Topics in Applied Optimization

Optimization Algorithms for ML and Data Sciences

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Example: Log-Determinant is a concave function

Log-determinant: $f(X) = \log\det X, \quad X > 0$ is a concave function

$X \in \mathbb{R}^{n\times n}$

$X \in S^n_+ \leftarrow$ Symm. & pos. def.

Proof:

- Consider arbitrary line $X = Z + tV, \quad Z, V \in \mathbb{S}^n$ and positive definite

Recall

$f: \mathbb{R}^n \to \mathbb{R}$ is convex iff f restricted on a arbitrary line in $\mathbb{R}^n$ is convex.

Example: Log-Determinant is a concave function

$g(0) = f(Z)$

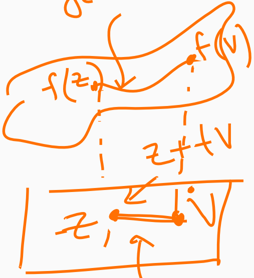
$g(t)$ $\quad g(1) = Z+V$

Log-determinant: $f(X) = \log\det X, \quad X > 0$ is a concave function

$\text{dom } f = S^n_+$

Proof:

- Consider arbitrary line $X = Z + tV, \quad Z, V \in \mathbb{S}^n$ and positive definite
- Consider $g(t) = f(Z + tV), \quad t \in [0, 1]$



Claim $Z + tV \in \text{dom } f$

$Z, V \in S^n_+$

$\Rightarrow Z + tV$ is also symm & positive definite

arb line joing two points

$$x^T (Z + tV) x = (x^T Z x) + t\,(x^T V x) > 0$$

$>0 \; + \; \geq 0 \quad \geq 0$

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Proof:

- Consider arbitrary line $X = Z + tV, \quad Z, V \in \mathbb{S}^n$ and positive definite
- Consider $g(t) = f(Z + tV), \quad t \in [0, 1]$
- We have

$$
\begin{aligned}
g(t) &= \log\det(Z + tV) \\
&= \log\det(Z^{1/2}(I + tZ^{-1/2}VZ^{-1/2})Z^{1/2}) \\
&= \sum_{i=1}^{n} \log(1 + t\lambda_i) + \log\det Z,
\end{aligned}
$$

Fact: If A is in S^n_+, then $A^{1/2}$ exists

analogy

Real: If $x > 0$, then $\sqrt{x}$ exist in real

Fact: $\det(AB) = \det(A)\det(B)$

$\log[\det(Z^{1/2}) \det(I + tZ^{-1/2}VZ^{-1/2}) \det(Z^{1/2})]$

$= \log[\det(I + tZ^{-1/2}VZ^{-1/2}) \det(Z)]$

$= \log[\det(I + tZ^{-1/2}VZ^{-1/2})] + \log(\det(Z)) \Rightarrow$

Let λ_i be eig. val of

Example: Log-Determinant is a concave function

Log-determinant: $f(X) = \log\det X, \quad X \succ 0$ is a concave function

Proof:

- Consider arbitrary line $X = Z + tV, \quad Z, V \in \mathbb{S}^n$ and positive definite
- Consider $g(t) = f(Z + tV), \quad t \in [0, 1]$
- We have

f restriction on line

$$\log(\det(I + tZ^{-1/2}VZ^{-1/2}))$$

$$= \log\left(\prod_{i=1}^{n}(1+t\lambda_i)\right)$$

$$= \sum_{i=1}^{n}(1+t\lambda_i)$$

$$\begin{aligned} g(t) &= \log\det(Z + tV) \\ &= \log\det(Z^{1/2}(I + tZ^{-1/2}VZ^{-1/2})Z^{1/2}) \\ &= \sum_{i=1}^{n}\log(1 + t\lambda_i) + \log\det Z, \end{aligned}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $Z^{-1/2}VZ^{-1/2}$. We have

$\Rightarrow$ eig. val of $1 + t\lambda_i$

Example: Log-Determinant is a concave function

Log-determinant: $f(X) = \log\det X, \quad X > 0$ is a concave function

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where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $Z^{-1/2}VZ^{-1/2}$. We have ≥ 0

$$
g'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i}, \qquad g''(t) = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + t\lambda_i)^2}
$$

≥ 0

≥ 0

≤ 0

Want to use 2nd derivative test for $g(t)$

Concave: f is concave if $f'' \leq 0$

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$$g'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i}, \quad g'' = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + t\lambda_i)^2}$$

- Hence $g''(t) < 0$, and f is concave!

Epigraph (means above the graph)

Graph: Graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\{(x, f(x)) \mid x \in \textbf{dom } f\}$$

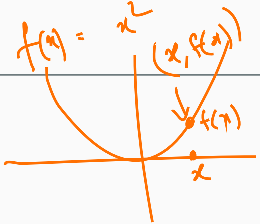

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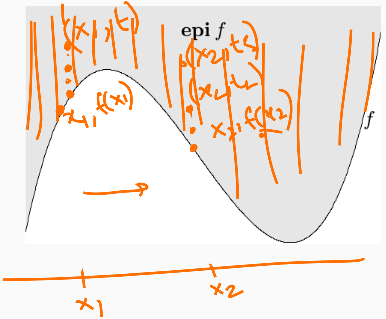
Figure 1: Epigraph



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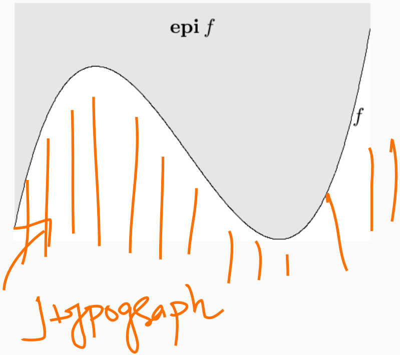

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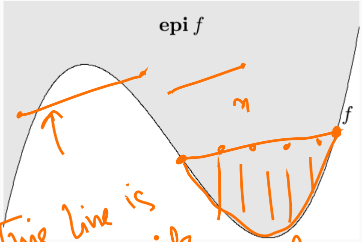

This line is not inside epi f

Fact: A function is convex if and only if its epigraph is convex

convexity of a function $\iff$ convexity of a set

$\Rightarrow$ f is not cvx fn

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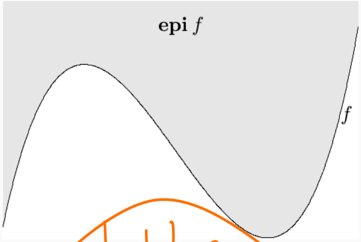

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Fact: A function is convex if and only if its epigraph is convex

Fact: A function is concave if and only if its hypograph is convex

Scratch Space

Scratch Space

Jensen's inequality

Jensen inequality: Given $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y), \quad x, y \in \mathbb{R}^n, \theta \in \mathbb{R}$$

It is extended to convex combination as follows

$$f(\theta x_1 + \cdots + \theta_k x_k) \leq \theta_1 f(x_1) + \cdots + \theta_k f(x_k),$$

$$x_1, x_2, \ldots, x_k \in \textbf{dom } f, \quad \theta_1, \theta_2, \ldots, \theta_k \geq 0, \quad \theta_1 + \cdots + \theta_k = 1$$

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Jensen inequality for Integrals: If $p(x) \geq 0$ on $S \subseteq \mathbf{dom}\, f$, $\int_S p(x)dx = 1$,

$$f\left(\int_S p(x)x\, dx\right) \leq \int_S f(x)p(x)\, dx$$

Jensen's inequality

- If $p(x)$ is the P.D.F of cont. r.v. X, then

$$\boxed{E[X] = \int p(x)\, x\, dx}$$

$$E[f(x)] = \int_S f(x) \cdot p(x)\, dx$$

Jensen inequality: Given $f : \mathbb{R}^n \to \mathbb{R}$,

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It is extended to convex combination as follows

$$f(\theta_1 x_1 + \cdots + \theta_k x_k) \le \theta_1 f(x_1) + \cdots + \theta_k f(x_k),$$

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$$f\left(\int_S p(x) x\, dx\right) \le \int_S f(x) p(x)\, dx$$

$$f(E[x]) \le E[f(x)]$$

Jensen inequality for expected values: If x is a random variable, $x \in \textbf{dom}\, f$, and f is convex, then

$$f(Ex) \le E\, f(x)$$

History of Jensen's inequality ...

Jensen's Original Inequality:

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \quad \rightarrow (*)$$

If a function is continuous, then Jensen's original inequality is necessary and sufficient condition for convexity. Why? Paper: *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, 1906

Figure 2: Jensen (1906)

f continuous & f satisfies Jensen's ineq $\Leftrightarrow$ f is convex

f is convex $\Rightarrow$ $f(\theta x+(1-\theta)y) \le \theta f(x)+(1-\theta)f(y)$

choose $\theta = \frac{1}{2}$ $\Rightarrow$ $(*)$

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176 J. L. W. V. Jensen.

J'introduirai la définition suivante. Lorsqu'une fonction $\varphi(x)$, réelle, finie et uniforme, de la variable réelle x, satisfait dans un certain intervalle à l'inégalité

$$\varphi(x)+\varphi(y) \geqq 2\varphi\left(\frac{x+y}{2}\right), \tag{1}$$

on dit que $\varphi(x)$ est une fonction convexe dans cet intervalle.

Si au contraire $\varphi(x)$ satisfait à l'inégalité

$$\varphi(x)+\varphi(y) \leqq 2\varphi\left(\frac{x+y}{2}\right), \tag{2}$$

on dit que $\varphi(x)$ est une fonction concave.

Figure 2: Jensen (1906)

How Jensen got motivated?

1. *Des fonctions convexes et concaves. Définition. Exemples.*

Dans sa célèbre Analyse algébrique (note II, pp. 457—59) Cauchy démontre que »la moyenne géométrique entre plusieurs nombres est toujours inférieure à leur moyenne algébrique». La méthode employée par Cauchy est extrêmement élégante, et elle à passé sans changement dans tous les traités d'analyse algébrique. Elle consiste, comme on sait, en ceci, que, de l'inégalité

$$\sqrt{ab} < \frac{1}{2}(a + b),$$

G.M ≤ A.M

où a et b sont des nombres positifs, on est conduit à l'inégalité analogue pour quatre nombres, savoir

$$\sqrt[4]{abcd} < \frac{1}{4}(a + b + c + d),$$

et aux suivantes, pour $8, 16, \ldots, 2^m$ nombres, après quoi ce nombre, par un artifice, est réduit à un nombre arbitraire inférieur, n. Cette méthode simple a été mon point de départ dans les recherches suivantes, qui conduisent, par une voie en réalité très simple et élémentaire, à des résultats généraux et non sans importance.

Quiz: **But how does this (AM-GM ineq.) relate to Jensen's inequality?**

Applications of Jensen's Inequality

AM-GM Inequality: Consider the AM-GM inequality

$$\sqrt{ab} \leq (a+b)/2, \quad a, b \geq 0$$

Choose $f(x) = -\log x$. This fn is convex.

$\Rightarrow f$ satisfies Jensen's ineq. with $\theta = 1/2$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(f(a) + f(b)\right)$$

$$\Rightarrow \cancel{-}\log\left(\frac{a+b}{2}\right) \geq \frac{-\log a - \log b}{2} = \cancel{-}\log(ab)^{1/2}$$

Taking exp. on both sides

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$\theta_1, \quad \theta_2$

$\theta_1 = \theta, \quad \theta_2 = 1-\theta$

$$f(\theta_1 a + \theta_2 b) \leq \theta_1 f(a) + \theta_2 f(b)$$

$$-\log(\theta a + (1-\theta)b) \leq -\theta\log a - (1-\theta)\log b$$

$$\log(\theta a + (1-\theta)b) \geq \theta\log a + (1-\theta)\log b = \log(a^{\theta} \cdot b^{1-\theta})$$

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Proof: To prove this using Jensen's inequality

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Proof: To prove this using Jensen's inequality (need suitable $f(x)$ and θ):

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- Choose $f(x) = -\log x$

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Applications of Jensen's Inequality

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Proof: To prove this using Jensen's inequality (need suitable $f(x)$ and θ):

- Choose $f(x) = -\log x$
- Choose $\theta = 1/2$

Holder's Inequality: Let $p > 1$, $1/p + 1/q = 1$, $x, y \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

Special case: Cauchy-Schwarz

$p = 2, \quad q = 2$

$$x^T y \le \sqrt{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}$$

Applications of Jensen's Inequality

AM-GM Inequality: Consider the AM-GM inequality

$$\sqrt{ab} \leq (a+b)/2, \quad a, b \geq 0$$

$a^{1/2} b^{1/2} \leq \frac{a+b}{2}$

If we choose general θ

$\Rightarrow a^{\theta} b^{1-\theta} \leq \theta a + (1-\theta) b$

Proof: To prove this using Jensen's inequality (need suitable $f(x)$ and θ):

- Choose $f(x) = -\log x$
- Choose $\theta = 1/2$

Holder's Inequality: Let $p > 1, 1/p + 1/q = 1, x, y \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^p \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}$$

Proof: To prove Holder's inequality:

- Choose $f(x) = -\log x$ which is convex
- For a general θ, we have

$$a^{\theta} b^{1-\theta} \leq \theta a + (1-\theta) b$$

Proof of Holder's Inequality...

$$a^{\theta} b^{1-\theta} = \left(\frac{|x_i|^p}{\sum |x_i|^p}\right)^{1/p} \left(\frac{|y_i|^q}{\sum |y_j|^q}\right)^{1/q} \le \frac{1}{p}\left(\frac{|x_i|^p}{\sum |x_i|^p}\right) + \left(\frac{p-1}{p}\right)\left(\frac{|y_i|^q}{\sum |y_i|^q}\right)$$

Then apply

$$\underbrace{a = \frac{|x_i|^p}{\sum_{j=1}^n |x_j|^p}}, \quad \underbrace{b = \frac{|y_i|^q}{\sum_{j=1}^n |y_j|^q}}, \quad \underbrace{\theta = 1/p},$$

$1 - 1/p$

We have

$$\underbrace{\left(\frac{|x_i|^p}{\sum_{j=1}^n |x_j|^p}\right)^{1/p} \left(\frac{|y_i|^q}{\sum_{j=1}^n |y_j|^q}\right)^{1/q}} \le \frac{|x_i|^p}{p \underbrace{\sum_{j=1}^n |x_j|^p}} + \frac{|y_i|^q}{q \sum_{j=1}^n |y_j|^q}$$

Summing the terms proves Holder's Inequality!

$$\sum \frac{x_i y_i}{\left(\sum |x_j|^p\right)^{1/p} \left(\sum |y_j|^q\right)^{1/q}} \le \frac{1}{p} \sum_i \cancel{\frac{|x_i|^p}{\sum_j |x_j|^p}}^{\,1} + \frac{1}{q} \sum_i \cancel{\frac{|y_i|^q}{\sum |y_i|^q}}^{\,1} = \frac{1}{p} + \frac{1}{q} = 1$$

Scratch Space

$$\Rightarrow \sum_i \frac{x_i y_i}{\left(\sum |x_j|^p\right)^{1/p} \left(\sum |y_j|^q\right)^{1/q}} \leq 1$$

$$\Rightarrow \sum x_i y_i \leq \left(\sum_j |x_j|^p\right)^{1/p} \left(\sum_j |y_j|^q\right)^{1/q}$$

$=$

Hölder's ineq.

Operations that Preserve Convexity

Non-Negative Weighted Sum: If $f_1, f_2, \ldots, f_m$ are convex, then their weighted sum

$$f = w_1 f_1 + \cdots + w_m f_m, \quad w_i \geq 0, \forall i$$

is convex. Note: Assume that $\mathbf{dom}\ f = \mathbf{dom}\ f_1 \cap \cdots \cap \mathbf{dom}\ f_m$

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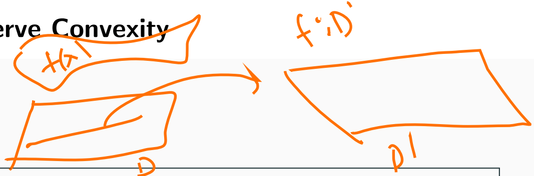

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Composition with Affine Map: Suppose $f : \mathbb{R}^n \to \mathbb{R}, A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by

$$g(x) = f(Ax + b),$$

$x \xrightarrow{Ax+b}$ $f(x)$

$f(Ax+b)$

with $\textbf{dom } g = \{x \mid Ax + b \in \textbf{dom } f\}$. Then if f is convex, then g is also convex.

Scratch Space

$\underline{g \text{ is convex}}$

$\theta g(x) + (1-\theta) g(y)$

$= \underline{\theta} f(\underbrace{Ax+b}_{u}) + (1-\theta) f(\underbrace{Ay+b}_{v})$

$\geq f(\theta u + (1-\theta) v)$

$= f(\theta(Ax+b) + (1-\theta)(Ay+b))$

$= f(\theta Ax + \cancel{\theta b} + (1-\theta)Ay + (1-\theta)b)$

$= f(A(\theta x + (1-\theta)y) + b) = g(x + (1-\theta)y)$

$\Rightarrow g(\underbrace{\theta x + (1-\theta)y}) = f(A(\theta x + (1-\theta)y) + b)$

$\Rightarrow g$ is cvx.

Scratch Space

Composition of Functions

Consider $h : \mathbb{R}^k \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f(x) = h(g(x)), \quad \textbf{dom } \{x \in \textbf{dom } g \mid g(x) \in \textbf{dom } h\}$$

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Consider $h : \mathbb{R}^k \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f(x) = h(g(x)), \quad \textbf{dom}\ \{x \in \textbf{dom}\ g \mid g(x) \in \textbf{dom}\ h\}$$

$g : \mathbb{R}^n \to \mathbb{R}$

$h : \mathbb{R} \to \mathbb{R}$

$f = h \circ g : \mathbb{R}^n \to \mathbb{R}$

- **Case-1: Scalar composition** Let $k = 1$, $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$
  - Set $n = 1$
  - Assume f, g are twice differentiable

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

$$f'(x) = \frac{d}{dg} h(g) \frac{dg}{dx} = \underbrace{h'(g(x))}_{1^{st}} \underbrace{g'(x)}_{2^{nd}}$$

$$f''(x) = h'(g(x))\, g''(x) + g'(x) \cdot h''(g(x)) \cdot g'(x)$$

Composition of Functions

Consider $h : \mathbb{R}^k \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f(x) = h(g(x)), \quad \textbf{dom}\ \{x \in \textbf{dom}\ g \mid g(x) \in \textbf{dom}\ h\}$$

- **Case-1: Scalar composition** Let $k = 1$, $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$
  - Set $n = 1$
  - Assume f, g are twice differentiable

$$f''(x) = \underbrace{\overset{\geq 0}{h''(g(x))}\,\overset{\geq 0}{g'(x)^2} + \overset{\geq 0}{h'(g(x))}\,\overset{\geq 0}{g''(x)}} \geq 0$$

We have:

- f is convex if h is convex and nondecreasing, and g is convex

$\Downarrow$ $\quad$ $\Downarrow$ $\quad$ $\Downarrow$

$h'' \geq 0$ $\quad$ $h' \geq 0$ $\quad$ $g'' \geq 0$

Composition of Functions

Consider $h : \mathbb{R}^k \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

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  - Assume f, g are twice differentiable

≥ 0

$\geq 0 \quad \geq 0 \quad \leq 0 \cdot \leq 0$

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x) \quad \geq 0$$

$\geq 0 \quad \geq 0$

We have:

- f is convex if h is convex and nondecreasing, and g is convex
- f is convex if h is convex and nonincreasing, and g is concave

$\Downarrow \quad \Downarrow \quad \Downarrow$

$h'' \geq 0 \quad h' \leq 0 \quad g'' \leq 0$

Composition of Functions

Consider $h : \mathbb{R}^k \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

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    - Set $n = 1$
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$\le 0 \quad \ge 0 \quad \ge 0 \quad \le 0$

$$f''(x) = \underbrace{h''(g(x))g'(x)^2}_{\le 0} + \underbrace{h'(g(x))g''(x)}_{\le 0} \qquad \le 0$$

We have:

- f is convex if h is convex and nondecreasing, and g is convex
- f is convex if h is convex and nonincreasing, and g is concave
- f is concave if h is concave and nondecreasing, and g is concave

$h'' \le 0 \qquad h' \ge 0 \qquad g'' \le 0$

Composition of Functions

Consider $h : \mathbb{R}^k \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

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Composition of Functions

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$$f(x) = h(g(x)), \quad \mathbf{dom}\ \{x \in \mathbf{dom}\ g \mid g(x) \in \mathbf{dom}\ h\}$$

$h : \mathbb{R} \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$h \circ g : \mathbb{R} \to \mathbb{R}$

- **Case-1: Scalar composition** Let $k = 1$, $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$
  - Set $n = 1$
  - Assume f, g are twice differentiable

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

$k = 1$, $n > 1$

$h : \mathbb{R} \to \mathbb{R}$

$g : \mathbb{R}^n \to \mathbb{R}$

$f = h \circ g : \mathbb{R}^n \to \mathbb{R}$.

We have:

  - f is convex if h is convex and nondecreasing, and g is convex
  - f is convex if h is convex and nonincreasing, and g is concave
  - f is concave if h is concave and nondecreasing, and g is concave
  - f is concave if h is concave and nonincreasing, and g is convex
- **Case-2:** $n > 1$. We have
  - f is convex if h is convex, $\tilde{h}$ is nondecreasing, and g is convex

Composition of Functions

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  - f is concave if h is concave and nondecreasing, and g is concave
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- **Case-2:** $n > 1$. We have
  - f is convex if h is convex, h is nondecreasing, and g is convex
  - f is convex if h is convex, h is nonincreasing, and g is concave

$f : [a, b] \to$

$f(-\infty, a) = \infty$

$f(b, \infty) = \infty$

Composition of Functions

Consider $h : \mathbb{R}^k \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

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Composition of Functions

Consider $h : \mathbb{R}^k \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^k$, and $f = h \circ g : \mathbb{R}^n \to \mathbb{R}$, defined by

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$\nabla^2 f \geq 0$

Boyd

- **Case-1: Scalar composition** Let $k = 1, h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$
  - Set $n = 1$
  - Assume f, g are twice differentiable

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

$h : \mathbb{R} \to \mathbb{R}$

$h'' \geq 0$

$h' > 0$

We have:

- f is convex if h is convex and nondecreasing, and g is convex
- f is convex if h is convex and nonincreasing, and g is concave
- f is concave if h is concave and nondecreasing, and g is concave
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  - f is concave if h is concave, $\tilde{h}$ is nondecreasing, and g is concave
  - f is concave if h is concave, $\tilde{h}$ is nonincreasing, and g is convex

$g : \mathbb{R}^n \to \mathbb{R}$

$\nabla^2 g \geq 0$

Note: $\tilde{h}$ is extended value function of h

Scratch Space

Conjugate Function

Conjugate Function: Let $f : \mathbb{R}^n \to \mathbb{R}$. The function $f^* : \mathbb{R}^n \to \mathbb{R}$ defined as

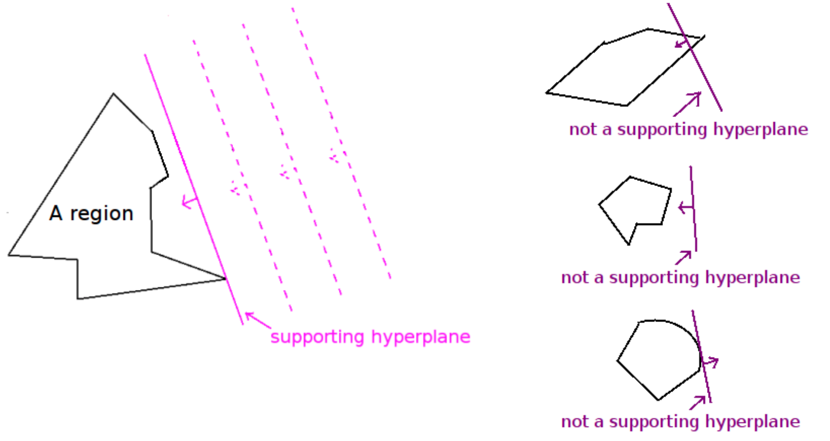
$$f^*(y) = \sup_{x \in \textbf{dom } f} (y^T x - f(x))$$

is called conjugate of the function f. The domain of the conjugate function consists of $y \in \mathbb{R}^n$ for which the sum is finite, i.e., the difference $y^T x - f(x)$ is bounded above.

- f^* is a convex function since it is a pointwise supremum of a family of convex functions
- f^* is convex regardless of whether f is convex or not

History and Geometric Intuition of Conjugates

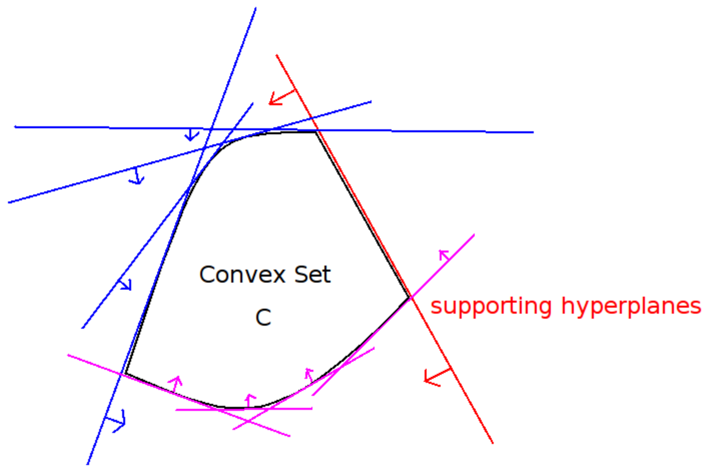
Recall supporting hyperplanes:



- Last one is not, because normal is pointing outwards

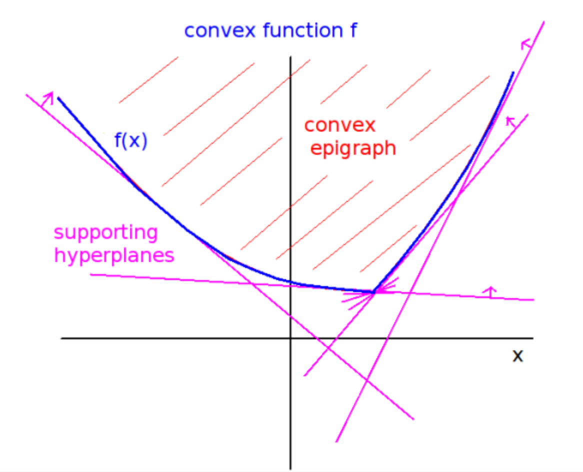
History and Geometric Intuition of Conjugates

A closed convex set can be represented by hyperplanes:



Note: At each point of a convex set, there is a unique supporting hyperplane

Supporting Hyperplanes for Convex Sets

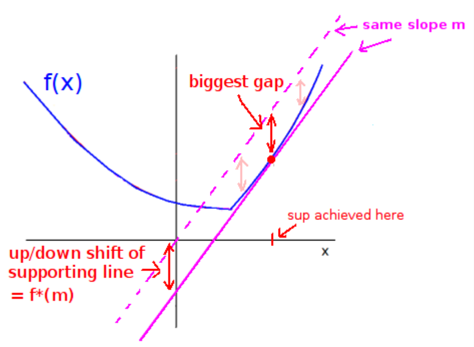


- A closed convex set is uniquely determined by lower hyperplanes

Geometric intuition of Fenchel/Legendre's Transform

In 1D, Fenchel/Legendre's tranform is:

$$f^*(m) = \sup_{x \in \mathbb{R}} (mx - f(x))$$



- Pick a plane with slope m and passing through origin
- Move the plane parallel to above plane until it becomes supporting hyperplane

Conjugates of Some Convex Functions on $\mathbb{R}$

Find the conjugates of the following functions:

- Affine function: $f(x) = ax + b$.
- Negative logarithm: $f(x) = -\log x$
- Exponential. $f(x) = e^x$
- Negative Entropy. $f(x) = x \log x$
- Inverse. $f(x) = 1/x$

See classnotes for solutions.

Scratch Space

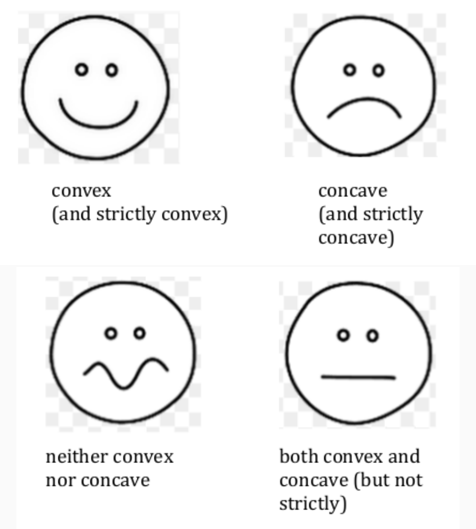
Scratch Space

Scratch Space

Scratch Space

Thats All for Convex Functions!

To summarize:



Convex Optimization Problems

Optimization problem:

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

Convex Optimization Problems

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- Find an x that minimizes $f_0(x)$ among all x that satisfy
  - $f_i(x) \leq 0, i = 1, \ldots, m$
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- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is called objective function or cost function

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- The inequalities $f_i(x) \le 0$ are called inequality constraints

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Convex Optimization Problems

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- The equations $h_i(x)$ are called equality constraints
- The functions $h_i : \mathbb{R}^n \to \mathbb{R}$ are called equality constraint functions

Convex Optimization Problems

Optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) && (1)\\
\text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m && (2)\\
& h_i(x) = 0, i = 1, \ldots, p && (3)
\end{aligned}
$$

Convex Optimization Problems

Optimization problem:

$$\begin{aligned} &\text{minimize} && f_0(x) && (1)\\ &\text{subject to} && f_i(x) \leq 0, i = 1, \ldots, m && (2)\\ & && h_i(x) = 0, i = 1, \ldots, p && (3) \end{aligned}$$

- If no constraints, i.e., $m = p = 0$, then called unconstrained problem

Convex Optimization Problems

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- Domain of opt. problem: where objective and constraint are defined

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i \quad \cap \quad \bigcap_{i=1}^{p} \mathbf{dom}\ h_i$$

Convex Optimization Problems

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- A point x is called feasible if it satisfies the constraints

$$\begin{aligned} f_i(x) &\leq 0, \quad i = 1, \ldots, m \\ h_i(x) &= 0, \quad i = 1, \ldots, p \end{aligned}$$

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$$\begin{aligned} f_i(x) \leq 0, & \quad i = 1, \ldots, m \\ h_i(x) = 0, & \quad i = 1, \ldots, p \end{aligned}$$

- The optimization problem is called feasible if there exists atleast one feasible point

Convex Optimization Problems

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- **Feasible set or constraint set:** set of all feasible points

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- **Feasible set or constraint set:** set of all feasible points
- **Optimal Value:** The optimal value p^* defined as

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p \}$$

Convex Optimization Problems

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- p^* is allowed to take extended values $\pm\infty$

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  - Note: we used the fact that $\inf \phi = \infty$

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- p^* is allowed to take extended values $\pm\infty$
- Infeasible problem: problem is called infeasible when $p^* = \infty$
  - Note: we used the fact that $\inf \phi = \infty$
- Unbounded below: Problem is unbounded below if $f_0(x_k) \to -\infty$ as $k \to \infty$

Optimal and locally optimal

Optimization problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

Optimal and locally optimal

Optimization problem:

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- Redundant constraint: A constraint is redundant if removing it does not change the feasible set

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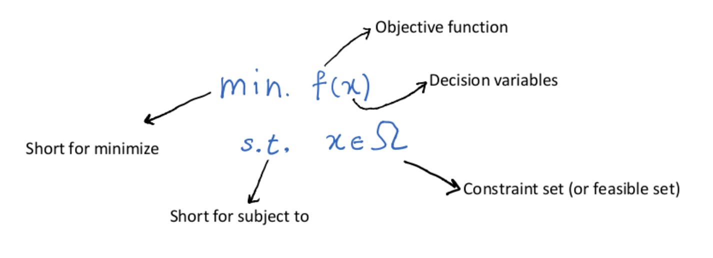
More compactly, we we can write:



Examples: $1/x$

Consider the optimization problem:

$$\begin{aligned} &\text{minimize } f_0(x) = 1/x, \\ &\text{subject to } x \in \mathbb{R} \end{aligned}$$

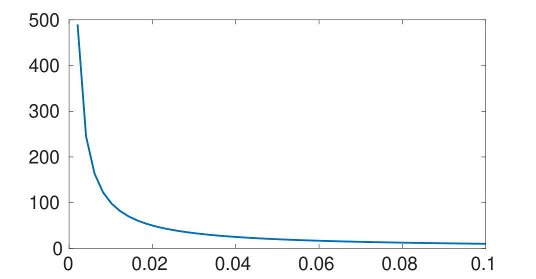
where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set?

Quiz: What is p^*?

Quiz: Is the optimal value acheived?

Figure 1: Plot of $1/x$



Examples: $-\log x$

Consider the optimization problem:

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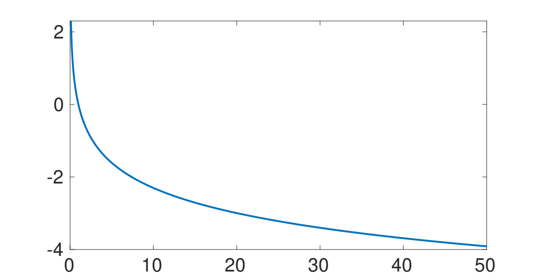
Figure 2: Plot of $-\log x$



Examples: $x \log x$

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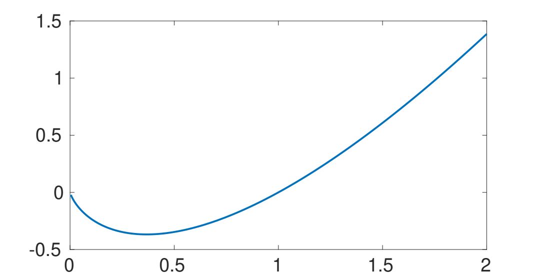
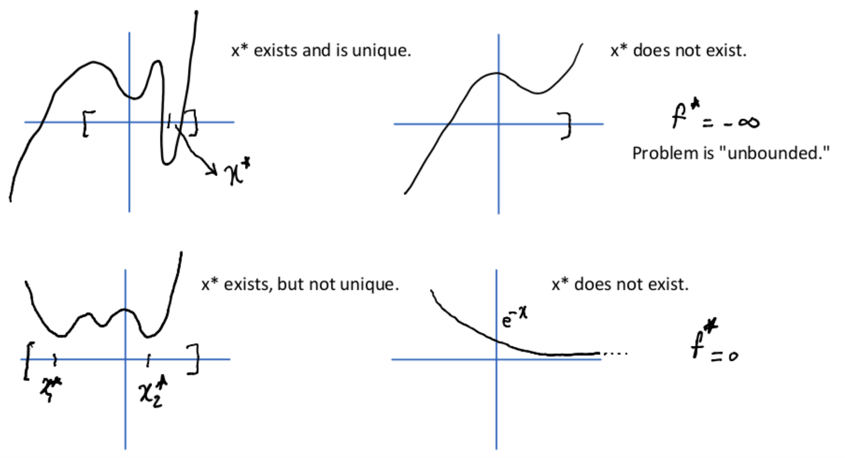
Quiz: Is this problem bounded below?
Quiz: What is optimal point?

Figure 3: Plot of $x \log x$



Examples: Graphically



Expressing Problems in Standard Form

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- $f_i(x) \geq 0$ as $-f_i(x) \leq 0$

(Box Constraints). Consider the following

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & l_i \leq x_i \leq u_i, \quad i = 1, \ldots, n \end{array}$$

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where $f_i(x) = l_i - x_i,\ i = 1, \ldots, n$ and $f_i(x) = x_{i-n} - u_{i-n},\ i = n+1, \ldots, 2n$

Maximization Problems Seen as Minimization Problems

Note: Maximization problem can be solved by minimization. Consider

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$$p^* = \sup \{ f_0(x) \mid \quad f_i(x) \leq 0,\ i = 1, \ldots, m, \quad h_i(x) = 0,\ i = 1, \ldots, p \}$$

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- If z solves above, then $x = \phi(z)$ solves the standard optimization problem.
- Similarly, if x solves original opt problem, then $z = \phi^{-1}(x)$ solves above

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Equivalent Problems: Transformation of objective and constraint function

- Suppose $\psi_0 : \mathbb{R} \to \mathbb{R}$ is monotone increasing
- $\psi_1, \ldots, \psi_m : \mathbb{R} \to \mathbb{R}$ satisfy

$$\psi_i(u) \le 0, \quad \text{if and only if } u \le 0$$

- $\psi_{m+1}, \ldots, \psi_{m+p} : \mathbb{R} \to \mathbb{R}$ satisfy

$$\psi_i(u) = 0, \quad \text{if and only if } u = 0$$

- Define $\tilde{f}_i$ as

$$\tilde{f}_i(x) = \psi_i(f_i(x)), \quad i = 0, \ldots, m$$

- Define $\tilde{h}_i$ as

$$\tilde{h}_i(x) = \psi_{m+i}(h_i(x)), \quad i = 1, \ldots, p$$

The associated problem is

$$\boxed{\begin{array}{rl} \text{minimize} & \tilde{f}_0(x) \\ \text{subject to} & \tilde{f}_i(x) \le 0, \quad i = 1, \ldots, m \\ & \tilde{h}_i(x) = 0, \quad i = 1, \ldots, p \end{array}}$$

Equivalent Problems: Slack Variables

Given the optimization problem in standard form

Optimization problem (Standard Form):

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

Quiz: Is it possible to replace inequality constraints by equality constraints and non-negaivity constraints?

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Quiz: Is it possible to replace inequality constraints by equality constraints and non-negaivity constraints?

Ans: Yes. Key observation is: $f_i(x) \leq 0$, if and only if there is an $s_i \geq 0$ such that $f_i(x) + s_i = 0$

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$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & s_i \geq 0, \quad i = 1, \ldots, m \\ & f_i(x) + s_i \leq 0, \; i = 1, \ldots, m \\ & h_i(x) = 0, \; i = 1, \ldots, p \end{array}$$

Note: Here s_i are called slack variable. Is this equivalent?

Convex Optimization Problem in Standard Form

Convex Optimization Problem (Standard Form):

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
 & a_i^T x = b_i, \quad i = 1, \ldots, p
\end{array}
$$

where $f_0, \ldots, f_m$ are convex functions.

Comparing this with the standard form

Optimization problem (Standard Form):

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 & h_i(x) = 0, i = 1, \ldots, p
\end{array}
$$

- objective function must be convex
- inequality constraint functions must be convex
- equality constraint functions $h_i(x) = a_i^T x - b_i$ must be affine

Convex Optimization Problem

Consider the following optimization problem with $x \in \mathbb{R}^2$

$$\begin{array}{rl} \text{maximize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0, \end{array}$$

which is in standard form.

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Quiz: Is this problem a convex optimization problem? Ans: No

Quiz: Can you rewrite this in convex optimization problem?

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which is in standard form.

Quiz: Is this problem a convex optimization problem? Ans: No

Quiz: Can you rewrite this in convex optimization problem? Ans: Yes

$$
\begin{aligned}
\text{maximize} \quad & f_0(x) = x_1^2 + x_2^2 \\
\text{subject to} \quad & f_1(x) = x_1 \leq 0 \\
& h_i(x) = x_1 + x_2 = 0,
\end{aligned}
$$

Note: This is now a convex optimization problem