Topics in Applied Optimization

Optimization Algorithms for ML and Data Sciences

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Conjugates of Some Convex Functions on $\mathbb{R}$

$f(x)$

$f^*(y) = \sup_x \left(y^T x - f(x) \right)$

f^* is the conjugate of f

Find the conjugates of the following functions:

- Affine function: $f(x) = ax + b$.
- Negative logarithm: $f(x) = -\log x$
- Exponential. $f(x) = e^x$
- Negative Entropy. $f(x) = x \log x$
- Inverse. $f(x) = 1/x$

See classnotes for solutions.

Solution to previous problem...

Find conjugate of the affine function $f(x) = ax + b$

done

Solution to previous problem...

Find conjugate of the negative logarithm $f(x) = -\log x$

done

What is $\underline{\max}\ g(x) = xy - e^x$?

$g'(x) = y - e^x = 0$

$\Rightarrow y = e^x$

$\Rightarrow \boxed{x = \log y}$.

$g''(x)\Big|_{x=\log y} = -e^x < 0$

$\Rightarrow x = \log y$ is $\underline{\text{maxima}}$.

$\underline{\text{For } y > 0}$

$xy - e^x$

as $x \to \infty \quad \infty (-) \to \infty$

$x \to -\infty \quad \left.\begin{matrix} xy \to -\infty \\ -e^x \to 0 \end{matrix}\right\} \quad xy - e^x \to -\infty$

$$\boxed{f^*(y) = \log y \cdot y - y}$$

Solution to previous problem...

Find conjugate of the exponential $f(x) = e^x$

$f : \mathbb{R} \to \mathbb{R}$

$$f^*(y) = \sup_x \left(y^T x - f(x) \right)$$

$$= \sup_x \left(yx - e^x \right)$$

$\operatorname{dom} f = \mathbb{R}$

Cases (i) $y > 0$

(ii) $y < 0$

(iii) $y = 0$

For $y < 0$

$g(x) := yx - e^x$

As $x \to \infty \Rightarrow yx \to -\infty$

& $-e^x \to -\infty$

As $x \to -\infty$

$yx \to +\infty$

$-e^x \to 0$

$g(x) \to +\infty$

For $y < 0$ $f^*(y)$ is not defined.

For $y > 0$:

Solution to previous problem...

Find conjugate of the negative entropy $f(x) = x \log x$

$f : \mathbb{R}_{++} \to \overline{\mathbb{R}}$

$\text{dom } f = \mathbb{R}_{++}$

$$f^*(y) = \sup_x \left(yx - x \log x \right)$$

$$g(x) = yx - x \log x$$

$y > 0$ as $x \to \infty$

$yx \to +\infty$

$x \log x \to +\infty$

$$g'(x) = y - \left[x \cdot \frac{1}{x} + \log x \cdot 1 \right]$$

$$= y - 1 - \log x \qquad \left. g''(x) \right|_{x = e^{y-1}} = -\frac{1}{x} < 0$$

$$g'(x) = 0 \Rightarrow y - 1 = \log x \Rightarrow x = e^{y-1}$$

$\Rightarrow x = e^{y-1}$ is a maxima.

$$\Rightarrow f^*(y) = \cancel{y \cdot e^{y-1}} - \cancel{e^{y-1}(y-1)}$$

$$= e^{y-1}$$

Solution to previous problem...

Find conjugate of the inverse function $f(x) = 1/x$

Thats All for Convex Functions!

To summarize:

Boyd Convex Opt.

convex
(and strictly convex)

concave
(and strictly
concave)

neither convex
nor concave

both convex and
concave (but not
strictly)

Convex Optimization Problems

Convex Optimization Problems

Optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{aligned}
$$

Convex Optimization Problems

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- Find an x that minimizes $f_0(x)$ among all x that satisfy
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- x is called optimization variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is called objective function or cost function loss

Convex Optimization Problems

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- The inequalities $f_i(x) \leq 0$ are called inequality constraints
- $f_i : \mathbb{R}^n \to \mathbb{R}$ are called inequality constraint functions $i \neq 0$

Convex Optimization Problems

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- The equations $h_i(x)$ are called equality constraints
- The functions $h_i : \mathbb{R}^n \to \mathbb{R}$ are called equality constraint functions

Convex Optimization Problems

Optimization problem:

$$\text{minimize} \quad f_0(x) \tag{1}$$

$$\text{subject to} \quad f_i(x) \leq 0, i = 1, \ldots, m \tag{2}$$

$$h_i(x) = 0, i = 1, \ldots, p \tag{3}$$

Convex Optimization Problems

Optimization problem:

$$\begin{aligned} &\text{minimize} && f_0(x) && (1)\\ &\text{subject to} && f_i(x) \leq 0, i = 1, \ldots, m && (2)\\ & && h_i(x) = 0, i = 1, \ldots, p && (3) \end{aligned}$$

- If no constraints, i.e., $m = p = 0$, then called unconstrained problem

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- Domain of opt. problem: where objective and constraint are defined

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i \quad \cap \quad \bigcap_{i=1}^{p} \mathbf{dom}\ h_i$$

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$$\begin{aligned} f_i(x) \leq 0, \quad & i = 1, \ldots, m \\ h_i(x) = 0, \quad & i = 1, \ldots, p \end{aligned}$$

- The optimization problem is called feasible if there exists atleast one feasible point

Convex Optimization Problems

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- Feasible set or constraint set: set of all feasible points

Convex Optimization Problems

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- Feasible set or constraint set: set of all feasible points
- Optimal Value: The optimal value p^* defined as

$$p^* = \inf \{ f_0(x) \mid f_i(x) \le 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p \}$$

Convex Optimization Problems

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- p^* is allowed to take extended values $\pm\infty$

Convex Optimization Problems

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infeasible

This set is ϕ.

$\inf(\phi) := \infty$

$\sup(\phi) := -\infty$

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- **Infeasible problem:** problem is called infeasible when $\boxed{p^* = \infty}$

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$\exists\, x_k$ seq.

- p^* is allowed to take extended values $\pm\infty$
- **Infeasible problem:** problem is called infeasible when $p^* = \infty$
  - Note: we used the fact that $\inf \phi = \infty$
- **Unbounded below:** Problem is unbounded below if $f_0(x_k) \to -\infty$ as $k \to \infty$

Optimal and locally optimal

Optimization problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned}$$

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- Optimal Point: x^* is called optimal if it solves the given optimization problem, i.e.,

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- Optimal Set: The set of all optimal points is called optimal set
- If there exists an optimal point, then we say that optimal value is achieved and the problem is solvable
- If the optimal set is empty, then we say that optimal value is not attained

Optimal and Locally Optimal Points

Optimization problem:

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Optimal and Locally Optimal Points

$\ell(f_{NN}(x) - y)$

$f_0(\theta) = \|f_{NN}(\theta) - y\|_2 \geq 0$

$\boxed{p^* = 0}$

Optimization problem:

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$p^* + \epsilon$

ϵ

$x_3 \; x_2 \; x_1$

p^*

- $\epsilon-$suboptimal: $\bar{x}$ is $\epsilon-$suboptimal if $f_0(\bar{x}) \leq p^* + \epsilon$, $\epsilon > 0$

min

$f_0(\bar{x})$

$f_0(\theta^k)$

ϵ

$=$

10^{-9}

Optimal and Locally Optimal Points

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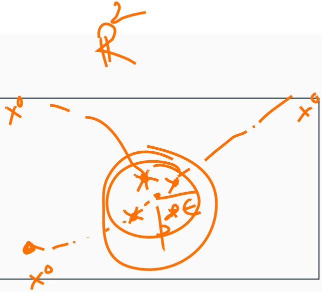

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- $\epsilon-$suboptimal set: set of all $\epsilon-$suboptimal points

Optimal and Locally Optimal Points

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- Active constraint: If x is feasible and $f_i(x) = 0$, then ith inequality is active

Optimal and Locally Optimal Points

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- Inactive constraint: If x is feasible and $f_i(x) < 0$, then this constraint is inactive

Optimal and Locally Optimal Points

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- $\epsilon-$suboptimal set: set of all $\epsilon-$suboptimal points
- Active constraint: If x is feasible and $f_i(x) = 0$, then ith inequality is active
- Inactive constraint: If x is feasible and $f_i(x) < 0$, then this constraint is inactive
- Redundant constraint: A constraint is redundant if removing it does not change the feasible set

Optimization problem:

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Define the feasible set

$$\Omega = \{x \mid f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad h_i(x) = 0 \quad i = 1, \ldots, p\}$$

Optimization problem:

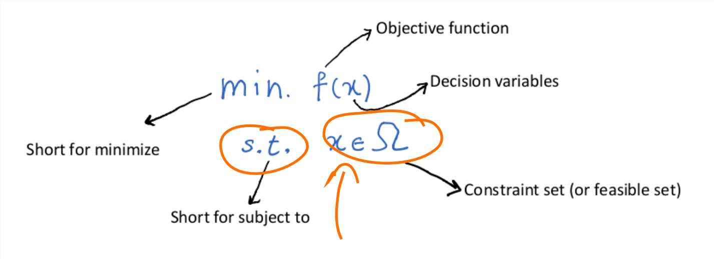
$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & \left\{ \begin{aligned} & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned} \right. \end{aligned}$$

wide all

Define the feasible set

$$\Omega = \{x \mid \underline{f_i(x) \leq 0}, \quad i = 1, \ldots, m, \quad \underline{h_i(x) = 0} \quad i = 1, \ldots, p\}$$

More compactly, we we can write:



Examples: $1/x$

Consider the optimization problem:

$$\text{minimize } f_0(x) = 1/x,$$
$$\text{subject to } x \in \mathbb{R}$$

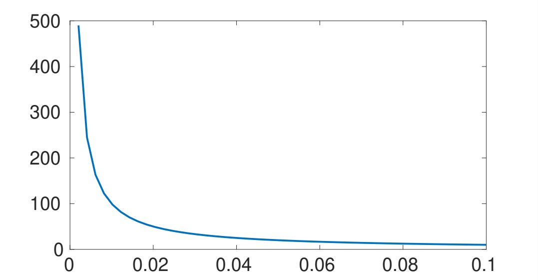
where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set? $\mathbb{R}_{++}$

Quiz: What is p^*? $p^* = 0$

Quiz: Is the optimal value acheived? No

Figure 1: Plot of $1/x$



Examples: $-\log x$

$x \in \mathbb{R}_{++}$

$f_0'(x) = -\frac{1}{x} < 0$

$\Rightarrow$ f_0 is monotonically decr.

Consider the optimization problem:

$$\text{minimize } f_0(x) = -\log x,$$

$$\text{subject to } x \in \mathbb{R}$$

where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

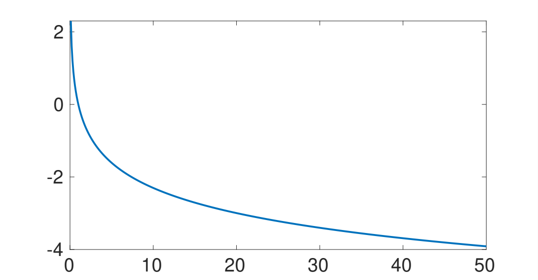
Quiz: What is feasible set? $\mathbb{R}_{++}$

Quiz: What is p^*? $\cancel{\emptyset}$ $-\infty$

Quiz: Is the optimal value acheived? No

Quiz: Is this problem bounded below? No

Figure 2: Plot of $-\log x$



Examples: $x \log x$

$f_0(x) = x \log x$

$f_0'(x) = x \cdot \frac{1}{x} + \log x \cdot 1 = 1 + \log x$

$f''(x) = \frac{1}{x} > 0 \quad \forall x > 0$

$f_0'(x) = 0$

$\Rightarrow 1 + \log x = 0$

$\Rightarrow \log x = -1$

$\Rightarrow x = e^{-1}$

Consider the optimization problem:

$$\text{minimize } f_0(x) = x \log x,$$
$$\text{subject to } x \in \mathbb{R}$$

where $f_0 : \mathbb{R}_{++} \to \mathbb{R}$

Quiz: What is feasible set? $\mathbb{R}_{++}$

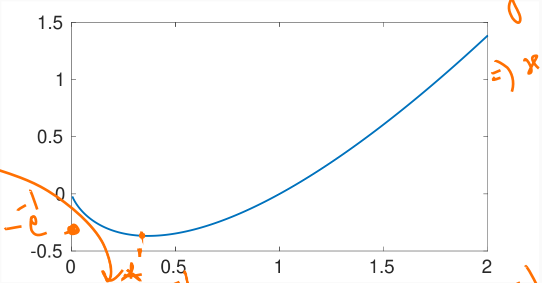
Quiz: What is p^*? $-e^{-1}$

Quiz: Is the optimal value acheived? yes x^*

Quiz: Is this problem bounded below? ✓

Quiz: What is optimal point? ✓

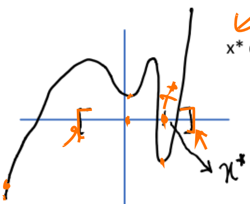
Figure 3: Plot of $x \log x$



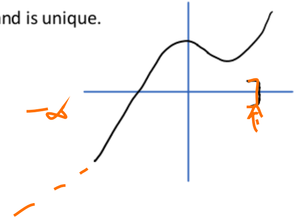
$\to x = e^{-1}$ is the min.

$f_0(x^*) = e^{-1} \log(e^{-1}) = -e^{-1}.$

Examples: Graphically



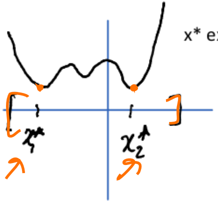
x* exists and is unique.



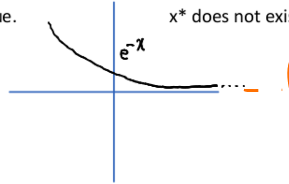
x* does not exist.

$f^* = -\infty$

Problem is "unbounded."



x* exists, but not unique.



x* does not exist.

$f^* = 0$

Expressing Problems in Standard Form

$\overbrace{x_1^2 + x_2^2}^{f_1} \leq x_3$ $\quad x$ $\quad \underbrace{x_1^2 + x_2^2 - x_3}_{f_1} \leq 0$

Optimization problem (Standard Form):

$$
\begin{array}{rl}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, i = 1, \ldots, m \\
 & h_i(x) = 0, i = 1, \ldots, p
\end{array}
$$

Expressing Problems in Standard Form

Optimization problem (Standard Form):

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, p \end{aligned}$$

Convention for standard form:

- righthand side of the inequality and equality constraints are zero

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Convention for standard form:

- righthand side of the inequality and equality constraints are zero
  - For example, $g_i(x) = \tilde{g}_i(x)$ can be written as $h_i(x) = 0$

$h_i = g_i - \tilde{g}_i$

$h_i = 0 \iff g_i = \tilde{g}_i$

Expressing Problems in Standard Form

$f_i(x) \geq 0$

$-f_i(x) \leq 0$

$\tilde{f}_i(x) \leq 0$

Optimization problem (Standard Form):

$$
\begin{array}{rl}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \ i = 1, \dots, m \\
& h_i(x) = 0, \ i = 1, \dots, p
\end{array}
$$

Convention for standard form:

- righthand side of the inequality and equality constraints are zero
  - For example, $g_i(x) = \tilde{g}_i(x)$ can be written as $h_i(x) = 0$
- $f_i(x) \geq 0$ as $-f_i(x) \leq 0$

(Box Constraints). Consider the following

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & \underbrace{l_i \le x_i \le u_i}, \quad \underbrace{i = 1, \dots, n} \end{aligned}$$

- constraints here are called variable bounds or box constraints

$f_i(x) \le 0$

$x \in \mathbb{R}^2$

$f_0(x)$

$a \le x_1 \le b$

$c \le x_2 \le d$

$2n$

$-x_1 + l_1 \le 0$

$x_1 - u_1 \le 0$

minimize $f_0(x)$

$\min f_0(x)$

s.t. $h_k(x) \le 0$

$k = 1, \dots, 2n$

$l_i \le x_i \le u_i$

h_i

$-x_i \le -l_i \Rightarrow \underbrace{-x_i + l_i \le 0}$, $i = 1, \dots n$

$k = i$

$\Rightarrow \begin{cases} x_i \ge l_i \Rightarrow \\ x_i \le u_i \Leftarrow \end{cases}$

$\Rightarrow \underbrace{x_i - u_i \le 0}$ $j = n+1 \dots 2n$

h_j $k = n+i$

$n+2, \dots$

$j = n - i$

$\cancel{k = n+i}$