11.09.2020

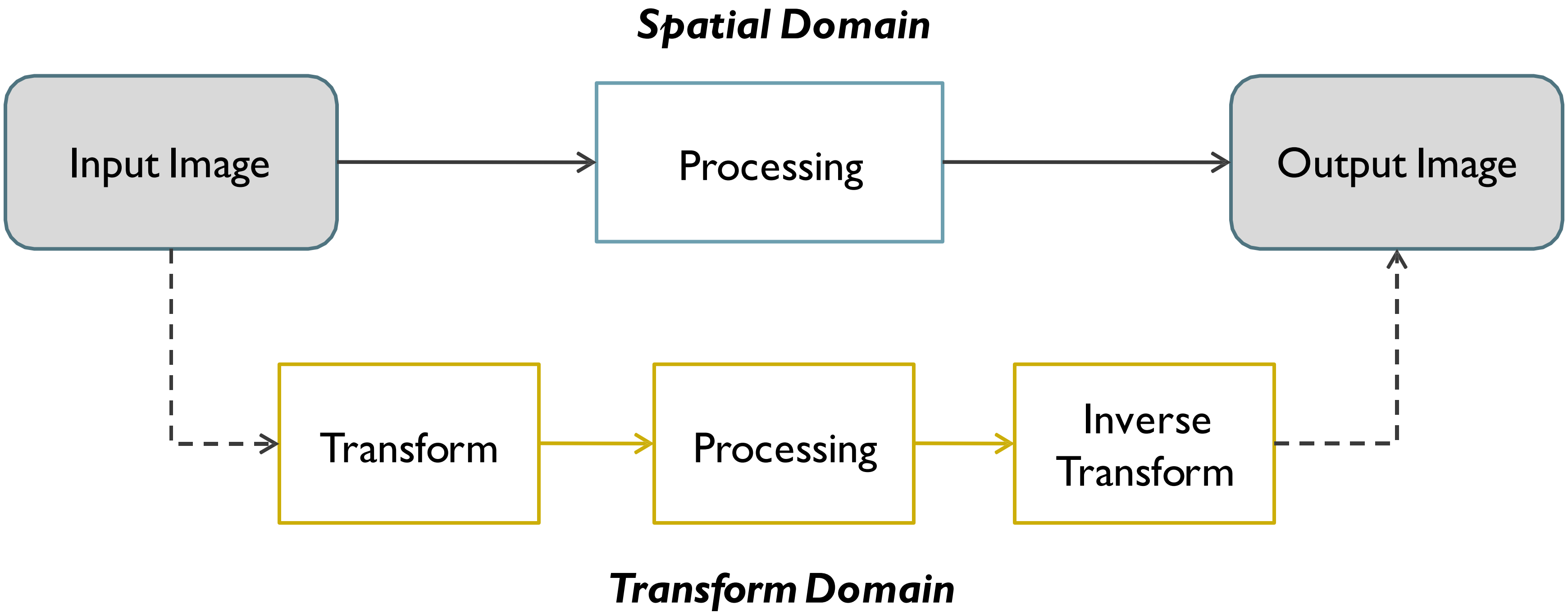
Digital Image Processing (CSE/ECE 478)

Lecture-10: Frequency Domain Processing

Ravi Kiran

Center for Visual Information Technology (CVIT), IIIT Hyderabad

Spatial vs. Transform Domain Processing



Fourier Transform

Approximate **non-periodic signals** with complex sinusoids

Intuition for FT

- *f(t)* = Single number
- How much of frequency ω signal is present for all values of t ?

$$F(\omega) = \int_{t=-\infty}^{t=\infty} f(t)e^{-i\omega t}dt$$

$$F(\omega) = \mathcal{F}[f(t)]$$

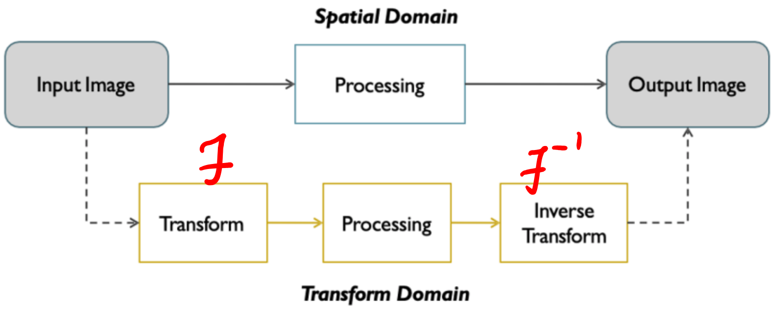
Fourier Transform and Inverse Fourier Transform

- Fourier Transform

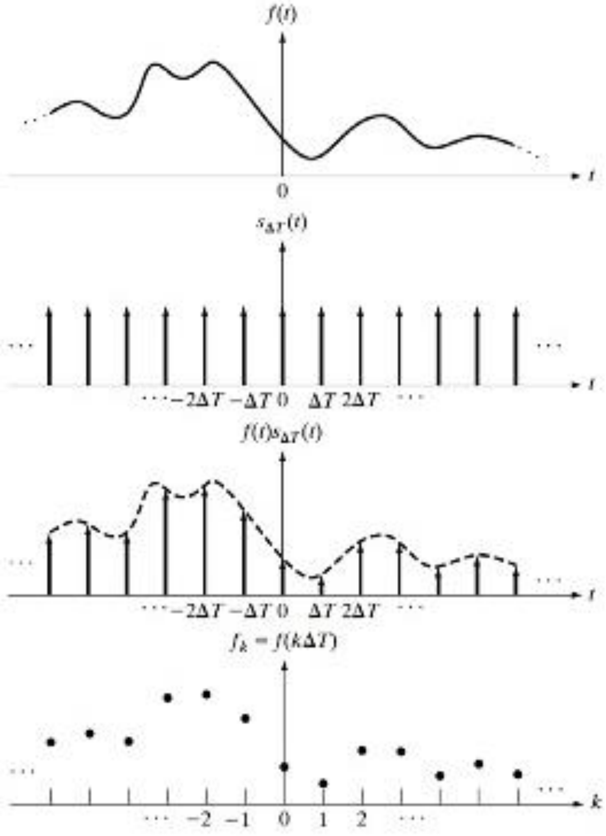
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad F(\omega) = \mathcal{F}[f(t)]$$

- Inverse Fourier Transform

$$f(t) = \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{2\pi} F(\omega) e^{i\omega t} d\omega \qquad f(t) = \mathcal{F}^{-1}[F(\omega)]$$



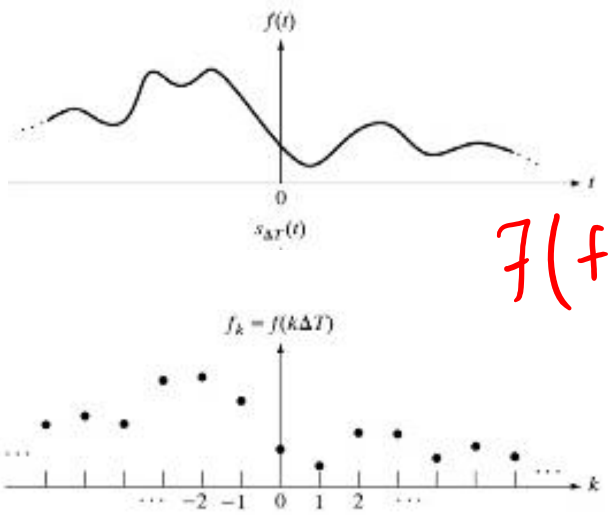
Sampling = f(t) x Impulse Train



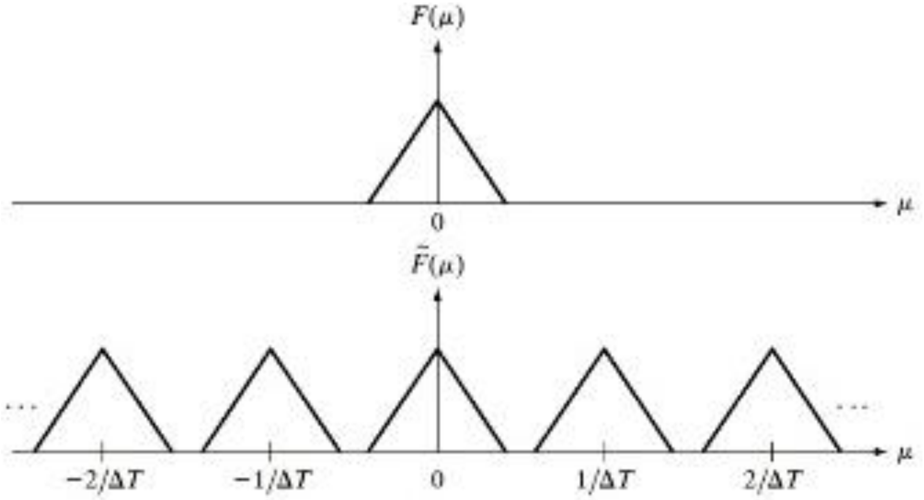
$$s_{\Delta T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\Delta T)$$

$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n \, \delta(t - n\Delta T)$$

FT of sampled function (G&W 4.2.4)



$\mathcal{F}(f(t)\,s_\Delta(t))$



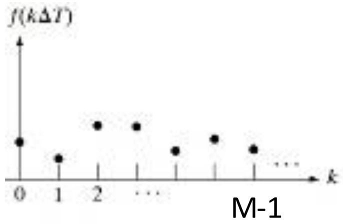
- Continuous
- Periodic (copies of f(t)'s FT)

$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n\,\delta(t - n\Delta T)$$

$$\tilde{F}(\mu) = \boxed{\frac{1}{\Delta T}} \sum_{n=-\infty}^{n=\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

$\frac{1}{\Delta T}$

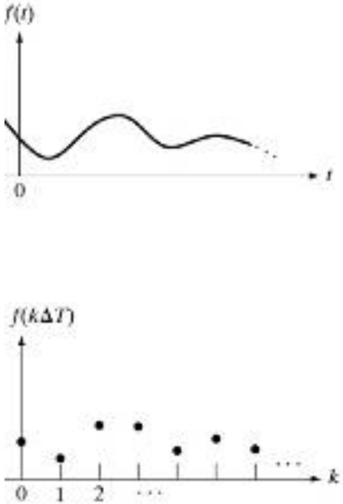
FT of sampled function

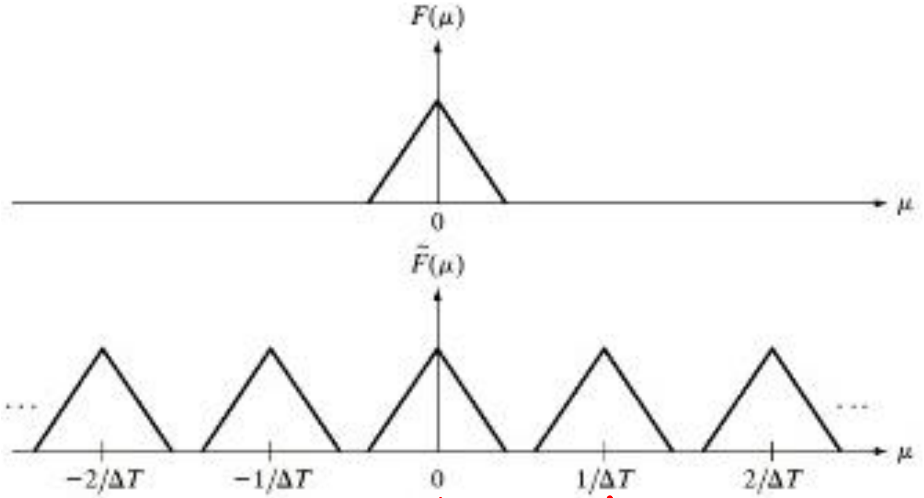


$$\tilde{f}(t) = \sum_{n=0}^{n=(M-1)\Delta T} f_n \, \delta(t - n\Delta T)$$

$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}$$

Digital processing of frequencies





$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n \, \delta(t - n\Delta T)$$

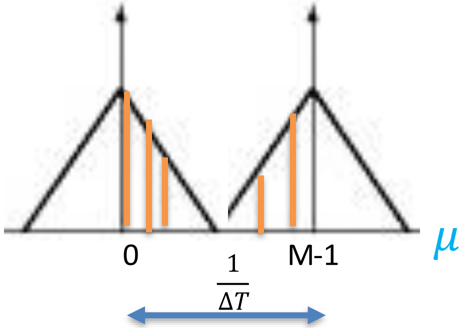
$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}$$

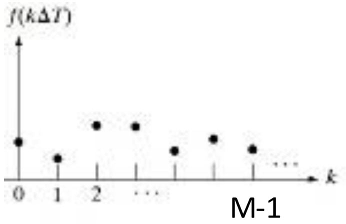
$\mu \in R$

- Need discrete frequency samples, but FT of sampled function is continuous
- OBSERVATION: Characterizing one period $(\frac{1}{\Delta T})$ is enough
- How do we get frequency 'samples' ?

FT of sampled function (G&W 4.4.1)

M sample



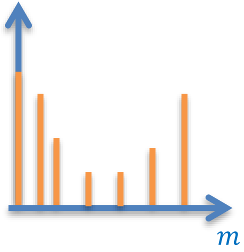


$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}, \mu \in R$$

$$\tilde{f}(t) = \sum_{n=0}^{n=(M-1)\Delta T} f_n\, \delta(t - n\Delta T)$$

- Substituting

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$



$$F[m] = \sum_{n=0}^{n=(M-1)} f_n\, e^{\frac{-j2\pi mn}{M}}, \mathrm{m} = 0,1, \ldots (\mathrm{M}-1)$$

DFT and IDFT

NOTE: No direct dependence on ΔT

$$F[m] = \sum_{n=0}^{n=(M-1)} f_n\, e^{\frac{-j2\pi nm}{M}}, \mathrm{m} = 0,1, \ldots (\mathrm{M}-1)$$

$$f_n = \frac{1}{M} \sum_{m=0}^{m=(M-1)} F_m\, e^{\frac{j2\pi nm}{M}}, \mathrm{n} = 0,1, \ldots (\mathrm{M}-1)$$

$Re\{F[m]\}$ $Im\{Fe[m]\}$

$F[m]$

- A complex value
- Represents amplitude, phase of function f[.]'s content at angular frequency 2πm/M

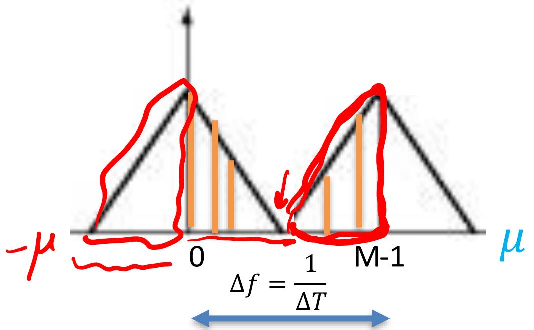
DFT: Record of 'energy' portion at various frequency bands present in input function f[.]

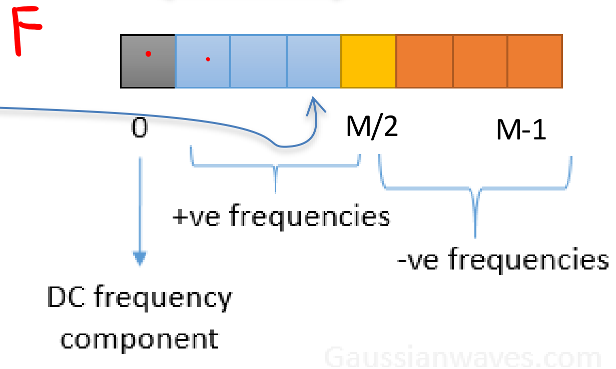
$C = A + iB$ $|c| = \sqrt{A^2 + B^2}$

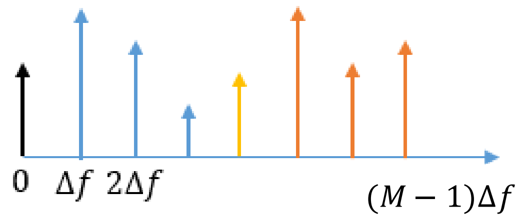
$\phi = \tan^{-1}\left(\frac{B}{A}\right)$

DFT (in practice)

$$F[m] = \sum_{n=0}^{n=(M-1)} f[n]\, e^{\frac{-j2\pi mn}{M}}, \mathrm{m} = 0,1, \dots (\mathrm{M}-1)$$



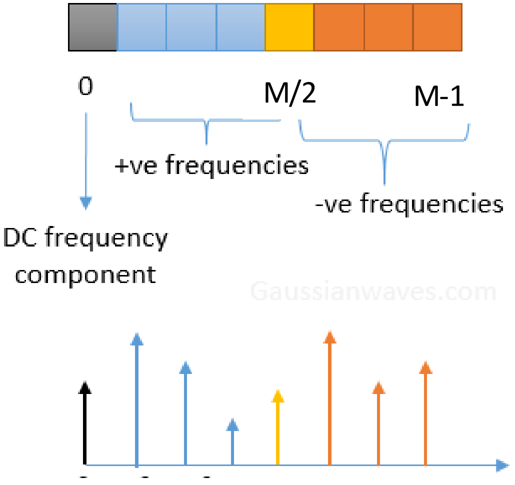


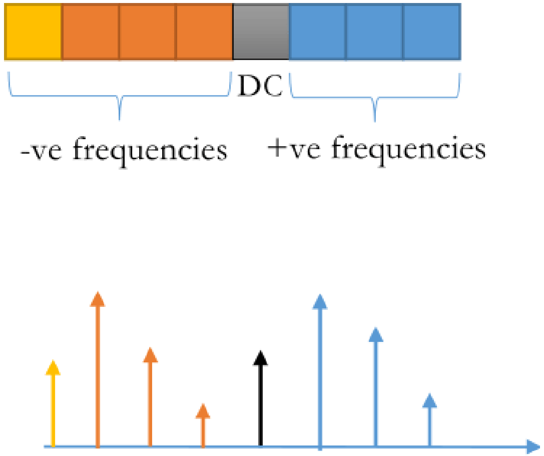


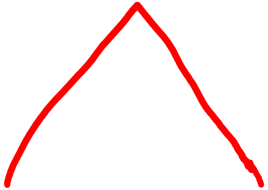
https://www.gaussianwaves.com/2015/11/interpreting-fft-results-complex-dft-frequency-bins-and-fftshift/

DFT – center shifted (for plotting)

$$F[m] = \sum_{n=0}^{n=(M-1)} f[n]\, e^{\frac{-j2\pi mn}{M}}, \mathrm{m} = 0,1, \ldots (\mathrm{M}-1)$$



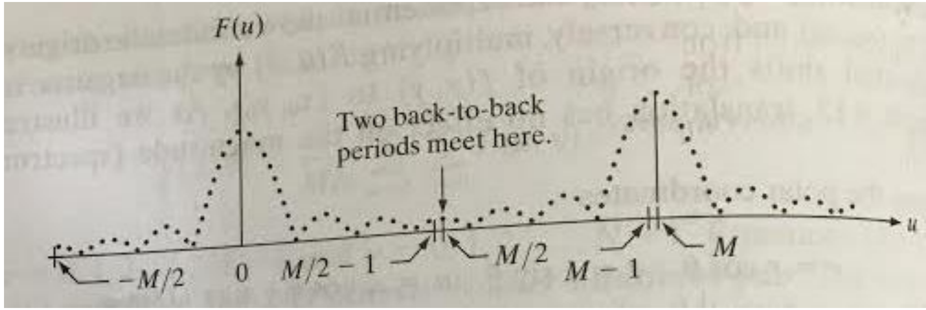


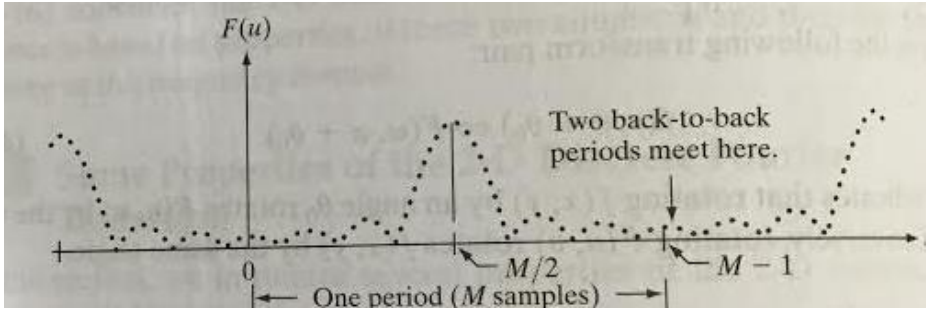


https://www.gaussianwaves.com/2015/11/interpreting-fft-results-complex-dft-frequency-bins-and-fftshift/

Shifting origin

1-D



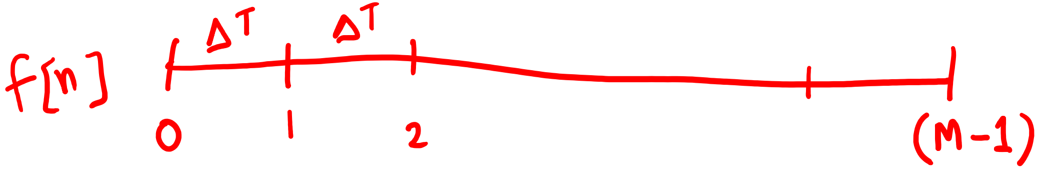


$F(u)$ $F(u-u_o)$

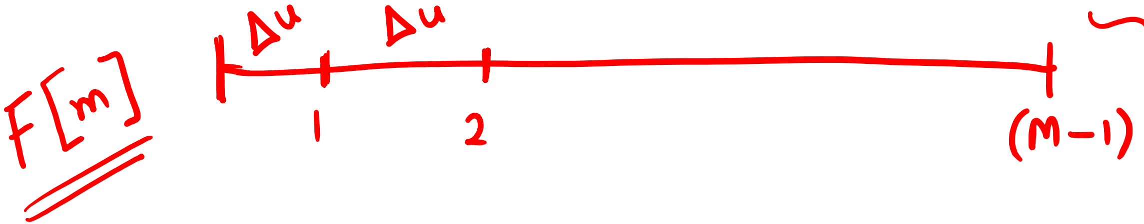
$$\underbrace{f[x]}\underbrace{e^{\frac{j2\pi u_O x}{M}}} \leftrightarrow \underline{F(u-u_o)}$$

$$\underbrace{f[n](-1)^n} \qquad u_o = \frac{M}{2}$$

Relationship between Sampling and Frequency Intervals



$$T = (M-1)\Delta T$$

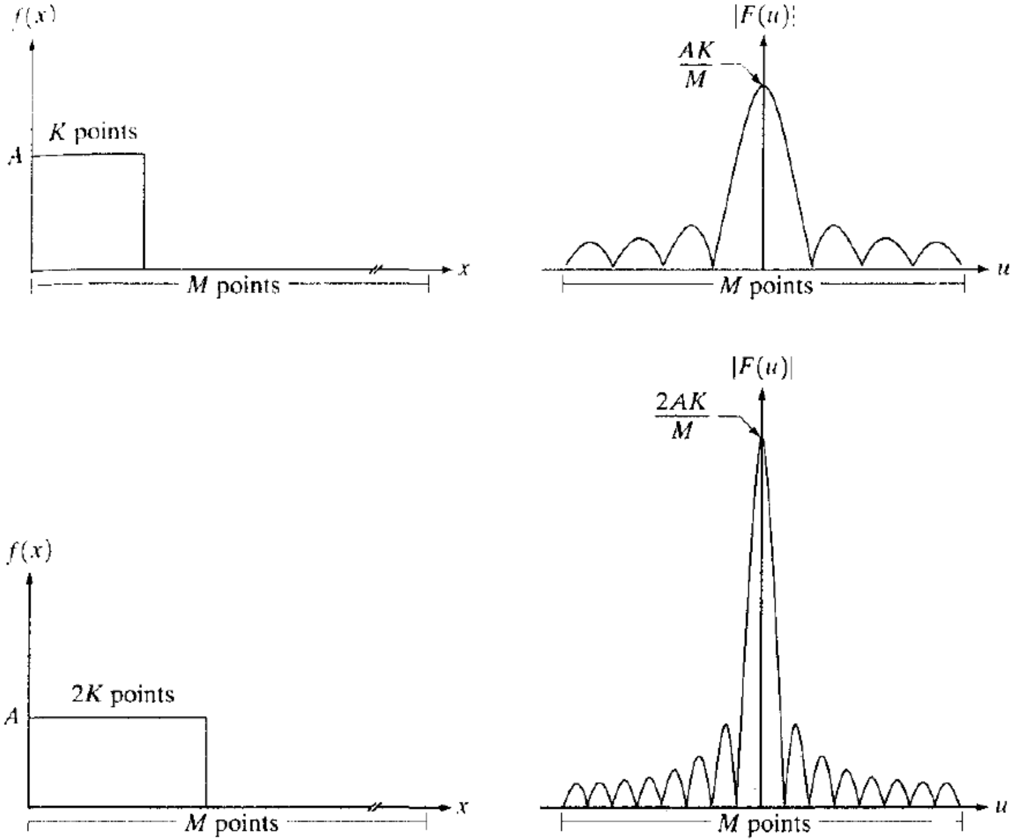


$$\underbrace{\Omega} = (M-1)\Delta u = \frac{1}{\underbrace{\Delta T}}$$

$$\Delta u = \frac{1}{(M-1)\Delta T} = \frac{1}{T}$$

- Ω (Range of frequencies) depends inversely on sampling interval ΔT
- Δu (Frequency Resolution of DFT) depends inversely on duration T over which $f(t)$ is sampled

Relationship between u and x



a b
c d

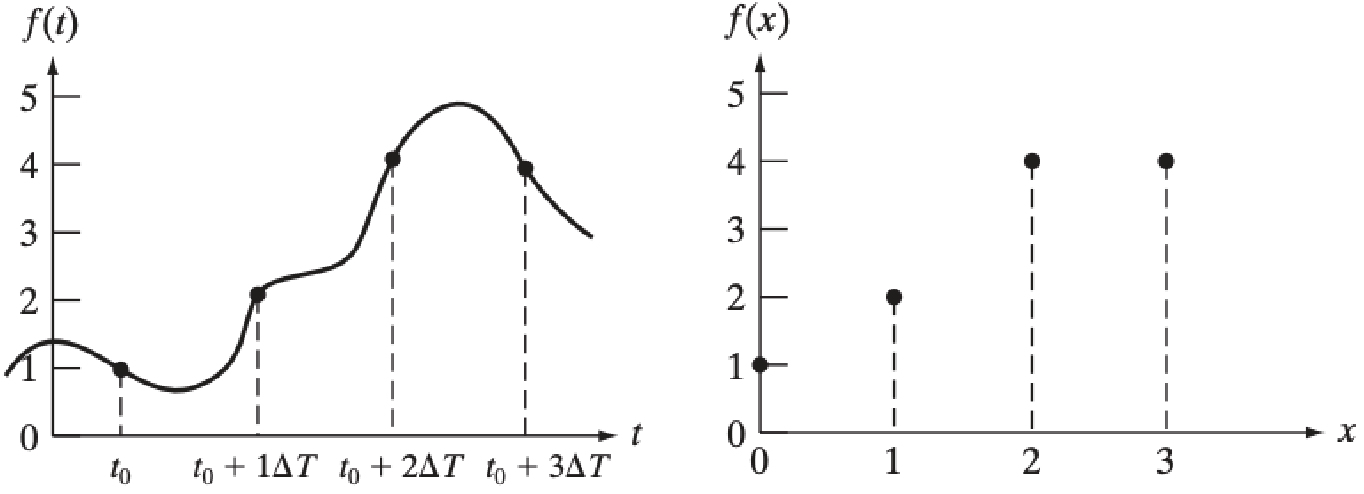
FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

$$\Delta u = \frac{1}{M\Delta x}$$

1-D DFT example

a b

FIGURE 4.11
(a) A function, and (b) samples in the *x*-domain. In (a), *t* is a continuous variable; in (b), *x* represents integer values.



$$F[m] = \sum_{n=0}^{n=(M-1)} f[n]\, e^{\frac{-j2\pi mn}{M}}, \mathrm{m} = 0,1, \ldots (\mathrm{M}-1)$$

$F[0] = 1+2+4+4$

$$F[1] = \sum_{n=0}^{3} f[n]\, e^{-j\, 2\pi \cancel{1} n/4}$$

$= -3+2j$

$0 \rightarrow e^{0} \rightarrow 1 \times 1$

$1 \rightarrow e^{-j2\pi/4} \rightarrow -j \times 2$

$2 \rightarrow e^{-j4\pi/4} \rightarrow -1 \times 4$

$3 \rightarrow e^{-j6\pi/4} \rightarrow +j \times 4$

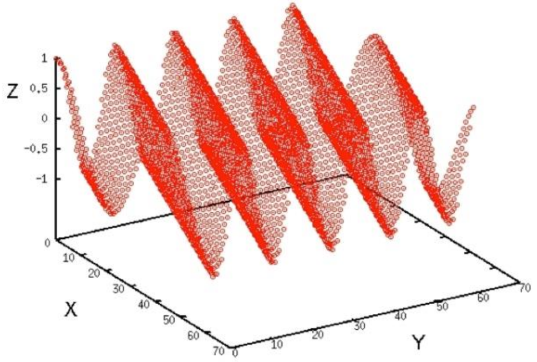
2D DFT and IDFT

$$\underline{F[m,n]} = \sum_{y=0}^{y=(N-1)} \sum_{x=0}^{x=(M-1)} \underline{f[x,y]}\, \underbrace{e^{-2\pi j\left(\frac{mx}{M}+\frac{ny}{N}\right)}}$$

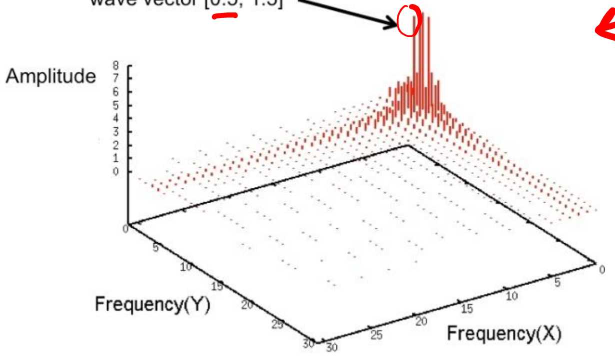
$F[3,3] =$

$\cos(0.5\pi x + 1.5\pi y)$

f_x f_y



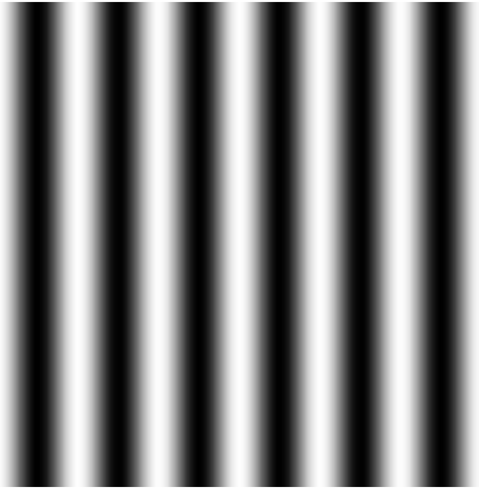
$\mathrm{FFT}[\cos(0.5\pi x + 1.5\pi y)]$

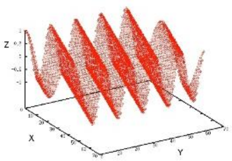


$$\underline{f[x,y]} = \frac{1}{MN} \sum_{n=0}^{n=(N-1)} \sum_{m=0}^{m=(M-1)} \underline{F[m,n]}\, e^{2\pi j\left(\frac{mx}{M}+\frac{ny}{N}\right)}$$

DFT for simple spatial patterns

Brightness Image

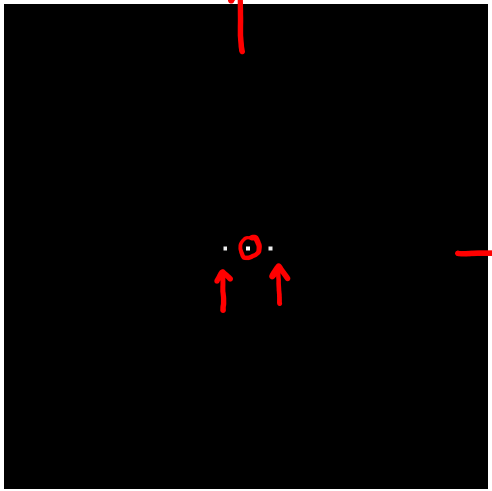




$|F(m,n)|$

Fourier transform spectrum

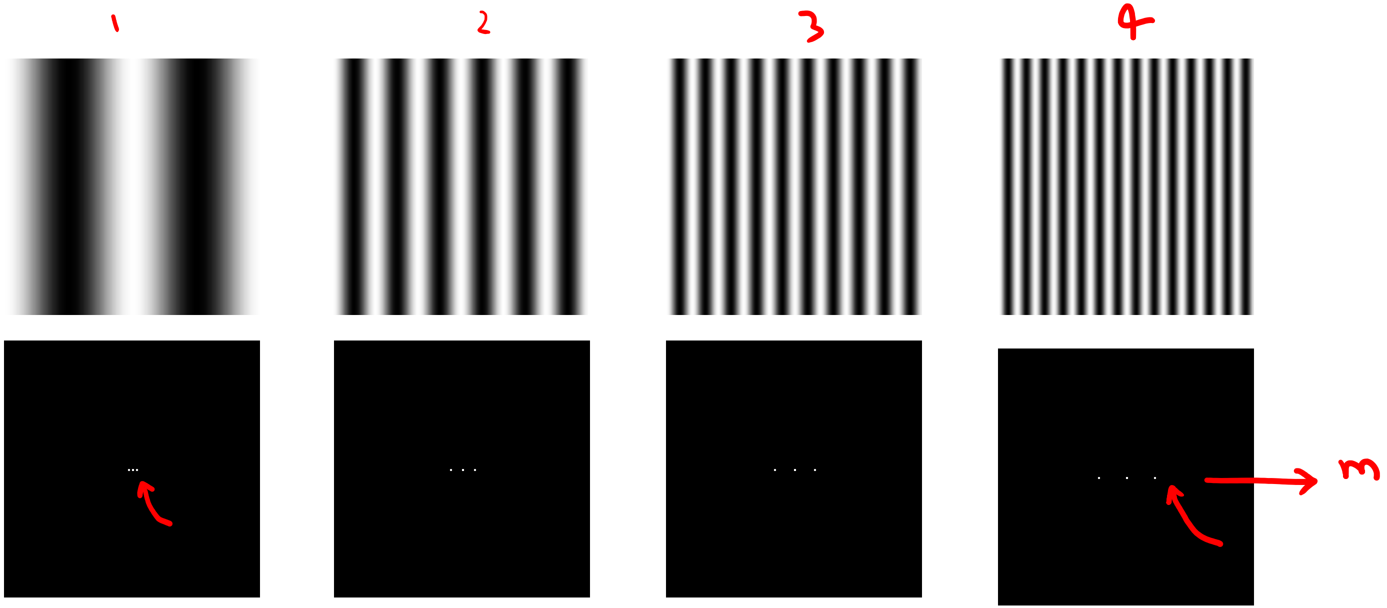
n



m

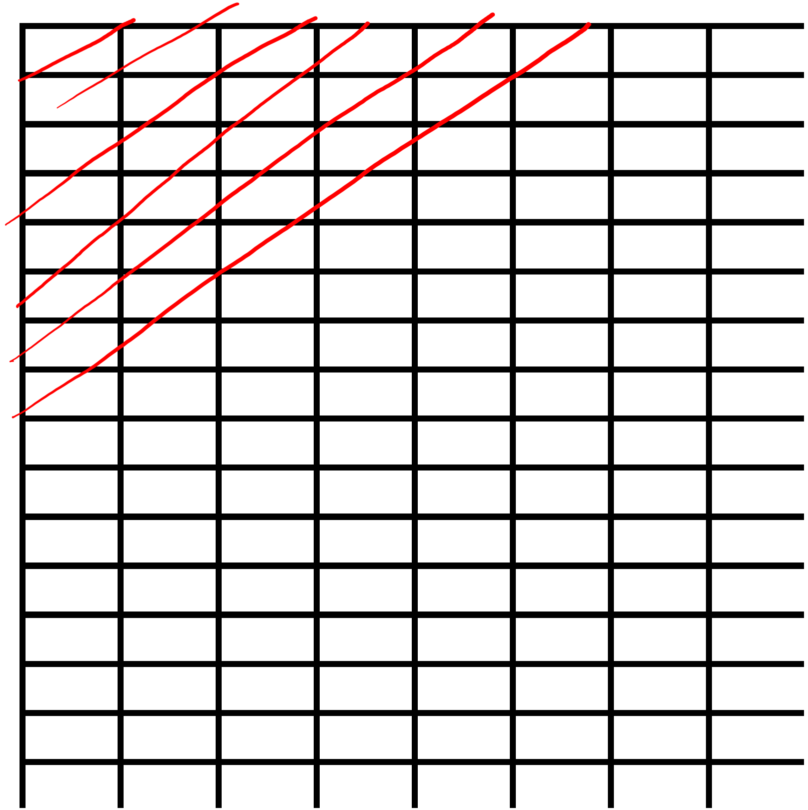
http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

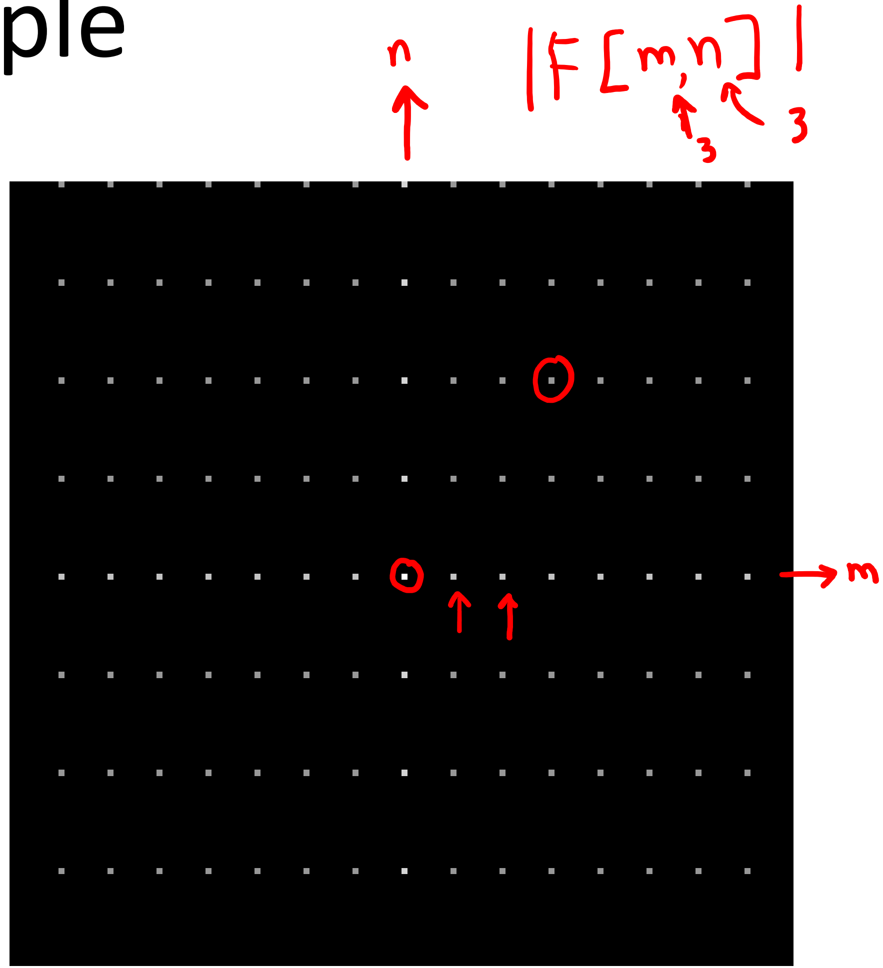
DFT Example



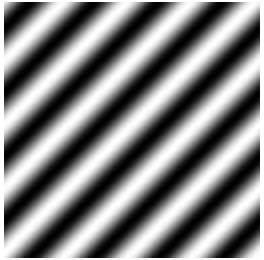
Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

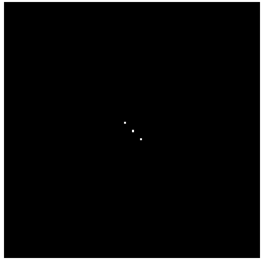
Example





DFT Example (Rotation)

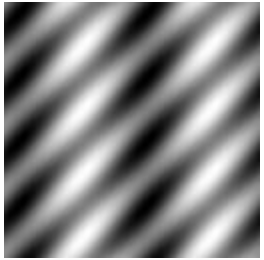


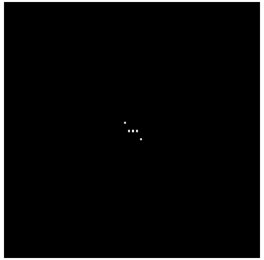


Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

DFT Example (Sum of Signals)

$\mathcal{F}(f_1)+\mathcal{F}(f_2)$



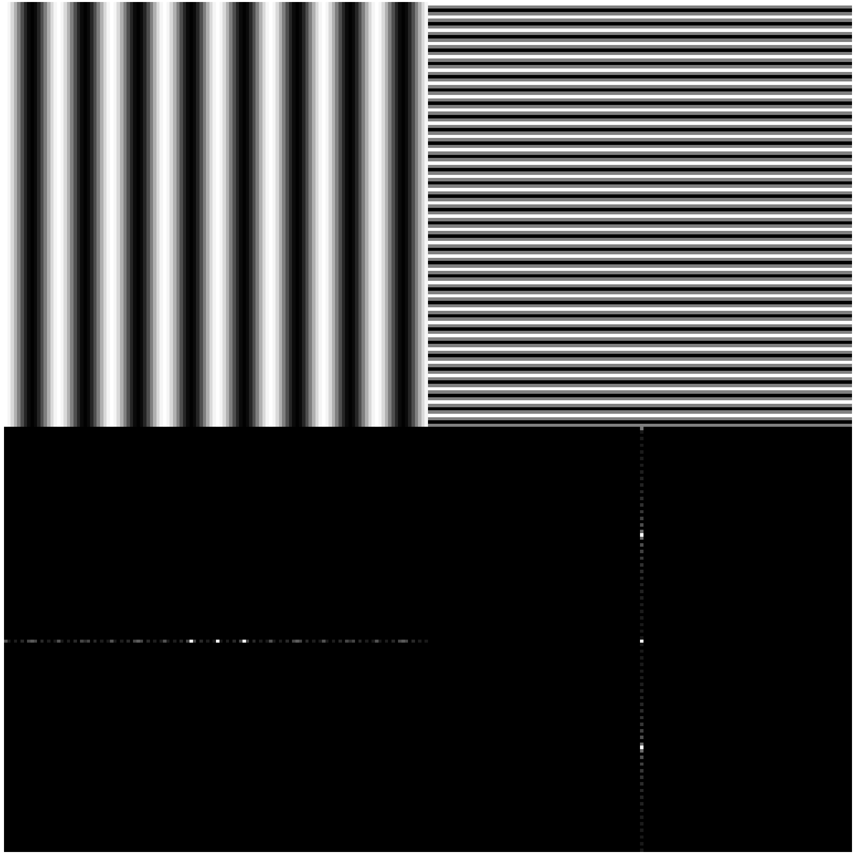


D + X J

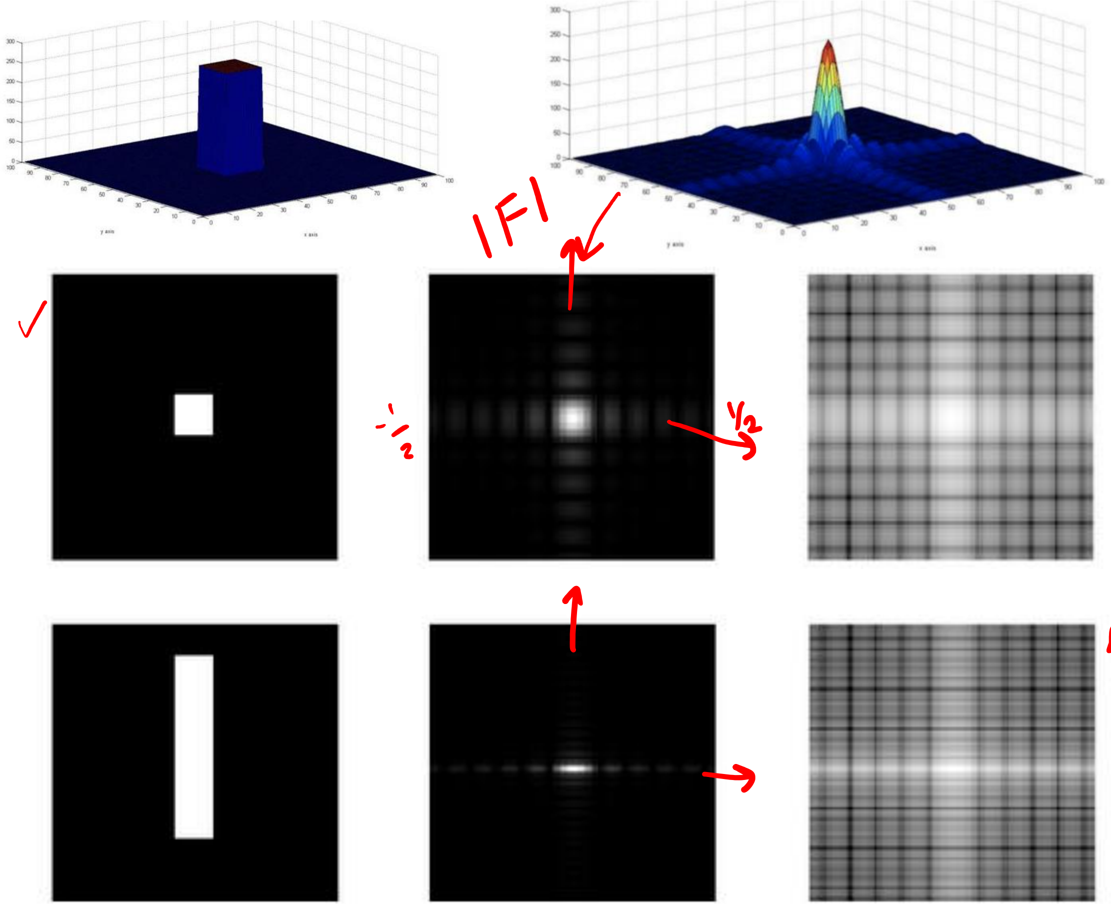
Courtesy: http://cns-alumni.bu.edu/~slehar/fourier/fourier.html

https://www.cs.unm.edu/~brayer/vision/fourier.html

DFT for simple ‘spatial’ patterns



DFT Example (Rect, Log Transformation)



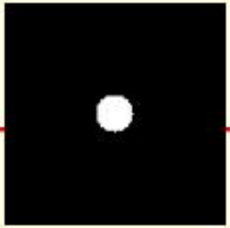
$\underline{10^{6} \text{ to } 10^{-2}} : V$

$\log(1+V)$

m, n

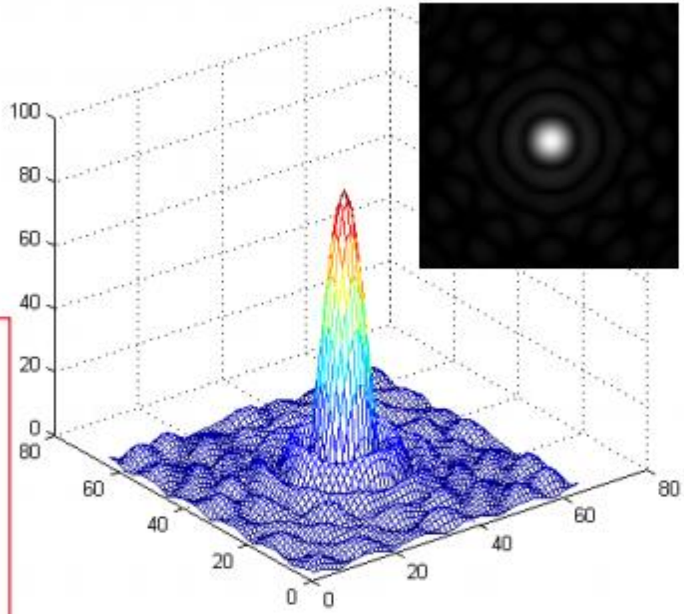
$0 \cdots \cancel{m}M-1$

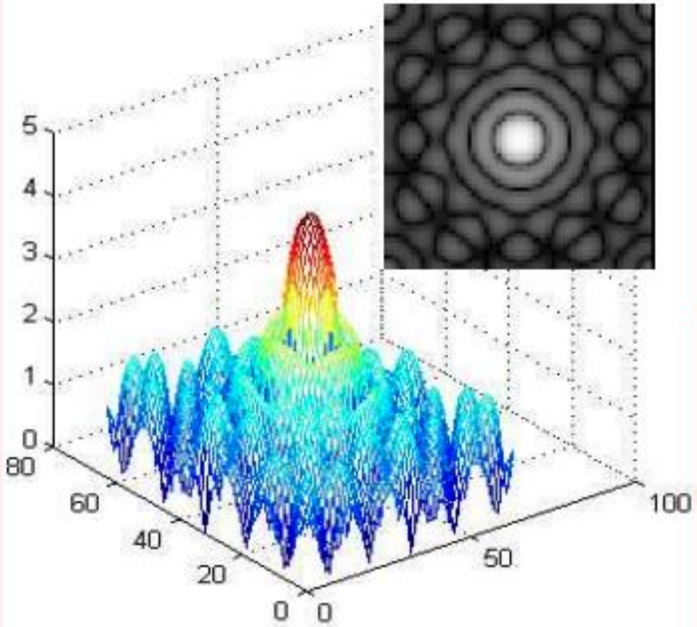
$0 \cdots N-1$



$f(x,y)$

(64x64)

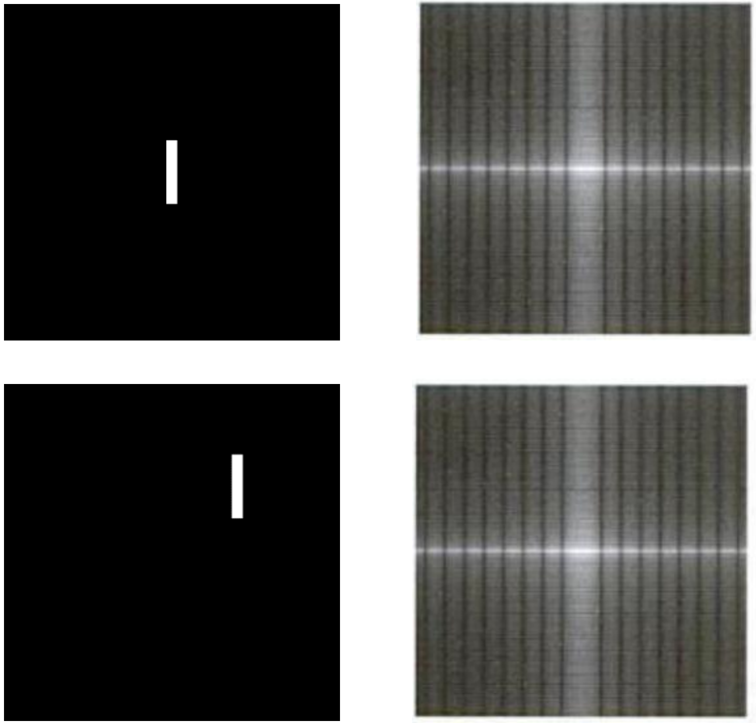




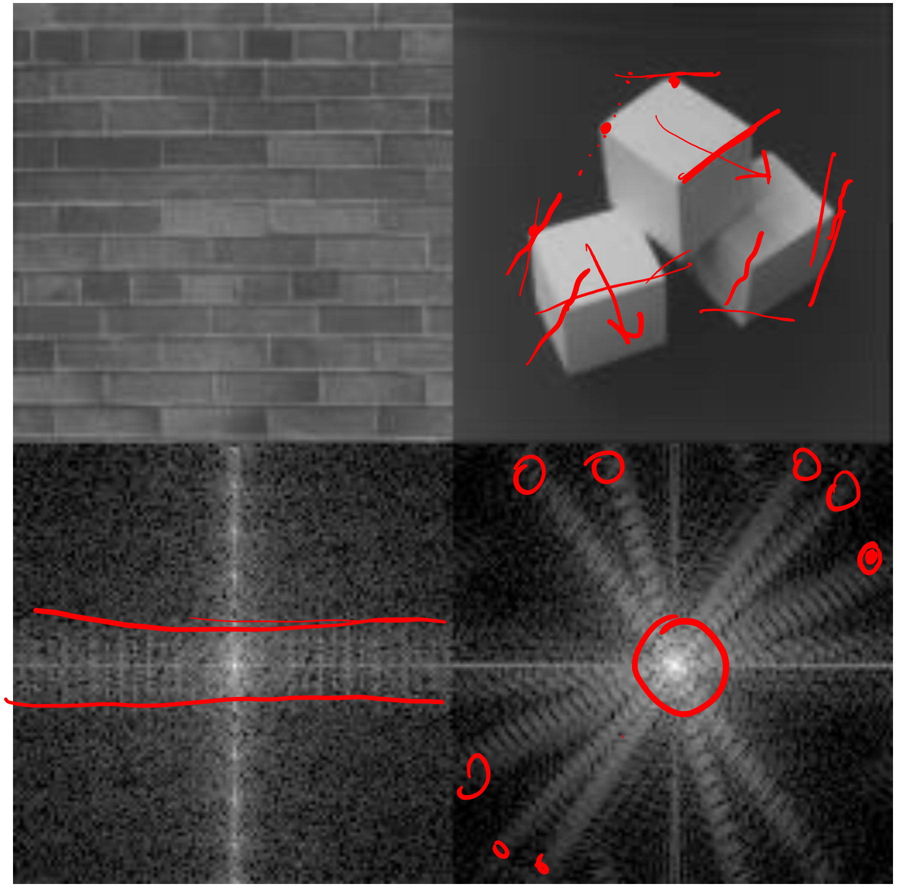
$|F(u,v)|$

$log(1+|F(u,v)|)$

DFT Example (Translation - Magnitude)



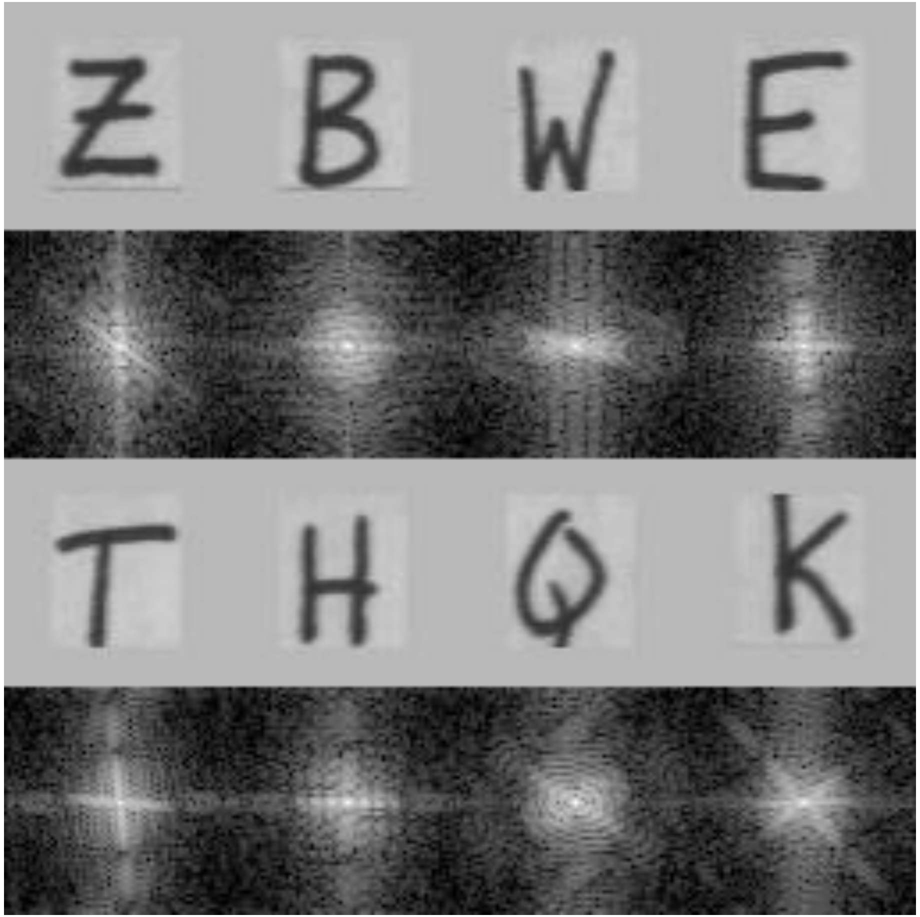
Some examples of images and spectra



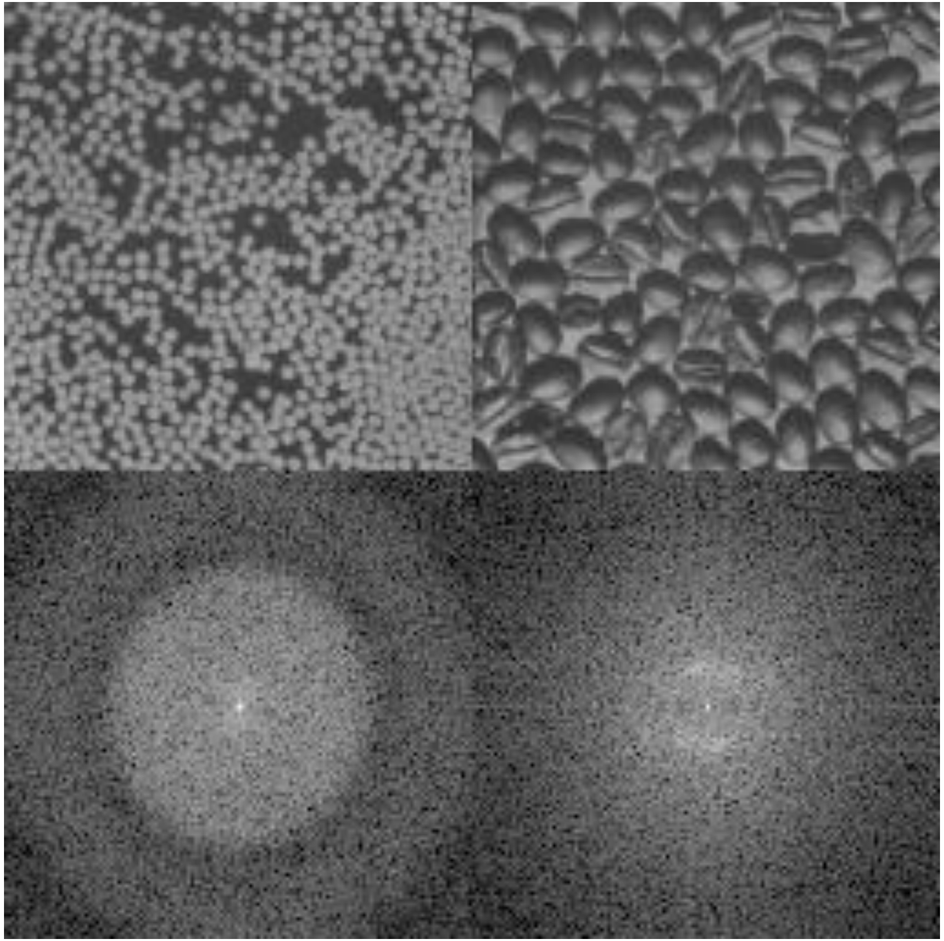
$F[m,n] =$

f

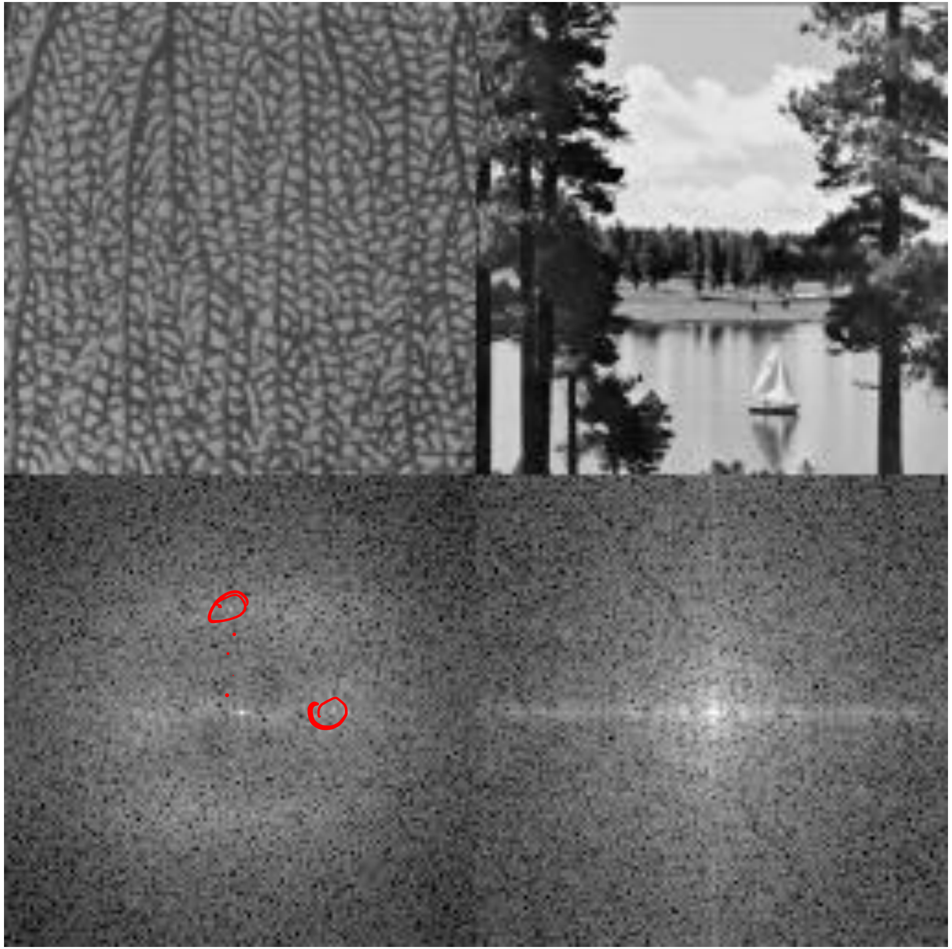
Some examples of images and spectra



Some examples of images and spectra



Some examples of images and spectra



Important Terms

- Magnitude spectrum

$$|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$$

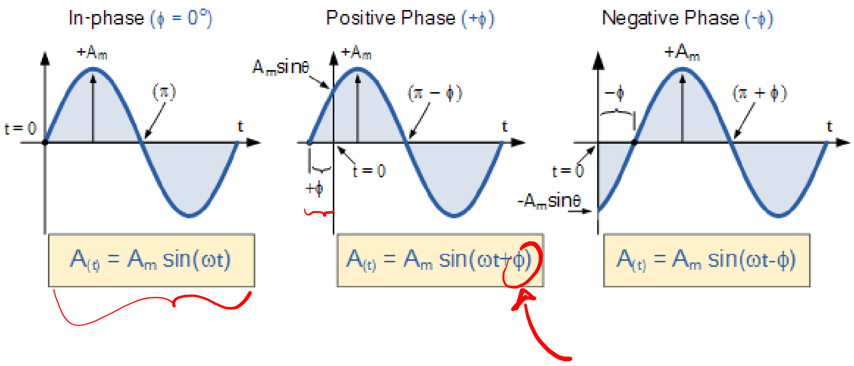
- Phase Spectrum

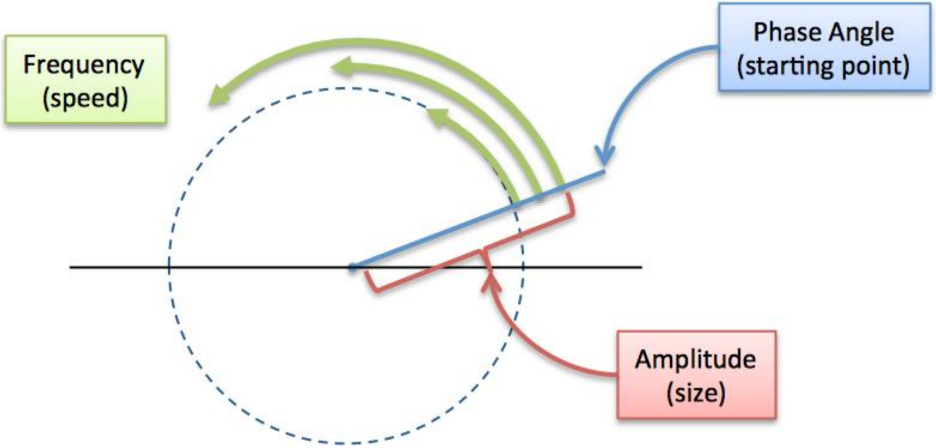
$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

- Power Spectrum

$$P(u) = |F(u)|^2$$

Phase





https://www.electronics-tutorials.ws/accircuits/phase-difference.html

http://betterexplained.com/wp-content/uploads/images//circular-path-parameters-20121201-203317.png

Magnitude and Phase Spectra

Figure 4a

Original

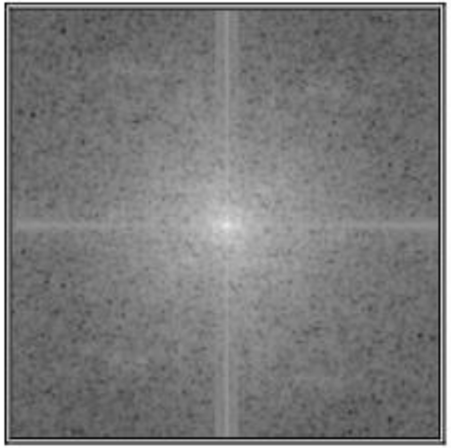


Figure 4b

$\log(|A(\Omega,\Psi)|)$

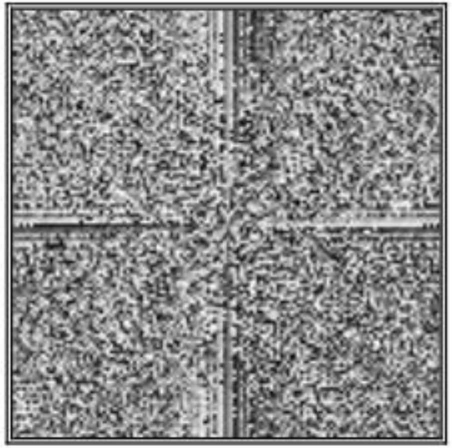


Figure 4c

$\varphi(\Omega,\Psi)$ $\tan^{-1}\left(\frac{Im}{Re}\right)$

We generally do not display PHASE images because most people who see them shortly thereafter succumb to hallucinogenics or end up in a Tibetan monastery – John Brayer

https://i.imgur.com/YidmGcQ.png

Magnitude and Phase Spectra
Both matter for reconstruction

Figure 4a

Original

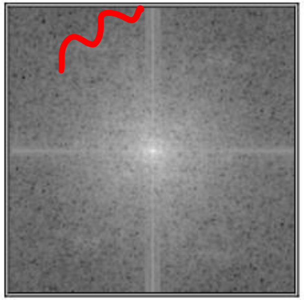


Figure 4b

$\log(|A(\Omega,\Psi)|)$

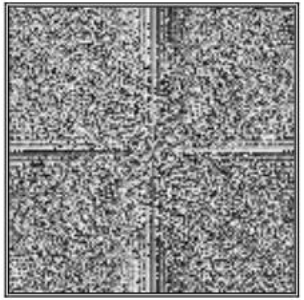


Figure 4c

$\varphi(\Omega,\Psi)$

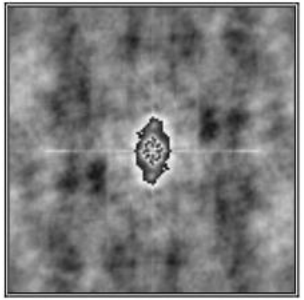


Figure 5a

$\varphi(\Omega,\Psi) = 0$

Figure 5b

$|A(\Omega,\Psi)| = \textit{constant}$ $\neq 0$

Mag

Phase

F

IDFT

0

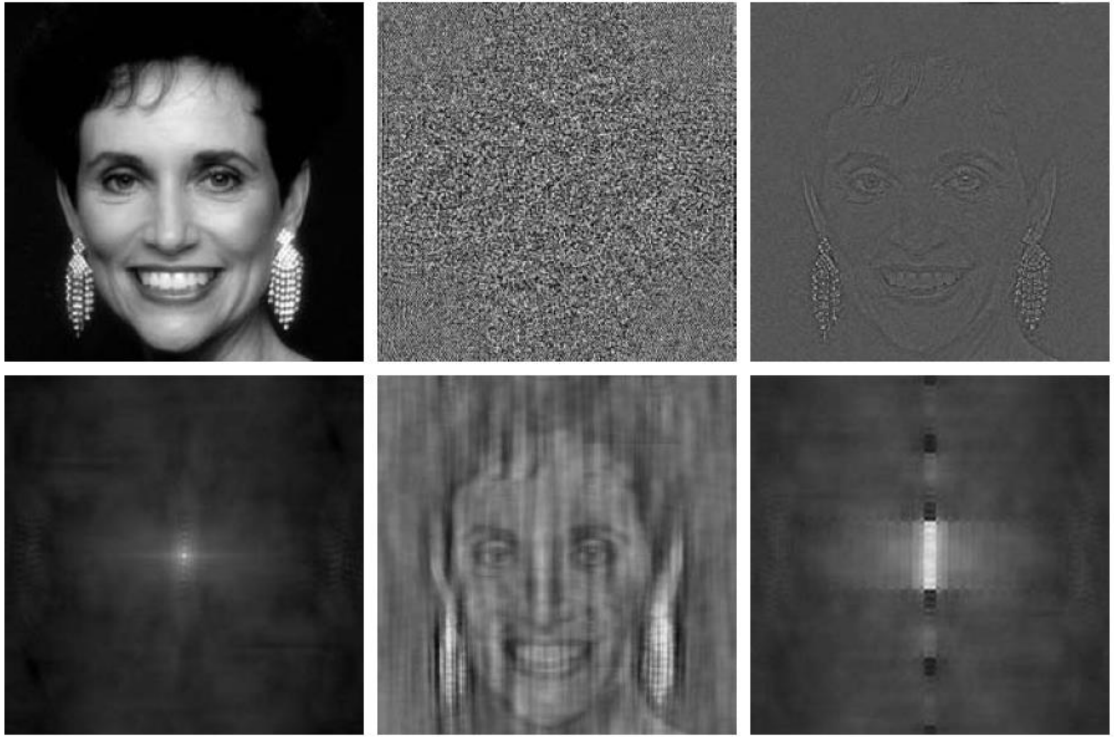
$F = Me^{-ph}$

$F = Me$

$F = A + iB$

$F = (mag, phase)$

https://i.imgur.com/YidmGcQ.png

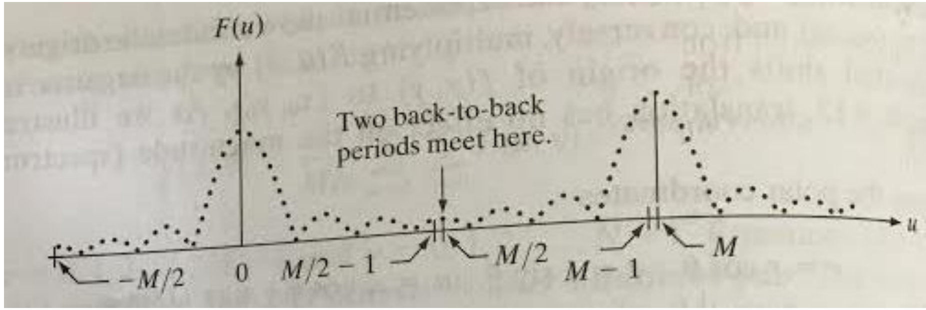


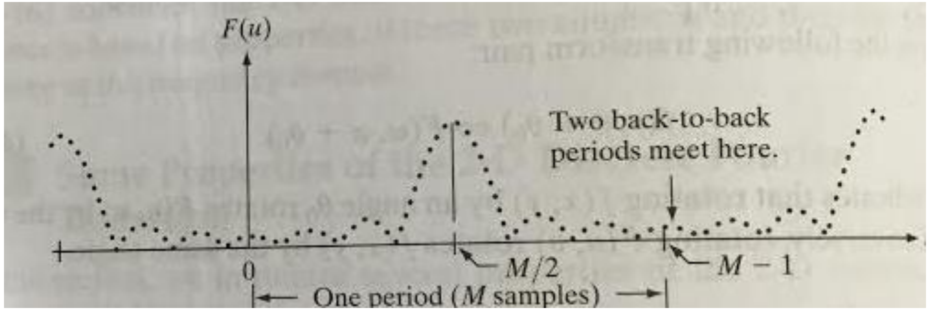
a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

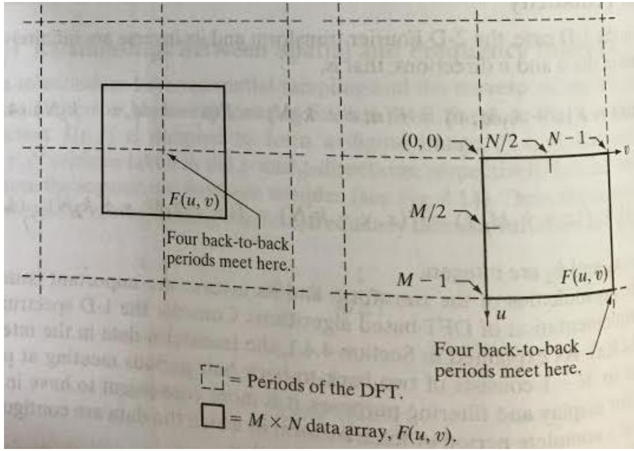
Shifting origin

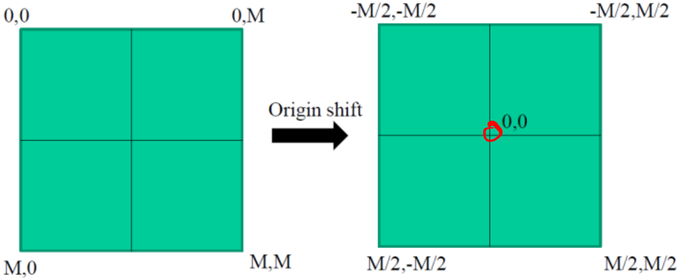
1-D





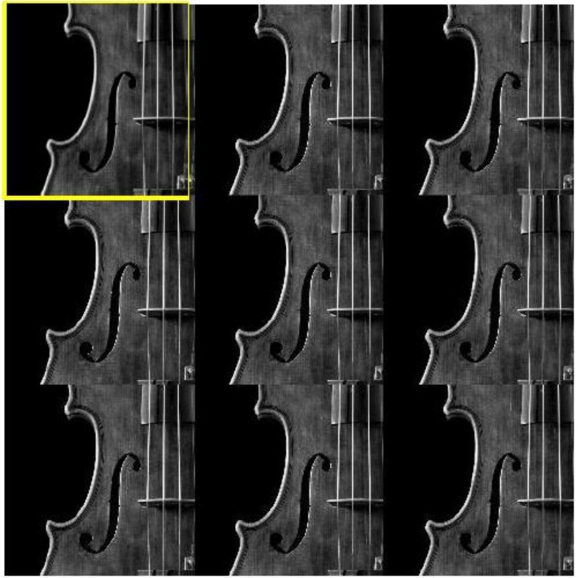
2-D

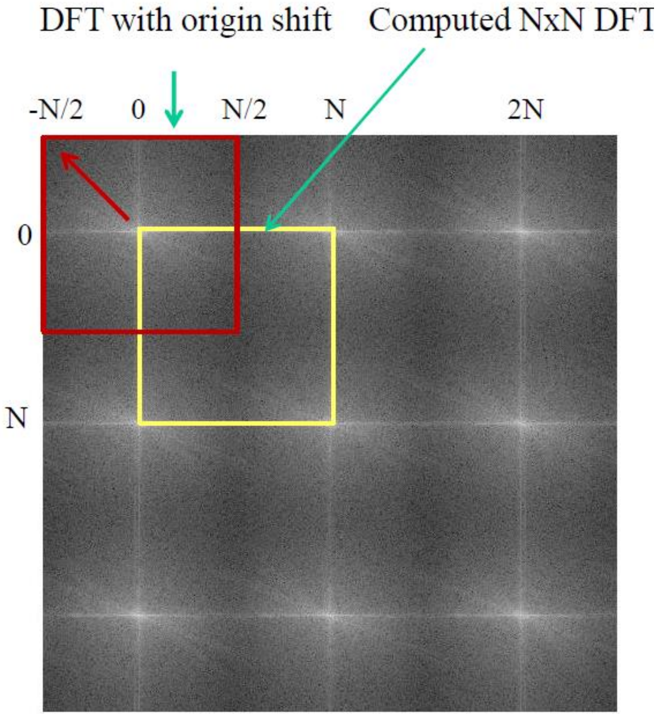


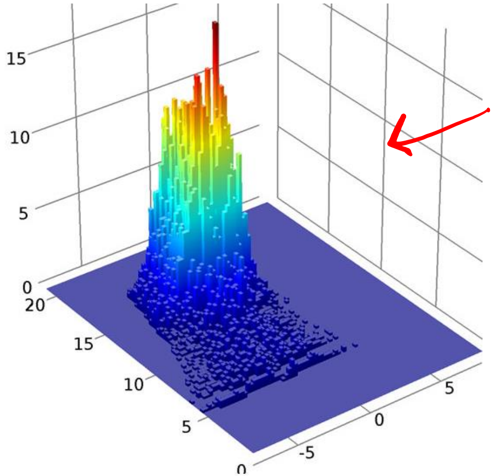


$$f[x,y]e^{j2\pi(\frac{u_O x}{M}+\frac{v_O y}{M})} \leftrightarrow F(u-u_o, v-v_0)$$

Shifting origin





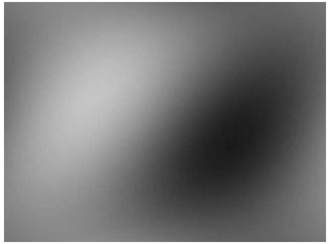


$$F[m,n] = \sum_{y=0}^{y=(N-1)} \sum_{x=0}^{x=(M-1)} f[x,y]\, e^{-2\pi j\left(\frac{mx}{M}+\frac{ny}{N}\right)}$$

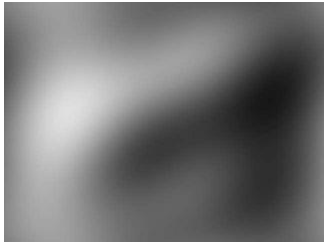
$$f[x,y] = \frac{1}{MN} \sum_{n=0}^{n=(N-1)} \sum_{m=0}^{m=(M-1)} F[m,n]\, e^{2\pi j\left(\frac{mx}{M}+\frac{ny}{N}\right)}$$

$$f[x,y] = \frac{1}{MN} \sum_{n=0}^{n=(N-1)} \sum_{m=0}^{m=(M-1)} F[m,n]\, e^{2\pi j\left(\frac{mx}{M}+\frac{ny}{N}\right)}$$

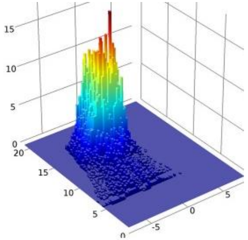
F[0,0]



F[1,0]



From 50% of the lowest frequencies



Adding up to 50% lowest frequencies

(Some) Properties of FT

	Name:	Condition:	Property:
1	Amplitude scaling	$f(t) \leftrightarrow F(\omega)$, constant K	$Kf(t) \leftrightarrow KF(\omega)$
2	Addition	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$, $\cdots$	$f(t) + g(t) + \cdots \leftrightarrow F(\omega) + G(\omega) + \cdots$
3	Hermitian	Real $f(t) \leftrightarrow F(\omega)$	$F(-\omega) = F^*(\omega)$
4	Even	Real and even $f(t)$	Real and even $F(\omega)$
5	Odd	Real and odd $f(t)$	Imaginary and odd $F(\omega)$
6	Symmetry	$f(t) \leftrightarrow F(\omega)$	$F(t) \leftrightarrow 2\pi f(-\omega)$
7	Time scaling	$f(t) \leftrightarrow F(\omega)$, real s	$f(st) \leftrightarrow \frac{1}{\lvert s \rvert} F(\frac{\omega}{s})$
8	Time shift	$f(t) \leftrightarrow F(\omega)$	$f(t - t_o) \leftrightarrow F(\omega) e^{-j\omega t_o}$
9	Frequency shift	$f(t) \leftrightarrow F(\omega)$	$f(t) e^{j\omega_o t} \leftrightarrow F(\omega - \omega_o)$
10	Modulation	$f(t) \leftrightarrow F(\omega)$	$f(t) \cos(\omega_o t) \leftrightarrow \frac{1}{2} F(\omega - \omega_o) + \frac{1}{2} F(\omega + \omega_o)$
11	Time derivative	Differentiable $f(t) \leftrightarrow F(\omega)$	$\frac{df}{dt} \leftrightarrow j\omega F(\omega)$
12	Freq derivative	$f(t) \leftrightarrow F(\omega)$	$-jtf(t) \leftrightarrow \frac{d}{d\omega} F(\omega)$
13	Time convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t) * g(t) \leftrightarrow F(\omega) G(\omega)$
14	Freq convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t) g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega)$
15	Compact form	Real $f(t)$	$f(t) = \frac{1}{2\pi} \int_0^\infty 2\lvert F(\omega) \rvert \cos(\omega t + \angle F(\omega)) d\omega$
16	Parseval, Energy W	$f(t) \leftrightarrow F(\omega)$	$W \equiv \int_{-\infty}^{\infty} \lvert f(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert F(\omega) \rvert^2 d\omega$

https://hkn.illinois.edu/wiki/_media/sidebar:screen_shot_2016-04-11_at_12.35.36_pm.png?w=800&tok=1f0b2e

- DFT has an efficient version called FFT (Fast Fourier Transform)

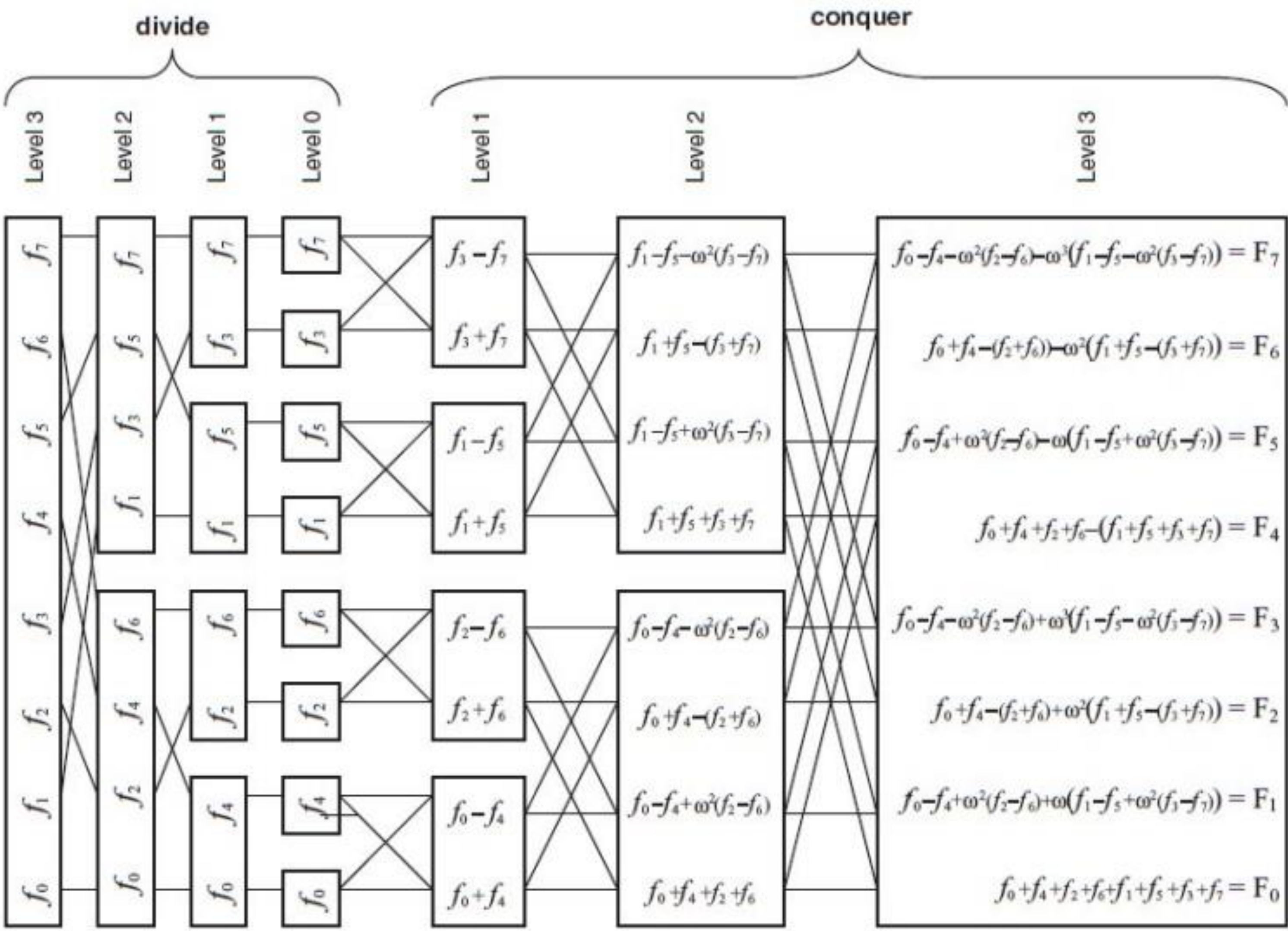
DFT vs FFT computation times

$O(N^2)$

$$F[m] = \sum_{n=0}^{n=(M-1)} f[n]\, e^{\frac{-j2\pi mn}{M}}, \mathrm{m} = 0,1, \ldots (\mathrm{M}-1)$$

n	$N = 2^n$	N^2	$N \log N$
10	1 024	1 048 576	10 240
12	4 096	16 777 216	49 152
14	16 384	268 435 456	229 376
16	65 536	4 294 967 296	1 048 576

FFT(n, [a_0, a_1, …, a_{n-1}]):
  if n=1: return a_0
  F_{even} = FFT(n/2, [a_0, a_2, …, a_{n-2}])
  F_{odd} = FFT(n/2, [a_1, a_3, …, a_{n-1}])
  for k = 0 to n/2 – 1:
    $\omega^k = e^{2\pi ik/n}$
    $y^k = F_{even\ k} + \omega^k F_{odd\ k}$
    $y^{k+n/2} = F_{even\ k} - \omega^k F_{odd\ k}$
  return [y_0, y_1, …, y_{n-1}]



References

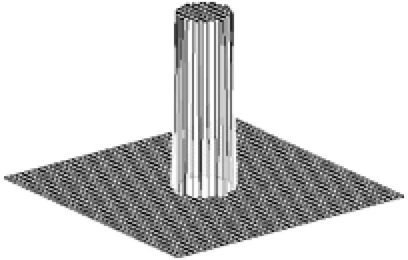
- http://cns-alumni.bu.edu/~slehar/fourier/fourier.html
- https://slideplayer.com/slide/5665338/
- https://2e.mindsmachine.com/asf07.02.html
- https://radiologykey.com/a-walk-through-the-spatial-frequency-domain/
- https://blogs.mathworks.com/steve/2009/12/04/fourier-transform-visualization-using-windowing/
- https://photo.stackexchange.com/questions/40401/what-does-frequency-mean-in-an-image
- http://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm
- http://paulbourke.net/miscellaneous/imagefilter
- https://www.cs.unm.edu/~brayer/vision/fourier.html

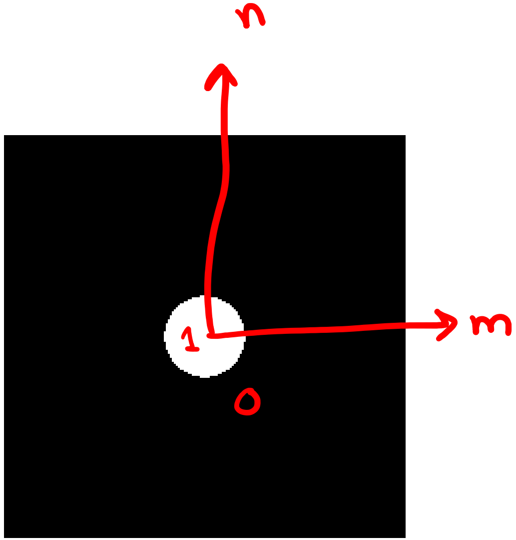
Image Enhancement and Filtering in Frequency Domain

Ideal Low Pass Filters

$$I \longrightarrow \underbrace{F .* H} \longrightarrow I$$

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$





where $D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$

$D_0 \rightarrow$ cut off frequency

Image courtesy: Gonzalez and Woods

Ideal Low Pass Filters

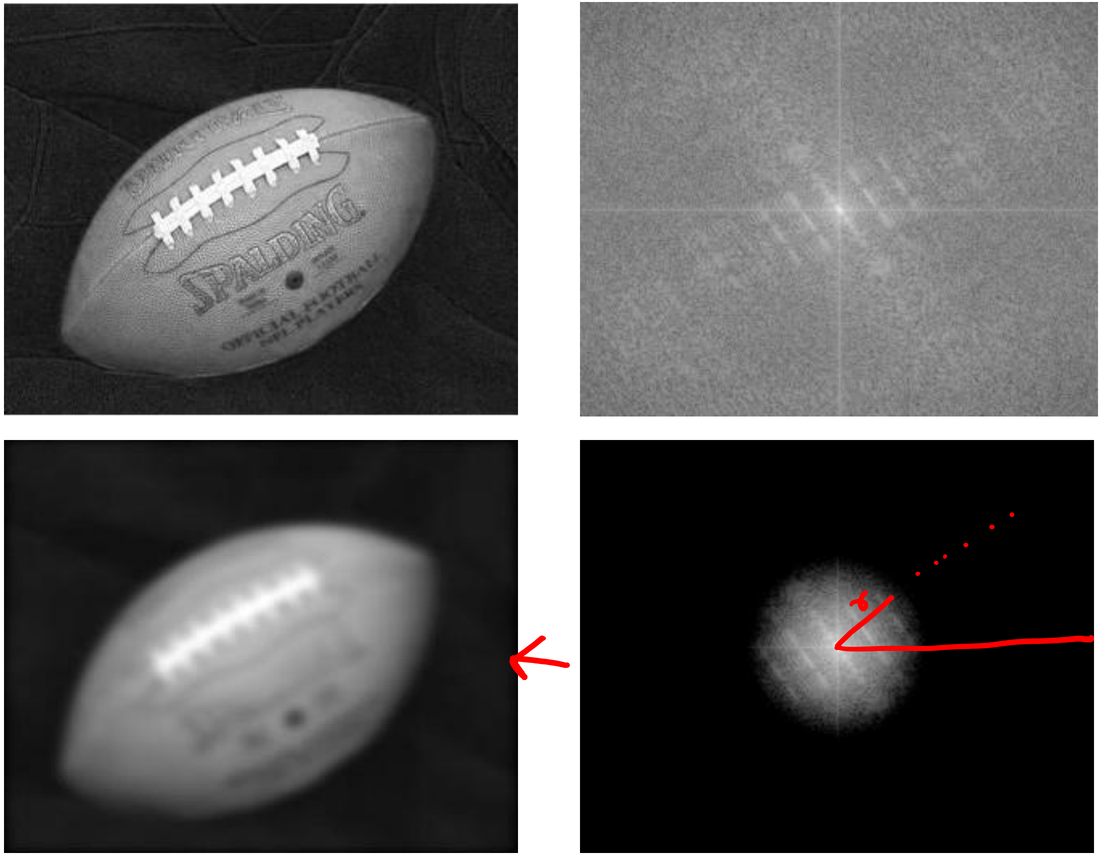
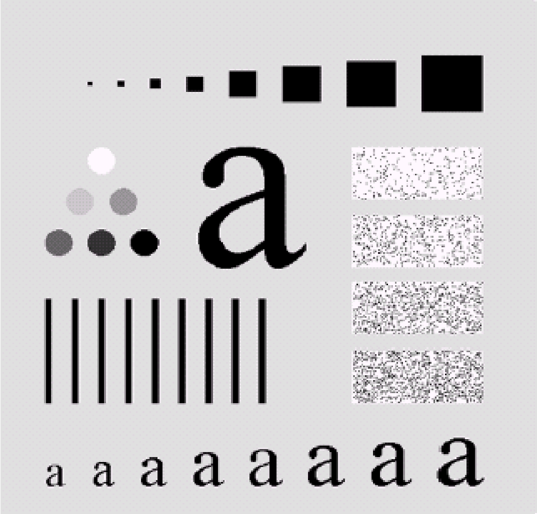
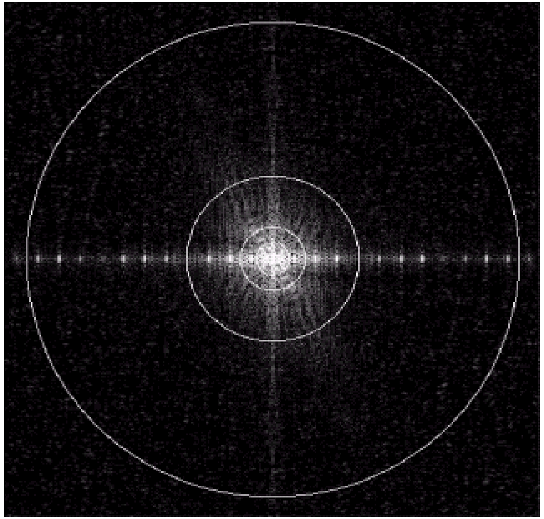


Image courtesy: cs.uregina.ca

Ideal Low Pass Filters

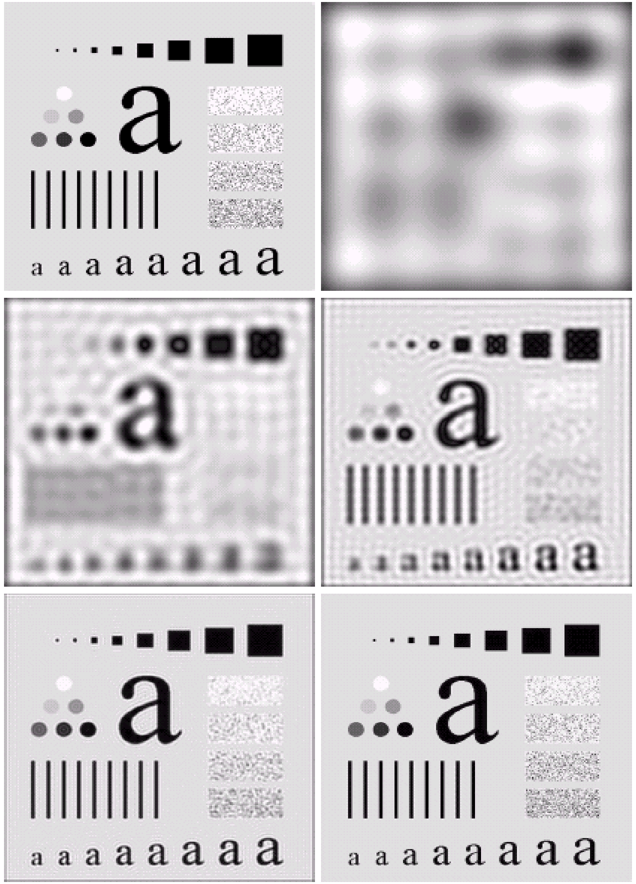




Radii 10,30,60,160 and 460 → power 87, 93.1, 95.7, 97.8 and 99..2

Image courtesy: Gonzalez and Woods

Ideal Low Pass Filters



ILPF radius 10

ILPF radius 30

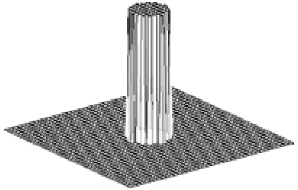
ILPF radius 60

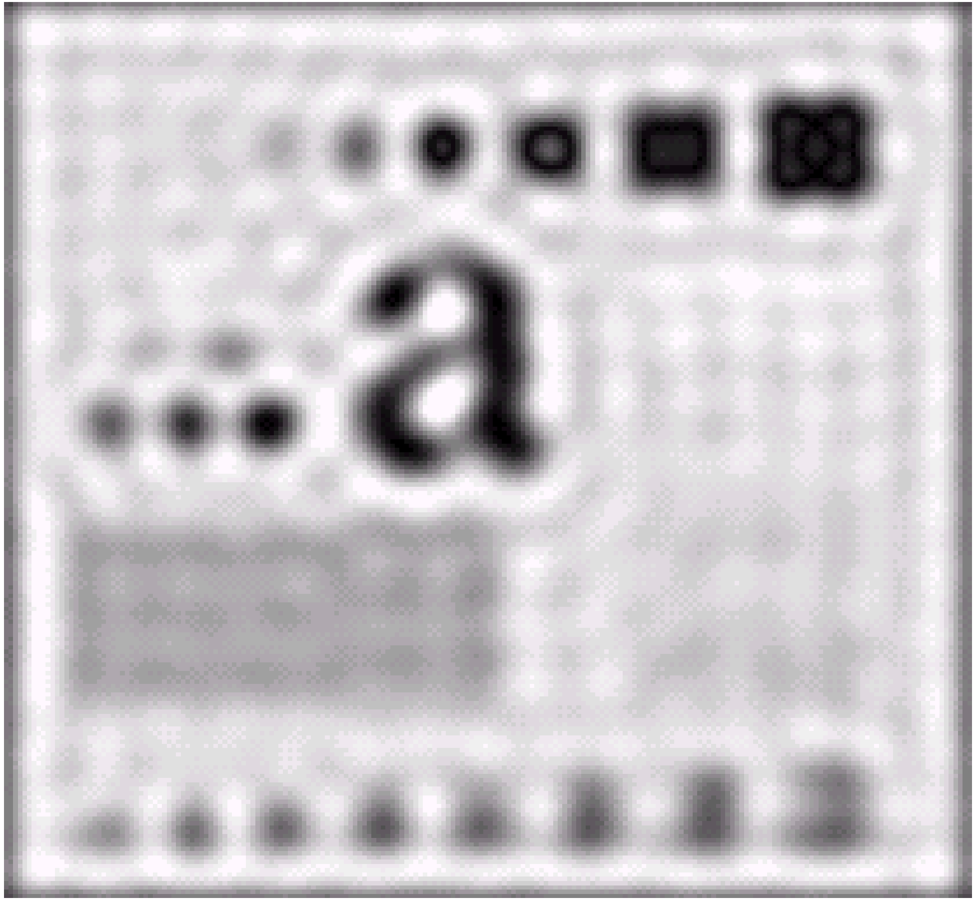
ILPF radius 160

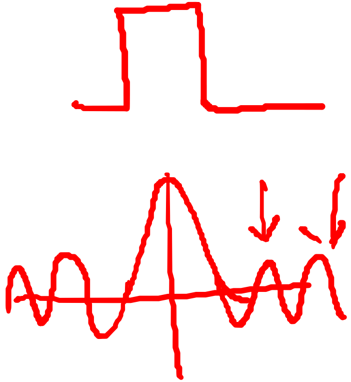
ILPF radius 460

Image courtesy: Gonzalez and Woods

Ideal Low Pass Filters



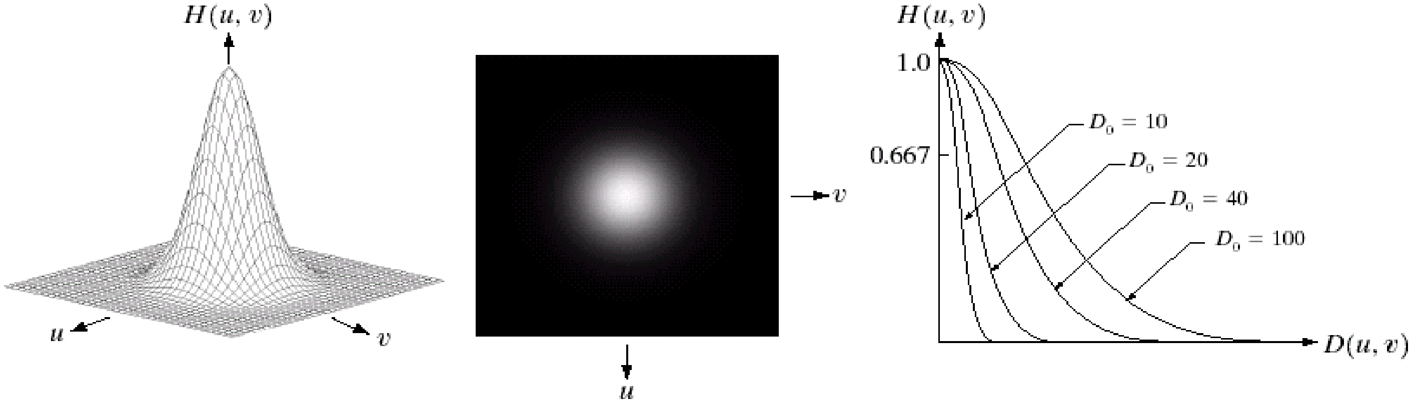




ILPF radius 30

Image courtesy: Gonzalez and Woods

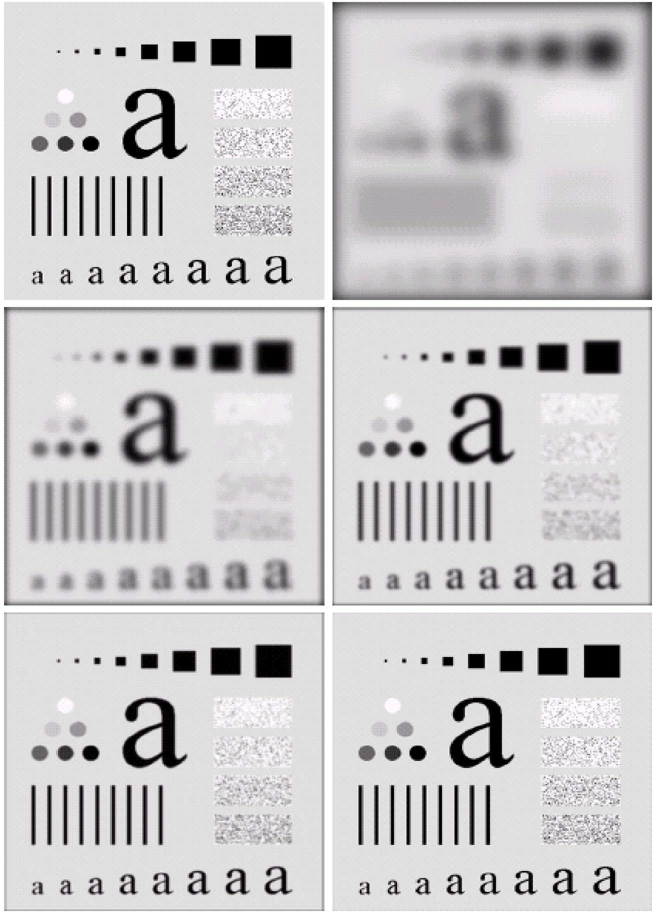
Gaussian Low Pass Filters



$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

Image courtesy: Gonzalez and Woods

Gaussian Low Pass Filters (GLPF)



GLPF cut off frequency 10

GLPF cut off frequency 30

GLPF cut off frequency 60

GLPF cut off frequency 160

GLPF cut off frequency 460

Image courtesy: Gonzalez and Woods

Comparison (ILPF, BLPF, GLPF)

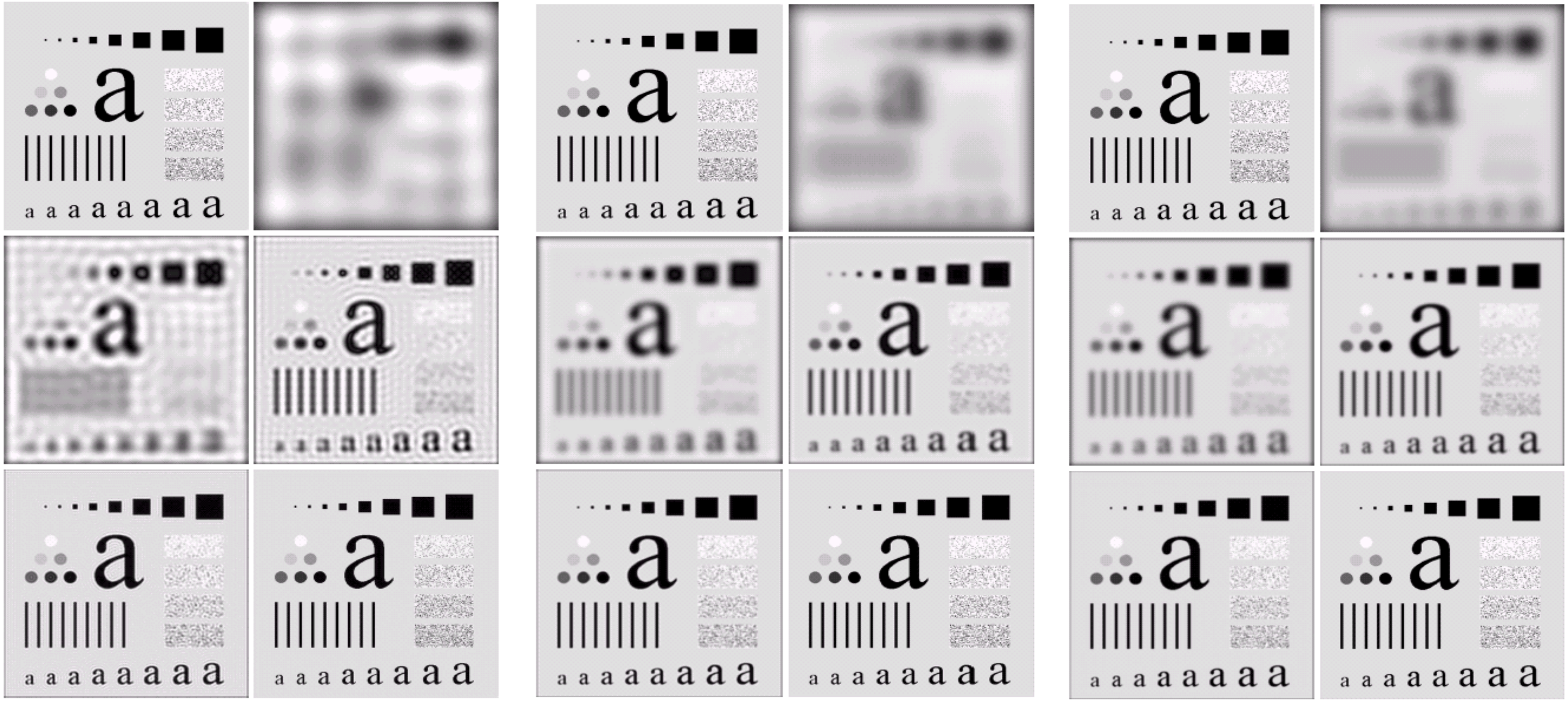


Image courtesy: Gonzalez and Woods

Low pass filtering application

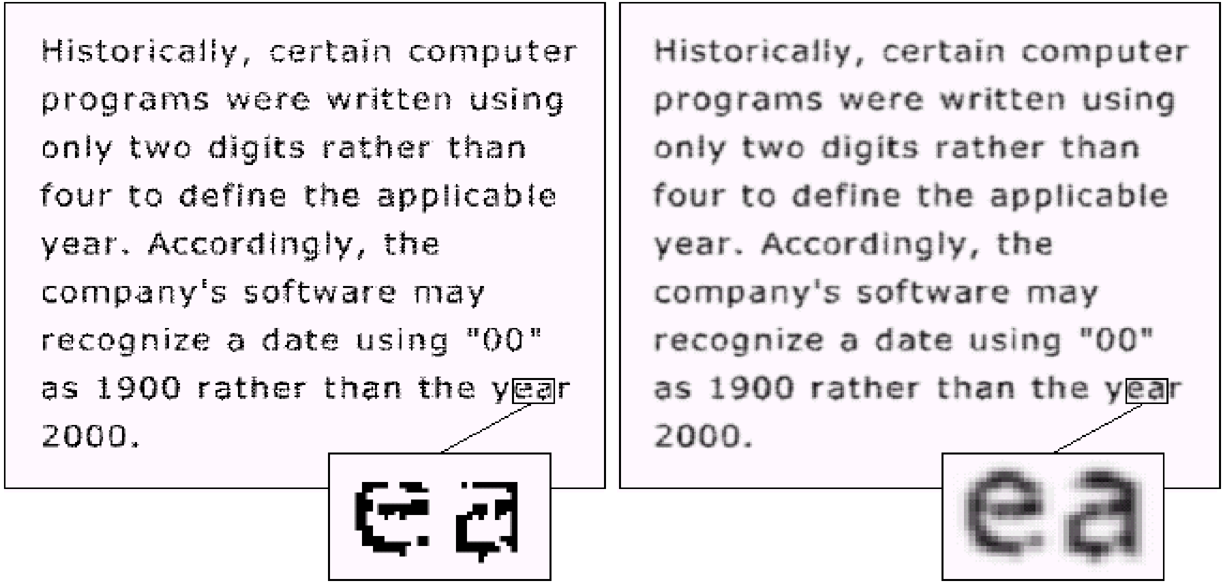


Image courtesy: Gonzalez and Woods

References

- G & W (4.5.1, 4.5.2, 4.5.5, 4.6 – 4.11)
- http://mstrzel.eletel.p.lodz.pl/mstrzel/pattern_rec/fft_ang.pdf

Scribe List

2018101097
2018101099
2018101106
2018101110
2018102003
2018102005