Recall Convex Sets! Let us Play with Examples!

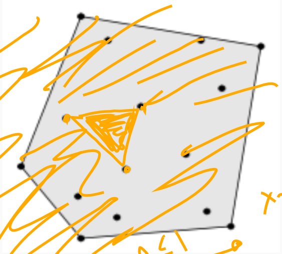
$$\boxed{y = x_2 + \theta(x_1 - x_2)}$$

Def. (Convex Set). A set is convex if every line between two points stays in the set.

$\theta_1 x_1 + \theta_2 x_2$

$$\theta x_1 + (1-\theta)x_2, \quad 0 \le \theta \le 1$$

$\theta \geq 0$



Another view: $y = x_2 + \theta(x_1 - x_2)$

- $y =$ base point x_2 going in the direction $x_1 - x_2$ with θ length
- $\theta = 0,\ y = ?, \quad \theta = 1,\ y = ?$ x_2 x_1

$\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3$

$x_2 + \frac{1}{2}(x_1 - x_2) = \frac{x_1 + x_2}{2}$

$\theta \le 1$

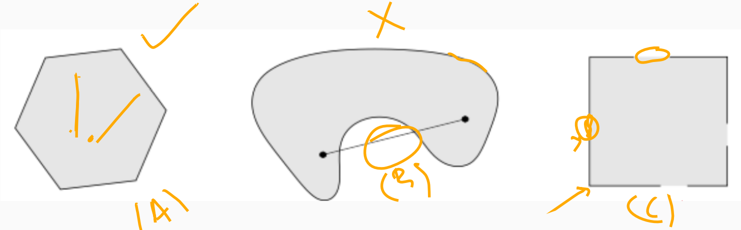
More generally: A set is convex if all convex combinations lie in the set

$$\sum_{i=1} \theta_i x_i = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \cdots \theta_n x_n, \quad \sum_{i=1} \theta_i = 1, \quad \theta_i \ge 0,\ \forall i$$

$\theta(x_1 - x_2)$

Test: Which of these are convex?

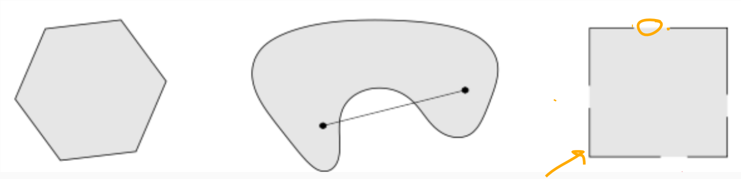
Quiz Which of the following are convex sets in $\mathbb{R}^2$?



Answer: yes Answer no Answer no

Test: Which of these are convex?

Quiz Which of the following are convex sets in $\mathbb{R}^2$?



Answer: Answer Answer

- What if the square was round?

circle

Can we make a non-convex set convex?

Figure 2: Convexification of convex sets

Can we make a non-convex set convex?

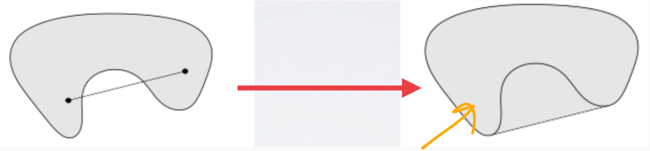


Figure 2: Convexification of convex sets

S Convexify T

Convexification:

- Consider arbitrary points x, y in convex set C
- Make sure that all the points on the line joining x and y is in C
- It is the smallest convex superset

T: smallest convex set contains S

Examples of Convex Sets: Hyperplane

Hyperplane: $\{x \mid a^T x = b\} = H$

a, b

$a_1x_1 + a_2x_2 = b$

$a_1x_1 + a_2x_2 + a_3x_3 = b$

let $x_1, x_2 \in H$ a, b

$0 \le \theta \le 1$

$y = \theta x_1 + (1-\theta)x_2 \in H$

Since $x_1, x_2 \in H$

$a^T x_1 = b, \quad a^T x_2 = b$

$a^T y = a^T[\theta x_1 + (1-\theta)x_2]$

$= \theta a^T x_1 + (1-\theta) a^T x_2$

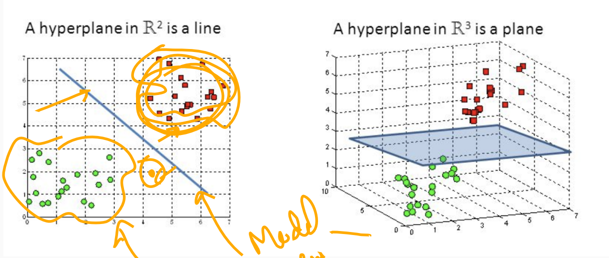
$= \theta b + (1-\theta) b$

$= b$

$\Rightarrow \boxed{y \in H}$

Examples of Convex Sets: Hyperplane

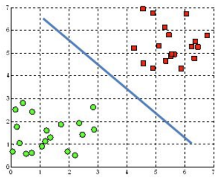
Hyperplane: $\{x \mid a^T x = b\}$

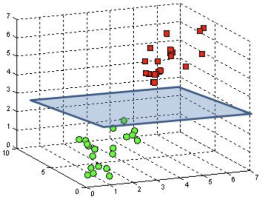


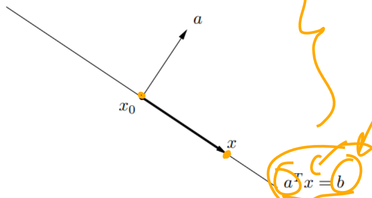
Model learn

Examples of Convex Sets: Hyperplane

Hyperplane: $\{x \mid a^T x = b\}$







$\vec{x}$ $\vec{n} = 0$

$\boxed{\vec{n} \cdot \vec{p} = a}$

b

Quiz: Is Hyperplane convex?

Example of Convex Set: Half Space

Half Space: $\{x \mid a^T x \geq b\} = H$

$x_1, x_2 \in H$

$a^T x_1 \geq b$

$a^T x_2 \geq b$

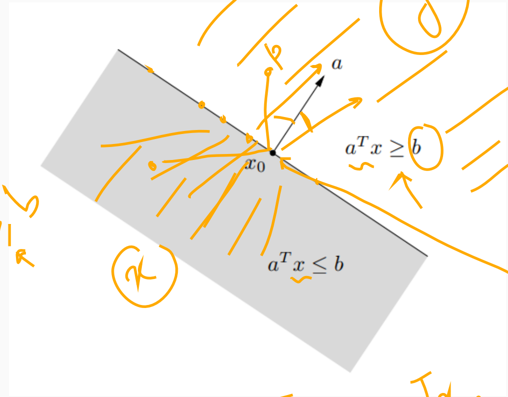
$a^T(\theta x_1 + (1-\theta) x_2) \geq b$

$= \theta a^T x + (1-\theta) a^T x_2$

$\geq \theta b + (1-\theta) b = b$

Example of Convex Set: Half Space

Half Space: $\{x \mid a^T x \geq b\}$



$a^T x = \langle a, x \rangle$

$a^T x = \|a\| \|x\| \cos\angle(a, x)$

$a^T x_0 = b$

$a^T x \geq a^T x_0$

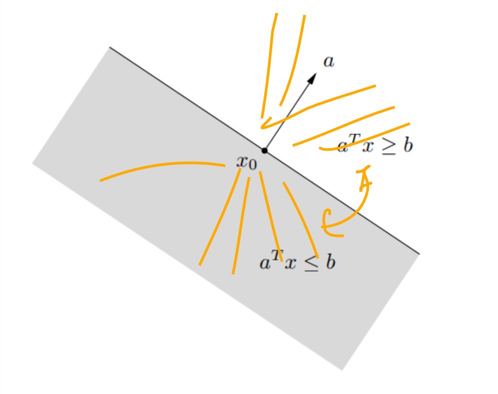
$\Rightarrow a^T(x - x_0) \geq 0$

$a^T(x - x_0) \leq 0$

$a^T(y - y_0) \geq 0$

Example of Convex Set: Half Space

Half Space: $\{x \mid a^T x \geq b\}$



$\mathbb{R}^n$

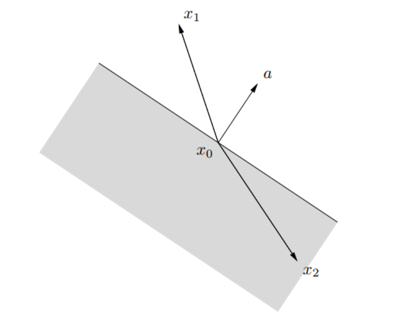
$\mathbb{R}^{n-1}$

Quiz: Is half space convex?

Quiz: What is the difference between half-space and hyperplane?

Which Points are in a Half space?

Half Space: $\{x \mid a^T(x - x_0) \leq 0\}$, $\quad b = a^T x_0$

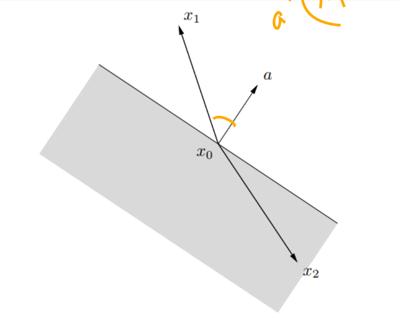


Quiz: Why x_1 is not in half-space?

Which Points are in a Half space?

Half Space: $\{x \mid a^T(x - x_0) \le 0\}, \quad b = a^T x_0$

$a^T(x_1 - x_0) > 0$

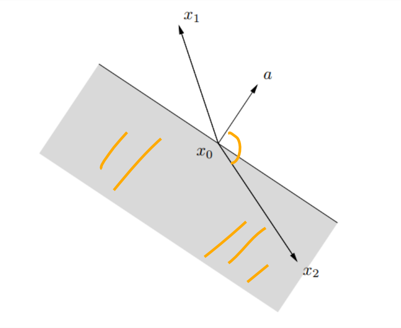


Quiz: Why x_1 is not in half-space? $x_1 - x_0$ makes acute angle with a

Quiz: Why x_2 is not in half-space?

Which Points are in a Half space?

Half Space: $\{x \mid a^T(x - x_0) \leq 0\}, \quad b = a^T x_0$

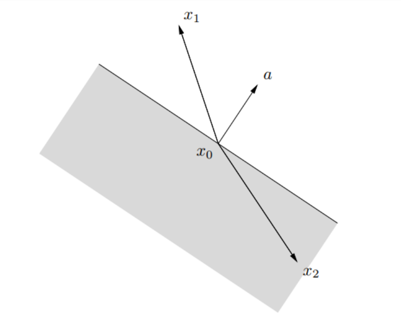


Quiz: Why x_1 is not in half-space? $x_1 - x_0$ makes acute angle with a

Quiz: Why x_2 is ~~not~~ in half-space? $x_2 - x_0$ makes obtuse angle with a

Which Points are in a Half space?

Half Space: $\{x \mid a^T(x - x_0) \leq 0\}, \quad b = a^T x_0$



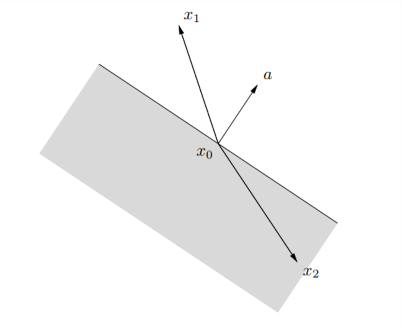
Quiz: Why x_1 is not in half-space? $x_1 - x_0$ makes acute angle with a

Quiz: Why x_2 is not in half-space? $x_2 - x_0$ makes obtuse angle with a

Conclusion: All vectors that make obtuse angle with a are in half space

Which Points are in a Half space?

Half Space: $\{x \mid a^T(x - x_0) \leq 0\}, \quad b = a^T x_0$



Quiz: Why x_1 is not in half-space? $x_1 - x_0$ makes acute angle with a

Quiz: Why x_2 is not in half-space? $x_2 - x_0$ makes obtuse angle with a

Conclusion: All vectors that make obtuse angle with a are in half space

Open Half space: The above is closed half space, the open half space is defined as

$$\{x \mid a^T(x - x_0) < 0\}$$

Example of Convex Set: Spheres

Norm Balls: Consider the norm balls

$$S = \{x \in \mathbb{R}^n \mid \|x\|_\ell = 1\}$$

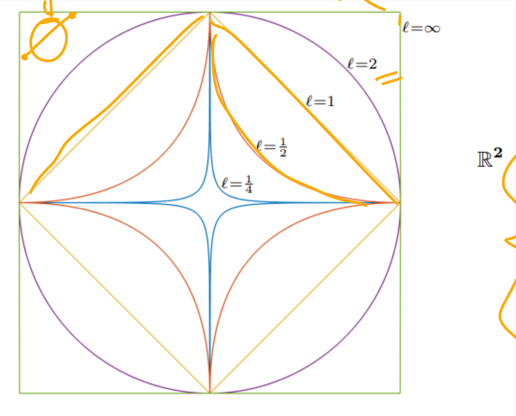
$\|x\|_2 = 1$

$\|x\|_1 = 1$

$x_1 + x_2 + \dots + x_n = 1$

$\|x\|_\ell = (x_1^\ell + x_2^\ell + \dots + x_n^\ell)^{1/\ell}$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$



$\mathbb{R}^2$

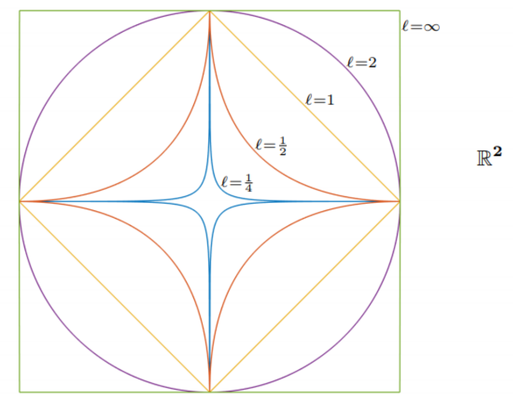
$$\begin{cases} \|x\|_\ell \geq 0 \\ \|x\|_\ell = 0 \text{ iff } x = 0 \\ \|x+y\|_\ell \leq \|x\| + \|y\| \end{cases}$$

Quiz: For which ℓ spheres are convex?

Example of Convex Set: Spheres

Norm Balls: Consider the norm balls

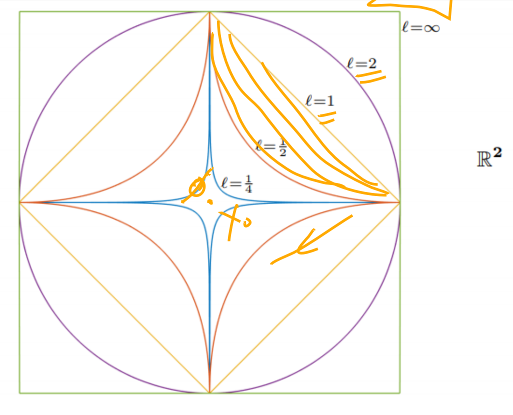
$$\{x \in \mathbb{R}^n \mid \|x\|_\ell = 1\}$$



Quiz: For which ℓ spheres are convex? Answer: None

Example of Convex Set: Norm Balls

Sphere: $\{x \in \mathbb{R}^n \mid \|x - x_0\| \leq 1\}$

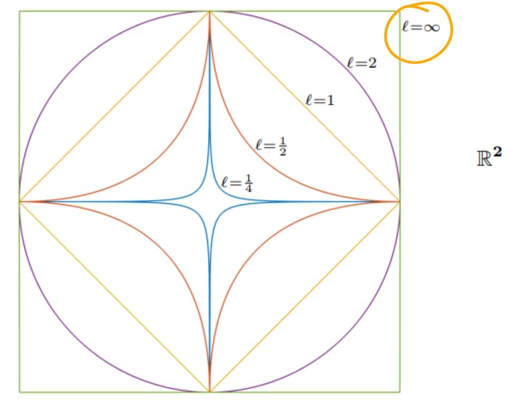


Quiz: For which ℓ, norm balls are convex?

Example of Convex Set: Norm Balls

$\|x\|_\infty = \max\{|x_1|, x_2, \cdots\}$

Sphere: $\{x \in \mathbb{R}^n \mid \|x - x_0\| \leq 1\}$



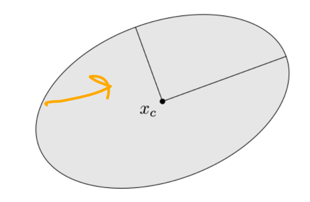
Quiz: For which ℓ, norm balls are convex? Answer: $\ell = 1, 2, 3 \ldots \cdots \infty$

Lecture-5: Gradients and Geometry-II

Example of convex set: Ellipsoid

Ellipsoid: $\{x \mid (x - x_c)^T P (x - x_c) \leq 1\}$

- P is symmetric positive definite



$x_1, x_2 \in \mathcal{E}$

$(x_1 - x_c)^T P (x_1 - x_c) \leq 1 \to 1$

$(x_2 - x_c)^T P (x_2 - x_c) \leq 1 \to 2$

$(\theta x_1 + (1-\theta)x_2 - x_c)^T P (\theta x_1 + (1-\theta)x_2 - x_c)$

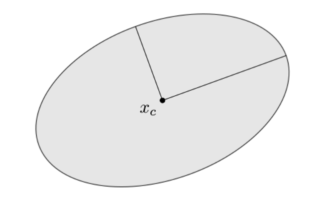
$=$

≤ 1

Example of convex set: Ellipsoid

Ellipsoid: $\{x \mid (x - x_c)^T P(x - x_c) \leq 1\}$

- P is symmetric positive definite



- Matrix P determines how far the ellipsoid extends in every direction from center x_c

Example of convex set: Ellipsoid

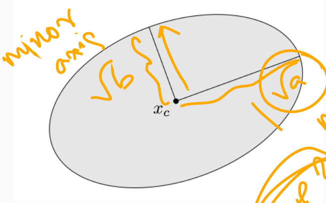
$P \in \mathbb{R}^{2\times 2}$ $\quad \boxed{\mathbb{R}^2}$ $\quad P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Ellipsoid: $\{x \mid (x - x_c)^T P (x - x_c) \le 1\}$

$x_c = 0$ $\quad P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

- P is symmetric positive definite

$\mathrm{Eig}(P) = \{a, b\}$



$\mathbb{R}^2$

$x^T P x$

$\frac{x_1^2}{(\sqrt{b})^2} + \frac{x_2^2}{(\sqrt{a})^2} = 1$

- Matrix P determines how far the ellipsoid extends in every direction from center x_c
- The length of semi-axes of ellipsoid are given by $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P

$a > b$

$\underbrace{[x_1, x_2]\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = 1 \Rightarrow ax_1^2 + bx_2^2 = 1$

$\Rightarrow \frac{x_1^2}{b} + \frac{x_2^2}{a} = 1$

Example of convex set: Ellipsoid

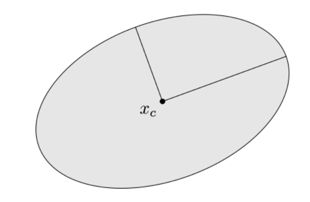
Ellipsoid: $\{x \mid (x - x_c)^T P(x - x_c) \leq 1\}$

- P is symmetric positive definite

$P = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} = a^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = a^2 I$

r^2

$P = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, r^2 $\boxed{a < 0}$ $\times$



- Matrix P determines how far the ellipsoid extends in every direction from center x_c
- The length of semi-axes of ellipsoid are given by $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P
- A ball is an ellipsoid with $P = r^2 I$

Example of convex set: Ellipsoid

Ellipsoid: $\{x \mid (x-x_c)^T P (x-x_c) \leq 1\}$

- P is symmetric positive definite

P is SPD

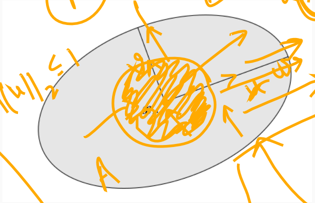
$\boxed{x^T P x \geq 0}$ $\forall x \neq 0$

$P = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ not SPD

$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$\|u\| \leq 1$

$\|u\| = 1$



- Matrix P determines how far the ellipsoid extends in every direction from center x_c
- The length of semi-axes of ellipsoid are given by $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P
- A ball is an ellipsoid with $P = r^2 I$
- Another representation of Ellipsoid

$$\{x_c + Au \mid \|u\|_2 \leq 1\}, \quad A \text{ is } SPD$$