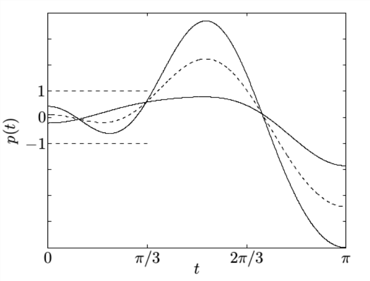

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Hint: Write $p(t)$ as $a^T y$? What would be a and y?

Fact: Every closed convex set is an intersection of halfspaces

Convexity is Preserved Under Affine Map

Affine Function: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **affine** if it is a sum of a linear function and a constant, i.e., it is of the form

$$f(x) = Ax + b, \quad A \in \mathbb{R}^{m\times n}, \quad b \in \mathbb{R}^m.$$

Convexity is Preserved Under Affine Map

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$$f(x) = Ax + b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m.$$

Fact: If $S \subseteq \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function. Then the image of S under f is

$$f(S) = \{f(x) \mid x \in S\}$$

is convex.

Similarly, if $f : \mathbb{R}^k \to \mathbb{R}^n$ is an affine function, then the inverse image of S under f,

$$f^{-1}S = \{x \mid f(x) \in S\}$$

is convex.

$S \to f(S)$

f^{-1}

$f^{-1}S$ S

cvx cvx

- This fact allows us to prove that variety of sets are convex!

f affine $\Rightarrow$ $f(x) = Ax + b$

$\to A(\theta s_1 + (1-\theta)s_2) + b$

$\to f(\theta s_1 + (1-\theta)s_2)$

$y = \boxed{\theta s_1 + (1-\theta)s_2}$

Since S is cvx &

$s_1, s_2 \in S$

S convex, f affine $\Rightarrow$ $f(S)$ convex

$f(S) = \{ f(s) \mid s \in S \}$

$\Rightarrow y \in S$. $\Rightarrow f(y) \in f(S)$

Claim $f(S)$ is convex

Let $x_1, x_2 \in f(S)$ $\Rightarrow$ $\exists$ $s_1, s_2 \in S$ s.t. $f(s_1) = x_1$

and $f(s_2) = x_2$

$\theta_1 + \theta_2 = 1$

$f(u+v) = f(u) + f(v)$

Claim $\boxed{\theta x_1 + (1-\theta)x_2} \in f(S)$

f - affine

$\theta f(s_1) + (1-\theta) f(s_2) = f(\theta s_1) + f((1-\theta)s_2)$

$= f(\theta s_1 + (1-\theta)s_2)$ $f(y)$

$\in S$

$\theta(As_1 + b) + (1-\theta)(As_2 + b)$

$= \theta A s_1 + (1-\theta) A s_2$

Scratch Space

$Ax+b$

f affine $\Rightarrow f^{-1}$ affine

If S convex $\Rightarrow f^{-1}(S)$ is cvx Given that f is affine.

Let $x_1, x_2 \overset{\text{are}}{\in} f^{-1}(S)$ we have to prove that $\theta x_1 + (1-\theta)x_2 \in f^{-1}(S)$

$0 \leq \theta \leq 1$

$f^{-1}(S)$

x_1, x_2

S

s_1, s_2

For $\theta x_1 + (1-\theta)x_2 \in f^{-1}(S)$

I must find $\boxed{z} \in S$ s.t

$f^{-1}(z) = \theta x_1 + (1-\theta)x_2$

We have $s_1, s_2 \in S$ s.t $x_1 = f^{-1}(s_1)$ & $x_2 = f^{-1}(s_2)$

$$\theta x_1 + (1-\theta)x_2 = \theta f^{-1}(s_1) + (1-\theta) f^{-1}(s_2)$$

$$= f^{-1}(\underbrace{\theta s_1 + (1-\theta)s_2}_{z \,\in\, S}) \in f^{-1}(S)$$

Examples: Convexity Preserved Under Affine Map

(Scaling) Let $S \subseteq \mathbb{R}^n$ be convex, $\alpha \in \mathbb{R}$. Is the following set convex?

$$\alpha S = \{\alpha x \mid x \in S\}$$

S $\xrightarrow{\text{affine}}$ αS

Quiz: Can you guess an affine map $f : S \to \alpha S$? ✓

$f(x) = \alpha x$

Examples: Convexity Preserved Under Affine Map

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(Translation) Let $S \subseteq \mathbb{R}^n$ be convex, $\alpha \in \mathbb{R}$. Is the following set convex?

$$S + a = \{x + a \mid x \in S\}$$

$$S \xrightarrow{f} S + a$$

$$f(x) = x + a$$

Quiz: Can you guess an affine function $f : S \to S + a$?

Examples: Convexity Preserved Under Affine Map

(Scaling) Let $S \subseteq \mathbb{R}^n$ be convex, $\alpha \in \mathbb{R}$. Is the following set convex?

$$\alpha S = \{\alpha x \mid x \in S\}$$

Quiz: Can you guess an affine map $f : S \to \alpha S$?

$f : \mathbb{R}^n \to \mathbb{R}^n$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ \vdots \\ x_n \end{bmatrix}\right) \to \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(Translation) Let $S \subseteq \mathbb{R}^n$ be convex, $\alpha \in \mathbb{R}$. Is the following set convex?

$$S + a = \{x + a \mid x \in S\}$$

Quiz: Can you guess an affine function $f : S \to S + a$?

$f(x) = Ax + 0$

$f(x) = Ax + b$

(Projection) Consider the projection of a convex set onto some of its coordinates

$$T = \{x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n\}.$$

Is T convex?

Quiz: Can you guess an affine map $f : S \to T$

$S \xrightarrow{\text{Proj} \equiv \text{affine}} f(S)$

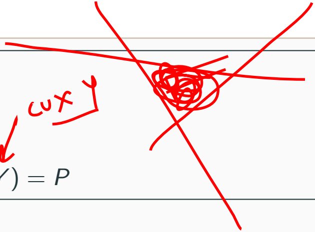
$$A = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$$

*Examples: Convexity Preserved Under Affine Maps

Polyhedron: The polyhedron

$$P = \{x \mid Ax \leq b, Cx = d\}$$

cvx



Quiz: Can you guess an affine function f and a set Y such that $f^{-1}(Y) = P$

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Polyhedron: The polyhedron

$$P = \{x \mid Ax \le b, Cx = d\}$$

Quiz: Can you guess an affine function f and a set Y such that $f^{-1}(Y) = P$

Answer: Consider the set

$$S = \{(x, 0) \mid x \ge 0\}, \quad \text{non-negative orthant}$$

*Examples: Convexity Preserved Under Affine Maps

Polyhedron: The polyhedron

$$P = \{x \mid Ax \le b, Cx = d\} = \{x \mid b - Ax \ge 0, \; Cx = d\}$$

Quiz: Can you guess an affine function f and a set Y such that $f^{-1}(Y) = P$

Answer: Consider the set

$$\cancel{S}\; Y = \{(x, 0) \mid x \ge 0\}, \quad \text{non-negative orthant} \quad = \{x \mid b - Ax \ge 0, \; d - Cx -$$

$$\boxed{f : X \to Y}$$

Consider the affine function

$$f(x) = (b - Ax, d - Cx) \in Y \implies b - Ax \ge 0 \;\&\; d - Cx = 0$$

Check

$$f^{-1}(\cancel{S}\,Y) = P \;?$$

$$f^{-1}(Y) = \{x \in X \mid f(x) \in Y\}$$

$$= \{x \in X \mid b - Ax \ge 0 \;\&\; d - Cx = 0\}$$

*Examples: Convexity Preserved Under Affine Maps

Polyhedron: The polyhedron

$$P = \{x \mid Ax \leq b, Cx = d\}$$

Quiz: Can you guess an affine function f and a set Y such that $f^{-1}(Y) = P$

Answer: Consider the set

$$S = \{(x, 0) \mid x \geq 0\}, \quad \text{non-negative orthant}$$

Consider the affine function

$$f(x) = (b - Ax, d - Cx)$$

Check that

$$f^{-1}(S) = P$$

Note: Invese image of a convex set under affine map is convex!

$10 + \frac{1}{2}$ $\quad 1, 2, \ldots$

$S = \{ 10 + \frac{1}{n} \mid n \in \mathbb{N} \}$

$\{ 11, 10 + \frac{1}{2}, 10 + \frac{1}{3}, \ldots \}$

$\sup(S) = 11$

$\inf(S) = 10$

Supremum: Let S be a set of real numbers. If there is a real number b such that $x \leq b$ for every $x \in S$, then b is called an upper bound for S and we say that S is bounded above by b.

- We say an upper bound, because any number greater than b will be an upper bound
- If b is also an element of S, then it is called maximum element of S
- A set with no upper bound is called unbounded above
- Supremum of set: $sup(S)$ = least of all upper bounds

$10 + (-1)^3 \cdot \frac{1}{3}$

$10 - \frac{1}{3}$

$\{ 10 + (-1)^n \cdot \frac{1}{n} \} = \{ 10 + (-1)^1 \cdot \frac{1}{1}, 10 + (-1)^2 \cdot \frac{1}{2}, \ldots \}$

$9 = \inf$

$10.5 = \sup$

$\{ 9, 10 + \frac{1}{2}, \ldots 10 - \frac{1}{3}, 10 + \frac{1}{4}, 10 - \frac{1}{4} \}$

$\le b_2 \le b_1 \le$ $S := \{ s_1, \ldots \}$ inf. $b-1,\ b-2$

Infimum: Let S be a set of real numbers. If there is a real number b such that $x \geq b$ for every $x \in S$, then b is called a lower bound for S and we say that S is bounded below by b.

- We say an lower bound, because any number less than b will be an lower bound
- If b is also an element of S, then it is called minimum element of S
- A set with no lower bound is called unbounded below
- Infimum of set: $inf(S)$ = greatest of all lower bounds

$1, 2$

$S := (0, 1) = \{ x \mid 0 < x < 1 \}$

$0 = \inf$

$1 = \sup$

$-1,\ 0$